COMBINATORIAL ARRANGEMENTS ANALOGOUS TO ORTHOGONAL ARRAYS

By C. RADHAKRISHNA RAO
Indian Statistical Institute

SUMMARY. In this paper are considered some combinatorial arrangements analogous to orthogonal arrays introduced by the author several years ago (Rao, 1946a). In an orthogonal array, all combinations of s elements taken d at a time allowing repetitions occur an equal number of times in every set of d rows of the array. By not allowing repetitions and relaxing the condition that the combinations should be ordered certain new arrangements called orthogonal arrays of Type I and Type II have been obtained. Their relationships with orthogonal latin squares and their use in the construction of BIB designs have been discussed.

1. Introduction

In a recent paper the author (Rao, 1961a) considered the construction of BIB designs with replications 11 to 15. Some solutions depended on certain types of arrangements analogous to orthogonal arrays introduced earlier (Rao, 1946a, p.134), developed and applied in the construction of confounded factorial designs (Rao, 1946b, 1947, 1948). The recent significant work of Bose, Shrikhande and Parker (1960) and Parker (1958) on the falsity of Euler's conjecture involved the concepts of orthogonal arrays and closely related arrangements which may be called orthogonal arrays of Type I. The object of the present paper is to examine certain other arrangements called orthogonal arrays of Type II which have been found useful in the construction of dicyclic solutions to BIB designs. (Rao, 1961a).

2. Different types of orthogonal arrays

Consider a set S of s symbols and a $t \times N$ matrix of elements of S. Such a matrix is called :

(i) *an orthogonal array* of strength d, constraints t and index λ, and represented by (N, t, s, d), if in every set of d rows, the N columns contain each of the s^d ordered combinations of *s elements taken d at a time allowing repetitions*, λ times;

(ii) *an orthogonal array of Type* I, strength d, constraints t and index λ, and represented by (N, t, s, d) : I, if in every set of d rows, the N columns contain each of the $s!/(s-d)!$ ordered combinations of *s elements taken d at a time without repetitions*, λ times;

(iii) *an orthogonal array of Type* II, strength d, constraints t and index λ, and represented by (N, t, s, d) :II, if in every set of d rows, the N columns contain each of the $s!/d!(s-d)!$ combinations of *s elements taken d at a time with order ignored and without repetitions*, λ times.

For example, when $s = 3$ orthogonal arrays of strength 2 are as follows :

	A	A	A	B	B	B	C	C	C
	A	B	C	A	B	C	A	B	C
(9, 4, 3, 2)	A	B	C	B	C	A	C	A	B
	A	B	C	C	A	B	B	C	A

	A	B	C	A	B	C
(6, 3, 3, 2) : I	B	C	A	C	A	B
	C	A	B	B	C	A

	A	B	C
(3, 3, 3, 2) : II	B	C	A
	C	A	B

For orthogonal arrays of strength 2, the minimum value of N and the maximum value of t, for given s, are

array		min N		max t
orthogonal		s^2		$s+1$
,,	type I	$s(s-1)$		s
,,	type II	$\frac{s(s-1)}{2}$	*for s odd*	s
		$s(s-1)$	*for s even*	

In the present paper it is shown that arrays of Type II and strength 2 can be constructed with the optimum values of N and t when a field with s elements, $GF(s)$ exists. For other values of s, the methods used by Parker (1958), and Bose, Shrikhande and Parker (1960) for the construction of orthogonal arrays may be employed.

3. Arrays of type I and strength 2

It is known (Rao, 1946a) that $(s^2, t, s, 2) \Longrightarrow$ the existence of $(t-2)$ mutually orthogonal latin squares (m.o.l.'s). Theorem 1 contains some results establishing the relationship between Type I arrays and m.o.l.'s.

Theorem 1 :
(i) $(t-1)$ *m.o.l.'s of size s* $\Longrightarrow (s(s-1), t, s, 2)$: I
(ii) $(s(s-1), t, s, 2)$: I $\Longrightarrow (t-2)$ *m.o.l.'s of size s*
(iii) $(s^2, t, s, 2)$ $\Longrightarrow (s(s-1), t-1, s, 2)$: I
(iv) $(t-2)$ *m.o.l.'s of size s with a common directrix* $\Longleftrightarrow (s(s-1), t, s, 2)$: I.

A latin square of size s is said to have a directrix if s different symbols can be found in s cells with the property that no two cells are in the same row or column. Result (iv) of Theorem 1 is important and has been used earlier in the construction of Quasi-factorial designs (Rao, 1956) and confounded designs for asymmetrical factorial experiments (Nair and Rao, 1948). In the latter paper it has been shown that (36, 3, 6, 2) : I exists although there are no orthogonal squares of size 6 since a latin square of size 6 with a directrix can be written down. It is interesting to note that the method of construction given by Bose, Shrikhande and Parker (1960) always leads to orthogonal squares with a common directrix.

4. Arrays of type II and strength 2

Theorem 2 contains the main result in the construction of orthogonal arrays of Type II.

Theorem 2 : *Existence of* $GF(s) \Longrightarrow (s(s-1)/2, s, s, 2)$: II *when* s *is odd.*

When s is even, the minimum value of N is $s(s-1)$ and a Type I array can be constructed with this value of N if $GF(s)$ exists, which is also a Type II array with index 2. The only interesting case is when s is odd.

Let $\alpha_0 = 0, \alpha_1, \ldots, \alpha_{s-1}$ be the elements of $GF(s)$. They can also be written as $\alpha_0 = 0, \beta_1, \ldots, \beta_{(s-1)/2}, -\beta_1, \ldots, -\beta_{(s-1)/2}$ when s is odd. Consider the vectors

$$(\beta_i \alpha_0, \beta_i \alpha_1, \ldots, \beta_i \alpha_{s-1}); \qquad i = 1, \ldots, (s-1)/2. \qquad \ldots \quad (4.1)$$

The symmetric differences of elements occurring in the $(r+1)$-th and $(u+1)$-th positions are

$$\beta_i(\alpha_r - \alpha_u), \; -\beta_i(\alpha_r - \alpha_u); \qquad i = 1, \ldots, (s-1)/2. \qquad \ldots \quad (4.2)$$

Since $(\alpha_r - \alpha_u) \neq 0$, the differences (4.2) include all the non-zero elements of $GF(s)$ exactly once each. Hence by an application of a theorem due to Bose (1939) it follows that the totality of sets

$$(\beta_i\alpha_0 + \alpha_j, \beta_i\alpha_1 + \alpha_j, \ldots, \beta_i\alpha_{s-1} + \alpha_j) \qquad \ldots \quad (4.3)$$

$$j = 0, \ldots, s-1; \qquad i = 1, \ldots, (s-1)/2$$

are such that in any two positions all combinations of the s elements taken two at a time occur exactly once each. The sets (4.3) written as columns provide $(s(s-1)/2, s, s, 2)$: II, which proves Theorem 2.

For example, if $s = 5$, the residue classes 0, 1, 2, 3, 4 (mod 5) form a field. We write the elements as 0, 1, 2, -1, -2, and hence the key sets (4.1) are

$$(0, \; 1, \; 2, \; 3, \; 4)$$
$$(0, \; 2, \; 4, \; 1, \; 3)$$

the latter being obtained from the former multiplying by 2. Writing them vertically (as shown below in blocks) and generating the other columns by addition of the elements of $GF(5)$ as indicated in (4.3) we obtain the 10 columns

0	1	2	3	4	0	1	2	3	4
1	2	3	4	0	2	3	4	0	1
2	3	4	0	1	4	0	1	2	3
3	4	0	1	2	1	2	3	4	0
4	0	1	2	3	3	4	0	1	2

which is (10, 5, 5, 2) : II. This also happens to be (10, 5, 5, 3) : II of strength 3, but a general method of constructing arrays of Type II and strength 3 has to be investigated. Considering only the first three rows we obtain (10, 3, 5, 2) : II, which has been used in deriving a cyclic solution to the *combinatorial assignment problem* with 36 officers (Rao, 1961b). Some other applications will be considered in a subsequent communication.

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