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AN EFFECTIVE SELECTION THEOREM

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§1. Introduction. A recent result of J.P. Burgess [1] states:

THEOREM 0. *Let F be a multifunction from an analytic subset T of a Polish space to a Polish space X. If F is Borel measurable,* Graph(F) *is coanalytic in $T \times X$ and $F(t)$ is nonmeager in its closure $\overline{F(t)}$ for each $t \in T$, then F admits a Borel measurable selector.*

The above result unifies and significantly extends earlier results of H. Sarbadhikari [8], S.M. Srivastava [9] and G. Debs (unpublished). The reader is referred to [1] for details.

The aim of this article is to give an effective version of Theorem 0. We do this by proving a basis theorem for Π_1^1 sets which are nonmeager in their closure and satisfy a local version of the measurability condition in Theorem 0. Our basis theorem generalizes a well-known result of P.G. Hinman [4] and S.K. Thomason [10] (see also [5] and [7, 4F.20]). Our methods are similar to those used by A. Louveau to prove that a Σ_1^1, σ-compact set is contained in a Δ_1^1, σ-compact set (see [7, 4F.18]).

The paper is organized as follows. §2 is devoted to preliminaries. In §3, we prove the basis theorem and deduce as a consequence an effective version of Theorem 0. We show in §4 how our methods can be used to give alternative proofs of some known results.

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§2. Preliminaries. The effective results will be established for the space $(\omega^\omega)^k \times \omega^l$, where $k \geq 1$. Since such a space is recursively isomorphic to ω^ω, we shall work in ω^ω. It should be mentioned that the results could be formulated and proved for the recursively presentable Polish spaces of Moschovakis [7], but we have not done so in order to keep the exposition simple.

We fix a base N_s for the topology of ω^ω, where

$$N_s = \{\alpha \in \omega^\omega : \bar{\alpha}(\text{lh}(s)) = s\}, \qquad s \in \omega.$$

If s and t are sequence numbers, we write $s < t$ if $s = t \restriction i$ for some i less than $\text{lh}(t)$; we write $s \leq t$ if $s < t$ or $s = t$. A tree on ω is identified with the set of sequence numbers of its elements. We say that $\alpha \in \omega^\omega$ is a *code* for a tree T on ω if $(\forall s)(\alpha(s) = 0 \leftrightarrow s \in T)$. Plainly, a tree T (as a subset of ω) is Δ_1^1 iff it has a Δ_1^1 code.

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If T is a tree on ω, then $[T]$, the body of T, is closed in ω^ω; conversely, if F is closed in ω^ω, then $F = [T]$ for some tree T on ω. If $B \subset \omega^\omega \times \omega^\omega$, we denote the projection of B to the first coordinate by $\pi[B]$.

Towards localizing the measurability condition in Theorem 0, we make the following definition.

DEFINITION. Let Γ be a pointclass. A set $A \subset \omega^\omega$ is *Γ-normal* if R_A, as a subset of ω, is in Γ, where

$$R_A(s) \leftrightarrow A \cap N_s \neq \emptyset.$$

We state some easily verifiable facts about Γ-normal sets, where Γ is the pointclass Δ_1^1 or the pointclass $\Delta_1^1(\alpha)$.

(a) P is Γ-normal iff $\bar{P}$, the closure of P, is Γ-normal.

(b) If P is Γ-normal, then $\bar{P}$ is in Γ.

(c) Any dense set is Γ-normal.

(d) Any open set in Γ is Γ-normal.

(e) Any σ-compact set in Γ is Γ-normal.

(f) There exist Π_1^0 sets which are not Δ_1^1-normal.

Our notation and terminology will closely follow [7]. The only result which we will use and which is not explicitly stated in [7] is the following observation of Louveau [6].

SELECTION LEMMA. *Let P be a Π_1^1 subset of $\omega^\omega \times \omega^\omega$. Let $B = \{\alpha \in \omega^\omega : (\exists \beta \in \Delta_1^1(\alpha))P(\alpha, \beta)\}$. If A is Σ_1^1 and $A \subset B$, then there exists a Δ_1^1-recursive function f: $\omega^\omega \to \omega^\omega$ such that* $(\forall \alpha \in A)\ P(\alpha, f(\alpha))$.

Here and in the sequel Δ_1^1-recursive functions are assumed to be total functions. We can do this because a partial function is Δ_1^1-recursive iff it is the restriction of a Δ_1^1-recursive total function to a Δ_1^1 set.

Relativized versions of our results will not be stated as they are easy to formulate and can be proved just like the absolute versions.

§3. Basis and selection theorems. In this section, E will be a fixed closed, Δ_1^1-normal subset of ω^ω. It follows by (b) of §2 that E is then a Δ_1^1 set. Also fix a Π_1^1-recursive partial function d: $\omega \to \omega^\omega$ which parametrizes points in $\Delta_1^1 \cap \omega^\omega$. This can be done by [7, 4D.2].

We next define some relations.

$$
\begin{aligned}
S_1(\alpha) &\overset{\text{def}}{\longleftrightarrow} \alpha \text{ codes some tree on } \omega\\
&\longleftrightarrow (\forall s)[\alpha(s) = 0 \to (\mathrm{Seq}(s)\ \&\ (\forall t)(\mathrm{Seq}(t)\ \&\ t \leq s\\
&\qquad\qquad \to \alpha(t) = 0))],\\
S_2(\alpha, \beta) &\overset{\text{def}}{\longleftrightarrow} (\forall n)(\alpha(\bar{\beta}(n)) = 0),\\
S_3(\alpha) &\overset{\text{def}}{\longleftrightarrow} \alpha \text{ codes some tree } T \text{ on } \omega \text{ such that } [T] \subset E \text{ and}\\
&\qquad [T] \text{ is nowhere dense in } E\\
&\longleftrightarrow S_1(\alpha)\ \&\ (\forall \beta)\,(S_2(\alpha, \beta) \to \beta \in E)\\
&\qquad \&\ (\forall s)(R_E(s) \to (\exists t)(R_E(t)\ \&\ s \leq t\ \&\ (\forall \beta)(S_2(\alpha, \beta) \to \beta \notin N_t))).
\end{aligned}
$$

Plainly, S_1 and S_2 are Π_1^0, while S_3 is Π_1^1.

LEMMA 1. *Suppose A is Σ_1^1, B is Π_1^1, M is closed, nowhere dense in E and $A \subset M \subset B \subset E$. Then there is a Δ_1^1 tree T^* on ω such that $[T^*]$ is nowhere dense in E and $A \subset [T^*] \subset B$.*

PROOF. By arguing as in the proof of [7, 4F.14], one can prove that there is $\partial \in \Delta_1^1 \cap \omega^\omega$ and a set F in $\Pi_1^0(\partial)$ such that $\bar{A} \subset F \subset B$. Now, by a relativized version of [7, 4A.1], there is a set $R \subset \omega$ such that R is recursive in ∂, Seq (s) & Seq(t) & $s < t$ & $R(t) \to R(s)$, and

$$\alpha \in F \leftrightarrow (\forall m) R(\bar{\alpha}(m)).$$

We define a tree T on ω by

$$T(s) \leftrightarrow \mathrm{Seq}(s) \ \&\ R(s).$$

Plainly, T is Δ_1^1 and $F = [T]$.

Next define

$$S(s, t) \leftrightarrow \neg R_E(s) \vee [\mathrm{Seq}(s) \ \&\ R_E(t) \ \&\ s < t \ \&\ \bar{A} \cap N_t = \varnothing].$$

Then S is Π_1^1. Moreover $(\forall s)(\exists t) S(s, t)$, since $\bar{A}$ is nowhere dense in E. Hence, by the Δ-selection principle [7, 4B.5], there is a Δ_1^1-recursive function $g: \omega \to \omega$ such that $(\forall s) S(s, g(s))$. Define

$$T^*(t) \leftrightarrow T(t) \ \&\ \neg(\exists u)(\exists v)(R_E(u) \ \&\ g(u) = v \ \&\ v \leq t).$$

Clearly, T^* is a Δ_1^1 tree on ω and $\bar{A} \subset [T^*] \subset [T] \subset B$. It remains only to argue that $[T^*]$ is nowhere dense in E. So assume $R_E(s)$ and put $t = g(s)$. Then $R_E(t)$, $s < t$ and $t \notin T^*$. It follows that $[T^*] \cap N_t = \varnothing$, which shows that $[T^*]$ is nowhere dense in E. This completes the proof.

LEMMA 2. *If R is Σ_1^1, $R \subset E$ and R is meager in E, then*

$$R(\alpha) \to (\exists T)(T \text{ is a } \Delta_1^1 \text{ tree on } \omega, [T] \subset E, [T] \text{ is nowhere dense in } E \ \&\ \alpha \in [T]).$$

PROOF. Fix a recursive function $F: \omega^\omega \to \omega^\omega$ and a Π_1^0 set $A \subset \omega^\omega$ such that $F(A) = R$. Define

$$A^* = \{\alpha \in \omega^\omega : (\exists s)(\alpha \in N_s \ \&\ (\forall \beta)(\beta \in A \cap N_s \to (\exists T)(T \text{ is a } \Delta_1^1 \text{ tree on } \omega, [T] \subset E, [T] \text{ is nowhere dense in } E \ \&\ F(\beta) \in [T])))\}.$$

Plainly, A^* is open. To see that A^* is Π_1^1, rewrite A^*:

$$\alpha \in A^* \leftrightarrow (\exists s)(\alpha \in N_s \ \&\ (\forall \beta)(\beta \in A \cap N_s \to (\exists n)(d(n)\downarrow \ \&\ S_3(d(n)) \ \&\ S_2(d(n), F(\beta))))).$$

To complete the proof, we need only show that $A \subset A^*$. Assume towards a contradiction that $B = A - A^* \neq \varnothing$. Note that B is closed and Σ_1^1. Since R is meager in E, there exist sets $K_n \subset E$ such that K_n is closed, nowhere dense in E and $R \subset \bigcup_n K_n$. It follows that $\varnothing \neq F(B) \subset \bigcup_n K_n$. Hence, by Kunugui's lemma [7, 4F.13], there exist $s_0 \in \omega$ and $n \in \omega$ such that $\varnothing \neq F(B \cap N_{s_0}) \subset K_n$. Since $F(B \cap N_{s_0})$ is a Σ_1^1 set, it follows from Lemma 1 that there is a Δ_1^1 tree T on ω such that $F(B \cap N_{s_0}) \subset [T] \subset E$ and $[T]$ is nowhere dense in E. Now fix $\alpha \in B \cap N_{s_0}$ and let $\beta \in A \cap N_{s_0}$. We now consider two cases: $\beta \in A^*$ and $\beta \notin A^*$. In either case, there is a Δ_1^1 tree T^* on ω such that $[T^*] \subset E$, $[T^*]$ is nowhere dense in E and $F(\beta) \in [T^*]$. In the first case, this follows from the fact that $\beta \in A$ and the definition of A^*; in the second case, this follows from the fact that $\beta \in B \cap N_{s_0}$ and our previous observation about $F(B \cap N_{s_0})$. Consequently, s_0 witnesses that $\alpha \in A^*$. But this contradicts $\alpha \in B$. This concludes the proof.

LEMMA 3. *If R is Σ_1^1, $R \subset E$ and R is meager in E, then there is $Q \subset \omega \times \omega^\omega$ such that Q is Δ_1^1, each n-section Q_n of Q is closed, nowhere dense in E and $(\forall\alpha)(R(\alpha) \rightarrow (\exists n)Q(n, \alpha))$.*

PROOF. Define

$$Q'(n, \alpha) \leftrightarrow d(n)\downarrow \;\&\; S_3(d(n)) \;\&\; S_2(d(n), \alpha)$$

and $R_1(\alpha) \leftrightarrow (\exists n)Q'(n, \alpha)$. Then Q' and R_1 are Π_1^1 sets. Now Lemma 2 implies that $R \subset R_1$. Hence, by the separation property of Σ_1^1 sets [7, 4B.11], there is a Δ_1^1 set R_2 with $R \subset R_2 \subset R_1$. Clearly $(\forall\alpha \in R_2)(\exists n)Q'(n, \alpha)$. So by the Δ-selection principle [7, 4B.5], there is a Δ_1^1-recursive function $g\colon \omega^\omega \rightarrow \omega$ such that $(\forall\alpha \in R_2)$ $Q'(g(\alpha), \alpha)$. Next define

$$A_1(n) \leftrightarrow (\exists\alpha \in R)(g(\alpha) = n),$$

$$A_2(n) \leftrightarrow d(n)\downarrow \;\&\; S_3(d(n)).$$

Then A_1 is Σ_1^1, A_2 is Π_1^1 and $A_1 \subset A_2$. Again by the separation property of Σ_1^1 sets, we can find a Δ_1^1 set A with $A_1 \subset A \subset A_2$. Finally, define

$$Q(n, \alpha) \leftrightarrow n \in A \;\&\; d(n)\downarrow \;\&\; S_2(d(n), \alpha).$$

Compute

$$\neg Q(n, \alpha) \leftrightarrow (n \notin A) \vee (d(n)\downarrow \;\&\; \neg S_2(d(n), \alpha)).$$

It follows that Q is Δ_1^1. The remaining assertions about Q made in the statement of Lemma 3 are now easy to verify. This completes the proof.

Kechris [5, Corollary 4.2.4] proved Lemma 3 above when $E = \omega^\omega$. However, his methods are quite different from ours.

LEMMA 4. *Assume that $E \neq \emptyset$, R is Σ_1^1, $R \subset E$ and R is meager in E. Then $E - R$ contains a Δ_1^1 point.*

PROOF. Let Q satisfy the assertion of Lemma 3. To exhibit a Δ_1^1 point in $E - R$ one need only give an effective proof of the Baire category theorem. So define

$$S(n, s, t) \leftrightarrow \neg R_E(s) \vee [\mathrm{Seq}(s) \;\&\; R_E(t) \;\&\; s < t \;\&\; (\forall\alpha)(Q(n, \alpha) \rightarrow \alpha \notin N_t)].$$

Then S is Π_1^1, and since each Q_n is nowhere dense in E, we have: $(\forall n)(\forall s)(\exists t)$ $S(n, s, t)$. By the Δ-selection principle [7, 4B.5], there exists a Δ_1^1-recursive function $g\colon \omega \times \omega \rightarrow \omega$ such that $(\forall n)(\forall s)S(n, s, g(n, s))$. We define $f\colon \omega \rightarrow \omega$ by primitive recursion: $f(0) = g(0, 1)$ (recall 1 is the sequence number of the empty sequence) $f(n + 1) = g(n + 1, f(n))$. According to [7, 7A.3], f is Π_1^1-recursive. Since f is a total function, it follows that f must be Δ_1^1-recursive.

Clearly, $\bigcap_{n=0}^{\infty} N_{f(n)}$ is a singleton, say $\{\alpha^*\}$, and $\alpha^* \in E$. Since $R \subset \bigcup_n Q_n$, $\alpha^* \notin R$. Finally,

$$s \in \mathscr{U}(\alpha^*) \overset{\text{def}}{\longleftrightarrow} \alpha^* \in N_s$$
$$\longleftrightarrow (\exists n)(\text{Seq}(s) \ \& \ s \leq f(n)).$$

It follows that α^* is a Δ_1^1 point in $E - R$. This completes the proof.

We now formulate our basis theorem.

THEOREM 1. *Let $P \subset \omega^\omega$ be a Π_1^1 set. If there exists $s_0 \in \omega$ such that $P \cap N_{s_0}$ is nonempty, Δ_1^1-normal and comeager in $\bar{P} \cap N_{s_0}$, then P contains a Δ_1^1 point.*

PROOF. In Lemma 4, take $E = \overline{P \cap N_{s_0}} = \bar{P} \cap N_{s_0}$ and $R = E - (P \cap N_{s_0})$. The theorem now easily falls out of Lemma 4.

An immediate consequence of Theorem 1 is the following result of Hinman and Thomason which we mentioned in the introduction.

COROLLARY 1. *If $P \subset \omega^\omega$ is Π_1^1 and nonmeager, then P contains a Δ_1^1 point.*

PROOF. Since P satisfies the Baire property, there is $s_0 \in \omega$ such that $P \cap N_{s_0}$ is comeager in N_{s_0}. Now apply Theorem 1.

We now use Theorem 1 to deduce an effective selection theorem.

THEOREM 2. *Let $P \subset \omega^\omega \times \omega^\omega$ be a Π_1^1 set. Let A be Σ_1^1 and suppose that $A \subset \pi[P]$. Assume that*

$$(\forall \alpha \in A)(\exists s)(P_\alpha \cap N_s \textit{ is nonempty, } \Delta_1^1(\alpha)\textit{-normal and comeager in } \bar{P}_\alpha \cap N_s).$$

Then there is a Δ_1^1-recursive function $f\colon \omega^\omega \to \omega^\omega$ such that $(\forall \alpha \in A)P(\alpha, f(\alpha))$.

PROOF. Let $B = \{\alpha \in \omega^\omega : (\exists \beta \in \Delta_1^1(\alpha))P(\alpha, \beta)\}$ In view of the hypotheses, a relativization of Theorem 1 implies that $A \subset B$. The selection lemma in §2 now does the rest.

Some of the more interesting consequences of Theorem 2 are incorporated in the next corollary.

COROLLARY 2. *Let $P \subset \omega^\omega \times \omega^\omega$ be a Π_1^1 set. Let A be Σ_1^1 and assume that $A \subset \pi[P]$. Suppose that one of the following conditions holds:*

(i) $(\forall \alpha \in A)(P_\alpha$ *is $\Delta_1^1(\alpha)$-normal and nonmeager in* $\bar{P}_\alpha)$.

(ii) $(\forall \alpha \in A)(P_\alpha$ *is $\Delta_1^1(\alpha)$-normal and* $\mathbf{\Pi}_2^0)$.

(iii) $(\forall \alpha \in A)(P_\alpha$ *is nonmeager*$)$.

Then there is a Δ_1^1-recursive function $f\colon \omega^\omega \to \omega^\omega$ such that $(\forall \alpha \in A)P(\alpha, f(\alpha))$.

The deduction of Corollary 2 from Theorem 2 is straightforward and is omitted. Corollary 2 (under condition (i)) can be viewed as an effective version of Theorem 0.

We conclude the section by deducing Theorem 0 from Corollary 2. Since any uncountable Polish space is Borel isomorphic to ω^ω, without loss of generality we may assume $T \subset \omega^\omega$. Since any Polish space is a continuous, open image of ω^ω, without loss of generality we may assume $X = \omega^\omega$. Let $P^1 = \text{Graph}(F)$. Find a $\mathbf{\Pi}_1^1$ set $P \subset \omega^\omega \times \omega^\omega$ such that $P^1 = P \cap (T \times \omega^\omega)$. As F is Borel measurable on T, for each $s \in \omega$, the set $H_s^1 = \{\alpha \in T : P_\alpha^1 \cap N_s \neq \varnothing\}$ is Borel in T, so there is a $\mathbf{\Delta}_1^1$ set $H_s \subset \omega^\omega$ such that $H_s^1 = H_s \cap T$. Define $H(\alpha, s) \leftrightarrow \alpha \in H_s$. It is easy to see that H is a $\mathbf{\Delta}_1^1$ subset of $\omega^\omega \times \omega$. Find z such that P is $\Pi_1^1(z)$, T is $\Sigma_1^1(z)$ and H is $\Delta_1^1(z)$. Now, for each $\alpha \in T$,

$$\begin{aligned} R_{P_\alpha}(s) &\leftrightarrow P_\alpha \dot{\cap} N_s \neq \varnothing \\ &\leftrightarrow P_\alpha^1 \cap N_s \neq \varnothing \\ &\leftrightarrow \alpha \in H_s^1 \\ &\leftrightarrow H(\alpha, s). \end{aligned}$$

Consequently, R_{P_α} is $\Delta_1^1(z, \alpha)$, hence P_α is $\Delta_1^1(z, \alpha)$-normal for $\alpha \in T$. Furthermore, P_α is nonmeager in $\bar{P}_\alpha$. So a relativized version of Corollary 2 applies to yield a $\Delta_1^1(z)$-recursive function $f: \omega^\omega \to \omega^\omega$ such that $(\forall \alpha \in T)(f(\alpha) \in F(\alpha))$. The restriction of f to T is a Borel selector for F.

§4. Further results. In this last section we show that the methods of §3 can be used to give alternative proofs of known results. Throughout this section, the set E fixed at the beginning of §3 will be taken to be ω^ω. The relations S_1, S_2, S_3 defined in §3 have the same meaning as before except that S_3 is defined with respect to ω^ω. Then as before S_1, S_2 are Π_1^0 sets, while S_3 is Π_1^1. By [7, 4D.2], we fix a Π_1^1-recursive partial function $d^*: \omega \times \omega^\omega \to \omega^\omega$ which parametrizes points in $\Delta_1^1(\alpha) \cap \omega^\omega$.

The next result was first proved by Kechris [5]; Vaught [11] independently proved the boldface version of the result. See also [7, 4F.19] and [2].

Theorem 3. *Let $P \subset \omega^\omega \times \omega^\omega$. If P is Σ_1^1 (Π_1^1), then $\{\alpha \in \omega^\omega: P_\alpha$ is nonmeager$\}$ is Σ_1^1 (Π_1^1). Similarly, if P is Σ_1^1 (Π_1^1), then $\{\alpha \in \omega^\omega: P_\alpha$ is comeager$\}$ is Σ_1^1 (Π_1^1).*

Proof. Let P be Σ_1^1. By a relativization of Lemma 2, we have:

$$P_\alpha \text{ is meager} \leftrightarrow (\forall \beta)[P(\alpha, \beta) \to (\exists n)(d^*(n, \alpha)\downarrow \;\&\; S_3(d^*(n, \alpha)) \;\&\; S_2(d^*(n, \alpha), \beta))].$$

It follows that $\{\alpha \in \omega^\omega: P_\alpha$ is nonmeager$\}$ is Σ_1^1.

Suppose next that P is Π_1^1. Since each P_α satisfies the Baire property, we have:

$$P_\alpha \text{ is nonmeager} \leftrightarrow (\exists s)(N_s - P_\alpha \text{ is meager}).$$

It follows from what we have just proved for Σ_1^1 sets that $\{\alpha \in \omega^\omega: P_\alpha$ is nonmeager$\}$ is Π_1^1.

The second assertion follows from the first.

Theorem 4. *If $P \subset \omega^\omega \times \omega^\omega$, P is Σ_1^1 and P_α is meager for each α, then there is $Q \subset \omega \times \omega^\omega \times \omega^\omega$ such that Q is Δ_1^1, each (n, α)-section $Q_{n,\alpha}$ of Q is closed, nowhere dense and*

$$(\forall \alpha)(\forall \beta)(P(\alpha, \beta) \to (\exists n)Q(n, \alpha, \beta)).$$

Proof. We have only to rewrite the proof of Lemma 3 uniformly in α. Define

$$Q'(n, \alpha, \beta) \leftrightarrow d^*(n, \alpha)\downarrow \;\&\; S_3(d^*(n, \alpha)) \;\&\; S_2(d^*(n, \alpha), \beta),$$

$$P_1(\alpha, \beta) \leftrightarrow (\exists n)Q'(n, \alpha, \beta).$$

Then Q' and P_1 are Π_1^1. Moreover, by a relativization of Lemma 2, $P \subset P_1$. Arguing as in the proof of Lemma 3, we get a Δ_1^1-recursive function $f: \omega^\omega \times \omega^\omega \to \omega$ such that $(\forall (\alpha, \beta) \in P)\; Q'(f(\alpha, \beta), \alpha, \beta)$. Next define

$$R_1(n, \alpha) \leftrightarrow (\exists \beta)(P(\alpha, \beta) \;\&\; f(\alpha, \beta) = n),$$

$$R_2(n, \alpha) \leftrightarrow d^*(n, \alpha) \downarrow \ \& \ S_3(d^*(n, \alpha)).$$

Then R_1 is Σ^1_1, R_2 is Π^1_1 and $R_1 \subset R_2$. By the separation property of Σ^1_1 sets, there is a Δ^1_1 set R such that $R_1 \subset R \subset R_2$. We now define

$$Q(n, \alpha, \beta) \leftrightarrow R(n, \alpha) \ \& \ d^*(n, \alpha) \downarrow \ \& \ S_2(d^*(n, \alpha), \beta).$$

It is easy to verify that Q has the desired properties. This completes the proof.

A boldface version of Theorem 4 has been obtained independently by Cenzer and Mauldin [2] and Hillard [3].

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