CONFLICTING CRITERIA OF 'GOODNESS' OF STATISTICS

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SUMMARY. In this note it is intended to show by an example that if B and C are two statistics unbiased for a parameter θ and C has uniformly smaller variance than B, then for testing $\theta = \theta_0$ against an alternative (say $\theta > \theta_0$) the test based on C need not have more power than B locally (i.e., near the null hypothesis).

Basu (1953) gave an example (also quoted by Rao (1952)) to show that there can exist estimates with uniformly lesser variance than maximum likelihood estimates. We use the same example here.

$x_1, x_2, \ldots, x_n$ are n independent observations from a rectangular distribution on $(\theta, 2\theta)$, $\theta > 0$. We wish to test the null hypothesis $\theta = \theta_0$ against the alternative $\theta > \theta_0$. Let us write

$$\xi = \max(x_1, x_2, \ldots, x_n), \; \eta = \min(x_1, x_2, \ldots, x_n), \; t = \frac{\eta}{\xi}$$

Consider the statistics

$$A = (\xi, \eta) \text{ or equivalently } (\xi, t)\,; \quad B = \xi\,; \quad C = 2\xi + \eta.$$

It is easy to verify that A is sufficient for θ but not complete; B is the maximum likelihood estimate for θ; $(n+1)C/(5n+4)$ is the minimum variance unbiased linear estimator of ξ and η for θ, and

$$V\left(\frac{n+1}{5n+4}\, C\right) = \frac{1}{(n+2)(5n+4)}\, \theta^2$$

$$E\left(\frac{n+1}{2n+1}\, B\right) = \theta, \; V\left(\frac{n+1}{2n+1}\, B\right) = \frac{n}{(n+2)(2n+1)^2}\, \theta^2$$

which show that, though B is the maximum likelihood estimate, there is another statistic with uniformly smaller variance.

Based on these statistics we can find (by an application of the Neyman-Pearson Lemma) uniformly most powerful tests A, B and C respectively for testing $\theta = \theta_0$, against $\theta > \theta_0$. The tests A, B and C of size α and their power functions β_A, β_B and β_C are given below as functions of $\lambda = \dfrac{\theta}{\theta_0}$.

Test A. Reject the null hypothesis if $\eta = \xi t \geqslant m$ and/or $\xi > 2\theta_0$

$$\mu = \frac{m}{\theta_0} = 2-\alpha^{1/n}$$

$$\beta_A(\lambda) \left\{ \begin{array}{ll} = 1-\frac{1}{\lambda^n}[(2-\lambda)^n-\alpha] & \text{if } 1 \leqslant \lambda \leqslant \mu \\ = 1 & \text{if } \mu < \lambda \end{array} \right\} \qquad \ldots (1)$$

Test B. Reject the null hypothesis if $\xi \geqslant m$

$$\mu = \frac{m}{\theta_0} = 1+(1-\alpha)^{1/n}$$

$$\beta_B(\lambda) \left\{ \begin{array}{ll} = 1-\frac{1}{\lambda^n}[1+(1-\alpha)^{1/n}-\lambda]^n & \text{if } 1 \leqslant \lambda \leqslant \mu \\ = 1 & \text{if } \mu < \lambda \end{array} \right\} \qquad \ldots (2)$$

Test C. Reject the null hypothesis if $2\xi+\eta \geqslant m$

$$\mu = \frac{m}{\theta_0} = 6-(3\alpha)^{1/n}.$$

$$\beta_C(\lambda) \left\{ \begin{array}{ll} = \frac{[6\lambda-6+(3\alpha)^{1/n}]^n}{3\lambda^n} & \text{if } 1 \leqslant \lambda \leqslant \frac{\mu}{5} \\ = 1-\frac{[6-(3\alpha)^{1/n}-3\lambda]^n}{3.2^{n-1}\lambda^n} & \text{if } \frac{\mu}{5}\, 6\lambda \leqslant \frac{\mu}{3} \\ = 1 & \text{if } \frac{\mu}{3} < \lambda \end{array} \right\} \qquad \ldots (3)$$

The numerical values of the powers of these tests (for the size 0.05) are given in the following Table for some selected values of λ and n.

The test A based on the sufficient statistic is naturally better than the other two tests. But it is surprising to find that test B is more powerful than test C in the neighbourhood of θ_0. The following Table shows that test C is poorer than test B in the neighbourhood of the null hypothesis, even though test C catches up and exceeds test B in power for higher values of λ. The practical statistician may note that in this region of large values of λ the power of both the tests is quite high that one would not be much the worse by using the poorer test B.

TABLE. POWER FUNCTION

null hypothesis: $\theta = \theta_0$ i.e., $\lambda = 1$; alternative: $\theta > \theta_0$ i.e., $\lambda > 1$. size of the test = 0.05

	test based on (ξ, η) the sufficient statistic					test based on ξ the m.l.e.					test based on $2\xi + \eta$ the u.m.v.u. linear estimate of ξ and η for θ.				
λ / n	1	2	5	10	20	1	2	5	10	20	1	2	5	10	20
1.0	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500	.0500
1.1	.2273	.3710	.6044	.8848	.9894	.2273	.3077	.6537	.8732	.9828	.2273	.2993	.5695	.8608	.9892
1.2	.3750	.5903	.8884	.9907	1.0000	.3750	.5832	.8765	.9837	.9997	.3750	.5322	.8756	.9900	1.0000*
1.3	.5000	.7390	.9682	1.0000	1.0000	.5000	.7306	.9579	.9981	1.0000*	.5000	.7207	.9680	.9995	1.0000*
1.4	.6071	.8418	.9948	1.0000	1.0000	.6071	.8315	.9867	.9998	1.0000*	.6071	.8303	.9933	1.0000*	1.0000*
1.5	.7000	.9111	1.0000	1.0000	1.0000	.7000	.8099	.9963	1.0000*	1.0000*	.7000	.9082	.9990	1.0000*	1.0000*
1.6	.7812	.9570	1.0000	1.0000	1.0000	.7812	.9452	.9991	1.0000*	1.0000*	.7812	.9570	1.0000*	1.0000*	1.0000*
1.7	.8529	.9862	1.0000	1.0000	1.0000	.8529	.9739	.9909	1.0000*	1.0000*	.8520	.9840	1.0000*	1.0000*	1.0000
1.8	.9167	1.0000	1.0000	1.0000	1.0000	.9167	.9894	1.0000*	1.0000*	1.0000*	.9167	.9979	1.0000	1.0000	1.0000
1.9	.9737	1.0000	1.0000	1.0000	1.0000	.9737	.9985	1.0000*	1.0000*	1.0000*	.9737	1.0000	1.0000	1.0000	1.0000
2.0	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000

* correct to four decimal places.

Differentiating the power function at $\lambda = 1$ we have

$$\left.\begin{aligned} \beta'_A(1) &= n(2-\alpha) \\ \beta'_B(1) &= n((1-\alpha)^{\frac{n-1}{n}}+1-\alpha) \sim n(2-2\alpha) \\ \beta'_C(1) &= n(2.(3\alpha)^{\frac{n-1}{n}}-\alpha) \sim n.\,5\alpha \end{aligned}\right\} \qquad \ldots \quad (4)$$

This clearly shows that test B is locally more powerful than test C.

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