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THE DISTANCE BETWEEN THE EIGENVALUES OF HERMITIAN MATRICES

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ABSTRACT. It is shown that the minmax principle of Ky Fan leads to a quick simple derivation of a recent inequality of V. S. Sunder giving a lower bound for the spectral distance between two Hermitian matrices. This brings out a striking parallel between this result and an earlier known upper bound for the spectral distance due to L. Mirsky.

Let A be a Hermitian matrix of order n and let $\lambda_\downarrow(A)$ denote the vector in $\mathbf{R}^n$ whose coordinates are the eigenvalues of A arranged as $\lambda_{[1]}(A) \geq \cdots \geq \lambda_{[n]}(A)$. Let $\lambda_{(1)}(A) \leq \cdots \leq \lambda_{(n)}(A)$ be the increasing rearrangement of these eigenvalues and $\lambda_\uparrow(A)$ the vector with coordinates $\lambda_{(j)}(A)$, $j = 1, 2, \ldots, n$. The same symbols $\lambda_\downarrow(A)$ and $\lambda_\uparrow(A)$ will also denote the diagonal matrices which have as their diagonal entries the components of the vectors $\lambda_\downarrow(A)$ and $\lambda_\uparrow(A)$, respectively. Let $\|\cdot\|$ denote any unitarily invariant norm on the space of matrices. (See [**4**].)

This note is concerned with the following result:

THEOREM. *Let A and B be Hermitian matrices. Then for every unitarily invariant norm we have*

$$\|\lambda_\downarrow(A) - \lambda_\downarrow(B)\| \leq \|A - B\| \leq \|\lambda_\downarrow(A) - \lambda_\uparrow(B)\|. \tag{1}$$

The first inequality in (1) appeared in a paper of Mirsky [**4**], who used a famous result of Lidskii and Wielandt to derive it. The second is proved in a recent paper of Sunder [**5**]. I give here another proof of the second inequality which has two attractive features: It is very short and it proceeds on exactly the same lines as the well-known proof of Lidskii, Wielandt and Mirsky for the first inequality. For illumination, I indicate how both inequalities follow from the same principle.

It is an easy consequence of the minmax principle of Wielandt that for any choice $1 \leq i_1 < \cdots < i_k \leq n$ of k indices we have

$$\sum_{j=1}^{k} \lambda_{[i_j]}(A+B) \leq \sum_{j=1}^{k} \lambda_{[j]}(A) + \sum_{j=1}^{k} \lambda_{[i_j]}(B) \tag{2}$$

for all $k = 1, 2, \ldots, n$, with equality holding for $k = n$. (See [**3**, p. 242].)

Writing $x \prec y$ to mean that the vector x is majorised by the vector y in $\mathbf{R}^n$ (see [**3**]), we get from inequalities (2)

$$\lambda_\downarrow(A+B) - \lambda_\downarrow(B) \prec \lambda_\downarrow(A). \tag{3}$$

With a change of variables, this gives

$$\lambda_\downarrow(A) - \lambda_\downarrow(B) \prec \lambda_\downarrow(A - B).$$

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Now the first part of the Theorem follows using standard characterisations of majorisation together with properties of symmetric gauge functions and unitarily invariant norms. This is the well-known proof of Mirsky [**4**].

Now note that from (2) we can also conclude

$$\lambda_{\downarrow}(A+B) \prec \lambda_{\downarrow}(A) + \lambda_{\downarrow}(B). \tag{4}$$

In fact, for this conclusion the full force of (2) is not needed. It suffices to use the special case $(i_1, \ldots, i_k) = (1, \ldots, k)$ which is much easier to prove using the minmax principle of Ky Fan [**2**].

Replace B by $-B$ in (4) and note that $\lambda_{\downarrow}(-B) = -\lambda_{\uparrow}(B)$. This gives

$$\lambda_{\downarrow}(A-B) \prec \lambda_{\downarrow}(A) - \lambda_{\uparrow}(B).$$

But this implies

$$\begin{aligned} &(|\lambda_{[1]}(A-B)|, \ldots, |\lambda_{[n]}(A-B)|) \\ &\qquad \prec_{\mathrm{w}} \; (|\lambda_{[1]}(A) - \lambda_{(1)}(B)|, \ldots, |\lambda_{[n]}(A) - \lambda_{(n)}(B)|) \end{aligned} \tag{5}$$

where $\prec_{\mathrm{w}}$ stands for weak majorisation [**3**, p. 116].

Let $s_{[j]}(A)$ denote the jth singular value of A. Let $\|A\|_k = s_{[1]}(A) + \cdots + s_{[k]}(A)$ for $k = 1, 2, \ldots, n$. Then (5) can be restated as $\|A-B\|_k \leq \|\lambda_{\downarrow}(A) - \lambda_{\uparrow}(B)\|_k$, $k = 1, 2, \ldots, n$. So the second inequality in (1) holds for this special class of norms and hence, by a well-known theorem of Ky Fan, for every unitarily invariant norm. (See [**4**].)

It should be remarked that Sunder's paper contains a stronger result in that it also establishes an analogue of the second inequality in (1) for the case when A, B and $A-B$ are all normal. Under these conditions an analogue of the first inequality in (1) has been established in [**1**].

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