PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 117, Number 1, January 1993

COMPLEXITY OF WINNING STRATEGIES FOR Δ_2^0 GAMES

RANA BARUA

(Communicated by Andreas R. Blass)

ABSTRACT. For a Δ_2^0 game played on ω, we show that the winning player has a winning strategy that is recursive in $\mathbb{E}_1$, where $\mathbb{E}_1$ is the total type-2 object that embodies operation $\mathscr{A}$.

0. INTRODUCTION

The complexity of winning strategies for certain definable games played on ω can, quite often, be expressed in terms of recursion in (appropriate) higher types. The simplest and the earliest known result is for Π_1^0 games. It is a well-known result in descriptive set theory that if player II wins a Π_1^0 game then she has a winning strategy that is Δ_1^1 recursive. Since the Δ_1^1 sets are precisely the sets that are recursive in ${}^2\mathbb{E}$, the Kleene's type-2 object that embodies countable $\cup$ (cf. [5]), player II has a winning strategy that is recursive in ${}^2\mathbb{E}$. For Σ_2^0 games, there is an analogous result. Solovay has shown that a set is $\mathfrak{G}\Sigma_2^0$ if and only if it is Σ_1^1-Ind, where $\mathfrak{G}$ denotes the game quantifier (see [10, 7C.10]). But the Σ_1^1-Ind sets are precisely the sets that are semi-recursive in $\mathbb{E}_1^\#$, by a result of Aczel [1], where $\mathbb{E}_1^\#$ is the partial type-2 object that embodies operation $\mathscr{A}$ (see also [5]). Thus $\mathfrak{G}\Sigma_2^0$ is the class of sets that are semirecursive in $\mathbb{E}_1^\#$. The Third Periodicity Theorem of Moschovakis [10] coupled with the above-mentioned fact shows that if player I wins a Σ_2^0 game then he has a winning strategy that is recursive in $\mathbb{E}_1^\#$. More recently John [7] showed that the complexity of winning strategies for Σ_3^0 games is related to recursion in Kolmogorov's operator $\mathbb{R}$. Quite naturally, one would like to have similar results for definable games that are in the higher levels of the arithmetical hierarchy. Such results, it appears, are quite difficult to obtain. In this short note we show that for Δ_2^0 games the result of Solovay-Aczel can be improved, i.e., we show that for Δ_2^0 games the winning strategy can be chosen to be recursive in $\mathbb{E}_1$, the (total) type-2 object that embodies operation $\mathscr{A}$. Although this result may not be of much significance, we feel that the proof, which is an adaptation of the techniques developed by Burgess in [4], could be useful in analysing Δ_n^0 games for $n \geq 3$.

Received by the editors December 20, 1990 and, in revised form, May 10, 1991.
1991 *Mathematics Subject Classification*. Primary 03D65, 03E15.

1. Notation

We denote the set of natural numbers by ω. The letters $i, j, k, m, n, \ldots$ will stand for natural numbers. SEQ will denote the set of sequence (or Gödel) numbers of finite sequences of natural numbers. The letters $s, t, \ldots$ will denote finite sequences of natural numbers as well as their sequence numbers. If $s, t \in SEQ$ then $s * t$ denotes the sequence number of the concatenation of s followed by t. If $s, t \in SEQ$ then we write $s \subseteq t$ if t extends s (both considered as finite sequences).

The set of infinite sequences of natural numbers will be denoted by ω^ω. Elements of ω^ω will be called reals and are denoted by $\alpha, \beta, \gamma, \delta, \ldots$. If $\alpha \in \omega^\omega$ and $n \in \omega$, then $\overline{\alpha}(n) = \langle \alpha(0), \alpha(1), \ldots, \alpha(n-1) \rangle$. WO denotes the (codes of) wellorderings of ω. If $\alpha \in \mathrm{WO}$, then $|\alpha|$ is the ordinal of the wellordering coded by α and $\leq_\alpha$ denotes the wellordering. If $P \subseteq X \times Y$, then for $x \in X$, P_x denotes the vertical section $\{y : P(x, y)\}$.

The symbol $\mathbb{E}_1$ stands for the type-2 object that embodies operation $\mathscr{A}$ and is defined by

$$\mathbb{E}_1(\alpha) = \begin{cases} 0 & \text{if } (\exists \beta)(\forall n)(\alpha(\overline{\beta}(n)) = 0), \\ 1 & \text{otherwise.} \end{cases}$$

The symbol $\mathbb{E}_1'$ stands for the type-2 object that embodies the open game quantifier. Thus

$$\mathbb{E}_1'(\alpha) = \begin{cases} 0 & \text{if } \exists a_0 \forall b_0 \exists a_1 \forall b_1 \cdots \exists n (\alpha(\langle a_0, b_0, \ldots, a_{n-1}, b_{n-1} \rangle) = 0), \\ 1 & \text{otherwise.} \end{cases}$$

The symbol $\mathbb{E}_1''$ is the type-2 object that embodies the closed game quantifier and is defined analogously. It is well known that $\mathbb{E}_1, \mathbb{E}_1', \mathbb{E}_1''$ are in the same degree, i.e., they are recursive in each other.

Unexplained notation and terminology from descriptive set theory are as in Moschovakis [10]; those from recursion theory are as in Hinman [5].

2. The main result

Fix a recursive function ϕ such that if $\alpha \in \mathrm{WO}$, the set of wellorderings of ω, then $\phi(\alpha, n)$ codes the strict initial segment of $\leq_\alpha$ with top n whenever $n \in \mathrm{Field}(\leq_\alpha)$; otherwise, it codes the empty relation.

The following is an effective version of a result of Hausdorff (cf. [8, §34VI]), due to Burgess, whose proof can be found in [2] (see also [9]).

Lemma. *Let E be a $\Delta_1^1 \cap \Delta_2^0$ subset of $\mathscr{Z} = \omega^l \times (\omega^\omega)^k$. Then there is a recursive ordinal μ such that for every recursive code α of μ there is a Δ_1^1, closed set $F \subseteq \omega \times \mathscr{Z}$ with the following properties:*

(a) $F_n \subseteq F_m$, *if* $m <_\alpha n$;
(b) $F_n = \bigcap_{n' <_\alpha n} F_{n'}$, *if* $|\phi(\alpha, n)|$ *is limit;*
(c) $E = \bigcup \{F_n - F_{n^+} : |\phi(\alpha, n)|$ *is even* & $n <_\alpha n^+$ & $|\phi(\alpha, n^+)| = |\phi(\alpha, n)| + 1\}$.

The proof of the next result is based on the techniques in [4]. The idea of the proof has been extracted from the Characterization Theorem in [4], in which Burgess proves that $\mathfrak{G}\Delta_3^0$ sets are the R-sets of Kolmogorov (see §12 of [4]).

Theorem 1. *If $A \subseteq \mathscr{Z}$, where $\mathscr{Z} = \omega^l \times (\omega^\omega)^k$, is in $\mathfrak{G}\Delta_2^0$ then A is recursive in $\mathbb{E}_1$.*

Proof. First observe that since a Δ_1^1, closed set is $\Pi_1^0(\gamma)$ for some Δ_1^1 real γ, we shall drop the parameter γ in our proof. Now, fix a Π_1^0 set $G \subseteq \omega \times \omega \times \omega^\omega \times \mathscr{Z}$ that is a good ω-universal for Π_1^0 subsets of $\omega \times \omega^\omega \times \mathscr{Z}$ (cf. [10]). We shall show that the following predicate is recursive in $\mathbb{E}_1$:

$$(*) \quad P(e, \alpha, x) \leftrightarrow \alpha \in \mathrm{WO}\ \&\ G_e \text{ satisfies (a) and (b) of the above lemma with respect to } \alpha\ \&\ \mathfrak{G}\beta D(e, \alpha, \beta, x),$$

where $\mathfrak{G}$ is the game quantifier, and

$$(\star) \quad D_{e,\alpha} = \bigcup\{G_{e,n} - G_{e,n^+} : |\phi(\alpha, n)| \text{ is even } \& \ |\phi(\alpha, n^+)| = |\phi(\alpha, n)| + 1\}.$$

Note that this will prove our theorem. To see this, observe that if $Q \subseteq \mathscr{Z}$ is $\mathfrak{G}\Delta_2^0$ then for some Δ_2^0 set $S \subseteq \omega^\omega \times \mathscr{Z}$ we have

$$Q(x) \leftrightarrow \mathfrak{G}\beta S(\beta, x).$$

Hence, by the lemma (ignoring the parameter γ that is involved), there is a recursive $\alpha \in \mathrm{WO}$ and $e \in \omega$ such that G_e satisfies properties (a) and (b) of the lemma with respect to α and

$$S = \bigcup\{G_{e,n} - G_{e,n^+} : |\phi(\alpha, n)| \text{ is even } \& \ |\phi(\alpha, n^+)| = |\phi(\alpha, n)| + 1\}.$$

Consequently, Q is the (e, α)-section of P (defined as in $(*)$ above) and, since α is recursive, Q is recursive in $\mathbb{E}_1$. Thus we only need to show that P is recursive in $\mathbb{E}_1$. To obtain this, fix a recursive set R such that

(i) $G(e, n, \beta, x) \leftrightarrow (\forall k) R(e, n, \overline{\beta}(k), x)$;
(ii) $s \in SEQ\ \&\ R(e, n, s, x)\ \&\ t \subseteq s \rightarrow R(e, n, t, x)$.

We now define a function $\psi(c, e, \alpha, x)$ as follows.

$$\begin{aligned}
\psi(c, e, \alpha, x) &\simeq \mathbb{E}_1''(\lambda s \cdot \chi_R(e, n^*, s, x)) \quad \text{if } |\alpha| = 1\ \&\ n^* \in \mathrm{Field}(\leq_\alpha);\\
&\simeq \mathbb{E}_1'(\lambda s \cdot {}^2\mathbb{E}(\lambda n \cdot \mathrm{Even}[\phi(\alpha, n), \{c\}^{\mathbb{E}_1}(\mathbb{S}(e^*, e, s), \phi(\alpha, n^+), x)]))\\
&\qquad \text{if } |\alpha| \text{ is even } \&\ G_e \text{ satisfies (a), (b) of the lemma w.r.t. } \alpha;\\
&\simeq \mathbb{E}_1'(\lambda s \cdot {}^2\mathbb{E}(\lambda n \cdot \mathbb{E}_1''(\lambda t \cdot \chi_R(e, n^*, s * t, x))\{c\}^{\mathbb{E}_1}(\mathbb{S}(e^*, e, s), \phi(\alpha, n^*), x)))\\
&\qquad \text{if } |\alpha| \text{ is odd } \&\ G_e \text{ satisfies (a), (b) of the lemma w.r.t. } \alpha;\\
&\simeq 1 \quad \text{otherwise},
\end{aligned}$$

where,

$$\mathrm{Even}(\alpha, n) = 0 \quad \text{iff} \quad \alpha \in \mathrm{WO}\ \&\ |\alpha| \text{ is even } \&\ n = 0,$$

n^* is the largest element of $\leq_\alpha$ when $|\alpha|$ is odd, and the recursive function $\mathbb{S}$ and the integer e^* are obtained as follows.

Put

(iii) $F(e, s, m, \beta, x) \leftrightarrow (\forall j) R(e, m, s * \overline{\beta}(j), x)$.

Clearly, F is Π_1^0. Hence there is an $e^* \in \omega$ such that

(iv) $F(e, s, m, \beta, x) \leftrightarrow \widetilde{G}(e^*, e, s, m, \beta, x) \leftrightarrow G(\mathbb{S}(e^*, e, s), m, \beta, x)$,

by the Good Parametrization Lemma [10, 3H.4]; $\widetilde{G}$ being a good ω-universal for Π_1^0 subsets of $(\omega)^3 \times \omega^\omega \times \mathscr{Z}$.

Since Π^1_1 sets are easily seen to be recursive in $\mathbb{E}_1$, it is easy to check that ψ is recursive in $\mathbb{E}_1$. Hence by the Recursion Theorem there is $c^* \in \omega$ such that

$$\psi(c^*, e, \alpha, x) \simeq \{c^*\}^{\mathbb{E}_1}(e, \alpha, x).$$

We shall prove by induction on $|\alpha|$ that $\{c^*\}^{\mathbb{E}_1}(e, \alpha, x)$ is the characteristic function of P. This will establish that P is recursive in $\mathbb{E}_1$.

If $|\alpha| = 1$, the result can be easily established. So assume that $|\alpha| > 1$ and G_e satisfies properties (a) and (b) of the lemma. To fix ideas, assume that $|\alpha|$ is an even ordinal. We shall show that

$$\begin{aligned} &\mathfrak{G}\beta D(e, \alpha, \beta, x) \\ &\quad \leftrightarrow (\exists a_0)(\forall b_0)(\exists a_1)(\forall b_1)\cdots(\exists i)(\exists n) \\ (**) &\qquad \mathfrak{G}\beta\{(\exists m \leq_\alpha n)(|\phi(\alpha, n)| \text{ is even } \& \ |\phi(\alpha, m)| \text{ is even} \\ &\qquad\quad \& \ (\forall j)R(e, m, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle * \overline{\beta}(j), x) \\ &\qquad\quad \& \ \sim(\forall k)R(e, m^+, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle * \overline{\beta}(k), x))\} \end{aligned}$$

where D is as in $(\star)$ above. Assume that we have obtained $(**)$. Then, in view of (iii) and (iv), $P(e, \alpha, x)$ holds iff

$$\begin{aligned} &(\exists a_0)(\forall b_0)\cdots(\exists i)(\exists n)\{|\phi(\alpha, n)| \text{ is even} \\ &\qquad \& \ P(\mathbb{S}(e^*, e, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle), \phi(\alpha, n^+), x)\}, \end{aligned}$$

i.e., iff

$$\begin{aligned} &(\exists a_0)(\forall b_0)\cdots(\exists i)(\exists n)\{|\phi(\alpha, n)| \text{ is even} \\ &\qquad \& \ \{c^*\}^{\mathbb{E}_1}(\mathbb{S}(e^*, e, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle), \phi(\alpha, n^+), x) = 0\}, \end{aligned}$$

by induction hypothesis, i.e., iff $\psi(c^*, e, \alpha, x) = 0$, by the definition of ψ. Thus it remains to show that $(**)$ is valid. Now observe that $(**)$ is plainly equivalent to the following:

$$\begin{aligned} &(\exists a_0)(\forall b_0)(\exists a_1)(\forall b_1)\cdots D(e, \alpha, (a_0, b_0, a_1, b_1, \dots), x) \\ &\quad \leftrightarrow (\exists a_0)(\forall b_0)(\exists a_1)(\forall b_1)\cdots(\exists i) \\ &\qquad \{(\exists n)(\exists a'_0)(\forall b'_0)(\exists a'_1)(\forall b'_1)\cdots(\exists m \leq_\alpha n) \\ &\qquad (|\phi(\alpha, n)| \text{ and } |\phi(\alpha, m)| \text{ are even} \\ (***) &\qquad \& \ (\forall j)R(e, m, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle \\ &\qquad\qquad * \langle a'_0, b'_0, \dots, a'_{j-1}, b'_{j-1}\rangle, x) \\ &\qquad \& \ \sim(\forall k)R(e, m^+, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle \\ &\qquad\qquad * \langle a'_0, b'_0, \dots, a'_{k-1}, b'_{k-1}\rangle, x))\}. \end{aligned}$$

The game on the right-hand side of $(***)$ will be called the $\mathscr{R}$-game and the game on the left will be called the $\mathscr{L}$-game. To see $(***)$, first assume that $\exists$ wins the $\mathscr{L}$-game with strategy τ. Now $\exists$ wins the $\mathscr{R}$-game cleverly using the strategy τ as described below. Observe that the $\mathscr{L}$-game is a game of length ω whereas the $\mathscr{R}$-game is, essentially, a game of length $\omega + \omega$. Thus, after ω many moves, i.e., after the end of the first *subgame*, $\exists$ is able to use the strategy τ provided he ignores all but the first finitely many moves of the subgame. This he is able to do since the winning condition of the $\mathscr{R}$-game depends only on the first $2i$ moves of the first subgame, for some i. Thus $\exists$ plays as follows. He plays (the first subgame) according to the strategy τ so that after ω moves the sequence $(a_0, b_0, \dots)$ is produced, where a_i's are the moves of $\exists$ as dictated by τ and b_i's are the moves of $\forall$. Since the sequence $(a_0, b_0, a_1, b_1, \dots)$ is consistent with τ and τ is a winning strategy for the $\mathscr{L}$-game we must have $D(e, \alpha, (a_0, b_0, \cdots), x)$. By the definition of D and (i), this implies that there is an $n \in \text{Field}(\leq_\alpha)$ such that, for some i,

$$-R(e, n^+, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle, x);$$

where $|\phi(\alpha, n)|$ is even and $|\phi(\alpha, n^+)| = |\phi(\alpha, n)| + 1$. Furthermore, we have

$$R(e, n, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle, x).$$

Thus after ω moves $\exists$ plays these integers i and n so obtained. To continue the $\mathscr{R}$-game using the strategy τ the player $\exists$ ignores the moves $a_i, b_i, a_{i+1}, b_{i+1}, \dots$, i.e., he assumes that the game has been played upto stage i producing the sequence of moves $a_0, b_0, \dots, a_{i-1}, b_{i-1}$ and then continues to play according to the strategy τ. This enables $\exists$ to play the second subgame using the strategy τ. Suppose the sequence $a_0, b_0, \dots, a_{i-1}, b_{i-1}, a'_0, b'_0, a'_1, b'_1, \dots$ of moves consistent with τ is produced—a'_j, b'_j being the moves of $\exists$ and $\forall$ in the second subgame. Since the above sequence is consistent with τ and τ is a winning strategy for $\exists$ in the $\mathscr{L}$-game, we have, for some m with $|\phi(\alpha, m)|$ even,

$$\begin{aligned}(\forall j)&R(e, m, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle * \langle a'_0, b'_0, \dots, a'_{j-1}, b'_{j-1}\rangle, x)\\ &\ \&\ \sim (\forall k)R(e, m^+, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle\\ &\qquad\qquad * \langle a'_0, b'_0, \dots, a'_{k-1}, b'_{k-1}\rangle, x).\end{aligned}$$

In particular, we have $R(e, m, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle, x)$. If $n <_\alpha m$ then $n^+ \leq_\alpha m$, and hence $R(e, n^+, \langle a_0, b_0, \dots, a_{i-1}, b_{i-1}\rangle, x)$ holds. This contradicts the choice of n. Thus $m \leq_\alpha n$ and so $\exists$ wins the $\mathscr{R}$-game.

Conversely, suppose $\exists$ wins the $\mathscr{R}$-game with strategy σ, say. To win the $\mathscr{L}$-game $\exists$ uses the strategy σ judiciously. If he simply plays as dictated by σ then it will amount to playing the first subgame of the $\mathscr{R}$-game and this alone will not ensure a winning condition for $\exists$ in the $\mathscr{L}$-game. So $\exists$ adopts the following tactics. He will play the $\mathscr{L}$-game as dictated by σ *until a stage* $i-1$ *is reached with the property that there exist moves* $b_i, b_{i+1}, \cdots$ *of* $\forall$ *at stages* $i, i+1$, *etc., such that if these are played by* $\forall$ *after the stage* $i-1$ *and* $\exists$ *plays with strategy* σ, *the strategy* σ *would have forced* $\exists$ *to play this integer* i satisfying the predicate within { } in $(***)$. Such an integer i exists. Otherwise, after every stage $i-1$, whatsoever the moves $b_i, b_{i+1}, \cdots$ of $\forall$ at stages $i, i+1, \dots$ are, with $\exists$ following the strategy σ, the condition within { } in $(***)$ will not be satisfied. Using the determinacy of Δ_2^0 games, this shows that $\forall$ is able to beat the strategy σ in the $\mathscr{R}$-game. This contradicts the fact that σ is a winning strategy for $\exists$ in the $\mathscr{R}$-game. Thus such an integer i exists. After this stage, $\exists$ pretends that he is playing the *second* subgame of the $\mathscr{R}$-game, the moves of the first subgame being

$$\mathbf{a}_0, \mathbf{b}_0, \dots, \mathbf{a}_{i-1}, \mathbf{b}_{i-1}, a_i, b_i, a_{i+1}, b_{i+1}, \dots, i, n.$$

Observe that the moves in boldface represent the moves that have *actually* taken place b_i, b_{i+1}, etc., the simulated moves of $\forall$ at positions i, $i+1$, etc. (in the first subgame of $\mathscr{R}$-game) whose existence we have already proved and a_i, a_{i+1}, etc. the moves of $\exists$ as dictated by σ. Now suppose $\mathbf{a}'_0, \mathbf{b}'_0, \mathbf{a}'_1, \mathbf{b}'_1, \ldots$ represent the moves in the second subgame of the simulated $\mathscr{R}$-game, where $\exists$ plays according to strategy σ. Thus $\exists$ plays the $\mathscr{L}$-game pretending that he is playing the $\mathscr{R}$-game where $\forall$ has made the moves $b_i, b_{i+1}, \ldots$ in the first subgame. Note that the sequence $\mathbf{a}_0, \mathbf{b}_0, \ldots, \mathbf{a}_{i-1}, \mathbf{b}_{i-1}, \mathbf{a}'_0, \mathbf{b}'_0, \mathbf{a}'_1, \mathbf{b}'_1, \ldots$ constitute a play in the $\mathscr{L}$-game, while the corresponding simulated play of the $\mathscr{R}$-game is

$$\begin{gathered}\mathbf{a}_0, \mathbf{b}_0, \ldots, \mathbf{a}_{i-1}, \mathbf{b}_{i-1}, a_i, b_i, a_{i+1}, b_{i+1}, \ldots, i, n \\ \mathbf{a}'_0, \mathbf{b}'_0, \mathbf{a}'_1, \mathbf{b}'_1, \ldots .\end{gathered}$$

Since in the above play $\exists$ plays according to σ and σ is a winning strategy in the $\mathscr{R}$-game, we have for some m such that $|\phi(\alpha, m)|$ is even

$$\begin{aligned}&(\forall j)R(e, m, \langle a_0, b_0, \ldots, a_{i-1}, b_{i-1}, a'_0, b'_0, \ldots, a'_{j-1}, b'_{j-1}\rangle, x) \\ &\qquad \& \sim (\forall k)R(e, m^+, \langle a_0, b_0, \ldots, a_{i-1}, b_{i-1}, a'_0, b'_0, \ldots, a'_{k-1}, b'_{k-1}\rangle, x).\end{aligned}$$

In view of (i) and (ii) this shows that $\exists$ wins the $\mathscr{L}$-game. Thus equivalence $(***)$ holds and the proof is complete.

By the Third Periodicity Theorem of Moschovakis [10, 6E], the winning player for a Δ^0_2 game has a winning strategy that is $\mathfrak{G}\Delta^0_2$-recursive, and thus we have

Theorem 2. *If* I *or* II *wins a* Δ^0_2 *game then he has a winning strategy that is recursive in* $\mathbb{E}_1$.

The next result shows that $\mathfrak{G}\Delta^0_2$ cannot exhaust ${}_2\mathrm{sc}(\mathbb{E}_1)$.

Theorem 3. *There is a set of reals recursive in* $\mathbb{E}_1$ *that is not* $\mathfrak{G}\Delta^0_2$.

Proof. First observe that by a result of Burgess [4], $\mathfrak{G}\Delta^0_2$ sets are $\mathscr{C}$-sets, where $\mathscr{C}$ is the smallest σ-field containing the closed sets and closed under operation $\mathscr{A}$. By [3] there is a set of reals recursive in $\mathbb{E}_1$ that is not in $\mathscr{C}$. The result now follows immediately.

Corollary. $\mathfrak{G}\Delta^0_2 \subsetneqq {}_2\mathrm{sc}(\mathbb{E}_1) \subsetneqq \mathfrak{G}\Sigma^0_2 \cap \mathfrak{G}\Pi^0_2$.

Proof. By the result of Solovay-Aczel, $\mathfrak{G}\Sigma^0_2 \cap \mathfrak{G}\Pi^0_2$ is precisely the set of reals that are recursive in $\mathbb{E}^\#_1$. Since there is a set recursive in $\mathbb{E}^\#_1$ that is not recursive in $\mathbb{E}_1$, the result follows from Theorems 1 and 3.

Remarks. 1. Notice that for sets of numbers, we have the inclusion $\mathfrak{G}\Delta^0_2 \subseteq {}_1\mathrm{sc}(\mathbb{E}_1)$. But sets of numbers recursive in $\mathbb{E}_1$ are precisely the effective $\mathscr{C}$-sets as was shown by Hinman in his fundamental paper [6]. Hence for sets of numbers the above method does not seem to work and we do not know whether $\mathfrak{G}\Delta^0_2$ exhausts sets of numbers recursive in $\mathbb{E}_1$, although we suspect this is not the case.

2. The result of Burgess mentioned in the proof of Theorem 3 is not explicitly proved in [4]. One direction of the result viz. $\mathfrak{G}\Delta^0_2 \subseteq \mathscr{C}$-sets, can be obtained

as in the proof of Theorem 1. The reverse inclusion can be proved in a much similar fashion.

ACKNOWLEDGMENT

We would like to thank the referee for his valuable suggestions.

REFERENCES

1. P. Aczel, *Representations of some systems of second order arithmetic*, Israel J. Math. **8** (1970), 309–328.
2. R. Barua, *Structure of hyperarithmetical sets of ambiguous Borel classes*, I. S. I. Tech. Report no. 15/85, 1985.
3. ——, *A note on* ${}_2\mathrm{sc}(\mathbb{E}_1)$, Proc. Amer. Math. Soc. **103** (1988), 921–925.
4. J. P. Burgess, *Classical hierarchies from a modern standpoint, Part* I: *$\mathscr{C}$-sets*, Fund. Math. **115** (1983), 81–95; *Part* II: *R-sets*, 97–105.
5. P. G. Hinman, *Recursion theoretic hierarchies*, Springer-Verlag, Berlin, Heidelberg, and New York, 1978.
6. ——, *Hierarchies of effective descriptive set theory*, Trans. Amer. Math. Soc. **142** (1969), 111–140.
7. T. John, *Recursion in Kolmogorov's $\mathbb{R}$-operator and the ordinal* σ_3, J. Symbolic Logic **51** (1986), 1–11.
8. K. Kuratowski, *Topology*, Vol. I, Academic Press, New York and London, 1966.
9. A. Louveau, *A separation theorem for* Σ_1^1 *sets*, Trans. Amer. Math. Soc. **260** (1980), 363–378.
10. Y. N. Moschovakis, *Descriptive set theory*, North-Holland, Amsterdam, New York, and Oxford, 1980.

DIVISION OF STATISTICS AND MATHEMATICS, INDIAN STATISTICAL INSTITUTE, 203 B. T. ROAD, CALCUTTA, INDIA

E-mail address: shakti!ncst.ernet.in!isical!rana