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A LIMIT THEOREM FOR THE IMBALANCE OF A ROTATING WHEEL

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SUMMARY. A rotating wheel fixed with n blades in a symmetrical fashion is subject to centrifugal forces exerted by the blades. We study the asymptotic nature of the resulting force acting on the wheel (as n increases) under the assumption that the blades come from a statistically controlled manufacturing process.

1. Introduction

Consider a rotating wheel with a specified number of blades fixed radially and symmetrically along the circumference of the wheel. The wheel is then subject to centrifugal forces exerted by the blades. The force exerted by each blade is proportional to its moment which is by definition the product of the blade weight and the distance between the wheel centre and the centre of the gravity of the blade. The magnitude of the net resulting force exerted by all the blades is called the *imbalance.* Due to process variability of the moments at the manufacturing stage, the imbalance is non-zero almost all the time. However, the engineering considerations require this to be small. This problem is not uncommon in engineering industry. A rotor of a steam turbine generator provides a good example in this regard.

The following combinatorial optimization problem is of considerable applicational interest. Given the specified number of blades of known moments, how to fix them at symmetric locations on the circumference of the wheel so that the imbalance is minimum? Murthy (1976), p. 416, has formulated this as a quadratic assignment problem. In this paper, we look at the wheel balancing problem in a different angle. We assume that the blades come from a manufacturing process that is under statistical control and they are fixed at the locations in a purely random order. Consequently, the imbalance is also random. We show that the asymptotic distribution of the imbalance (with a change of scale)

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is a χ-distribution with 2 degrees of freedom. In many practical situations, the number of blades in a rotor ranges from 150 to 250 and therefore the asymptotic approximation is very much valid.

2. The statistical model

To model the physical system described above, we first make a simplifying assumption that all the forces act in the same plane. This enables us to find the net resultant force, that is, the imbalance by resolving each of the forces in two specified mutually perpendicular directions. Let ρ denote the radius of the wheel and n the specified number of blades. Mark n symmetric positions on the circumference of the wheel as 1,2, ..., n. Note that any two adjacent positions make an angle of $2\pi/n$ radians at the centre of the wheel. Take the centre of the wheel as the origin and the line passing through the origin and position n as x-axis. Take the line perpendicular to x-axis as y-axis. Suppose the blades are fixed to all the n positions of the wheel in a random order. Index the blade fixed to r−th position $(1 \leq r \leq n)$ as r and let w_r be the weight of blade r and c_r the distance between its fixing end and its centre of gravity. Assume that the centre of gravity of blade r lies on the line joining the origin and position r. The centrifugal force exerted by blade r is proportional to $Z_r = (\rho + c_r)w_r$. To keep the description simple, we treat Z_r itself as the centrifugal force exerted by blade r.

Let $\alpha_{nr} = \cos\,(2\pi r/n)$ and $\beta_{nr} = sin(2\pi r/n)$ for $r = 1$ to n. The centrifugal force Z_r of blade r can be resolved into components $\alpha_{nr}Z_r$ and $\beta_{nr}Z_r$ along x-axis and y-axis respectively. Thus the components of the resulting force along x-axis and y-axis are

$$X_n = \sum_{r=1}^{n} \alpha_{nr} Z_r$$

$$Y_n = \sum_{r=1}^{n} \beta_{nr} Z_r$$

and the imbalance is $\sqrt{X_n^2 + Y_n^2}$.

In this paper, we shall investigate the asymptotic distribution of $\sqrt{X_n^2 + Y_n^2}$.

3. Notation and preliminary results

Since all the blades come from a statistically controlled manufacturing process, $Z_1, Z_2, ..., Z_n$ are independent and identically distributed non-negative random variables. Assume that the first three moments exist and let $\mu = E(Z_r)$, $\sigma^2 = V(Z_r) \neq 0$ and $\gamma = E \mid Z_r - \mu \mid^3$. This assumption usuallly holds in practical situations.

Lemma 1.

(i) $\sum_{r=1}^{n} \alpha_{nr} = \sum_{r=1}^{n} \beta_{nr} = 0 \; for \; n \geq 2$

(ii) $\sum_{r=1}^{n} \alpha_{nr}\beta_{nr} = 0 \; for \; n \geq 2$

(iii) $\sum_{r=1}^{n} \alpha_{nr}^2 = \sum_{r=1}^{n} \beta_{nr}^2 = n/2 \; for \; n \geq 3.$

Proof. The proof makes use of the fact that the sum and the sum of squares of all n-th roots of unity are equal to zero for $n \geq 3$. For reference, see John (1980, 201-202).

Theorem 1. *For* $n \geq 3$,

(i) $E(X_n) = E(Y_n) = 0$

(ii) $E(X_n^2) = E(Y_n^2) = n\sigma^2/2$

(iii) $E(X_nY_n) = 0.$

Proof. Trivial consequence of Lemma 1.

Let a and b be two real numbers such that $(a, b) \neq (0, 0)$ and let

$$c_{nr} = \frac{1}{\sqrt{a^2+b^2}} \frac{\sqrt{2}}{\sigma\sqrt{n}}(a\alpha_{nr} + b\beta_{nr}) \qquad \ldots(1)$$

and

$$Z_{nr} = c_{nr}(Z_r - \mu) \qquad \ldots(2)$$

for $n \geq 1$ and $1 \leq r \leq n$. Let $F_{nr}(x)$ denote the distribution function of Z_{nr}, that is, $F_{nr}(x) = P(Z_{nr} < x)$ and let $F(x)$ be the distribution function of $(Z_r - \mu)$. Define

$$W_n = \sum_{r=1}^{n} Z_{nr} \text{ for } n \geq 1.$$

The double sequence $\{Z_{nr}, 1 \leq r \leq n, \quad n \geq 1\}$ is said to be an elementary system if it satisfies the following conditions.

(1) $Z_{n1}, Z_{n2}, \ldots, Z_{nn}$ are independent r.v.'s for any fixed n;
(2) $V(Z_{nr})$ is finite;
(3) $V(W_n)$ is bounded by a constant C not dependent on n;
(4) $\max_{1\leq r\leq n} V(Z_{nr}) \to 0$ as $n \to \infty$.

If the sequence satisfies, in addition, two more conditions
(5) $E(Z_{nr}) = 0$ for $1 \leq r \leq n$;
(6) $V(W_n) = 1$;

for $n \geq 1$, then for sequence is said to be a normalised elementary system.

Theorem 2. (Gnedenko, 1978, p. 284). *A necessary and sufficient condition for the convergence of sums W_n's of a normalised elementary system to $N(0,1)$ in distribution is*

$$\sum_{r=1}^{n} \int_{|x|>\tau} x^2 dF_{nr}(x) \to 0 \text{ as } n \to \infty \text{ for any } \tau > 0 \qquad \ldots(3)$$

The condition (3) is called Lindberg condition.

4. Asymptotic distribution of the imbalance

We now show that the asymptotic distribution of $(\sqrt{2/n\sigma^2}X_n, \sqrt{2/n\sigma^2}Y_n)$ is a bivariate normal distribution and that of $(\frac{2}{n\sigma^2} X_n^2 + \frac{2}{n\sigma^2} Y_n^2)$ is a χ^2-distribution with 2 degrees of freedom.

Theorem 3. *The double sequence $\{Z_{nr}, 1 \leq r \leq n, n \geq 1\}$ is a normalised elementary system satisfying the Lindberg condition.*

Proof. Note that for any fixed n, the variables $Z_{n1}, Z_{n2}, ..., Z_{nn}$ are independent with

$$E(Z_{nr}) = 0 \text{ and } V(Z_{nr}) = \frac{2(a\alpha_{nr} + b\beta_{nr})^2}{n(a^2+b^2)} \qquad \ldots(4)$$

It is obvious that for any finite a and b,

$$V(Z_{nr}) \leq \frac{2(|a| + |b|)^2}{n(a^2+b^2)} < \infty$$

and

$$\max_{1 \leq r \leq n} V(Z_{nr}) \to 0 \text{ as } n \to \infty. \qquad \ldots(5)$$

We have

$$\begin{aligned} V(W_n) &= \sum_{r=1}^{n} V(Z_{nr}) \\ &= \tfrac{2}{n(a^2+b^2)} \sum_{r=1}^{n} (a\alpha_{nr} + b\beta_{nr})^2 \\ &= \tfrac{2}{n(a^2+b^2)} (na^2/2 + nb^2/2) \end{aligned}$$

The last equality holds due to Lemma 1. Thus we have

$$V(W_n) = 1. \qquad \ldots(6)$$

It follows from equations (4) to (6) that the double sequence $\{z_{nr'}, 1 \leq r \leq n, n \geq 1\}$ is a normalised elementary system.

We now show that the Lindberg condition also holds. It can be easily seen that

$$\int_{|x|>\tau} x^2 dF_{nr}(x) = c_{nr}^2 \int_{|x|>\tau/|c_{nr}|} x^2 dF(x)$$

and

$$\begin{aligned}\sum_{r=1}^{n} \int_{|x|>\tau} x^2 dF_{nr}(x) &= \sum_{r=1}^{n} c_{nr}^2 \Big(\int_{|x|>\tau/|c_{nr}|} x^2 dF(x)\Big) \\ &\leq \sum_{r=1}^{n} c_{nr}^2 \mid \frac{c_{nr}}{\tau} \mid \Big(\int_{|x|>\tau/|c_{nr}|} \mid x^3 \mid dF(x)\Big) \\ &\leq \tfrac{1}{\tau}\Big(\sum_{r=1}^{n} \mid c_{nr} \mid^3\Big) E \mid Z_n - \mu \mid^3\end{aligned}$$

It is easy to verify that

$$\sum_{r=1}^{n} \mid c_{nr} \mid^3 \leq \frac{1}{\sqrt{n}} \left[\frac{\sqrt{2}(\mid a \mid + \mid b \mid)}{\sigma\sqrt{a^2+b^2}}\right]^3$$

Therefore,

$$\sum_{r=1}^{n} \int_{|x|>\tau} x^2 dF_{nr}(x) \leq \frac{1}{\tau\sqrt{n}} \left[\frac{\sqrt{2}(\mid a \mid + \mid b \mid)}{\sigma\sqrt{a^2+b^2}}\right]^3 \gamma.$$

Now the Lindberg condition holds since the value $\left[\frac{\sqrt{2}(|a|+|b|)}{\sigma\sqrt{a^2+b^2}}\right]^3 \gamma$ is finite and invariant of n.

It follows from Theorems 2 and 3 that for any real numbers a and b, $(a, b) \neq (0, 0)$, $W_n \xrightarrow{L} N(0, 1)$, that is,

$$a\frac{\sqrt{2}}{\sqrt{n}}X_n + b\frac{\sqrt{2}}{\sqrt{n}}Y_n \xrightarrow{L} N(0, \sigma^2(a^2+b^2)).$$

Thus we have for any real a and b

$$a\frac{\sqrt{2}}{\sigma\sqrt{n}}X_n + b\frac{\sqrt{2}}{\sigma\sqrt{n}}Y_n \xrightarrow{L} aX + bY \qquad \ldots(7)$$

where X and Y are two independent standard normal variables.

Theorem 4. (Rao, 1974, p. 123). *Let* $\{X_n^{(1)}, X_n^{(2)} \ldots, X_n^{(k)}\}$, $n = 1, 2, \ldots$ *be a sequence of vector random variables and let, for any real* $\lambda_1, \lambda_2, \ldots \lambda_k$,

$$\lambda_1 X_n^{(1)} + \ldots + \lambda_k X_n^{(k)} \xrightarrow{L} \lambda_1 X^{(1)} + \ldots + \lambda_k X^{(k)}$$

where $X^{(1)}, \ldots X^{(k)}$ have a joint distribution $F(x_1, \ldots, x_k)$. Then the limiting joint distribution function of $X_n^{(1)}, \ldots X_n^{(k)}$ *exists and is equal to* $F(x_1, \ldots, x_k)$.

From the convergence (7) and the above theorem, it follows that the limiting joint distribution of $(\frac{\sqrt{2}}{\sigma\sqrt{n}}X_n, \frac{\sqrt{2}}{\sigma\sqrt{n}}Y_n)$ is a bivariate normal distribution with mean vector $\mathbf{0}$ and dispersion matrix I. Further, the asymptotic distribution of $\frac{2}{\sigma^2 n}X_n^2 + \frac{2}{\sigma^2 n}Y_n^2$ is a χ^2-distribution with 2 degrees of freedom since the function $g(x, y) = x^2 + y^2$ is a continuous real valued Borel function. For reference, see the theorem on page 24 of Serfling (1980).

Therefore, the magnitude of the imbalance multiplied by $\sqrt{2}/(\sigma\sqrt{n})$ follows asymptotically χ-distribution with 2 degrees of freedom. Since the mean of a χ distribution with 2 degrees of freedom is $\sqrt{\pi/2}$, we can write for large n

$$E(\sqrt{X_n^2 + Y_n^2}) \approx (\sigma\sqrt{n\pi})/2. \qquad \ldots(8)$$

We also have from Theorem 1

$$E(X_n^2 + Y_n^2) \approx n\sigma^2 \qquad \ldots(9)$$

for $n \geq 3$. Therefore, we have

$$V(\sqrt{X_n^2 + Y_n^2}) \approx (1 - \pi/4)n\sigma^2 \qquad \ldots(10)$$

for large n.

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