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CHARACTERIZATION OF STOCHASTIC PROCESSES BY STOCHASTIC INTEGRALS

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Abstract

Let $\{X(t), t \in T\}$ be a continuous homogeneous stochastic process with independent increments. A review of the recent work on the characterization of Wiener and stable processes and connected results through stochastic integrals is presented. No proofs are given but appropriate references are mentioned.

WIENER PROCESS; STABLE PROCESSES; DOUBLE STOCHASTIC INTEGRAL; PROCESSES DETERMINED BY THEIR STOCHASTIC INTEGRALS; CHARACTERIZATION

0. Introduction

Our aim in this paper is to review the recent work in the area of characterization of stochastic processes by stochastic integrals. We have stated only the main theorems and indicated the references where the proofs of these results can be found. For an earlier survey paper in this area, see Lukacs (1970b).

Section 1 contains some definitions. Stochastic integrals are discussed in Section 2. Characterizations for Wiener process and stable processes through identically distributed stochastic integrals are given in Section 3. Characterization theorems for the Wiener process taking values in a Hilbert space are also presented in this section. Section 4 contains characterization theorems for Wiener processes through the property of independence of two stochastic integrals. Characterizations through properties of the conditional distribution of one stochastic integral with respect to another like symmetry of the conditional distribution or linearity or constancy of the regression are studied in Section 5. Here characterization theorems for Wiener and stable processes are given. In Section 6, sufficient conditions for the determination of a stochastic process by stochastic integrals are developed.

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1. Definitions

Let $T=[A, B]$. A stochastic process $\{X(t), t\in T\}$ is said to be a homogeneous process with independent increments if the distribution of the increment $X(t+h)-X(t)$, t, $t+h\in T$ depends only on h but not on t and if the increments over non-overlapping intervals are stochastically independent. The process is said to be continuous if $X(t)$ converges in probability to $X(s)$ as t tends to s for every $s\in T$. Unless otherwise stated we shall only consider *continuous homogeneous process with independent increments* throughout this paper. If $\theta(u; h)$ denotes the characteristic function of $X(t+h)-X(t)$, $t, t+h\in T$, it is well known that $\theta(u; h)$ is infinitely divisible. In fact $\theta(u; h)=[\theta(u; 1)]^h$ if $t, t+1\in T$ (cf. Lukacs (1975)).

2. Stochastic integrals

Let $\{X(t), t\in T\}$ be a continuous homogeneous process with independent increments. Suppose $a(t)$ is a continuous function defined on $[A, B]\subset T$. Stochastic integrals of the form

$$\int_A^B a(t)\, dX(t)$$

can be defined either in the sense of convergence in probability or in the sense of quadratic mean depending on the properties of the process $\{X(t), t\in T\}$. For details, see Lukacs (1975).

Let $b(t)$ and $w(t)$ be functions defined on $[A, B]\subset T=[0, \infty)$ and $w(t)$ be non-negative. Let

$$D_n: A=t_{n,0}<t_{n,1}<\cdots<t_{n,n}=B, \qquad n\geq 1$$

be a sequence of subdivisions of the interval $[A, B]$ such that

$$\lim_{n\to\infty} \max_{1\leq k\leq n} (t_{n,k}-t_{n,k-1})=0.$$

Select $t^*_{n,k}\in[t_{n,k-1}, t_{n,k}]$ and construct the sum

$$S_n=\sum_{k=1}^{n} b(t^*_{n,k})[X(w(t_{n,k}))-X(w(t_{n,k-1}))].$$

If the sequence S_n converges in probability to a random variable S and if this limit is independent of the choice of the subdivision and the points $t^*_{n,k}$, then we say that S exists in probability and it is denoted by

$$\int_A^B b(t)\, dX(w(t)).$$

If the limit exists in quadratic mean, then the integral is said to exist in quadratic mean.

These integrals were studied in Riedel (1980a). Ramachandran and Rao (1970) (cf. Kagan, Linnik and Rao (1973), Chapter 13) discussed similar integrals under slightly different conditions. We follow Riedel (1980a).

Suppose $w(t)$ is non-decreasing, non-negative and left continuous on $[A, B]$. Then it is known that there exists a finite Borel measure V on the real line such that

$$V[(-\infty, t)] = \begin{cases} 0 & \text{if} \quad t < A \\ w(t) - w(A) & \text{if} \quad A \leqq t < B \\ w(B) - w(A) & \text{if} \quad t > B. \end{cases}$$

Suppose further that $b(t)$ is continuous on $[A, B]$. Define

$$w_b(t) = V[\{s : b(s) \leqq t\}].$$

Then $w_b(t)$ is non-decreasing, non-negative and left continuous.

The following theorem is due to Riedel (1980a).

Theorem 2.1 (Riedel). Let $b(t)$ be a continuous function on $[A, B]$ and $w(t)$ be a non-decreasing, non-negative and left-continuous function on $[A, B]$. Define

$$C = \min_{A \leqq t \leqq B} b(t), \qquad D = \max_{A \leqq t \leqq B} b(t).$$

Then the integrals

$$Y = \int_A^B b(t)\, dX(w(t)) \quad \text{and} \quad Z = \int_C^D t\, dX(w_b(t))$$

exist in the sense of convergence in probability and they are identically distributed. Furthermore the characteristic function ϕ of the random variable Y is given by

$$\log \phi(u) = \int_A^B \log \psi[ub(t)]\, dw(t) = \int_C^D \log \psi(ut)\, dw_b(t)$$

where $\psi(u)$ is the logarithm of the characteristic function of $X(t+1) - X(t)$, $t, t+1 \in [A, B]$.

It is well known (cf. Lukacs (1970a)) that a characteristic function g is an infinitely divisible characteristic function if and only if it can be written in the form

$$\log g(u) = iau - \frac{\sigma^2}{2} u^2 + \int_{-\infty}^{0-} r(u, x)\, dM(x) + \int_{0+}^{\infty} r(u, x)\, dN(x)$$

where a is a real constant, $\sigma \geqq 0$, M and N are non-decreasing in the intervals $(-\infty, 0)$ and $(0, \infty)$ respectively with

$$M(-\infty) = N(\infty) = 0,$$

$$\int_{-\varepsilon}^{0-} x^2\, dM(x) < \infty \quad \text{and} \quad \int_{0+}^{\varepsilon} x^2\, dN(x) < \infty \quad \text{for every} \quad \varepsilon > 0,$$

and

$$r(u, x) = e^{iux} - 1 - \frac{iux}{1+x^2}.$$

Riedel (1980a) obtained the above Lévy–Khintchin canonical representation for the characteristic function of the stochastic integral

$$\int_A^B t\, dX(w(t)).$$

Theorem 2.2 (Riedel). Let $w(t)$ be as in Theorem 2.1. Let the canonical representation for the characteristic function of $X(t+1) - X(t)$ be given by a, σ, M and N as defined above. Then the Lévy–Khintchin canonical representation for the characteristic function of the stochastic integral

$$\int_A^B t\, dX(w(t))$$

is given by a_w, σ_w, M_w and N_w where

$$a_w = \int_A^B \left\{ ta + t(1-t^2) \int_{0+}^{\infty} \frac{x^3}{(1+(tx)^2)(1+x^2)}\, d(M(-x)+N(x)) \right\} dw(t),$$

$$\sigma_w^2 = \sigma^2 \int_A^B t^2\, dw(t),$$

$$M_w(x) = \int_{\min(A,0)}^{\min(B,0)} -N\left(\frac{x}{t}\right) dw(t) + \int_{\max(A,0)}^{\max(B,0)} M\left(\frac{x}{t}\right) dw(t), \qquad x < 0,$$

$$N_w(x) = \int_{\min(A,0)}^{\min(B,0)} -M\left(\frac{x}{t}\right) dw(t) + \int_{\max(A,0)}^{\max(B,0)} N\left(\frac{x}{t}\right) dw(t), \qquad x > 0.$$

Wang (1975) obtained sufficient conditions for the existence of double stochastic integrals of the form

$$\int_A^B \int_A^B g(s, t) X(ds) X(dt)$$

in the sense of convergence in quadratic mean. We shall briefly discuss his result. For $i = 1, 2$, let

$$D_i : A = t_{i0} < t_{i1} < \cdots < t_{in_i} = B$$

be a subdivision of $[A, B]$ and define

$$S(n_1, n_2) = \sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} g(t^*_{1j_1}, t^*_{2j_2}) X(\Delta t_{1j_1}, \Delta t_{2j_2})$$

where $t_{i,j_i-1} \leqq t^*_{ij_i} \leqq t_{ij_i}$ and

$$X(\Delta t_{1j_1}, \Delta t_{2j_2}) = \prod_{i=1}^{2} [X(t_{ij_i}) - X(t_{i,j_i-1})].$$

Suppose that $n_1 \to \infty$ and $n_2 \to \infty$ such that

$$\max_{j_1, j_2} \left\{ \prod_{i=1}^{2} (t_{ij_i} - t_{ij_{i-1}}) \right\} \to 0$$

and $S(n_1, n_2)$ converges in probability (or in quadratic mean) to a limiting random variable S independent of the sequence of subdivisions D_i, $i=1, 2$ and the intermediate points $\{t^*_{ij_i}\}$. Then the limit S is called a double stochastic integral and it is denoted by

$$\int_A^B \int_A^B g(t_1, t_2)\, dX(t_1)\, dX(t_2).$$

The integral is said to exist in probability or in quadratic mean depending upon the type of convergence to S.

A function $\gamma(t_1, \cdots, t_l)$, $l \geqq 1$ is said to be of *bounded variation* on $[A, B]$ if there exists $0 < M < \infty$ independent of the subdivisions $D_1, \cdots, D_l$ of $[A, B]$ but possibly depending on $[A, B]$ such that

$$\sum_{i_1=1}^{n_1} \cdots \sum_{i_l=1}^{n_l} \Delta\gamma(t_{i_1}, \cdots, t_{i_l}) < M$$

where

$$\Delta\gamma(t_{i_1}, \cdots, t_{i_l}) = \prod |\gamma(t_{i_1}, \cdots, t_{i_l}) - \gamma(t_{i_{1'}}, \cdots, t_{i_{l'}})|$$

and the product is taken over all $(i_{1'}, \cdots, i_{l'})$ such that exactly one coordinate of $(i_{1'}, \cdots, i_{l'})$ is equal to the corresponding one of $(i_1 - 1, \cdots, i_l - 1)$ and the other $l-1$ are equal to those of $(i_1, \cdots, i_l)$.

Theorem 2.3 (Wang). Suppose g is continuous on $[A, B] \times [A, B]$ and the function $\gamma(t_1, t_2; s_1, s_2) = E[X(t_1)X(t_2)X(s_1)X(s_2)]$ is of bounded variation on $[A, B]$. Then the stochastic integral

$$\int_A^B \int_A^B g(t_1, t_2)\, dX(t_1)\, dX(t_2)$$

exists in quadratic mean.

In fact, if $E|X(t)|^4<\infty$, then under the conditions stated in Theorem 2.3, the double stochastic integral exists if and only if the Riemann–Stieltjes integral

$$\int_A^B \int_A^B g(t_1, t_2)g(s_1, s_2)\gamma(dt_1, dt_2, ds_1, ds_2)$$

exists. The reader is referred to Wang (1975) for the proof of Theorem 2.3. Wang (1975) proved the theorem for the k-dimensional stochastic integral, $k \geqq 2$. It is not known whether the double stochastic integral exists under weaker conditions in the sense of convergence in probability.

Before we conclude this section, we introduce another stochastic integral where the integrand is also a random process.

Suppose $\{R(t), t\in T\}$ is a random process with continuous sample paths and the process is independent of the process $\{X(t), t\in T\}$. One can define stochastic integrals of the form

$$S \equiv \int_A^B R(t)\, dX(t)$$

in the sense of convergence in probability through approximating sums as described above. It can be shown that the characteristic function of S when it is well defined is given by

$$E[e^{iuS}] = E\left[\exp\left\{\int_A^B \log \psi(uR(t))\, dt\right\}\right]$$

for u real, where $\psi(u)$ is the characteristic function of $X(t+1)-X(t)$, $t, t+1\in T$. For details, see Prakasa Rao (1982).

3. Characterization through identical distribution of two stochastic integrals

Let $\{X(t), t\in T\}$ be a continuous homogeneous process with independent increments. The process is called a *Wiener process* if the increments $X(t)-X(s)$ are normally distributed with variance proportional to $|t-s|$. The process is called a *stable process* with exponent γ if the increments of the process have stable distribution with exponent γ (cf. Lukacs (1970a)). It is said to be *symmetric stable* if the increments have symmetric stable distributions.

Characterization of the Wiener process

Theorem 3.1 (Laha and Lukacs). Let $T=[A, B]$. Suppose the process $\{X(t), t\in T\}$ has moments of all orders and let $a(t)$ and $b(t)$ be two continuous functions on $[A, B]$ such that

$$\max_{A\leqq t\leqq B} |a(t)| \neq \max_{A\leqq t\leqq B} |b(t)|.$$

Let
$$Y=\int_A^B a(t)\,dX(t) \quad \text{and} \quad Z=\int_A^B b(t)\,dX(t)$$
be two stochastic integrals defined as limits in quadratic mean. Then Y and Z are identically distributed if and only if (i) the process $\{X(t), t\in T\}$ is a Wiener process with linear mean function, (ii) either $\int_B^A a(t)\,dt=\int_A^B b(t)\,dt$ or the mean function is 0 and (iii) $\int_A^B a^2(t)\,dt=\int_A^B b^2(t)\,dt$.

The following theorem can be proved relaxing the assumption on the existence of moments of the process $\{X(t), t\in T\}$.

Theorem 3.2 (Laha and Lukacs). Let $T=[A, B]$ and $a(t)$ be continuous but not constant on $[A, B]$. Let $\alpha\neq 0$ be real such that either

(a) $\max_{A\leq t\leq B}|a(t)|<|\alpha|$ and $B-A>1$ or

(b) $\max_{A\leq t\leq B}|a(t)|>|\alpha|$ and $B-A<1$ holds. Let
$$Y=\int_A^B a(t)\,dX(t)$$
be defined in the sense of convergence in probability. Then $\{X(t), t\in T\}$ is a Wiener process with linear mean function if and only if

(i) Y is identically distributed as $\alpha[X(t+1)-X(t)]$,

(ii) either $\int_A^B a(t)\,dt=\alpha$ or the mean function is 0, and

(iii) $\int_A^B a^2(t)\,dt=\alpha^2$.

Proofs of Theorem 3.1 and 3.2 are given in Lukacs (1975). Other characterizations of the Wiener process through identically distributed stochastic integrals have been studied by Laha and Lukacs (cf. Lukacs (1975), Chapter 7), Ramachandran and Rao (1970) (cf. Kagan, Linnik and Rao (1973), Chapter 13) under slightly different conditions.

Theorem 3.3 (Ramachandran and Rao). Let $T=[A, B]$. Let $w(t)$ be a non-constant, non-decreasing, right-continuous function defined on a compact interval $[a, b]$ with $w(a)=A$ and $w(b)=B$. Let $g(t)$ be continuous on $[a, b]$ such that either (i) $|g(t)|<1$ for all t in $[a, b]$ and g has a finite number of zeros on $[a, b]$ or (ii) $|g(t)|\geqq 1$ for all t in $[a, b]$. Suppose
$$Y=\int_a^b g(t)\,dX(w(t))$$
(defined in the sense of convergence in probability) has the same distribution as the sum of n independent random variables each distributed as $X(t+(1/n))-X(t)$, $A\leqq t<t+(1/n)\leqq B$ for some $n\geqq 1/(B-A)$. Then $\{X(t), t\in[A, B]\}$ is a Wiener process with linear mean function if and only if
$$\int_a^b g^2(t)\,dw(t)=1.$$

Further, in that case,

$$\int_a^b g(t)\, dw(t) = 1$$

or the mean function is 0.

Theorem 3.4 (Ramachandran and Rao). Let $T=[A,B]$ and $w(t)$ be as defined in Theorem 3.3 and g and h be continuous functions on $[a,b]$ such that $\max|g(t)| \neq \max|h(t)|$ in $[a,b]$. Suppose the process $\{X(t), t \in T\}$ has moments of all orders. Let

$$Y = \int_a^b g(t)\, dX(w(t)) \quad \text{and} \quad Z = \int_a h(t)\, dX(w(t))$$

be defined as limits in quadratic mean. Then Y and Z are identically distributed if and only if

(i) the process $\{X(t), t \in [A,B]\}$ is a Wiener process with linear mean function

(ii) $\int_a^b g(t)\, dw(t) = \int_a^b h(t)\, dw(t)$ or the mean function is 0, and

(iii) $\int_a^b g^2(t)\, dw(t) = \int_a^b h^2(t)\, dw(t)$.

For proofs of Theorems 3.3 and 3.4, see Ramachandran and Rao (1970) and Kagan, Linnik and Rao (1973), Chapter 13.

Recently Riedel (1980b) obtained the following characterization theorems for the Wiener process as special cases of his general results for stable processes.

Theorem 3.5 (Riedel). Let $T=[0,\infty)$. Let $b_j(t)$ be continuous on $[A_j, B_j]$ and $w_j(t)$ be non-decreasing, non-negative and left continuous on $[A_j, B_j] \subset T$, $j=1,2$. Suppose $E[X(1)]^2 < \infty$. Then

$$\int_{A_1}^{B_1} b_1(t)\, dX(w_1(t)) \quad \text{and} \quad \int_{A_2}^{B_2} b_2(t)\, dX(w_2(t)) + q$$

are identically distributed for some real q if and only if $\{X(t), t \geqq 0\}$ is a Wiener process with linear mean function.

Let $b_j(t)$ and $w_j(t)$, $j=1,2$ be as defined above. For $\operatorname{Re}(z) \geqq 0$, define

$$S(z) = \int_{A_1}^{B_1} |b_1(t)|^z\, dw_1(t) - \int_{A_2}^{B_2} |b_2(t)|^z\, dw_2(t)$$

and

$$\hat{S}(z) = \int_{A_1}^{B_1} |b_1(t)|^{z-1}\, b_1(t)\, dw_1(t) - \int_{A_2}^{B_2} |b_2(t)|^{2-1} b_2(t)\, dw_2(t)$$

where $[A_j, B_j] \subset T$, $j=1,2$. Then $S(z)$ and $\hat{S}(z)$ are analytic in $\operatorname{Re}(z) > 0$ and continuous in $\operatorname{Re}(z) \geqq 0$.

Theorem 3.6 (Riedel). Define $S(z)$ and $\hat{S}(z)$ as given above. Suppose that $z=0$ is not an accumulation point of zeros of $S(\cdot)\hat{S}(\cdot)$ and

$$\limsup_{x\to 0+} x \log|S(x)\hat{S}(x)| = 0$$

where $z=x+iy$. Then the properties (i) $\{X(t), t\in T\}$ is a Wiener process with linear mean function $m(t)$ and

(ii) $$\int_{A_1}^{B_1} b_1(t)\, dX(w_1(t)) \quad \text{and} \quad \int_{A_2}^{B_2} b_2(t)\, dX(w_2(t))$$

are identically distributed are equivalent if and only if

(a) $S(2)=0$,

(b) $S(z)\neq 0$ for $0<\operatorname{Re} z<2$, $\operatorname{Im} z=0$,

(c) $\hat{S}(1)=0$ or $m(t)=0$.

Characterization of stable processes

Theorem 3.7 (Lukacs). Let $T=[0,\infty)$. Suppose the increments of the process have a symmetric distribution and $X(0)=0$. Then the process $\{X(t), t\geqq 0\}$ is a symmetric stable process if and only if there exists a function $t(y)$ such that (i) $t(y)>0$ for $y>0$ and (ii) the stochastic integral

$$\int_0^y (y-t)\, dX(t)$$

has the same distribution as the random variable $X(t(y))$ for each $y>0$. (Here the stochastic integral is defined in the sense of convergence in probability.)

The above theorem has been extended to stable processes in general in Lukacs (1969) and this characterization is given below.

Theorem 3.8 (Lukacs). Let $T=[0,\infty)$. Suppose the distribution of $X(t)$ is non-degenerate for every $t>0$ and $X(0)=0$. Then the process $\{X(t), t\geqq 0\}$ is a stable process if and only if there exist two function $t(y)$ and $s(y)$ such that (i) $t(y)>0$ for all $y>0$ and (ii) the stochastic integral

$$\int_0^y (y-t)\, dX(t)$$

(defined in the sense of convergence in probability) has the same distribution as $X(t(y))+s(y)$ for all $y>0$.

The following theorem is due to Riedel (1980b).

Theorem 3.9 (Riedel). Let $T=[0,\infty)$. Define $S(\cdot)$ and $\hat{S}(\cdot)$ as given earlier. Suppose $z=0$ is not an accumulation point of zeros of $S(\cdot)\hat{S}(\cdot)$ and

$$\limsup_{x\to 0+} x \log|S(x)\hat{S}(x)| = 0$$

where $z=x+iy$. Then the properties (i) $\{X(t), t\in T\}$ is a stable process with exponent α and

(ii) $$\int_{A_1}^{B_1} b_1(t)\,dX(w_1(t)) \quad\text{and}\quad \int_{A_2}^{B_2} b_2(t)\,dX(w_2(t))+q$$

are identically distributed for some real q are equivalent if and only if

(a) there exists a unique real zero α of $S(\cdot)$, $0<\alpha\leqq 2$; in case $\alpha<2$, its multiplicity is not higher than 2;

(b) $\hat{S}(\alpha)(2-\alpha)=0$;

(c) if $\alpha<2$, then $S(z)\hat{S}(z)\neq 0$ for $\operatorname{Re} z=\alpha$, $z\neq\alpha$.

For a proof of this theorem and related results, see Riedel (1980b). This theorem generalizes Theorem 3.8.

Theorem 3.10 (Prakasa Rao). Let $T=[0,1]$. Suppose $\{X(t), t\in T\}$ is a symmetric stable process with exponent α. Then

$$S=\int_0^1 R(t)\,dX(t) \quad\text{and}\quad X(1)-X(0)$$

are identically distributed for every random process $\{R(t), 0\leqq t\leqq 1\}$ with non-negative continuous sample paths independent of $\{X(t), 0\leqq t\leqq 1\}$ such that

$$\int_0^1 R(t)^\alpha\,dt=1 \text{ a.s.}$$

Conversely, suppose the increments of process $\{X(t), 0\leqq t\leqq 1\}$ have symmetric non-degenerate distributions. Then $\{X(t), 0\leqq t\leqq 1\}$ is a symmetric stable process with exponent α if and only if

$$\int_0^1 (2t)^{1/\alpha}\,dX(t)$$

and $X(1)-X(0)$ are identically distributed.

For a proof of this theorem, see Prakasa Rao (1982).

Characterization of a Wiener process taking values in a Hilbert space

Let Λ be the interval $[0,1]$ and $\mathscr{B}$ denote the σ-algebra of Borel subsets of $[0,1]$. For each $\Delta\in B$, let $\phi(\Delta)$ be a random element taking values in a real separable Hilbert space H. Suppose $\phi(\Delta)$ satisfies the following properties; (i) if Δ and Δ' are disjoint Borel subsets of $[0,1]$, then $\phi(\Delta)$ and $\phi(\Delta')$ are independent and $\phi(\Delta\cup\Delta')=\phi(\Delta)+\phi(\Delta')$ (ii) $\phi(\Delta)$ has stationary increments i.e., $\phi(\Delta)$ and $\phi(\Delta')$ are identically distributed if Δ and Δ' have the same Lebesgue measure (iii) if μ_t denotes the probability measure of $\phi([0,t])$, then μ_t converges weakly to the distribution degenerate at the origin as $t\to 0$.

A process ϕ on Λ with properties (i), (ii) and (iii) as stated above is said to be a *homogeneous process with independent increments.*

A homogeneous process ϕ on Λ with independent increments is said to be a *Wiener process with mean* 0 if the characteristic functional $\hat{\mu}_t(y)$ of $\phi([0, t])$ has the representation

$$\hat{\mu}_t(y) = \exp\{-\tfrac{1}{2}t(Sy, y)\}$$

where S is an S-operator.

For more details on probability measures on a Hilbert space, the reader is referred to Parthasarathy (1967).

Let ϕ be a homogeneous process with independent increments on Λ with mean 0 and with $E_\mu[\|X\|]^2 < \infty$ where μ is the distribution of $X = \phi([0, 1])$. Let S denote the S-operator associated with ϕ. For any bounded linear operator A, define

$$n(A) = [\mathrm{Tr}\,(ASA')]^{\frac{1}{2}} + [\mathrm{Tr}\,(A'SA)]^{\frac{1}{2}}.$$

Then the set $\{A : n(A) = 0\}$ is a linear semigroup in the linear group of all bounded linear operators A. The function n is a norm in the corresponding factor group. We shall not distinguish between a coset and the individual operator in the coset. In this sense, n is a norm in the linear set of all bounded linear operators. Let $\mathcal{A}_S$ denote the completion of this set in the norm n. Consider the space $L_2 = L_2(\Lambda, \mathcal{B}, m, \mathcal{A}_S)$ of functions $A(\cdot)$ with values in $\mathcal{A}_S$ which are strongly measurable and such that

$$|A|^2 = \int_\Lambda n^2(A(\cdot))\, dm < \infty$$

where m is the Lebesgue measure on Λ. Vakhaniya and Kandelski (1967) have defined stochastic integrals of the form

$$J = \int_\Lambda A(\lambda)\phi(d\lambda)$$

for functions $A(\cdot)$ in L_2. Under this setup, the following characterization theorem for the Wiener process taking values in a real separable Hilbert space H is proved in Prakasa Rao (1971).

Theorem 3.11 (Prakasa Rao). Suppose ϕ is a homogeneous process with independent increments on Λ with mean 0 and finite associated S-operator S. Let $A(\cdot)$ and $B(\cdot)$ be functions in L_2 satisfying the following properties:

(i) $a = \sup_\lambda \|A(\lambda)\| < \infty$; $b = \sup_\lambda \|B(\lambda)\| < \infty$;

(ii) $H_\lambda^A = H_\lambda^B = H$ for all $\lambda \in \Lambda$ where H_λ^A denotes the subspace spanned by the operator $A(\lambda)$ etc.;

(iii) $\int_\Lambda [\|A(\lambda)x\|^2 - \|B(\lambda)x\|^2]\, d\lambda$

is either strictly greater than 0 or strictly less than 0 for all $x \in H-\{0\}$. Then

$$\int_\Lambda A(\lambda)\phi(d\lambda) \quad \text{and} \quad \int_\Lambda B(\lambda)\phi(d\lambda)$$

are identically distributed if and only if ϕ is a Wiener process and $A(\cdot)$ and $B(\cdot)$ satisfy the relation

$$\int_\Lambda A(\lambda)SA'(\lambda)\, d = \int_\Lambda B(\lambda)SB'(\lambda)\, d\lambda.$$

Analogous results characterizing a Wiener process taking values in a Hilbert space are derived in Kannan (1972b) using the operator-valued stochastic integrals developed by Kannan and Bharucha-Reid (1971). We shall not discuss them here.

4. Characterization through independence

Let $\{X(t), t \in T\}$ be a continuous homogeneous process with independent increments. The following theorem gives a characterization of the Wiener process through independence of stochastic integrals.

Theorem 4.1 (Skitovich). Let $T=[A, B]$. Suppose $a(t)$ and $b(t)$ are continuous functions defined in $[A, B]$ which are not identically 0 in $[A, B]$ such that for each t either $a(t)$ or $b(t)$ does not vanish in $[A, B]$. Let

$$Y = \int_A^B a(t)\, dX(t) \quad \text{and} \quad Z = \int_A^B b(t)\, dX(t)$$

be stochastic integrals defined in the sense of convergence in probability. Then $\{X(t), t \in T\}$ is a Wiener process with linear mean function if and only if

(i) Y and Z are stochastically independent, and

(ii) $\int_A^B a(t)b(t)\, dt = 0$.

This theorem is a modified version of a theorem due to Skitovich (1956). For an indication of the proof of this theorem, see Lukacs (1975). A generalization of this theorem is given in Ramachandran and Rao (1970) (cf. Kagan, Linnik and Rao (1973), Chapter 13). We now state their result.

Theorem 4.2 (Ramachandran and Rao). Let $T=[A, B]$. Let $w(t)$ be a non-constant, non-decreasing, right-continuous function defined on a compact interval $[a, b]$ with $w(a)=A$ and $w(b)=B$. Let g and h be continuous functions on $[a, b]$, at least one of them non-vanishing and the other non-vanishing on a set of positive w-measure (the measure induced by $w(\cdot)$ on

$[a, b]$). Then the stochastic integrals

$$Y = \int_a^b g(t)\, dX(w(t)) \quad \text{and} \quad Z = \int_a^b h(t)\, dX(w(t))$$

exist in the sense of convergence in probability and Y and Z are independent if and only if

(i) $\{X(t), t \in T\}$ is a Wiener process with linear mean function, and

(ii) $\int_a^b g(t)h(t)\, dw(t) = 0$ if $X(t)$ is a non-degenerate process.

5. Characterization through properties of conditional distribution of one stochastic integral with respect to another stochastic integral

In this section we shall study characterizations of the Wiener process and stable processes in general either through regression of one stochastic integral with respect to another or through the symmetry of the conditional distribution of one stochastic integral with respect to the other.

Characterization of the Wiener process

Theorem 5.1 (Laha and Lukacs). Let $T = [A, B]$. Suppose $\{X(t), t \in T\}$ is a second-order process and that its mean function and covariance function are of bounded variation in $[A, B]$. Suppose $a(t)$ and $b(t)$ are two continuous functions defined in $[A, B]$ such that $a(t)b(t) \neq 0$ for $t \in [A_1, B_1]$ where $A \leqq A_1 < B_1 \leqq B$ and $a(t)$ is not proportional to $b(t)$. Let

$$Y = \int_A^B a(t)\, dX(t) \quad \text{and} \quad Z = \int_A^B b(t)\, dX(t)$$

be two stochastic integrals defined as limits in quadratic mean. Then the process $\{X(t), t \in T\}$ is a Wiener process with linear mean function if and only if Y has linear regression and constant scatter on Z (i.e. the regression of Y on Z is linear and homoscedastic).

For a proof of this theorem, see Lukacs (1975). Lukacs (1977) studied the stability of the above characterization of the Wiener process. A slight generalization of Theorem 5.1 is due to Ramachandran and Rao (1970). We omit their result. The next theorem gives another characterization by constant regression of one stochastic integral on another stochastic integral.

Theorem 5.2 (Prakasa Rao). Let $T = [A, B]$. Suppose the process $\{X(t), t \in T\}$ possesses moments of all orders, its mean function and covariance function are of bounded variation in $[A, B]$ and the increments of the process have non-degenerate distributions. Let $g(t)$ and $h(t)$ be continuous functions defined on $[A, B]$ with the property that

$$\int_A^B g(t)h(t)\,dt = 0$$

implies that

$$\int_A^B g(t)[h(t)]^k\,dt \neq 0$$

for all $k > 1$. Let

$$Y = \int_A^B g(t)\,dX(t) \quad \text{and} \quad Z = \int_A^B h(t)\,dX(t)$$

be stochastic integrals defined in quadratic mean. Then Y has constant regression Z, that is $E(Y \mid Z) = E(Z)$ a.e. if and only if the process $\{X(t), t \in T\}$ is a Wiener process with linear mean function and

$$\int_A^B g(t)h(t)\,dt = 0.$$

For a proof of this theorem, see Prakasa Rao (1970).

The next theorem gives a characterization of the Wiener process based on the symmetry of the conditional distribution of one stochastic integral with respect to another stochastic integral.

Theorem 5.3 (Prakasa Rao). Let $T = [A, B]$. Suppose the increments of the process $\{X(t), t \in T\}$ have non-degenerate distributions. Let $g(t)$ and $h(t)$ be continuous functions defined on $[A, B]$ with the property that

$$\int_A^B \frac{h^3(t)}{g(t)}\,dt \neq 0 \quad \text{and} \quad \int_A^B \left|\frac{h^3(t)}{g(t)}\right|\,dt < \infty.$$

Let

$$Y = \int_A^B g(t)\,dX(t) \quad \text{and} \quad Z = \int_A^B h(t)\,dX(t)$$

be stochastic integrals defined in the sense of convergence in probability. Then the conditional distribution of Y given Z is symmetric if and only if the process $\{X(t), t \in T\}$ is a Wiener process with linear mean function $m(t) = \lambda t$ and $g(t)$ and $h(t)$ satisfy the relation

$$\lambda \int_A^B g(t)\,dt = 0 \quad \text{and} \quad \int_A^B g(t)h(t)\,dt = 0.$$

Proof of Theorem 5.3 can be found in Prakasa Rao (1972). The following result gives a characterization of the Wiener process and it is based on the regression properties of one double stochastic integral on another stochastic integral.

Theorem 5.4 (Wang). Let $T=[A, B]$ and $\{X(t), t\in T\}$ be a second-order process with independent increments. Suppose the stochastic integrals

$$Y_1=\int_A^B h(t)\,dX(t), \qquad Y_2=\int_A^B\int_A^B g(s,t)\,dX(s)\,dX(t)$$

exist in the sense of convergence in quadratic mean. Then

$$E(Y_2 \mid Y_1)=\beta \text{ a.e.}$$

for some real β if and only if the process $\{X(t), t\in T\}$ is a Wiener process with linear mean function.

See Wang (1975) for a proof of Theorem 5.4. For related results, see Wang (1974).

Characterization of stable processes

Theorem 5.5 (Prakasa Rao). Let $T=[0, 1]$, $X(0)=0$, $E[X(t)]=0$ for all t and the increments of the process $\{X(t), t\in T\}$ have non-degenerate symmetric distributions. Let

$$Y_\lambda=\int_0^1 t^\lambda\,dX(t)$$

for any $\lambda>0$. Then Y_λ is defined in the sense of convergence in probability and the process $\{X(t), t\in T\}$ is a symmetric stable process with exponent $\gamma>1$ if and only if for some positive numbers λ and μ, $\lambda\neq\mu$,

$$E(Y_\lambda \mid Y_\mu)=\beta Y_\mu \text{ a.e.}$$

for some real constant β depending on λ and μ. Furthermore γ, λ, μ and β are connected by the relation

$$\mu\gamma+1=\beta(\lambda-\mu+\mu\gamma+1).$$

For the proof of Theorem 5.5, see Prakasa Rao (1968).

6. Determination of a stochastic process by stochastic integrals

Let $\{X(t), t\geqq 0\}$ be a continuous homogeneous process with independent increments as before. We now obtain conditions under which the stochastic integrals formed by the process $\{X(t), t\in T\}$ completely determine the process. It is clear that the process $\{X(t), t\in T\}$ is uniquely determined by the characteristic function of $X(0)$ and the characteristic function of $X(t+1)-X(t)$. We say that process $\{X(t), t\in T\}$ is *determined up to shift* if $\{Y(t), t\in T\}$ is another stochastic process satisfying the same properties as the process $\{X(t), t\in T\}$, then $X(t)=Y(t)+ct$ a.s for some constant c independent of t and for all t. Such a process is said to be *completely determined* if $c=0$.

The following theorem is due to Prakasa Rao (1975).

Theorem 6.1 (Prakasa Rao). Let $\{X(t), t \in T\}$, $T=[A, D]$ be a continuous homogeneous process with independent increments. Suppose the process has moments of all orders and its mean function and covariance function are of bounded variation. Suppose $a(t)$ and $b(t)$ are continuous functions on $[A, B] \subset T$ and $[C, D] \subset T$ such that $A<C<B<D$. Further suppose that either

$$\int_A^B a^k(t)\, dt \neq 0$$

for all $k \geqq 2$ or

$$\int_C^D b^k(t)\, dt \neq 0$$

for all $k \geqq 2$. Let

$$Y = \int_A^B a(t)\, dX(t) \quad \text{and} \quad Z = \int_C^D b(t)\, dX(t)$$

be stochastic integrals defined in the sense of convergence in quadratic mean. Then, the joint distribution of (Y, Z) determines the process $\{X(t), t \in T\}$ up to shift provided the characteristic function of $X(t+1)-X(t)$, t, $t+1 \in T$ is entire. In such an event either

$$\int_A^B a(t)\, dt = \int_C^D b(t)\, dt = 0$$

or there is no shift.

This theorem has been generalized recently by Riedel (1980c).

Let $b(t)$ be continuous and $w(t)$ be a non-negative, non-decreasing and left-continuous function on $[A, B]$, as defined in Section 2. For $\operatorname{Re}(z) \geqq 0$, define

$$S(z) = \int_A^B |b(t)|^z\, dw(t)$$

and

$$\hat{S}(z) = \int_A^B |b(t)|^{z-1}\, b(t)\, dw(t)$$

as in Section 3.

Theorem 6.2 (Riedel). Suppose $X(0)=0$ and that $E\,|X(1)|^\lambda < \infty$ for some $0<\lambda<2$. Then the stochastic integral

$$Y \equiv \int_A^B b(t)\, dX(w(t))$$

defined in the sense of convergence in probability determines the process $\{X(t), t \in T\}$ completely if and only if the following conditions are satisfied:

(a) $S(z) \neq 0$, $\lambda \leqq \text{Re}\, z < 2$,

(b) $\hat{S}(z) \neq 0$, $\lambda \leqq \text{Re}\, z < 2$,

(c) $\hat{S}(1) \neq 0$.

Theorem 6.3 (Riedel). Suppose $EX(1)^2 < \infty$. Then the stochastic integral Y given above determines the process $\{X(t), t \in T\}$ completely if and only if

$$\hat{S}(1) = \int_A^B b(t)\, dw(t) \neq 0.$$

For other versions of Theorems 6.2 and 6.3, see Riedel (1980c).

7. Open problems

Zinger and Linnik (1970) have extended the characterization theorems for the normal distribution through independent linear forms to linear forms with random coefficients. It would be interesting to find whether the Wiener process can be characterized by the independence of stochastic integrals having random processes as integrands and integrators. Such stochastic integrals can be defined under some conditions (for instance, see Section 2). In general, it seems to be hard to obtain the characteristic function of such a stochastic integral in a closed form. Theorem 5.5 gives a characterization of symmetric stable processes. It is reasonable to ask whether this theorem can be extended to stochastic integrals of the type

$$Y = \int_0^1 a(t)\, dX(t) \quad \text{and} \quad Z = \int_0^1 b(t)\, dX(t),$$

where $a(t)$ and $b(t)$ are functions other than powers of t, using the recent work of Riedel (1980a). Recently several people have studied stable distributions on Hilbert spaces. It would be nice to obtain results generalizing the work of Lukacs (1969) and Prakasa Rao (1968), (1982) for stable processes taking values in a Hilbert space as was done for the Wiener process taking values in a Hilbert space by Prakasa Rao (1971) and Kannan (1972b). Stability of the characterization results discussed in this paper is of extreme interest. The only result in this direction is on a characterization for the Wiener process due to Lukacs (1977).

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