ON A LOCAL PROPERTY OF COMBINED INTER- AND INTRA-BLOCK ESTIMATORS

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SUMMARY. Roy and Shah (1962) proved the unbiasedness of a combined intra- and inter-block estimator of a treatment contrast in an incomplete block design. This result is valid for any treatment contrast in any incomplete block design and for a general class of procedures for combined estimation which includes the usual Yates-Rao procedure. In the present paper it is shown that a combined estimator obtained by any of the above procedures has variance smaller than that of the corresponding intra-block estimator provided that the ratio of inter-to intra-block variance does not exceed two.

1. Notation and preliminaries

Consider an incomplete block design with b blocks of k plots each involving v treatments, each replicated r times, having the $v \times b$ matrix $N = ((n_{ij}))$ as the incidence matrix. A linear function of observations which is orthogonal to each of the block totals will be called an intra-block contrast. Obviously, we can construct $b(k-1)$ mutually orthogonal intra-block constrasts. Also, if a contrast in observations is a function of block totals only, we shall call it an inter-block contrast. We can construct $(b-1)$ mutually orthogonal inter-block contrasts. Without loss of generality we may assume that these contrasts are normalised, i.e., the sum of squares of coefficients is unity.

The joint distribution of these $bk-1$ contrasts is assumed to be multivariate normal where the expected value of any contrast is obtained by replacing in the contrast every observation by the corresponding treatment parameter, the variance of any intra (inter)-block contrast is $\sigma_0^2(\sigma_1^2)$ and the covariances are all zero. Thus we may call $\sigma_0^2(\sigma_1^2)$ intra (inter) block variance per plot or simply intra (inter)-block variance. We shall assume that $\rho = \sigma_1^2/\sigma_0^2 \geqslant 1$.

Let $B = (B_1, B_2, ..., B_b)$, $T = (T_1, T_2, ..., T_v)$ and $\theta = (\theta_1, \theta_2, ..., \theta_v)$ be the row-vectors of block totals, treatment totals and treatment parameters respectively. By G, we shall denote the total of all observations. Let further,

$$Q = T - \frac{1}{k} BN' \qquad Q_1 = \frac{1}{k} BN' - \frac{rG}{bk} E_{v,1}$$

$$C = rI - \frac{1}{k} NN' \qquad C_1 = \frac{1}{k} NN' - \frac{r^2}{bk} E_{v,v} \qquad \dots \quad (1.1)$$

where $E_{m,n}$ denotes a $(m \times n)$ matrix with all elements unity.

We shall consider connected designs only, i.e., the designs for which the matrix C has exactly one latent root zero. For these designs the matrix NN' has a latent root rk with multiplicity one and all other latent roots will be smaller than rk. Suppose the rank of NN' is q. Let p_s, $s = 1, 2, ..., q$ be a set of orthonormal latent vectors of

NN' corresponding to the q positive latent roots $\phi_1, \phi_2, \ldots, \phi_q$, all smaller than rk. Let p_s, $s = q+1, q+2, \ldots, v-1$, be a set of $v-1-q$ normalised orthogonal vectors each orthogonal to $p_1, p_2, \ldots, p_q$ and also to $E_{v,1}$. Now we define, as in Roy and Shah (1962),

$$x_{0s} = \begin{cases} k^{\frac{1}{2}}(rk-\phi_s)^{-\frac{1}{2}}\, Qp_s' & s = 1, 2, \ldots, q \\ r^{-\frac{1}{2}}\, Qp_s' & s = q+1, \ldots, v-1. \end{cases} \qquad \ldots \quad (1.2)$$

It can be seen that x_{0s}, $s = 1, 2, \ldots, v-1$ are mutually orthogonal normalised intra-block contrasts. Hence we can find $e_0 = b(k-1)-v+1$ mutually orthogonal normalised intra-block contrasts each orthogonal to $x_{01}, x_{02}, \ldots, x_{0, v-1}$. These we may denote by z_{0s}, $s = 1, 2, \ldots, e_0$. We also define

$$x_{1s} = (k\phi_s)^{-\frac{1}{2}}\, BN'p_s' \qquad s = 1, 2, \ldots, q. \qquad \ldots \quad (1.3)$$

These can be seen to be mutually orthogonal normalised inter-block contrasts and hence we can find $e_1 = b-1-q$ mutually orthogonal normalised inter-block contrasts each orthogonal to $x_{11}, \ldots, x_{1q}$. We denote these by z_{1s}, $s = 1, 2, \ldots, e_1$.

If follows from our assumptions that

$$\begin{aligned} E(x_{0s}) &= a_{0s}\,\tau_s & s &= 1, 2, \ldots, v-1 \\ E(x_{1s}) &= a_{1s}\,\tau_s & s &= 1, 2, \ldots, q \\ E(z_{0s}) &= 0 & s &= 1, 2, \ldots, e_0 \\ E(z_{1s}) &= 0 & s &= 1, 2, \ldots, e_1. \end{aligned} \qquad \ldots \quad (1.4)$$

where

$$a_{0s} = \begin{cases} (r-\phi_s/k)^{\frac{1}{2}} & \text{for } s = 1, 2, \ldots, q \\ r^{\frac{1}{2}} & \text{for } s = q+1, \ldots, v-1 \end{cases}$$

and

$$\begin{aligned} a_{1s} &= (\phi_s/k)^{\frac{1}{2}} & s &= 1, 2, \ldots, q \\ \tau_s &= \theta p_s' & s &= 1, 2, \ldots, v-1. \end{aligned} \qquad \ldots \quad (1.5)$$

It also follows from our assumptions that x_{0s}, $s = 1, 2, \ldots, v-1$ and z_{0s}, $s = 1, 2, \ldots, e_0$ are all uncorrelated each having variance σ_0^2 and x_{1s}, $s = 1, 2, \ldots, q$ and z_{1s}, $s = 1, 2, \ldots, e_1$ are all uncorrelated each with variance σ_1^2.

We note that $\tau_1, \tau_2, \ldots, \tau_{v-1}$ are linearly independent parametric contrasts. If one uses the intra-block contrasts only, the minimum variance unbiased linear estimator of τ_s is given by

$$t_s = x_{0s}/a_{0s} \qquad s = 1, 2, \ldots, v-1. \qquad \ldots \quad (1.6)$$

This may be called the intra-block estimator of τ_s.

When ρ is known, inter-block contrasts may also be used in addition to the intra-block contrasts. In this case, the minimum variance linear unbiased estimator of τ_s is given by

$$t_s(\rho) = \begin{cases} (\rho a_{0s}\, x_{0s} + a_{1s} x_{1s})/(\rho a_{0s}^2 + a_{1s}^2) & s = 1, 2, \ldots, q \\ x_{0s}/a_{0s} & s = q+1, \ldots, v-1. \end{cases} \qquad \ldots \quad (1.7)$$

This may be called the combined estimator of τ_s.

It is easily verified that

$$V(t_s) = \sigma_0^2/a_{0s}^2 \qquad s = 1, 2, \ldots, v-1$$

and
$$V(\bar{t}_s(\rho)) = \begin{cases} \rho\sigma_0^2/(\rho a_{0s}^2 + a_{1s}^2) & s = 1, 2, \ldots, q \\ \sigma_0^2/a_{0s}^2 & s = q+1, \ldots, v-1. \end{cases} \qquad \ldots \quad (1.8)$$

The procedure commonly adopted is to substitute for ρ in (1.7) some estimate ρ^* which is a function of observations. Such an estimate of τ_s will be denoted by $\bar{t}_s(\rho^*)$. In this case, the variance of $\bar{t}_s(\rho^*)$ will depend upon the estimate ρ^* used.

2. Comparison of variances of the intra-block estimator and the combined estimator based on an estimate of ρ.

In this section, we shall consider a class of estimators of ρ and for an estimator ρ^* belonging to this class we shall compare the variance of $\bar{t}_s(\rho^*)$ with that of t_s.

Consider a statistic P of the form

$$P = \frac{aS_1 + \sum_{s=1}^{q} b_s z_s^2}{S_0} + d \qquad \ldots \quad (2.1)$$

where
$$z_s = x_{0s} - a_{0s}\, x_{1s}/a_{1s} \qquad s = 1, 2, \ldots, q$$

$$S_0 = \sum_{s=1}^{e_0} z_{0s}^2$$

$$S_1 = \sum_{s=1}^{e_1} z_{1s}^2 \qquad \ldots \quad (2.2)$$

and $a, b_1, b_2, \ldots, b_q, d$ are some constants. One may choose these constants suitably and define

$$\rho^* = \begin{cases} P & \text{if } P > 1 \\ 1 & \text{otherwise.} \end{cases} \qquad \ldots \quad (2.3)$$

It is easily seen that for $\quad s = 1, 2, \ldots, q$

$$\bar{t}_s(\rho^*) - \bar{t}_s(\rho) = \frac{c_s}{a_{0s}(1+\rho c_s)}\, w_s$$

where
$$c_s = a_{0s}^2/a_{1s}^2 = (rk - \phi_s)/\phi_s$$

and
$$w_s = \frac{\rho^* - \rho}{1 + c_s \rho^*}\, z_s. \qquad \ldots \quad (2.4)$$

We note that $V(z_s) = \sigma_0^2 + c_s \sigma_1^2$ for $s = 1, 2, \ldots, q$.

It is shown in Roy and Shah (1962) that if ρ^* is of the form (2.3), $\hat{l}_s(\rho^*)$ is unbiased for τ_s and its variance is given by

$$V\{\hat{l}_s(\rho^*)\} = V\{\hat{l}_s(\rho)\} + \frac{c_s^2}{a_{0s}^2(1+\rho c_s)^2} E(w_s^2). \quad \ldots \quad (2.5)$$

It is also shown in Roy and Shah (1962) that in this case, the combined estimators of τ_s and $\tau_{s'}$ ($s \neq s'$) are uncorrelated. Now, any treatment contrast τ can be expressed as

$$\tau = \sum_{s=1}^{v-1} m_s \tau_s \quad \ldots \quad (2.6)$$

where $m_1, m_2, \ldots, m_{v-1}$ are some constants. Let $\hat{l}(\rho^*) = \Sigma m_s \hat{l}_s(\rho^*)$ denote the combined estimator of τ when ρ^* is used as an estimate for ρ. If ρ^* is of the form (2.3), $V\{\hat{l}(\rho^*)\}$ is given by

$$V\{\hat{l}(\rho^*)\} = V\{\hat{l}(\rho)\} + \sum_{s=1}^{q} \frac{c_s^2 m_s^2 E(w_s^2)}{a_{0s}^2(1+\rho c_s)^2} \quad \ldots \quad (2.7)$$

where $\hat{l}(\rho) = \Sigma m_s \hat{l}_s(\rho)$ denotes the combined estimator of τ when ρ is known.

The intra-block estimator of τ is given by $t = \Sigma m_s t_s$ and its variance is given by

$$V(t) = \Sigma m_s^2 V(t_s). \quad \ldots \quad (2.8)$$

It is easy to check that the variance of t_s can be expressed in the form

$$V(t_s) = V\{\hat{l}_s(\rho)\} + \frac{c_s^2}{a_{0s}^2(1+\rho c_s)^2} E\left(\frac{z_s^2}{c_s^2}\right), \quad s = 1, 2, \ldots, q. \quad \ldots \quad (2.9)$$

This exceeds $V\{\hat{l}_s(\rho^*)\}$ given by (2.5) if

$$E\left\{\frac{z_s^2}{c_s^2} - \frac{(\rho^*-\rho)^2 z_s^2}{(1+\rho^* c_s)^2}\right\} > 0. \quad \ldots \quad (2.10)$$

It is readily seen that the term in the parenthesis is positive if

$$\rho < 2\rho^* + \frac{1}{c_s}. \quad \ldots \quad (2.11)$$

Since ρ^* defined by (2.3) is truncated from below at unity (2.10) is satisfied provided that

$$\rho < 2 + \frac{1}{c_s}. \quad \ldots \quad (2.12)$$

In view of (2.7) and (2.8), for an arbitrary treatment contrast τ a similar argument leads to the following result.

For any treatment contrast τ the combined estimator $\hat{l}(\rho^*)$ will have variance not larger than that of t, the intra-block estimator, provided that $\rho \leqslant 2$ and ρ^* is of the form (2.3).

REFERENCE

Roy, J. and Shah, K. R. (1962): Recovery of inter-block information. *Sankhyā*, Series A, 24, 269-280.

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