*Sankhyā: The Indian Journal of Statistics*
1996, Volume 58, Series A, Pt. 3, pp. 347-381

# TRANSITION DENSITIES OF REFLECTING DIFFUSIONS

*By* S. RAMASUBRAMANIAN
*Indian Statistical Institute*

*SUMMARY.* Using a parametrix method, localization procedure and probabilistic arguments we construct the transition density function for a nondegenerate diffusion process in a smooth domain, with oblique reflection at the boundary and with "killing" terms in the interior and on the boundary; (our method obtains the density also on the boundary). In the case of half space, a Gaussian type upper bound and minimality are established for the transition density. We also prove some auxiliary results concerning Green function and Poisson kernel for a second order parabolic operator with mixed boundary conditions in certain domains.

## 1. INTRODUCTION

The aim of this article is to construct the transition density function for a nondegenerate diffusion process in a smooth domain with oblique reflection at the boundary and with "killing" terms in the interior and on the boundary; our method enables to get the density function also on the boundary.

When the direction of reflection is the conormal direction such densities (equivalently, fundamental solutions for the corresponding parabolic equations with Neumann boundary conditions) have been constructed by S. Ito (1957) using the parametrix method; see his recent monograph (1992) for a lucid presentation. This has been used extensively by probabilists, especially in the context of probabilistic approach to the Neumann problem (for the Laplacian); (see Hsu (1985), Papanicolaou (1990), Ramasubramanian (1993) and the references given therein). If in addition the domain is a half space or an orthant, Bhattacharya and Waymire (1992) present an elementary way of getting the densities by the method of images.

The situation seems to be far from clear in the case of oblique reflection (that is, when the direction of reflection is not the conormal direction). For the half space, when the diffusion coefficients are constants, the direction of reflection is constant, and the "generator" does not have first order derivatives in the normal direction, Keller (1981) has given a construction by generalising the method of

Paper received. December 1994.
*AMS* (1991) *subject classifications.* Primary 60J60, secondary 35K15, 35K20.
*Key words and phrases.* Fundamental solution for parabolic equations, oblique reflection, parametrix method, Green function, Poisson kernel, exit time, strong Markov property, boundary local time.

images; (in fact, this is the starting point of our analysis). Bismut (1985) has briefly indicated a method of establishing the existence of a smooth density in a half space by means of Malliavin calculus and excursion theory, when the coefficients are infinitely differentiable, but again the "generator" not having any first order derivatives in the normal direction. For a general smooth domain Anderson and Orey (1976, p. 212) allude to the existence of a continuous density in the *interior* of the domain; however, no references are given. Existence of a fundamental solution in the *generalised sense* can perhaps be established by functional analytic methods (and perhaps known to experts); but this would not be adequate for many purposes.

In view of the above, it seems worth the effort to present an exposition on the construction of transition densities of diffusions with boundary conditions.

Here's a brief outline of the contents of the paper. In Section 2, starting with Keller's result, we use suitable transformations and parametrix method to construct the fundamental solution in the closed half space. We also establish that the fundamental solution is dominated by the transition probability density of a reflecting Brownian motion (in the half space) with normal reflection; such a Gaussian type estimate may be of independent interest. Moreover a regularity result (needed for Section 3) is proved under stronger differentiability assumptions on the coefficients.

In Section 3 we establish some auxiliary results concerning Green function and Poisson kernel for "mixed problem" for the parabolic operator in certain domains with oblique reflection on part of the boundary and Dirichlet condition on rest of the boundary.

The general case of a bounded domain with smooth boundary is considered in Section 4. A localization procedure and probabilistic arguments are used.

Finally in Section 5 we prove that the transition density constructed in Section 2 is minimal.

## 2. Transition densities in the half space

Let $D = \{x \in \mathbb{R}^d : x_1 > 0\}$ with $d \geq 2$. The operators $L$ and $J$ respectively on $D$ and $\partial D$ are defined by

$$Lf(s,x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(s,x)\frac{\partial^2 f(s,x)}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(s,x)\frac{\partial f(s,x)}{\partial x_i} + c(s,x)f(s,x) \qquad \ldots (2.1)$$

for $s \geq 0, x \in D$, and

$$Jf(s,x) = \sum_{i=1}^{d} \gamma_i(s,x)\frac{\partial f}{\partial x_i}(s,x) + \rho(s,x)f(s,x), \qquad \ldots (2.2)$$

for $s \geq 0, x \in \partial D$, where the coefficients $a, b, c, \gamma, \rho$ satisfy the following :

(A1) : For each $s \geq 0, x \in \overline{D}, a(s,x) := ((a_{ij}(s,x)))$ is a $d \times d$ real symmetric positive definite matrix; each $a_{ij}$ is bounded and uniformly Lipschitz continuous. There exist $0 < \lambda_1 \leq \lambda_2 < \infty$ such that for any $s, x$ any eigenvalue of $a(s,x) \in [\lambda_1, \lambda_2]$.

(A2) : For each $s \geq 0, x \in \overline{D}, b(s,x) := (b_1(s,x), \ldots, b_d(s,x))$; each $b_i$ is bounded and uniformly Lipschitz continuous.

(A3) : $c$ is a bounded uniformly Lipschitz continuous function on $[0,\infty] \times \overline{D}$.

(A4) : There exists $\beta_0 > 0$ such that for each $s \geq 0, x \in \partial D, < \gamma(s,x), n(x) > \geq \beta_0$, where $\gamma(s,x) := (\gamma_1(s,x), \ldots, \gamma_d(s,x))$ and $n(x)$ is the inward unit normal; each $\gamma_i \in C_b^{2,3}([0,\infty) \times \partial D)$.

(A5) : $\rho$ is a bounded continuous function on $[0,\infty) \times \partial D$.

A word now about the notation. For $x = (x_1, x_2, \ldots, x_d)$ we write $\overline{x} = (x_2, \ldots, x_d)$ and $x^* = (-x_1, x_2, \ldots, x_d) = (-x_1, \overline{x})$. For a matrix $a = ((a_{ij}))_{1 \leq i,j \leq d}$ we write $\overline{a} = ((a_{ij})_{2 \leq i,j \leq d}$. The subscript $x$ in the operators $D_x^\alpha, \nabla_x, L_x, J_x$, etc. denotes that differentiation is in the $x$-variables.

For $x, z \in \overline{D}$ note that

$$|z - x| \leq |z^* - x| \qquad \ldots(2.3)$$

In this section we construct the fundamental solution for $(\partial/\partial s) + L$ in $[0,\infty) \times D$ with reflecting boundary condition given by $J$, using a parametrix method.

We shall denote by $p_0$ the transition probability density function of Brownian motion in $\overline{D}$ with mean zero, covariance matrix $C_2 I$ (with $C_2$ being a positive constant and $I$ the $(d \times d)$ identity matrix) and *normal* reflection at the boundary; the constant $C_2$ may differ in different contexts.

Theorem 2.1. *Assume (A1) - (A5). Then there exists a function $p(s,x;t,z)$ defined for $0 \leq s < t, x, z \in \overline{D}$ satisfying the following :*

(i) *for any fixed $t > 0, z \in \overline{D}$ the function $(s,x) \mapsto p(s,x;t,z)$ is in $C^{1,2}([0,t) \times D) \cap C([0,t) \times \overline{D})$; moreover, for any $T > 0$, there exist constants $C_1, C_2 > 0$ such that*

$$|D_x^\alpha p(s,x;t,z)| \leq C_1 (t-s)^{-|\alpha|/2} p_0(s,x;t,z) \qquad \ldots(2.4)$$

*for all $x \in \overline{D}, z \in \overline{D}, 0 \leq s < t$ with $(t-s) \leq T, \alpha$ a multiindex with $|\alpha| \leq 1$, where $p_0$ is as described above;*

(ii) *for any bounded continuous function $f$ on $\overline{D}$, fixed $t > 0$, define*

$$u(s,x) = \int_{\overline{D}} f(z) p(s,x;t,z) dz \qquad \ldots(2.5)$$

*for* $0 \leq s < t, x \in \overline{D}$; *then*

$$\left.\begin{array}{lcl} ((\partial/\partial s) + L_x)u(s,x) & = & 0,\ 0 \leq s < t, x \in D, \\ J_x u(s,x) & = & 0,\ \ 0 \leq s < t, x \in \partial D, \\ \lim_{s\uparrow t} u(s,x) & = & f(x),\ x \in \overline{D} \end{array}\right\} \qquad \ldots (2.6)$$

*Note.* $x \mapsto p(s,x;t,z)$ is differentiable over $\overline{D}$; however the derivative may fail to be continuous at the boundary, but the estimate (2.4) nevertheless holds; see the note following Lemma 2.5.

*Proof.* The proof is in several steps. Without loss of generality we may take $\gamma_1 \equiv 1$.

*Step* 1. Assume that $b \equiv 0, c \equiv 0, \rho \equiv 0, a_{ij}, \gamma_i$ are constants; $a_{1i} \equiv a_{i1} \equiv 0, i \geq 2$. Let $\Gamma$ denote the transition probability density function of the $L$-diffusion (which is just the $d$-dimensional Brownian motion with constant covariance matrix $a$) in $\mathbb{R}^d$. Note that there exist constants $C_1, C_2$ such that

$$|D_x^\alpha \Gamma(s,x;t,z)| \leq C_1(t-s)^{-(|\alpha|+d)/2} exp\left\{-\frac{C_2|z-x|^2}{2(t-s)}\right\} \qquad \ldots (2.7)$$

for all $x, z \in \mathbb{R}^d, 0 \leq s < t$ and any multiindex $\alpha$.

Define for $0 \leq s < t, x, z \in \overline{D}$,

$$q(s,x;t,z) = \Gamma(s,x;t,z) - \Gamma(s,x;t,z^*) - 2\frac{\partial}{\partial x_1}\int_0^\infty \Gamma(s,x;t,z^* - r\gamma)dr \ \ldots (2.8)$$

It has been proved by Keller (1981) that $q$ is the transition probability density function for the diffusion in $\overline{D}$ with generator $L$, and with oblique reflection at the boundary given by $\gamma$.

*Remark* 2.2. Let $T$ be the linear transformation on $\mathbb{R}^d$ given by

$$T(z_1, z_2, \ldots, z_d) = (z_1, z_2, \ldots, z_d) - z_1(0, \gamma_2, \ldots, \gamma_d) \qquad \ldots (2.9)$$

Clearly $T$ is invertible, one-one on any $\{z_1 = \text{ constant}\}$, and $T\gamma = (1, 0, \ldots, 0)$. If $z_1 \geq 0, r \geq 0$ (or if $z_1 \leq 0, r \leq 0$) note that $[(Tz)_1 + r]^2 \geq (Tz)_1^2 + r^2$. Hence for any $z \in \mathbb{R}^d$ such that $z_1 \geq 0$ and any $r \geq 0$ (or any $z$ such that $z_1 \leq 0$ and any $r \leq 0$) we have

$$\begin{array}{rcl} -|z + r\gamma|^2 & \leq & -\|T\|^{-2}|T(z+r\gamma)|^2 \\ & = & -\|T\|^{-2}\{[(Tz)_1 + r]^2 + \sum_{i=2}^d (Tz)_i^2\} \\ & \leq & -\|T\|^{-2}\{|Tz|^2 + r^2\} \end{array} \qquad \ldots (2.10)$$

□

We continue with Step 1. By the estimate (2.7) and the above remark we obtain

$$\left|\frac{\partial^2}{\partial x_i \partial x_j}\frac{\partial}{\partial x_1}\int_0^\infty \Gamma(s,x;t,z^* - r\gamma)dr\right|$$

$$\le C(t-s)^{-(d+3)/2}\int_0^\infty \exp\left\{-\frac{C\|T\|^{-2}}{(t-s)}|T(z^*-x)|^2\right\}\cdot$$

$$\exp\left\{-\frac{C\|T\|^{-2}}{(t-s)}r^2\right\}dr \qquad \ldots(2.11)$$

$$\le C(t-s)^{-(d+2)/2} \quad \exp\left\{-\frac{C\|T\|^{-2}\|T^{-1}\|^{-2}}{(t-s)}|z^*-x|^2\right\}.$$

From (2.7), (2.8), (2.11) it is clear that the estimate (2.4) holds for $q$ in the case $|\alpha| = 2$, in fact for all $x, z \in \overline{D}$; the other cases can be handled similarly.

*Step* 2 : We now drop the assumption $a_{i1} \equiv a_{1i} \equiv 0, i \ge 2$, but the other assumptions of Step 1 are retained.

Define the transformation $T(x_1, x_2, \ldots, x_d) = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_d)$ on $\overline{D}$ by $\tilde{x}_1 = x_1$, $\tilde{x}_i = x_i - (a_{1i}/a_{11})x_1, i \ge 2$. Note that $T$ is invertible on $\overline{D}$ and is identity on $\partial D$. Put $\tilde{a} = TaT^*$, $\tilde{\gamma} = T\gamma$, note that $\tilde{a}_{1i} \equiv \tilde{a}_{i1} \equiv 0, i \ge 2$. Set

$$\begin{aligned}\tilde{L}f(s,\tilde{x}) &= \frac{1}{2}\sum_{i,j=1}^{d}\tilde{a}_{ij}\frac{\partial^2}{\partial\tilde{x}_i\partial\tilde{x}_j}f(s,\tilde{x}), \ s \ge 0, \tilde{x} \in D\\ \tilde{J}f(s,\tilde{x}) &= \frac{\partial}{\partial\tilde{x}_1}f(s,\tilde{x}) + \sum_{i=2}^{d}\tilde{\gamma}_i\frac{\partial f}{\partial\tilde{x}_i}(s,\tilde{x}), \ s \ge 0, \tilde{x} \in \partial D.\end{aligned}$$

By Step 1 there is a transition probability density function $\tilde{q}$ for the $(\tilde{L}, \tilde{J})$-diffusion in $\overline{D}$. As $\det T = 1$, observe that $q$ given by

$$q(s,x;t,z) = \tilde{q}(s,\tilde{x};t,\tilde{z}), \quad 0 \le s < t, x, z \in \overline{D} \qquad \ldots(2.12)$$

is the transition probability density function for the $(L, J)$-diffusion in $\overline{D}$. It is also clear that the estimate (2.4) holds even for $|\alpha| = 2, x, z \in \overline{D}$.

Thus the theorem holds in the case $b \equiv 0, c \equiv 0, \rho \equiv 0, a_{ij}, \gamma i$ being constants. In fact, from the proof it is clear that the transition density is infinitely differentiable at any $s < t, x, z \in \overline{D}$. □

*Remark* 2.3. Let $a(\cdot,\cdot)$ satisfy (A1); assume $\gamma \equiv (1, 0, \ldots 0)$. For $t > 0, z \in \overline{D}$ put

$$\tilde{\gamma}_i(t,z) = -\frac{a_{1i}(t,z)}{a_{11}(t,z)}, \quad i \ge 2; \qquad \ldots(2.13)$$

define $T_{(t,z)}$ on $\overline{D}$ by

$$(T_{(t,z)}x)_1 = x_1, \quad (T_{(t,z)}x)_i = x_i + \tilde{\gamma}_i(t,z)x_1, \quad i \ge 2 \qquad \ldots(2.14)$$

Put

$$\tilde{a}(t,z) = T_{(t,z)} a(t,z) T^*_{(t,z)} \qquad \ldots (2.15)$$

$$A_{(t,z)} = ((A^{ij}_{(t,z)}))_{1 \le i,j \le d} = [\tilde{a}(t,z)]^{-1} \qquad \ldots (2.16)$$

Observe that

$$\tilde{a}_{11}(t,z) = a_{11}(t,z); \quad \tilde{a}_{1i}(t,z) \equiv \tilde{a}_{i1}(t,z) \equiv 0, \quad i \ge 2 \qquad \ldots (2.17)$$

Hence

$$A^{1j}_{(t,z)} = A^{j1}_{(t,z)} = 0, \quad j \ge 2 \qquad \ldots (2.18)$$

Define

$$\eta(s,x;t,z) = < [T_{(t,z)} z - T_{(t,z)} x], \tilde{a}(t,z)^{-1} [T_{(t,z)} z - T_{(t,z)} x] > \qquad \ldots (2.19)$$

$$\zeta(s,x;t,z) = < [(T_{(t,z)} z)^* - T_{(t,z)} x], \tilde{a}(t,z)^{-1} [(T_{(t,z)} z)^* - T_{(t,z)} x] > \qquad \ldots (2.20)$$

$$\psi_1(s,x;t,z) = (2\pi(t-s))^{-d/2} (det\ \tilde{a}(t,z))^{-1/2} \exp\left\{ -\frac{\eta(s,x;t,z)}{2(t-s)} \right\} \qquad \ldots (2.21)$$

$$\psi_2(s,x;t,z) = (2\pi(t-s))^{-d/2} (det\ \tilde{a}(t,z))^{-1/2} \exp\left\{ -\frac{\zeta(s,x;t,z)}{2(t-s)} \right\} \qquad \ldots (2.22)$$

for $0 \le s < t, x, z \in \overline{D}$. It is easily seen that

$$\begin{aligned} \eta(s,x;t,z) &= \sum_{i,j=1}^{d} A^{ij}_{(t,z)} (z_i - x_i)(z_j - x_j) \\ &\quad + (z_1 - x_1)^2 \sum_{i,j=2}^{d} A^{ij}_{(t,z)} \tilde{\gamma}_i(t,z) \tilde{\gamma}_j(t,z) \\ &\quad + 2(z_1 - x_1) \sum_{i,j=2}^{d} A^{ij}_{(t,z)} (z_i - x_i) \tilde{\gamma}_i(t,z) \end{aligned} \qquad \ldots (2.23)$$

$$\begin{aligned} \zeta(s,x;t,z) &= \sum_{i,j=1}^{d} A^{ij}_{(t,z)} (z^*_i - x_i)(z^*_j - x_j) \\ &\quad + (z_1 - x_1)^2 \sum_{i,j=2}^{d} A^{ij}_{(t,z)} \tilde{\gamma}_i(t,z) \tilde{\gamma}_j(t,z) \\ &\quad + 2(z_1 - x_1) \sum_{i,j=2}^{d} A^{ij}_{(t,z)} (z^*_i - x_i) \tilde{\gamma}_i(t,z) \end{aligned} \qquad \ldots (2.24)$$

If $z \in \partial D$, then $(T_{(t,z)} z)^* = T_{(t,z)} z = z$ and hence

$$\psi_1(s,x;t,z) = \psi_2(s,x;t,z) \qquad \ldots (2.25)$$

for $0 \le s, t, x \in \overline{D}, z \in \partial D$. Similarly, when $x \in \partial D$ we have $\zeta(s,x;t,z) = \eta(s,x;t,z)$ and hence

$$\psi_1(s,x;t,z) = \psi_2(s,x;t,z) \qquad \ldots(2.26)$$

for $0 \le s < t, x \in \partial D, z \in \overline{D}$. □

*Step* 3. We now take $a(\cdot,\cdot)$ satisfying (A1) and $\gamma(\cdot,\cdot) \equiv (1,0,\ldots,0)$. For fixed $t > 0, z \in \overline{D}$ set

$$L^{(0)}_{(t,z)} = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(t,z)\frac{\partial^2}{\partial x_i \partial x_j} \qquad \ldots(2.27)$$

Note that $L^{(0)}_{(t,z)}$ is an operator with constant coefficients. Let $q_{(t,z)}$ denote the transition probability density function of the $\left(L^{(0)}_{(t,z)}, \frac{\partial}{\partial x_1}\right)$-diffusion in $\overline{D}$.

For any $0 \le s < t, x \in \overline{D}$ define

$$Q(s,x;t,z) = q_{(t,z)}(s,x;t,z) \qquad \ldots(2.28)$$

In this step we prove some results concerning $Q$. Observe first that

$$\begin{aligned} Q(s,x;t,z) &= \psi_1(s,x;t,z) - \psi_2(s,x;t,z) \\ &+2\int_0^\infty \frac{(z_1+\alpha+x_1)}{a_{11}(t,z)(t-s)}\psi_2(s,(x+\alpha e_1);t,z)d\alpha \end{aligned} \qquad \ldots(2.29)$$

where $e_1 = (1,0,\ldots,0)$. Also by (2.28) and Step 2

$$|D_x^\alpha Q(s,x;t,z)| \le C_1(t-s)^{-|\alpha|/2}p_0(s,x;t,z) \qquad \ldots(2.30)$$

for all $0 \le s < t$ with $(t-s) \le T, x, z \in \overline{D}$ and any multindex $\alpha$; because of the hypothesis (A1) it is clear that $C_1, C_2$ can be taken to be independent of $(t,z)$.

Lemma 2.4. *For any bounded continuous function $f$ on $[0,\infty) \times \overline{D}$,*

$$\lim_{t \downarrow s} \int_{\overline{D}} f(t,z)Q(s,x;t,z)dz = f(s,x) \qquad \ldots(2.31)$$

*for $s \ge 0, x \in \overline{D}$, and*

$$\lim_{s \uparrow t} \int_{\overline{D}} f(s,z)Q(s,x;t,z)dz = f(t,x) \qquad \ldots(2.32)$$

*for $t > 0, x \in \overline{D}$.*

*Proof.* We will prove only (2.31); the proof of (2.32) is similar.

We first consider the case when $x \in D$: that is $x_1 > 0$. Note that

$$\left|\int_{\overline{D}} f(t,z)\psi_2(s,x;t,z)dz\right| \le C\int_{x_1/\sqrt{t-s}}^\infty \exp(-\xi^2)d\xi \to 0 \qquad \ldots(2.33)$$

as $(t-s)\downarrow 0$.

Next, as $x \in D$, suitably extending $a_{ij}, f$ (by symmetry) to $[0,\infty)\times\mathbb{R}^d$ so that continuity, same ellipticity constant and bounds hold, it can be seen that

$$\lim_{t\downarrow s}\int_{\overline{D}} f(t,z)\psi_1(s,x;t,z)dz = f(s,x) \qquad \ldots(2.34)$$

Also, as $x_1>0$ we have

$$\begin{aligned}&\left|\int_{\overline{D}}\int_0^\infty f(t,z)\frac{(z_1+\alpha+x_1)}{a_{11}(t,z)(t-s)}\psi_2(s,(x+\alpha e_1);t,z)d\alpha dz\right|\\ &\quad\le C\int_0^\infty\int_0^\infty \frac{(z_1+\alpha+x_1)}{(t-s)}\frac{1}{\sqrt{t-s}}\exp\left\{-\frac{(z_1+\alpha+x_1)^2}{(t-s)}\right\}d\alpha dz_1\\ &\quad= C\int_0^\infty\frac{1}{\sqrt{t-s}}\exp\left\{-\frac{(\alpha+x_1)^2}{(t-s)}\right\}d\alpha \to 0\end{aligned} \qquad \ldots(2.35)$$

as $(t-s)\downarrow 0$. From (2.29), (2.33) - (2.35) it follows that (2.31) holds when $x\in D$.

Now let $x\in\partial D$. Then by (2.26) for all $s,t$

$$\int_{\overline{D}} f(t,z)[\psi_1(s,x;t,z)-\psi_2(s,x;t,z)]dz = 0 \qquad \ldots(2.36)$$

To complete the proof we have to show that

$$\lim_{t\downarrow s}\int_{\overline{D}}\int_0^\infty f(t,z)\frac{(z_1+\alpha)}{a_{11}(t,z)(t-s)}\psi_2(s,((0,\overline{x})+\alpha e_1);t,z)d\alpha dz = \frac{1}{2}f(s,x) \qquad \ldots(2.37)$$

First change variables by $z'=(z-x)/\sqrt{t-s}$; since $x_1=0$, note that $z'$ varies over $\overline{D}$ as $z$ varies over $\overline{D}$. Next put $\alpha'=\alpha/\sqrt{t-s}$ and finally apply the $(d-1)$ dimensional transformation

$$\overline{w} = [\overline{\tilde{a}}(s,x)]^{-1/2}[\overline{z'}-(z_1'+\alpha')\overline{\tilde{\gamma}}(s,x)],$$

keeping $z_1'$ fixed, where $\tilde{\gamma}$ is defined by (2.13). It is then easily seen that

$$\begin{aligned}&l.h.s.of\ (2.37)\\ &= f(s,x)\int_0^\infty\int_0^\infty\frac{1}{\sqrt{2\pi}}\left(\frac{1}{a_{11}(s,x)}\right)^{3/2}(z_1'+\alpha')\exp\left\{-\frac{(z_1'+\alpha')^2}{2a_{11}(s,x)}\right\}dz_1'd\alpha'\\ &= \frac{1}{2}f(s,x).\end{aligned}$$

□

Lemma 2.5. *Let $t>0$ and $f$ a continuous function on $[0,t)\times\partial D$ such that*

$$|f(r,\eta)|\le K[1+(t-r)^{-d/2}exp\left\{-\frac{k|\eta|^2}{(t-r)}\right\}] \qquad \ldots(2.38)$$

*for* $0 \le r < t, \eta \in \partial D$, *where* $k, K$ *are positive constants. Set*

$$v(s,x) = \frac{1}{2}\int_s^t \int_{\partial D} a_{11}(r,\eta) f(r,\eta) Q(s,x;r,\eta) d\sigma(\eta) dr \qquad \ldots (2.39)$$

*for* $0 \le s < t, x \in \overline{D}$ *where* $d\sigma(\cdot)$ *denotes the* $(d-1)$ *dimensional Lebesgue measure on* $\partial D$. *Then* $v$ *is continuous on* $[0,t) \times \overline{D}$, *and*

$$\frac{\partial}{\partial x_1} v(s,x) = -f(s,x), \quad x \in \partial D, s < t \qquad \ldots (2.40)$$

*Note.* It can be shown that for $2 \le i \le d$

$$\frac{\partial}{\partial x_i} v(s,x) = \frac{1}{2}\int_s^t \int_{\partial D} a_{11}(r,\eta) f(r,\eta) \frac{\partial Q}{\partial x_i}(s,x;r,\eta) d\sigma(\eta) dr$$

for any $x \in \overline{D}$ and that $\partial v/\partial x_i$ is continuous over $\overline{D}$. Hence by Lemma 2.5 it follows that $v$ is differentiable in $x$ over $\overline{D}$; however, as is clear from Lemma 2.5, $\partial v/\partial x_1$ is not continuous at the boundary.

*Proof.* By (2.30), (2.38), dominated convergence theorem and the Chapman-Kolmogorov equation for Gaussian densities it is clear that $v$ is well defined and is continuous on $[0,t) \times \overline{D}$; (by putting $v(t,\cdot) \equiv 0$ it is also continuous on $[0, t \times \overline{D})$. Differentiability at any $x \in D$ is also similarly dealt with; in such a case one can differentiate under the integral, and in fact the derivative is continuous at any $x \in D$.

By (2.29), (2.25), (2.26), (2.39) for any $x \in \partial D, h > 0$ we have

$$\frac{1}{h}[v(s,x+he_1) - v(s,x)]$$

$$= -\frac{1}{h}\int_s^t \int_{\partial D} \int_0^h f(r,\eta) \frac{\alpha}{(r-s)} \psi_2(s,(x+\alpha e_1);r,\eta) d\alpha d\sigma(\eta) dr. \qquad \ldots (2.41)$$

Introduce the variables $\alpha' = \alpha/h, \eta' = (\eta - x)/h, r' = (r-s)/h^2$; since $x \in \partial D$ note that $\eta'$ varies over $\partial D$ as $\eta$ varies over $\partial D$. Take the limit in (2.41) as $h \downarrow 0$, and finally apply the $(d-1)$ dimensional transformation $\eta'' = \tilde{\tilde{a}}(s,x)^{-\frac{1}{2}}[\eta' - \alpha'\tilde{\tilde{\gamma}}(s,x)]$ to get

$$\begin{aligned}
&\lim_{h \downarrow 0} \frac{1}{h}\,[v(s,x+he_1) - v(s,x)] \\
&= -f(s,x) \int_0^\infty \int_0^1 \frac{\alpha'}{r'} \frac{1}{\sqrt{2\pi r'}} \frac{1}{\sqrt{a_{11}(s,x)}} \exp\left\{ -\frac{(\alpha')^2}{2r' a_{11}(s,x)} \right\} d\alpha' dr' \\
&= -f(s,x) \frac{1}{\sqrt{\pi}} \int_0^1 \int_0^\infty \xi^{-1/2} e^{-\xi} d\xi d\alpha' = -f(s,x).
\end{aligned}$$

□

Lemma 2.6. (i) *Let $t > 0, z \in \overline{D}$ be fixed. Let $\Phi$ be a continuous function on $[0,t) \times \overline{D}$ such that*

$$|\Phi(r,\eta)| \le K[1 + (t-r)^{-1/2} p_0(r,\eta;t,z)] \qquad \ldots (2.42)$$

*for $r < t, \eta \in \overline{D}$. Set*

$$V(s,x) = \int_s^t \int_{\overline{D}} \Phi(r,\eta) Q(s,x;r,\eta)\, d\eta\, dr \qquad \ldots (2.43)$$

*for $0 \le s < t, x \in \overline{D}$. Then $V$ is continuous on $[0,t) \times \overline{D}$ and is once continuously differentiable in $x$ over $\overline{D}$.*

*(ii) Let $t, z, \Phi, V$ be as above; in addition let $\Phi$ satisfy*

$$|\Phi(r,\eta) - \Phi(r,\eta')| \le K|\eta - \eta'|^{1/2} (t-r)^{-3/4} [p_0(r,\eta;t,z) + p_0(r,\eta';t,z)] \qquad \ldots (2.44)$$

*for any $r < t, \eta, \eta' \in D$. Then $(s,x) \mapsto V(s,x)$ is in $C^{1,2}([0,t) \times D)$.*

*Proof.* (i) In view of (2.30), (2.42), Lemma 2.4, dominated convergence theorem and the Chapman-Kolmogorov equation for $p_0$, the first assertion can be proved easily.

(ii) This can be proved as in Theorems 4 and 5 of Chap. I, Sec. 3 of Friedman (1983), pp. 9-13; note that integrability of the terms involved in the proof follow from (2.44) and the Chapman-Kolmogorov equation for $p_0$. □

*Step* 4. We now consider $a(\cdot,\cdot), b(\cdot,\cdot), c(\cdot,\cdot)$ satisfying the hypothesis (A1) - (A3); assume $\gamma(\cdot,\cdot) \equiv (1,0,\ldots,0)$ and $\rho \equiv 0$.

Let $Q$ be given by (2.28). For fixed $t > 0, z \in \overline{D}$ we have

$$\begin{aligned} MQ(s,x;t,z) &:= \left(\frac{\partial}{\partial s} + L_x\right) Q(s,x;t,z) \\ &= \frac{1}{2}\sum_{i,j=1}^{d} (a_{ij}(s,x) - a_{ij}(t,z)) \frac{\partial^2}{\partial x_i \partial x_j} q_{(t,z)}(s,x;t,z) \\ &\quad + \sum_{i=1}^{d} b_i(s,x) \frac{\partial}{\partial x_i} q_{(t,z)}(s,x;t,z) + c(s,x) q_{(t,z)}(s,x;t,z) \end{aligned} \qquad \ldots (2.45)$$

for $0 \le s < t, x \in \overline{D}$. By the estimate (2.30), Lipschitz continuity of $a_{ij}$ and boundedness of $b_i, c$ we get

$$|MQ(s,x;t,z)| \le C[1 + (t-s)^{-1/2}] p_0(s,x;t,z) \qquad \ldots (2.46)$$

for $0 \le s < t$ such that $(t-s) \le 1$ and all $x, z \in \overline{D}$.

Now put $(MQ)_1(s,x;t,z) = MQ(s,x;t,z)$ and define inductively for $n = 2,3,\ldots$

$$(MQ)_{n+1}(s,x;t,z) = \int_s^t \int_{\overline{D}} MQ(s,x;r,\eta)(MQ)_n(r,\eta;t,z)\, d\eta\, dr \qquad \ldots (2.47)$$

Using the estimate (2.46), Chapman-Kolmogorov equation for $p_0$, and properties of beta integrals, we get

$$|(MQ)_n(s,x;t,z)| \leq C^n[\Gamma\left(\frac{n}{2}\right)]^{-1}p_0(s,x;t,z) \qquad \ldots (2.48)$$

for $x, z \in \overline{D}, 0 < (t-s) \leq 1$. Define

$$\Phi(s,x;t,z) = \sum_{n=1}^{\infty}(MQ)_n(s,x;t,z) \qquad \ldots (2.49)$$

for $x, z \in \overline{D}, 0 < (t-s) \leq 1$. From (2.48) it is clear that the series in (2.49) converges uniformly over $x, z \in \overline{D}, 0 < t-s \leq 1$; consequently $\Phi$ is continuous and

$$|\Phi(s,x;t,z)| \leq C(t-s)^{-1/2}p_0(s,x;t,z) \qquad \ldots (2.50)$$

for $x, z \in \overline{D}, 0 < t-s \leq 1$. Moreover from (2.47), (2.49) it is easily seen that $\Phi$ satisfies the integral equation

$$\Phi(s,x;t,z) = MQ(s,x;t,z) + \int_s^t \int_{\overline{D}} MQ(s,x;r,\eta)\Phi(r,\eta;t,z)d\eta dr \qquad \ldots (2.51)$$

for $x, z \in \overline{D}, 0 < t-s \leq 1$.

Using arguments similar to those on pp. 16-17 (in the proof of Theorem 7, Chap I, Sec. 4) of Friedman (1983), and the estimates (2.30), (2.46) and (2.3) we can show that

$$|MQ(s,x;t,z) - MQ(s,x',t,z)| \leq K|x-x'|(t-s)^{-3/4}[p_0(s,x;t,z) + p_0(s,x';t,z)] \qquad \ldots (2.52)$$

and consequently by (2.51)

$$|\Phi(s,x;t,z) - \Phi(s,x';t,z)| \leq K|x-x'|(t-s)^{-3/4}[p_0(s,x;t,z) + p_0(s,x';t,z)] \qquad \ldots (2.53)$$

for $x, x' \in \overline{D}, 0 < t-s \leq 1$.

For $0 < t-s \leq 1, x, z \in \overline{D}$ define

$$k(s,x;t,z) := Q(s,x;t,z) + \int_s^t \int_{\overline{D}} Q(s,x;r,\eta)\Phi(r,\eta;t,z)d\eta dr \qquad \ldots (2.54)$$

By Lemma 2.6, Lemma 2.4 and (2.51) it is seen that

$$\left(\frac{\partial}{\partial s} + L_x\right) k(s,x;t,z) = 0, \quad x \in D, z \in \overline{D} \qquad \ldots (2.55)$$

By (2.28), (2.54)

$$\frac{\partial}{\partial x_1} k(s,x;t,z) = 0, \quad x \in \partial D, z \in \overline{D} \qquad \ldots (2.56)$$

Now let $f$ be a bounded continuous function on $\overline{D}$. Then by (2.50), (2.30) we get

$$|\int_{\overline{D}}\int_s^t\int_{\overline{D}} f(z)Q(s,x;r,\eta)\Phi(r,\eta;t,z)d\eta drdz| \leq C(t-s)^{1/2}$$

Therefore by (2.32)

$$\lim_{s\uparrow t}\int_{\overline{D}} f(z)k(s,x;t,z)dz = f(x) \qquad \ldots(2.57)$$

for $t > 0, x \in \overline{D}$. From (2.55) - (2.57) it follows that $k$ given by (2.54) is the fundamental solution for $((\partial/\partial s)+L)$ with the boundary condition $\frac{\partial}{\partial x_1} = 0$, in any time interval of length $< 1$.

If $(t-s) > 1$, then $k$ can be extended in an obvious manner using the Chapman-Kolmogorov equation. It is also clear that $k$ has the required regularity and satisfies (2.4). Thus the theorem has been proved for the case $\gamma \equiv (1,0,\ldots,0), \rho \equiv 0$. □

*Step* 5. Take $a,b,c$ satisfying the hypotheses (A1)-(A3); let $\gamma$ satisfy (A4), and $\rho \equiv 0$.

Since $\rho \equiv 0$, the fundamental solution for $((\partial/\partial s)+L, J)$ is also that for $((\partial/\partial s)+L, \nu J)$ for any constant $\nu$. So without loss of generality we may take $|D^\alpha_{s,x}\gamma_i| \leq \mu < \lambda_1/(8d^3\lambda_2)$, where $\alpha$ is a multi-index such that $|\alpha| \leq 3$. Consequently,

$$|\gamma(s,\overline{x}) - \gamma(s,\overline{x'})|^2 \leq (d-1)\mu^2|x-x'|^2 \qquad \ldots(2.58)$$

for $s \geq 0, \overline{x}, \overline{x'} \in \partial D$. Let $0 < \theta < 1/(2d\mu)$; let $\phi$ be a smooth function on $(-1,\infty)$ such that $\phi$ is non decreasing, $|\phi'| \leq 1, \phi(r) = r$ if $r \leq \frac{1}{2}\theta$ and $\phi(r) = \theta$ if $r \geq \theta$. Define $T : [0,\infty)\times\overline{D} \to [0,\infty)\times\overline{D}$ by

$$(\tilde{s},\tilde{x}_1,\tilde{x}_2,\ldots,\tilde{x}_d) = T(s,x_1,x_2,\ldots,x_d)$$

$$= (s,x_1,x_2,\ldots,x_d) + \phi(x_1)(0,0,\gamma_2(s,\overline{x}),\ldots,\gamma_d(s,\overline{x})) \qquad \ldots(2.59)$$

In view of (2.58) it can be shown that $T$ is a $C^2$-diffeomorphism.

Define $(d+1)\times(d+1)$ matrix $A(\cdot,\cdot)$ by $A_{0i}(s,x) \equiv A_{i0}(s,x) \equiv 0, 0 \leq i \leq d, A_{ij}(s,x) = a_{ij}(s,x)$, $1 \leq i,j \leq d$. Put

$$\tilde{A}(\tilde{s},\tilde{x}) = j_T^*(T^{-1}(\tilde{s},\tilde{x}))A(T^{-1}(\tilde{s},\tilde{x}))j_T(T^{-1}(\tilde{s},\tilde{x})), \qquad \ldots(2.60)$$

where $j_T$ is the Jacobian of the transformation $T$. Note that $\tilde{A}_{0i}(\cdot,\cdot) \equiv \tilde{A}_{i0}(\cdot,\cdot) \equiv 0, 0 \leq i \leq d$. Thus it is easily seen that the operator $(\frac{\partial}{\partial s}+L)$ in the $(s,x)$-variables is transformed to

$$\frac{\partial}{\partial\tilde{s}} + \tilde{L} = \frac{\partial}{\partial\tilde{s}} + \frac{1}{2}\sum_{i,j=1}^d \tilde{a}_{ij}(\tilde{s},\tilde{x})\frac{\partial^2}{\partial\tilde{x}_i\partial\tilde{x}_j} + \sum_{i=1}^d \tilde{b}_i(\tilde{s},\tilde{x})\frac{\partial}{\partial\tilde{x}_i} + \tilde{c}(\tilde{s},\tilde{x}), \quad \ldots(2.61)$$

with $\tilde{a}, \tilde{b}, \tilde{c}$ satisfying the hypotheses (A1) - (A3). It is also clear that the boundary operator $J$ is transformed to $\partial/\partial\tilde{x}_1$. (The proof in Ramasubramanian (1986, 1988) for the time homogeneous case can easily be extended to the present case.)

In view of Step 4, the above transformation and as the first three derivatives of $\tilde{x}$-variables with respect to $x$-variables are bounded, it now follows that the theorem holds in this case. □

Lemma 2.7. *Let $a, b, c, \gamma$ satisfy the hypotheses (A1) - (A4); let $\rho \equiv 0$. Let $k$ be the fundamental solution in this case. Let $f$ be a continuous function on $[0,t) \times \partial D$ satisfying (2.38). Set*

$$v(s,x) = \frac{1}{2}\int_s^t \int_{\partial D} a_{11}(r,\eta) f(r,\eta) k(s,x;r,\eta) d\sigma(\eta) dr \qquad \ldots (2.62)$$

*for $0 \le s < t, x \in \overline{D}$. Then $v$ is continuous on $[0,t) \times \overline{D}$ and $Jv(s,x) = -f(s,x), 0 \le s < t, x \in \partial D$.*

*Proof.* In Step 5 note that $\tilde{a}_{11}(s,x) = a_{11}(s,x)$ for any $x \in \partial D$. Therefore it is enough to consider the case $J = \frac{\partial}{\partial x_1}$; in such a case $k$ is given by (2.54)

Now by (2.30), (2.38), (2.50) note that

$$\int_s^t \int_{\partial D} \int_s^r \int_{\overline{D}} |a_{11}(r,\eta) f(r,\eta) \frac{\partial Q}{\partial x_1}(s,x';\alpha,\xi)\Phi(\alpha,\xi;r,\eta)| d\xi d\alpha d\sigma(\eta) dr$$

$$\begin{aligned} &\le K \int_s^t \int_{\partial D} \left[1 + (t-r)^{-d/2} e^{-\frac{K\eta_1^2}{(t-r)}}\right] p_0(s,x';r,\eta) d\sigma(\eta) dr \\ &\le K \left[(t-s)^{1/2} + p_0(s,(0,\overline{x});t,(0,0))\right] < \infty \end{aligned} \qquad \ldots (2.63)$$

for all $x'$ sufficiently close to $x = (0,\overline{x}) \in \partial D$; (in the above we have used Chapman-Kolmogorov equations for the $(d-1)$ dimensional Brownian motion) From (2.63), dominated convergence theorem and Lemma 2.5 the required conclusion now follows. □

*Note.* Lemma 2.7 is a slight generalisation of Theorem 1, Chapter 5 (p. 137) of Friedman (1983) concerning single layer potentials.

*Step* 6. We now consider the general situation in the half space. Let $k$ denote the fundamental solution for $\left(\left(\frac{\partial}{\partial s} + L\right), \Sigma\gamma_i(\cdot,\cdot)\frac{\partial}{\partial x_i}\right)$. Note that (2.4) holds for $k$; (that is, with $p$ replaced by $k$).

Put $(Nk)_1(s,x;t,z) := Nk(s,x;t,z) := \rho(s,\overline{x})k(s,x;t,z)$, and inductively for $n = 2,3,\ldots$.

$$(Nk)_{n+1}(s,x;t,z) = \frac{1}{2}\int_s^t \int_{\partial D} Nk(s,x;r,\eta) a_{11}(r,\eta) (Nk)_n(r,\eta;t,z) d\sigma(\eta) dr \qquad \ldots (2.64)$$

For $\alpha, \beta \geq 0, 0 \leq s < t$ note that

$$\exp\left[-\left\{\frac{\alpha^2}{2(r-s)}+\frac{\beta^2}{2(t-r)}\right\}\right] \leq \exp\left[-\frac{(\alpha-\beta)^2}{2(t-s)}\right] \qquad \ldots(2.65)$$

Observe that

$$p_0(s,x;t,z) = p_0^{(1)}(s,x_1;t,z_1)p_0^{(2)}(s,\bar{x};t,\bar{z}) \qquad \ldots(2.66)$$

where $p_0^{(1)}$ is the transition probability density of the reflecting Brownian motion in $[0,\infty)$, and $p_0^{(2)}$ is that of the $(d-1)$-dimensional Brownian motion. Consequently using (2.65), (2.66) and the estimate (2.4) for $k$, one can prove that for $0 < (t-s) \leq 1$,

$$|(Nk)_2(s,x;t,z)| \leq C(t-s)^{1/2}p_0(s,x;t,z).$$

Proceeding inductively one can show that

$$|(Nk)_n(s,x;t,z)| \leq C^n\left[\Gamma(\frac{n}{2})\right]^{-1} p_0(s,x;t,z) \qquad \ldots(2.67)$$

Put

$$\Psi(s,x;t,z) = \sum_{n=1}^{\infty}(Nk)_n(s,x;t,z), \quad 0 \leq s < t, x,z \in \overline{D} \qquad \ldots(2.68)$$

Note that $\Psi$ is well defined and that by (2.64)

$$\Psi(s,x;t,z) = Nk(s,x;t,z) + \frac{1}{2}\int_s^t\int_{\partial D} Nk(s,x;r,\eta)a_{11}(r,\eta)\Psi(r,\eta;t,z)d\sigma(\eta)dr \qquad \ldots(2.69)$$

Now define for $0 < t-s \leq 1, x,z \in \overline{D}$,

$$p(s,x;t,z) = k(s,x;t,z) + \frac{1}{2}\int_s^t\int_{\partial D} k(s,x;r,\eta)a_{11}(r,\eta)\Psi(r,\eta;t,z)d\sigma(\eta)dr \qquad \ldots(2.70)$$

By (2.69) and Lemma 2.7 it is clear that $J_x p(s,x;t,z) = 0, x \in \partial D$. Using the estimate (2.4) for $k$ and proceeding as in Step 4, the proof can be completed. Thus Theorem 2.1 is now proved. □

Our next objective is to prove a regularity theorem; for this we need a few lemmas.

Lemma 2.8. *Let $T_{(t,z)}$ be defined by (2.13), (2.14) as in Remark 2.3 Suppose*

$$\sum_{i=2}^{d}\left|\frac{a_{1i}(t,z)}{a_{11}(t,z)}\right|^2 < \frac{1}{24} \qquad \ldots(2.71)$$

*for all $t \geq 0, x,z \in \overline{D}$. Then*

$$|(T_{(t,z)}z)^* - T_{(t,z)}x|^2 \geq \frac{1}{2}\left|T_{(t,z)}(z^*-x)\right|^2 \qquad \ldots(2.72)$$

*for all* $t \geq 0, x, z \in \overline{D}$.

*Proof.* Observe that

$$\left|(T_{(t,z)}z)^* - T_{(t,z)}x\right|^2 = \frac{1}{2}\left|T_{(t,z)}(z^* - x)\right|^2 + f(\xi_1, \overline{\xi}; z)$$

where $\xi_1 = (z_1 + x_1), \xi_i = (z_i - x_i), i \geq 2, \overline{\xi} = (\xi_2, \ldots, \xi_d)$ and

$$f(\xi_1, \overline{\xi}; z) = \frac{1}{2}\xi_1^2 + \frac{1}{2}|\overline{\xi}|^2 + \frac{1}{2}\xi_1^2|\overline{\tilde{\gamma}}|^2 + \xi_1 < \overline{\tilde{\gamma}}, \overline{\xi} > + [4z_1^2 + 4z_1\xi_1]|\overline{\tilde{\gamma}}|^2 + 4z_1 < \overline{\tilde{\gamma}}, \overline{\xi} >$$

with $\overline{\tilde{\gamma}} = \overline{\tilde{\gamma}(t,z)}$. To prove the lemma, it is enough to show that $f(\xi_1, \overline{\xi}; z) \geq 0$ for all $x_1 \geq 0, (x_2, \ldots, x_d) \in \mathbb{R}^{d-1}$, for any fixed $z_1 \geq 0, (z_2, \ldots, z_d) \in \mathbb{R}^{d-1}, t \geq 0$.

Fix $z \in \overline{D}, t \geq 0$. Then $\overline{\tilde{\gamma}}(t,z)$ can be taken to be a fixed vector $\overline{\gamma} \in \mathbb{R}^{d-1}$ with $|\overline{\gamma}|^2 < \frac{1}{24}$.

If $< \overline{\gamma}, \overline{\xi} > \geq 0$, there is nothing to prove. Set

$$g(\xi_1, \overline{\xi}) = \frac{1}{2}\xi_1^2 + \frac{1}{2}|\overline{\xi}|^2 + \frac{1}{2}|\overline{\gamma}|^2\xi_1^2 + 5\xi_1 < \overline{\gamma}, \overline{\xi} >$$

Observe that

$$f(\xi_1, \overline{\xi}) = g(\xi_1, \overline{\xi}) + [4z_1^2 + 4z_1\xi_1]|\overline{\gamma}|^2 - 4x_1 < \overline{\gamma}, \overline{\xi} >$$

If $< \overline{\gamma}, \overline{\xi} >< 0$, then it is enough to prove $g(\xi_1, \overline{\xi}) \geq 0$ for any $\xi_1 \geq 0, \overline{\xi} \in \mathbb{R}^{d-1}$.

Under the assumption $|\overline{\gamma}|^2 < \frac{1}{24}$ it is easy to see that $\xi = 0$ is the only stationary point of $g$, that the matrix of the second derivatives of $g$ is positive definite. And since $g(0) = 0$, the conclusion follows. □

We will make the following assumption for the remainder of this section. It may be mentioned that our hypothesis is stronger than necessary, especially concerning smoothness of $b, c, \rho$.

(A6) : $a_{ij}, b_i, c, \gamma_i, \rho$ are $C_b^4$-functions on their respective domains.

Lemma 2.9. *Let* $a(\cdot,\cdot)$ *satisfy (A1), (A6), (2.71). Let* $Q$ *be given by (2.28). Then for any* $t_0 > 0$ *there exist constants* $C_1, C_2$ *such that*

$$\begin{aligned} |D_x^\alpha D_z^\beta Q(s,x;t,z)| &= |D_z^\beta D_x^\alpha Q(s,x;t,z)| \\ &\leq C_1(t-s)^{-(|\alpha|+|\beta|)/2} p_0(s,x;t,z) \end{aligned} \qquad \ldots (2.73)$$

*for* $0 \leq s < t$ *with* $(t-s) \leq t_0, x, z \in \overline{D}$, *any multiindex* $\alpha$ *and multiindex* $\beta$ *with* $|\beta| \leq 2$.

*Proof.* Without loss of generality take $t_0 = 1$. Let $\eta, \zeta, \psi_1, \psi_2$ be as in Remark 2.3. Note that

$$(T_{(t,z)}z)^* - T_{(t,z)}x = z^* - x + (z_1 - x_1)\overline{\tilde{\gamma}}(t,z) \qquad \ldots (2.74)$$

Using (2.3), (2.24), (2.74) it is easily seen that

$$\left|\frac{\partial \zeta}{\partial z_k}(s,x;t,z)\right| \le K(|z^* - x| + |z^* - x|^2) \qquad \ldots (2.75)$$

$$\left|\frac{\partial^2}{\partial x_i \partial z_k}\zeta(s,x;t,z)\right| \le K(1 + |z^* - x|) \qquad \ldots (2.76)$$

$$\left|\frac{\partial^3}{\partial x_i \partial x_j \partial z_k}\zeta(s,x;t,z)\right| \le K \qquad \ldots (2.77)$$

for $0 \le s < t, x, z \in \overline{D}, 1 \le i,j,k \le d$. Also $D_x^\alpha \frac{\partial}{\partial z_k}\zeta \equiv 0$ for $|\alpha| \ge 3$.

By the preceding lemma, (2.22), (2.75) - (2.77) it follows that

$$|\psi_2(s,x;t,z)| \le C_1(t-s)^{-d/2}\exp\left(-\frac{C_2|z^* - x|^2}{(t-s)}\right) \qquad \ldots (2.78)$$

$$\begin{aligned}\left|D_x^\alpha \frac{\partial}{\partial z_k}\psi_2(s,x;t,z)\right| &= \left|\frac{\partial}{\partial z_k}D_x^\alpha \psi_2(s,x;t,z)\right| \\ &\le C_1(t-s)^{-(|\alpha|+d+1)/2}\exp\left\{-\frac{C_2|z^* - x|^2}{(t-s)}\right\}\end{aligned} \qquad \ldots (2.79)$$

for $0 \le s < t$, multiindex $\alpha, x, z \in \overline{D}, k = 1, 2, \ldots, d$.

In a similar fashion we get

$$|\psi_1(s,x;t,z)| \le C_1(t-s)^{-d/2}\exp\left(-\frac{C_2|z - x|^2}{(t-s)}\right) \qquad \ldots (2.80)$$

$$\begin{aligned}\left|D_x^\alpha \frac{\partial}{\partial z_k}\psi_1(s,x;t,z)\right| &= \left|\frac{\partial}{\partial z_k}D_x^\alpha \psi_1(s,x;t,z)\right| \\ &\le C_1(t-s)^{-(|\alpha|+d+1)/2}\exp\left\{-\frac{C_2|z - x|^2}{(t-s)}\right\}\end{aligned} \qquad \ldots (2.81)$$

with the same notation as before.

As $z_1 \ge 0, x_1 \ge 0, r \ge 0$ using (2.78), (2.79) it can be proved that

$$\left|D_x^\alpha \frac{\partial}{\partial z_k}\int_0^\infty \frac{(z_1 + r + x_1)}{a_{11}(t,z)(t-s)}\psi_2(s,(x + re_1);t,z)dr\right|$$

$$\le C_1(t-s)^{-(|\alpha|+d+1)/2}\exp\left\{-\frac{C_2|z^* - x|^2}{(t-s)}\right\} \qquad (2.82)$$

with the usual notation. From (2.29), (2.79), (2.81), (2.82) it is clear that (2.73) holds when $|\beta| = 1$. The case $|\beta| = 2$ is treated similarly.

Theorem 2.10. *Let (A1) - (A6) hold. (i) Then $p(s,x;t,z)$ is continuously differentiable in z-variables over D, and for any $T > 0$ and any compact set H in D there is a constant C such that*

$$|D_x^\alpha D_z^\beta p(s,x;t,z)| \le C(t-s)^{-(|\alpha|+|\beta|)/2}p_0(s,x;t,z) \qquad \ldots (2.83)$$

*for* $0 \le s < t$ *with* $t - s \le T, x \in D$ *(resp.* $x \in \overline{D}$*),* $z \in H, \alpha$ *a multiindex with* $|\alpha| \le 2$ *(resp.* $|\alpha| \le 1$*),* $\beta$ *a multiindex with* $|\beta| \le 1$.

*(ii) Moreover* $p(s,x;t,z)$ *is thrice continuously differentiable in* $x$*-variables over* $D$*, and for* $T > 0$ *and compact set* $H$ *in* $D$ *there is a constant* $C$ *such that*

$$|D_x^\alpha p(s,x;t,z)| \le C(t-s)^{-3/2} p_0(s,x;t,z) \qquad \ldots (2.84)$$

*for* $s, t$ *as before,* $x \in H, z \in \overline{D}$ *and any multiindex* $\alpha$ *with* $|\alpha| = 3$.

*Note.* The variance parameter $C_2$ of $p_0$ does not depend on $H, s, t, x, z$.

*Proof.* (i) Without loss of generality we may assume (2.71) holds; for otherwise transform the variables $(x_1, x_2, \ldots, x_d) \mapsto (Kx_1, (x_2/K), \ldots, (x_d/K))$ for a suitable constant $K$.

Therefore by the preceding lemma, (2.3), (2.45), Lipschitz continuity of $a_{ij}$ we get

$$\left| \frac{\partial}{\partial z_k} MQ(s,x;t,z) \right| \le C(t-s)^{-1} p_0(s,x;t,z) \qquad \ldots (2.85)$$

for $0 < t - s \le 1, x \in \overline{D}, z \in \overline{D}$.

We first consider the case $\gamma \equiv (1, 0, \ldots, 0), \rho \equiv 0$. Observe that

$$\begin{aligned}
&\int_s^t \int_{\overline{D}} MQ(s,x;r,\eta) \frac{\partial}{\partial z_k} MQ(r,\eta;t,z) d\eta dr \\
= \ & \int_s^{(t+s)/2} \int_{\overline{D}} MQ(s,x;r,\eta) \frac{\partial}{\partial z_k} MQ(r,\eta;t,z) d\eta dr \\
&+ \int_{(t+s)/2}^t \int_{\overline{D}} MQ(s,x;r,\eta) c(r,\eta) \frac{\partial Q}{\partial z_k}(r,\eta;t,z) d\eta dr \\
&+ \sum_{i=1}^d \int_{(t+s)/2}^t \int_{\overline{D}} MQ(s,x;r,\eta) b_i(r,\eta) \frac{\partial^2}{\partial \eta_i \partial z_k} Q(r,\eta;t,z) d\eta dr \\
&- \frac{1}{2} \sum_{i,j=1}^d \int_{(t+s)/2}^t \int_{\overline{D}} MQ(s,x;r,\eta) \left[ \frac{\partial a_{ij}}{\partial z_k}(t,z) \right] \frac{\partial^2}{\partial \eta_i \partial \eta_j} Q(r,\eta;t,z) d\eta dr \\
&+ \frac{1}{2} \sum_{i,j=1}^d \int_{(t+s)/2}^t \int_{\overline{D}} MQ(s,x;r,\eta) [a_{ij}(r,\eta) - a_{ij}(t,z)] \\
&\qquad \frac{\partial^3}{\partial \eta_i \partial \eta_j \partial z_k} Q(r,\eta;t,z) d\eta dr \\
= \ & I_1 + I_2 + I_3 + I_4 + I_5
\end{aligned} \qquad \ldots (2.86)$$

By (2.46), (2.73), (2.85) it is easily seen that

$$|I_1| \le C(t-s)^{-1/2} p_0(s,x;t,z) \qquad \ldots (2.87)$$

$$|I_2| \le C p_0(s,x;t,z) \qquad (2.88)$$

Let $H$ be a compact subset of $D$. Note that

$$|D_\eta^\alpha D_z^\beta Q(r,\eta;t,z)| \leq C p_0(r,\eta;t,z) \qquad \ldots(2.89)$$

for $0 \leq r < t, z \in H, \eta \in \partial D$ any multiindices $\alpha, \beta$.

By (2.73), (2.85), (2.89) and the divergence theorem

$$\begin{aligned} |I_3| &\leq \int_{(t+s)/2}^t \int_D \left| \sum_i \frac{\partial}{\partial \eta_i}[b_i(r,\eta) MQ(s,x;r,\eta)] \right| \cdot \left| \frac{\partial}{\partial z_k} Q(r,\eta;t,z) \right| d\eta dr \\ &\quad + \sum_i \int_{(t+s)/2}^t \int_{\partial D} |MQ(s,x;r,\eta) b_i(r,\eta)| \cdot \left| \frac{\partial}{\partial z_k} Q(r,\eta;t,z) \right| d\sigma(\eta) dr \\ &\leq C p_0(s,x;t,z) \end{aligned} \qquad \ldots(2.90)$$

In a similar fashion it can be shown that

$$|I_4| \leq C p_0(s,x;t,z) \qquad \ldots(2.91)$$

By an analogous argument, but with divergence theorem applied twice and using Lipschitz continuity of $a_{ij}$, we get

$$|I_5| \leq C(t-s)^{-1/2} p_0(s,x;t,z) \qquad \ldots(2.92)$$

From (2.86) - (2.92) it follows that $(MQ)_2$ is continuously differentiable in $z$ and that

$$\begin{aligned} \left| \frac{\partial}{\partial z_k}(MQ)_2(s,x;t,z) \right| &= \left| \int_s^t \int_{\overline{D}} MQ(s,x;r,\eta) \frac{\partial}{\partial z_k} MQ(r,\eta,t,z) d\eta dr \right| \\ &\leq C(t-s)^{-1/2} p_0(s,x;t,z) \end{aligned} \qquad \ldots(2.93)$$

for any $0 < t-s \leq 1, x \in \overline{D}, z \in H$. By (2.93) and iterating it is easily seen that $(MQ)_n$ is continuously differentiable in $z$ and that

$$\left| \frac{\partial}{\partial z_k}(MQ)_n(s,x;t,z) \right| \leq C^n \left( \Gamma\left(\frac{n}{2}\right) \right)^{-1} p_0(s,x;t,z) \qquad \ldots(2.94)$$

for $s,t,x,z$ as before and $n \geq 3$. Consequently by (2.49), (2.85), (2.93), (2.94)

$$\begin{aligned} &\left| \frac{\partial \Phi}{\partial z_k}(s,x;t,z) \right| \\ &= \left| \frac{\partial MQ}{\partial z_k}(s,x;t,z) + \frac{\partial (MQ)_2}{\partial z_k}(s,x;t,z) + \sum_{n=3}^\infty \frac{\partial}{\partial z_k}(MQ)_n(s,x;t,z) \right| \\ &\leq C\left[ (t-s)^{-1} + (t-s)^{-1/2} + 1 \right] p_0(s,x;t,z) \end{aligned} \qquad \ldots(2.95)$$

Next, observe that by an argument similar to the derivation of (2.93) it can be shown that

$$\int_s^t \int_{\overline{D}} \left| Q(s,x;r,\eta) \frac{\partial}{\partial z_k} MQ(r,\eta;t,z) \right| d\eta dr \leq C(t-s)^{-1/2} p_0(s,x;t,z) \quad \ldots (2.96)$$

Therefore by (2.30), (2.54), (2.95), (2.96) it follows that

$$\begin{aligned} &\left| \frac{\partial k}{\partial z_i}(s,x;t,z) \right| \\ &= \left| \frac{\partial}{\partial z_i} Q(s,x;t,z) + \int_s^t \int_{\overline{D}} Q(s,x;r,\eta) \sum_{n=1}^{\infty} \frac{\partial (MQ)_n}{\partial z_i}(r,\eta;t,z) d\eta dr \right| \\ &\leq C(t-s)^{-1/2} p_0(s,x;t,z) \end{aligned} \quad \ldots (2.97)$$

for $0 < t-s \leq 1, x \in \overline{D}, z \in H$.

Because of our hypothesis (A6) a tedious but routine argument gives

$$D_x^{\alpha} \psi_l(s,x;t,z) = \pm D_z^{\alpha} \psi_l(s,x;t,z) + \sum^{(1)} D_z^{\alpha'} \psi_l^{(\alpha')}(s,x;t,z) \quad \ldots (2.98)$$

where

$$|\psi_l^{(\alpha')}(s,x;t,z)| \leq K[1 + (t-s)^{-1/2}] p_0(s,x;t,z),$$

$\sum^{(1)}$ indicates summation over multiindices $\alpha'$ which are subsets of $\alpha$ with $|\alpha'| \leq |\alpha| - 1$, for $l = 1,2, x \in \overline{D}, z \in D, 0 \leq s < t$ and multiindex $\alpha$ with $1 \leq |\alpha| \leq 4$. Therefore

$$D_x^{\alpha} M\psi_l(s,x;t,z) = \sum^{(2)} f_\beta D_z^{\beta} \psi_l^{(\beta)}(s,x;t,z) \quad \ldots (2.99)$$

where $\sum^{(2)}$ indicates summation over multiindices $\beta$ such that $|\beta| \leq |\alpha| + 2$, $f_\beta$ and their appropriate derivatives are bounded functions and

$$|\psi_l^{(\beta)}(s,x;t,z)| \leq K[1 + (t-s)^{-1/2}] p_0(s,x;t,z)$$

for $s,t,x,z,l$ as above, and $|\alpha| = 1,2$.

It is also easily seen that

$$|D_x^{\alpha} M\psi_l(s,x;t,z)| \leq K(t-s)^{-(|\alpha|+1)/2} p_0(s,x;t,z) \quad \ldots (2.100)$$

for $s,t,x,z,l,\alpha$ as above. (For proving (2.100) one does not need (2.98) or (2.99); also twice differentiability of $a_{ij}, b_i$ would do.)

Using (2.98) - (2.100), divergence theorem and arguments similar to the derivation of (2.95) - (2.97) we get

$$|D_x^{\alpha} D_z^{\beta} k(s,x;t,z)| \leq C(t-s)^{-(|\alpha|+|\beta|)/2} p_0(s,x;t,z) \quad \ldots (2.101)$$

for $0 < t - s \leq 1, x \in D$ (resp. $\overline{D}$), $z \in H, \alpha, \beta$ multiindices with $|\beta| \leq 1, |\alpha| \leq 2$ (resp. $|\alpha| \leq 1$).

We now assume that $\gamma \equiv (1, 0, \ldots, 0)$ but $\rho$ is no longer zero. Define $(Nk)_n$ by (2.64).

As $z \in H \subset D, \eta \in \partial D$, using (2.97) and arguments similar to the derivation of (2.93) - (2.95) it can be shown that

$$\left| \frac{\partial}{\partial z_i} \Psi(s, x; t, z) \right| \leq K p_0(s, x; t, z) \qquad \ldots (2.102)$$

and consequently

$$\left| \frac{\partial}{\partial z_i} p(s, x; t, z) \right| \leq K(t - s)^{-1/2} p_0(s, x; t, z) \qquad \ldots (2.103)$$

for $0 < t - s \leq 1, x \in \overline{D}, z \in H$.

Now suppose $x \in D$. If $D_x^2$ involves only tangential derivatives, then $D_x^2 Nk(s, x; r, \eta)$ can be expressed as a sum of terms of the type $D_\eta^\alpha \zeta^{(\alpha)}$ with $|\alpha| \leq 2, D_\eta^\alpha$ involving only tangential derivatives,

$$|\zeta^{(\alpha)}(s, x; r, \eta)| \leq K[1 + (r - s)^{-1/2}] p_0(s, x; r, \eta)$$

Applying divergence theorem in $\partial D = \mathbb{R}^{d-1}$ and arguing as before we get

$$\left| \frac{\partial^2}{\partial x_i \partial x_j} \frac{\partial}{\partial z_k} p(s, x; t, z) \right| \leq K(t - s)^{-3/2} p_0(s, x; t, z) \qquad \ldots (2.104)$$

Suppose $D_x^2$ involves $\partial / \partial x_1$ or $\partial^2 / \partial x_1^2$, then again argue as before, applying the divergence theorem in $D$; once again the estimate (2.104) is obtained.

For $x \in \partial D$, by Lemma 2.7

$$\frac{\partial}{\partial x_1} \left[ \frac{1}{2} \int_s^t \int_{\partial D} Nk(s, x; r, \eta) a_{11}(r, \eta) \frac{\partial}{\partial z_k} Nk(r, \eta; t, z) d\sigma(\eta) dr \right]$$

$$= -\rho(s, \overline{x}) \frac{\partial}{\partial z_k} \rho(s, \overline{x}) k(s, x; t, z) \qquad \ldots (2.105)$$

To estimate the tangential derivatives the procedure given in the preceding paragraph can be used. It is now not difficult (but tedious) to see that arguments as given in the earlier paragraphs would lead to the required estimate (2.83); the details are omitted.

The general case can be reduced to the case of $\gamma \equiv (1, 0, \ldots, 0)$ as in Step 5 by a suitable sufficiently smooth diffeomorphism.

Proof of (ii) is similar. □

*Remark* 2.11. Nonnegativity of $p$ can be proved as in Section 8 of Ito (1992), with the obvious modifications needed for considering the backward parabolic equations. Strict positivity can be established using the maximum principles on pp. 173-174 of Protter and Weinberger (1984).

Proceeding as in the proof of Theorem 16, Chap I, Section 9 of Friedman (1983) it can be shown that a unique solution exists for problem (2.6) in the class of functions satisfying

$$\int_0^T \int_{\overline{D}} |u(s,x)| \exp(-K|x|^2) dx ds < \infty$$

for some $K > 0$. Using this uniqueness property it can be proved that $p$ satisfies the Chapman-Kolmogorov equation. We omit the details.

## 3. On mixed problem

In this section we obtain some auxiliary results concerning Green function and Poisson kernel for $(\partial/\partial s) + L$ with mixed boundary conditions in certain bounded domains in the half space.

Fix $T > 0$. We consider the mixed problem in $[0,T] \times G$ where $G = B(0 : R) \cap D$, or $G$ is the diffeomorphic image of $B(0 : R) \cap D$ under a diffeomorphism as in Step 5 of previous section, with $R > 0$ and $D$ denoting the half space. Let $\partial_1 G = \partial G \cap \partial D, \partial_2 G = \partial G \backslash \partial D$. (Such a diffeomorphism maps $D$ onto $D$ and is identity on $\partial D$.) We make the following assumption.

(A7) : The coefficients of $L$ (resp. $J$) are restrictions to $[0,T] \times \overline{G}$ (resp. $[0,T] \times \partial_1 G$) of $a, b, c$ (resp. $\gamma, \rho$) defined on $[0,\infty) \times \overline{D}$ (resp. $[0,\infty) \times \partial D$) satisfying (A1) - (A6).

*Remark* 3.1. For applications in Section 4, the coefficients $a, b, c$ may be taken to be defined on $[0, T+\epsilon] \times B(0 : R+\epsilon)$ satisfying (A1) - (A6). In such a case one can define $\widehat{a}, \widehat{b}, \widehat{c}$ on $[0,\infty) \times \overline{D}$ so that $\widehat{a}, \widehat{b}, \widehat{c}$ agree with $a, b, c$ respectively on $[0,T] \times B(0 : R)$ and satisfy (A1) - (A6) on $[0,\infty) \times \overline{D}$; this can be done using the procedure given on $p$. 81 of Friedman (1983). Similar comments apply to $\gamma, \rho$. Thus (A7) is not a restrictive hypothesis for our purposes.

Lemma 3.2. *Let $G$ be a diffeomorphic image of $B(0 : R) \cap D$ as described above. Let $T > 0$ be fixed. Let $\gamma(\cdot,\cdot) \equiv (1, 0, \ldots, 0)$. Assume (A1) - (A7). Let $f$ be a bounded continuous function. For $s < T, x' \in \overline{G} \backslash (\overline{\partial_2 G})$ define*

$$w(s,x') = \int_s^T \int_{\partial_2 G} f(t,z) D_{\nu(z)} p(s,x';t,z) d\sigma(z) dt \qquad \ldots (3.1)$$

*where $d\sigma(\cdot)$ is the surface area measure on $\partial_2 G$, and*

$$D_{\nu(z)} p(s,x';t,z) = < a(t,z) n(z), \ \nabla_z p(s,x';t,z) > \qquad \ldots (3.2)$$

*is the derivative in the inward conormal direction (at $z \in \partial_2 G$) in $z$-variables. Let $x \in \partial_2 G$ be such that $x_1 > 0$. Then for $x' \to x$ with $x' \in G$*

*we have*

$$\lim_{x'\to x} w(s,x') = \int_s^T \int_{\partial_2 G} f(t,z) D_{\nu(z)} p(s,x;t,z) d\sigma(z) dt - \frac{1}{2} f(s,x) \qquad \ldots (3.3)$$

*Proof.* For $x' \in \overline{G} \backslash \overline{(\partial_2 G)}$ note that $d(x', \partial_2 G) > 0$ and hence (3.1) is well defined.

Let $x \in \partial_2 G$ with $x_1 > 0$ be fixed and $x' \to x$ with $x' \in D$. We can assume that there is a compact neighbourhood $\widehat{H}$ of $x$ such that $x', x \in \widehat{H}, \widehat{H} \subset D$. Let $H = \widehat{H} \cap \partial_2 G$ and $H^c = (\partial_2 G) \backslash H$. Then

$$\begin{aligned} w(s,x') &= \int_s^T \int_H f(t,z) D_{\nu(z)} p(s,x';t,z) d\sigma(z) dt \\ &\quad + \int_s^T \int_{H^c} f(t,z) D_{\nu(z)} p(s,x';t,z) d\sigma(z) dt \\ &\equiv w_0(s,x') + w_c(s,x') \end{aligned} \qquad \ldots (3.4)$$

As sup $\{|D_{\nu(z)} p(s,x';t,z)| : 0 \le s < t, z \in H^c\} < \infty$, we get

$$\lim_{x'\to x} w_c(s,x') = \int_s^T \int_{H^c} f(t,z) D_{\nu(z)} p(s,x;t,z) d\sigma(z) dt \qquad \ldots (3.5)$$

Now by (2.29), (2.54), (2.70) we have

$$\begin{aligned} &w_0(s,x') \\ &= \int\int f(t,z) D_{\nu(z)} \psi_1(s,x';t,z) d\sigma(z) dt \\ &\quad - \int\int f(t,z) D_{\nu(z)} \psi_2(s,x';t,z) d\sigma(z) dt \\ &\quad + 2\int\int f(t,z) D_{\nu(z)} \left\{ \int_0^\infty \frac{(z_1+\alpha+x_1')}{a_{11}(t,z)(t-s)} \psi_2(s,(x'+\alpha e_1);t,z) d\alpha \right\} d\sigma(z) dt \\ &\quad + \int\int f(t,z) D_{\nu(z)} \left\{ \int_s^t \int_{\overline{D}} F_2(s,x';\alpha,\xi;t,z) d\xi d\alpha \right\} d\sigma(z) dt \\ &\quad + \int\int f(t,z) D_{\nu(z)} \left\{ \int_s^t \int_{\partial D} F_3(s,x';r,\eta;t,z) d\sigma(\eta) dr \right\} d\sigma(z) dt \\ &\quad + \int\int f(t,z) D_{\nu(z)} \left\{ \int_s^t \int_{\partial D} \int_s^r \int_{\overline{D}} F_4(s,x';\alpha,\xi;r,\eta;t,z) d\xi d\alpha d\sigma(\eta) dr \right\} d\sigma(z) dt \\ &\equiv J_1(x') - J_2(x') + J_3(x') + I_2(x') + I_3(x') + I_4(x') \end{aligned} \qquad \ldots (3.6)$$

where

$$\begin{aligned} F_2(s,x';\alpha,\xi;t,z) &= Q(s,x';\alpha,\xi)\Phi(\alpha,\xi;t,z), \\ F_3(s,x';r,\eta;t,z) &= \frac{1}{2} Q(s,x';r,\eta) a_{11}(r,\eta) \Psi(r,\eta;t,z), \\ F_4(s,x';\alpha,\xi;r,\eta;t,z) &= \frac{1}{2} Q(s,x';\alpha,\xi) \Phi(\alpha,\xi;r,\eta) a_{11}(r,\eta).\Psi(r,\eta;t,z), \end{aligned}$$

and $\int\int$ denotes integration over $[s,T]\times H$ w.r.t. $dtd\sigma(z)$.

Note that by taking local coordinates at $x$

$$\int_{\partial_2 G} p_0(s,x';t,z)d\sigma(z) \le C[1+(t-s)^{-1/2}] \qquad \ldots(3.7)$$

for all $x'$ sufficiently close to $x$, where $p_0$ is as in Theorem 2.1.

By (2.65), (2.102), (3.7) we get

$$\begin{aligned}
&\int\int\int_s^t\int_{\partial D} |f(t,z)D_{\nu(z)}F_3(s,x';r,\eta;t,z)|d\sigma(\eta)drd\sigma(z)dt \\
&\le K\int\int\int_s^t\int_{\partial D} p_0(s,x';r,\eta)p_0(r,\eta;t,z)d\sigma(\eta)drd\sigma(z)dt \\
&\le K\int\int p_0^{(2)}(s,\bar{x}';t,z)\int_s^t(r-s)^{-1/2}(t-r)^{-1/2} \\
&\qquad \exp\left\{-\frac{(x_1')^2}{4(r-s)}-\frac{z_1^2}{4(t-r)}\right\}drd\sigma(z)dt \\
&\le K[(T-s)^{3/2}+(T-s)]
\end{aligned}$$

where $\int\int$ denotes integration over $[s,T]\times H$ w.r.t. $dtd\sigma(z)$. Consequently

$$\lim_{x'\to x} I_3(x') = \int\int\int_s^t\int_{\partial D} f(t,z)D_{\nu(z)}F_3(s,x;r,\eta;t,z)d\sigma(\eta)drd\sigma(z)dt \quad \ldots(3.8)$$

Observe that we may write

$$\begin{aligned}
&I_2(x') \\
&= \int_s^T\int_H\int_s^t\int_{\overline{D}} f(t,z)Q(s,x';\alpha,\xi)D_{\nu(z)}\left[\sum_{j=2}^{\infty}(MQ)_j(\alpha,\xi;t,z)\right]d\xi d\alpha.d\sigma(z)dt \\
&\quad + \int_s^T\int_H\int_s^{(s+t)/2}\int_{\overline{D}} f(t,z)Q(s,x';\alpha,\xi)D_{\nu(z)}MQ(\alpha,\xi;t,z)d\xi d\alpha d\sigma(z)dt \\
&\quad + \int_s^T\int_H\int_{(s+t)/2}^t\int_{\overline{D}} f(t,z)Q(s,x';\alpha,\xi)D_{\nu(z)}MQ(\alpha,\xi;t,z)d\xi d\alpha d\sigma(z)dt \\
&\equiv I_{21}(x')+I_{22}(x')+I_{23}(x')
\end{aligned} \qquad \ldots(3.9)$$

Using (2.93), (2.94), (3.7) in the case of $I_{21}$, and using (2.85), (3.7) in the case of $I_{22}$ it can be seen that

$$\lim_{x'\to x} I_{21}(x')+I_{22}(x')$$

$$= \int_s^T\int_H\int_s^t\int_{\overline{D}} f(t,z)Q(s,x;\alpha,\xi)D_{\nu(z)}\left[\sum_{j=2}^{\infty}(MQ)_j(\alpha,\xi;t,z)\right]d\xi d\alpha d\sigma(z)dt$$

$$+\int_s^T \int_H \int_s^{(t+s)/2} \int_{\overline{D}} f(t,z)Q(s,x;\alpha,\xi)D_{\nu(z)}MQ(\alpha,\xi;t,z)d\xi d\alpha d\sigma(z)dt \qquad \ldots(3.10)$$

Using (2.99), divergence theorem and estimates for derivatives of $Q$ we get

$$\lim_{x'\to x} I_{23}(x')$$

$$= \int_s^T \int_H \int_{(s+t)/2}^{t} \int_{\overline{D}} f(t,z)Q(s,x;\alpha,\xi)D_{\nu(z)}MQ(\alpha,\xi;t,z)d\xi d\alpha d\sigma(z)dt \qquad \ldots(3.11)$$

In an analogous manner it can be shown that

$$\lim_{x'\to x} I_4(x')$$

$$= \int_s^T \int_H \int_s^t \int_{\partial D} \int_s^r \int_{\overline{D}} f(t,z)D_{\nu(z)}F_4(s,x;\alpha,\xi;r,\eta;t,z)d\xi d\alpha d\sigma(\eta)dr d\sigma(z)dt \qquad \ldots(3.12)$$

Next, observe that

$$\int_s^T \int_H |f(t,z)D_{\nu(z)}\psi_2(s,x';t,z)|d\sigma(z)dt$$

$$\leq K \int_s^T \int_H (t-s)^{-(d+1)/2} \exp\left\{-\frac{C}{(t-s)}|\xi(z)-x'|^2\right\} d\sigma(z)dt$$

where $\xi(z)$ varies over a compact set contained in $D^c$ as $z$ varies over $H$. Hence the integrand in the above is bounded (as a function of $(t,z)$) uniformly over $x'$ varying over a neighbourhood of $x$. Therefore

$$\lim_{x'\to x} J_2(x') = \int_s^T \int_H f(t,z)D_{\nu(z)}\psi_2(s,x;t,z)d\sigma(z)dt \qquad \ldots(3.13)$$

In a similar manner

$$\lim_{x'\to x} J_3(x')$$

$$= \int_s^T \int_H \int_0^\infty f(t,z)D_{\nu(z)}\left[\frac{(z_1+\alpha+x_1)}{a_{11}(t,z)(t-s)}\psi_2(s,(x+\alpha e_1);t,z)\right] d\alpha d\sigma(z)dt \qquad \ldots(3.14)$$

It is also clear that the r.h.s. of (3.8) - (3.14) are well defined

As $\gamma \equiv (1,0,\ldots,0)$ note that

$$\psi_1(s,\xi;t,z) = \left(\frac{1}{2\pi(t-s)}\right)^{d/2} \frac{1}{\sqrt{\det a(t,z)}} \exp\left\{-\frac{<(z-\xi),a(t,z)^{-1}(z-\xi)>}{2(t-s)}\right\}$$

Since $a(t,z)^{-1}$ has bounded derivatives, an argument as in p. 396 of Ladyzenskaja, Solonnikov and Uralceva (1968) give that for $x \in \partial_2 G, z \in \partial_2 G$,

$$|D_{\nu(z)}\psi_1(s,x;t,z)| \leq K(t-s)^{(\delta-\frac{1}{2})}p_0(s,x;t,z) \qquad \ldots(3.15)$$

for any sufficiently small $\delta > 0$. As $x \in D$, by the jump relation for single layer potential (see Sections 15 and 16, Chap. IV of Ladyzenskaja *et al.* (1968)) it now follows that

$$\lim_{x' \to x} J_1(x') = \int_s^T \int_H f(t,z) D_{\nu(z)}\psi_1(s,x;t,z)d\sigma(z)dt - \frac{1}{2}f(s,x) \qquad \ldots(3.16)$$

Note that (3.7), (3.15) imply well definedness of r.h.s. of (3.16). Now well definedness and validity of (3.3) follow from (3.4) - (3.16). □

Lemma 3.3. *Assume (A1) - (A7); let $G, \gamma$ be as in the preceeding lemma. For $x \in \partial_2 G$ with $x_1 > 0$ and $z \in \partial_2 G$*

$$|D_{\nu(z)}p(s,x;t,z)| \leq K_1(t-s)^{(\delta-\frac{1}{2})}p_0(s,x;t,z) + K_2 \qquad \ldots(3.17)$$

*for any sufficiently small $\delta > 0$.*

*Proof.* Follows from (3.15) and the arguments of the preceding lemma. □

Let $G, \gamma$ be as in the preceding lemmas. Let $t > 0, z \in \overline{G} \backslash \overline{(\partial_2 G)}$ be fixed. Put

$$\mu_1(s,x;r,\eta) = -2p(s,x;r,\eta) \quad x,\eta \in \overline{G}.$$

Define inductively for $n = 2, 3, \ldots, 0 \leq s < t, x \in \partial_2 G$,

$$\mu_{n+1}(s,x;t,z) = -\int_s^t \int_{\partial_2 G} \left[D_{\nu(\eta)}\mu_1(s,x;r,\eta)\right] \mu_n(r,\eta;t,z)d\sigma(\eta)dr \qquad \ldots(3.18)$$

As $z \notin \overline{\partial_2 G}$, note that $\sup\{p_0(r,\eta;t,z) : s < r < t, \eta \in \partial_2 G\} < \infty$. Consequently by Lemma 3.3 and (3.7) it follows that $|\mu_2(s,x;t,z)| \leq K(t-s)^\delta$ for all $x \in \partial_2 G, s < t$. Proceeding inductively it can be shown that

$$|\mu_n(s,x;t,z)| \leq \frac{C^n(t-s)^{n\delta}}{\Gamma(n\delta+1)} \qquad \ldots(3.19)$$

for all $x \in \partial_2 G, s < t$. Define

$$\mu(s,x;t,z) = \sum_{n=1}^{\infty} \mu_n(s,x;t,z), \quad s < t, x \in \partial_2 G \qquad \ldots(3.20)$$

By (3.18), (3.20) it is clear that $\mu$ satisfies the integral equation

$$\mu(s,x;t,z) = -2p(s,x;t,z) + 2\int_s^t \int_{\partial_2 G} \left[D_{\nu(\eta)}p(s,x;r,\eta)\right] \mu(r,\eta;t,z)d\sigma(\eta)dr \qquad \ldots(3.21)$$

for all $0 \leq s < t, x \in \partial_2 G$.

We are now in a position to prove the following result concerning a mixed boundary value problem.

Theorem 3.4. *Assume (A1) - (A7); let* $G = B(0 : R) \cap D$. *Then there exists a function* $p_G(s, x; t, z)$ *defined for* $0 \leq s < t, x \in \overline{G}, z \in \overline{G} \backslash \overline{(\partial_2 G)}$ *satisfying the following :*

*(i)* $p_G$ *is jointly continuous in its arguments;*

*(ii)* $(s, x) \mapsto p_G(s, x; t, z)$ *is in* $C^{1,2}((0, t) \times G)$ *for fixed* $t, z$*;*

*(iii)* $p_G$ *is continuously differentiable at any* $x \in \partial_2 G$, *for* $s < t, z \in \overline{G} \backslash \overline{(\partial_2 G)}$*;*

*(iv) for any bounded continuous function* $f$ *whose support is contained in* $\overline{G} \backslash \overline{(\partial_2 G)}$ *let*

$$u(s, x) = \int_{\overline{G}} f(z) p_G(s, x; t, z) dz; \qquad \ldots (3.22)$$

*then* $u \in C^{1,2}((0, t) \times G) \cap C_b([0, t] \times \overline{G})$ *and is the unique classical solution to the problem*

$$\left.\begin{array}{lll} \left(\frac{\partial}{\partial s} + L\right) u(s, x) & = & 0, x \in G, s < t \\ J_x u(s, x) & = & 0, x \in \partial_1 G, s < t \\ u(s, x) & = & 0, x \in \partial_2 G, s < t \\ \lim\limits_{s \uparrow t} u(s, x) & = & f(x), x \in \overline{G} \end{array}\right\} \qquad \ldots (3.23)$$

*Moreover such a function* $p_G$ *is unique.*

*Proof.* Let $G$ denote a diffeomorphic image of $B(0 : R) \cap D$ as in Lemma 3.2; let $\gamma \equiv (1, 0, \ldots, 0)$. For $0 \leq s < t, x, z \in \overline{G} \backslash \overline{(\partial_2 G)}$ define

$$v(s, x; t, z) = \int_s^t \int_{\partial_2 G} \left[D_{\nu(\eta)} p(s, x; r, \eta)\right] \mu(r, \eta; t, z) d\sigma(\eta) dr \qquad \ldots (3.24)$$

where $\mu$ is given by (3.18), (3.20). For fixed $t > 0, z \notin \overline{(\partial_2 G)}$, by (3.19) note that $\sup \{\mu(r, \eta; t, z) : s < r < t, \eta \in \partial_2 G\} < \infty$. Hence by (3.21) and Lemma 3.2 it follows that

$$\lim_{x \to x_0, x \in G} v(s, x; t, z) = p(s, x_0; t, z) \qquad \ldots (3.25)$$

for $0 \leq s < t, x_0 \in \partial_2 G$.

It is not difficult to verify that

$$\left(\frac{\partial}{\partial s} + L\right) v(s, x; t, z) = 0, x \in G, s < t \qquad \ldots (3.26)$$

$$J_x v(s, x; t, z) = 0, \ x \in \partial_1 G, s < t \qquad \ldots (3.27)$$

$$\lim_{s \uparrow t} v(s, x; t, z) = 0, x \in \overline{G} \qquad \ldots (3.28)$$

Now define $p_G$ by

$$p_G(s,x;t,z) = p(s,x;t,z) - v(s,x;t,z) \qquad \ldots (3.29)$$

for $0 \le s < t, x \in \overline{G}, z \in \overline{G}\backslash\overline{(\partial_2 G)}$. It is clear that $p_G$ satisfies (i) and (ii); also $u$ defined by (3.22) satisfies (3.23). As the domain $G$ is bounded, by appropriate maximum principle (see Protter and Weinberger (1984)) uniqueness of the solution to (3.23), and consequently uniqueness of $p_G$ follow.

We will now prove (iii); it is enough to show that $v$ defined by (3.24), (3.25) is continuously differentiable at any $x \in \partial_2 G$, for $s < t, z \in \overline{G}\backslash\overline{(\partial_2 G)}$. Set

$$\begin{aligned} v_0(s,x) &= v(s,x;t,z), \quad x \in \partial G\backslash\overline{(\partial_2 G)}, s < t, \\ &= p(s,x;t,z), \quad x \in \partial G \cap \overline{(\partial_2 G)}, s < t \end{aligned} \qquad \ldots (3.30)$$

By (3.25) - (3.28) note that $v$ is the solution for the first boundary value problem for $\left(\frac{\partial}{\partial s} + L\right)$ with boundary data $v_0$. By Theorem 2.10 note that $v_0$ is thrice continously differentiable at any $x \in \partial_2 G$; (such an $x$ will have $x_1 > 0$). Also observe that the proof of smoothness at a boundary point in Theorem 7, Chap. 4 (pp. 127-128) of Friedman (1983) involves only a neighbourhood of that point. Consequently from the proofs of Theorem 7, Chap. 4 and Theorem 7, Chap. 3 of Friedman (1983), it now follows that $v$ is continuously differentiable at any $x \in \partial_2 G$.

Thus the theorem is proved in this case. The general case can be reduced to this case by an appropriate diffeomorphism as in Step 5 of the proof of Theorem 2.1. □

Lemma 3.5. *Assume (A1) - (A7); let $G$ be as in the preceding theorem. Then $p_G(s,x;t,z)$ can be defined for $z \in \overline{G}, 0 \le s < t, x \in \overline{G}\backslash\overline{(\partial_2 G)}$; moreover $(t,z) \mapsto p_G(s,x;t,z)$ is in $C^{1,2}((s,\infty)\times G)$ for fixed $s \ge 0, x \in \overline{G}\backslash\overline{(\partial_2 G)}$, and $p_G$ is continuously differentiable at any $z \in \partial_2 G$ for fixed $s < t, x \in \overline{G}\backslash\overline{(\partial_2 G)}$*

*Proof.* Observe that the formal adjoint of $L$ is given by

$$\begin{aligned} L^* g(t,z) &= \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial^2}{\partial z_i \partial z_j}(a_{ij}(t,z) g(t,z)) \\ &\quad - \sum_{i=1}^{d} \frac{\partial}{\partial z_i}(b_i(t,z) g(t,z)) + c(t,z) g(t,z) \end{aligned} \qquad \ldots (3.31)$$

Set

$$\begin{aligned} b_i^*(t,z) &= b_i(t,z) - \sum_{j=1}^{d} \frac{\partial}{\partial z_j} a_{ij}(t,z), \quad i = 1,2,\ldots,d, \\ a^{(\gamma)}(t,z) &= \frac{< a(t,z)n(z), n(z) >}{2 < n(z), \gamma(t,z) >} = \frac{1}{2} a_{11}(t,z), \end{aligned}$$

where $n(z)$ is the unit inward normal; (we assume without loss of generality that $\gamma_1 \equiv 1$). Put

$$\alpha(t,z) = -\frac{1}{2} a(t,z) n(z) + a^{(\gamma)}(t,z)\gamma(t,z).$$

Clearly $\alpha_1 \equiv 0$; putting $\widehat{\alpha}(t,z) = \sum_{i=2}^{d} \frac{\partial}{\partial z_i}\alpha_i(t,z)$ we see that

$$\int_{\partial D} < \alpha(t,z), \nabla f(z) > d\sigma(z) = -\int_{\partial D} f(z)\widehat{\alpha}(t,z)d\sigma(z)$$

for all smooth functions $f$ with compact support; (remember that by (A7) all the quantities concerned are defined on $\overline{D}$ or $\partial D$). Define

$$\begin{aligned} J^* g(t,z) &= < a(t,z)n(z), \nabla_z g(t,z) > - < a^{(\gamma)}(t,z)\gamma(t,z), \nabla_z g(t,z) > \\ &\quad + \left[a^{(\gamma)}(t,z)\rho(t,z) - < b^*(t,z), n(z) > -\widehat{\alpha}(t,z)\right] g(t,z) \end{aligned} \qquad \ldots(3.32)$$

Note that the coefficients of $(-(\partial/\partial t) + L^*)$ and $J^*$ satisfy the hypotheses of Section 2; hence there exists a fundamental solution $p^*$ for $(-(\partial/\partial t) + L^*, J^*)$. Consequently by Theorem 3.4 a unique Green function $p_G^*$ for $(-(\partial/\partial t) + L^*)$, with reflecting boundary condition on $\partial_1 G$ specified by $J^*$ and Dirichlet boundary condition on $\partial_2 G$, exists; also $(t,z) \mapsto p_G^*(t,z;s,x)$ is in $C^{1,2}([s,\infty)\times G)$ and $p_G^*$ is continuously differentiable at any $z \in \partial_2 G$, for any $s \geq 0, x \in \overline{G}\backslash\overline{(\partial_2 G)}$.

Now by an argument using Green's formula it can be shown (as in Theorem 17 on p.84 of Friedman (1983)) that $p_G^*(t,z;s,x) = p_G(s,x;t,z)$ for $s < t, x, z \in G$. By continuity the same is true for $s < t, x \in \overline{G}\backslash\overline{(\partial_2 G)}, z \in \overline{G}$. This completes the proof. □

Our next result is

Theorem 3.6. *Assume (A1) - (A7);let G be as in Theorem 3.4. Let* $t > 0$ *be fixed; let* $f$ *be a* $C_b^{1,2}$*-function on* $[0,t]\times\partial_2 G$ *such that* $\lim_{r\uparrow t} D_r^\beta D_z^\alpha f(r,z) = 0$ *for* $\beta \leq 1, |\alpha| \leq 2$. *Define*

$$u(s,x) = \int_s^t \int_{\partial_2 G} f(r,z) D_{\nu(z)} p_G(s,x;r,z) d\sigma(z) dr \qquad \ldots(3.33)$$

*where* $D_{\nu(z)}$ *denotes derivative in the conormal direction at* $z \in \partial_2 G$. *Then* $u$ *is the solution for the mixed problem :*

$$\left.\begin{aligned} ((\partial/\partial s) + L)u(s,x) &= 0,\ x \in G, s < t, \\ \lim_{s\uparrow t} u(s,x) &= 0, x \in \overline{G} \\ J_x u(s,x) &= 0, x \in \partial_1 G, s < t \\ \lim_{x\to x_0, x\notin\partial_2 G} u(s,x) &= f(s,x_0),\ x_0 \in \partial_2 G, s < t \end{aligned}\right\} \qquad \ldots(3.34)$$

*Proof.* By the proof of the preceding lemma we have

$$\left.\begin{aligned} (-(\partial/\partial r) + L_z^*)p_G(s,x;r,z) &= 0,\ z \in G, r > s, \\ J_z^* p_G(s,x;r,z) &= 0,\ z \in \partial_1 G, r > s, \\ p_G(s,x;r,z) &= 0,\ z \in \partial_2 G, r > s \end{aligned}\right\} \qquad \ldots(3.35)$$

for $x \in \overline{G}\backslash\overline{(\partial_2 G)}$. Note also that

$$\lim_{r\downarrow s} \int_G f(r,z) p_G(s,x;r,z) dz = f(s,x), \quad x \notin \overline{\partial_2 G}.$$

Therefore (as $\partial G$ is piecewise smooth) by Green's formula we get for $x \in \overline{G}\backslash\overline{(\partial_2 G)}$,

$$\begin{aligned} u(s,x) &= \int_s^t \int_{\partial_2 G} f(r,z) D_{\nu(z)} p_G(s,x;r,z) d\sigma(z) dr \\ &\quad + \int_s^t \int_{\partial_1 G} f(r,z) J_z^* p_G(s,x;r,z) d\sigma(z) dr \\ &= \int_s^t \int_G p_G(s,x;r,z) \left(\frac{\partial}{\partial r} + L_z\right) f(r,z) dz dr \\ &\quad + \int_s^t \int_{\partial_1 G} p_G(s,x;r,z) J_z f(r,z) d\sigma(z) dr + f(s,x) \end{aligned} \qquad \ldots (3.36)$$

It is now clear that $u$ is well defined, bounded and satisfies (3.34). □

## 4. General case

We now consider the general case. Let $D$ be a bounded domain with a $C^4$-boundary. We assume that the coefficients $a, b, c, \gamma, \rho$ satisfy (A1) - (A6). Define the generator $L_0$ and the boundary operator $J_0$ by

$$L_0 f(s,x) = \frac{1}{2} \sum_{i,j=1}^d a_{ij}(s,x) \frac{\partial^2 f(s,x)}{\partial x_i \partial x_j} + \sum_{i=1}^d b_i(s,x) \frac{\partial f(s,x)}{\partial x_i} \qquad \ldots (4.1)$$

for $s \geq 0, x \in D$, and

$$J_0 f(s,x) = \sum_{i=1}^d \gamma_i(s,x) \frac{\partial}{\partial x_i} f(s,x) \qquad \ldots (4.2)$$

for $s \geq 0, x \in \partial D$. Let $\{P_{s,x} : s \geq 0, x \in \overline{D}\}$ denote the reflecting diffusion process corresponding to $L_0, J_0$; we will call it the $(L_0, J_0)$-diffusion. Let $X(t)$ denote the $t$-th coordinate projection on $\Omega := C([0,\infty) : \overline{D})$; let $\mathcal{B}_t := \sigma\{X(r) : 0 \leq r \leq t\}$ be the usual filtration. For any stopping time $\tau$ relative to $\mathcal{B}_t$, let $\mathcal{B}_\tau$ denote the associated $\sigma$-algebra. Let $\{\xi(t) : t \geq 0\}$ denote the boundary local time of the $(L_0, J_0)$-diffusion. (See Friedlin (1985) or Ikeda and Watanabe (1981) for details). $E_{s,x}[g : A]$ shall denote the integral of $g$ w.r.t. the probability measure $P_{s,x}$ over the set $A$. Put

$$\left.\begin{aligned} e(t;s) &= \exp\left(\int_s^t c(r, X(r)) dr\right), \\ \widehat{e}(t;s) &= \exp\left(\int_s^t \rho(r, X(r)) d\xi(r)\right). \end{aligned}\right\} \qquad \ldots (4.3)$$

Define the evolution

$$T_t^s f(x) = E_{s,x}[e(t;s)\widehat{e}(t;s) f(X(t))] \qquad \ldots (4.4)$$

for $0 \leq s < t, x \in \overline{D}$ and bounded measurable function $f$. Observe that the required fundamental solution is just the integral kernel for the evolution $\{T_t^s\}$.

For $z_0 \in \partial D$ there exist $r, \delta > 0$ such that $B := B(z_0 : r) \cap D$ and $B_1 := B(z_0 : r + \delta) \cap D$ are respectively $C^4$ diffeomorphic to $G := \{y : y_1 > 0, |y| < R\}$ and $G_1 := \{y : y_1 > 0, |y| < R+\epsilon\}$ for some $R > 0, \epsilon > 0$. Under this diffeomorphism the diffusion coefficients are transformed to $\hat{a}, \hat{b}, \hat{c}, \hat{\gamma}, \hat{\rho}$, which satisfy (A1) - (A7) on $[0,T] \times \overline{G}$ (or $[0,T] \times \partial_1 G$ as the case may be). Put $\partial_1 B = \partial B \cap \partial D$ and $\partial_2 B = \partial B \backslash \partial D$; note that $\partial_i B$ is diffeomorphic to $\partial_i G, i = 1, 2$. Let $\hat{p}_G$ be the Green function (corresponding to $\hat{a}, \hat{b}, \hat{c}, \hat{\gamma}, \hat{\rho}$), guaranteed by Theorem 3.4. Define

$$p_B(s, x; t, z) = \hat{p}_G(s, \hat{x}; t, \hat{z}) \qquad \ldots (4.5)$$

for $s < t, x \in \overline{B}, z \in \overline{B}\backslash\overline{(\partial_2 B)}$ or $x \in \overline{B}\backslash\overline{(\partial_2 B)}, z \in \overline{B}$, where $\hat{x}, \hat{z}$ are the images of $x, z$ under the above diffeomorphism. Then $p_B$ has the appropriate smoothness properties; that is, analogues of Theorem 3.4, Lemma 3.5 and Theorem 3.6 hold for $p_B$. It is the Green function for the problem

$$\left.\begin{array}{lcl} ((\partial/\partial s) + L)p_B(s, x; t, z) & = & \delta(t-s)\delta(z-x), \\ J_x p_B(s, x; t, z) & = & 0, x \in \partial_1 B, \\ p_B(s, x; t, z) & = & 0, x \in \partial_2 B \end{array}\right\} \qquad \ldots (4.6)$$

for any fixed $t > 0, z \in \overline{B}\backslash\overline{(\partial_2 B)}$. Also $D_{\nu(z)} p_B(s, x; t, z), x \in \overline{B}\backslash(\partial_2 B), z \in \partial_2 B, s < t$ is the Poisson kernel for the mixed problem for $((\partial/\partial s) + L)$ in $B$ with Dirichlet boundary condition on $\partial_2 B$ and reflecting boundary condition (given by $J$) on $\partial_1 B$.

For $z_0 \in D$, there is $r > 0$ such that $\overline{B} := \overline{B(z_0 : r)} \subset D$. In such a case, let $p_B$ denote the Green function for $((\partial/\partial s) + L)$ in $B$ with Dirichlet boundary condition on $\partial B (= \partial_2 B$; in this case $\partial_1 B = \phi)$; $p_B$ has the required smoothness properties and the conormal derivative of $p_B$ is the Poisson kernel for $((\partial/\partial s) + L)$ in $B$ with Dirichlet boundary condition on $\partial B$.

In view of the above the following can be proved using stochastic calculus.

Lemma 4.1. *Let (A1) - (A6) hold. Let $B, p_B$ be as above. Let $\tau$ denote the exit time from $B$ for the $(L_0, J_0)$-diffusion; that is, $\tau$ is the time of hitting $\partial_2 B$.*

*(i) Let $f$ be a bounded continuous function whose support is contained in $\overline{B}\backslash\overline{(\partial_2 B)}$. Then for $s < t, x \in \overline{B}$,*

$$E_{s,x}[e(t; s)\hat{e}(t; s) f(X(t)) : \{\tau > t\}] = \int_{\overline{B}} f(z) p_B(s, x; t, z) dz \qquad \ldots (4.7)$$

*(ii) Let $g$ be a smooth function on $[0, t] \times \partial_2 B$ such that $\lim_{r \uparrow t} D_r^\beta D_z^\alpha g(r, z) = 0, \beta \leq 1, |\alpha| \leq 2$. Then*

$$E_{s,x}[e(\tau; s)\hat{e}(\tau; s) g(\tau, X(\tau)) : \{\tau < t\}] = \int_s^t \int_{\partial_2 B} g(r, z) D_{\nu(z)} p_B(s, x; r, z) d\sigma(z) dr \qquad \ldots (4.8)$$

*Remark* 4.2. If $c \equiv 0, \rho \equiv 0$ then

$$D_{\nu(z)} p_B(s,x;t,z) d\sigma(z) dt, \quad t > s, z \in \partial_2 B$$

gives the exit distribution $P_{s,x}(\tau, X(\tau))^{-1}$, for fixed $s \geq 0, x \in \overline{B} \backslash \overline{(\partial_2 B)}$. (As $\partial D \cap \overline{(\partial_2 B)}$ is a $(d-2)$ dimensional submanifold note that the probability of hitting $\partial D \cap \overline{(\partial_2 B)}$ is zero; see Ramasubramanian (1988)). See Hsu (1986) for an analogue of the above result when $L = \frac{1}{2}\Delta, \partial_1 B = \phi$; that is, reflecting boundary condition is absent. □

We will now use a probabilistic method for obtaining the integral kernel for $\{T_t^s\}$ from Green functions described above.

Let $0 \leq s < t, z \in \overline{D}$ be fixed. We can find $0 < r_1 < r_2$ such that $p_A, p_B$ (described as above) exist where $A = B(z : r_1) \cap D, B = B(z : r_2) \cap D$. Set

$$\begin{aligned} \tau_{2j+1} &= \inf\{r \geq \tau_{2j} : X(r) \in \overline{\partial_2 A}\}, \quad j = 0, 1, 2, \ldots \\ \tau_{2j} &= \inf\{r \geq \tau_{2j-1} : X(r) \in \overline{\partial_2 B}\}, j = 1, 2, \ldots \end{aligned}$$

As $\overline{\partial_2 A} \cap \overline{\partial_2 B} = \phi$ note that $\tau_j \uparrow \infty$ a.s. $P_{s,x}$ for any $x \in \overline{D}$. Define for $0 \leq s < t, x \in \overline{D}$,

$$\begin{aligned} p(s,x;t,z) &= I_{\overline{A}}(x) p_A(s,x;t,z) \\ &\quad + \sum_{j=1}^{\infty} E_{s,x}[e(\tau_{2j-1}; s)\hat{e}(\tau_{2j-1}; s) p_B(\tau_{2j-1}, X(\tau_{2j-1}); t, z) : H_j] \end{aligned} \quad \ldots (4.9)$$

where $H_j = \{\tau_{2j-1} < t < \tau_{2j}\}$.

Lemma 4.3. *For any bounded measurable function $f$ on $\overline{D}$,*

$$T_t^s f(x) = \int_{\overline{D}} f(z) p(s,x;t,z) dz \quad \ldots (4.10)$$

*Proof.* It is enough to prove (4.10) for $f = I_V$ where $V$ is a Borel set such that $V \subset A \subset \overline{A} \subset B$ with $A, B$ being as above; this will also show that $p$ defined by (4.9) is independent of $A, B$.

By Lemma 4.1 and the strong Markov property of the $(L_0, J_0)$-diffusion note that

$$\begin{aligned} T_t^s I_V(x) &= E_{s,x}[e(t;s)\hat{e}(t;s) I_V(X(t)) : \{\tau_1 > t\}] \\ &\quad + \sum_{j=1}^{\infty} E_{s,x}[e(t;s)\hat{e}(t;s) I_V(X(t)) : \{\tau_{2j-1} < t < \tau_{2j}\}] \\ &= \int_{\overline{D}} I_V(z) I_{\overline{A}}(x) p_A(s,x;t,z) dz \\ &\quad + \sum_{j=1}^{\infty} E_{s,x}\left[e_j \hat{e}_j \int_B I_V(z) p_B(\tau_{2j-1}, X(\tau_{2j-1}), t, z) dz : H_j^*\right] \\ &= \int_{\overline{D}} I_V(z) p(s,x;t,z) dz, \end{aligned}$$

where $e_j = e(\tau_{2j-1}; s), \hat{e}_j = \hat{e}(\tau_{2j-1}; s), H_j^* = \{\tau_{2j-1} < t\}$. □

Here is our main result.

Theorem 4.4. *Let $D$ be a bounded domain with $C^4$-boundary; let $a, b, c, \gamma, \rho$ satisfy (A1) - (A6). Define $\{T_t^s\}$ by (4.4). Then there exists a strictly positive function $p(s, x; t, z), 0 \leq s < t, x, z \in \overline{D}$ such that*

(i) $T_t^s f(x) = \int_{\overline{D}} f(z) p(s, x; t, z) dz, \quad s < t, x \in \overline{D}$;
(ii) $(s, x) \mapsto p(s, x; t, z)$ *is in* $C^{1,2}((0, t) \times D) \cap C([0, t) \times \overline{D})$;
(iii) $J_x p(s, x; t, z) = 0, \ x \in \partial D, s < t$;
(iv) $lim_{s \uparrow t} T_t^s f(x) = f(x), \quad x \in \overline{D}$.

*Proof.* Let $p$ be defined by (4.9); the first assertion is proved in the preceding lemma.

Let $s < t, x, z \in \overline{D}$ be fixed. Choose a bounded neighbourhood $E$ (in $\overline{D}$) of $x$ such that $z \notin \overline{(\partial_2 E)}$ and $p_E$ exists. We claim that $(r, \eta) \mapsto p(r, \eta; t, z)$ is a bounded measurable function on $[s, t] \times \partial_2 E$. From (3.25) - (3.28), Theorem 3.6 and Lemma 4.1 (ii) it follows that $v$ in the proof of Theorem 3.4 is nonnegative. Consequently by (3.29) it follows that the Gaussian type bound (2.4) holds for $p_G$; hence (4.5), (4.9) now yield the claim; (recall that the symbol $p$ is used in different contexts in (3.29) and (4.9); as it is unlikely to cause confusion, we persist with this !)

By Lemma 4.1 (i), (ii) and strong Markov property it follows that

$$\begin{aligned} & p(s, x; t, z) \\ = \ & p_E(s, x; t, z) I_E(z) + E_{s,x}[e(\tau_E; s)\hat{e}(\tau_E; s) p(\tau_E, X(\tau_E); t, z) : \{\tau_E < t\}] \\ = \ & p_E(s, x; t, z) I_E(z) + \int_s^t \int_{\partial_2 E} p(r, \eta; t, z) D_{\nu(\eta)} p_E(s, x; r, \eta) d\sigma(\eta) dr \end{aligned} \quad \ldots (4.11)$$

As $x, z \notin \overline{(\partial_2 E)}$ required smoothness in $(s, x)$ follows by (4.11). Assertion (iii) also follows similarly; (iv) is clear from (4.11) and the first equation in (4.6). Strict positivity of $p$ can be proved using the maximum principles on pp. 173-174 of Protter and Weinberger (1984). This completes the proof. □

*Remark* 4.5. For a bounded domain $D$ it can be shown that the fundamental solution given in the preceding theorem is unique. This can be done as in Theorem 8.1 of Ito (1992) with appropriate modifications; or using uniqueness of solution to initial - boundary value problem for $(\partial/\partial s) + L$ with boundary condition given by $J$ (which in turn can be proved by stochastic representation of a solution or using maximum principles); we omit the details. □

## 5. MINIMALITY

We conclude with a result concerning minimality of fundamental solution in the half space.

Theorem 5.1. *Let $D$ denote the half space; assume (A1) - (A6). Let $p$ be as in Theorem 2.10. Then $p$ is the minimal fundamental solution in the sense that, if $q(s,x;t,z)$ is a continuous nonnegative fundamental solution for $((\partial/\partial s)+L,J)$ then*

$$p(s,x;t,z) \le q(s,x;t,z), \quad \forall 0 \le s < t, x, z \in \overline{D} \qquad \ldots(5.1)$$

*Proof.* For $n = 1,2,\ldots$ let $B_n = D \cap B(0:n)$ and $p_n(s,x;t,z) = p_{B_n}(s,x;t,z)$. By (3.25) - (3.29), Step 5 of the proof of Theorem 2.1, Theorem 3.6 and Lemma 4.1 (ii) we have

$$p_n(s,x;t,z) = p(s,x;t,z) - E_{s,x}[e(\tau_n;s)\hat{e}(\tau_n;s)p(\tau_n,X(\tau_n);t,z):\{\tau_n < t\}] \qquad \ldots(5.2)$$

where $\tau_n$ is the time of hitting $\partial_2 B_n$ (for the $(L_0,J_0)$-diffusion) and $E_{s,x}$ denotes expectation w.r.t. the $(L_0,J_0)$-diffusion in $\overline{D}$ starting at $(s,x)$. See Port and Stone (1978) for a similar expression in the context of Brownian motion with absorbing boundary.

As the coefficients are bounded note that $(L_0,J_0)$-diffusion is conservative; (that is, the measures $P_{s,x}$ are supported on $C([0,\infty):\overline{D})$). Hence $\tau_n \to \infty$ a.s. $P_{s,x}$. For any $k$ note that

$$\sup\{|I_{(\tau_n<t\}}p(\tau_n,X(\tau_n);t,z)| : |z| \le k, n \ge k+1\} < \infty$$

Also exp $(\alpha\xi(t))$ is integrable for any constant $\alpha \ge 0$ and $t > 0$. Therefore by (5.2) it follows that

$$p_n(s,x;t,z) \to p(s,x;t,z), \quad n \to \infty \qquad \ldots(5.3)$$

uniformly over $(s,(x,z))$ varying on compact sets.

Let $q$ be another continuous nonnegative fundamental solution. Fix $n$ and let $f$ be a nonnegative continuous function with compact support in $B_n$. Put

$$v_n(s,x) = \int_{\overline{B_n}} f(z)q(s,x;t,z)dz \qquad \ldots(5.4)$$

Let $g(s,x) = v_n(s,x), 0 \le s < t, x \in \partial_2 B_n$; observe that $g$ is a nonnegative function on $[0,t) \times \partial_2 B_n$. Clearly, $v_n$ solves the problem

$$\left.\begin{array}{lcl} ((\partial/\partial s)+L)v_n(s,x) & = & 0, x \in B_n, s < t, \\ J_x v_n(s,x) & = & 0, x \in \partial_1 B_n, s < t, \\ v_n(s,x) & = & g(s,x),\ x \in \partial_2 B_n, s < t, \\ \lim_{s\uparrow t} v_n(s,x) & = & f(x),\ x \in \overline{B_n} \end{array}\right\} \qquad \ldots(5.5)$$

As $p_n$ is the unique nonnegative fundamental solution in the bounded domain $B_n$, by Theorems 3.4 and 3.6 it can be shown that

$$v_n(s,x) = \int_{B_n} f(z)p_n(s,x;t,z)dz + \int_s^t \int_{\partial_2 B_n} g(r,z)D_{\nu(z)}p_n(s,x;r,z)d\sigma(z)dr \quad \ldots (5.6)$$

By maximum principle, $D_{\nu(z)}p_n \geq 0$ on $\partial_2 B_n$; hence the second term on the r.h.s. of (5.6) is nonnegative. (One can use Lemma 4.1 (ii) for another derivation of this fact.) And as $f$ is arbitrary, by (5.4) and (5.6) it now follows that

$$p_n(s,x;t,z) \leq q(s,x;t,z) \quad \ldots (5.7)$$

for $0 \leq s < t, x, z \in \overline{B}_n$. From (5.3), (5.7), the required conclusion (5.1) now follows. □

*Remark.* For unbounded domains with sufficiently smooth boundary, our analysis of Sections 4 and 5 can be carried through. □

*Remark.* It would be interesting to prove the results of Sections 4 and 5 just under the hypotheses (A1) - (A5) as in Theorem 2.1. The additional hypotheses are needed for considering the adjoint problem, and to ensure that $p$ has the necessary smoothness so that Green function and Poisson kernel can be defined in Section 3. It may be noted that such a situation arises even in the case of the conormal reflection; see Ito (1957), especially the last paragraph of §3 on p. 66. □

*Acknowledgement.* The author wishes to thank the referee for encouraging comments.

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INDIAN STATISTICAL INSTITUTE
BANGALORE CENTER
8TH MILE, MYSORE ROAD
BANGALORE - 560059
INDIA