THE PROBLEM OF TESTING LINEAR HYPOTHESIS ABOUT POPULATION MEANS WHEN THE POPULATION VARIANCES ARE NOT EQUAL AND M-TEST

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SUMMARY. Given k independent samples of n_i units from k populations $N(m_i, \sigma_i^2)$ $(i = 1, 2, \ldots, k)$ a test statistic for testing a hypothesis H_0 about s $(s \leq k)$ linear functions of k population means without any *a priori* knowledge of population variances or the ratio of the variances is of interest. A new test statistic called M statistic is defined for testing such hypothesis where any prior knowledge about the population variances is not available. The error of the first kind (probability of rejection of the hypothesis when true) of the test statistic depends on the unknown population variances but the test statistic is so defined that for all possible values of population variances the error of the first kind is less than or equal to some pre-assigned probability α. It is shown that critical values of the test statistic for testing a hypothesis about two linear functions of k population means with $\alpha = 0.05, 0.02, 0.01$, etc., can all be obtained from tabulated values of F-table. A numerical example for testing equality of three population means has been considered. It is also shown that the test statistic can be used in multivariate problems as well. An analysis of Barnard's data (Barnard, 1935) has been considered.

1. Introduction

1.1. Given k samples of n_i units from k normal populations $N_i(m_i, \sigma_i^2)$ $(i = 1, 2, \ldots, k)$ having equal variances or the ratio of the variances known *a priori* any hypothesis about any linear function $\sum_1^k c_i m_i$ of population means (where c_i $(i = 1, 2, \ldots, k)$ are known coefficients) can be tested by the t-statistic. Also, any hypothesis about more than one linear function of population means can be tested by F-statistic or F-ratio. If the population variances are not equal or the ratio of the variances are not known *a priori* it is possible to test (Banerjee, 1961) any hypothesis about any single linear function of population means. Also, any hypothesis about more than one linear function of population means can be tested by a new statistic hereinafter called M-statistic or M-ratio.

2. Samples from heteroscedastic populations

2.1. Let $\bar{x}_i$, s_i^2 $(i = 1, 2, \ldots, k)$ be sample estimates of population means and variances of k samples of n_i-units drawn drom k normal population $N_i(m_i, \sigma_i^2)$ $(i = 1, 2, \ldots, k)$. Suppose it is required to test the hypothesis that

$$H_0 \begin{cases} c_{11}m_1 + c_{12}m_2 + \ldots + c_{1k}m_k = M_1 & \ldots \quad (2.1.1) \\ c_{21}m_1 + c_{22}m_2 + \ldots + c_{2k}m_k = M_2 & \ldots \quad (2.1.2) \\ \vdots & \\ c_{s1}m_1 + c_{s2}m_2 + \ldots + c_{sk}m_k = M_s & \ldots \quad (2.1.s) \end{cases}$$

where c_{ij} ($i = 1, 2, ..., s$; $j = 1, 2, ..., k$) and M_j ($j = 1, 2, ..., s$) are known constants. It is assumed without any loss of generality that the relations (2.1.1), (2.1.2), ... (2.1.3) are mutually consistent and independent. It is also assumed that $s < k$ for if $s = k$ the relations (2.1.1), (2.1.2), ... (2.1.3) can be replaced by

$$m_i = M'_i \quad (i = 1, 2, ..., k)$$

and H_0 can be tested by the statistic

$$T = \sum_1^k \left[\frac{\bar{x}_i - M'_i}{s_i\sqrt{n}} \right]^2 = \sum_1^k t_i^2$$

whose percentage points, although not tabulated, can be evaluated as each t_i ($i = 1, 2, ..., k$) would be independently distributed as a Student's t-variate if the hypothesis be true.

2.2. Let test variates $U_1, U_2, ..., U_s$ be defined as

$$U_i = \sum_{j=1} c_{ij}\bar{x}_j. \quad (i = 1, 2, ..., s). \qquad ... \quad (2.2.1)$$

The test variates $U_1, U_2, ..., U_s$ as defined in (2.2.1) are stochastic variates jointly distributed in a multivariate normal form.

2.3. Now let us consider the probability of the inequality

$$\sum_{i=1}^s (U_i - M_i)^2 \geqslant \sum_{j=1}^k A_j C_j \frac{s_j^2}{n_j}$$

where $C_1, C_2, ..., C_k$ are defined as

$$C_j = \sum_{i=1}^s c_{ij}^2; \quad (j = 1, 2, ..., k)$$

and A_j ($j = 1, 2, ..., k$) are positive constants to be suitably determined in a manner as discussed later.

2.4. Let $M'_1, M'_2, ..., M'_s$ be respectively means of test variates $U_1, U_2, ..., U_s$ whereas by hypothesis H_0 the means are $M_1, M_2, ..., M_s$. Let variates u_i ($i = 1, 2, ..., s$) be defined to

$$u_i = U_i - M'_i; \quad (i = 1, 2, ..., s) \qquad ... \quad (2.4.1)$$

u_i ($i = 1, 2, ..., s$) as defined in (2.4.1) follow a multivariate normal distribution with zero mean with, say, disperson matrix R. Now consider a further transformation (Ferrar, 1053) to variates v_i ($i = 1, 2, ..., s$) so that

$$\left.\begin{aligned} \sum_1^s u_i^2 &= \sum_1^s v_i^2 \\ \text{and} \quad uR^{-1}u' &= \lambda_1 v_1^2 + \lambda_2 v_2^2 + ... + \lambda_s v_s^2 \end{aligned}\right\} \qquad ... \quad (2.4.2)$$

where u is a row matrix $(u_1, u_2, ..., u_s)$,

u' is a transpose u

and R^{-1} is a $s \times s$ matrix reciprocal to the dispersion matrix R.

The transformed variates v_i ($i = 1, 2, ..., s$) are independently normally distributed with zero mean and variance, say, σ_{vi}^2 ($i = 1, 2, ..., s$).

2.5. Now by virtue of (2.4.1) and (2.4.2)

$$\sum_1^s (U_i-M_i)^2 = \sum_1^s (u_i-M_i+M_i')^2 = \sum_1^s (u_i-d_i)^2 = \sum_1^s (v_i-d_i')^2 \quad \ldots \quad (2.5.1)$$

$\left(\text{where } \sum_1^s d_i^2 = \sum_1^s d_i'^2\right)$.

Also, by virtue of (2.4.1)

$$\sum_1^s V(U_i) = \sum_1^s V(u_i) = \sum_1^s E(u_i^2) = \sum_1^s E(v_i^2) = \sum_1^s \sigma^2_{v_i}. \quad \ldots \quad (2.5.2)$$

2.6. From (2.5.1) and (2.5.2) the probability of the inequality

$$\sum_1^s (U_i-M_i)^2 > \sum_1^k A_j C_j \frac{s_j^2}{n_j}$$

is equal to

$$\frac{\sum_1^s (v_i-d_i')^2}{\sum_1^s \sigma^2_{v_i}} \left\{= \frac{\sum_{i=1}^s (U_i-M_i)^2}{\sum_{i=1}^s V(U_i)}\right\} > \frac{\sum_{j=1}^k A_j C_j \frac{s_j^2}{n_j}}{\sum_{i=1}^s V(U_i)}$$

which is equal to

$$\sum_{i=1}^s \beta_i \chi_{1i}'^2 > \sum_{j=1}^k A_j \omega_j \frac{\chi_j^2}{\nu_j}$$

where

$\chi_{1i}'^2$ are non-central χ^2-variates with 1 d.f. $(i = 1, 2, \ldots, s)$

χ_j^2 are χ^2—variates with ν_j d.f. $(\nu_j = n_j-1)$, $(j = 1, 2, \ldots, k)$

β_j and ω_j are positive weights defined as .

$$\beta_i = \frac{\sigma^2_{v_i}}{\sum_1^s \sigma^2_{v_i}} \; ; \; \omega_j = \frac{C_j^2\sigma_j^2/n_j}{\sum_1^k C_j^2\sigma_j^2/n_j}. \quad (i = 1, 2, \ldots, s; \; j = 1, 2, \ldots, k)$$

If the hypothesis H_0 is true $\chi_{1i}'^2 (i = 1, 2, \ldots, s)$ are, however, distributed as χ^2-variates with 1 d.f.

2.7. The crux of the problem of having a test statistic for testing hypothesis H_0 based on test variates U_i (i = 1, 2, ..., s) therefore boils down to finding positive constants A_j $(j = 1, 2, \ldots, k)$ so that

$$\text{prob}\left[\sum_{i=1}^s \beta_i \chi_{1i}^2 > \sum_{j=1}^k A_j \omega_j \frac{\chi_j^2}{\nu_j}\right] < \alpha \quad \ldots \quad (2.7.1)$$

where χ_{1i}^2 $(i = 1, 2, \ldots, s)$ and χ_j^2 $(j = 1, 2, \ldots, k)$ are all independently distributed χ^2-variates with respectively 1 and ν_j $(j = 1, 2, \ldots, k)$ d.f. and β_i and ω_j are positive weights adding up to unity. First, it has, however, to be proved that it is *at all possible* to find finite positive constants A_j $(j = 1, 2, \ldots, k)$ so that given some pre-assigned α (2.8.1) would be satisfied.

2.8. Theorem : *Let U_i $(i = 1, 2, \ldots, s)$ be a set of stochastic variates (not necessarily independently distributed) which satisfy the relation that*

$$prob\,[U_i < 0] < \alpha_i \qquad (i = 1, 2, \ldots s). \quad \ldots \quad (2.8.1)$$

Now if β_i $(i = 1, 2, ..., s)$ *be a set of arbitrary positive weights adding up to unity* (i.e. $\sum_1^s \beta_i = 1$), *then*

$$prob\ [\sum_1^s \beta_i U_i < 0] \leqslant \sum_1^s \alpha\ . \qquad ... \quad (2.8.1)$$

Proof: First, let us consider the case of only two variates $-U_1$ and U_2. Now if β_1 and β_2 be two positive weights adding up to unity

$$\text{prob}\ [\beta_1 u_1 + \beta_2 u_2 \leqslant 0] < \text{prob}\ [U_1 < 0] + \text{prob}\ [U_2 < 0] < \alpha_1 + \alpha_2.$$

Also, similarly it can be proved that

$$\text{prob}\ [\sum_{i=1}^s \beta_i U_i < 0] \leqslant \sum_{i=1}^s \alpha_i. \qquad ... \quad (2.8.2)$$

2.9. Now let $U_i\,(i = 1, 2, ..., s)$ be defined as

$$\sum_{j=1}^s A_j \omega_j \frac{\chi_j^2}{\nu_j} - \chi_{1i}^2 \qquad (i = 1, 2, ..., s) \qquad ... \quad (2.9.1)$$

where χ_{1i}^2 $(i, 1. 2. ..., s)$ and χ_j^2 $(j = 1, 2, ..., k)$ are all independently distributed χ^2-variates with respectively 1 and ν_j $(j = 1, 2, ..., k)$ d.f. and $A_j\,(j = 1, 2, ..., k)$ are 100. α/s percentile point of Student's t-table of d.f. $\nu_j\,(j = 1, 2, ..., k)$ so that (Banerjee, 1960)

$$\text{prob}\ [U_i \leqslant 0] < \alpha/s. \qquad ... \quad (2.9.2)$$

From (2.8.1) and (2.8.2) it follows

$$\text{prob}\ [\sum_1^s \beta_i \chi_{1i}^2 \geqslant \sum_1^s A_j \omega_j \chi_j^2/\nu_j] \leqslant \alpha. \qquad ... \quad (2.9.3)$$

3. Statement of the statistic

3.1. Let $M_{s,k}$-statistic (M after Mahalanobis) for testing hypothesis about s linear functions of population means without any *a priori* knowledge of population variances of size α (or with maximum value of error of the first kind α) be defined as

$$\frac{\sum_{i=1}^s \beta_i \chi_{1i}^2}{\sum_{j=1}^k A_j \omega_j \frac{\chi_j^2}{\nu_j}}.$$

where χ_{1i}^2 $(i = 1, 2, ..., s)$ and χ_j^2 $(j = 1, 2, ..., k)$ are independently distributed χ^2-variates with respectively 1 and $\nu_j\,(j = 1, 2, ..., k)$ d.f. and β_i and ω_j $(i = 1, 2, ..., s;\ j = 1, 2, ..., k)$ are a set of positive weights adding up to unity and A_s $(j = 1, 2, ..., k)$ are irreducible positive constants which have been so determined so that

$$\text{prob}\ \left[\ [\sum_1^s \beta_i \chi_{1i}^2 \geqslant \sum_1^k A_j \omega_j \frac{\chi_j^2}{\nu_j}\right]$$

is less than or equal to α for all possible values of β_i and ω_j $(i = 1, 2, ..., s; j=1, 2, ..., k)$.

4. Critical values of M-statistic

4.1. Let us consider the case of finding critical values of M-statistic for the case $s = 2$ and any k. The problem of finding critical values of $M_{2,k}$ amounts to finding minimum possible numerical values of A_j ($j = 1, 2, \ldots, k$) so that

$$\text{prob}\left[\sum_1^s \beta_i \chi_{1i}^2 > \sum_1^k A_j \omega_j \frac{\chi_j^2}{\nu_j}\right] < \alpha. \qquad \ldots \quad (4.1.1)$$

If P denotes the probability of the inequality

$$\sum_1^2 \beta_i \chi_{1i}^2 > \sum_1^k A_j \omega_j \frac{\chi_j^2}{\nu_j} \qquad \ldots \quad (4.1.2)$$

we have

$$1-P = \int_0^\infty \int_0^\infty \cdots \int_0^\infty \prod_{i=1}^k f(\chi_j^2)\left[\int_0^{T/\beta_1} h(\chi_{11}^2)\left\{\int_0^{(T-\beta_1\chi_{11}^2)/\beta_2} h(\chi_{12}^2) d\chi_{12}^2\right\} d\chi_{11}^2\right] d\chi_1^2 \chi_2^2 \ldots d\chi_k^2 \ldots \quad (4.1.3)$$

where $h(\chi_{1i}^2)$ ($i = 1, 2$) denotes frequency function of a χ^2-variate with 1d.f. ($i = 1, 2$) $f(\chi_j^2)$ denotes frequency functions of χ^2-variate with ν_j d.f. ($j = 1, 2, \ldots, k$) and $T = \sum_{j=1}^k A_j \omega_j \frac{\chi_j^2}{\nu_j}$.

4.2. The integral $\int_0^{z/\beta_1} h(\chi_{11}^2)\left\{\int_0^{z(-\beta_1\chi_{11}^2)/\beta_2} h(\chi_{12}^2) d\chi_{12}^2\right\} d\chi_{11}^2$ is an upward convex function of z (Courant, 1957) (details in Appendix A.1) so that

$$\int_0^{\sum^k \omega_i z_i/\beta_1} h(\chi_{11}^2)\left\{\int_0^{(\sum_1^k \omega_i z_i-\beta_1\chi_{11}^2)/\beta_2} h(\chi_{12}^2) d\chi_{12}^2\right\} d\chi_{11}^2$$

$$\geqslant \sum_1^k \omega_i \int_0^{z_i/\beta_1} h(\chi_{11}^2)\left\{\int_0^{(z_i-\beta_1\chi_{11}^2)/\beta_2} h(\chi_{12}^2) d\chi_{12}^2\right\} d\chi_{11}^2. \qquad \ldots \quad (4.2.1)$$

From (4.1.1), (4.1.2) and (4.1.3) it follows

$$P < \sum_{j=1}^k \omega_j P_j \qquad \ldots \quad (4.2.2)$$

where
$$P_j = \int_0^\infty f(\chi_j^2)\left[\int_{T_1/\beta_1}^\infty h(\chi_{11}^2)\left\{\int_{(T_1-\beta_1\chi_{11}^2)/\beta_2}^\infty h(\chi_{12}^2) d\chi_{1\cdot}^2\right\} d\chi_{11}^2\right] d\chi_j^2$$

$$= \int_0^\infty \int_0^\infty h(\chi_{11}^2) h(\chi_{12}^2)\left\{\int_0^{T_2} f(\chi_j^2) d\chi_j^2\right\} d\chi_{11}^2 d\chi_{12}^2 \qquad \ldots \quad (4.2.3)$$

where
$$T_1 = \frac{A_j \chi_j^2}{\nu_j}$$

and
$$T_2 = \frac{\beta_1\chi_{11}^2 + \beta_2\chi_{12}^2}{\frac{A_j}{\nu_j}}.$$

4.3. Now, for degrees of freedom of χ_j^2 equal to 1 or 2, the integral

$$\int_0^\infty \int_0^\infty h(\chi_{i1}^2)h(\chi_{i2}^2)\left\{\int_0^{T_3} f(\chi^2)d\chi_j^2\right\}d\chi_{i1}^2 d\chi_{i2}^2 \quad \ldots \quad (4.3.1)$$

where
$$T_3 = \frac{\beta_1\chi_{i1}^2+\beta_2\chi_{i2}^2}{F_{2,\nu_j,\alpha}/\nu_j}$$

for variation in β_1 and β_2 is always less than or equal to α, where $F_{2,\nu_j,\alpha}$ is tabulated F-value of F-table corresponding to $100\,\alpha$ percentage point and d.f. of greater mean square 2 and d.f. of smaller mean square ($\nu_j = 1,2$). (Details in Appendix A.2).

4.4. Also, for the case $\nu_j \geqslant 3$ and $\alpha = 0.05, 0.02, 0.01$, etc., the integral

$$\int_0^\infty \int_0^\infty h(\chi_{i1}^2)\, h(\chi_{i2}^2)\{\int_0^{T_4} f(\chi_j^2)d\chi_j^2\}\, d\chi_{i1}^2 . d\chi_{i2}^2 \quad \ldots \quad (4.4.1)$$

where
$$T_4 = \frac{\beta_1\chi_{i1}^2+\beta_2\chi_{i2}^2}{F_{1,\nu_j,\alpha}/\nu_j}$$

for variation in β_1 and β_2 is always less than or equal to α, where $F_{1,\nu_j,\alpha}$ is tabulated F-value of F-table corresponding to 100α percentage point and d.f. of greater mean square 1 and d.f. of smaller mean square ($\nu_j \geqslant 3$). (Details in Appendix A.2).

4.5. Numerical values of A_j of $M_{2,k}$ test can thus be determined from tabulated values F-table. Table 1 below gives numerical values of A_j of $M_{2,k}$ test of size 0.05 and d.f. $\nu_j = 1, 2, \ldots, 20$. The values have been taken from F-table.

TABLE 1. NUMERICAL VALUES OF A_j OF M_{2k} TEST OF SIZE 0.05

ν_j	A_j	ν_j	A_j
1	200.00	11	4.84
2	19.00	12	4.75
3	10.13	13	4.67
4	7.71	14	4.60
5	6.61	15	4.54
6	5.99	16	4.49
7	5.59	16	4.45
8	5.32	18	4.41
9	5.12	19	4.38
10	4.96	20	4.35

5. Testing equality of population means

5.1. Given k samples from k normal populations $N_i(m_i, \sigma_i^2)$ to test the equality of population means $k-1$ mutually independent linear functions L_i $(i = 1, 2, \ldots, k-1)$ of population means and associated test variates may be defined as

$$L_i = \sum_{j=1}^{k} c_{ij} m_j \; ; \qquad U_i = \sum_{j=1}^{k} c_{ij} \, . \, x_j \; ; \qquad \ldots \quad (5.1.1)$$

$$(i = 1, 2, \ldots, k-1).$$

where $\sum_{j=1}^{k} c_{ij} = 0$. If s_i^2 denotes estimate of population variance of the i-th population $(i = 1, 2, \ldots, k)$ $M_{k-1,k}$-statistic may be computed as

$$\frac{\sum_{i=1}^{k-1} U_i^2}{\sum_{j=1}^{k} A_j C_j \frac{s_j^2}{n_j}} \qquad \ldots \quad (5.1.2)$$

(where $C_j = \sum_{i=1}^{k} c_{ij}^2$; $j = 1, 2, \ldots, k$) with suitable choice of A_j $(j = 1, 2, \ldots, k)$ and the hypothesis would be rejected if the numerical value of $M_{k-1,k}$ as defined in (5.1.2) exceeded unity.

6. Numerical example

6.1. Three samples from three populations supply the following estimates.

TABLE 2

	population		
	I	II	II
sample mean $\bar{x}_i$	5.0	20.0	10.0
sample variance s_i^2	18.0	5.5	20.0
sample size n_i	3	11	21

Defining test variates U_1, and U_2 as

$$U_1 = \frac{1}{\sqrt{2}}(\bar{x}_1 - \bar{x}_2) = \frac{1}{\sqrt{2}}(5-20)$$

$$U_2 = \frac{1}{\sqrt{6}}(\bar{x}_1 + \bar{x}_2 - 2\bar{x}_3) = \frac{1}{\sqrt{6}}(25-20).$$

$M_{2,3}$-statistic of size .05 may be computed as

$$M_{2\cdot 3} = \frac{U_1^2+U_2^2}{\frac{2}{3}\left[\frac{A_1s_1^2}{n_1}+\frac{A_2s_2^2}{n_2}+\frac{A_3s_3^2}{n_3}\right]}$$

$$= \frac{\frac{1}{2}[5-20]^2+\frac{1}{6}[25-20]^2}{\frac{2}{3}\left[10.00\times\frac{18}{3}+4.00\times\frac{5.5}{11}+4.35\times\frac{20}{21}\right]}$$

$$= \frac{\frac{225}{2}+\frac{25}{6}}{\frac{2}{3}[114.00+2.48+4.14]}$$

$$= \frac{116.67}{80.41} = 1.45$$

where numerical values of $A_j (j = 1, 2, 3)$ have been taken from Table 1 above. Since $M_{2\cdot 3}$ is greater than unity any hypothesis about equality of means is rejected.

7. The case of multivariate population

7.1. Let k samples of N_i $(i = 1, 2, ..., k)$ units be drawn from k, p-variate normal populations having dispersion matrices Σ_i $(i = 1, 2, ..., k)$ which are not necessarily equal. Let $\bar{x}_{ij}$ and m_{ij} denote sample mean and population mean of j-th character of i-th population. Also let s_{ij} and σ_{ij} denote sample and population variance of j-th character of i-th population. To test the hypothesis that

$$\sum_{i=1}^{k} c_{ij}m_{ij} = \lambda_j \qquad (j = 1, 2, ..., p). \qquad \text{... (7.1.1)}$$

$M_{p,pk}$-statistic may be defined as

$$\frac{\sum_{j=1}^{p}\left\{\sum_{i=1}^{k} c_{ij}\bar{x}_{ij}-\lambda_j\right\}^2}{\sum_{i=1}^{k}\frac{A_i}{N_i}\sum_{j=1}^{p} c_{ij}^2 s_{ij}^2} \qquad \text{... (7.1.2)}$$

with suitable choice of A_i $(i = 1, 2, ..., k)$ depending upon the size of the test. It can be shown that $M_{p,pk}$ as defined in (7.1.2) is equal to

$$\frac{\sum_{1}^{p} \beta_i\chi_{1i}^2}{\sum_{i=1}^{k} A_i \sum_{j=1}^{p} \omega_{ij}\frac{\chi_{ij}^2}{\nu_{ij}}} \qquad \text{... (7.1.3)}$$

where χ_{1i}^2 $(i = 1, 2, ..., p)$ and χ_{ij}^2 $(i = 1, 2, ..., k; j = 1, 2, ..., p)$ are independently distributed χ^2-variates, χ_{1i}^2 being distributed with 1 and χ_{ij}^2 being distributed with N_i-1 d.f. and β_i and ω_{ij} are a set of positive weights adding up to unity i.e. $\sum_{i=1}^{p} \beta_i = 1$ and $\sum_{i=1}^{k}\sum_{j=1}^{p} \omega_{ij} = 1$.

8. Further numerical example

8.1. As an example of likely use of M-statistic in multivariate problems let us consider Barnard's data on Egyptian skulls. Four measurements on four populations are summarised as

TABLE 3. MEAN VALUES OF FOUR CHARACTERS

	character			
	I	IV	VI	VII
population I	133.583	98.308	50.835	133.000
II	134.205	96.463	51.148	134.883
III	134.371	95.857	50.100	133.643
IV	135.307	95.040	52.093	131.467

with numbers of observations as $N_1 = 91$, $N_2 = 162$, $N_3 = 70$ and $N_4 = 75$ and pooled corrected sum of squares of the four characters as (i) 9661.997, (ii) 9073.115, (iii) 3938.320 and (iv) 8741.509. Let $\bar{x}_{ij}$ and m_{ij} denote sample mean and population mean of j-th character of the i-th population ($i, j = 1, 2, 3, 4$). Also let s_j^2 and σ_j^2 denote sample and population variances of the j-th character. (Here the dispersion matrices of the populations have been assumed to be equal.) To test the hypothesis that

$$m_{1j} = m_{2j} = m_{3j} = m_{4j} \quad (j = 1, 2, 3, 4).$$

Let test variates U_{jk} ($j = 1, 2, 3, 4$; $k = 1, 2, 3$) be defined be

$$\left.\begin{aligned} U_{j1} &= \frac{1}{\sqrt{2}} \{\bar{x}_{1j}-\bar{x}_{2j}\} \\ U_{j2} &= \frac{1}{\sqrt{2}} \{\bar{x}_{3j}-\bar{x}_{4j}\} \\ U_{j3} &= \frac{1}{\sqrt{4}} \{\bar{x}_{1j}+\bar{x}_{2j}-\bar{x}_{3j}-\bar{x}_{4j}\} \end{aligned}\right\} \quad \ldots \quad (8.1.1)$$

$$(j = 1, 2, 3, 4).$$

On the basis of test variates U_{jk} ($j = 1, 2, 3, 4$; $k = 1, 2, 3$) $M_{12 \cdot 4}$-statistic may be computed as

$$\frac{\sum\limits_{j=1}^{4} \sum\limits_{k=1}^{3} U_{jk}^2}{\frac{3}{4} A \sum\limits_{j=1}^{4} s_j^2 \left\{\frac{1}{N_1}+\frac{1}{N_2}+\frac{1}{N_3}+\frac{1}{N_4}\right\}} \quad \ldots \quad (8.1.2)$$

with suitable choice of A depending on the size of the test. Taking numerical value of A equal to 3.86 (value taken from tabulated 5 p.c. point of F-table corresponding to $\nu_1 = 1$ and $\nu_2 = 400$) *approximate* numerical value of $M_{12 \cdot 4}$-statistic comes out as 1.49 . Since numerical value of $M_{12 \cdot 4}$-statistic exceeds unity the hypothesis cannot be accepted.

Appendix A.1

Let

$$F(z) = \int_0^{z/\beta_1} e^{-x_1} x_1^{-\frac{1}{2}} \left\{ \int_0^{(z-\beta_1 x_1)/\beta_2} e^{-x_2} x_2^{-\frac{1}{2}} dx_2 \right\} dx_1. \quad \ldots \text{(A.1.1)}$$

where $\beta_1+\beta_2 = 1$; $\beta_1, \beta_2 > 0$ and $\beta_2 > \beta_1$.

We have

$$\frac{d}{dz} F(z) = \int_0^{z/\beta_1} e^{-x_1} x_1^{-\frac{1}{2}} e^{-(z-\beta_1 x_1)/(1-\beta_1)} \left\{ \frac{z-\beta_1 x_1}{1-\beta_1} \right\}^{-\frac{1}{2}} \left\{ \frac{1}{1-\beta_1} \right\} dx_1$$

$$= K. \int_0^z e^{-x/\beta_1} e^{-(z-x)/(1-\beta_1)} \left\{ z-x \right\}^{-\frac{1}{2}} x^{-\frac{1}{2}} dx = I_1+I_2 \quad \ldots \text{(A.1.2)}$$

where

$$I_1 = K. e^{-x/\beta_1} e^{-(z-x)/(1-\beta)} \int x^{-\frac{1}{2}} (z-x)^{-\frac{1}{2}} dx \Big]_0^z$$

$$= 2K. e^{-x/\beta_1} e^{-(z-x)/(1-\beta_1)} \sin^{-1} \sqrt{\frac{x}{z}} \Big]_0^z$$

$$= 2K. e^{-z/\beta_1} \frac{\pi}{2} = K. e^{-z/\beta_1} \quad \ldots \text{(A.1.3)}$$

and

$$I_2 = -K \int_0^z e^{-x/\beta_1} e^{-(z-x)/(1-\beta_1)} \left\{ -\left(\frac{1}{\beta_1} - \frac{1}{1-\beta_1} \right) \right\} \times 2 \sin \sqrt{\frac{x}{z}} dx. \quad \ldots \text{(A.1.4)}$$

Now

$$\frac{d}{dz} I_1 = \pi e^{-z/\beta_1} \left\{ -\frac{1}{\beta_1} \right\} \quad \ldots \text{(A.1.5)}$$

and

$$\frac{d}{dz} I_2 = I_{21}+I_{22} \quad \ldots \text{(A.1.6)}$$

where

$$I_{21} = 2K e^{-z/\beta_1} \left\{ \frac{1}{\beta_1} - \frac{1}{1-\beta_1} \right\} \frac{\pi}{2}$$

$$= K. e^{-z/\beta} \left\{ \frac{1}{\beta_1} - \frac{1}{1-\beta_1} \right\} \pi$$

and

$$I_{22} = K \int_0^z e^{-x/\beta_1} \left\{ \frac{1}{\beta_1} - \frac{1}{1-\beta_1} \right\} e^{-x/(1-\beta_1)}$$

$$\times \frac{d}{dz} \left[e^{-z/(1-\beta_1)} \sin^{-1} \sqrt{\frac{x}{z}} \right] dx. \quad \ldots \text{(A.1.7)}$$

As

$$\frac{d}{dz} \left\{ e^{-z/(1-\beta_1)} \sin^{-1} \sqrt{\frac{x}{z}} \right\} = e^{-z/(1-\beta_1)} \sin^{-1} \sqrt{\frac{x}{z}} \left\{ -\frac{1}{1-\beta_1} \right\}$$

$$+ e^{-z/(1-\beta_1)} \left\{ \frac{1}{2} \frac{z}{\sqrt{x}\sqrt{z-x}} \left(-\frac{x}{z^2} \right) \right\},$$

$$I_{22} = K \int_0^z e^{-x/\beta_1} e^{-(z-x)/(1-\beta_1)} \left\{ \frac{1}{\beta_1} - \frac{1}{1-\beta_1} \right\} \times \left[\sin^{-1} \sqrt{\frac{x}{z}} \left(-\frac{1}{1-\beta_1} \right) - \frac{x^{\frac{1}{2}}}{2z\sqrt{z-x}} \right] dx \quad \ldots \text{(A.1.8)}$$

From (A.1.1), (A.1.2), ... (A.1.8) it follows

$$\frac{d^2}{dz^2} F(z) = -Ke^{-z/\beta_1} \left\{ \frac{1}{1-\beta_1} \right\} + I_{22}.$$

As I_{22} is negative, $\frac{d^2}{dz^2} F(z)$ is negative, so that $F(z)$ is an upward convex function of z.

Appendix A.2

Let

$$F(\beta_1,\beta_2) = \int_0^\infty \int_0^\infty e^{-x_1-x_2}\, x_1^{-\frac{1}{2}}\, x_2^{-\frac{1}{2}} \left\{ \int_0^T e^{-y} (y_1)^{\nu/2-1}\, dy \right\} dx_1 dx_2 \qquad \ldots \quad (A.2.1)$$

where $T = (\beta_1 x_1+\beta_2 x_2)/A'$, $A' = A/\nu$. $\beta_1+\beta_2 = 1$ and $\beta_1, \beta_2 > 0$,

$$\frac{d}{d\beta_1} F(\beta_1, \beta_2) = K_1 \int_0^\infty \int_0^\infty e^{-x_1(1+\beta_1/A')-x_2(1+\beta_2/A')}\, x_1^{-\frac{1}{2}}\, {}_2^{-\frac{1}{2}}\, (\beta_1 x_1+\beta_2 x_2)^{\nu/2-1} (x_1-x_2) dx_1 dx_2 \qquad \ldots \quad (A.2.2)$$

$$= K_2 \int_0^\infty \int_0^\infty e^{-u_1-u_2}\, u_1^{-\frac{1}{2}}\, u_2^{-\frac{1}{2}}\, (c_1 u_1+c_2 u_2)^{\nu/2-1} \times (u_1(1+a_1)^{-1} - u_2(1+a_2)^{-1})\, du_1 du_2 \qquad \ldots \quad (A.2.3)$$

where

$$a_1 = \beta_1/A';\; a_2 = \beta_2/A';$$
$$c_1 = \beta_1/(1+a_1);\; c_2 = \beta_2/(1+a_2).$$

Sub-case 1: For $\nu = 2$, from (A.2.3.)

$$\frac{d}{d\beta_1} F(\beta_1, \beta_2) = I_1 - I_2. \qquad \ldots \quad (A.2.4)$$

where

$$I_1 = (1+a_1)^{-1} \int_0^\infty \int_0^\infty e^{-u_1-u_2}\, u_1^{-\frac{1}{2}}\, u_2^{-\frac{1}{2}}\, u_1 du_1 du_2$$

$$I_2 = (1+a_2)^{-1} \int_0^\infty \int_0^\infty e^{-u_1-u_2}\, u_1^{-\frac{1}{2}}\, u_2^{-\frac{1}{2}}\, u_2 du_1 du_2$$

From (A.2.4) $\qquad I_2/I_1 = (1+a_1)/(1+a_2) = (A'+\beta_1)/(A'+\beta_2)$

which is less than unity if $\beta_1<\beta_2$, so that $F(\beta_1, \beta_2)$ increases as β_1 increases $(\beta_1<\beta_2)$. It can be similarly shown that if $\beta_1>\beta_2$ $F(\beta_1, \beta_2)$ decreases as β_1 increases and the function $F(\beta_1, \beta_2)$ has a maximum value at $\beta_1 = \beta_2 = 1/2$.

Sub-case 2: For $\nu=1$, from (A.2.3) for $\beta_1, \beta_2 > \epsilon > 0$,

$$\frac{d}{d\beta_1} F(\beta_1, \beta_2) = I_1 - I_2 \qquad \ldots \quad (A.2.5)$$

where

$$I_1 = K_2 (1+a_1)^{-1} \int_0^\infty \int_0^\infty e^{-u_1-u_2}\, u_1^{-\frac{1}{2}}\, u_2^{-\frac{1}{2}}\, (c_1 u_1+c_2 u_2)^{-\frac{1}{2}}\, u_1 du_1 du_2$$

and

$$I_2 = K_2 (1+a_2)^{-1} \int_0^\infty \int_0^\infty e^{-u_1-u_2}\, u_1^{-\frac{1}{2}}\, u_2^{-\frac{1}{2}}\, (c_1 u_1+c_2 u_2)^{-\frac{1}{2}}\, u_2 du_1 du_2.$$

Defining variates $V_1 = u_1$ and $V_2 = u_2/u_1$, it can be shown that

$$I_1 = K_3 (1+a_1)^{-1} \int_0^\infty V_2^{-\frac{1}{2}} (1+c_2 V_2/c_1)^{-\frac{1}{2}} (1+V_2)^{3/2}\, dV_2. \qquad \ldots \quad (A.2.6)$$

For $\beta_1 < \beta_2$, defining $Z = 1/(1+V_2)$, it can be shown that

$$I_1 = K_4 (1+a_1)^{-1} F(1/2, 3/2;\, 2;\, \lambda_1) \qquad \ldots \quad (A.2.7)$$

where $\lambda_1 = A'(A'+\beta_1)^{-1}(\beta_2-\beta_1)/\beta_2$.

Also, for $\beta_1 < \beta_2$ it can be shown that

$$I_2 = K_4 (1+a_2)^{-1} F(1/2, 1/2;\, 2;\, \lambda_1). \qquad \ldots \quad (A.2.8)$$

From (A.2.7) and (A.2.8) it thus follows that for $\beta_1 < \beta_2$

$$I_2/I_1 = (A'+\beta_1)(A'+\beta_2)^{-1}\; F(1/2, 1/2;\, 2;\, \lambda_1)/F(1/2, 3/2;\, 2;\, \lambda_1) \qquad \ldots \quad (A.2.9)$$

For $\beta_1 < \beta_2$ thus $F(\beta_1, \beta_2)$ increases as β_1 increases. It can also be similarly shown that for $\beta_1 > \beta_2$, $F(\beta_1, \beta_2)$ decreases as β_1 increases and the function has a maximum value at $\beta_1 = \beta_2 = 1/2$.

Sub-case 3: For $\nu > 3$, we have from (A.2.3)

$$\frac{d}{d\beta_1} F(\beta_1, \beta_2) = I_1 - I_2 \qquad \ldots \quad (A.2.10)$$

where $$I_1 = K_2(1+a_1)^{-1} \int_0^\infty \int_0^\infty e^{-u_1-u_2} u_1^{-\frac{1}{2}} u_2^{-\frac{1}{2}} \times (c_1u_1+c_2u_2)^{\nu/2-1} u_1 du_1 du_2 \qquad \ldots \quad (A.2.11)$$

and $$I_2 = K_2(1+a_2)^{-1} \int_0^\infty \int_0^\infty e^{-u_1-u_2} u_1^{-\frac{1}{2}} u_2^{-\frac{1}{2}} \times (c_1u_1+c_2u_2)^{\nu/2-1} u_2 du_1 du_2. \qquad \ldots \quad (A.2.12)$$

For $\beta = 0$, from (A.2.11) and (A.2.12)

$$I_1 = K_3 . \Gamma(3/2) \ \Gamma(\nu/2-\tfrac{1}{2})/(1+a_1),$$
$$I_2 = K_3 . \Gamma(\tfrac{1}{2}) \ \Gamma(\nu/2+1-\tfrac{1}{2})/(1+a_2),$$

so that $$I_2/I_1 = (1+a_1)(1+a_2)^{-1} (\nu-1) = (A'+\beta_1)(A'+\beta_2)^{-1} (\nu-1). \qquad \ldots \quad (A.2.13)$$

From (A.2.13), I_2 would be greater than I_1 if

$$A'\nu = A > \nu/(\nu-2). \qquad \ldots \quad (A.2.14)$$

Now for $0 < \beta_1 < \beta_2$, defining variates $V_2 = u_2$ and $V_1 = u_1/u_2$, it can be shown from (A.2.11) that

$$I_1 = K_4 (1+a_1)^{-1} \int_0^\infty V_1^{\frac{1}{2}} (1+D_1V_1)^p (1+V_1)^{-(p+2)} \ dV_1 \qquad \ldots \quad (A.2.15)$$

where $$D_1 = c_1/c_2 \text{ and } p = \nu/2-1.$$

From (A.2.15) it can be shown that

$$I_1 = K_5 \ (1+a_1)^{-1} D_1^{-\frac{1}{2}} F(\nu/2+1, \tfrac{1}{2} ; 2 ; \lambda_2) \qquad \ldots \quad (A.2.16)$$

where $$\lambda_2 = 1-D_1 = 1-\beta_1 \beta_2^{-1} (A'+\beta_2)/(A'+\beta_1).$$

It can also be shown from (A.2.12) that for $0 < \beta_1 < \beta_2$

$$I_2 = K_5 (1+a_2)^{-1} D_1^{\frac{1}{2}} F(\nu/2+1, 3/2 ; 2 ; \lambda_2). \qquad \ldots \quad (A.2.17)$$

From (A.2.16) and (A.2.17) we thus have

$$I_2/I_1 = D_1 (A'+\beta_1)(A'+\beta_2)^{-1} \ F(\nu/2+1, 3/2 ; 2 ; \lambda_2)/F(\nu/2+1, 1/2 ; 2 ; \lambda_2)$$
$$= (1-\lambda_2)(A'+\beta_1)(A'+\beta_2)^{-1} \ F(\nu/2+1, 3/2 ; 2 ; \lambda_2)/F(\nu/2+1, 1/2 ; 2 ; \lambda_2) \qquad \ldots \quad (A.2.18)$$

Now according to algebraic relations due to Gauss (Erdelyi, 1953) satisfied by contiguous hypergeometric functions,

$$(1-z) \frac{F(a, b+1 ; c ; z)}{F(a, b ; c ; z)} = 1+z \ \frac{a-c}{c} \ \frac{F(a, b+1 ; c+1 ; z)}{F(a, b ; c ; z)}. \qquad \ldots \quad (A.2.19)$$

From (A.2.18) and (A.2.19) I_2 would be greater than I_1 if

$$(A'+\beta_1)(A'+\beta_2)^{-1} \ (1+\lambda_2 E) > 1 \qquad \ldots \quad (A.2.20)$$

where $$E_1 = (a-c) \ c^{-1} \ F(a, b+1 ; c+1 ; \lambda_2)/F(a, b ; c ; \lambda_2)$$
$$a = \nu/2+1 ; \ b = \tfrac{1}{2} ; \ c = 2 ;$$

and $$\lambda_2 = 1-\beta_1 . \beta_2^{-1} \ (A'+\beta_2)/(A'+\beta_1)^{-1}.$$

From (A.2.20), I_2 would be greater than I_1 if

$$(A'+\beta_1) . (A'+\beta_2)^{-1} (1+E) > 1+\beta_1 \ \beta_2^{-1} \ E,$$

or, if $$A'(\beta_2-\beta_1)E > \beta_2(\beta_2-\beta_1),$$

or, if $$A \nu^{-1} E > \beta_2 \qquad \ldots \quad (A.2.21)$$

Now λ_2 and β_2 are connected as

$\lambda_2 = 1-\beta_1.\ \beta_1^{-1}(A'+\beta_1)(A'+\beta_1)^{-1} = A'\beta_1^{-1}(\beta_2-\beta_1)(A'+\beta_1)^{-1}$ so that for $1 > \beta_2 > \epsilon > \frac{1}{2}$,

$$\lambda_2 > A'\{A'+(1-\epsilon)\}^{-1}(2\epsilon-1)\epsilon^{-1}. \qquad \ldots \quad (A.2.22)$$

For clarity of exposition (A.2.21) would be considered under two heads:

Sub-case 1 : β_2 lies in the range $3/4 > \beta_2 > \frac{1}{2}$.

Sub-case 2 : β_2 lies in the range $1 > \beta_2 > 3/4$.

For sub-case 1, from (A.2.21) it follows that since $F(a,b+1;c+1;\lambda_2)/F(a,b;c;\lambda_2)$ is greater than or equal to unity, (A.2.21) would be satisfied if

$$A' = A/\nu > c\,(a-c)^{-1}\ 3/4 = 3/(\nu-2). \qquad \ldots \quad (A.2.23)$$

For sub-case 2, since λ_2 from (A.2.22) would be greater than or equal to $A'(A'+1/4)^{-1}.\ 2/3$, (A.2.21) would be satisfied if

$$A'\left[\frac{1+a(b+1)\lambda_0/(c+1)}{1+ab\lambda_0/c}\right] > c/(a-c) = 4/(\nu-2) \qquad \ldots \quad (A.2.24)$$

where
$$\lambda_0 = A'(A'+1/4)^{-1}\,2/3.$$

From (A.2.24) it follows that I_2 would be greater that I_1 if

$$A'\frac{A'+1/4+aA'/3}{A'+1/4+aA'/6} > 4/(\nu-2). \qquad \ldots \quad (A.2.25)$$

Considering (A.2.25) the following auxiliary function U may be considered:

$$U = A'\{(A'+1/4)+aA'/3\} - 4(\nu-2)^{-1}(A'+1/4+aA'/6). \qquad \ldots \quad (A.2.26)$$

In (A.2.26) substituting $K/(\nu-2)$ for A' we get

$$(\nu-2)\,U = (A'+1/4)(K-4)+aA'(2K-4)/6$$
$$= A'(K-4+(\nu+2)(K-2)/6)+(K-4)/4$$

or,

$$12(\nu-2)^2U = K\{12(K-4)+2(\nu+2)(K-2)\}+3(K-4)(\nu-2)$$
$$= K^2(2\nu+16)-K(\nu+62)-12(\nu-2). \qquad \ldots \quad (A.2.27)$$

Since the co-efficient of K^2 of the quadratic on the R H S of (A.2.27) is positive, for some value of $K > K_0$ numerical value of the quadratic and as such numerical value of U is positive. Let the roots of the quadratic

$$(2\nu+16)K^2-K(\nu+62)-12(\nu-2) = 0 \qquad \ldots \quad (A.2.28)$$

be K_1 and K_2 (where $K_2 > K_1$). Now it can be shown that for $\nu > 3$,

$$K_2 < \frac{(\nu+62)+(10\nu+43)}{4\nu+32} = \frac{11\nu+105}{4\nu+32}. \qquad \ldots \quad (A.2.29)$$

Since the expression on the R H S of (A.2.29) for $\nu > 3$ is less than 16/5, it follows that U would be positive for $K > 16/5$, which means that (A.2.25) or (A.2.24) would be satisfied for

$$A'(=A/\nu) > 3.2/(\nu-2)$$

or,

$$A > 3.2\nu/(\nu-2). \qquad \ldots \quad (A.2.30)$$

From (A.2.14), (A.2.23) and (A.2.25) it thus follows that for $0 < \beta_1 < \beta_2$, I_2 would be greater than I_1 if

$$A > 3.2\nu/(\nu-2). \qquad (A.2.31)$$

The function $F(\beta_1, \beta_2)$ thus decreases as β_1 increases for $\beta_1 < \beta_2$, if A is greater than or equal to $3.2\nu/(\nu-2)$. It can also be similarly shown that for $\beta_1 > \beta_2$, $F(\beta_1, \beta_2)$ increases as β_1 increases if A is numerically greater than or equal to $3.2\nu/(\nu\text{-}2)$ and the function has a minimum value at $\beta_1 = \beta_2 = 1/2$ and maximum value at $\beta_1 = 0$ and $\beta_1 = 1$.

Since critical values of F-table for 1 and $\nu (\nu > 3)$ d.f. for 5 p.c., 2 p.c., 1 p.c. etc. level of significance are all greater than $3.2\nu/(\nu-2)$ [a relation which can be proved using the algebraic relation due to Fisher (1941, page 151 middle)] the relation :

$$\beta_1 \chi^2_{1,1} + \beta_2 \chi^2_{1,2} > A\chi^2_{\nu}\ \nu$$

would be satisfied with probability less than or equal to α for $\alpha = 0.05, 0.02, 0.01$, etc. and $\nu > 3$, if A is taken from F-table corresponding to 1 and ν d.f. for given α.

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