SANKHYĀ

THE INDIAN JOURNAL OF STATISTICS

Editors: P. C. MAHALANOBIS, C. R. RAO

SERIES B, VOL. 26 | NOVEMBER 1964 | PARTS 1 & 2

ON SOME VARIABLE ELASTICITY ENGEL CURVE FORMS[1]

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SUMMARY: Some variable elasticity Engel curve forms are examined in this paper from two different points of view. First is considered the problem of projecting consumer demand when the income distribution[2] changes, prices and preferences remaining unchanged (vide Roy and Laha, 1960; Iyengar, 1960a). The estimation of Engel elasticities from the concentration curves for income (or total expenditure) and for item consumption is taken up at the second stage (Iyengar, 1960b). Results are obtained for a number of Engel curve forms which are analogous to those available for the constant elasticity form. Most of the results assume that the income distribution[2] is lognormal, which is quite realistic in India, but a few approximate results for projection of demand seem to be distribution-free.

1. INTRODUCTION

1.1. Although the constant elasticity Engel curve is often criticised as not completely satisfactory from both theoretical and empirical points of view, the various forms of variable elasticity Engel curves are not widely used in practice. Thus, in the Indian work on projection of demand, it is the constant elasticity curve which has been almost invariably used even after the possibility of a better fit was realised (e.g., in Roy and Dhar, 1960). And in one study which emphasised the need of a variable elasticity Engel curve, the curve was actually obtained by freehand smoothing, and the subsequent work was done by numerical methods.

[1] Some results for the semi-log form have been obtained independently by N. S. Iyengar (1964): A consistent method of estimating the Engel curve from grouped survey data. *Econometrica*, Vol. 32, No. 4, pp. 591–618.

[2] Per capita income or consumer expenditure, to be precise.

1.2. The comparative neglect of the variable elasticity forms seems to be partly due to the feeling that they are intractable and inconvenient in subsequent handling. One of the purposes of this paper is to point out that this impression is not wholly correct.

1.3. Consider first the problem of projecting consumer demand as formulated in Iyengar (1960a). Let x denote per capita income or consumer expenditure and y the per capita consumption (quantity or value) of a particular commodity, both defined over the population of persons. Let $E(y\,|\,x) = \psi(x)$ be the Engel curve, assumed unchanged over time; prices and preferences are thereby assumed to be fixed. Let $g(x)$ and $g^*(x)$ be the marginal distributions of x in the base and the future periods. Since population growth can be projected separately, the problem is to predict the future per capita demand

$$E^*(y) = \int_0^\infty E(y\,|\,x)\, g^*(x)\, dx \qquad \ldots \quad (1)$$

when $g^*(x)$ deviates in a postulated manner from $g(x)$, and $g(x)$ and $E(y\,|\,x)$ are estimated from family budget data for the base period.

1.4. It is wellknown that (vide Roy and Laha, 1960) if

$$E(y\,|\,x) = Ax^\eta \qquad \ldots \quad (2)$$

and if further

$$g^*(x) = g\left(\frac{x}{c}\right)\frac{1}{c} \qquad \ldots \quad (3)$$

that is, if the distribution of x shifts by a constant proportion c, then whatever be the algebraic form of $g(x)$,

$$E^*(y) = E(y)c^\eta. \qquad \ldots \quad (4)$$

Here mean of x increases c times but concentration remains unchanged.

1.5. Iyengar (1960a) generalised this result by giving a prediction formula for the case where the Engel curve is of the constant elasticity form and where $g^*(x)$ deviates from $g(x)$ to specified extents both in the average as well as in concentration; he, however, assumed that both $g(x)$ and $g^*(x)$ are of the lognormal form.[3]

1.6. In Section 2 we derive some *approximate* but distribution-free results analogous to (4) above but relating to certain variable elasticity Engel curves. Iyengar's approach has also been followed up and some results analogous to Iyengar's are also given in the same section.

1.7. For the second part of the paper (Section 3) on the estimation of elasticity from concentration curves, we need consider only one of the two periods. Choosing the base period, for example, we get the concentration curve for total expenditure

[3] The lognormal distribution gives a very close fit to the distribution of per capita expenditure emanating from the Indian National Sample Survey [see Roy and Dhar, 1960; also Iyengar, 1960b]

(or income) x by plotting

$$Q_0(x) = \frac{\int_0^x ug(u)\,du}{\int_0^\infty ug(u)\,du} = \frac{\int_0^x ug(u)\,du}{E(x)} \qquad \dots \quad (5)$$

against

$$P(x) = \int_0^x g(u)\,du. \qquad \dots \quad (6)$$

The concentration coefficient is given by

$$L_0 = 1-2\int_0^1 Q_0\,dP. \qquad \dots \quad (7)$$

1.8. The specific concentration curve for item consumption y is obtained by plotting

$$Q(x) = \frac{\int_0^x E(y\,|\,u)\,g(u)\,du}{\int_0^\infty E(y\,|\,u)\,g(u)\,du} = \frac{\int_0^x E(y\,|\,u)\,g(u)\,du}{E(y)} \qquad \dots \quad (8)$$

against $P(x)$ given by (6). The specific concentration coefficient is given by

$$L = 1-2\int_0^1 QdP. \qquad \dots \quad (9)$$

1.9. Such concentration curves abound in the work of Mahalanobis on Indian family budget data (see Mahalanobis, 1960; see also, Iyengar, 1960b). Aitchison and Brown (1957) gave the equation of the concentration curve for x when $g(x)$ is lognormal. Iyengar (1960b) gave the equation of the specific concentration curve for y assuming that $E(y\,|\,x) = Ax^\eta$ and also that $g(x)$ is lognormal. These two curves have many elegant properties; and what is more important, it is possible to estimate the Engel elasticity η when these two curves are given.[4]

1.10. In Section 3 are given the expressions for specific concentration curves and specific concentration coefficients for cases where $g(x)$ is lognormal but where the Engel curve assumes certain variable elasticity forms. These may be used for estimating the variable Engel elasticity in a manner analogous to Iyengar's (1960b).

1.11. The variable elasticity forms examined are: (i) the semi-log form $E(y\,|\,x) = \alpha+\beta \log x$ $(\beta > 0)$, (ii) the hyperbolic form $E(y\,|\,x) = \alpha-\beta/x (\alpha, \beta > 0)$, (iii) the form $E(y\,|\,x) = k\Lambda(x\,|\,\mu, \sigma)$, where $\Lambda(x\,|\,\mu, \sigma)$ is the distribution function of a lognormal distribution with parameters μ and σ, and (iv) the polynomial form $E(y\,|\,x) = \sum_{r=0}^{k} a_r x^r$. Forms (i) and (ii) were used by Prais and Houthakker (1955) on British data, and the former was found to be generally superior to the constant

[4] The general problem of studying Engel elasticity when these two curves are given was studied by Roy, Chakravarti and Laha (1960).

elasticity Engel curve for food items. Form (iii) was found useful for British data by Aitchison and Brown (1957). Form (iv) includes the straight line of Allen and Bowley; whatever may be the theoretical objections, the polynomial seems to have considerable possibilities in Engel curve graduation and subsequent manipulations.

1.12. The Tornquist's forms (Wold, 1953) and the log-inverse form of Prais and Houthakker (1955) proved to be completely intractable in both the problems under study. Also, Form (iii) could not be tackled in the second problem.

1.13. Section 4 illustrates the application of the new results with some data from the Indian National Sample Survey.

2. Projection of consumer demand

2.1. *Approximate distribution-free results.* We first present some approximate results analogous to (4) above. The underlying assumptions are given in paragraphs 1.3-1.4.

2.2. A complete specification of the semi-log form would be

$$\left.\begin{aligned} E(y|x) &= \alpha+\beta\log x \quad \text{when } x \geqslant e^{-\alpha/\beta} = \gamma \text{ say } (\gamma > 0) \\ &= 0 \qquad\qquad \text{when } 0 \leqslant x \leqslant \gamma. \end{aligned}\right\} \quad \dots \quad (10)$$

Now assume that

$$g^*(x) = g\left(\frac{x}{c}\right)\frac{1}{c}$$

as before, (vide paragraph 1.4). Then[5]

$$E(y) \sim \int_0^\infty (\alpha+\beta\log x)g(x)dx = \alpha+\beta\log G \quad \dots \quad (11)$$

where G is the geometric mean of x, and

$$E^*(y) \sim \int_0^\infty (\alpha+\beta\log x)g^*(x)dx = \alpha+\beta\log G+\beta\log c \quad \dots \quad (12)$$

so that

$$E^*(y)-E(y) \sim \beta\log c \quad \dots \quad (13)$$

simply. This can be easily used since β can be estimated from family budget data.

2.3. For the form $E(y|x) = \alpha-\dfrac{\beta}{x}$, the threshold is at $\dfrac{\beta}{\alpha}$. Assuming that the proportion of cases below this is negligible according to $g(x)$ and $g^*(x)$, we get

$$E(y) \sim \alpha-\beta E\left(\frac{1}{x}\right) = \alpha-\frac{\beta}{H} \quad \dots \quad (14)$$

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[5] It is assumed here that the proportion of cases with x below γ (the threshold) is negligible both according to $g(x)$ and $g^*(x)$. This assumption is not unrealistic. For this form with declining elasticity is meant for necessaries and we may assume that for all observed values of x, $E(y|x) > 0$.

say, where H is the harmonic mean and

$$E^*(y) \sim \alpha - \frac{\beta}{cH} \qquad \dots \quad (15)$$

so that

$$E^*(y) \sim \alpha - \frac{1}{c}\,[x - E(y)]. \qquad \dots \quad (16)$$

Here α can be estimated from family budget data.

2.4. No such result could be derived for the form $E(y|x) = k\Lambda(x|\mu, \sigma)$, but for the polynomial $E(y|x) = \sum_0^k a_r x^r$, we get, using similar assumptions

$$E(y) \sim \sum_0^k a_r\, E(x^r) \qquad \dots \quad (17)$$

and

$$E^*(y) \sim \sum_0^k a_r\, c^r E(x^r). \qquad \dots \quad (18)$$

These relations may be used for approximate calculations; the case $k = 1$ is extremely simple.

2.5. It may be noted that for the semi-log form, the hyperbolic form and the linear form, average per capita demand depends respectively on the geometric mean, the harmonic mean and the arithmetic mean of x, and in a certain sense, changes in concentration do not matter in projection. However, these properties of the first two forms cannot be used in practice, and the linear form rarely gives a good fit.

2.6. *Results assuming lognormality of $g(x)$ and $g^*(x)$.*[*] Under this assumption, it is possible to take account of changes in concentration, apart from changes in the average, of the distribution of x. Suppose the distribution function of x is $\Lambda(x|\theta, \lambda)$ in the base period and $\Lambda(x|\theta^*, \lambda^*)$ in the future period. We shall utilise the expressions for $E(y)$ derived in Section 3.

2.7. For the semi-log form, we have

$$E(y) = (\alpha + \beta\theta)[1 - \Phi(\gamma')] + \beta\lambda\Phi'(\gamma') \qquad \dots \quad (19)$$

where $\gamma' = \dfrac{\log \gamma - \theta}{\lambda}$ (with $\gamma = e^{-\alpha/\beta}$, the threshold) $= -\dfrac{\alpha + \beta\theta}{\beta\lambda}$, and Φ and Φ' denote respectively the distribution function and the frequency function of the standard normal distribution. One can estimate θ and λ from the marginal distribution of x, and α and β by Engel curve-fitting. It is then a simple matter to get $E^*(y)$ by putting θ^* and λ^* instead of θ and λ in (19). Of course, the value of γ' has to be

[*] The other underlying assumptions are those of paragraph 1.3.

re-calculated. Since the mean and the concentration coefficient of the lognormal distribution $g(x)$ are given by

$$E(x) = e^{\theta+\lambda^2/2} \quad \text{and} \quad L_0 = 2\Phi\left(\frac{\lambda}{\sqrt{2}}\right)-1$$

one can get θ^* and λ^* by requiring that mean and concentration coefficient of the distribution $g^*(x)$ will differ by specified extents from those of $g(x)$.

2.8. Instead of estimating both α and β by Engel curve-fitting, one can estimate γ' or $\frac{\alpha}{\beta}$ from concentration curves (vide next section), and the absolute values of α and β by using equation (19). Actually, however, $E^*(y)/E(y)$ depends on γ' or $\frac{\alpha}{\beta}$ only, and not on absolute values of α and β.

2.9. In the special case when $\Phi(\gamma')$ and $\Phi'(\gamma')$ can be neglected, that is, when the proportion of x-values below the threshold is very small, we get

$$E(y) \sim \alpha+\beta\theta \quad \text{and} \quad E^*(y) \sim \alpha+\beta\theta^*$$

so that changes in concentration are not important.

2.10. For the hyperbolic form, we get

$$E(y) = \alpha[1-\Phi(\gamma')]-\beta e^{-\theta+\lambda^2/2}[1-\Phi(\gamma'+\lambda)] \qquad \ldots \quad (20)$$

where $\gamma' = \frac{\log\gamma-\theta}{\lambda}$ with $\gamma = \beta/\alpha$, the threshold. As before, θ and λ can be estimated from the marginal distribution of x, and α and β by methods of Engel curve-fitting; alternatively γ' or β/α can be estimated by using concentration curves (vide next section) and the actual values of α and β by using equation (20). The projected demand $E^*(y)$ can then be estimated by putting θ^* and λ^* in place of θ and λ in (20). Here again, the ratio $E^*(y)/E(y)$ depends on β/α and not on α and β separately.

2.11. When $\Phi(\gamma')$ and $\Phi(\gamma'+\lambda)$ can be neglected,

$$E(y) \sim \alpha-\beta e^{-\theta+\lambda^2/2} \quad \text{and} \quad E^*(y) \sim \alpha-\beta e^{-\theta^*+\lambda^{*2}/2}$$

2.12. For the form $E(y|x) = k\Lambda(x|\mu, \sigma)$, the problem of threshold does not arise, and we get

$$E(y) = \int_0^\infty k\Lambda(x|\mu, \sigma)d\Lambda(x|\theta, \lambda) = k \text{ prob } \{x_1 < x_2\}$$

(where x_1 is $\Lambda(x|\mu, \sigma)$ and x_2 is $\Lambda(x|\theta, \lambda)$ and x_1 and x_2 are independent)

$$= k\Phi\left(\frac{\theta-\mu}{\sqrt{\lambda^2+\sigma^2}}\right). \qquad \ldots \quad (21)$$

Now suppose θ and λ are estimated from the marginal distribution of x, and k, μ and σ by methods of Engel curve-fitting (Aitchison and Brown, 1957). It is then a simple matter to find $E^*(y)$ by putting θ^* and λ^* in place of θ and λ in (21).

2.13. For this form, we could not give any method of estimation based on the concentration curve. One parameter can, however, be estimated by using equation (21); or alternatively, we can use the fact that $E^*(y)/E(y)$ does not depend on k.

2.14. For the polynomial form, we get, ignoring the point that the polynomial can give negative values,

$$E(y) = \int_0^\infty (\Sigma a_r x^r) d\Lambda(x \mid \theta, \lambda) = \Sigma a_r e^{r\theta + \frac{1}{2}r^2\lambda^2}. \quad \dots \quad (22)$$

When the parameters θ and λ have been estimated from the marginal distribution of x, and the parameters a_r by Engel curve-fitting, it is easy to get $E^*(y)$ by putting θ^* and λ^* instead of θ and λ in (22).

2.15. The relative values of the parameters $a_0 : a_1 : a_2 : \dots : a_l$ can also be estimated from the concentration curves (vide next section), and the actual values of these a_r's by using (22) for $E(y)$. One may note, however, that $E^*(y)/E(y)$ depends only on the relative values of the a_r's.

3. Estimation of Engel elasticity from concentration curves[7]

3.1. Consider family budget data for one period, say the base period. Assume that the marginal distribution of x is lognormal, i.e., let $g(x) = \frac{d\Lambda(x \mid \theta, \lambda)}{dx}$. The concentration curve for x and the specific concentration curve for y have been defined in Section 1. In the lognormal case, the concentration curve for x was obtained as (Aitchison and Brown, 1957)

$$t_{Q_0} = t_P - \lambda \quad \dots \quad (23)$$

where $\Phi(t_k) = k$. Assuming lognormality of $g(x)$ and $E(y \mid x) = Ax^\eta$, Iyengar (1960a) obtained the equation of the concentration curve for y as

$$t_Q = t_P - \eta\lambda \quad \dots \quad (24)$$

and utilised equations (23) and (24) for estimating Engel elasticity η. Properties of these two curves were also studied.

3.2. Below we present the equations of the specific concentration curve for y assuming lognormality of $g(x)$, but using some variable elasticity forms for $E(y \mid x)$. We also consider the estimation of variable Engel elasticities from concentration and specific concentration curves.

3.3. *The semi-log form.* Consider first the semi-log form with the threshold at $x = e^{-\alpha/\beta} = \gamma$. We have, of course,

$$P(x) = \Lambda(x \mid \theta, \lambda) = \Phi\left(\frac{\log x - \theta}{\lambda}\right) = \Phi(x') \quad \dots \quad (25)$$

[7] All results in this section are based on the assumption that $g(x)$ is lognormal.

say for all the forms of Engel curves.

$$Q(x) = \int_{\gamma}^{x} (\alpha+\beta \log x) d\Lambda(x \mid \theta, \lambda) / \int_{\gamma}^{\infty} (\alpha+\beta \log x) d\Lambda(x \mid \theta, \lambda)$$

if $x > \gamma$, that is, if $P(x) > \Phi(\gamma')$,

$$= 0 \text{ if } P(x) \leqslant \Phi(\gamma'), \text{ where } \gamma' = \frac{\log \gamma - \theta}{\lambda} = -\frac{\alpha+\beta\theta}{\beta\lambda}.$$

If $x > \gamma$, the numerator of $Q(x)$ simplifies to

$$(\alpha+\beta\theta)\,[\Phi(x')-\Phi(\gamma')]-\beta\lambda[\Phi'(x')-\Phi'(\gamma')] \qquad \dots \quad (26)$$

and the denominator is, consequently, the limit of this as $x' \to \infty$, i.e.

$$E(y) = (\alpha+\beta\theta)\,[1-\Phi(\gamma')]+\beta\lambda\Phi'(\gamma'). \qquad \dots \quad (27)$$

Thus we got the following equation of the specific concentration curve

$$Q = \frac{(\alpha+\beta\theta)\,[P-\Phi(\gamma')]-\beta\lambda\,[\Phi'(t_P)-\Phi'(\gamma')]}{(\alpha+\beta\theta)\,[1-\Phi(\gamma')]+\beta\lambda\Phi'(\gamma')}$$

$$= \frac{\left(\frac{\alpha}{\beta}+\theta\right)[P-\Phi(\gamma')]-\lambda\,[\Phi'(t_P)-\Phi'(\gamma')]}{\left(\frac{\alpha}{\beta}+\theta\right)[1-\Phi(\gamma')]+\lambda\Phi'(\gamma')}$$

$$= \frac{\gamma'[P-\Phi(\gamma')]+[\Phi'(t_P)-\Phi'(\gamma')]}{\gamma'[1-\Phi(\gamma')]-\Phi'(\gamma')} \quad \text{if } P \geqslant \Phi(\gamma')$$

and $$Q = 0 \quad \text{if} \quad P < \Phi(\gamma'). \qquad \dots \quad (28)$$

3.4. Now for this semi-log form of the Engel curve we have for the variable elasticity

$$\left.\begin{aligned} \eta_x &= \frac{d \log E(y \mid x)}{d \log x} = \frac{\beta}{E(y \mid x)} = \frac{1}{\frac{\alpha}{\beta}+\log x} = \frac{1}{\log \frac{x}{\gamma}} \text{ when } x > \gamma \\ \text{and} \quad \eta_x &= 0 \text{ for } x < \gamma. \end{aligned}\right\} \qquad \dots \quad (29)$$

Thus the variable elasticity depends on the ratio α/β and hence on γ. For x between γ and γ_e, $\eta_x > 1$, and beyond this $\eta_x < 1$.

3.5. Any point (P, Q) on the specific concentration curve with $P > \Phi(\gamma')$ i.e. $Q > 0$ gives an equation for determining γ' (hence α/β when θ and λ are known). It can be shown that the solution is unique, if we use the obvious condition that $\gamma' < t_P$. Actually, for a given P, Q is a decreasing function of γ', falling from P to 0 as γ' rises from $-\infty$ to t_p.[8] For values of γ' lying between $-\infty$ and $+\infty$, the specific concentration curves defined by (28) form a family of curves, the lower curves corresponding to higher values of γ'. [Note that $\gamma' \to -\infty$ means $\beta \to 0$, or the Engel curve becomes horizontal.] But the solution has to be found by trial and error. Also both θ and λ must be known if one wants to find γ or α/β from the estimate of γ'; and only λ can be estimated from the concentration curve for x.

3.6. The specific concentration coefficient comes out as

$$L = 1-2\int_0^1 Q\,dP = \frac{\gamma'\Phi(1-\Phi)+\Phi'(1-2\Phi)-\frac{1}{\sqrt{\pi}}\left[1-\Phi(\sqrt{2}\gamma')\right]}{\gamma'(1-\Phi)-\Phi'} \qquad \ldots \quad (30)$$

where Φ and Φ' stand for $\Phi(\gamma')$ and $\Phi'(\gamma')$ respectively. This equation can also yield a unique estimate of γ' if some trial values are examined.

3.7. Now suppose the proportion of x-values below γ is negligible. Specifically, let us suppose that $\Phi(\gamma')$, $\Phi(\sqrt{2}\gamma')$ and $\Phi'(\gamma')$ are negligible, though γ' may have a finite negative value (say -4).[9] This special case is realistic since the concentration curves for most necessaries seem to rise above the P-axis straight from the origin $(P, Q) = (0, 0)$. Then we get the following special relations:

$$\text{(a)} \quad Q \sim P+\Phi'(t_P)/\gamma' = P-\frac{1}{\alpha/\beta+\theta}\lambda\Phi'(t_p) = P-\eta_m\lambda\Phi'(t_p) \text{ for all } P \qquad \ldots \quad (31)$$

where $\quad \eta_m = \dfrac{1}{\theta-\log\gamma} = -\dfrac{1}{\lambda\gamma'}$ is the elasticity at the median of x, viz. e^{θ},

$$\text{and} \quad \text{(b)} \quad L \sim -\frac{1}{\sqrt{\pi}\gamma'} = \frac{\lambda}{\sqrt{\pi}(\alpha/\beta+\theta)} = \frac{\eta_m\lambda}{\sqrt{\pi}}. \qquad \ldots \quad (32)$$

The approximate relation (31) has the defect that Q is negative, though small, in the neighbourhood of $P = \Phi(\gamma')$; but $Q \to 0$ as $P \to 0$. Actually $\dfrac{dQ}{dP} = 0$ at $P = \Phi(\gamma')$, but below this $\dfrac{dQ}{dP} < 0$ and above this $\dfrac{dQ}{dP} > 0$. This seems to be a minor point against the advantages of this special case. Equation (31) defines a family of non-intersecting curves where the lower curve corresponds to a higher value of γ'.

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[8] For a given P, $\dfrac{dQ}{d\gamma'} = \dfrac{\left[1-\Phi(t_p)\right]\left[1-\Phi(\gamma')\right]\left[\dfrac{\Phi'(\gamma')}{1-\Phi(\gamma')}-\dfrac{\Phi'(t_p)}{1-\Phi(t_p)}\right]}{\left[\gamma'(1-\Phi(\gamma'))-\Phi'(\gamma')\right]^2}$

which is negative for all $\gamma' < t_p$ since $\dfrac{\Phi'(x)}{1-\Phi(x)}$ is a monotone increasing function of x.

[9] This further assumption is made for all the results in paragraphs 3.7—3.10. The basic assumption of lognormality is, of course, there.

3.8. Under the assumption stated in paragraph 3.7, one can estimate γ' from the relation

$$\gamma' = -\frac{1}{\sqrt{L}} \quad \ldots \quad (33)$$

and hence get γ or α/β from a knowledge of θ and λ. But only λ can be estimated from the concentration curve for x. If one is interested in η_m, however, the knowledge of θ is not necessary, for

$$\eta_m = \frac{L\sqrt{\pi}}{\lambda}. \quad \ldots \quad (34)$$

Any ordinate of the concentration curve can give an estimate of γ' or η_m. Thus, using the value of Q for $P = 0.5$, we get

$$\gamma' = \frac{1}{\sqrt{2\pi}(Q_{0\cdot 5}-0.5)} \quad \text{and} \quad \eta_m = \frac{\sqrt{2\pi}(0.5-Q_{0\cdot 5})}{\lambda}. \quad \ldots \quad (35)$$

One should not, however, use the ordinates in the neighbourhood of $P = \Phi(\gamma')$. The estimate of γ' obtained from this simpler form may be used as a first approximation in solving equation (28) or (30) for the general case.

3.9. Since $P-Q = \eta_m \lambda \Phi'(t_P)$, the vertical distance of the curve from the egalitarian line is the same for P and $1-P$. This distance is 0 for $P = 0$ or 1, and increases towards a maximum of $\eta_m \lambda/\sqrt{2\pi}$ at $P = \frac{1}{2}$. This symmetry means that if the bottom 100 P% enjoy 100Q% of the item, the top 100P% enjoy 100(2$P-Q$)%. So these two groups together get their *due* share in the consumption of the item, and so do the persons occupying the middle range. Note that the 100P% point and the 100(1$-P$)% points on the distribution of x have e^{θ}, the median, as their geometric mean.

3.10. This symmetry presents an interesting contrast with the symmetry of the concentration curve for x viz., $t_{Q_x} = t_P-\lambda$, and of the specific concentration curve for y, when $E(y|x) = Ax^{\eta}$ viz., $t_Q = t_P-\lambda\eta$ [see Aitchison and Brown, 1957; Iyengar, 1960a]. In these cases, if the bottom 100P% get a 100Q% share, the top 100Q% get 100P%; so these two groups together get their due share. Note first that the bottom 100P% plus the top 100P% get more than 100(2P)%, their due share. Secondly, the 100P% point and the 100(1$-Q$)% points on the distribution of x have the arithmetic mean $e^{\theta+\lambda^2/2}$ as their geometric mean.

3.11. *The hyperbolic form.* Now consider the hyperbolic form $E(y|x) = \alpha-\frac{\beta}{x}$ ($\alpha, \beta > 0$), for necessaries. The threshold is at $x = \frac{\beta}{\alpha} = \gamma$ say ($\gamma > 0$). The variable elasticity is given by

$$\left.\begin{aligned} \eta_x &= \frac{d \log E(y|x)}{d \log x} = \frac{1}{\frac{x}{\gamma}-1} \qquad \text{for } x > \gamma \\ \text{and} \quad \eta_x &= 0 \quad \text{for } x < \gamma \end{aligned}\right\} \quad \ldots \quad (36)$$

So for x between γ and 2γ, $\eta_x > 1$, and beyond $x = 2\gamma$, $\eta_x < 1$.

3.12. As before, the abscissa of the specific concentration curve is given by

$$P(x) = \Phi(x \mid \theta, \lambda) = \Phi\left(\frac{\log x - \theta}{\lambda}\right) = \Phi(x')$$

say. The ordinate $Q(x) = 0$ upto $x = \gamma$, that is, upto $P = \Phi(\gamma')$,

where $$\gamma' = \frac{\log \gamma - \theta}{\lambda}.$$

If $P > \Phi(\gamma')$, that is, if $x > \gamma$, we get

$$Q(x) = \int_{\gamma}^{x}\left(\alpha - \frac{\beta}{x}\right) d\Lambda(x \mid \theta, \lambda) / \int_{\gamma}^{\infty}\left(\alpha - \frac{\beta}{x}\right) d\Lambda(x \mid \theta, \lambda).$$

The numerator comes out as

$$\alpha[\Phi(x') - \Phi(\gamma')] - \beta e^{-\theta + \lambda^2/2}[\Phi(x' + \lambda) - \Phi(\gamma' + \lambda)]. \qquad \ldots \quad (37)$$

The denominator, which is $E(y)$, is the limit of the numerator as $x' \to \infty$.

Thus $$E(y) = \alpha[1 - \Phi(\gamma')] - \beta e^{-\theta + \lambda^2/2}[1 - \Phi(\gamma' + \lambda)]. \qquad \ldots \quad (38)$$

The equation of the specific concentration curve is

$$Q = \frac{\alpha[P - \Phi(\gamma')] - \beta e^{-\theta + \lambda^2/2}[\Phi(t_p + \lambda) - \Phi(\gamma' + \lambda)]}{\alpha[1 - \Phi(\gamma')] - \beta e^{-\theta + \lambda^2/2}[1 - \Phi(\gamma' + \lambda)]}$$

$$= \frac{P - \Phi(\gamma') - \frac{\gamma}{H}[\Phi(t_p + \lambda) - \Phi(\gamma' + \lambda)]}{1 - \Phi(\gamma') - \frac{\gamma}{H}[1 - \Phi(\gamma' + \lambda)]} \quad \text{for } P \geqslant \Phi(\gamma')$$

$$= 0 \text{ for } P < \Phi(\gamma'). \qquad \ldots \quad (39)$$

Here H is the harmonic mean of x, i.e. $\frac{1}{H} = e^{-\theta + \lambda^2/2}$. Then H is the geometric mean of median and mode. One can put $\gamma = e^{\theta + \lambda\gamma'}$ in the above relation if necessary, so that $\gamma/H = e^{\lambda\gamma' + \lambda^2/2}$.

3.13. For a fixed P, Q decreases from P to 0 as γ' rises from $-\infty$ to t_p. Therefore every non-zero ordinate of this specific concentration curve gives a unique estimate of γ' when λ is known. Hence γ or $\frac{\beta}{\alpha}$ can be estimated if θ and λ are known. [Here again as γ' varies between $-\infty$ and $+\infty$, one gets a family of non-intersecting curves where the lower curves correspond to the higher values of γ'.]

3.14. The specific concentration coefficient involves an integral $\int_{\Phi(\gamma')}^{1} \Phi(t_P+\lambda)dP$ which could not be expressed in a simple form.

3.15. Suppose now that $\Phi(\gamma')$ and $\Phi(\gamma'+\lambda)$ are negligible, but γ' has some finite negative value.[10] We then get the following approximations :

$$\left.\begin{aligned} Q &= \frac{P-\dfrac{\gamma}{H}\,\Phi(t_P+\lambda)}{1-\gamma/H} \\ &= \frac{P-e^{\lambda\gamma'+\lambda^2/2}\Phi(t_P+\lambda)}{1-e^{\lambda\gamma'+\lambda^2/2}} \end{aligned}\right\} \text{ for all } P. \qquad \ldots \quad (40)$$

This approximation (40) also has the defect that Q has negative values, though small, for P in the neighbourhood of $\Phi(\gamma')$; however, as $P\to 0$, $Q\to 0$. Actually $\frac{dQ}{dP} = 0$ at $P = \Phi(\gamma')$; below this $\frac{dQ}{dP} < 0$ and above this $\frac{dQ}{dP} > 0$.

3.16. The specific concentration coefficient is obtained as

$$L = \frac{\dfrac{\gamma}{H}\left[2\Phi\left(\dfrac{\lambda}{\sqrt{2}}\right)-1\right]}{1-\gamma/H}. \qquad \ldots \quad (41)$$

Since $L_0 = 2\Phi\left(\frac{\lambda}{\sqrt{2}}\right)-1$, we get $L = \frac{\frac{\gamma}{H}L_0}{1-\gamma/H} \lesseqgtr L_0$ according as $\frac{\gamma}{H} \lesseqgtr \frac{1}{2}$. Naturally we expect $L < L_0$. In any case, we get

$$\frac{\gamma}{H} = e^{\lambda\gamma'+\lambda^2/2} = \frac{L}{L+L_0} \qquad \ldots \quad (42)$$

which gives an estimate of γ' when λ is known from the concentration curve for x.

3.17. Any ordinate of the curve also gives an estimate of γ' when λ is known. The estimate of γ or β/α requires further knowledge of θ. The estimate obtained from this special case can be used as a first approximation for solving the equation for the general case.

3.18. No difficulty about threshold arises with the form $E(y\,|\,x) = k\Lambda(x\,|\,\mu,\sigma)$, but the integral $\int_0^x \Lambda(x\,|\,\mu,\sigma)d\Lambda(x\,|\,\theta,\lambda)$ in the numerator of $Q(x)$ could not be obtained in a simple form. Neither could the specific concentration coefficient be evaluated directly.

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[10] This further assumption is made for results in paragraphs 3.15—3.17. The basic assumption of lognormality is, of course, there.

3.19. *The polynomial form.* Lastly, for the polynomial $E(y|x) = \sum_0^k a_r x^r$, if $\sum_0^k a_r x^r \geqslant 0$ for all x or if the ranges where $\sum a_r x^r$ is negative can be ignored, we get $P(x) = \Phi(x')$ as before and

$$Q(x) = \frac{\int_0^x (\Sigma a_r x^r)\, d\Lambda\,(x|\theta, \lambda)}{\int_0^\infty (\Sigma a_r x^r)\, d\Lambda\,(x|\theta, \lambda)}$$

$$= \frac{\Sigma a_r e^{r\theta + \frac{r^2}{2}\lambda^2}\, \Phi\,(x' - r\lambda)}{\Sigma a_r e^{r\theta + \frac{r^2}{2}\lambda^2}}$$

using the moment distribution property of the lognormal distribution (Aitchison and Brown, 1957) and the fact that $\int_0^\infty x^j d\Lambda\,(x|\theta, \lambda) = e^{j\theta + j^2\lambda^2/2}$. Thus, the equation of the specific concentration curve is

$$Q = \frac{\sum_0^k a_r e^{r\theta + \frac{r^2\lambda^2}{2}}\, \Phi\,(t_p - r\lambda)}{\sum_0^k a_r e^{r\theta + \frac{r^2}{2}\lambda^2}}\,. \qquad \ldots \quad (43)$$

Once θ and λ are known, any k ordinates of the curve give the ratios $a_0 : a_1 : \ldots : a_k$ on which the curve depends. The variable elasticity also depends on these relative magnitudes, for $\eta_x = \frac{\Sigma r a_r x^r}{\Sigma a_r x^r}$. The absolute values of the a_r's can be estimated by using equation (22) for $E(y)$.

3.20. The specific concentration coefficient is given by

$$L = \frac{\Sigma a_r e^{r\theta + \frac{r^2\lambda^2}{2}} \left[2\Phi\left(\frac{r\lambda}{\sqrt{2}}\right) - 1 \right]}{\Sigma a_r e^{r\theta + r^2\lambda^2/2}}. \qquad \ldots \quad (44)$$

4. An empirical illustration

4.1. The following data are based on the family budget enquiry carried out in the 7th Round of the Indian National Sample Survey (October 1953-March 1954). The coverage was all-India excluding Jammu and Kashmir, Andaman and Nicobar Islands and Sikkim. The data presented relate to the rural sector of all-India and are based on 1413 households selected from 954 villages. Foodgrains included cereals like rice, wheat, jowar etc., cereal products like chira, muri etc., and cereal substitutes (e.g. tapioca and peas).

	monthly per capita consumer expenditure (in Rs.) on all items												
	0-8	8-11	11-13	13-15	15-18	18-21	21-24	24-28	28-34	34-43	43-55	55-	all classes
1. estimated p.c. of population	15.47	17.80	12.04	10.31	10.83	8.75	6.94	5.77	4.82	3.73	1.97	1.57	100.00
2. consumer expenditure (in Rs.) per person per month on													
(a) foodgrains (y)	3.88	5.24	6.60	7.08	7.82	8.14	9.05	8.58	8.98	9.43	12.70	11.15	6.95
(b) all items (x)	6.20	9.63	11.92	13.96	16.21	19.06	22.26	26.06	30.53	36.89	48.98	80.05	17.24

4.2. The above forms a part of the material analyzed in Roy and Dhar (1960) where it was shown that the distribution of persons by monthly per capita expenditure follows the lognormal distribution to a close approximation. The concentration coefficient for total expenditure is found to be 0.3333. Using the relations

$$E(x) = e^{\theta+\lambda^2/2} \text{ and } L_0 = 2\Phi\left[\frac{\lambda}{\sqrt{2}}\right]-1,$$

we get the estimates $\theta = 2.6611$ and $\lambda = 0.6091$.

4.3. The Engel curve for foodgrains seems to be of the semi-log form; it shows appreciable curvature on the double-log scale. The specific concentration coefficient is found to be 0.1603. Using equation (33), we obtain $\gamma' = -3.5196$. The same value exactly satisfies the more general equation (30). Using θ and λ, we get $\log_e\gamma = -\dfrac{\alpha}{\beta} = 0.5173$ and $\gamma =$ Rs. 1.6775, which seems to be extremely reasonable. The elasticity at any point may now be determined from (29).

4.4. Using (34), we get the elasticity at median (Rs. 14.31) as 0.466 and the elasticity at mean (Rs. 17.24) comes out as 0.429. Roy and Dhar (1960) fitted the constant elasticity form by weighted least squares and obtained the constant elasticity as 0.516; the Tornquist form for necessaries gave a better fit and yielded an elasticity of 0.448 at the mean.

4.5. Equation (19) for $E(y)$ gives the estimates $\alpha = -1.6770$ and $\beta = 3.2419$.

4.6. Now suppose the distribution of x changes to another lognormal distribution with the mean higher by 10% and the concentration coefficient lower by 10% than those of the observed distribution. Putting

$$L_0^* = 2\Phi\left[\frac{\lambda^*}{\sqrt{2}}\right]-1 = 0{\cdot}9\, L_0 = 0{\cdot}9\,(0{\cdot}3333).$$

and $$E^*(x) = e^{\theta^* + \lambda^{*2}/2} = 1 \cdot 1\, E(x) = 1 \cdot 1(17 \cdot 24)$$

we got $\theta^* = 2.7934$ and $\lambda^* = 0.5449$. The recalculated γ' comes out as -4.1772. Substituting these in equation (19) we get the percentage increase $100\left[\frac{E^*(y)}{E(y)} - 1\right]$ as 6.2%.

4.7. If we ignore the change in concentration and use (13), we get this percentage as 4.45% only.

4.8. Suppose we try to use the constant elasticity form $E(y|x) = Ax^{\eta}$ to the Engel curve for foodgrains. We then get the estimate of η as 0.470 using the following equations[see Iyengar (1960b)]

$$L_0 = 0 \cdot 3333 = 2\Phi\left[\frac{\lambda}{\sqrt{2}}\right] - 1 \text{ and } L = 0 \cdot 1603 = 2\Phi\left[\frac{\eta\lambda}{\sqrt{2}}\right] - 1.$$

Next using the prediction formula given by Iyengar (1960a)

$$\frac{E^*(y)}{E(y)} = e^{\eta(\theta^* - \theta) + \frac{1}{2}\eta^2(\lambda^{*2} - \lambda^2)}$$

we get the percentage increase in per capita demand as 5.55%.

4.9. It appears that even when the Engel curve is not linear on the double-log scale, the use of the constant elasticity Engel curve would give fairly good projections of consumer demand. The variable elasticity forms, however, give a more reasonable picture of the variation of elasticity with income, and may give greater insight into the nature of the item by suggesting a threshold or a satiety level.

5. Acknowledgment

The author is indebted to Dr. J. Joy and to Shri Taresh Moitra for helpful discussions.

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Paper received : May, 1963.