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CONNECTIONS WITH PRESCRIBED FIRST PONTRJAGIN FORM

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ABSTRACT. Let P be a principal $O(n)$ bundle over a C^∞ manifold M of dimension m. If $n \geq 5m + 4 + 4\binom{m+1}{4}$, then we prove that every differential 4-form representing the first Pontrjagin class of P is the Pontrjagin form of some connection on P.

1. INTRODUCTION

Let P be a principal $O(n)$ bundle over a C^∞ manifold M of dimension m, and let $p_i \in H^{4i}(M)$ denote the i-dimensional Pontrjagin class of P. We address the question whether a $4i$-form representing the class p_i is a Pontrjagin form of some connection on P. In [1] we considered the top-dimensional Pontrjagin class p_d of a principal $O(n)$ bundle P over a $4d$-dimensional *open* manifold M for $n \geq 2d$, and we gave a homotopy classification of connections α on P that satisfy $p_d(\alpha) = \omega$, where ω is a volume form on M. In this paper, we take up the case of the first Pontrjagin form and prove the following result.

Theorem 1.1. *If $n \geq 5m + 4 + 4\binom{m+1}{4}$, then every differential 4-form representing the first Pontrjagin class p_1 is the Pontrjagin form of some connection on P. Moreover, when M is a closed manifold, the same is true for $n \geq 5m + 4\binom{m}{4}$.*

Here $\binom{m}{k}$ denotes the integer $\frac{m!}{k!(m-k)!}$.

We observe that when $n > m$, then P reduces to the direct sum $P_1 \oplus P_2$ of two principal bundles, where P_1 is an $O(m)$ bundle and P_2 is the trivial $O(n-m)$ bundle on M. Since the Pontrjagin form is additive, the above observation reduces the problem to finding a connection on a trivial principal bundle with a given exact form as its Pontrjagin form.

Now, if an exact 4-form on M can be expressed as the sum of q primary monomials of the form $df_1 \wedge df_2 \wedge df_3 \wedge df_4$, where the f_i's are smooth functions on M, then we can explicitly construct a connection on the trivial principal $O(2q)$-bundle over M by taking a 2×2 block

$$\alpha = \begin{pmatrix} 0 & f_1\, df_2 - f_3\, df_4 \\ -f_1\, df_2 + f_3\, df_4 & 0 \end{pmatrix}$$

along the principal diagonal for each such monomial. It can be seen easily that the Pontrjagin form of such a connection is the given exact form on M. Indeed, we can prove the following result (compare ([2], 3.4.1 (B$'$))).

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Theorem 1.2. *Every exact 4-form $d\omega$ on M can be expressed as the sum of q primary monomials for $q \geq 2(m+1) + 2\binom{m+1}{4}$. Furthermore, if M is a closed manifold, then the same is true for $q \geq 2m + 2\binom{m}{4}$.*

In view of the above discussion it is easy to see that Theorem 1.1 follows from Theorem 1.2.

We employ the sheaf-theoretic and analytic techniques of the theory of the h-principle [2] to prove the above result. We observe that an exact 4-form $d\omega$ can be expressed as the sum of q primary monomials if and only if there is a map $f : M \longrightarrow \mathbb{R}^{4q}$ such that

$$d\omega = \sum_{i=1}^{q} f_i^* \sigma,$$

where σ is the canonical volume form on $\mathbb{R}^4$ and $f_i : M \longrightarrow \mathbb{R}^4$, $i = 1, 2, \ldots, q$, are components of f. The maps characterized by the above equation are solutions to a certain first-order partial differential equation. The associated partial differential operator being infinitesimally invertible on an open subset, we apply Gromov's formulation of the Implicit Function Theorem in the infinite-dimensional setup to make way for the sheaf techniques.

In Section 2, following Gromov [2], we briefly describe the notion of infinitesimal inversion of partial differential operators and state some results relating to the solution sheaf of infinitesimally invertible operators. We shall assume that the reader is familiar with the language of the h-principle, in particular with the terms: partial differential relations, holonomic section, the h-principle, (micro)flexible sheaf and sharply moving diffeotopy. For a brief review of terminology and sheaf techniques in the h-principle we refer to the Appendix of [1]. In section 3 we consider immersions in a manifold N with a fixed k-form σ and prove the h-principle for "σ-regular" immersions that induce a given k-form on the source manifold. This has been shown by observing that the relevant differential operator is infinitesimally invertible on the space of "σ-regular" immersions. In section 4 we prove that if (N, σ) is the q-fold product of the k-dimensional Euclidean space with canonical volume form, then σ-regular immersions are generic for q sufficiently large. Using genericity of σ-regular maps, we then prove the second part of Theorem 1.2. Finally, by applying the h-principle of σ-regular maps (Section 3), we prove the full form of Theorem 1.2 and the main result of this paper.

2. Infinitesimal inversion of differential operators

Let $X \longrightarrow M$ be a C^∞ fibration and $G \longrightarrow M$ be a C^∞ vector bundle over a manifold M. We denote by $\mathcal{X}^\alpha$ and $\mathcal{G}^\alpha$ respectively the spaces of C^α sections of X and G with the fine C^α topology, for $\alpha = 1, 2, \ldots, \infty$. Let $\mathcal{D} : \mathcal{X}^r \longrightarrow \mathcal{G}^0$ be a C^∞ differential operator of order r, so that if x is a $C^{\alpha+r}$ section of X, then $\mathcal{D}(x)$ is a C^α section of G for $\alpha = 1, 2, \ldots, \infty$.

Let $T_{vert}(X) \subset TX$ denote the subspace of vertical vectors (i.e., tangent to the fibres of the fibration $X \longrightarrow M$) in the tangent bundle TX of X. For a section $x : M \longrightarrow X$, we denote the induced vector bundle $x^*T_{vert}(X)$ by Y_x. When x is C^α, this bundle is C^β-smooth for $\beta \leq \alpha$ and we denote by $\mathcal{Y}_x^\beta$ the space of C^β sections of this induced bundle. The space $\mathcal{Y}_x^\alpha$ can be realized as the infinite-dimensional tangent space of $\mathcal{X}^\alpha$ at x. We define the linearization L_x of the operator

$\mathcal{D}$ at x as follows:

$$L_x : \mathcal{Y}_x^r \longrightarrow \mathcal{G}^0,$$

$$L_x(y) = \tfrac{d}{dt}\mathcal{D}(x_t)|_{t=0},$$

where $\{x_t : t \geq 0\}$ is a 1-parameter family of sections of X with $x_0 = x$ and $\frac{dx_t}{dt}|_{t=0} = y$. Clearly, L_x is a linear differential operator of order r in y and $L(x,y) = L_x(y)$ is a differential operator of order r in both x and y.

Let $A \subset X^{(d)}$ be an open subset of the d-jet space of sections of X for some $d \geq r$. Following Gromov, we shall call such a subset an open *differential relation* of order d. A solution of A will also be referred to as an A-regular section of X. Let $\mathcal{A}$ denote the space of solutions of the relation A. Clearly, $\mathcal{A}$ is contained in $\mathcal{X}^d$, and $\mathcal{A}^{\alpha+d} = \mathcal{A} \cap \mathcal{X}^{\alpha+d}$ is an open subset of $\mathcal{X}^{\alpha+d}$ in the fine $C^{\alpha+d}$ topology.

$\mathcal{D}$ is said to be *infinitesimally invertible* over the subset $\mathcal{A} \subset \mathcal{X}$ if for every $x \in \mathcal{A}$ there is a linear differential operator $M_x : \mathcal{G}^s \longrightarrow \mathcal{Y}_x^0$ of a certain order s (independent of x) such that the following properties are satisfied:

(1) The global operator
$$M : \mathcal{A}^d \times \mathcal{G}^s \longrightarrow T(\mathcal{X}^0)$$
is a differential operator that is given by a C^∞ map $A \oplus G^{(s)} \longrightarrow T_{vert}(X)$, where $T(\mathcal{X}^0)$ denotes the tangent bundle of $\mathcal{X}^0$.

(2) $L(x, M(x,g)) = g$ for all $x \in \mathcal{A}^{d+r}$ and $g \in \mathcal{G}^{r+s}$. In other words, M_x is a right inverse of L_x.

The integer d is called the *defect* of the infinitesimal inversion M ([2], 2.3.1).

We now state an infinite-dimensional Implicit Function Theorem due to Gromov which generalizes Nash's theory in the context of differential operators.

Let $\mathcal{D}$ be a C^∞ differential operator of order r. Suppose $\mathcal{D}$ admits an infinitesimal inversion of order s and of defect d. Let us fix a Riemannian metric on M. Let $\alpha > \max(d, 2r+s)$.

Theorem 2.1 ([2], 2.3.2). *For every $x \in \mathcal{A}^\infty$ there exists a fine $C^{\alpha+s}$ neighbourhood $\mathcal{B}_x$ of the zero section in the space $\mathcal{G}^{\alpha+s}$ and an operator $\mathcal{D}_x^{-1} : \mathcal{B}_x \longrightarrow \mathcal{A}^\alpha$ such that*

(1) $\mathcal{D}_x^{-1}(0) = x$.
(2) *(Inversion property)* $\mathcal{D}(\mathcal{D}_x^{-1}(g)) = \mathcal{D}(x) + g$.
(3) *If $g \in \mathcal{B}_x$ is $C^{\beta+s}$-smooth, for $\beta \geq \alpha$, then $\mathcal{D}_x^{-1}(g)$ is C^β-smooth.*
(4) *(Locality) The value of $\mathcal{D}_x^{-1}(g)$ at any point $v \in M$ does not depend on the behaviour of x and g outside the unit ball $B_v(1)$ in M with centre v relative to the fixed metric on M.*

In particular, the operator $\mathcal{D} : \mathcal{A}^\infty \longrightarrow \mathcal{G}^\infty$ is an open map in the respective fine C^∞ topologies.

It is to be noted that the local inverse $\mathcal{D}_x^{-1}$ depends on the Riemannian metric on M. If we choose an appropriate Riemannian metric on M, then applying the locality property of the inverse in Theorem 2.1 we can prove

Proposition 2.2 ([2], 2.3.2). *If $\mathcal{D}$ is infinitesimally invertible, then the sheaf of A-regular solutions of the differential equation $\mathcal{D}(x) = g$ is microflexible.*

We now consider some partial differential relations which have the same C^∞ solutions, namely the solutions to the equation $\mathcal{D}(x) = g$. Let $\mathcal{R}^\alpha \subset X^{(\alpha+r)}$

consist of $(\alpha + r)$-jets of infinitesimal solutions of $\mathcal{D} = g$ of order α and let $\mathcal{R}^0$ be denoted as $\mathcal{R}$. Recall that x is an infinitesimal solution of $\mathcal{D} = g$ of order α if $\mathcal{D}(x) - g$ has zero α-jet. Define

$$\mathcal{R}_\alpha = \mathcal{R}^\alpha \cap (p_d^{\alpha+r})^{-1}(A),$$

where $p_d^{\alpha+r} : X^{(\alpha+r)} \longrightarrow X^{(d)}$ is the canonical projection map for $\alpha \geq d - r$. The relations $\mathcal{R}_\alpha$ have the same C^∞ solutions for all $\alpha \geq d-r$, namely the C^∞ solutions of the equation $\mathcal{D}(x) = g$ in $\mathcal{A}$ (such a solution, from now on, will be referred to as an A-regular solution of the equation).

Let Φ denote the sheaf of A-regular solutions of the equation $\mathcal{D}(x) = g$ with the C^∞ compact open toplogy and let Ψ_α be the sheaf of sections of $\mathcal{R}_\alpha$ with C^0 compact open topology. It is a consequence of Theorem 2.1 that

Proposition 2.3 ([2], 2.3.2). *If $\alpha \geq \max(d + s, 2r + 2s)$, then an infinitesimal solution of $\mathcal{R}_\alpha$ can be deformed to a local solution. Furthermore, the map $J : \Phi \longrightarrow \Psi_\alpha$, defined by $J(\phi) = j_\phi^{r+\alpha}$, is a local weak homotopy equivalence. In other words, $\mathcal{R}_\alpha$ satisfies the local h-principle.*

3. The h-principle of isometric σ-regular maps

We start with the following definition.

Definition 3.1 ([2], 3.4.1). *Let (N, σ) be a smooth manifold with a closed k-form σ. A smooth map $f : M \longrightarrow N$ is said to be* σ-regular *if for each $x \in M$, the map*

$$\begin{array}{rcl} I_\sigma : T_{f(x)}N & \longrightarrow & \Lambda^{k-1}(T_xM), \\ \partial & \mapsto & f^*(\partial.\sigma) \end{array}$$

is surjective for all $x \in M$.

A σ-regular map is necessarily an immersion.

Let ω be a given k-form on M for $k \geq 2$. We call a map $f : (M, \omega) \longrightarrow (N, \sigma)$ *isometric* if $f^*\sigma = \omega$. In this section we shall prove the h-principle for σ-regular isometric maps $(M, \omega) \longrightarrow (N, \sigma)$ in the following situation:

(1) both σ and ω are exact;
(2) $M = M_0 \times \mathbb{R}$;
(3) ω is induced from a k-form on M_0 by the projection map $p : M_0 \times \mathbb{R} \longrightarrow M_0$.

Let $\mathcal{D} : C^\infty(M, N) \longrightarrow \Omega^k(M)$ denote the first-order differential operator on the space of C^∞ maps $f : M \longrightarrow N$ with values in the space of k-forms $\Omega^k(M)$ defined by $\mathcal{D}(f) = f^*\sigma$. Since σ is a closed form, the sheaf of solutions of $\mathcal{D} = \omega$ is not microflexible ([2], 3.4.1). Now, suppose that $\sigma = d\sigma_1$ and $\omega = d\omega_1$ for some $(k-1)$-forms σ_1 and ω_1 on N and M respectively. If f is a smooth immersion such that $f^*\sigma = \omega$, then locally on any contractible set the above equation reduces to $f^*\sigma_1 + d\phi = \omega_1$ for some $(k-2)$-form ϕ on M. Conversely, if (f, ϕ) is a pair satisfying $f^*\sigma_1 + d\phi = \omega_1$, then $f^*\sigma = \omega$. Let

$$\bar{\mathcal{D}} : C^\infty(M, N) \times \Omega^{k-2}(M) \longrightarrow \Omega^{k-1}(M)$$

denote the differential operator defined by $\bar{\mathcal{D}}(f, \phi) = f^*\sigma_1 + d\phi$, where $f : M \longrightarrow N$ is a smooth map and ϕ is a differential $(k-2)$-form on M. Note that the pairs (f, ϕ) can be realized as sections of the fibre bundle $(M \times N) \oplus \Lambda^{k-2}(M)$ over M which will be denoted by E for future reference.

The linearization $L_{(f,\phi)}$ of the operator $\bar{\mathcal{D}}$ at (f,ϕ) can be obtained as follows: Consider a smooth 1-parameter family of sections $\{(f_t,\phi_t)\}$ in E such that $(f_0,\phi_0)=(f,\phi)$. Then

$$L_{(f,\phi)}(\partial,\tilde{\phi}) = \frac{d}{dt}\mathcal{D}(f_t,\phi_t)|_{t=0},$$

where $\partial = \frac{df_t}{dt}|_{t=0}$ and $\frac{d\phi_t}{dt}|_{t=0}=\tilde{\phi}$. Hence,

$$L_{(f,\phi)}(\partial,\tilde{\phi}) = f^*d(\partial.\sigma_1)+f^*(\partial.d\sigma_1)+d\tilde{\phi},$$

where ∂ is a vector field on N along f and $\tilde{\phi}$ is a $(k-2)$-form on M. The equation $L_{(f,\phi)}=\omega_1$ can be solved for $(\partial,\tilde{\phi})$ if the following system has a solution:

$$f^*(\partial.d\sigma_1)=\omega_1,$$
$$f^*(\partial.\sigma_1)+\tilde{\phi}=0.$$

Now the above system of equations is solvable for $(\partial,\tilde{\phi})$ if f is a σ-regular map. Thus the operator $\bar{\mathcal{D}}$ is infinitesimally invertible on all those (f,ϕ) for which f is σ-regular ([2]). Since σ-regularity is an open condition and depends only on the first jet of a map, the space of pairs (f,ϕ) for which f is σ-regular corresponds to the solution space of an open differential relation $A\subset E^{(1)}$, where $E^{(1)}$ is the 1-jet bundle of sections of the fibre bundle E mentioned above. Hence the operator $\bar{\mathcal{D}}$ has the zeroth-order inversion (i.e., $s=0$, where s is defined as in Section 2) with defect $d=1$.

Let Φ be the sheaf of σ-regular solutions of the equation $\mathcal{D}(f)=\omega$ and let $\bar{\Phi}$ be the sheaf of pairs (f,ϕ) satisfying the equation $\bar{\mathcal{D}}=\omega_1$ where f is σ-regular. There is a canonical map $\bar{\Phi}\longrightarrow\Phi$ that takes a pair (f,ϕ) onto f. Furthermore, $\Phi(x)$ has the same homotopy type as the space $\bar{\Phi}(x)$.

Let $\bar{\mathcal{R}}^\alpha\subset E^{(\alpha+1)}$ consist of $(\alpha+1)$-jets of infinitesimal solutions of $\bar{\mathcal{D}}=\omega_1$ of order α and let $\bar{\mathcal{R}}_\alpha=\bar{\mathcal{R}}^\alpha\cap(p_1^{\alpha+1})^{-1}(A)$, where $p_1^{\alpha+1}:E^{(\alpha+1)}\longrightarrow E^{(1)}$ is the canonical projection. The following proposition is a direct consequence of Proposition 2.2 and Proposition 2.3.

Proposition 3.2. (*i*) *The solution sheaf $\bar{\Phi}$ of $\bar{\mathcal{D}}=\omega_1$ is microflexible.*

(*ii*) *The 3-jet map $j^3:\bar{\Phi}(x)\longrightarrow\bar{\Psi}(x)$ is a weak homotopy equivalence for every $x\in M$, where $\bar{\Psi}$ is the sheaf of sections of $\bar{\mathcal{R}}_2$. In particular, if (f,ϕ) is an infinitesimal solution of order 2 of $\bar{\mathcal{D}}=\omega_1$ where f is also σ-regular, then (f,ϕ) can be homotoped to a local solution of the equation.*

Theorem 3.3. *Let σ be an exact k-form on N as above and let $M=M_0\times\mathbb{R}$. If the form $\omega=d\omega_1$ on M is induced from an exact k-form on M_0 by the projection map $p:M_0\times\mathbb{R}\longrightarrow M_0$, then every section of $\bar{\mathcal{R}}_2$ is homotopic to a holonomic section (in the space of continuous sections of $\bar{\mathcal{R}}_2$ with C^0 compact open topology).*

Proof. Let $\bar{\Psi}$ denote the sheaf of sections of the jet bundle $E^{(3)}$ with images in $\bar{\mathcal{R}}_2$. We shall prove that

$$j^3:\bar{\Phi}|_{M_0}\longrightarrow\bar{\Psi}|_{M_0}$$

is a weak homotopy equivalence.

First observe that the fibre-preserving diffeomorphisms of $M_0\times\mathbb{R}$ act on the sheaf $\bar{\Phi}$. To see this take a smooth immersion $f:M_0\times\mathbb{R}\longrightarrow N$ and a $(k-2)$-form ϕ such that $f^*\sigma_1+d\phi=\omega_1$, where $\omega_1=p^*\omega_0$ for some $(k-1)$-form on M_0. Let

$\delta : M_0 \times \mathbb{R} \longrightarrow M_0 \times \mathbb{R}$ be a fibre-preserving diffeomorphism so that $p \circ \delta = p$. Define the action of δ by

$$\delta.(f, \phi) \mapsto (f \circ \delta, \delta^* \phi).$$

Then,

$$(f \circ \delta)^* \sigma_1 + d(\delta^* \phi) = \delta^*(f^* \sigma_1 + d\phi) = \delta^* \omega_1 = \delta^* p^* \omega_0 = p^* \omega_0 = \omega_1.$$

Also, if f is σ-regular, then so is $f \circ \sigma$.

On the other hand, the fibre-preserving diffeotopies sharply move M_0 in $M_0 \times \mathbb{R}$ ([2], [1]). Since the sheaf $\bar{\Phi}$ is microflexible (Proposition 3.2), we conclude that the restriction of $\bar{\Phi}$ to M_0 is flexible ([2], 2.3.2,[1]).

A standard argument proves that the sheaf $\bar{\Psi}$ is flexible ([2], 1.4.2 (A′)) and Proposition 3.2 (ii) says that

$$j^3 : \bar{\Phi}(x) \longrightarrow \bar{\Psi}(x)$$

is a weak homotopy equivalence for every $x \in M$. Then by the Sheaf Homomorphism Theorem ([2], 2.2.1 (B))

$$j^3 : \bar{\Phi}|_{M_0} \longrightarrow \bar{\Psi}|_{M_0}$$

is a weak homotopy equivalence.

Finally, the theorem follows from the observation that $M_0 \times \mathbb{R}$ can be deformed into an arbitrary small neighbourhood of M_0 by means of fibre-preserving diffeomorphisms of M. □

Let $\mathcal{R}_1$ consist of 2-jets of σ-regular infinitesimal solutions of order 1 of the equation $\mathcal{D} = \omega$ and let $\Gamma(\mathcal{R}_1)$ denote the space of continuous sections of $\mathcal{R}_1$ with C^0 compact open topology. Then we have the following.

Corollary 3.4. *An arbitrary section of $\mathcal{R}_1 \subset J^2(M, N)$ is homotopic to a holonomic section in $\Gamma(\mathcal{R}_1)$. Hence, the σ-regular isometric C^∞ immersions $f : (M_0 \times \mathbb{R},\ d\omega_1 = d\omega_0 \oplus 0) \longrightarrow (N, d\sigma_1)$ satisfy the h-principle. Furthermore, if M is an open manifold, then σ-regular isotropic immersions satisfy the h-principle.*

Proof. Let (f, ϕ) be a second-order infinitesimal solution at x of the equation $\bar{\mathcal{D}} = \omega_1$. Then $j^2_{(f^*\sigma_1 - d\phi)} = j^2_{\omega_1}$ at x. There is a bundle map $\Delta_{k-1} : (\Lambda^{k-1}(M))^{(2)} \longrightarrow \Lambda^k(M)^{(1)}$ associated to the exterior differential operator d such that $\Delta_{k-1}(j^2_\tau) = j^1_{d\tau}$. Then applying Δ_{k-1} on the preceding equation we get $j^1_{f^*\sigma}(x) = j^1_\omega(x)$. Thus f is an infinitesimal solution of order 1 of the equation $\mathcal{D} = \omega$. Hence we have the canonical map $p : \bar{\mathcal{R}}_2 \longrightarrow \mathcal{R}_1$ that maps $(j^3_f(x), j^3_\phi(x))$ onto $j^2_f(x)$. We shall prove that this map is surjective and that fibres of p are affine subspaces. This would imply that p has a section, and then the first part of the corollary would follow from the above theorem.

To prove that p is surjective, consider the following sequence of vector bundles:

$$\ldots \longrightarrow \left(\Lambda^{k-2}(M)\right)^{(3)} \xrightarrow{\Delta_{k-2}} \left(\Lambda^{k-1}(M)\right)^{(2)} \xrightarrow{\Delta_{k-1}} \left(\Lambda^{k}(M)\right)^{(1)} \longrightarrow \ldots$$

where the bundle maps Δ_k are induced by the exterior differential operator d as $\Delta_k \circ j^i_\tau = j^{i-1}_{d\tau}$. By the formal Poincaré Lemma this sequence is exact.

Let f be a first-order infinitesimal solution of $\mathcal{D} = \omega$ at $x \in M$, which is also σ-regular, so that $j^2_f(x) \in \mathcal{R}_1$. Then $j^1_{f^*\sigma}(x) = j^1_\omega(x)$ and consequently $j^2_{f^*\sigma_1}(x) - j^2_{\omega_1}(x)$ is in $\ker \Delta_{k-1}$. Hence there exists a 3-jet $j^3_\phi(x)$ such that

$$j^2_{f^*\sigma_1}(x) - j^2_{\omega_1}(x) = \Delta_{k-2}(j^3_\phi(x)) = j^2_{d\phi}(x).$$

Therefore, $(j^3_f(x), j^3_\phi(x)) \in \bar{\mathcal{R}}_2$ and p is surjective.

Now let $j^2_f(x) \in \mathcal{R}_1$. Then $p^{-1}(j^2_f(x))$ consists of all pairs $(j^3_g(x), j^3_\phi(x)) \in E^{(3)}$ such that $j^2_g(x) = j^2_f(x)$ and $j^2_{d\phi}(x) = j^2_{g^*\sigma_1}(x) - j^2_{\omega_1}(x)$, equivalently, $j^3_\phi(x) \in \Delta^{-1}_{k-2}(j^2_{g^*\sigma_1}(x) - j^2_{\omega_1}(x))$. This shows that the fibres of p are affine subspaces and that $p : \bar{\mathcal{R}}_2 \longrightarrow \mathcal{R}_1$ is an affine bundle. This proves the first part of the corollary.

To prove the second part, one has to note, in addition, that the zero form is invariant under any diffeomorphism of M, and M can be deformed into an arbitrary small neighbourhood of its $(m-1)$-skeleton by an isotopy. $\square$

4. Existence of σ-regular immersions inducing ω

Let σ_0 be a closed k-form on a manifold N_0, and let N be the q-fold Cartesian product of N_0 with the k-form $\sigma = \sum_{i=1}^q \pi_i^*\sigma_0$, where $\pi_i : N \longrightarrow N_0$ is the projection onto the i-th factor. We first determine when the σ-regular maps exist generically and then prove the existence of isometric maps, applying the results obtained in the previous sections.

Definition 4.1. An immersion $f = (f_1, f_2, \dots, f_q) : M \longrightarrow N$ is said to be σ_0-*large* if $f_1^*\sigma_0, \dots, f_q^*\sigma_0$ span the k-th exterior bundle $\Lambda^k(M)$; this means, for every k-form ω on M, there exist continuous functions $\beta_i : M \longrightarrow \mathbb{R}$, $i = 1, \dots, q$, such that

$$\omega = \sum_{i=1}^q \beta_i f_i^*\sigma_0.$$

Let

$$\tilde{\mathcal{A}} = \{(\ell_1, \dots, \ell_q) \in J^1_x(M, N) : \ell_1^*\sigma_0, \dots, \ell_q^*\sigma_0 \text{ span } \Lambda^k(T_xM), x \in M\}.$$

If $f = (f_1, \dots, f_q)$ is a solution of $\tilde{\mathcal{A}}$, then $f_1^*\sigma_0(x), \dots, f_q^*\sigma_0(x)$ span $\Lambda^k(T_xM)$ for each $x \in M$. Moreover, it follows from the lemma below that the σ_0-large maps are precisely the solutions of the relation $\tilde{\mathcal{A}}$.

Lemma 4.2. *Let $\omega_1, \dots, \omega_q$ be k-forms on M such that for each $x \in M$, $\omega_1(x), \dots,$ $\omega_q(x)$ span $\Lambda^k(T_xM)$. Then $\omega_1, \dots, \omega_q$ span the space of k-forms $\Omega^k(M)$ over the ring of continuous functions on M.*

Since $\tilde{\mathcal{A}}$ is an open relation, the σ_0-large immersions form an open set in the fine C^∞ topology. Next we observe that

Proposition 4.3. *If $f = (f_1, \dots, f_q) : M \longrightarrow N$ is a σ_0-large immersion, then f is σ-regular.*

Proof. Let $f = (f_1, \dots, f_q)$ be a σ_0-large immersion of M into the q-fold product of N_0. If $\partial_1, \partial_2, \dots \partial_q$ are vector fields on M, then we have the relation

$$\sum_{i=1}^q \partial_i . f_i^*\sigma_0 = \sum_{i=1}^q f_i^*(\bar{\partial}_i . \sigma_0) = f^*((\bar{\partial}_1, \dots, \bar{\partial}_q).\sigma),$$

where $\bar{\partial}_i = (f_i)_*\partial_i$ is a vector field on N along f_i. The proposition now follows from the following simple observation.

Lemma 4.4. *If $\omega_1, \ldots, \omega_q$ are linear k-forms on $\mathbb{R}^m$ spanning $\Lambda^k(\mathbb{R}^m)$, then the linear map*

$$\begin{array}{rcl} \mathbb{R}^m \times \cdots \times \mathbb{R}^m & \longrightarrow & \Lambda^k(\mathbb{R}^{m-1}), \\ \partial_1, \ldots, \partial_q & \mapsto & \sum_{i=1}^q \partial_i.\omega_i \end{array}$$

is surjective.

In the rest of this article, M and (N, σ) will be as follows:

(1) M will denote a manifold of dimension m;
(2) N will denote the q-fold Cartesian product of the Euclidean space $\mathbb{R}^k$;
(3) σ will denote the k-form obtained by summing the q canonical volume forms $\sigma_k := dy_1 \wedge \cdots \wedge dy_k$ on each $\mathbb{R}^k$ factor, where $y_1, y_2, \ldots, y_k$ are the canonical coordinates on $\mathbb{R}^k$.

Proposition 4.5. *If $q \geq m + \binom{m}{k}$, then $f_1^*\sigma_k, \ldots, f_q^*\sigma_k$ span the k-th exterior bundle of M for generic $(f_1, \ldots, f_q) : M \longrightarrow N$. Consequently, if $q \geq 2m + 2\binom{m}{k}$, there exists a σ_k-large immersion $f : M \longrightarrow (N, \sigma)$ such that $f^*(\sigma) = 0$.*

Proof. Here $N = \mathbb{R}^{qk}$ and $\sigma = \bigoplus_{i=1}^q \sigma_k$. Fix a basis $e_1, e_2, \ldots, e_m$ for $\mathbb{R}^m$. Let L be a linear map from $\mathbb{R}^m$ to $\mathbb{R}^{qk}$. Then L can be expressed as $L = (L_1, L_2, \ldots, L_q)$, where L_i is the projection of L onto the i-th copy of $\mathbb{R}^k$.

If L is σ_k-large, then the forms $L_1^*\sigma_k, L_2^*\sigma_k, \ldots, L_q^*\sigma_k$ span the bundle $\Lambda^k(\mathbb{R}^m)$. Note that the $k \times k$ cofactors of L_i correspond to the values of $L_i^*\sigma_k$ on the k-tuples of basis vectors $(e_{i_1}, \ldots, e_{i_k})$, where $\{i_1, i_2, \ldots, i_k\}$ is an ordered subset of $\{1, 2, \ldots, m\}$. If $\bar{L}_i$ denotes the column vector formed by the $k \times k$ cofactors of the matrix L_i, then by a σ-large condition on L is meant that $\bar{L} = (\bar{L}_1, \ldots, \bar{L}_q)$ has the maximum rank. Let Σ' consist of all linear maps $L = (L_1, \ldots, L_q) : \mathbb{R}^m \longrightarrow \mathbb{R}^{qk}$ such that $\operatorname{rank} \bar{L}$ is strictly less than $l = \binom{m}{k}$; in other words, any $l \times l$ cofactor of $\bar{L}$ is zero. Therefore, Σ' is semialgebraic and hence stratified ([2], 1.3.1). Moreover, the codimension of Σ' in $L(\mathbb{R}^m, \mathbb{R}^{qk})$ is $q - \binom{m}{k} + 1$.

Let Σ be the subset of the 1-jet space $J^1(M, N)$ consisting of all 1-jets $j_f^1(x)$ such that $\{f_i^*\sigma_k : i = 1, 2, \ldots, q\}$ do not span $\Lambda_x^k(M)$. Hence a map $f : M \longrightarrow N$ is σ_k-large if its 1-jet map misses the set Σ. Since σ has global symmetry, the singular set Σ in the 1-jet space fibres over M and therefore it is stratified with codimension $q - \binom{m}{k} + 1$. Hence by the Thom Transversality Theorem, a generic map is σ_k-large if $q - \binom{m}{k} \geq m$.

Now, let $f = (f_1, \ldots, f_q) : M \longrightarrow \mathbb{R}^{qk}$ be a σ_k-large immersion; then define $\bar{f} = (\bar{f}_1, \ldots, \bar{f}_q)$ as follows:

$$\bar{f}_i = (f_{i2}, f_{i1}, f_{i3}, \ldots, f_{ik}),$$

where $f_i = (f_{i1}, f_{i2}, f_{i3}, \ldots, f_{ik}) : M \longrightarrow \mathbb{R}^k$. Note that $\bar{f}_i^*\sigma_k = -f_i^*\sigma_k$ for every i. Hence $(f, \bar{f}) : M \longrightarrow \mathbb{R}^{qk} \times \mathbb{R}^{qk}$ is a σ_k-large immersion of M into $\mathbb{R}^{2qk}$ that pulls back $\sigma \oplus \sigma$ onto the zero form on M. $\square$

Theorem 4.6. *Let M be a closed manifold. If $q \geq 2m + 2\binom{m}{k}$, then every exact form on M can be induced from σ by a σ-regular immersion $f : M \longrightarrow N$. Consequently, every exact k-form on a closed m-dimensional manifold is expressible as the sum of q primary monomials for $q \geq 2m + 2\binom{m}{k}$.*

Proof. Let $\tau = y_1\, dy_2 \wedge \cdots \wedge dy_k$ so that $\sigma_k = d\tau$. It follows from Section 3 and Proposition 4.5 that the operator

$$\bar{\mathcal{D}} : (f_1, f_2, \ldots, f_q, \phi) \mapsto \sum_{i=1}^{q} f_i^* \tau + d\phi$$

is infinitesimally invertible on σ_k-large immersions which exist generically for $q \geq m + \binom{m}{k}$. Hence by Theorem 2.1, the image of σ_k-large immersions under $\bar{\mathcal{D}}$ is a nonempty open set in the fine C^∞ topology for $q \geq m + \binom{m}{k}$. Moreover, when $q \geq 2m + 2\binom{m}{k}$, there exists a σ_k-large immersion $f = (f_1, \ldots, f_q) : M \longrightarrow \mathbb{R}^{kq}$ such that $\sum_{i=1}^{q} f_i^* \sigma_k = 0$, which implies that $\sum_{i=1}^{q} f_i^* \tau$ is a closed form. As a consequence, Image $\bar{\mathcal{D}}$ contains a closed $(k-1)$-form c.

Let M now be closed and let $\omega = d\alpha$ be exact. Then for sufficiently small $\lambda > 0$, $c + \lambda\alpha \in$ Image $\bar{\mathcal{D}}$. In other words, there exists a σ_k-large immersion $(g_1, g_2, \ldots, g_q)$ and a $(k-2)$-form ψ such that

$$c + \lambda\alpha = \sum_{i=1}^{q} g_i^* \tau + d\psi,$$

and therefore

$$\omega = \sum_{i=1}^{q} \left(\frac{1}{\sqrt[k]{\lambda}} g_i\right)^* \sigma_k.$$

Clearly, $(\frac{1}{\sqrt[k]{\lambda}} g_1, \frac{1}{\sqrt[k]{\lambda}} g_2, \ldots, \frac{1}{\sqrt[k]{\lambda}} g_q)$ is σ-regular and this completes the proof of the theorem. □

The next result is an immediate consequence of the above theorem.

Corollary 4.7. *If M is arbitrary, then every compactly supported exact k-form on M can be induced by a σ-regular immersion $f : M \longrightarrow (N, \sigma)$ for $q \geq 2m + 2\binom{m}{k}$.*

Corollary 4.8. *If P is a principal $O(n)$ bundle over a closed manifold M, then every compactly supported 4-form on M representing the first Pontrjagin class of P is the Pontrjagin form of some connection on P, for $n \geq 5m + 4\binom{m}{4}$.*

The proof of the above corollary will be similar to that of Corollary 4.10 and we omit it here.

Theorem 4.9 ([2], 3.4.1 (B′)). *Let (N, σ) be the q-fold product of $(\mathbb{R}^k, \sigma_k)$ for $k \geq 2$. If $q \geq 2(m+1) + 2\binom{m+1}{k}$, then an arbitrary exact k-form on M can be induced by a σ-regular immersion $f : M \longrightarrow N$. Therefore, every exact k-form on an m-dimensional manifold is expressible as the sum of q primary monomials for $q \geq 2(m+1) + 2\binom{m+1}{k}$.*

Proof. Let $x_1, x_2, \ldots, x_m$ denote a local coordinate system on M. Then a k-form ω on M can be represented locally as:

$$\omega = \sum_I \omega_I dx_I,$$

where I runs over all multi-indices $(i_1, i_2, \ldots, i_k)$ for $1 \leq i_1 < i_2 < \cdots < i_k \leq m$, $\omega_I = \omega_{i_1, i_2, \ldots, i_k}$ are smooth functions defined locally on M, and $dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k}$.

Recall that $\sigma = \bigoplus_{i=1}^{q} \sigma_k$, where σ_k is the canonical volume form on $\mathbb{R}^k$. If $f = (f_1, \ldots, f_q) : M \longrightarrow \mathbb{R}^{qk}$ is a smooth map, then $f^*\sigma = \omega$ defines for each multi-index $I = (i_1, i_2, \ldots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k \le m$ an equation E_I:

$$\sum_{j=1}^{q} \det \begin{pmatrix} \frac{\partial f_j}{\partial x_{i_1}} & \frac{\partial f_j}{\partial x_{i_2}} & \cdots & \frac{\partial f_j}{\partial x_{i_k}} \end{pmatrix}$$
$$= \sum_{j} \sum_{\alpha} (-1)^{\operatorname{sgn} \alpha} \frac{\partial f_{j\alpha_1}}{\partial x_{i_1}} \frac{\partial f_{j\alpha_2}}{\partial x_{i_2}} \cdots \frac{\partial f_{j\alpha_k}}{\partial x_{i_k}} = \omega_I,$$

where α represents an element of the symmetry group S_k on k letters $\{1, 2, \ldots, k\}$, and $\{f_{j\alpha}\}$ denote the components of f_j.

Differentiating E_I with respect to x_p, $p \in \{1, 2, \ldots, m\}$, we get an equation E_I^p:

$$\sum_{j=1}^{q} \sum_{\pi, \alpha} (-1)^{\operatorname{sgn} \alpha} \frac{\partial^2 f_{j\alpha_1}}{\partial x_p \partial x_{i_{\pi(1)}}} \frac{\partial f_{j\alpha_2}}{\partial x_{i_{\pi(2)}}} \cdots \frac{\partial f_{j\alpha_k}}{\partial x_{i_{\pi(k)}}} = \frac{\partial \omega_I}{\partial x_p},$$

where π is an element of the symmetry group S_k on k letters $\{1, 2, \ldots, k\}$.

The collection $\{f_j(x), \frac{\partial f_j}{\partial x_i}(x), \frac{\partial^2 f_j}{\partial x_i \partial x_p}(x)\}$ defines the 2-jet of the function $f : M \longrightarrow \mathbb{R}^{qk}$ at x. If f satisfies the equation $f^*\sigma = \omega$, then its 2-jet map satisfies the above system of equations.

Replacing the partial derivatives in the above equations by ordinary variables, namely substituting

$$\frac{\partial f_{j\alpha}}{\partial x_i} = v_i^{j\alpha}, \quad \frac{\partial^2 f_{j\alpha}}{\partial x_i \partial x_p} = v_{ip}^{j\alpha},$$

we obtain a system of equations $\{\bar{E}_I, \bar{E}_I^p\}$, where I runs over all multi-indices $(i_1, i_2, \ldots, i_k)$ with $1 \le i_1 < i_2 < \cdots < i_k \le m$. It can be verified that this system of equations is independent of coordinate transformation and defines the relation $\mathcal{R}^1$ in the 2-jet space.

Note that $\{\bar{E}_I^p\}$ is a system of $m\binom{m}{k}$ equations that are linear in the variables $v_{ip}^{j\alpha}$, the total number of which is $kqm(m+1)/2$. Let A denote the coefficient matrix of the vector $\{v_{ip}^{j\alpha}\}$ in the system $\{\bar{E}_I^p\}$. The system of equations $\{\bar{E}_I^p\}$ has a solution if the matrix A has the maximum rank everywhere. Since $k \ge 2$, the condition "rank $A <$ maximum" defines a stratified subset Σ in the 1-jet space $J^1(M, \mathbb{R}^{qk})$. If q is such that $kq(m+1)/2 \ge \binom{m}{k} + 1$, then codim $\Sigma > m$ and hence by the Thom Transversality Theorem, j_f^1 misses Σ for generic f. In other words, we get a map f for which the following system of equations has a solution for each $x \in M$:

$$\sum_{j=1}^{q} \sum_{\pi, \alpha} (-1)^{\operatorname{sgn} \alpha} \frac{\partial f_{j\alpha_2}}{\partial x_{i_{\pi(2)}}} \cdots \frac{\partial f_{j\alpha_k}}{\partial x_{i_{\pi(k)}}} v_{pi_{\pi(1)}}^{j\alpha_1} = \frac{\partial \omega_I}{\partial x_p}. \tag{1}$$

Moreover, the space of solutions is an affine subspace in $\mathbb{R}^d$ of codimension $m\binom{m}{k}$, where $d = kqm(m+1)/2$. Therefore, if $q \ge \binom{m+1}{k} + m + 1$, then there exists a map $f : M \times \mathbb{R} \longrightarrow (N, \sigma)$ that is a σ_k-large immersion and for which the system of equations (1) has a solution, say $v_{ip}^{j\alpha} = \bar{v}_{ip}^{j\alpha}$. Since f is σ_k-large, there exist continuous real-valued functions β_i on M such that $\omega = \sum_{i=1}^{q} \beta_i L_i^* \sigma_k$, where

L_i denotes the derivative map df_i. Define for each $i = 1, 2, \dots, q$, a bundle map $\bar{L}_i : TM \longrightarrow T\mathbb{R}^k$ by

$$\bar{L}_i(x) = \begin{cases} df_i(x) & \text{if } \beta_i(x) > 1, \\ d\bar{f}_i(x) & \text{if } \beta_i(x) < 1, \end{cases}$$

where $\bar{f}_i$ is obtained from f_i by interchanging the first two component functions. Take $T = (L_1, \dots, L_q, \bar{\beta}_1 \bar{L}_1, \dots, \bar{\beta}_q \bar{L}_q)$, where $\bar{\beta}_i = |\beta_i - 1|^{\frac{1}{k}}$. Note that T_i extends continuously over all of M if we define it to be identically zero on the set $\beta_i^{-1}(1)$. Thus we get a σ-regular bundle map $T : TM \longrightarrow T\mathbb{R}^{2q}$ such that $T^*\sigma = \omega$. We extend this (locally) to a section of $\mathcal{R}_1$ by taking $v_{ip}^{j\alpha} = \bar{v}_{ip}^{j\alpha}$ for $j \le q$ and $v_{ip}^{j\alpha} = 0$ for $j > q$. These local solutions finally define a global section of $\mathcal{R}_1$ if we patch them together by a partition of unity. (Note that the system of equations (1) is linear in $v_{ip}^{j\alpha}$.) We now conclude the existence of an isometric immersion by Theorem 3.3. $\square$

Theorems 4.9 and 4.6 prove Theorem 1.2.

Corollary 4.10. *Let P be a principal $O(n)$ bundle over a manifold M of dimension m, and let $n \ge 5m + 4 + 4\binom{m+1}{4}$. Then every 4-form on M representing the first Pontrjagin class of P is the Pontrjagin form of some connection on P.*

Proof. If $n > \dim M$, then P can be reduced to $P_1 \oplus P_2$, where P_1 is a principal $O(m)$ bundle and P_2 is the trivial $O(n-m)$ bundle over M. This may be seen easily if we view a principal $O(n)$ bundle as a frame bundle associated to some vector bundle of rank n. Moreover, we have a canonical inclusion $Q = P_1 \oplus P_2 \xrightarrow{i} P$ that takes the fibres of $P_1 \oplus P_2$ canonically into the fibres of P. Now we prove that the Pontrjagin forms of the bundles P and Q are the same. It is a standard fact that a connection α_Q on Q can be extended uniquely to a connection α_P on P such that $i^*\alpha_P = \alpha_Q$. We shall show that $p_1(\alpha_Q) = p_1(\alpha_P)$. We recall that the first Pontrjagin form $p_1(\alpha_Q)$ is uniquely determined by the equation

$$\pi_Q^* p_1(\alpha_Q) = \operatorname{trace}(D\alpha_Q \wedge D\alpha_Q), \tag{2}$$

where D stands for the covariant differentiation and π_Q denotes the projection map $Q \longrightarrow M$. Similarly, $\pi_P^* p_1(\alpha_P) = \operatorname{trace}(D\alpha_P \wedge D\alpha_P)$ ([3]). Taking pull-back by i we get $i^*\pi_P^* p_1(\alpha_P) = \operatorname{trace}(D\alpha_Q \wedge D\alpha_Q)$. Since $\pi_P \circ i = \pi_Q$, the left-hand side is equal to $\pi_Q^* p_1(\alpha_P)$. Hence by equation (2) and the uniqueness property, $p_1(\alpha_P) = p_1(\alpha_Q)$. Moreover, the Pontrjagin form is additive, so that if α_1 and α_2 are connections on P_1 and P_2, respectively, then $p_1(\alpha_1 \oplus \alpha_2) = p_1(\alpha_1) + p_1(\alpha_2)$. In view of the above observation it is enough to show that every exact form on M is the Pontrjagin form of some connection on the trivial principal $O(n)$ bundle for $n \ge 4(m+1) + 4\binom{m+1}{4}$.

Let $d\omega$ be an exact 4-form on M. We have proved in Theorem 4.9 that an exact 4-form on a manifold of dimension m can be expressed as the sum of q primary monomials for $q \ge 2(m+1) + 2\binom{m+1}{4}$. Let $d\omega = 2\sum_{i=1}^{q} df_{i1} \wedge df_{i2} \wedge df_{i3} \wedge df_{i4}$, where f_{ij} are smooth functions on M and where q satisfies the above relation. Now consider an $\mathfrak{o}(2q)$-valued 1-form α on M such that corresponding to each monomial $df_{i1} \wedge df_{i2} \wedge df_{i3} \wedge df_{i4}$ there exists a 2×2 block

$$\alpha_i = \begin{pmatrix} 0 & f_{i1}\, df_{i2} - f_{i3}\, df_{i4} \\ -f_{i1}\, df_{i2} + f_{i3}\, df_{i4} & 0 \end{pmatrix}$$

along the principal diagonal, all other elements being zero. Clearly α is a connection on the trivial principal $O(2q)$-bundle over M and its first Pontrjagin form is

$$\begin{aligned} p_1(\alpha) &= \sum_{i=1}^{q} p_1(\alpha_i) \\ &= \sum_{i=1}^{q} \operatorname{trace}(D\alpha_i \wedge D\alpha_i) \\ &= \sum_{i=1}^{q} \operatorname{trace}(d\alpha_i \wedge d\alpha_i) = d\omega. \end{aligned}$$

This completes the proof. □

Corollaries 4.8 and 4.10 prove Theorem 1.2.

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