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CONVERGENCE OF LOWER RECORDS AND INFINITE DIVISIBILITY

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Abstract

We study the properties of sums of lower records from a distribution on $[0, \infty)$ which is either continuous, except possibly at the origin, or has support contained in the set of nonnegative integers. We find a necessary and sufficient condition for the partial sums of lower records to converge almost surely to a proper random variable. An explicit formula for the Laplace transform of the limit is derived. This limit is infinitely divisible and we show that all infinitely divisible random variables with continuous Lévy measure on $[0, \infty)$ originate as infinite sums of lower records.

Keywords: Lower record; geometric distribution; infinite divisibility; asymptotic distribution; Lévy measure; self-decomposable; generalized gamma convolution

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1. Introduction

Let F be any distribution function on $[0, \infty)$. Let $\{Z_n : n \geq 0\}$ be a sequence of independent and identically distributed (i.i.d.) observations from F. We say that Z_j is a *lower record value* if

$$Z_j \leq \min\{Z_0, Z_1, \ldots, Z_{j-1}\}.$$

By convention, Z_0 is a record. (The *upper records* may be defined likewise by reversing the inequality above.) We note here that this definition is slightly different from the usual definition where a strict inequality is used. However, when F is continuous, the two definitions are equivalent. Furthermore, if F is continuous on $(0, \infty)$ and $F(0) > 0$, our lower records will be strictly decreasing, until they hit 0. Thereafter, with our definition, all subsequent records will assume the value 0. This contrasts with the usual definition, where there can be no new record. Since we are interested in the infinite sum of records, both the concepts yield the same results.

Define $L_0 := 1$ and, for $n \geq 1$,

$$L_n := \min\{j > L_{n-1} : Z_j \leq Z_{L_{n-1}}\}.$$

Define $X_n = Z_{L_n}$. Thus, $\{X_n : n \geq 0\}$ is the sequence of lower records of i.i.d. observations from the distribution F. Note that $\mathrm{P}(x, \mathrm{d}u) = \mathbf{1}_{\{u \leq x\}} F(\mathrm{d}u)/F(x)$ defines a transition function.

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It may be observed that $\{X_n : n \geq 0\}$ is a Markov process with this transition function and the initial distribution F.

The interest in asymptotic behaviour of record statistics can be traced back to Gnedenko [5]. Later, a thorough investigation was made by Resnick [8] who derived the class of all possible limit distributions of the nth upper record statistic as $n \to \infty$. His work has been followed by many others. In a recent work, Arnold and Villaseñor [1] derived some asymptotic properties of partial sums of the first n upper records. Bose *et al.* [3] in a subsequent article also dealt with sums of upper records and settled some of the questions raised in [1]. In this paper we study the properties of sums of lower records.

In Section 2, F is assumed to be continuous on $(0, \infty)$. In this case we obtain a necessary and sufficient condition for the almost sure convergence of the sum of the records. The limit turns out to be infinitely divisible. We obtain an explicit relation between the Lévy measure of the limit and the parent distribution F. This has several interesting consequences. First, we show how specific classes of infinitely divisible distributions, such as the self-decomposable class or the class of generalized gamma convolutions, arise from different classes of F. Furthermore, any infinitely divisible distribution on $[0, \infty)$, under a mild restriction, arises as a limit of the sums of lower records from a suitable F. It thus gives a method of simulating observations from a given infinitely divisible distribution which is known only through its Lévy measure. We are also able to derive conditions on F which guarantee the absolute continuity of the limit with respect to the Lebesgue measure.

In Section 3, the underlying distribution F is assumed to be discrete with support in $\mathbb{N}_0 := \{0, 1, 2, \dots\}$ and results similar to those in Section 2 are derived.

2. The continuous case

In this section, we assume that F is continuous on $(0, \infty)$. Note that 0 is the only possible point of discontinuity of F. The main results are Theorem 1 and Theorem 2 in Subsection 2.1. They provide necessary and sufficient conditions for the finiteness of $\sum_{n=0}^{\infty} X_n$ and establish the Laplace transform of the limit. Further properties of the limit, including its connection to the class of infinitely divisible distributions, are given in Subsection 2.2.

2.1. Convergence results

Obviously, X_n decreases as n increases, so it has a limit, almost surely, which can easily be shown to be degenerate at the infimum of the support of F. This is stated in the following proposition. Its proof is a rather simple consequence of the Borel–Cantelli lemma.

Proposition 1. *Let* $c_0 = \inf\{u : u \in \operatorname{supp}(F)\}$. *Then, as* $n \to \infty$, $X_n \downarrow c_0$ *almost surely.*

We are interested in the convergence of an infinite sum of the sequence of records $\{X_n : n \geq 0\}$. Therefore, we shall require in the sequel that $c_0 = 0$. Define $T_n := \sum_{k=0}^{n} X_k$. Since $X_n \geq 0$ for all $n \geq 0$, T_n is nondecreasing as n increases.

We now derive a necessary and sufficient condition for $T_n \to T$ such that $T < \infty$ almost surely. If $F(0) > 0$, define $N := \min\{n \geq 0 : X_n = 0\}$. Then N will be dominated by a geometric random variable with probability of success being $F(0)$, and so N is finite almost surely. Therefore, the infinite sum becomes a finite sum and hence $T_n \to T$ where $T < \infty$ almost surely. Thus, the more interesting case is when $F(0) = 0$ but $F(x) > 0$ for all $x > 0$. *Henceforth, we assume this.*

First, we note that, since T_n is nondecreasing in n, it must have a limit, albeit ∞. Therefore, $T_n \to T$ almost surely where T may take the value $+\infty$ with positive probability. If we show that $T_n \Rightarrow V_F$ where $V_F < \infty$ almost surely, we will have shown that $T \stackrel{\text{D}}{=} V_F$ and therefore $T < \infty$ almost surely. Here, $\Rightarrow$ denotes weak convergence and $\stackrel{\text{D}}{=}$ denotes equality in distribution.

For fixed $t > 0$, define

$$\psi_t^{(n)}(x) := \mathrm{E}(\exp(-tT_n) \mid X_0 \le x) = \mathrm{E}\biggl(\exp\biggl(-t\sum_{k=0}^{n} X_k\biggr) \biggm| X_0 \le x\biggr) \quad \text{for } x > 0.$$

We first show that, for fixed $n \ge 0$, $\psi_t^{(n)}(x)$ is decreasing in x for $x > 0$. First, note that, when $x_1 > x_2 > 0$,

$$\begin{aligned}
&F(x_1)F(x_2)[\psi_t^{(0)}(x_1) - \psi_t^{(0)}(x_2)]\\
&\quad= \biggl[F(x_2)\int_0^{x_1} \exp(-tu)F(\mathrm{d}u) - F(x_1)\int_0^{x_2} \exp(-tu)F(\mathrm{d}u)\biggr]\\
&\quad= \biggl[F(x_2)\int_{x_2}^{x_1} \exp(-tu)F(\mathrm{d}u) - (F(x_1) - F(x_2))\int_0^{x_2} \exp(-tu)F(\mathrm{d}u)\biggr]\\
&\quad\le \biggl[F(x_2)\exp(-tx_2)\int_{x_2}^{x_1} F(\mathrm{d}u) - (F(x_1) - F(x_2))\exp(-tx_2)\int_0^{x_2} F(\mathrm{d}u)\biggr]\\
&\quad\le 0.
\end{aligned}$$

Thus, $\psi_t^{(0)}$ is decreasing. To apply induction, assume that $\psi_t^{(n)}$ is decreasing. Then, when $x_1 > x_2 > 0$,

$$\begin{aligned}
&F(x_1)F(x_2)\bigl[\psi_t^{(n+1)}(x_1) - \psi_t^{(n+1)}(x_2)\bigr]\\
&\quad= \biggl[F(x_2)\int_0^{x_1} \exp(-tu)\psi_t^{(n)}(u)F(\mathrm{d}u) - F(x_1)\int_0^{x_2} \exp(-tu)\psi_t^{(n)}(u)F(\mathrm{d}u)\biggr]\\
&\quad= \biggl[F(x_2)\int_{x_2}^{x_1} \exp(-tu)\psi_t^{(n)}(u)F(\mathrm{d}u)\\
&\qquad\quad - (F(x_1) - F(x_2))\int_0^{x_2} \exp(-tu)\psi_t^{(n)}(u)F(\mathrm{d}u)\biggr]\\
&\quad\le \biggl[F(x_2)\exp(-tx_2)\psi_t^{(n)}(x_2)\int_{x_2}^{x_1} F(\mathrm{d}u)\\
&\qquad\quad - (F(x_1) - F(x_2))\exp(-tx_2)\psi_t^{(n)}(x_2)\int_0^{x_2} F(\mathrm{d}u)\biggr]\\
&\quad\le 0.
\end{aligned}$$

Hence, $\psi_t^{(n)}$ is a decreasing function of x for each n.

Since $X_n \ge 0$, it follows that, for every fixed x, $\psi_t^{(n)}(x)$ is decreasing in n. Hence, for fixed $x > 0$, the limit

$$\psi_t(x) := \lim_{n\to\infty} \psi_t^{(n)}(x)$$

exists and lies between 0 and 1. Furthermore, since each $\psi_t^{(n)}$ is decreasing in x, $\psi_t(x)$ is also a decreasing function of x. Define

$$\psi_t(\infty) := \lim_{x\uparrow\infty} \psi_t(x).$$

Next, we claim that

$$\lim_{n\to\infty} \mathrm{E}(\exp(-tT_n)) = \psi_t(\infty).$$

To prove this, fix any $\varepsilon > 0$ and choose y so large that $F(y) > 1-\varepsilon$ and $|\psi_t(\infty) - \psi_t(y)| < \varepsilon$. Then,

$$\begin{aligned}
&|\,\mathrm{E}(\exp(-tT_n)) - \psi_t(\infty)| \\
&\quad\le |\,\mathrm{E}(\exp(-tT_n)\,\mathbf{1}_{\{X_0\le y\}}) - \psi_t(y)| + \mathrm{P}(X_0 > y) + |\psi_t(\infty) - \psi_t(y)| \\
&\quad\le F(y)|\psi_t^{(n)}(y) - \psi_t(y)| + \psi_t(y)|1 - F(y)| + 2\varepsilon.
\end{aligned}$$

Now, for this fixed y, choose N so large that $|\psi_t^{(n)}(y) - \psi_t(y)| \le \varepsilon$ for all $n \ge N$. Therefore, for all $n \ge N$, we have $|\,\mathrm{E}(\exp(-tT_n)) - \psi_t(\infty)| \le 4\varepsilon$, proving the claim.

Now, conditioning on X_0 and remembering that $\{X_n : n \ge 0\}$ is a Markov process with transition probabilities $\mathrm{P}(x, \mathrm{d}u) = \mathbf{1}_{\{u\le x\}}\, F(\mathrm{d}u)/F(x)$, we have

$$\begin{aligned}
\psi_t^{(n)}(x) &= \frac{\int_0^x \mathrm{E}(\exp(-t\sum_{j=0}^n X_j) \mid X_0 = u)F(\mathrm{d}u)}{F(x)} \\
&= \frac{\int_0^x \mathrm{E}(\exp(-tu - t\sum_{j=1}^n X_j) \mid X_1 \le u)F(\mathrm{d}u)}{F(x)} \\
&= \frac{\int_0^x \exp(-tu)\psi_t^{(n-1)}(u)F(\mathrm{d}u)}{F(x)}. \qquad (1)
\end{aligned}$$

Letting $n \to \infty$ and applying the dominated convergence theorem on the right-hand side of (1), we see that ψ_t satisfies the following integral equation:

$$F(x)\xi(x) = \int_0^x \exp(-tu)\xi(u)F(\mathrm{d}u). \qquad (2)$$

Since F is continuous on $(0, \infty)$ and ψ_t is bounded, from the above equation it easily follows that ψ_t is continuous on $(0, \infty)$. Furthermore, by setting

$$\psi_t(0) := \lim_{x\downarrow 0} \psi_t(x),$$

we see that ψ_t is a continuous function on $[0, \infty)$ taking values in $[0, 1]$. Henceforth, when we talk about solutions to this and similar integral equations, we shall always restrict to *bounded continuous* solutions.

We are now in a position to establish the following result.

Theorem 1. *Suppose that $F(0) = 0$, $F(x) > 0$ for $x > 0$, F is continuous on $(0, \infty)$ and*

$$\int_0^1 \frac{uF(\mathrm{d}u)}{F(u)} < \infty. \qquad (3)$$

Then

$$\psi_t(x) = \exp\biggl(\int_0^x \frac{(\exp(-tu) - 1)F(\mathrm{d}u)}{F(u)}\biggr).$$

Furthermore, $T_n \Rightarrow V_F$, where V_F is a nonnegative proper random variable whose Laplace transform for all $t > 0$ is given by

$$\phi_F(-t) := \mathrm{E}(\exp(-tV_F)) = \exp\biggl(\int_0^\infty \frac{(\exp(-tu) - 1)F(\mathrm{d}u)}{F(u)}\biggr). \qquad (4)$$

Proof. To prove the result, it suffices to show that

(i) $\psi_t(\infty)$ is given by (4), and

(ii) as a function of $t > 0$, $\psi_t(\infty)$ is the Laplace transform of a nonnegative random variable whose distribution is proper; that is, without mass at infinity.

We first assume (i) and show (ii). By [2, p. 8], for instance, it suffices to show that

(a) $\psi_t(\infty)$ is completely monotone, that is,

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}\psi_t(\infty) \geq 0 \quad \text{for } n \geq 1,$$

(b) $\lim_{t\to 0}\psi_t(\infty) = 1$.

To show (a), differentiating under the integral, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(\infty) = -\psi_t(\infty)\int_0^\infty \frac{\exp(-tu)uF(\mathrm{d}u)}{F(u)}.$$

Note that this is permissible when the integral $\int_0^\infty F(\mathrm{d}u)\exp(-tu)u/F(u)$ is finite. But, since $\exp(-tu)u \to 0$ as $u \to \infty$, we can say that $\exp(-tu)u \leq C_1\min(u, 1)$ for some $C_1 > 0$. Thus, using the condition (3), we have $\int_0^\infty F(\mathrm{d}u)\exp(-tu)u/F(u) < \infty$.

Now, consider the function $s(t) = \int_0^\infty \exp(-tu)uF(\mathrm{d}u)/F(u)$. It is easy to see that $s(t)$ is infinitely differentiable using similar arguments. Furthermore, for all $n \geq 0$,

$$(-1)^n \frac{\mathrm{d}^n}{\mathrm{d}t^n}s(t) = \int_0^\infty \frac{\exp(-tu)u^{n+1}F(\mathrm{d}u)}{F(u)} \geq 0.$$

Now assume that

$$(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi_t(\infty) \geq 0 \quad \text{for } k = 1, 2, \dots, n.$$

We then have

$$\begin{aligned}(-1)^{n+1}\frac{\mathrm{d}^{n+1}}{\mathrm{d}t^{n+1}}\psi_t(\infty) &= (-1)^n\frac{\mathrm{d}^n}{\mathrm{d}t^n}(\psi_t(\infty)s(t)) \\ &= (-1)^n\sum_{k=0}^{n}\binom{n}{k}\frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi_t(\infty)\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}s(t) \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{\mathrm{d}^k}{\mathrm{d}t^k}\psi_t(\infty)(-1)^{n-k}\frac{\mathrm{d}^{n-k}}{\mathrm{d}t^{n-k}}s(t) \\ &\geq 0.\end{aligned}$$

We now show (b). Clearly, as $t \to 0$, we have $(1-\exp(-tu))/F(u) \to 0$. For $t < 1$, we have $1 - \exp(-tu) \leq \min(1, u)$. From the condition (3), we get that $\int_0^\infty \min(1, u)F(\mathrm{d}u)/F(u) < \infty$. Therefore, by the dominated convergence theorem, we obtain that $\psi_t(\infty) \to 1$ as $t \to 0$.

Finally we show (i). For all $x \geq 0$, define

$$g(x) := \exp\biggl(-\int_0^x \frac{(1-\exp(-tu))F(\mathrm{d}u)}{F(u)}\biggr).$$

Note that to show (i), it is enough to show that $\psi_t = g$. Towards this end, we show that

(A) all solutions of (2) which are continuous on $[0, \infty)$ and assume values in $[0, 1]$ are multiples of g, and

(B) ψ_t is the *largest* among all such solutions.

It is easier to establish (B). If η is any such solution of (2), then

$$\begin{aligned}\psi_t^{(0)}(x) &= \frac{1}{F(x)}\int_0^x \exp(-tu)F(\mathrm{d}u)\\ &\geq \frac{1}{F(x)}\int_0^x \exp(-tu)\eta(u)F(\mathrm{d}u)\\ &= \eta(x).\end{aligned}$$

Using (1) and repeating the above arguments, it follows that $\psi_t^{(n)} \geq \eta$ for every $n \geq 0$. Thus, $\psi_t \geq \eta$.

We now establish (A). By using (3) it is easy to see that the nonnegative expression g is continuous, decreasing in x and takes values in $[0, 1]$. Note that $g(\mathrm{d}u) = -F(\mathrm{d}u)g(u) \times (1 - \exp(-tu))/F(u)$. Now, using the integration by parts formula, we have

$$\begin{aligned}&\frac{1}{F(x)}\int_0^x \exp(-tu)g(u)F(\mathrm{d}u)\\ &\quad= \frac{1}{F(x)}\bigg[\int_0^x F(u)\frac{(\exp(-tu)-1)g(u)F(\mathrm{d}u)}{F(u)} + \int_0^x g(u)F(\mathrm{d}u)\bigg]\\ &\quad= \frac{1}{F(x)}\bigg[\int_0^x F(u)g(\mathrm{d}u) + \int_0^x g(u)F(\mathrm{d}u)\bigg]\\ &\quad= \frac{1}{F(x)}(g(x)F(x) - g(0)F(0))\\ &\quad= g(x)\end{aligned}$$

since $F(0) = 0$. Thus, g is a solution of (2).

Now, we prove that all bounded continuous solutions of (2) are given by constant multiples of g. Indeed, it is easy to see that all constant multiples of g are solutions of (2). Conversely, suppose that η is a solution of (2). First assume that $\eta(0) = 0$. Now fix any $\varepsilon \in (0, 1)$. Using the continuity of η at 0, choose $\delta > 0$ such that $|\eta(x)| < \varepsilon\|\eta\|$ whenever $0 \leq x \leq \delta$, where $\|\eta\| = \sup\{|\eta(x)| : x \in [0, \infty)\}$. Now, for any $x > \delta$,

$$\begin{aligned}|\eta(x)| &= \frac{1}{F(x)}\bigg|\int_0^x \exp(-tu)(\eta(u))F(\mathrm{d}u)\bigg|\\ &\leq \frac{1}{F(x)}\int_0^x \exp(-tu)|\eta(u)|F(\mathrm{d}u)\\ &= \frac{1}{F(x)}\bigg[\int_0^\delta \exp(-tu)|\eta(u)|F(\mathrm{d}u) + \int_\delta^x \exp(-tu)|\eta(u)|F(\mathrm{d}u)\bigg]\\ &\leq \frac{1}{F(x)}[\varepsilon\|\eta\|F(\delta) + \exp(-t\delta)\|\eta\|(F(x) - F(\delta))]\\ &\leq \max(\varepsilon, \exp(-t\delta))\|\eta\|.\end{aligned}$$

Thus,

$$\|\eta\| = \sup\{|\eta(x)| : x \in [0, \infty)\} \le \max(\varepsilon, \exp(-t\delta))\|\eta\|.$$

Since $\max(\varepsilon, \exp(-t\delta)) < 1$, this yields that $\|\eta\| = 0$.

Now assume that η is a solution with $\eta(0) \neq 0$. Then $\eta - \eta(0)g$ is also a solution of (2) and $(\eta - \eta(0)g)(0) = 0$. Therefore, from the previous argument, we must have $\eta - \eta(0)g \equiv 0$. In other words, $\eta = \eta(0)g$. This establishes the result completely.

Remark 1. So far we have assumed that $F(0) = 0$. Suppose now that $F(0) > 0$. We have already remarked that $T < \infty$ almost surely in this case. Also, note that the condition (3) is also satisfied. Denoting by $\psi_t(x)$ the conditional Laplace transformation of T given that $X_0 \le x$, we note that $\psi_t(0) = 1$ and, for any $x > 0$,

$$\psi_t(x) = \frac{1}{F(x)}\bigg[\int_{(0,x]} \exp(-tu)\psi_t(u)F(\mathrm{d}u) + F(0)\psi_t(0)\bigg].$$

Following the same method, we can also solve this integral equation to show that

$$\psi_t(x) = \exp\bigg(-\int_{(0,x]} \frac{(1-\exp(-tu))F(\mathrm{d}u)}{F(u)}\bigg) = \exp\bigg(-\int_0^x \frac{(1-\exp(-tu))F(\mathrm{d}u)}{F(u)}\bigg).$$

Remark 2. Let $\xi(t) = \mathrm{E}(\exp(\mathrm{i}tT))$ be the characteristic function of T. It is easy to prove that, if (3) holds, then

$$\xi(t) = \exp\bigg(\int_0^\infty \frac{(\exp(\mathrm{i}tu) - 1)F(\mathrm{d}u)}{F(u)}\bigg).$$

We now prove the 'converse' of Theorem 1.

Theorem 2. *Suppose that $F(0) = 0$, $F(x) > 0$ for $x > 0$, F is continuous on $(0, \infty)$ and*

$$\int_0^1 \frac{uF(\mathrm{d}u)}{F(u)} = \infty. \tag{5}$$

Then

$$T_n \to \infty \quad \text{as } n \to \infty \text{ almost surely.}$$

Proof. As discussed earlier, it is enough to show that $T_n \Rightarrow \infty$. For this, it is enough to show that the only bounded continuous solution of the integral equation (2) is zero. We follow arguments similar to those in the proof of Theorem 1 and use similar integral equations.

Let η be any bounded solution of (2). Now, fix $\delta > 0$ and choose $x_0 \in (0, \delta)$. Consider the following integral equation: for $x > x_0$,

$$\xi(x) = \frac{1}{F(x)}\bigg[\int_{x_0}^x \exp(-tu)\xi(u)F(\mathrm{d}u) + F(x_0)\xi(x_0)\bigg] \tag{6}$$

with the boundary condition $\xi(x_0) = \eta(x_0)$. Clearly, η restricted to $[x_0, \infty)$ is a solution of (6).

The uniqueness argument in the previous theorem can be easily modified to prove that (6) has a unique solution.

Now consider the continuous function

$$g_{x_0} : [x_0, \infty) \to [0, 1]; \quad x \mapsto \eta(x_0)\exp\bigg(-\int_{x_0}^x \frac{(1-\exp(-tu))F(\mathrm{d}u)}{F(u)}\bigg).$$

Following arguments given before, it is easy to verify that g_{x_0} satisfies (6). Therefore, $\eta(x) = g_{x_0}(x)$ for all $x > x_0$, and thus

$$\eta(\delta) = \eta(x_0)\exp\biggl(-\int_{x_0}^{\delta}\frac{(1-\exp(-tu))F(\mathrm{d}u)}{F(u)}\biggr). \tag{7}$$

This is true whenever $0 < x_0 < \delta$. Now, let $x_0 \to 0$. Then, by the condition (5),

$$\int_{x_0}^{\delta}\frac{(1-\exp(-tu))F(\mathrm{d}u)}{F(u)} \to \infty$$

while $\eta(x_0)$ remains bounded. Hence, the right-hand side of (7) tends to zero. This implies that $\eta(\delta) = 0$. But δ was arbitrary, so $\eta \equiv 0$, proving the theorem.

Example 1. The condition (3) is quite easy to verify for a large class of distributions. For example, if F admits a density f in a neighbourhood of the origin such that, for $u > 0$, $C_2u^{\gamma-1} \le f(u) \le C_3u^{\gamma-1}$ where $0 < C_2 \le C_3 < \infty$ and $\gamma > 0$, then $T < \infty$. Similarly, if $F(u) = \exp(-C_4u^{-\gamma})$ for $u > 0$ with $F(0) = 0$ where C_4 and $\gamma > 0$, then $T < \infty$ if and only if $\gamma < 1$.

Remark 3. Monotone transforms will preserve the records. In other words, if h is a continuous, strictly increasing function on $[0, \infty)$ with $h(0) = 0$, then the sequence $\{h(X_n) : n \ge 0\}$ will represent the sequence of records from the distribution $F \circ h^{-1}$. Using this identification, we can obtain that $\sum_{n=0}^{\infty} h(X_n) < \infty$ almost surely if and only if

$$\int_0^1 \frac{h(u)F(\mathrm{d}u)}{F(u)} < \infty.$$

2.2. Properties of the infinite sum

In this subsection, we will consider the properties of the distribution of T. So, we will assume throughout the subsection that the convergence criterion is satisfied, i.e. $\int_0^1 uF(\mathrm{d}u)/F(u) < \infty$. From (2.3.1) of [2], a random variable taking values in $\mathbb{R}_+$ is infinitely divisible if and only if the Laplace transform can be expressed as

$$m(-t) = \exp\biggl(-at + \int_{(0,\infty)}(\exp(-tu) - 1)L(\mathrm{d}u)\biggr),$$

where $t > 0$, $a \ge 0$ and L, the Lévy measure, is nonnegative and satisfies

$$\int_{(0,\infty)} \min(1, u)L(\mathrm{d}u) < \infty.$$

Setting $a = 0$ and $L(\mathrm{d}u) = F(\mathrm{d}u)/F(u) = \log F(\mathrm{d}u)$, we can write the Laplace transform of T in (4) in the above form. Thus, we have the following result.

Proposition 2. *The distribution of T is infinitely divisible.*

One question that arises now is the following: what is the class of infinitely divisible distributions which arise in this way? To answer this question, we observe first that, for any $x > 0$, the function $h_L : (0, \infty) \to \mathbb{R}_+$ defined by

$$h_L(x) := L([x, \infty)) = \int_x^{\infty}\frac{F(\mathrm{d}u)}{F(u)} = -\log F(x) \tag{8}$$

is a continuous function of x.

Now, suppose that a given infinitely divisible distribution has L as its Lévy measure. Then it satisfies the condition that $\int_{(0,\infty)} \min(1, u)L(\mathrm{d}u) < \infty$. This implies that $L([x, \infty)) < \infty$ for any $x > 0$. Furthermore, suppose that L satisfies the property that the function $h_L(x) = L([x, \infty))$ is continuous in x. Given any such L, define $F : [0, \infty) \to [0, 1]$ by (8), i.e. $F(0) = \exp(-L((0, \infty)))$ and, for any $x > 0$,

$$F(x) = \exp(-h_L(x)) = \exp(-L([x, \infty))). \tag{9}$$

Define $F(x) = 0$ for all $x < 0$.

Clearly, F is a nondecreasing function. Since h_L is continuous on $(0, \infty)$, so is F on $(0, \infty)$. Furthermore, $\lim_{x\to\infty} F(x) = \lim_{x\to\infty} \exp(-h_L(x)) = 1$. Also, note that F is right-continuous at 0 since $\lim_{x\downarrow 0} L([x, \infty)) = L((0, \infty))$. The only possible point of discontinuity of F is at 0. Clearly, if the mass of L is infinite, i.e. if $L((0, \infty)) = \infty$, then $F(0) = 0$; hence F is continuous. However, if $L((0, \infty)) < \infty$, we have that $F(0) > 0$ and hence F admits a point of discontinuity at 0.

Now, suppose that we have a sequence of records $\{X_n : n \geq 0\}$ from the distribution F. First, we note that, from (9), $L(\mathrm{d}u) = F(\mathrm{d}u)/F(u)$ on the set $\{u > 0\}$. Thus,

$$\begin{aligned}
\int_0^1 \frac{uF(\mathrm{d}u)}{F(u)} &= \int_{(0,1)} u \log F(\mathrm{d}u) \\
&\leq \int_{(0,1]} u \log F(\mathrm{d}u) + \int_1^\infty \log F(\mathrm{d}u) \\
&= \int_{(0,\infty)} \min(1, u) \log F(\mathrm{d}u) \\
&= \int_{(0,\infty)} \min(1, u) L(\mathrm{d}u) \\
&< \infty.
\end{aligned}$$

Thus, $\sum_{n=0}^{\infty} X_n < \infty$ almost surely. Furthermore, it is obvious from the above that the Laplace transform of $\sum_{n=0}^{\infty} X_n$ and the given infinitely divisible random variable match. Thus, we have proved the following characterization theorem.

Theorem 3. *If K is an infinitely divisible random variable on $[0, \infty)$ such that its Lévy measure L has the property that $h_L(x) = L([x, \infty))$ is a continuous function of x on $(0, \infty)$, then there exists a distribution F with 0 as its only possible point of discontinuity such that*

$$\sum_{n=0}^{\infty} X_n \stackrel{\mathrm{D}}{=} K, \tag{10}$$

where X_n is the $(n + 1)$th record from the distribution F. Moreover, F is given by (9).

Remark 4. Consider a random variable K with distribution G and Lévy measure L. The distribution G is in principle completely determined by its Lévy measure. However, it is difficult to implement this to simulate observations from G whenever it is not known in a closed form. The equation (10) provides a way of *simulating* the distribution of K. Using the representation above, we can approximate K by $\sum_{n=0}^{N} X_n$, where $\{X_n : n \geq 0\}$ are the lower records from F defined as in (9) and N is a sufficiently large integer. The records can be simulated easily since they form a Markov chain with a known transition probability and an initial distribution F. If the Lévy measure has finite mass, we can obtain the exact distribution by choosing N as a random integer with $N = \min\{n \geq 0 : X_n = 0\}$. In the case when L has infinite mass, it will be interesting to study the rate of convergence of the sum $\sum_{n=0}^{N} X_n$ to $\sum_{n=0}^{\infty} X_n$ as $N \to \infty$.

Observe that, if F has density f, then the Lévy density of T is given by $f(y)/F(y)$. A natural question is: under what conditions on F will the distribution of T belong to specified *subclasses* of the class of infinitely divisible laws? For a comprehensive description of interesting subclasses of infinitely divisible laws, see [2]. We give below three such classes.

2.2.1. *The class $\mathcal{L}$.* We first consider the class of *self-decomposable* laws, the so-called class $\mathcal{L}$. It consists of distributions of those random variables X for which, for every $c \in (0, 1]$, there exists a random variable ε_c independent of X such that $X \stackrel{\mathrm{D}}{=} cX + \varepsilon_c$. The proof of the following proposition is straightforward from [2, p. 18] and is omitted.

Proposition 3. *If F admits a density f such that $\int_0^1 (uf(u)/F(u))\,\mathrm{d}u < \infty$ and $uf(u)/F(u)$ is decreasing in u, then T is self-decomposable. Conversely, for any self-decomposable random variable T, there exists a density f with $\int_0^1 (uf(u)/F(u))\,\mathrm{d}u < \infty$ and $uf(u)/F(u)$ is decreasing in u such that $\sum_{n=0}^{\infty} X_n \stackrel{\mathrm{D}}{=} T$, where $\{X_n : n \geq 0\}$ is the sequence of lower records from the distribution with density f.*

2.2.2. *The subclass $\mathcal{T}_2$.* The class $\mathcal{T}_2$ consists of *generalized mixtures of exponentials* arising as weak limits of mixtures of exponentials and is characterized by the complete monotonicity of the Lévy density (see [2, p. 138]). Using this and noting that $-\log(F(\cdot))$ is completely monotone if and only if $f(y)/F(y)$ is, we derive the following proposition.

Proposition 4. *The random variable T belongs to $\mathcal{T}_2$ if and only if $-\log(F(\cdot))$ is completely monotone.*

As a specific case, note that the Lévy density $l(y) = \beta y^{-1} \exp(-ty)\,\mathbf{1}_{\{y>0\}}$ characterizes the gamma(β, t) distribution where $\beta, t > 0$. This will arise as the distribution of T if

$$F(x) = \exp\biggl(-\beta \int_x^{\infty} y^{-1} \exp(-ty)\,\mathrm{d}y\biggr).$$

2.2.3. *The class $\mathcal{T}$.* The class $\mathcal{T}$ is a subclass of both $\mathcal{L}$ and $\mathcal{T}_2$ and consists of *generalized gamma convolutions* defined as weak limits of finite convolutions of gamma distributions. By Theorem 3.1.1 of [2], a distribution G which gives full mass to the set of nonnegative real numbers belongs to the class $\mathcal{T}$ if and only if its Lévy density $l(\cdot)$ satisfies

$$yl(y) = \int_0^{\infty} \exp(-yt) U(\mathrm{d}t)$$

for all $y > 0$ and for some measure U on $(0, \infty)$ with $\int_0^1 |\log t| U(\mathrm{d}t) < \infty$ and $\int_1^{\infty} t^{-1} U(\mathrm{d}t) < \infty$. Equivalently, this holds if $yl(y)$ is completely monotone on $(0, \infty)$. Thus, we have the following result.

Proposition 5. *Suppose that F is a distribution such that $\int_0^1 (uf(u)/F(u))\,\mathrm{d}u < \infty$ and $uf(u)/F(u)$ is completely monotone. Then the random variable T belongs to $\mathcal{T}$. Conversely, if T is a random variable in $\mathcal{T}$, then there exists an F such that $T = \sum_{n=0}^{\infty} X_n$, where $\{X_n : n \geq 0\}$ is the sequence of lower records from F. Furthermore, F admits a density which satisfies the above conditions.*

The proof of the above proposition is straightforward, by noting that $l(y)$ is completely monotone if $yl(y)$ is completely monotone. As an example of the above, the positive stable distributions have $l(y) \propto y^{-\alpha-1}$ for $y > 0$ where $0 < \alpha < 1$. These arise from $F(x) = \exp(-cx^{-\alpha})$, $x > 0$.

When does the distribution of T admit a density? Using the infinite divisibility, we may obtain an answer to this question.

Proposition 6. *Suppose that $F(0) = 0$, F admits a density f on the whole of $(0, \infty)$ such that $\int_0^1 x \log F(\mathrm{d}x) < \infty$ and $\int_0^\infty x F(\mathrm{d}x) < \infty$. Then T is absolutely continuous with respect to the Lebesgue measure.*

Proof. Hudson and Tucker [6] obtained sufficient conditions for an infinitely divisible distribution to be equivalent to the Lebesgue measure. Using Theorem 1 of [6], it is enough to verify that the Lévy measure is absolutely continuous with respect to the Lebesgue measure, has infinite mass and finite first moment.

In our case, we have the Lévy measure as $f(x)/F(x)\, \mathbf{1}_{\{x>0\}}\, \mathrm{d}x$, which satisfies the properties of absolute continuity and infinite mass (see the discussion before Theorem 3). It is clear that

$$\begin{aligned}\int_0^\infty \frac{xf(x)}{F(x)}\,\mathrm{d}x &\le \int_0^1 \frac{xf(x)}{F(x)}\,\mathrm{d}x + \int_1^\infty \frac{xf(x)}{F(x)}\,\mathrm{d}x \\ &\le \int_0^1 \frac{xf(x)}{F(x)}\,\mathrm{d}x + \frac{1}{F(1)}\int_1^\infty xf(x)\,\mathrm{d}x \\ &< \infty.\end{aligned}$$

This proves the result.

From our discussion so far about the behaviour of T, it is clear that the behaviour of F near the origin plays a crucial role. Therefore, we might expect that, if F admits a density only near the origin, T might still admit a density. We give a partial answer to the above question using sufficient conditions available in terms of the characteristic function.

Proposition 7. *Suppose that F admits a density f in a neighbourhood of the origin such that, for some $\gamma > 0$,*

$$\liminf_{u\to 0} \frac{f(u)}{u^{\gamma-1}} = C_5, \qquad \limsup_{u\to 0} \frac{f(u)}{u^{\gamma-1}} = C_6$$

and $2C_5\gamma > C_6$. Then T admits a density.

Proof. Fix an $\varepsilon \in (0, C_5)$ such that $c(\varepsilon) := (C_5 - \varepsilon)\gamma/(C_6 + \varepsilon) > \frac{1}{2}$ and choose a $\delta > 0$ such that $(C_5 - \varepsilon)x^{\gamma-1} \le f(x) \le (C_6 + \varepsilon)x^{\gamma-1}$ for all $x \le \delta$. Then, for all $x \le \delta$,

$$F(x) \le \frac{(C_6 + \varepsilon)x^\gamma}{\gamma}.$$

Let $\xi(t) = \mathrm{E}(\exp(\mathrm{i}tT))$ be the characteristic function of T. Hence, for any $t \in \mathbb{R}$,

$$\begin{aligned}|\xi(t)| &= \left|\exp\left(\int_0^\infty \frac{(\cos(tu) - 1)}{F(u)} F(\mathrm{d}u) + \mathrm{i}\int_0^\infty \frac{\sin(tu)}{F(u)} F(\mathrm{d}u)\right)\right| \\ &= \exp\left(-\int_0^\infty \frac{(1 - \cos(tu))}{F(u)} F(\mathrm{d}u)\right) \\ &\le \exp\left(-\int_0^\delta \frac{(1 - \cos(tu))}{F(u)} F(\mathrm{d}u)\right).\end{aligned}$$

Now, for any $t > 0$,

$$\begin{aligned}
\int_0^{\delta} \frac{(1-\cos(tu))}{F(u)} F(\mathrm{d}u) &\geq \int_0^{\delta} \frac{(1-\cos(tu))(C_5-\varepsilon)u^{\gamma-1}\gamma}{(C_6+\varepsilon)u^{\gamma}} \mathrm{d}u \\
&= c(\varepsilon) \int_0^{\delta} \frac{(1-\cos(tu))\,\mathrm{d}u}{u} \\
&= c(\varepsilon) \int_0^{t\delta} \frac{(1-\cos(u))\,\mathrm{d}u}{u} \\
&\geq c(\varepsilon) \sum_{j=0}^{\lfloor t\delta/\pi \rfloor -1} \int_{j\pi}^{(j+1)\pi} \frac{(1-\cos(u))\,\mathrm{d}u}{u} \\
&\geq c(\varepsilon) \sum_{j=0}^{\lfloor t\delta/\pi \rfloor -1} \frac{1}{(j+1)\pi} \int_{j\pi}^{(j+1)\pi} (1-\cos(u))\,\mathrm{d}u \\
&\geq c(\varepsilon) \sum_{j=0}^{\lfloor t\delta/\pi \rfloor -1} \frac{1}{j+1} \\
&\geq c(\varepsilon) \log(1 + \lfloor t\delta/\pi \rfloor) \\
&\geq c(\varepsilon) \log(t\delta/\pi),
\end{aligned}$$

where $\lfloor \cdot \rfloor$ denotes the integer-part function.

Thus, for $t > 0$,

$$|\xi(t)| \leq \exp(-c(\varepsilon)\log(t\delta/\pi)) = C_7 t^{-c(\varepsilon)},$$

where $C_7 = 1/(\delta/\pi)^{c(\varepsilon)}$. Again, for $t < 0$, since cosine is an even function, we have the same estimate. Now, noting that $c(\varepsilon) > \frac{1}{2}$, we have that $\int_{-\infty}^{\infty} |\xi(t)|^2\,\mathrm{d}t < \infty$. Therefore, using Exercise 11 of [4, p. 159], we have shown that T admits a density.

3. The discrete case

In this section, we will derive similar results for the case when F is concentrated on nonnegative integers. In this case, with our definition, we may have repetition of records. Let $\mathbb{N}_0 = \{0, 1, \dots\}$. Suppose that F is a distribution with mass only on $\mathbb{N}_0$. Define $p_k = F(k) - F(k-)$ where $F(k-)$ is the left limit as $u \uparrow k$. Then, we have $0 \leq p_k \leq 1$, $\sum_{k=0}^{\infty} p_k = 1$ and $F(k) = \sum_{j=0}^{k} p_j$ for $k = 0, 1, \dots$.

Let $X_0 \sim F$. Then $\{X_n : n \geq 0\}$ is a sequence of *lower record values* if it is a Markov chain with the stationary transition probabilities given by the truncated distribution

$$\mathrm{P}(X_{n+1} = j \mid X_n = i) = \frac{p_j}{\sum_{l=0}^{i} p_l} \quad \text{if } j \leq i.$$

Again, it is clear that X_n converges almost surely to the constant $\min\{j : p_j > 0\}$. In case this minimum is 0, we can then naturally raise the question of the convergence of the sequence of partial sums of these records. As we have already discussed in Subsection 2.1, the minimum is actually hit in a finite time and the sum is finite with probability 1.

Let $T = \sum_{n=0}^{\infty} X_n$. It is easy to see that $T < \infty$ almost surely if and only if $p_0 > 0$. *Henceforth, we will assume that* $p_0 > 0$. Now, we turn to the distributional properties of T.

First, let us obtain a formula for its characteristic function by using our familiar conditioning argument. This formula readily yields the infinite divisibility of T.

Let us fix any $t \in \mathbb{R}$, and define

$$\phi_k(t) = \mathrm{E}(\exp(\mathrm{i}tT) \mid X_0 \le k).$$

Clearly, if $k = 0$, then $T = 0$ almost surely. Thus, $\phi_0(t) = 1$ for all $t \in \mathbb{R}$.

Proposition 8. *The characteristic function of T is given by*

$$\phi(t) = \lim_{k\to\infty} \phi_k(t) = \prod_{j=1}^{\infty} \frac{F(j-1)}{F(j) - p_j \exp(\mathrm{i}tj)}. \tag{11}$$

Note that the product on the right-hand side of (11) is nonzero since

$$\sum_{j=1}^{\infty} \left| 1 - \frac{F(j-1)}{F(j) - p_j \exp(\mathrm{i}tj)} \right| < C_8 \sum_{j=1}^{\infty} p_j < \infty,$$

where $C_8 > 0$ is a constant.

Proof. For the first equality, note that $\phi_k(t) = \mathrm{E}(\exp(\mathrm{i}tT)\,\mathbf{1}_{\{X_0 \le k\}})/F(k)$. Now, as $k \to \infty$, $F(k) \to 1$, and also $\mathbf{1}_{\{X_0 \le k\}} \to 1$ almost surely. Thus, by the dominated convergence theorem, we have $\phi_k(t) \to \phi(t)$.

Now, for any $k \ge 0$,

$$\begin{aligned}
\phi_k(t) &= \mathrm{E}(\exp(\mathrm{i}tT) \mid X_0 \le k) \\
&= \frac{\sum_{j=0}^{k} \mathrm{E}(\exp(\mathrm{i}tT)\,\mathbf{1}_{\{X_0=j\}})}{F(k)} \\
&= \frac{\sum_{j=0}^{k} \mathrm{E}(\exp(\mathrm{i}t(j + \sum_{n=1}^{\infty} X_n))\,\mathbf{1}_{\{X_0=j\}})}{F(k)} \\
&= \frac{\sum_{j=0}^{k} \exp(\mathrm{i}tj) p_j\, \mathrm{E}(\exp(\mathrm{i}t \sum_{n=1}^{\infty} X_n) \mid X_1 \le j)}{F(k)} \\
&= \frac{\sum_{j=0}^{k} \exp(\mathrm{i}tj) p_j \phi_j(t)}{F(k)}.
\end{aligned}$$

Thus,

$$\phi_k(t) F(k) = \sum_{j=0}^{k} \exp(\mathrm{i}tj) p_j \phi_j(t).$$

Therefore, for $k \ge 1$,

$$\phi_k(t)F(k) - \phi_{k-1}(t)F(k-1) = \exp(\mathrm{i}tk) p_k \phi_k(t).$$

This implies that, for $k \ge 1$,

$$\phi_k(t) = \frac{\phi_{k-1}(t)}{1 + p_k(1 - \exp(\mathrm{i}tk))/F(k-1)}. \tag{12}$$

Thus, from the fact that $\phi_0(t) = 1$ for all $t \in \mathbb{R}$ and using the recursion formula (12), we have

$$\phi_k(t) = \prod_{j=1}^{k} \frac{1}{1 + p_j(1 - \exp(itj))/F(j-1)}.$$

Hence, letting $k \to \infty$ in the above expression, the characteristic function of T is given by

$$\phi(t) = \prod_{j=1}^{\infty} \frac{1}{1 + p_j(1 - \exp(itj))/F(j-1)},$$

which simplifies to (11).

In order to obtain the representation for T that yields infinite divisibility, we start with a definition.

Definition 1. A random variable G taking values in $\mathbb{N}_0 = \{0, 1, \dots\}$ will be said to have a geometric distribution with multiplicity k and parameter α (denoted by $G \sim \text{Geo}(k, \alpha)$) if

$$\mathrm{P}(G = n) = \begin{cases} (1-\alpha)\alpha^j & \text{if } n = jk, \\ 0 & \text{otherwise,} \end{cases}$$

where $k \geq 1$ and $0 < \alpha < 1$.

In other words, $G \sim \text{Geo}(k, \alpha)$ if and only if G/k follows the usual geometric distribution with parameter α. In this case, the characteristic function of G can be written as

$$\begin{aligned} \phi_G(t) &= \frac{1-\alpha}{1 - \alpha \exp(itk)} \\ &= \exp\biggl(\sum_{n=1}^{\infty} (\exp(itkn) - 1)\frac{\alpha^n}{n}\biggr) \\ &= \exp\biggl(\sum_{n=1}^{\infty} (\exp(itn) - 1)\mu'(n; k, \alpha)\biggr), \end{aligned}$$

where

$$\mu'(n; k, \alpha) = \begin{cases} \dfrac{\alpha^j}{j} & \text{if } n = jk, \\ 0 & \text{otherwise.} \end{cases}$$

It is clear from the above that this characteristic function is infinitely divisible. Indeed, we can write it down in the Lévy–Khinchin form (see e.g. [7, Theorem 5.5.1]). Taking the logarithm of $\phi_G(t)$, we have

$$\log \phi_G(t) = ib_G t + \sum_{n=1}^{\infty} \biggl(\exp(itn) - 1 - \frac{itn}{1+n^2}\biggr)\frac{1+n^2}{n^2}\mu(n; k, \alpha),$$

where

$$\mu(n; k, \alpha) = \frac{n^2}{1+n^2}\mu'(n; k, \alpha) = \begin{cases} \dfrac{k^2 j\alpha^j}{1+(jk)^2} & \text{if } n = jk,\ j \geq 1, \\ 0 & \text{otherwise,} \end{cases} \tag{13}$$

and

$$b_G = \sum_{n=1}^{\infty} \frac{n}{1+n^2}\mu'(n; k, \alpha) = k \sum_{j=1}^{\infty} \frac{\alpha^j}{1+(jk)^2} < \infty. \tag{14}$$

It is also clear that $\sum_{n=1}^{\infty} \mu(n; k, \alpha) < \infty$.

Proposition 9. *Let $\{G_n : n \geq 1\}$ be a sequence of independent variables with G_n having* $\mathrm{Geo}(k_n, \alpha_n)$ *distribution. Then $G = \sum_{n=1}^{\infty} G_n < \infty$ almost surely if and only if $\sum_{n=1}^{\infty} \alpha_n < \infty$ and, in that case, G is infinitely divisible.*

Proof. Since each G_n takes values in $\mathbb{N}_0$, we have $G < \infty$ almost surely if and only if $\mathrm{P}(\limsup_{n\to\infty}\{G_n > 0\}) = 0$. Indeed, if $\omega \in \limsup_{n\to\infty}\{G_n > 0\}$, then there exists a sequence $n_j \uparrow \infty$ such that $\omega \in \{G_{n_j} > 0\}$ for every $j \geq 1$. Owing to the discrete nature of the G_n, this implies that $G = \infty$. Conversely, if $\omega \notin \limsup_{n\to\infty}\{G_n > 0\}$, then there exists an n_0 such that $G_n = 0$ for all $n > n_0$, which implies that $G = \sum_{n=1}^{n_0} G_n < \infty$.

Now, for the 'if' part. Since $\sum_{n=1}^{\infty} \mathrm{P}(G_n > 0) = \sum_{n=1}^{\infty} \alpha_n < \infty$, by the first Borel–Cantelli lemma, we have $\mathrm{P}(\limsup_{n\to\infty}\{G_n > 0\}) = 0$. Conversely, if $\sum_{n=1}^{\infty} \alpha_n = \infty$, then from the independence of the events $\{G_n > 0\}$, we have, by the second Borel–Cantelli lemma, $\mathrm{P}(\limsup_{n\to\infty}\{G_n > 0\}) = 1$. This proves the 'if' part. To show that G is infinitely divisible, define $H_n := \sum_{j=1}^{n} G_j$. Since each G_j is infinitely divisible, so is H_n. Now $H_n \to G$ almost surely, and hence also in distribution. Thus, by Theorem 7.6.5 of [4, p. 244], G is infinitely divisible.

Remark 5. The Lévy measure of G can be obtained from (13) and (14). Indeed, the logarithm of the characteristic function of G can be written as

$$\log \phi_{\sum_{n=1}^{\infty} G_n}(t) = \mathrm{i}bt + \sum_{j=1}^{\infty}\left(\exp(\mathrm{i}tj) - 1 - \frac{\mathrm{i}tj}{1+j^2}\right)\frac{1+j^2}{j^2}\mu_j,$$

where

$$b = \sum_{n=1}^{\infty} b_n \quad \text{and} \quad \mu_j = \sum_{n=1}^{\infty} \mu(j; k_n, \alpha_n).$$

It is fairly easy to verify that $b < \infty$ and $\sum_{j=1}^{\infty} \mu_j < \infty$.

We may now state the representation for T.

Theorem 4. (Representation theorem.) *For any F having support in $\mathbb{N}_0$ with $F(0) > 0$, we have*

$$T \stackrel{\mathrm{D}}{=} \sum_{k=1}^{\infty} G_k,$$

where $\{G_k : k \geq 1\}$ is a sequence of independent random variables with G_k having the distribution $\mathrm{Geo}(k, p_k/F(k))$. *Hence, T is infinitely divisible. Conversely, given any infinite convolution of* $\{\mathrm{Geo}(k, \alpha_k) : k \geq 1\}$ *such that $\sum_{k=1}^{\infty} \alpha_k < \infty$, there exists an F having support in $\mathbb{N}_0$ such that the above representation holds.*

Proof. From the facts that $\sum_{k=0}^{\infty} p_k = 1$ and $F(k) \to 1$ as $k \to \infty$, it readily follows that $\sum_{k=1}^{\infty} p_k/F(k) < \infty$. Hence, from Proposition 9, $G = \sum_{k=1}^{\infty} G_k$ is finite and infinitely divisible.

Now note that T and G have the same characteristic function. This proves the representation and that T is infinitely divisible.

For the last part, given any $\{\alpha_k : k \geq 1\}$ such that $\sum_{k=1}^{\infty} \alpha_k < \infty$, define

$$F(k) := \prod_{j=k+1}^{\infty} (1 - \alpha_j)$$

for any $k \geq 0$. Since $\sum_{k=1}^{\infty} \alpha_k < \infty$, we have that $F(0) > 0$ and $F(k) \uparrow 1$ as $k \to \infty$. Clearly, for this distribution function, we will have $\alpha_k = 1 - F(k-1)/F(k)$ for all $k \geq 1$. This proves the result.

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