Economic Theory **21**, 843–869 (2003)

Economic Theory

Individual welfare, social deprivation and income taxation★

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Received: September 20, 1999; revised version: March 6, 2002

Summary. In a homogeneous framework where individuals can only be distinguished on the basis of their incomes, we examine the incidence of taxation on the amount of deprivation felt in the society. We conceive deprivation in terms of utility or well-being rather than just in terms of income and we measure it by comparing the deprivation profiles arising in the different situations. We identify the restrictions to be imposed on the utility function which guarantee that a more progressive system of taxes always results in less social deprivation. We show that, in general, it is not possible to get an equivalence and realize a social improvement in terms of social deprivation by substituting a more progressive system of taxes for a less progressive one.

Keywords and Phrases: Welfarism, Taxation, Progressivity, Deprivation, Individual welfare.

JEL Classification Numbers: D31, D63, H24.

★ This paper forms part of the research programme of the TMR network Living Standards, Inequality and Taxation [Contract No. ERBFMRXCT 980248] of the European Communities whose financial support is gratefully acknowledged. Chakravarty wishes to express his sincere gratitude to the French Ministry of Education for financing his stay at DELTA during which this paper was written. The authors would like to thank Stephen Bazen, Nicolas Gravel and an anonymous referee, whose comments have helped to improve the paper, but they retain sole responsibility for remaining deficiencies.

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1 Introduction and summary

1.1 Motivation and relationship to the literature

In a homogeneous environment, progressive taxation has long been known to cut income differentials and therefore to imply a reduction of inequality [1]. Precisely, Jakobsson [18] has shown that a non-decreasing average tax rate is a necessary and sufficient condition for after tax incomes to be judged as no less unequally distributed than before tax incomes according to the relative Lorenz criterion. This implies in particular that any relative Lorenz consistent index cannot display an increase in inequality when tax-units are subjected to a progressive – in the above sense – tax-schedule. Analogous results can be obtained by substituting alternative Lorenz criteria for the standard relative Lorenz quasi-ordering provided that one adapts in a suitable manner the definition of a progressive tax-schedule. For instance, it can be shown that a non-decreasing tax-liability – the minimal progressivity notion introduced by Fei [15] – is a necessary and sufficient condition for after tax incomes to be no less unequally distributed than before tax ones according to the absolute Lorenz criterion (see e.g. Moyes [28]). It follows that a minimally progressive tax-schedule will be considered as inequality non-increasing by any supporter of Kolm's [21] position with respect to inequality measurement. Less extreme views along the lines of Bossert and Pfingsten [7] would yield results in the same vein by adapting the notion of progressivity in an appropriate way.

So far the literature has mainly concentrated on the way the distribution of individual incomes are affected by the tax-system. However it is typically assumed in the economic literature that what matters is not really the individual's consumption but rather the utility that the consumer derives from her consumption. According to welfarism, the distributions of agents' utilities constitute the only relevant information when comparing alternative situations or social states. Typically, a situation will be judged socially preferable to another situation if the distribution of utilities it generates is considered as being better than the distribution of utilities resulting from the latter situation according to some ordering defined on the utility space. In other words, the ranking of social states is uniquely determined by the ranking of the associated utility distributions. It is therefore tempting to base the analysis of the redistributive effect of alternative tax proposals on the comparisons of the corresponding after tax utility profiles.

1.2 The theoretical approach developed in the paper

The question immediately arises of what is the proper criterion to be used for making meaningful comparisons of utility distributions. A longstanding tradition initiated by the work of Kolm [20] and Atkinson [3] recommends to make distributive judgements on the basis of Lorenz consistent measures emphasizing the

[1] The situation is far less clearcut in the heterogeneous case, where households differ in other respects than income, for instance, family size and composition (see e.g. Moyes and Shorrocks [34], Ebert and Moyes [12]).

normative properties of the Lorenz quasi-ordering. The appealing property of the Lorenz quasi-ordering in this framework is that it records a social welfare improvement when utility is transferred from a better-off individual to a less well-off individual and the positions of individuals in terms of well-being are not affected by such a transformation. Though it is conceivable to apply the standard Lorenz inequality criteria for evaluating alternative utility profiles, the generalization introduced by Shorrocks [38] might seem more appropriate as it incorporates some concern for efficiency in addition to equity considerations and therefore allows to pass judgements when the sum of utilities are different. An interesting implication of basing judgements on the comparisons of the so-called generalised Lorenz curves of the utility profiles is that any conclusive verdict translates to any social welfare functional which incorporates equity and efficiency considerations such as the Utilitarian, Ralwsian or Leximin ones. Suppose we are interested in the implications of a modification of the tax-system that shifts the burden of the tax from low income earners to high income earners in such a way that the tax-revenue is not affected. Formally, the distribution of tax-liabilities is made more unequal in the sense that the former distribution of taxes can be obtained from the latter by means of a finite sequence of progressive transfers. Then it can be shown that such a tax reform will improve the distribution of individual utilities according to the generalised Lorenz criterion – and thus by any equity and efficiency oriented social welfare functional – provided that the common individual utility function be concave[2]. Clearly, the practical implications of this result are limited since the usual notion of progressivity is distinct from the general equalizing shift we have considered above [3]. In addition, this result tells nothing about the welfare incidence of a tax reform that implies an increase in the tax-liability of every income-unit since the generalised Lorenz criterion has the property that efficiency always overcomes equity.

More fundamentally, the approach based on the Lorenz criterion is not immune to criticism. Indeed, whereas most of the literature on inequality and welfare measurement imposes the principle of transfers, one may however raise doubts about the ability of such a condition to capture the very idea of inequality in general. Though a progressive transfer unambiguously reduces inequality between the individuals involved in the transfer, it is far from being obvious that everyone would agree on the fact that inequality on the whole has declined as a result. The fact that progressive transfers are not universally approved has been confirmed by recent experimental studies (see e.g. Amiel and Cowell [2], Ballano and Ruiz-Castillo [6], Harrison and Seidl [16]).

It seems that attitudes such as envy and deprivation are important components of individual judgements that have to be taken into account as far as distributive justice is concerned. In addition, there is some evidence that the social status of an individual – approximated by her position on the social hierarchy – plays an important role in the determination of her well-being (see e.g. Weiss and Fershtman

[2] In Appendix A we provide a sketch of the proof of this claim, which is a direct consequence of results on the preservation and/or conversion of the Lorenz type quasi-orderings by means of individualistic and symmetric transformations (see e.g. Moyes [29], Moyes and Shorrocks [33]).

[3] Actually, it could be shown that an increase in progressivity consistent with a fixed tax-revenue cannot be welfare improving according to the generalised Lorenz criterion.

[41]). The notion of individual deprivation originating in the work of Runciman [36] precisely accommodates these views making the individual's assessment of a given social state depend on her situation compared with the situations of individuals more favourably treated than her. The deprivation profile, which indicates the level of deprivation felt by each individual, constitutes therefore the basis of social judgement. Drawing upon previous work by Yitzhaki [39,40], Hey and Lambert [17], Kakwani [19], Chakravarty, Chattopadhyay and Majumder [9], and Chakravarty [8], one can propose two deprivation quasi-orderings depending on the way individual deprivation is defined. Individual deprivation in a given state formally ressembles the aggregate poverty gap where the poverty line is set equal to the individual's income[4]. So stated, one may conceive of absolute individual deprivation, which is simply the sum of the gaps between the individual's income and the incomes of all individuals richer than her, and relative deprivation, where the income gaps are deflated by the individual's income. Then the deprivation quasi-orderings are based on the comparisons of the individual deprivation curves and social deprivation unambiguously decreases as the individual deprivation curve is moving downwards.

Following a suggestion by Hey and Lambert [17], we apply these two deprivation quasi-orderings to the utility deprivation profiles rather than to the income deprivation profiles since the distribution of utility or well-being is ultimately what matters in the welfarist framework. The problem is that utilities – and therefore their distributions across individuals – cannot be observed so that it is impossible to decide whether one situation is worse than another in terms of utility deprivation. On the contrary, the distributions of income among individuals are known perfectly and we can use this information in order to deduce the ranking of the corresponding utility distributions on the basis of deprivation. To do this, one has to place appropriate restrictions on the individual's utility function and we seek to identify what these are. More precisely we investigate the class of utility functions which have the property that, whatever the member of the class one selects, progressive taxation implies or is equivalent to a reduction of deprivation in society. Intuitively, the classes of utility functions will depend on the choice of the concept of progressivity as well as the concept of deprivation. Our analysis is related to Chakravarty and Mukherjee [10] who established that a constant relative risk aversion utility function is sufficient to guarantee that a progressive tax-schedule – in the sense of a non-decreasing average tax rate – implies less absolute deprivation in terms of utilities. In this paper, we strenghen and extend Chakravarty and Mukherjee's [10] result focusing on basic principles. For instance, the analysis is framed into taxation programmes rather than tax-functions and we provide results involving equivalence rather than just implications. It appears that this more demanding structure actually does not restrict further the class of utility functions so that the class of constant risk aversion utility functions still emerges even in this more demanding framework.

[4] Most scholars take for granted that individual deprivation is simply the sum – possibly normalized in a suitable way – of the income gaps between the individual's income and the incomes of all individuals richer than her. An axiomatic characterization of the absolute deprivation profile is provided by Ebert and Moyes [13]

The results extend easily to the case where one considers tax-schedules rather just taxation programmes.

1.3 Organization of the paper

We introduce in Section 2 our basic framework and the criteria we will use later on in order to assess the redistributive impact of taxation. After having defined our two deprivation quasi-orderings, we present the differential quasi-orderings which are closely related to the concepts of progressivity used in the literature. Technical results linking these two different families are presented then. Section 3 contains our main results identifying the classes of utility functions such that the progressivity of the taxation programme implies – or is equivalent to – a reduction of social deprivation if and only if one measures individual welfare before and after tax using a member of this class. In Section 4 we hint at the results one arrives at when relative deprivation is substituted for absolute deprivation and provide a catalogue of results depending on the concepts of progressivity and deprivation one chooses. We also indicate how our results generalise to the case where the analysis is framed in terms of tax-schedules. Finally Section 5 concludes the paper hinting at some directions for further research. Appendix A contains a formal presentation of some arguments concerning the welfare implications of less equally distributed taxes we discussed above, while Appendix B contains the proofs of our results.

2 Notation and preliminary definitions and results

2.1 The basic framework

We consider a fixed *population* or *society* $S := \{1, 2, \ldots, n\}$ consisting of n homogeneous households ($n \geq 3$) and we assume that incomes are drawn from a nondegenerate real interval $D := (\underline{v}, +\infty)$. It may be helpful to regard D as the set of positive reals $\mathbb{R}_{++}$, but in general the income range will depend on the context and incomes will not be necessarily positive. A typical *income distribution* is a vector $\mathbf{x} := (x_1, \ldots, x_n)$, where $x_i \in D$ is the income of individual i, and we denote the *arithmetic mean* of distribution $\mathbf{x}$ by $\mu(\mathbf{x})$[5]. To simplify the presentation we will assume without loss of generality that incomes are non-decreasingly arranged so that $x_1 \leq x_2 \leq \cdots \leq x_n$ and we will denote the set of income distributions by $\mathcal{Y}(D)$. So far a social state is assimilated with an income distribution that may be viewed as the result of some institutional arrangement such as the Walrasian mechanism, for instance. Because by definition individuals are identical in all respects other than income, we assume the same *utility function* U for all individuals. One may conveniently think of U as a *social norm* expressing the way society or some impartial observer values the welfare achieved by every individual depending on the income she receives. We let $\mathbf{U}(D) := \{U : D \rightarrow \mathbb{R} \mid U \text{ is continuous and increasing}\}$

[5] Given a subset $\mathbf{A} \subseteq \mathbb{R}^m$ ($m \geq 2$) and two vectors $\mathbf{u} := (u_1, \ldots, u_m), \mathbf{v} := (v_1, \ldots, v_m) \in \mathbf{A}$, we write $\mathbf{u} \geq \mathbf{v}$ to mean that $u_i \geq v_i$, for all $i = 1, 2, \ldots, m$.

represent the class of admissible utility functions. Given a distribution $\mathbf{x} \in \mathcal{Y}(D)$ and a utility function $U \in \mathbf{U}(D)$, we denote by $U(\mathbf{x}) := (U(x_1), \ldots, U(x_n))$ the corresponding *distribution of utilities* or shortly *imputation*. It follows from the definitions above that $U(x_1) \leq U(x_2) \leq \cdots \leq U(x_n)$. Given a utility function $U \in \mathbf{U}(D)$ and a subset $\mathbf{A} \subseteq \mathcal{Y}(D)$, we let $\mathbf{U}(\mathbf{A}) := \{\mathbf{u} := (u_1, \ldots, u_n) \in \mathbb{R}^n \mid \exists \mathbf{x} \in \mathbf{A} : u_i = U(x_i), \forall i \in S\}$ represent the set of feasible imputations. According to welfarism (see e.g. Sen [37]), distribution $\mathbf{x} \in \mathcal{Y}(D)$ will be socially preferred to distribution $\mathbf{y} \in \mathcal{Y}(D)$ if the imputation $\mathbf{v} := U(\mathbf{x})$ is ranked above the imputation $\mathbf{u} := U(\mathbf{y})$ according to some ordering defined on the utility space. More precisely, one defines an ordering R° on $\mathcal{Y}(D)$ starting from an ordering R^* on $\mathbb{R}^n$ so that:

$$\forall\, \mathbf{x}, \mathbf{y} \in \mathcal{Y}(D) : \mathbf{x}\, R^\circ\, \mathbf{y} \text{ if and only if } U(\mathbf{x})\, R^*\, U(\mathbf{y}). \tag{2.1}$$

The utility functions are defined up to an increasing transformation ϕ which preserves the ranking of distributions on the utility space i.e.,

$$U(\mathbf{x})\, R^*\, U(\mathbf{y}) \text{ if and only if } \phi(U(\mathbf{x}))\ R^*\ \phi(U(\mathbf{y})),\ \forall\, \mathbf{x}, \mathbf{y} \in \mathcal{Y}(D), \tag{2.2}$$

where $\phi(U(\mathbf{x})) := (\phi(U(x_1)), \ldots, \phi(U(x_n)))$. Clearly, the transformation ϕ will depend on the ordering R^* one defines on the utility space.

2.2 The normative criteria used in the paper

Because the quasi-orderings we introduce below will be applied indistinctly to income distributions and/or utility distributions, it is convenient to frame the presentation in terms of comparisons of *situations*, a broader term that may be interpretated in both ways. It is a longstanding tradition in normative analysis to appeal to Lorenz consistent measures when comparing alternative situations. However the approach based on the Lorenz criterion is not immune to criticism. Indeed, although most of the literature on inequality and welfare measurement imposes the principle of transfers, one may however raise doubts about the ability of such a condition to capture the very idea of inequality in general. The fact that progressive transfers are not necessarily universally approved is confirmed by recent experimental studies (see e.g. Amiel and Cowell [2], Ballano and Ruiz-Castillo [6], Harrison and Seidl [16]). In this paper, we will refer to two families of less debatable criteria: the deprivation quasi-orderings and the differentials quasi-orderings.

The Lorenz quasi-ordering does not seem to be compatible with attitudes such as envy and resentment which, according to experimental studies, seem to be important components of individual judgements. The notion of individual deprivation originating in the work of Runciman [36] accommodates these views, making the individual's assessment of a given social state depend on her situation compared to the situations of individuals more favourably treated than her. Drawing upon previous work by Hey and Lambert [17], Kakwani [19], Chakravarty, Chattopadhyay and Majumder [9], and Chakravarty [8], one can propose two deprivation quasi-orderings depending on the way individual deprivation is defined. Given the situation $\mathbf{u} := (u_1, \ldots, u_n) \in \mathbb{R}^n$ where $u_1 \leq u_2 \leq \cdots \leq u_n$, we let

$$ADP(k; \mathbf{u}) := \frac{1}{n} \sum_{j=k}^{n} (u_j - u_k), \tag{2.3}$$

measure the absolute deprivation of individual $k \in S$. This is consistent with the view that the degree of deprivation is the *intensity* with which it is felt by the individual (see Runciman [35, p. 10]). By definition, the most well-off individual is never deprived and $ADP(n; \mathbf{u}) = 0$, for all $\mathbf{u} \in \mathbb{R}^n$. The absolute deprivation profile in situation $\mathbf{u} \in \mathbb{R}^n$ is then given by $ADP(\mathbf{u}) := (ADP(1; \mathbf{u}), \ldots, ADP(n-1; \mathbf{u}))$. Following Chakravarty, Chattopadhyay and Majumder [9], and Chakravarty [8], given two situations $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we will say that *there is no more absolute deprivation in situation* $\mathbf{u}$ *than in situation* $\mathbf{v}$, which we write $\mathbf{u} \geq_{ADP} \mathbf{v}$, if and only if

$$ADP(k; \mathbf{u}) \leq ADP(k; \mathbf{v}), \ \forall \ k = 1, 2, \ldots, n-1. \tag{2.4}$$

Instead of focusing on absolute utility differences, one may rather conceive deprivation as arising from relative losses and let

$$RDP(k; \mathbf{u}) := \frac{1}{n} \sum_{j=k}^{n} \left(\frac{u_j - u_k}{u_k} \right) \tag{2.5}$$

measure the relative deprivation of individual $k \in S$ in situation $\mathbf{u} \in \mathbb{R}^n_{++}$[6]. We let $RDP(\mathbf{u}) := (RDP(1; \mathbf{u}), \ldots, RDP(n-1; \mathbf{u}))$ represent the relative deprivation profile in situation $\mathbf{u} \in \mathbb{R}^n_{++}$ and we will say that *there is no more relative deprivation in situation* $\mathbf{u}$ *than in situation* $\mathbf{v}$, which we write $\mathbf{u} \geq_{RDP} \mathbf{v}$, if and only if

$$RDP(k; \mathbf{u}) \leq RDP(k; \mathbf{v}), \ \forall \ k = 1, 2, \ldots, n-1. \tag{2.6}$$

The absolute and relative deprivation quasi-orderings remain invariant under increasing affine and increasing linear transformations of utilities respectively. When one assimilates a situation with a distribution of individual utilities, our deprivation quasi-orderings on the utility space are analogous to Meyer [27]'s stochastic dominance with respect to a function. In order to relate Meyer's stochastic dominance criteria to inequality, the income relative and absolute Lorenz curves need to be replaced by the utility relative and absolute Lorenz curves respectively (see Lambert and Hey [17]). Analogously, in order to look at the welfare implications of utility based deprivation, we have to use the utility deprivation curve rather than the income deprivation curve. In order to show that these two deprivation quasi-orderings are logically independent, consider population $S = \{1, 2, 3, 4\}$ and the situations defined in the table below where $u < v < w < t$, $0 \leq \Delta \leq (t-w)/2$, $(t-w)/(t-w-2\Delta) \leq \lambda$, and $2\Delta w/(t-w) \leq \gamma$. The presence of a "1" means

6 This concept of individual relative deprivation is at variance with the definition of Kakwani [19] (see also Chakravarty, Chattopadhyay and Majumder [9], and Chakravarty [8]), where individual deprivation is assimilated with the aggregate income share shortfall.

Table 1. Independence of the absolute and relative deprivation quasi-orderings

	$\mathbf{u} \leq_{ADP} \mathbf{v}$	$\mathbf{u} \leq_{RDP} \mathbf{v}$
$\mathbf{v} := (u, v, w, t); \mathbf{u}^{(1)} := (u, v, w + \Delta, t - \Delta)$	1	1
$\mathbf{v} := (u, v, w, t); \mathbf{u}^{(2)} := (\lambda u, \lambda v, \lambda(w + \Delta), \lambda(t - \Delta))$	0	1
$\mathbf{v} := (u, v, w, t); \mathbf{u}^{(3)} := (u - \gamma, v - \gamma, (w + \Delta), \gamma, (t - \Delta) - \gamma)$	1	0
$\mathbf{v} := (u, v, w, t); \mathbf{u}^{(4)} := (u, v, w + \Delta, t - \Delta)$	0	0

that "there is no more deprivation in situation $\mathbf{u}$ than in situation $\mathbf{v}$" and that of a "0" means that "situations $\mathbf{u}$ and $\mathbf{v}$ are not comparable according to the deprivation quasi-ordering".

Though the deprivation quasi-orderings defined above are to a certain extent more acceptable than the Lorenz criteria, they may still come into conflict with very basic notions of inequality reduction. For instance, there is unambiguously less deprivation – both in relative and absolute terms – in situation $\mathbf{u}^{(1)}$ than in situation $\mathbf{v}$, and clearly inequality decreases for any pair of individuals $\{1, 4\}$, $\{2, 4\}$, $\{3, 4\}$. However it is equally true that inequality increases as far as pairs $\{1, 3\}$ and $\{2, 3\}$ are concerned, so that the net effet on overall inequality of the progressive transfer needed in order to obtain $\mathbf{u}^{(1)}$ from $\mathbf{v}$ is ambiguous.

The quasi-orderings we consider next are more demanding as they require that all pairwise inequalities be less in situation $\mathbf{u}$ than in situation $\mathbf{v}$ for the former situation to be declared as less unequal than the latter. We say that *situation* $\mathbf{u}$ *dominates situation* $\mathbf{v}$ *in absolute differentials*, which we write $\mathbf{u} \geq_{AD} \mathbf{v}$, if and only if

$$u_i - v_i \geq u_{i+1} - v_{i+1}, \ \forall\, i = 1, 2, \ldots, n - 1. \tag{2.7}$$

This simply means that the differences between any two individual utilities taken in increasing order are less in situation $\mathbf{u}$ than in situation $\mathbf{v}$. Similarly, we will say that *situation* $\mathbf{u}$ *dominates situation* $\mathbf{v}$ *in relative differentials*, which we write $\mathbf{u} \geq_{RD} \mathbf{v}$, if and only if

$$u_i / v_i \geq u_{i+1} / v_{i+1}, \ \forall\, i = 1, 2, \ldots, n - 1. \tag{2.8}$$

Thus it is the ratio of any two utilities arranged in increasing order that must be less in situation $\mathbf{u}$ than in situation $\mathbf{v}$. The absolute and relative differentials quasi-orderings are invariant with respect to respectively increasing affine and increasing linear transformations of utilities. These two quasi-orderings, first introduced by Marshall et al. [26] in the fields of majorization, may be considered as suitable inequality criteria (see also Moyes [30,31] and Preston [35]) [7]. The following result will be useful later on:

[7] Though there exists at present no theoretical basis for the differentials quasi-orderings comparable to what has been achieved in the case of the Lorenz quasi-orderings, there are however some partial results. For instance, it can be shown that there exists no utility function such that social welfare always increases as a result of an improvement as measured by the absolute or relative differentials quasi-ordering according to the utilitarian rule.

Lemma 2.1. *The two following statements are true but not their converse unless* $n = 2$.

(a) *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$: $\mathbf{u} \geq_{AD} \mathbf{v}$ *implies* $\mathbf{u} \geq_{ADP} \mathbf{v}$.
(b) *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{++}$: $\mathbf{u} \geq_{RD} \mathbf{v}$ *implies* $\mathbf{u} \geq_{RDP} \mathbf{v}$.

2.3 Income taxation and progressivity

For the sake of generality, it is convenient to avoid any reference to a tax-schedule and frame the analysis in terms of comparisons of alternative systems of individual taxes. Formally, our framework is borrowed from Fei [15] who introduced the notion of a *taxation programme* i.e., an ordered pair $(\mathbf{x}; \mathbf{y})$, where $\mathbf{y} := (y_1, \ldots, y_n)$ and $\mathbf{x} := (x_1, \ldots, x_n)$ are respectively the pre-tax and post-tax distributions. In the particular case where the post-tax distribution is derived from the pre-tax distribution by means of a net income-schedule f, we will have $x_i = f(y_i)$, for all $i \in S$. More generally, one may think of $\mathbf{x}$ as resulting from $\mathbf{y}$ by means of a complex system involving commodities taxes in addition to direct taxes and transfer-payments. To fix the ideas, it is convenient to assume that taxes are imposed in order to raise some revenue $B \geq 0$ so that the post-tax distribution $\mathbf{x}$ verifies: (i) $x_1 \leq x_2 \leq \cdots \leq x_n$ and (ii) $\sum_{i=1}^{n} [y_i - x_i] = B$. In what follows, we are interested in ranking taxation programmes $(\mathbf{x}^\circ; \mathbf{y})$ and $(\mathbf{x}^*; \mathbf{y})$ which will be done by comparing the post-tax distributions $\mathbf{x}^\circ$ and $\mathbf{x}^*$ on the basis of the utility distributions they generate. This means in particular that no attention will be attached to the taxation process which implies that the pre-tax distribution we start from will play no role in the subsequent analysis, something which is clearly debatable [8]. We will assume that the post-tax distributions we are comparing are derived from the same pre-tax distribution and for convenience, for any $\mathbf{y} \in \mathcal{Y}(D)$, we will write

$$\mathcal{Z}(\mathbf{y}) := \left\{\mathbf{x} := (x_1, \ldots, x_n) \in \mathcal{Y}(D) \mid \sum_{i=1}^{n} (x_i - y_i) \leq 0\right\} \tag{2.9}$$

for the set of post-tax distributions that can be generated starting from $\mathbf{y}$. Since by definition $\mathbf{y} \in \mathcal{Z}(\mathbf{y})$, our subsequent results can be equally interpretated in terms of comparisons of alternative post-tax distributions arising from the same pre-tax distribution or in terms of comparisons of pre-tax and post-tax distributions.

There are typically two competing notions of schedule-progressivity in the literature (see Lambert [22, Chap. 6] for an extensive review of the different measures of progression). Minimal progressivity in the sense of Fei [15] requires that the tax-liability be non-decreasing with income, while the standard concept of progressivity insists that it is the average tax rate that must be non-decreasing with income. There is a close connection between these two concepts of progressivity and the differentials quasi-orderings we introduced above which is best exemplified

[8] To the best of our knowledge there is surprisingly no work concerned with the comparison of taxation programmes $(\mathbf{x}^\circ; \mathbf{y}^\circ)$ and $(\mathbf{x}^*; \mathbf{y}^*)$, where the pre-tax situations of the tax-units possibly differ. This might be of particular relevance for comparisons of progressivity in empirical work, where the available data consist only of the pre-tax and post-tax distributions.

when one considers taxation programmes. Indeed, given the taxation programme $(\mathbf{y};\mathbf{x})$, we can interpret the difference $y_i - x_i$, which may be positive or negative as the tax-liability of individual $i \in S$. By analogy with the definitions of schedule-progressivity, we will say that $(\mathbf{y};\mathbf{x})$ is *minimally progressive* if $y_i - x_i$ is non-decreasing in i which is equivalent to the requirement that $\mathbf{x} \geq_{AD} \mathbf{y}$. Similarly, the programme $(\mathbf{y};\mathbf{x})$ will be said to be *average rate progressive* if $(y_i - x_i)/y_i$ is non-decreasing in i, or equivalently $\mathbf{x} \geq_{RD} \mathbf{y}$. Because we are *in fine* interested in the comparison of alternative post-tax distributions originating from the same pre-tax distribution, we will assimilate a more progressive taxation programme with more equally distributed after tax distributions according to the differentials quasi-orderings. Consequently, we will say that the taxation programme $(\mathbf{y};\mathbf{x}^*)$ is *more minimally progressive* than the taxation programme $(\mathbf{y};\mathbf{x}^\circ)$ if $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$. Similarly, the taxation programme $(\mathbf{y};\mathbf{x}^*)$ will said to be *more average rate progressive* than the taxation programme $(\mathbf{y};\mathbf{x}^\circ)$ if $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$.

3 The absolute redistributive effect of progressive taxation in the individual welfare space

Here we are interested in the implications of progressivity on the distribution of after tax utilities. More precisely we would like to know to what extent a more progressive taxation programme is sufficient and/or necessary for reducing absolute social deprivation in terms of well-being. We will examine successively the concepts of minimal progressivity and average rate progressivity which we assimilate with absolute and relative differentials dominance respectively. Precisely we investigate the class of utility functions which have the property that, whatever be the member of the class one selects, progressive taxation implies or is equivalent to a reduction of absolute deprivation in the society. Intuitively, the classes of utility functions will depend on both the chosen concepts of progressivity and deprivation. Given the key role played by the differentials quasi-orderings, we also examine the way increased progression affects after tax utility differentials.

It may be worthwhile to summarize the main findings of the current section. The first result establishes that more minimally progressive taxation programmes is equivalent to welfare improvement in the sense of the absolute differentials quasi-ordering if and only if the utility function is affine. Then we come out with an impossibility result, which demonstrates that there exists no utility function for which equivalence between absolute deprivation quasi-ordering of the utility profiles and absolute differentials quasi-ordering of the post-tax distribution holds. A natural relative counterpart to the first result shows that for a one-to-one correspondence between more average rate progressive taxation programmes and welfare improvements in the sense of absolute differentials quasi-ordering to hold, it is necessary and sufficient that the utility function be logarithmic. Finally, as a relative counterpart to the second result it emerges that there exists no utility function that can make a decrease in social absolute deprivation equivalent to more equally distributed post-tax incomes according to the relative differentials quasi-ordering.

3.1 Absolute progressive taxation and absolute deprivation in well-being

Does a more minimally progressive taxation programme always lead to an improvement in the distribution of after tax well-being according to our absolute deprivation quasi-ordering? It is interesting to consider successively (i) the case where the tax reform affects in the same way all tax-liabilities – all tax-units experience an increase [decrease] in after tax income – and (ii) the informationally less demanding case where nothing is known about the way tax-liabilities are affected. As it will become clear below, these informational assumptions have important consequences for underlying utility functions.

The result below identifies the class of utility functions that guarantees that a more minimally progressive taxation programme – equivalently, more equally distributed after tax incomes according to the absolute differentials quasi-ordering – always implies a reduction in absolute deprivation in terms of well-being taking into account the fact that all tax-liabilities increase or decrease.

Proposition 3.1.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution,* $U \in \mathrm{U}(D)$ *be a utility function, and* $K = AD, ADP$*. Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$*:* $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ *and* $\mathbf{x}^* \geq [\leq] \, \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$*.*
(b) $U(y+\Delta) - U(y)$ *is non-increasing [non-decreasing] in y, for all* $\Delta > 0$ *and all* $y \in D$*.*

The concavity [convexity] of the utility function is needed for smaller [larger] and more minimally progressive taxes to imply a reduction in absolute social deprivation. It is interesting to note that the result is not affected if one substitutes the absolute differentials quasi-ordering for the absolute deprivation quasi-ordering when assessing the impact of taxation on the distribution of after tax utilities. Therefore, so far as one is concerned with the implications of minimal progressivity on the distribution of well-being, it becomes immaterial whether one focuses on utility deprivation or on utility differentials. As we will see in a while, this is no longer true when one argues the other way round and looks for the possibility to improve the distribution of after tax welfare by substituting more progressive taxes for less progressive ones.

Now we turn to the more general case where the increase in progression does not necessarily go with a uniform increase or decrease in tax-liabilities. As the next proposition shows, the only class of utility functions which guarantees a reduction in after tax utility deprivation is the class of affine utility functions. In order to simplify the proof of the next proposition we find convenient to introduce a technical result.

Lemma 3.1.1 (Aczel [1, Theorem 1, p. 142]). *Let* $\phi : D \to D$ *be continuous and increasing. Then, the only solution to the functional equation*

$$\phi(u+\Delta) - \phi(u) = \phi(v+\Delta) - \phi(v), \ \forall \ u, v \in D \ (u < v), \ \forall \ \Delta > 0 \quad (3.1)$$

is $\phi(y) := \alpha + \beta y \ (\beta > 0)$*, for all* $y \in D$*.*

We are now in a position to state the announced result, according to which only affine utility functions guarantee that more minimally progressive taxes reduce absolute social deprivation and absolute utility differentials.

Proposition 3.1.2. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) *For all* $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$*:* $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$.
(b) $U(y) := \alpha + \beta y$ $(\beta > 0)$*, for all* $y \in D$.

So far we have concentrated on the implications of substituting a more minimally progressive taxation programme for a less progressive one on the distribution of well-being in the society. We have shown that under suitable restrictions on the utility function absolute deprivation is always reduced as a result of more equally distributed – according to the absolute differentials quasi-ordering – after tax incomes.

A related but different question would be to know if it would have been possible to improve upon the existing distribution of after tax utilities by substituting a more minimally progressive vector of taxes for the current one. Actually, we have the following result, the proof of which is omitted as it is similar to the proof of Proposition 3.1.2 above.

Proposition 3.1.3. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) *For all* $\mathbf{u}^\circ, \mathbf{u}^* \in \mathbf{U}(\mathcal{Z}(\mathbf{y}))$*:* $\mathbf{u}^* \geq_{AD} \mathbf{u}^\circ$ *implies* $U^{-1}(\mathbf{u}^*) \geq_K U^{-1}(\mathbf{u}^\circ)$.
(b) $U^{-1}(u) := a + bu$ $(a \in \mathbb{R}, b > 0)$*, for all* $u \in \mathbb{R}$.

Thus an affine utility function guarantees that any reduction in absolute utility differentials can be implemented by modifying the taxation system in such a way that after tax incomes become more evenly distributed according to the absolute deprivation quasi-ordering or the absolute differentials quasi-ordering. However, it is important to note that the above result does not tell that less deprivation in social welfare can be achieved by such a tax reform.

Combining Propositions 3.1.2 and 3.1.3, we deduce immediately that only affine utility functions guarantee a one-to-one correspondance between more minimally progressive taxation programmes and welfare improvements as measured by the absolute differentials quasi-ordering.

Theorem 3.1.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution and* $U \in \mathbf{U}(D)$ *be a utility function. Then, the following statements are equivalent:*

(a) *For all* $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$*:* $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ *if and only if* $U(\mathbf{x}^*) \geq_{AD} U(\mathbf{x}^\circ)$.
(b) $U(y) := \alpha + \beta y$ $(\beta > 0)$*, for all* $y \in D$.

Theorem 3.1.1 does not tell anything about the possibility of reducing absolute deprivation in well-being by a modification of the tax-system when the utility function is affine. Actually, Proposition 3.1.3 above raises some doubts about such a possibility as it demonstrates that affine utility functions are already necessary to transform the absolute differentials quasi-ordering of the utility profiles into the absolute differentials quasi-ordering of post-tax distributions. Because it is more demanding to require dominance in terms of absolute differentials than in terms of absolute deprivation, one may intuitively expect that the set of utility functions that guarantee the equivalence between the absolute differentials quasi-ordering of post-tax distributions and the absolute deprivation quasi-ordering of the utility profiles will be a subset of the class of affine utility functions. The next result makes this intuition clear indicating that the class of such utility functions is empty. Precisely:

Theorem 3.1.2. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution and consider the class* $\mathbf{U}(D)$ *of utility functions. Then, there is no* $U \in \mathbf{U}(D)$ *such that:*

$$\forall\, \mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y}) : \mathbf{x}^* \geq_{AD} \mathbf{x}^\circ \text{ if and only if } U(\mathbf{x}^*) \geq_{ADP} U(\mathbf{x}^\circ) . \qquad (3.2)$$

Remark 3.1. In the proof of Theorem 3.1.2, we implicitly dispensed with the restriction that $\mathbf{x}^* \geq \mathbf{x}^\circ$, which is legitimate since we did not impose this further condition when we stated Theorem 3.1.2. One might be interested to know if the introduction of such a restriction modifies our result. Actually it is a straightforward exercise to adapt the argument in order to deal with such a case. Let $u > v$, $\xi > (v-u)/2$, and choose the profiles $\mathbf{u}^\circ := (u, \ldots, u, u, v)$ and $\mathbf{u}^* := (u+\xi, \ldots, u+\xi, (u+v)/2+\xi, (u+v)/2+\xi)$. By definition $\mathbf{u}^* \geq \mathbf{u}^\circ$, $\mathbf{u}^* \geq_{ADP} \mathbf{u}^\circ$ but, since U is increasing so is its inverse U^{-1}, and we obtain

$$U^{-1}\left(u^*_{n-2}\right) - U^{-1}\left(u^\circ_{n-2}\right) < U^{-1}\left(u^*_{n-1}\right) - U^{-1}\left(u^\circ_{n-1}\right) . \qquad (3.3)$$

Choosing next $\mathbf{u}^\circ := (u, \ldots, u, u, v)$ and $\mathbf{u}^* := (u-\xi, \ldots, u-\xi, (u+v)/2-\xi, (u+v)/2-\xi)$, we have $\mathbf{u}^* \leq \mathbf{u}^\circ$ and $\mathbf{u}^* \geq_{ADP} \mathbf{u}^\circ$, hence condition (3.3) holds again. Therefore we conclude that $\neg\left[U^{-1}(\mathbf{u}^*) \geq_{AD} U^{-1}(\mathbf{u}^\circ)\right]$.

Whereas a more minimally progressive taxation programme reduces absolute social deprivation under certain conditions relative to the utility function, the converse does not hold as one cannot find a utility function $U \in \mathbf{U}(D)$ such that a reduction in social absolute deprivation can be achieved through an increase in minimal progression.

3.2 Relative progressive taxation and absolute deprivation in well-being

Though minimal progressivity is an intuitively appealing notion, it is fair to admit that the literature on income taxation has mainly focused on the alternative notion of average rate progressivity. In our particular framework where we are interested in the distributional incidence of taxation programmes with the same pre-tax distribution and non-negative tax-revenue, neither concept of progressivity appears to be stronger than the other one [9].

[9] Actually one notion of progressivity can be shown to imply the other in particular situations. Given a pre-tax distribution $\mathbf{y} \in \mathcal{Y}(D)$, consider two after tax distributions $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$ such that $x^*_i \geq x^\circ_i$,

We first investigate the class of utility functions that guarantee that a more average rate progressive taxation programme – equivalently, more equally distributed after tax incomes according to the relative differentials quasi-ordering – always results in a reduction in absolute social deprivation taking into account the fact that tax-liabilities increase or decrease uniformly.

Proposition 3.2.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$: $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$ *and* $\mathbf{x}^* \geq [\leq] \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$.
(b) $U(\lambda y) - U(y)$ *is non-increasing [non-decreasing] in* y, *for all* $\lambda > 1$ *and all* $y \in D$.

Condition (b) in Proposition 3.2.1 is a particular case of the concept of $(f\text{-}g)$-concavity introduced by Avriel [5] which encompasses a number of well-known properties [10]. Usual concavity is obtained when $f = g = I$ and condition (b) above follows from letting $f = \ln$ and $g = I$, where I represents the identity function defined by $I(y) := y$, for all $y \in \mathbb{R}$. When the utility function is differentiable, condition (b) can be shown to be equivalent to

$$U'(y)y \text{ is non-increasing in } y, \ \forall\ y \in D, \tag{3.4}$$

which means that U is concave in the logarithm of income (see Moyes [32] for more details on $(f\text{-}g)$-concavity and an application to minimal equal sacrifice taxation). In words, this says that the absolute gain in utility caused by a proportional increase in income is non-increasing with income, whatever be the factor of proportionality. Therefore $(\ln\text{-}I)$-concavity – or shortly concavity in the logarithms – is the condition to be imposed on the social norm if we want that smaller and more average rate progressive taxes imply a reduction in absolute social deprivation. As it was the case for Proposition 3.1.1, we insist on the fact that this result is not affected if one substitutes the absolute differentials quasi-ordering for the absolute deprivation quasi-ordering when evaluating the distributional impact of taxation on after tax utilities.

We consider next the case where the increase in average rate progression is not necessarily accompanied by a uniform increase or decrease in tax-liabilities. As it can be anticipated from former results, the fact that we impose no condition other than an increase in average rate progression will narrow down the set of utility functions. In order to simplify the proof of the next proposition we find convenient to introduce the following technical result.

for all $i = 1, 2, \ldots, n$. Then, it can be easily checked that (i) $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ whenever $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$, and (ii) $\mathbf{x}^\circ \geq_{RD} \mathbf{x}^*$ whenever $\mathbf{x}^\circ \geq_{AD} \mathbf{x}^*$.

[10] Given two continuous and increasing functions $f, g : \mathbb{R} \to \mathbb{R}$, we will say that $\phi : D \subseteq \mathbb{R} \to \mathbb{R}$ is *$(f\text{-}g)$-concave* if $\phi \circ f^{-1}((1-\lambda)f(u) + \lambda f(v)) \geq g^{-1}((1-\lambda)g \circ \phi(u) + \lambda g \circ \phi(v))$, for all $u, v \in D$ and all $\lambda \in [0,1]$. One would similarly define *$(f\text{-}g)$-convexity* by reversing the inequality sign in the previous condition. A function which is $(f\text{-}g)$-concave and $(f\text{-}g)$-convex is said to be *$(f\text{-}g)$-affine*.

Lemma 3.2.1 (Aczel [1, Theorem 4, p. 144]). *Let $D \subseteq \mathbb{R}_{++}$ and $\phi : D \to D$ be continuous and increasing. Then, the only solution to the functional equation*

$$\phi(\lambda u) - \phi(u) = \phi(\lambda v) - \phi(v), \ \forall \ u, v \in D \ (u < v), \ \forall \ \lambda > 1 \qquad (3.5)$$

is $\phi(y) := \alpha + \beta \ln y$ $(\beta > 0)$, for all $y \in D$.

We are now in a position to state the announced result according to which only affine in the logarithms utility functions guarantee that more average rate progressive taxes reduce absolute social deprivation and absolute utility differentials.

Proposition 3.2.2. *Let $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ be an arbitrary before tax distribution, $U \in \mathbf{U}(D)$ be a utility function, and $K = AD, ADP$. Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$: $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$ implies $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$.
(b) $U(y) := \alpha + \beta \ln y$ $(\beta > 0)$, for all $y \in D$.

The preceeding results have concentrated on the implications of substituting a more average rate progressive taxation programme for a less progressive one on the distribution of well-being in the society. We have shown that under suitable restrictions on the utility function, absolute deprivation always decreases when the distribution of after tax incomes is becoming more equal in the sense that all pairwise relative income differentials are made smaller.

An important question from a practical point of view would be to know if it is possible to improve upon the current distribution of after tax utilities by substituting a more average rate progressive vector of taxes for the existing one. The lemma below will be needed for establishing our next result:

Lemma 3.2.2 (Aczel [1, Theorem 2, p. 143]). *Let $D \subseteq \mathbb{R}_{++}$ and $\phi : D \to D$ be continuous and increasing. Then, the only solution to the functional equation*

$$\frac{\phi(u + \Delta)}{\phi(u)} = \frac{\phi(v + \Delta)}{\phi(v)}, \ \forall \ u, v \in D \ (u < v), \ \forall \ \Delta > 0 \qquad (3.6)$$

is $\phi(y) := \gamma \exp(\beta y)$ $(\gamma, \beta > 0)$, for all $y \in D$.

The following proposition indicates that it is possible to reduce the absolute differentials in utilities by increasing average rate progression provided that the inverse of the utility function is (I-exp)-affine, or equivalently, the utility function function is affine in the logarithms.

Proposition 3.2.3. *Let $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ be an arbitrary before tax distribution, $U \in \mathbf{U}(D)$ be a utility function, and $K = RD, RDP$. Then, the following statements are equivalent:*

(a) For all $\mathbf{u}^\circ, \mathbf{u}^* \in \mathbf{U}(\mathcal{Z}(\mathbf{y}))$: $\mathbf{u}^* \geq_{AD} \mathbf{u}^\circ$ implies $U^{-1}(\mathbf{u}^*) \geq_K U^{-1}(\mathbf{u}^\circ)$.
(b) $U^{-1}(u) := \gamma \exp(\beta u)$ $(\gamma, \beta > 0)$, for all $u \in \mathbb{R}$.

Thus an affine in the logarithms utility function guarantees that any reduction in absolute utility differentials can be implemented by modifying the taxation system in such a way that after tax incomes become more evenly distributed according to the relative deprivation quasi-ordering or the relative differentials quasi-ordering. However, it not clear whether less deprivation in social welfare can be achieved by such a modification of the tax structure.

Before investigating this issue, we deduce immediately from Propositions 3.2.2 and 3.2.4 that affine in the logarithms utility functions guarantee the existence of a one-to-one correspondance between more average rate progressive taxation programmes and welfare improvements as measured by the absolute differentials quasi-ordering.

Theorem 3.2.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ *be an arbitrary before tax distribution and* $U \in \mathbf{U}(D)$ *be a utility function. Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$*:* $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$ *if and only if* $U(\mathbf{x}^*) \geq_{AD} U(\mathbf{x}^\circ)$.
(b) $U(y) := \alpha + \beta \ln y$ $(\beta > 0)$*, for all* $y \in D$.

Theorem 3.2.1 does not tell anything about the possibility of reducing absolute deprivation in well-being by a modification of the tax-system under affine in the logarithms utility functions. Theorem 3.2.2, which can be readily anticipated from Proposition 3.2.4, makes clear that it is not always possible to reduce social deprivation by means of an increase in average rate progression. The proof of this result, which exploits the fact that the utility function is increasing, is omitted as it parallels the proof of Theorem 3.1.2.

Theorem 3.2.2. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ *be an arbitrary before tax distribution and consider the class* $\mathbf{U}(D)$ *of utility functions. Then, there is no* $U \in \mathbf{U}(D)$ *such that:*

$$\forall\, \mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y}) : \mathbf{x}^* \geq_{RD} \mathbf{x}^\circ \text{ if and only if } U(\mathbf{x}^*) \geq_{ADP} U(\mathbf{x}^\circ) . \quad (3.7)$$

Formally, the above result shows that it is not possible to establish an equivalence between an increase in average rate progression – equivalently, more equally distributed after tax incomes according to the relative differentials quasi-ordering – and a decrease in social absolute deprivation. From Proposition 3.2.2 we know that an increase in average rate progression always yields a reduction of deprivation. Now, Theorem 3.2.2 makes clear that increasing average rate progression is not the only means for improving the distribution of well-being according to our absolute deprivation quasi-ordering. To some extent, this result goes against Chakravarty and Mukherjee's [10] claim according to which dominance in terms of social absolute deprivation and increased average rate progression do coincide when the utility function is affine in the logarithms. One may conceivably object that our argument does not apply because our analysis is not framed in terms of net income-schedules as it is done in the above mentioned paper. However the distinction is immaterial as we will demonstrate in a while.

4 Extensions and discussion of related results in the literature

4.1 The relative redistributive effect of progressive taxation in the individual welfare space

In the preceeding section we have examined the implications of substituting a more progressive taxation programme for a less progressive one on absolute social deprivation. We have identified two classes of utility functions – conditional upon the chosen progressivity concept – which guarantee that, whatever be the member of the class selected, the absolute deprivation profile moves downwards as progressivity increases. One might rather conceive individual deprivation as a relative notion and can therefore restrict attention to the relative deprivation quasi-ordering defined by (2.6). The question that naturally arises here is: what is the class of utility functions that will guarantee a reduction in relative social deprivation as taxes become more progressive? Actually all the preceeding results extend easily if (i) we restrict attention to the subset $\mathbf{U}(D)_{++}$, which consists of all utility functions $U \in \mathbf{U}(D)$ such that $U(y) > 0$, for all $y \in D$, and (ii) we substitute the logarithmic transformation of the utility functions for the utility functions we obtained. Because the proofs of all the results are *mutatis mutandis* the same, we do not repeat the arguments and summarize the findings by means of Table 2, which indicates the conditions to be met by the social norm for an increase in progression – accompanied or not by a uniform decrease in tax-liabilities – to imply a reduction in social deprivation. It is interesting to note that we obtain exactly the same classes of utility functions that arise in the case of minimal equal sacrifice taxation (see Moyes [32]). This was already emphasized by Chakravarty and Mukherjee [10] in the particular case of average rate progression and absolute social deprivation.

Table 2. Progression and social deprivation

Tax progression	Social deprivation	
	$U(\mathbf{x}^*) \geq ADPU(\mathbf{x}^\circ)$	$U(\mathbf{x}^*) \geq RDPU(\mathbf{x}^\circ)$
$\mathbf{x}^* \geq AD\mathbf{x}^\circ$ and $\mathbf{x}^* \geq \mathbf{x}^\circ$	(*I*-*I*)-concavity $U'(y) \downarrow y$	(ln-*I*)-concavity $U'(y)y \downarrow y$
$\mathbf{x}^* \geq AD\mathbf{x}^\circ$	(*I*-*I*)-affine $U(y) = \alpha + \beta y (\beta > 0)$	(ln-*I*)-affine $U(y) = \alpha + \beta \ln y (\beta > 0)$
$\mathbf{x}^* \geq RD\mathbf{x}^\circ$ and $\mathbf{x}^* \geq \mathbf{x}^\circ$	(*I*-ln)-concavity $U'(y)/U(y) \downarrow y$	/ln-ln)-concavity $U'(y)y/U(y) \downarrow y$
$\mathbf{x}^* \geq RD\mathbf{x}^\circ$	(*I*-ln)-affine $U(y) = \exp(\alpha + \beta y)(\beta > 0)$	(ln-ln)-affine $U(y) = \beta y^\eta (\beta, \eta > 0)$

4.2 Social deprivation and the progressivity of the tax-schedule

As we already insisted in the Introduction, our analysis was partially motivated by former results by Chakravarty and Mukherjee [10] derived in a slightly different context. Indeed, these authors framed their analysis of the distributional impact of progressive taxation on well-being in terms of net income-schedules rather than in terms of taxation programmes as it is done in this paper. Because, the introduction of a function actually places more structure on the problem to be analysed, one may wonder whether our results hold in their framework. This is in particular crucial for Theorems 3.1.2 and 3.2.2 since we got an impossibility, which apparently contradicts a claim by Chakravarty and Mukherjee [10].

To make things precise, we consider the natural equivalent of Proposition 3.2.1 when the analysis is framed in terms of tax-schedules rather than taxation programmes. However, before presenting the analogue to Proposition 3.2.1, we need to introduce a further piece of notation. Given $D \subseteq \mathbb{R}_{++}$, we let $\mathcal{F}(D) := \{f : D \to D \mid f \text{ is continuous and non-decreasing}\}$ represent the set of net income-schedules. Given two net income-schedules $f^\circ, f^* \in \mathcal{F}(D)$, we will say that f^* *is more average rate progressive than* f°, which we write $f^\circ \geq_{ARP} f^*$, if

$$\frac{f^*(y)}{f^\circ(y)} \text{ is non-increasing in } y,\ \forall\ y \in D. \tag{4.1}$$

We will also write $f^* \geq [\leq]\ f^\circ$ to mean that $f^*(y) \geq [\leq]\ f^\circ(y)$, for all $y \in D$ [11]. Finally, given the net income-schedule $f \in \mathcal{F}(D)$ and the pre-tax distribution $\mathbf{y} \in \mathcal{Y}(D)$, we let $f(\mathbf{y}) := (f(y_1), \ldots, f(y_n))$ represent the resulting post-tax distribution. Then the analogue to Proposition 3.2.1, when net income-schedules are involved, becomes:

Proposition 4.1. *Let $D \subseteq \mathbb{R}_{++}$ be an arbitrary interval, $U \in \mathbf{U}(D)$ be a utility function, and $K = AD, ADP$. Then, the following statements are equivalent:*

(a) *For all $f^\circ, f^* \in \mathcal{F}(D) : f^* \geq_{ARP} f^\circ$ and $f^* \geq f^\circ \implies [\forall \mathbf{y} \in \mathcal{Y}(D) : U(f^*(\mathbf{y})) \geq_K U(f^\circ(\mathbf{y}))]$.*
(b) *$U(\lambda y) - U(y)$ is non-increasing [non-decreasing] in y, for all $\lambda > 1$ and all $y \in D$.*

Arguing as above, one can check that all the results we derived for taxation programmes actually extend to net income-schedules. However, it is important to stress that statement (a) in the adapted propositions has to be rewritten in an appropriate way. In particular, one needs here the "for all $f^\circ, f^* \in \mathcal{F}(D)$" clause, which allows to choose freely the net income-schedules in the necessity part of the relevant proofs. To convince the reader that Theorem 3.2.2 still holds when applied to net income-schedule, we have to show that a (ln-I)-affine utility function does not guarantee that, for all $f^\circ, f^* \in \mathcal{F}(D)$:

$$[\forall \mathbf{y} \in \mathcal{Y}(D) : U(f^*(\mathbf{y})) \geq_{ADP} U(f^\circ(\mathbf{y}))] \implies f^* \geq_{ARP} f^\circ. \tag{4.2}$$

[11] Actually (4.1) is equivalent to the condition that $f^{*\prime}(y)y/f^*(y) \leq f^{\circ\prime}(y)y/f^\circ(y)$, for all $y \in D$, i.e., the elasticity of f^* is nowhere greater than the elasticity f^* (see e.g. Le Breton et al. [24]).

Given $D \subseteq \mathbb{R}_{++}$ and the utility function $U(y) := \alpha + \beta \ln y$ ($\beta > 0$), we claim that it is possible to find two schedules $f^\circ, f^* \in \mathcal{F}(D)$ such that $U(f^*(\mathbf{y})) \geq_{ADP} U(f^\circ(\mathbf{y}))$, for all $\mathbf{y} \in \mathcal{Y}(D)$, and f^* is not more average rate progressive than f°. The idea is to choose f° and f^* such that (i) $f^* \leq f^\circ$, and (ii) f^* is more minimally progressive than f°. Then, it follows that $U(f^*(\mathbf{y})) \leq U(f^\circ(\mathbf{y}))$ and $U(f^*(\mathbf{y})) \geq_{ADP} U(f^\circ(\mathbf{y}))$, for all $\mathbf{y} \in \mathcal{Y}(D)$. However, this is perfectly compatible with the fact $f^*(u)/f^\circ(u) < f^*(v)/f^\circ(v)$, for some $u < v$ ($u, v \in D$): take for instance $f^\circ(y) := y$ and $f^*(y) := \max\{y/2, y-1\}$, for all $y \in D := (0, \infty)$.

5 Summary and concluding remarks

We have investigated the classes of utility functions which ensure that progressive taxation always implies an improvement in the distribution of well-being as measured by the amount of reduction in deprivation felt in the society. Depending on the concepts of progressivity and deprivation the planner chooses, one gets particular elements of the general family of (f-g)-concave utility functions. Usual concavity is obtained when progressivity and deprivation are conceived in an absolute way. Concavity in the logarithms is the condition needed for a more average rate progressive taxation programme to imply less absolute social deprivation. These results extend to relative social deprivation by simply considering the logarithmic transformations of the above mentioned classes of utility functions. It is not possible to obtain an equivalence between an increase in progression and a reduction in social deprivation. All the results obtained for taxation programmes carry over to tax-schedules under minor restrictions.

Our analysis is concerned with homogeneous populations where the tax-units differ only with respect to income. In practical situations however it is possible to distinguish households on the basis of differing needs and these are generally taken into account by the tax authorities when computing the tax-liabilities of the tax-units. The analysis would be enriched – and consequently made more operational – by incorporating the possibility of differentiating households according to a number of characteristics such as family size and composition. There are at least two possible avenues by which our results can be generalised to the case of heterogeneous individuals or households. The first option is to follow the usual practice which consists in converting multidimensional distributions into unidimensional distributions for a virtual population of homogeneous units by means of equivalence scales. For instance one may be willing to build on the results by Ebert and Moyes [12] on the taxation of heterogeneous households. However, as shown by Ebert and Moyes [14], there are problems with the standard equivalence scale technique which may undermine the significance of the normative conclusions to which they give rise. A more attractive possibility is to pursue the approach of Atkinson and Bourguignon [4] and work directly with multidimensional criteria. The main difficulty here is that there exists at the moment no straightforward generalisation of the criteria we used in the paper to a multidimensional setting. A way of simplifying things is to confine ourselves to considering purely redistributive taxes, in which case the Atkinson and Bourguignon's [4] dominance criteria are natural candidates for measuring *distributional progressivity*. Investigations along these lines would certainly enrich the analysis of the redistributive effects of taxation in a heterogeneous environment and they are left for future research.

Appendix A. Welfare implications of progressive taxes

We are interested in the implications of a modification of the tax-system that shifts the burden of the tax from low income earners to high income earners in such a way that the tax-revenue is not affected. Then it can be shown that such a tax reform will improve the distribution of individual utilities according to the generalised Lorenz criterion – and thus by any equity and efficiency oriented social welfare functional – provided that the common individual utility function be concave. Given two non-decreasingly arranged vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$, we will say that $\mathbf{u}$ *generalised Lorenz dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_{GL} \mathbf{v}$, if and only if:

$$\sum_{j=1}^{k} u_i \geq \sum_{j=1}^{k} v_i, \ \forall\ k = 1, 2, \ldots, n \tag{A.1}$$

(see Shorrocks [38]). If (A.1) holds with an equality for $k = n$, we will say that $\mathbf{u}$ *Lorenz dominates* $\mathbf{v}$, which we write $\mathbf{u} \geq_L \mathbf{v}$. Lorenz dominance is a subrelation of generalised Lorenz dominance as the former implies the latter but not the converse. Now we have the following result:

Proposition A.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution and* $U \in \mathbf{U}(D)$ *be a utility function. Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$*:* $\mathbf{x}^* \geq_L \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_{GL} U(\mathbf{x}^\circ)$.
(b) U *is concave.*

Proof.
$(a) \Longrightarrow (b)$: Suppose that U is not concave in which case there exists $w, t \in D$ $(w < t)$ such that $U((w+t)/2) < (U(w)+U(t))/2$. Choose $\mathbf{x}^\circ := (w, \ldots, w, w, t)$ and $\mathbf{x}^* := (w, \ldots, w, (w+t)/2, (w+t)/2)$. Then clearly $\mathbf{x}^* \geq_L \mathbf{x}^\circ$ but $\sum_{i=1}^{n} U(x_i^*) < \sum_{i=1}^{n} U(x_i^\circ)$, which implies that $Not\,[U(\mathbf{x}^*) \geq_{GL} U(\mathbf{x}^\circ)]$.
$(b) \Longrightarrow (a)$: It is well-know that $\mathbf{x}^* \geq_L \mathbf{x}^\circ$ is equivalent to the fact fact that $\mathbf{x}^*$ is obtained from $\mathbf{x}^\circ$ by means of a finite sequence of progressive transfers (see e.g. Dasgupta, Sen and Starrett [11], Marshall and Olkin [25]). Without loss of generality, suppose that $\mathbf{x}^*$ is obtained from $\mathbf{x}^\circ$ by a rank-preserving progressive transfer so that $\mathbf{x}^* := \left(x_1^\circ, \ldots, x_i^\circ + \Delta, \ldots, x_j^\circ - \Delta, \ldots, x_n^\circ\right)$, where $0 < \Delta \leq (x_j^\circ - x_i^\circ)/2$. Then it can be checked directly (see e.g. Moyes [29]) that the concavity of U implies that $U(\mathbf{x}^*) \geq_{GL} U(\mathbf{x}^\circ)$. □

An interesting implication of Proposition A.1 is that the concavity of the utility function is sufficient [and necessary] for a Lorenz improving redistribution of taxes between individuals to imply a social welfare enhancement according to any equity oriented social welfare functional.

Appendix B. Proofs of the results

Lemma 2.1. *The two following statements are true but not their converse unless* $n = 2$.

(a) *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$: $\mathbf{u} \geq_{AD} \mathbf{v}$ *implies* $\mathbf{u} \geq_{ADP} \mathbf{v}$.
(b) *For all* $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_{++}$: $\mathbf{u} \geq_{RD} \mathbf{v}$ *implies* $\mathbf{u} \geq_{RDP} \mathbf{v}$.

Proof. Statement (a): Suppose that $\mathbf{u} \geq_{ADP} \mathbf{v}$, which is by definition equivalent to

$$\sum_{j=k+1}^{n} (u_j - u_k) \leq \sum_{j=k+1}^{n} (v_j - v_k), \ \forall\ k = 1, 2, \ldots, n-1. \tag{B.1}$$

A sufficient condition for (B.1) to hold is that every term within brackets on the left hand side is greater than the corresponding term on the right hand side, which upon decomposition is equivalent to

$$(u_j - u_{j-1}) + \cdots + (u_{k+1} - u_k) \leq (v_j - v_{j-1}) + \cdots + (v_{k+1} - v_k). \tag{B.2}$$

Now a sufficient condition for (B.2) to hold is that $u_{i+1} - u_i \leq v_{i+1} - v_i$, for all $i = j-1, j, \ldots, k$, which is implied by $\mathbf{u} \geq_{AD} \mathbf{v}$.

Statement (b): Suppose that $\mathbf{u} \geq_{RDP} \mathbf{v}$, which upon manipulation, is equivalent to

$$\sum_{j=k}^{n} (u_j - u_k)/u_k \leq \sum_{j=k}^{n} (v_j - v_k)/v_k, \ \forall\ k = 1, 2, \ldots, n-1. \tag{B.3}$$

A sufficient condition for (B.3) to hold is that

$$u_j/u_{j-1} \times \cdots \times u_{k+1}/u_k \leq v_j/v_{j-1} \times \cdots \times v_{k+1}/v_k. \tag{B.4}$$

This is verified as soon as $u_{i+1}/u_i \leq v_{i+1}/v_i$, for all $i = j-1, j, \ldots, k$, which is implied by $\mathbf{u} \geq_{RD} \mathbf{v}$.

To prove that the converse implications are false whenever $n > 2$, consider the situations $\mathbf{u} = (u, \ldots, u, (u+v)/2, (u+v)/2)$ and $\mathbf{v} = (u, \ldots, u, u, v)$ where $u < v$. Then we have $\mathbf{u} \geq_{RDP} \mathbf{v}$ and $\mathbf{u} \geq_{ADP} \mathbf{v}$, but $\neg [\mathbf{u} \geq_{RD} \mathbf{v}]$ and $\neg [\mathbf{u} \geq_{AD} \mathbf{v}]$. □

Proposition 3.1.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) *For all* $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$: $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ *and* $\mathbf{x}^* \geq [\leq] \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$.
(b) $U(y + \Delta) - U(y)$ *is non-increasing* [*non-decreasing*] *in* y, *for all* $\Delta > 0$ *and all* $y \in D$.

Proof. We consider only the situation where every tax-unit pays less taxes in $(\mathbf{y}; \mathbf{x}^*)$ than in $(\mathbf{y}; \mathbf{x}^\circ)$, that is, where $\mathbf{x}^* \geq \mathbf{x}^\circ$. The proof in the other case follows similarly.

$(b) \Longrightarrow (a)$: Choose any $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$ such that $x_i^* - x_i^\circ \geq x_{i+1}^* - x_{i+1}^\circ$, for all $i = 1, 2, \ldots, n-1$, and $x_i^* \geq x_i^\circ$, for all $i = 1, 2, \ldots, n$. Using condition (b) and the fact that U is increasing, we have

$$U(x_i^*) - U(x_i^\circ) = U(x_i^\circ + (x_i^* - x_i^\circ)) - U(x_i^\circ) \geq \tag{B.5}$$
$$U\left(x_{i+1}^\circ + (x_i^* - x_i^\circ)\right) - U\left(x_{i+1}^\circ\right) \geq U\left(x_{i+1}^\circ + \left(x_{i+1}^* - x_{i+1}^\circ\right)\right) - U\left(x_{i+1}^\circ\right) =$$
$$U\left(x_{i+1}^*\right) - U\left(x_{i+1}^\circ\right), \ \forall\, i = 1, 2, \ldots, n-1,$$

which along with Lemma 2.1 implies that $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$, for $K = AD, ADP$.

$(a) \Longrightarrow (b)$: Take any $u < v$ and $\Delta > 0$, and choose $\mathbf{x}^\circ = (u, u + \Delta, \ldots, u + \Delta)$ and $\mathbf{x}^* = (v, v + \Delta, \ldots, v + \Delta)$, so that $\mathbf{x}^* \geq \mathbf{x}^\circ$ and $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$. If statement (a) holds, then in particular $U(\mathbf{x}^*) \geq_{AD} U(\mathbf{x}^\circ)$, and therefore

$$U(u + \Delta) - U(u) \geq U(v + \Delta) - U(v), \tag{B.6}$$

which, since this holds true for all $u < v$ and $\Delta > 0$, implies condition (b). □

Proposition 3.1.2. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$: $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$.
(b) $U(y) := \alpha + \beta y$ $(\beta > 0)$, *for all* $y \in D$.

Proof.
$(b) \Longrightarrow (a)$: Choose any $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$ such that $x_i^* - x_i^\circ \geq x_{i+1}^* - x_{i+1}^\circ$, for all $i = 1, 2, \ldots, n-1$. Given any $\alpha \in \mathbb{R}$ and $\beta > 0$, this implies that

$$(\alpha + \beta x_i^*) - (\alpha + \beta x_i^\circ) \geq \left(\alpha + \beta x_{i+1}^*\right) - \left(\alpha + \beta x_{i+1}^\circ\right), \forall\, i = 1, 2, \ldots, n-1. \tag{B.7}$$

Choosing $U(y) = \alpha + \beta y$, for all $y \in D$, we finally obtain

$$U(x_i^*) - U(x_i^\circ) \geq U\left(x_{i+1}^*\right) - U\left(x_{i+1}^\circ\right), \ \forall\, i = 1, 2, \ldots, n-1, \tag{B.8}$$

so that condition (a) holds.

$(a) \Longrightarrow (b)$: Take any $u < v$ and $\Delta > 0$, and choose $\mathbf{x}^\circ = (u, u + \Delta, \ldots, u + \Delta)$ and $\mathbf{x}^* = (v, v + \Delta, \ldots, v + \Delta)$, so that $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$. If statement (a) holds, then in particular $U(\mathbf{x}^*) \geq_{AD} U(\mathbf{x}^\circ)$, and therefore

$$U(u + \Delta) - U(u) \geq U(v + \Delta) - U(v). \tag{B.9}$$

Choosing now $\mathbf{x}^\circ = (v, v + \Delta, \ldots, v + \Delta)$ and $\mathbf{x}^* = (u, u + \Delta, \ldots, u + \Delta)$, we have $\mathbf{x}^* \geq_{AD} \mathbf{x}^\circ$ and we obtain along a similar reasoning

$$U(u + \Delta) - U(u) \leq U(v + \Delta) - U(v). \tag{B.10}$$

Since (B.9) and (B.10) hold for all $u < v$ and $\Delta > 0$, we can combine them and employ Lemma 3.1.1 to get condition (b). □

Theorem 3.1.2. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ *be an arbitrary before tax distribution and consider the class* $\mathbf{U}(D)$ *of utility functions. Then, there is no* $U \in \mathbf{U}(D)$ *such that:*

$$\forall\ \mathbf{x}^{\circ}, \mathbf{x}^{*} \in \mathcal{Z}(\mathbf{y}) : \mathbf{x}^{*} \geq_{AD} \mathbf{x}^{\circ} \text{ if and only if } U(\mathbf{x}^{*}) \geq_{ADP} U(\mathbf{x}^{\circ}) . \qquad (3.2)$$

Proof. To establish the impossibility, it suffices to show that for any utility function $U \in \mathbf{U}(D)$ the following condition cannot hold:

$$\forall\ \mathbf{u}^{\circ}, \mathbf{u}^{*} \in \mathbf{U}(\mathcal{Z}(\mathbf{y})) : \mathbf{u}^{*} \geq_{ADP} \mathbf{u}^{\circ} \implies U^{-1}(\mathbf{u}^{*}) \geq_{AD} U^{-1}(\mathbf{u}^{\circ}) . \qquad \text{(B.11)}$$

Choose $\mathbf{u}^{\circ} := (u, \ldots, u, u, v)$ and $\mathbf{u}^{*} := (u, \ldots, u, (u+v)/2, (u+v)/2)$, so that $\mathbf{u}^{*} \geq_{ADP} \mathbf{u}^{\circ}$. The corresponding post-tax distributions are given by $\mathbf{x}^{\circ} := U^{-1}(\mathbf{u}^{\circ})$ and $\mathbf{x}^{*} := U^{-1}(\mathbf{u}^{*})$. Since by definition U is increasing so is its inverse U^{-1} and it follows that

$$U^{-1}\left(u^{*}_{n-2}\right) - U^{-1}\left(u^{\circ}_{n-2}\right) = 0 < U^{-1}\left(u^{*}_{n-1}\right) - U^{-1}\left(u^{\circ}_{n-1}\right) , \qquad \text{(B.12)}$$

which implies that $\neg \left[U^{-1}(\mathbf{u}^{*}) \geq_{AD} U^{-1}(\mathbf{u}^{\circ})\right]$. □

Proposition 3.2.1. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) *For all* $\mathbf{x}^{\circ}, \mathbf{x}^{*} \in \mathcal{Z}(\mathbf{y})$: $\mathbf{x}^{*} \geq_{RD} \mathbf{x}^{\circ}$ *and* $\mathbf{x}^{*} \geq [\leq]\ \mathbf{x}^{\circ}$ *implies* $U(\mathbf{x}^{*}) \geq_{K} U(\mathbf{x}^{\circ})$.
(b) $U(\lambda y) - U(y)$ *is non-increasing* [*non-decreasing*] *in* y, *for all* $\lambda > 1$ *and all* $y \in D$.

Proof. We consider only the situation where every tax-unit pays less taxes in $(\mathbf{y}; \mathbf{x}^{*})$ than in $(\mathbf{y}; \mathbf{x}^{\circ})$ since the proof is similar in either case.
$(b) \implies (a)$: Choose any $\mathbf{x}^{\circ}, \mathbf{x}^{*} \in \mathcal{Z}(\mathbf{y})$ such that $x^{*}_{i}/x^{\circ}_{i} \geq x^{*}_{i+1}/x^{\circ}_{i+1}$, for all $i = 1, 2, \ldots, n-1$, and $x^{*}_{i} \geq x^{\circ}_{i}$, for all $i = 1, 2, \ldots, n$. Using condition (b) and the fact that U is increasing, we have

$$\begin{aligned} &U\left(x^{*}_{i}\right) - U\left(x^{\circ}_{i}\right) = U\left(\left(\frac{x^{*}_{i}}{x^{\circ}_{i}}\right) x^{\circ}_{i}\right) - U\left(x^{\circ}_{i}\right) \geq \qquad \text{(B.13)} \\ &U\left(\left(\frac{x^{*}_{i}}{x^{\circ}_{i}}\right) x^{\circ}_{i+1}\right) - U\left(x^{\circ}_{i+1}\right) \geq U\left(\left(\frac{x^{*}_{i+1}}{x^{\circ}_{i+1}}\right) x^{\circ}_{i+1}\right) - U\left(x^{\circ}_{i+1}\right) = \\ &U\left(x^{*}_{i+1}\right) - U\left(x^{\circ}_{i+1}\right), \ \forall\, i = 1, 2, \ldots, n-1, \end{aligned}$$

which along with Lemma 2.1 implies that $U(\mathbf{x}^{*}) \geq_{K} U(\mathbf{x}^{\circ})$, for $K = AD, ADP$.

$(a) \Longrightarrow (b)$: Take any $u < v$ and $\lambda > 1$, and choose $\mathbf{x}^\circ = (u, \lambda u, \ldots, \lambda u)$ and $\mathbf{x}^* = (v, \lambda v, \ldots, \lambda v)$, so that $\mathbf{x}^* \geq \mathbf{x}^\circ$ and $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$. If statement (a) holds, then in particular $U(\mathbf{x}^*) \geq_{AD} U(\mathbf{x}^\circ)$, and therefore

$$U(\lambda u) - U(u) \geq U(\lambda v) - U(v), \tag{B.14}$$

which, since this holds true for all $u < v$ and $\lambda > 1$, implies condition (b). □

Proposition 3.2.2. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = AD, ADP$. *Then, the following statements are equivalent:*

(a) For all $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$*:* $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$ *implies* $U(\mathbf{x}^*) \geq_K U(\mathbf{x}^\circ)$.
(b) $U(y) := \alpha + \beta \ln y$ $(\beta > 0)$*, for all* $y \in D$.

Proof.
$(b) \Longrightarrow (a)$: Choose any $\mathbf{x}^\circ, \mathbf{x}^* \in \mathcal{Z}(\mathbf{y})$ such that $x_i^*/x_i^\circ \geq x_{i+1}^*/x_{i+1}^\circ$, for all $i = 1, 2, \ldots, n-1$. Given any $\beta, \eta > 0$, this implies

$$\eta (x_i^*)^\beta / \eta (x_i^\circ)^\beta \geq \eta \left(x_{i+1}^*\right)^\beta / \eta \left(x_{i+1}^\circ\right)^\beta, \; \forall\, i = 1, 2, \ldots, n-1. \tag{B.15}$$

Taking the logarithm of (B.15) and choosing $U(y) = \alpha + \beta \ln y$ $(\alpha = \ln \eta)$, for all $y \in D$, we finally obtain

$$U(x_i^*) - U(x_i^\circ) \geq U\left(x_{i+1}^*\right) - U\left(x_{i+1}^\circ\right), \; \forall\, i = 1, 2, \ldots, n-1, \tag{B.16}$$

so that condition (a) holds.
$(a) \Longrightarrow (b)$: Take any $u < v$ and $\lambda > 1$, and choose $\mathbf{x}^\circ = (u, \lambda u, \ldots, \lambda u)$ and $\mathbf{x}^* = (v, \lambda v, \ldots, \lambda v)$, so that $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$. If statement (a) holds, then in particular $U(\mathbf{x}^*) \geq_{AD} U(\mathbf{x}^\circ)$, and therefore

$$U(\lambda u) - U(u) \geq U(\lambda v) - U(v). \tag{B.17}$$

Choosing now $\mathbf{x}^\circ = (v, \lambda v, \ldots, \lambda v)$ and $\mathbf{x}^* = (u, \lambda u, \ldots, \lambda u)$, we have $\mathbf{x}^* \geq_{RD} \mathbf{x}^\circ$ and obtain along a similar reasoning

$$U(\lambda u) - U(u) \leq U(\lambda v) - U(v), \tag{B.18}$$

Now (B.17) and (B.18) hold for all $u < v$ and $\lambda > 1$. Therefore, we can combine them and employ Lemma 3.2.1 to get condition (b). □

Proposition 3.2.3. *Let* $\mathbf{y} \in \mathcal{Y}(D)$ $(D \subseteq \mathbb{R}_{++})$ *be an arbitrary before tax distribution,* $U \in \mathbf{U}(D)$ *be a utility function, and* $K = RD, RDP$. *Then, the following statements are equivalent:*

(a) For all $\mathbf{u}^\circ, \mathbf{u}^* \in \mathbf{U}(\mathcal{Z}(\mathbf{y}))$*:* $\mathbf{u}^* \geq_{AD} \mathbf{u}^\circ$ *implies* $U^{-1}(\mathbf{u}^*) \geq_K U^{-1}(\mathbf{u}^\circ)$.

(b) $U^{-1}(u) := \gamma \exp(\beta u)$ $(\gamma, \beta > 0)$, *for all* $u \in \mathbb{R}$.

Proof.
$(b) \Longrightarrow (a)$: Choose any $\mathbf{u}^\circ, \mathbf{u}^* \in \mathbf{U}(\mathcal{Z}(\mathbf{y}))$ such that $u_i^* - u_i^\circ \geq u_{i+1}^* - u_{i+1}^\circ$, for all $i = 1, 2, \ldots, n-1$. Given any $\alpha \in \mathbb{R}$ and $\beta > 0$, this implies that

$$(\alpha + \beta u_i^*) - (\alpha + \beta u_i^\circ) \geq \left(\alpha + \beta u_{i+1}^*\right) - \left(\alpha + \beta u_{i+1}^\circ\right), \forall\, i = 1, 2, \ldots, n-1. \tag{B.19}$$

Taking the exponential of (B.19) and choosing $U^{-1}(u) = \gamma \exp(\beta u)$ $(\gamma = \exp\alpha)$, for all $u \in \mathbb{R}_{++}$, we finally obtain

$$U^{-1}\left(u_i^*\right)/U^{-1}\left(u_i^\circ\right) \geq U^{-1}\left(u_{i+1}^*\right)/U^{-1}\left(u_{i+1}^\circ\right),\ \forall\, i = 1, 2, \ldots, n-1, \tag{B.20}$$

so that condition (a) holds.
$(a) \Longrightarrow (b)$: Take any $u < v$ and $\Delta > 0$, and choose $\mathbf{u}^\circ = (u, u+\Delta, \ldots, u+\Delta)$ and $\mathbf{u}^* = (v, v+\Delta, \ldots, v+\Delta)$, so that $\mathbf{u}^* \geq_{AD} \mathbf{u}^\circ$. If statement (a) holds, then in particular $U^{-1}(\mathbf{u}^*) \geq_{RD} U^{-1}(\mathbf{u}^\circ)$, and therefore

$$U^{-1}(u+\Delta)/U^{-1}(u) \geq U^{-1}(v+\Delta)/U^{-1}(v)\,. \tag{B.21}$$

Choosing now $\mathbf{u}^\circ = (v, v+\Delta, \ldots, v+\Delta)$ and $\mathbf{u}^* = (u, u+\Delta, \ldots, u+\Delta)$, we have $\mathbf{u}^* \geq_{AD} \mathbf{u}^\circ$ and we obtain along a similar reasoning

$$U^{-1}(u+\Delta)/U^{-1}(u) \leq U^{-1}(v+\Delta)/U^{-1}(v)\,. \tag{B.22}$$

Since (B.21) and (B.22) hold for all $u < v$ and $\Delta > 0$, we can combine them and invoke Lemma 3.2.2 to get condition (b). □

Proposition 4.1. *Let $D \subseteq \mathbb{R}_{++}$ be an arbitrary interval, $U \in \mathbf{U}(D)$ be a utility function, and $K = AD, ADP$. Then, the following statements are equivalent:*

(a) *For all* $f^\circ, f^* \in \mathcal{F}(D) : f^* \geq_{ARP} f^\circ$ *and* $f^* \geq f^\circ \Longrightarrow [\forall \mathbf{y} \in \mathcal{Y}(D) : U(f^*(\mathbf{y})) \geq_K U(f^\circ(\mathbf{y}))]$.
(b) $U(\lambda y) - U(y)$ *is non-increasing [non-decreasing] in* y, *for all* $\lambda > 1$ *and all* $y \in D$.

Proof.
$(b) \Longrightarrow (a)$: The proof is analogous to the proof of sufficiency for Proposition 3.2.1 letting $\mathbf{x}^* = f^*(\mathbf{y})$ and $\mathbf{x}^\circ = f^\circ(\mathbf{y})$.
$(a) \Longrightarrow (b)$: Suppose that $U(\lambda u) - U(u) < U(\lambda v) - U(v)$, for some $u < v$ and some $\lambda > 1$. Choose $\mathbf{y} := (u, \ldots, u, v)$, $f^*(y) := \lambda y$ and $f^\circ(y) := y$, for all $y \in D$. Then we have $f^* \geq_{ARP} f^\circ$ and $f^* \geq f^\circ$. However, we get

$$U\left(f^*\left(y_i\right)\right) - U\left(f^\circ\left(y_i\right)\right) < U\left(f^*\left(y_n\right)\right) - U\left(f^\circ\left(y_n\right)\right),\ \forall\, i = 1, 2, \ldots, n-1, \tag{B.23}$$

which implies that $\neg\,[U\,(f^*(\mathbf{y})) \geq_K U\,(f^\circ(\mathbf{y}))]$.
The proof in the case where every tax-unit pays more taxes under f^* than under f°, i.e., $f^* \leq f^\circ$, is similar and it is therefore omitted. □

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