When expanding out the product, only the terms $1 \cdot 1 \cdot 1 \cdots 1$ and $\left(\frac{t_1}{q}\right) \cdot \left(\frac{-t_2}{q}\right) \cdot \left(\frac{t_3}{q}\right) \cdots \left(\frac{t_p}{q}\right)$ should be taken into consideration; the other terms disappear because Legendre symbols sum up to zero, i.e., $\sum_{t \in \mathbb{Z}/(q)} \left(\frac{t}{q}\right) = 0$. Therefore, the above expression simplifies to

$$q^{p-1} + \left(\frac{(-1)^{\frac{p-1}{2}}}{q}\right) \sum_{t_1+\cdots+t_p=1} \left(\frac{t_1 t_2 t_3 \cdots t_p}{q}\right).$$

Now modulo p, the latter sum almost completely vanishes, since the tuples $(t_1, \ldots, t_p)$ satisfying $t_1 + \cdots + t_p = 1$ with not all t_i equal to p^{-1} can be collected in groups of size p by cyclic permutation. Note that p is indeed a multiplicative unit in $\mathbb{Z}/(q)$. We thus obtain

$$N_p \equiv 1 + \left(\frac{(-1)^{\frac{p-1}{2}}}{q}\right) \left(\frac{p^{-p}}{q}\right) \equiv 1 + (-1)^{\frac{p-1}{2}\frac{q-1}{2}} \left(\frac{p}{q}\right) \mod p.$$

The last congruence follows from the well-known formula $\left(\frac{a}{q}\right) \equiv a^{\frac{q-1}{2}} \bmod q$ (which in the case $a = \pm 1$ becomes an exact equality) and the obvious observation that p^{-p} is a square in $\mathbb{Z}/(q)$ if and only if p is a square in $\mathbb{Z}/(q)$.

Comparing our two formulas for N_p (mod p), the reciprocity law follows.

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Infinite Divisibility of GCD Matrices

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In 1876 H. J. S. Smith [**8**] showed that the determinant of the $n \times n$ matrix A whose entries are given by $a_{ij} = (i, j)$, where (i, j) stands for the greatest common divisor (GCD) of the positive integers i and j, has an interesting formula:

$$\det A = \varphi(1)\varphi(2) \cdots \varphi(n).$$

Here φ represents the Euler totient function. Such matrices, and their generalisations, have since been studied by several authors. See, for example, the references [**5, 7**].

Let $\{x_1, \dots, x_n\}$ be any set of distinct positive integers. The *GCD matrix* G associated with this set is the $n \times n$ matrix whose entries are $g_{ij} = (x_i, x_j)$. One striking property of this matrix is that it is *infinitely divisible*, which, by definition, means that for every nonnegative real number r, the matrix $[g_{ij}^r]$ is positive semidefinite. A proof of this fact was given by K. Bourque and S. Ligh [**4**], and is reproduced in the article [**1**] where several other examples of infinitely divisible matrices are discussed. In this note we present a simple proof that derives this property of GCD matrices from the positive semidefiniteness of the closely related min matrices.

Let $\lambda_1, \dots, \lambda_n$ be nonnegative real numbers. The *min matrix* M associated with them is the matrix with entries $m_{ij} = \lambda_i \wedge \lambda_j = \min(\lambda_i, \lambda_j)$. It is very easy to see that this matrix is positive semidefinite. One proof of this is indicated in Exercise 18, p. 401 of [**6**]. Another simple proof, which also shows M is infinitely divisible, is presented in [**1**].

If $A = [a_{ij}]$ and $B = [b_{ij}]$ are any two $n \times n$ matrices, we denote by $A \circ B$ their *Hadamard product*, or the *entrywise product* $[a_{ij}\, b_{ij}]$. If $a_{ij} \geq 0$ for all i, j, and r is any nonnegative real number, we denote by $A^{\circ r}$ the *Hadamard power* $[a_{ij}^r]$. If A and B are positive semidefinite, then by a well-known theorem of Schur, the product $A \circ B$ is positive semidefinite, and so are the powers $A^{\circ m}$ for $m = 1, 2, \dots$.

Now suppose $x_1, \dots, x_n$ are (distinct) positive integers. Let $p_1, \dots, p_k$ be prime numbers such that

$$x_i = p_1^{m_{i_1}} p_2^{m_{i_2}} \dots p_k^{m_{i_k}}, \quad 1 \leq i \leq n.$$

Then we have

$$(x_i, x_j) = p_1^{m_{i_1} \wedge m_{j_1}} p_2^{m_{i_2} \wedge m_{j_2}} \dots p_k^{m_{i_k} \wedge m_{j_k}}.$$

For $1 \leq \ell \leq k$, let P_ℓ be the matrix whose i, j entry is equal to $p_\ell^{m_{i_\ell} \wedge m_{j_\ell}}$. Then the GCD matrix G with entries $g_{ij} = (x_i, x_j)$ can be expressed as a Hadamard product

$$G = P_1 \circ P_2 \circ \cdots \circ P_k.$$

If we show that each of the matrices P_ℓ is infinitely divisible, then by Schur's theorem G will also be infinitely divisible.

Suppose $A = [a_{ij}]$ is any matrix with nonnegative entries and q is any real number greater than 1. Let E be the matrix that has all its entries equal to 1. Then the matrix $[q^{a_{ij}}]$ can be represented by the series

$$E + (\log q)A + \frac{(\log q)^2}{2!} A^{\circ 2} + \frac{(\log q)^3}{3!} A^{\circ 3} + \cdots.$$

If A is positive semidefinite, then each of these summands, and hence their sum, is positive semidefinite.

This observation, together with the fact that min matrices are positive semidefinite, shows that for each $r \geq 0$ and each ℓ satisfying $1 \leq \ell \leq k$ the matrix

$$P_\ell^{\circ r} = \left[p_\ell^{r(m_{i_\ell} \wedge m_{j_\ell})} \right]$$

is positive semidefinite. In other words, P_ℓ is infinitely divisible as we claimed.

Closely related to the GCD matrix is the Hadamard reciprocal of the LCM matrix. Let ℓ_{ij} be the least common multiple of the positive integers x_i and x_j. Since

$$\frac{1}{\ell_{ij}} = \frac{(x_i, x_j)}{x_i\, x_j},$$

the matrix L whose i, j entry is $1/\ell_{ij}$ can be expressed as XGX, where X is the diagonal matrix whose diagonal entries are the positive numbers $1/x_i$, $1 \leq i \leq n$. Hence the infinite divisibility of G implies the same property for L.

Infinitely divisible matrices arise in several different contexts, and are discussed in [**1**], [**2**], and [**3**].

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Borsuk-Ulam Implies Brouwer: A Direct Construction Revisited

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1. INTRODUCTION. In the MONTHLY article [**3**] Su showed by a direct construction that the Borsuk-Ulam theorem implies the Brouwer fixed point theorem. Su used a cubical ball in his proof. In [**4**] the author using a simple construction for a Euclidean ball showed that the multi-valued generalization of the Borsuk-Ulam theorem implies the multi-valued generalization of the Brouwer theorem, in particular the Kakutani theorem.

The purpose of this note is to present a very simple and short proof by a direct construction that the Schauder fixed point theorem follows from the generalization of the Borsuk-Ulam theorem for Banach spaces due to B. Gel'man [**1**]. In the finite-dimensional case it turns into the proof that the Borsuk-Ulam theorem implies the Brouwer theorem. In Section 4, a class of similar constructions is developed and in Section 5 we show that Su's construction (considered for an arbitrary ball) can be used in this argument as well.

We refer the reader to the paper of Su [**3**] for historical and other comments.