DENSITY IN THE LIGHT OF PROBABILITY THEORY-II*

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SUMMARY. Let $\{X_n\}$ be a sequence of abstract spaces, each X_n consisting of the points 0, 1, 2, ... At the point r in X_n, we place probability $1/q_n^r(1-1/q_n)$, q_n being the n-th prime number. Let X be the product space $X_1 X_2 \ldots$ and let P be the product measure.

Let J be a sequence (j_m) of positive integers. Let S be any set of positive integers. $M_J^U(S)$ is the set of vectors $(x_1, x_2, \ldots) \in X$ such that $2^{x_1} \ldots q_n^{x_n} \in S$ for infinitely many $n \in J$. $M_J^L(S)$ is the set of vectors $(x_1, x_2, \ldots) \in X$ such that $2^{x_1} \ldots q_n^{x_n} \in S$ for all sufficiently large $n \in J$. We prove that $P[M_J^L(S)] \leqslant \delta_L(S) \leqslant \delta^U(S) \leqslant P[M_J^U(S)]$ for all sets S if and only if $\frac{\log j_{m+1}}{\log j_m}$ is bounded as $m \to \infty$. δ_L and δ^U stand for lower and upper logarithmic densities, respectively.

Let f be a finite function defined on the set of positive integers. Suppose for a J satisfying the condition above, $\lim_{m \to \infty} f\left(2^{x_1} \ldots q_{j_m}^{x_{j_m}}\right) = g(x)$ exists with probability 1. Then f has a distribution and this is the same as that of $g(x)$; we employ logarithmic density.

GENERALIZATION OF THE MAGNIFICATION THEOREM

We now generalize the magnification theorem in the case of the special example discussed in the previous paper (Paul, 1962). Let J be a class of positive integers. Let S be an arbitrary set of positive integers. We define the *upper J-magnification* of S, $M_J^U(S)$, to be the set of vectors $(x_1, x_2, \ldots)$ such that $\left(2^{x_1} 3^{x_2} \ldots q_n^{x_n}\right) \epsilon S$ for infinitely many values of $n \epsilon J$. The *lower J-magnification* of S, $M_J^L(S)$, is defined to be the set of vectors $(x_1, x_2, \ldots)$ such that $\left(2^{x_1} 3^{x_2} \ldots q_n^{x_n}\right) \epsilon S$ for all sufficiently large values of n in J. Obviously, $M^L(S) \leqslant M_J^L(S) \leqslant M_J^U(S) \leqslant M^U(S)$. This raises the question of obtaining sharper estimates for lower and upper logarithmic densities.

Let J consist of $j_1, j_2, \ldots$, in ascending order. We shall prove the following theorem.

Theorem : $P[M_J^L(S)] \leqslant \delta^L(S) \leqslant \delta^U(S) \leqslant P[M_J^U(S)]$ *for all sets S if and only if* $\left(\frac{\log j_{n+1}}{\log j_n}\right)$ *remains bounded as* $n \to \infty$.

The proof of the 'if' part is similar to the proof given by the author (Paul, 1962). Let us call the space $X_1 X_2 \ldots X_{j_1}$ by the name Y_1 and $X_{(j_1+1)} X_{(j_1+2)} \ldots X_{j_2}$ by the name $Y_2 \ldots$. In each space X_n, let us introduce the

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measure described earlier by the author (Paul, 1962) and in each space Y_m let us introduce the product measure. X may be looked upon as the space $Y_1 Y_2 Y_3 \ldots$. Instead of the spaces $X_1, X_2, \ldots$ (Paul, 1962; Section 2) we now have $Y_1, Y_2, \ldots$. We treat the point $(0, 0, \ldots, 0)$ of Y_n as the element 0 of X_n. Let $(x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in I \subset X$. We associate with it the number $q_1^{x_1} \ldots q_n^{x_n}$. If $\sigma \subset I$, we define $\delta^U(\sigma)$ to be the upper logarithmic density of the corresponding set of positive integers. The space $Y_1 \; Y_2 \ldots$ and δ satisfy Postulates (A) to (F) of Section 2 and condition G of Section 3 of the previous paper (Paul, 1962). The proof that condition H also holds is similar to the proof given in Section 6 of the previous paper (Paul, 1962) but requires a little explanation. Let B be a right-complete set in $I \subset Y_1 Y_2 \ldots$. Let $(x_1, \ldots, x_m, 0, 0, \ldots)$ be a basic vector of B and let $x_m > 0$. Let $j_n < m \leqslant j_{(n+1)}$. Let

$$f_n(s) = \frac{(1-1/q_1^s)(1-1/q_2^s)\ldots(1-1/q^s_{j(n+1)})}{\left(2^{x_1} \ldots q_m^{x_m}\right)^s}$$

We are interested in proving that $\sum_n f_n(s)$, over all basic vectors, is continuous on [1, 2]. Since m may be $< j_{(n+1)}$, our previous argument does not go through directly. So we introduce

$$\phi_n(s) = \frac{(1-1/q_1^s)\ldots(1-1/q_m^s)}{\left(2^{x_1} \ldots q_m^{x_m}\right)^s}.$$

Then
$$\frac{f_n(s)}{\phi_n(s)} \geqslant \left(1-\frac{1}{q_{j_n}}\right)\cdots\left(1-\frac{1}{q_{j_{n+1}}}\right) > \frac{\log q_{j_n}}{2 \log q_{j_{(n+1)}}},$$

by Merten's theorem, $\geqslant \alpha > 0$, by hypothesis on J.

We now apply the argument given in the previous paper (Paul, 1962) and prove that $\sum_n \phi_n(s)$ is continuous on [1, 2]. Continuity of $\sum_n f_n(s)$ on [1, 2] follows immediately, and the 'if' part is proved.

Before proving the 'only if' part, we give an example of a J and a right-complete set in $I \subset Y_1 Y_2 \ldots$ such that condition H (Paul, 1962) is violated. Of course, in this case $\left(\frac{\log j_{n+1}}{\log j_n}\right)$ will be unbounded. Let us take a fixed number < 1, say $\frac{3}{4}$. Also, let us take the sequence $\frac{9}{10}, \frac{10}{11}, \frac{11}{12}, \ldots \to 1$.

Let $j_1 = 2$, so that the first block of primes is 2, 3. Let us declare $(0, 1, 0, 0, \ldots)$ and $(2, 1, 0, 0, \ldots)$ as basic vectors. The cylinder sets whose bases are the points (0, 1) and (2, 1) carry probability

$$\beta_1 = \frac{(1-\frac{1}{2})(1-\frac{1}{3})}{3} + \frac{(1-\frac{1}{2})(1-\frac{1}{3})}{2^2.3} = \frac{5}{36}.$$

We now determine j_2. The set of numbers of the form $2^{n_1}\,3^{n_2}$ has density zero. Thus the complementary set C_1 has density 1. We now take numbers 5, 7, 10, 11, 13, 14, 15, 17, ..., M_2 of C_1 so that

$$\frac{\Sigma \frac{1}{n} \text{ of these number}}{\sum\limits_{n=1}^{M_2} \frac{1}{n}} > \frac{9}{10}.$$

Let $$\phi(m) = (1-1/q_1)\ldots(1-1/q_m)$$

We take a j_2 so large that

$$\beta_2 = \frac{5}{36}+\phi(j_2).\left\{1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{M_2}\right\} < \frac{3}{4}.$$

We now introduce basic vectors so that 5, 7, 10, ..., M_2 all become members of our right-complete set. In order to admit 5, we declare (0, 0, 1, 0, 0, ...) as a basic vector. In order to admit 7, we declare (0, 0, 0, 1, 0, 0, ...) as a basic vector. For 10, we declare (1, 0, 1, 0, 0, ...), and proceed like this until M_2 gains entry into our right-complete set. Of course, we make j_2 so large that $q_{j_2} > M_2$.

Let C_2 be the complement of the set of numbers of the form $2^{n_1}\ldots q_{j_2}^{n_{j_2}}$. We choose an M_3 so large that

$$\frac{\left\{\sum\limits_{\substack{(j_2+1) < n < M_3, \\ n\epsilon C_2}} \frac{1}{n}\right\}}{\sum\limits_{1}^{M_3} \frac{1}{n}} > \frac{10}{11}.$$

We choose a j_3 so large that $\beta_3 = \beta_2+\phi(j_3)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{M_3}\right) < \frac{3}{4}$.

We then admit basic vectors so that all n in $\underset{n}{E}\left\{n\epsilon C_2,\ q_{(j_2+1)} \leqslant n \leqslant M_3\right\}$ gain entry into our right-complete set. Proceeding like this, we construct a right-complete (with respect to J) set whose upper logarithmic density is $= 1$ but whose magnification has measure $< \frac{3}{4}$.

Now, let J be any given sequence such that $\frac{\log j_{n+1}}{\log j_n}$ is unbounded. The counter example given above can be modified so as to prove the 'only if' part, as follows. Suppose $q_{k+1}, \ldots, q_l$ is a block of consecutive primes. Let M be such that $q_{k+1} < M < q_l$. Consider the set of numbers all of whose prime factors are exclusively from among $q_1, q_2, \ldots, q_k$, let C_k be the complement of this set. Consider the quantity

$$\frac{\left\{\sum\limits_{\substack{n\epsilon C_k \\ q_{k+1} < n < M}} \frac{1}{n}\right\}}{\sum\limits_{1}^{M} \frac{1}{n}} \geqslant \frac{\log M - e^{\gamma}\log q_k}{\log M}$$

approximately (γ denotes Euler's constant),

$$= 1-e^{\gamma} \cdot \frac{\log q_k}{\log M}.$$

We now use the following lemma :

Lemma : *Let $a_1, a_2, \ldots$ be increasing sequence of positive integers such that $\frac{\log a_{n+1}}{\log a_n}$ is unbounded. Take any $\epsilon > 0$, $\delta > 0$. We can determine an n and a positive, integer M such that*

$$a_n < M < a_{n+1} \quad \textit{and} \quad \frac{\log a_n}{\log M} < \epsilon \quad \textit{and} \quad \frac{\log M}{\log a_{n+1}} < \delta.$$

Rigorizing the nonrigorous part above is trivial.

Corollary : *Let $f(n)$ be a finite real-valued function defined on the set of positive integers. Suppose there is a sequence J of positive integers j_n such that $\frac{\log j_{n+1}}{\log j_n}$ is bounded and $f(2^{x_1} \ldots q_{j_n}^{x_{j_n}})$ converges with probability 1 to a random variable $g(x)$, as $n \to \infty$. Then f has a distribution and this is the same as the distribution of $g(x)$; we use logarithmic density.*

Reference

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