INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination

First semester

M. Math - First year 2016-2017

Analysis of Several Variables

Date: September 5, 2016 — Maximum Marks: 40

Duration: 2 hours

Answer all questions.

You must state clearly any result you use.

(1) (a) Suppose that $f : U \to \mathbb{R}^n$ is a differentiable function defined on an open subset U of $\mathbb{R}^m$. Prove that for every $x \in U$ and every unit vector $v \in \mathbb{R}^m$, the directional derivative $D_v f(x)$ equals $Df_x(v)$.

(b) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}$ be two differentiable maps and $h = g \circ f$. Prove the following relation:

$$\frac{\partial h}{\partial x_i}(x) = \sum_{j=1}^{n} \frac{\partial g}{\partial y_j}(f(x)) \frac{\partial f_j}{\partial x_i}(x)$$

where $f = (f_1, f_2, \ldots, f_n)$, $x_1, \ldots, x_m$ and $y_1, \ldots, y_n$ are respectively the canonical coordinates on $\mathbb{R}^m$ and $\mathbb{R}^n$.

5+3

(2) Let $h : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$h(x,y) = \begin{cases} \frac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

Compute the partial derivatives at the origin. Show that h is not differentiable at $(0,0)$. 6

(3) (a) Let $\gamma : (-1,1) \to O(n)$ be a differentiable curve such that $\gamma(0) = I_n$ (I_n denotes the $n \times n$ Identity matrix). Prove that $\gamma'(0)$ is a skew-symmetric matrix.

(b) Let A be a skew-symmetric matrix. Describe a curve $\gamma : (-1,1) \to O(n)$ such that $\gamma(0) = I_n$ and $\gamma'(0) = A$. Give complete justification. 4+5

P. T. O

(4) If $f : U \to V$ is a diffeomorphism between open subsets U, V of $\mathbb{R}^n$ then show that $Df_x : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism for all $x \in U$. 4

(5) Suppose that U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}$ is differentiable. Prove that if f has a local maxima at $p \in U$ then $Df_p = 0$. 6

(6) Let $L(\mathbb{R}^m, \mathbb{R}^n)$ be the vector space of all linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Consider the function $F : L(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^m \to \mathbb{R}^n$ defined by $F(\ell, v) = \ell(v)$ for all $(\ell, v) \in L(\mathbb{R}^m, \mathbb{R}^n) \times \mathbb{R}^m$. Compute the derivative of F at a general point. 6

(7) Justify the following statement: The system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z. 6

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination: 2016–17 (First Semester)

M. MATH. I YEAR
Algebra I

Date: 7.9.2016 Maximum Marks: 60 Duration: $3\frac{1}{2}$ Hours

Attempt Question 7 and ANY FOUR from the rest.

1. (i) Let K be an infinite field and $f(X_1, \dots, X_n)$ be a nonzero element of the polynomial ring $K[X_1, \dots, X_n]$. Show that there exist $a_1, \dots, a_n$ in K such that $f(a_1, \dots, a_n) \neq 0$.

   (ii) Let $k = \mathbb{Z}/3\mathbb{Z}$. Construct a nonzero polynomial $f(X) \in k[X]$ such that $f(a) = 0$ for every $a \in k$.

   (iii) Prove that any maximal ideal of $\mathbb{C}[X]$ is of the form $(X - \lambda)\mathbb{C}[X]$ for some $\lambda \in \mathbb{C}$.

   [7+2+3=12]

2. (i) Prove that any nonzero nonunit in a Noetherian domain can be expressed as a product of irreducible elements.

   (ii) Show that, over any field k, the polynomial ring $k[X]$ has infinitely many distinct irreducible monic polynomials. [6+6=12]

3. Let $R = \mathbb{Z}[\sqrt{-5}]$.

   (i) Prove that $R \cong \mathbb{Z}[X]/(X^2 + 5)$.

   (ii) Prove that $R/3R \cong \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$.

   (iii) Demonstrate an element in R whose image in $R/3R$ is a nontrivial idempotent.

   [5+5+2=12]

4. Let $A = \mathbb{C}[X, Y, Z]/(XY - Z^2)$. Let x, y, z denote the images of X, Y, Z in A.

   (i) Which of the ideals xA, $(x, y)A$ and $(x, z)A$ are prime ideals of A? (Justify.)

   (ii) Explicitly describe three maximal ideals of A. [6+6=12]

5. (i) Let N be a normal subgroup of a group G such that $|N|$ is coprime to $|G/N|$. Prove that N is the unique subgroup of G of order $|N|$.

   (ii) Prove that the ring $A = \mathbb{C}[T^2, T^3]$ is isomorphic to a quotient of the ring $B = \mathbb{C}[X, Y, Z]/(X^3 - YZ^2)$. [6+6=12]

6. Prove ANY THREE of the following statements.

   (i) If H is a subgroup of a group G containing the commutator subgroup $[G, G]$, then H is a normal subgroup of G.

   (ii) A local ring (i.e., a ring with exactly one maximal ideal) cannot have any idempotent other than 0 and 1.

   (iii) Any ideal I of a product ring $R_1 \times \cdots \times R_n$ is of the form $I_1 \times \cdots \times I_n$, for some ideal I_j of R_j, $1 \leq j \leq n$.

   (iv) For any ideal I of a Noetherian ring R, there exists an integer m such that $(\sqrt{I})^m \subseteq I$. (Recall that $\sqrt{I} = \{f \in R \mid f^n \in I \text{ for some } n \geq 1\}$.) $[4 + 4 + 4 = 12]$

7. State whether the following statements are TRUE or FALSE with brief justification. Attempt ANY FIVE.

(i) If $K \subset H \subset G$ are groups such that G and K are isomorphic as groups, then G and H are isomorphic as groups.

(ii) The ring of real-valued continuous functions on $[0, 1]$ has no nonzero nilpotents.

(iii) The image of X in $\mathbb{Z}[X, Y]/(XY - 1)$ generates a prime ideal of $\mathbb{Z}[X, Y]/(XY - 1)$.

(iv) Any subring of a PID is a PID.

(v) A non-constant polynomial in $R[X]$ over a commutative ring R can have only finitely many roots in R.

(vi) If $R \subset A \subset R[X]$ are integral domains then any irreducible element in R remains irreducible in A.

(vii) For $r \in \mathbb{R}$, $(X - r)\mathbb{R}[[X]]$ is a proper ideal of $\mathbb{R}[[X]]$ if and only if $r = 0$. [5×3 = 15]

Indian Statistical Institute
Mid Semestral Examination: 2016-17

Course Name: M. Math, 1st year
Subject Name : Linear Algebra
Date: 08.09.2016, Maximum Marks: 40, Duration: Two and a half hours

Answer all questions. All vector spaces are over the complex field $\mathbb{C}$. In case you use any matrix corresponding to a linear transformation in your solutions, please indicate the basis w.r.t which the matrix is defined. If you do not specify the matrix, you will not get any marks.

1. Let V be a finite dimensional inner product space. Let w_1, w_2 be vectors in V. Define a linear map $T : V \to V$ by the formula

$$T(v) = \langle v, w_1 \rangle \, w_2.$$

a. Compute $T^*(v)$.

b. Prove that given any linear map $S : V \to V$ such that $\dim(\mathrm{Ran}(T)) = 1$, there exist $w_1, w_2 \in V$ such that

$$\forall \, v \in V, \; S(v) = \langle v, w_1 \rangle \, w_2.$$

(**Hint:** Recall how a rank one operator in a vector space looks like) **4 + 6 = 10**

2. Let V be a finite dimensional inner product space and let $T : V \to V$ be linear.

a. Prove that there exist a positive operator N and a unitary operator U such that $T = NU$. (You do not need to prove any theorem).

b. Let $\sigma(T)$ denote the set of all eigenvalues of T. If T is normal and $f : \sigma(T) \to \mathbb{C}$ a function, prove that

$$\sigma(f(T)) = f(\sigma(T)) \text{ as sets.}$$

Here, $f(\sigma(T))$ denotes the range of the function f.

c. Using part **b.** or otherwise, prove that if S is a self adjoint linear map on V, then there exists a real number c such that $S + cI$ is positive.

P.T.O

(**Hint:** You can use the fact that a self adjoint linear map in a finite dimensional inner product space is positive if and only if all its eigenvalues are positive.) **3 + 5 + 7 = 15**

3. **a.** Prove that any finite dimensional normed linear subspace of $\mathbb{C}^n$ is closed.

   b. Let E be an idempotent in a finite dimensional vector space. Compute $(I+E)^{-1}$.

   c. Let V be an n-dimensional complex vector space and N a nilpotent operator on V such that $N^{n-1} = 0$. If $v \in V$ is such that $N^{n-1}(v) \neq 0$, prove that $\text{Span}\{f(N)(v) : f \text{ is a real valued polynomial}\} = V$. **3 + 5 + 7 = 15**

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination

First semester

M. Math First year, 2016

Measure Theoretic Probability

Date: 9th September, 2016 | Maximum Marks: 30

Duration: 2 hours

Anybody caught using unfair means will immediately get 0. Please try to explain every step. Only handwritten class notes are allowed in the exam.

(1) State the following is **true or false** with appropriate reasons.

(a) Let X_n and Y_n be random variables on a common probability space such that $\{\mu_{X_n}\}$ and $\{\mu_{Y_n}\}$ are tight families. Assume $Y_n > 0$ almost surely. Let us define $W_n = X_n/Y_n$. Then $\{\mu_{W_n}\}$ is a tight family.

(b) Suppose $\{\mu_n\}_{n\geq 1}$ and μ be probability measures on $\mathbb{R}$ such that $\mu_n \xrightarrow{d} \mu$. Then for any closed set A of $\mathbb{R}$,

$$\limsup_{n\to\infty} \mu_n(A) \leq \mu(A).$$

(c) There is a countable infinite σ-algebra.

(d) Suppose Ω is an infinite set and $\mathcal{F}$ be the finite-co-finite σ-field on it. Define the following set function

$$P(A) = \begin{cases} 0 & \text{if } |A| < \infty \\ 1 & \text{otherwise} \end{cases}$$

for $A \in \mathcal{F}$. Then P is countably additive.

(e) If $X_n \geq 0$ and $X_n \downarrow X$ almost surely and $\mathsf{E}[X_k] < \infty$ for some k then $\mathsf{E}[X_n] \to \mathsf{E}[X]$ as $n \to \infty$.

[3 × 5=15 points]

(2) Let $\mu_n = \frac{1}{n}\sum_{k=1}^{n} \delta_{\frac{k}{n}}$ and μ be the uniform probability measure on $[0,1]$. Show directly by definition that $d(\mu_n, \mu) \to 0$ as $n \to \infty$. [5 points]

(3) Suppose μ_n and μ be probability measures on $\mathbb{R}$. Suppose that $F_{\mu_n}(x) \to F_\mu(x)$ for all x in some dense set D on $\mathbb{R}$. Show that $\mu_n \xrightarrow{d} \mu$. [5 points]

(4) Let μ be a probability measure on $(\Omega, \mathcal{F})$ where $\mathcal{F} = \sigma(\mathcal{A})$ for some field $\mathcal{A}$. Show that for each $A \in \mathcal{F}$, and $\varepsilon > 0$, there exists $B \in \mathcal{A}$ such that $\mu(A\Delta B) < \varepsilon$. [5 points]

INDIAN STATISTICAL INSTITUTE

Semestral Examination
First semester

M. Math First year, 2016

Measure Theoretic Probability

Date: 15th November, 2016 Maximum Marks: 70

Duration: 4 hours

- The question paper is of 75 marks but the maximum you can score is 70.
- Please try to explain every step.
- Keep all electronic devices in your bags. Only handwritten class notes and handwritten solved assignments are allowed in the exam. No photocopied notes or books are allowed.
- Anybody caught using unfair means will immediately get 0.

(1) Let X and Y be random variables such that $|X| \leq M$ and $|Y| \leq M$ for some constant $M > 0$. Suppose that $\mathsf{E}[X^r] = \mathsf{E}[Y^r]$ for all $r = 1, 2, \cdots$.
  (a) Show that $\mathsf{E}[f(X)] = \mathsf{E}[f(Y)]$ for any continuous function $f : [-M, M] \to \mathbb{R}$.
  (b) Deduce that $X \stackrel{d}{=} Y$.

[5+5=10]

(2) State whether the following are true or false and provide proofs or counterexamples accordingly.

(a) Let $f : [0,1] \to [0,1]$ be Borel measurable function. Define its graph as $G_f := \{(x, f(x)) : x \in [0,1]\}$. Then $\lambda_2(G_f) = 0$ where λ_2 is the Lebesgue measure on $\mathbb{R}^2$.

(b) Let μ be $N(0,1)$ distribution. Then there exists a unique measurable function $S : [0,1] \to \mathbb{R}$ such that $\lambda \circ S^{-1} = \mu$, where λ is the Lebesgue measure on $[0,1]$.

(c) If X_n's are independent random variables and $X_n \xrightarrow{P} X$ then X is constant almost surely.

(d) $U[1,3]$ is absolutely continuous with respect to $U[2,4]$. (Here $U[a,b]$ denotes the measure induced by Uniform distribution on $[a,b]$). [5 × 4=20]

(3) Consider a coin in which head appears with probability p and tail with probability $1-p$ (where $p \in (0,1)$). Suppose this coin is flipped independently infinitely many

times. Let T_n be the minimum number of flips to get n heads. Show that there exists a constant $c > 0$ such that $T_n/n \underset{n\to\infty}{\to} c$ almost surely. Compute c.

[9+1=10]

(4) Let I be a countable set. Suppose $(\Omega_i, \mathcal{F}_i, P_i)_{i\in I}$ be probability spaces and let $\Omega = \times_i \Omega_i$ and $\mathcal{F} = \otimes_i \mathcal{F}_i$ and $P = \otimes_i P_i$. If $A \in \mathcal{F}$ then show that for any $\epsilon > 0$, there is a cylinder set B such that $P(A\Delta B) < \epsilon$.

[10]

(5) Given a distribution function $F : \mathbb{R} \to [0,1]$. Show that there is a unique way to write it as $F = \alpha F_d + (1-\alpha)F_c$ where $0 \le \alpha \le 1$, F_d, F_c are distribution functions, F_c is continuous and F_d is discontinuous that increases only in jumps (that is, if $[a, b]$ contains no discontinuities of F_d, then $F_d(b) = F_d(a)$). [10]

(6) Let $X_1, X_2, \cdots$ be independent with mean zero and suppose $\sup_{n\ge 1} E[X_n^4]$ is bounded. If $S_n = X_1 + \cdots + X_n$ then show that $\frac{S_n}{n} \to 0$ as $n \to \infty$ almost surely.

Caution: X_n's need not be identically distributed. Hint: Estimate $E[S_n^4]$.

[15]

INDIAN STATISTICAL INSTITUTE

Semestral Examination

First semester

M. Math - First year 2016-2017

Analysis of Several Variables

Date: 18 November, 2016 | Maximum Marks: 60

Duration: 3 hours

Answer all questions.

You must state clearly any result you use.

(1) Consider the 2-form

$$\omega = x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy$$

on $\mathbb{R}^3$.

(a) Prove that $\int_\sigma \omega \neq 0$, where $\sigma : [0, 2\pi] \times [-\pi/2, \pi/2] \to \mathbb{R}^3$ is defined by $\sigma(u, v) = (\cos u \cos v, \sin u \cos v, \sin v)$.

(b) Write down the oriented boundary of σ as a singular 1-chain.

(c) Is ω an exact form? Justify your answer. 8+7+7

(2) Let $\omega = M\,dx + N\,dy$ be a smooth 1-form on $\mathbb{R}^2$, where M and N satisfy the relation $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$. Define a function $f : \mathbb{R}^2 \to R$ as follows:

$$f(x, y) = \int_0^y N(0, t)\,dt + \int_0^x M(s, y)\,ds.$$

Prove that $df = \omega$. 6

(3) Let $X : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field defined as $X(x, y) = (y, -x)$. Find the integral curve $\gamma : \mathbb{R} \to \mathbb{R}^2$ of X which satisfies the initial condition $\gamma(0) = (1, 0)$. 10

(4) Assume that $A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear map. Let $A^{\#}$ denote the induced map $\wedge^n(\mathbb{R}^{n*}) \to \wedge^n(\mathbb{R}^{n*})$. Define $\det A$, the determinant of A. Prove that $A^{\#}\varphi = (\det A)\varphi$ for any antisymmetric n-multilinear map φ on $\mathbb{R}^n$. 6

P. T. O.

(5) Evaluate the integral

$$\int_R \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right) dx\,dy\,dz,$$

where R is the region in $\mathbb{R}^3$ defined by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\leq 1$ and $x\geq 0, y\geq 0, z\geq 0$.

8

(6) Suppose that f is a continuous function defined on $\mathbb{R}^2$. Give a detailed proof of the equality

$$\int_R f(y)\,dy = \int_{T(R)} f(T(x))|J_T(x)|dx$$

in the following two cases:

(a) $T(u,v)=(v,u)$ for all $(u,v)\in\mathbb{R}^2$.

(b) $T(u,v)=(u+a,v+b)$ for all $(u,v)\in\mathbb{R}^2$ ($(a,b)\in\mathbb{R}^2$ is a fixed vector),

where J_T is the Jacobian of T. 5+5

INDIAN STATISTICAL INSTITUTE
Semestral Examination: 2016–17 (First Semester)

M. MATH. I YEAR
Algebra I

Date: 21.11.2016 Maximum Marks: 70 Duration: 4 Hours

Attempt Question 1 and ANY FIVE from the rest.

1. State whether the following statements are TRUE or FALSE with brief justification. Attempt ANY FIVE.

(i) $8 = 1 + 1 + 2 + 4$ cannot be the class equation of a group.

(ii) If a simple group G of order 60 has a non-trivial action on a set X then X must have at least 5 elements.

(iii) If R is any commutative ring then units in $R[X]$ are precisely the units in R.

(iv) $\mathbb{R}[X, Y, Z]/(X^2 + Y^2 + Z^2 - 1)$ is not an Euclidean domain.

(v) In $R = \mathbb{Z}[X]$, the ideal $(4X, 9X)$ is a free R-module.

(vi) If N and P are internal direct summands of a module M over a commutative ring R, then $N + P$ is necessarily an internal direct summand of the R-module M.

(vii) If R is a subring of a Noetherian ring A such that A is a subring of the polynomial ring $R[X_1, \ldots, X_n]$ over R, then R must be a Noetherian ring.

(viii) $\mathbb{Z}/4\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} \cong \mathbb{Z}/12\mathbb{Z}$. [5 × 3 = 15]

2. Prove ANY THREE of the following statements:

(i) If G is a finite group acting transitively on a finite set X which has at least two elements, then G necessarily has an element g which has no fixed point.

(ii) The rings $\mathbb{C}[X, Y]/(XY - 1)$ and $\mathbb{C}[X, Y]/(X^2 + Y^2 - 1)$ are isomorphic.

(iii) The polynomial $X^3 + 3X^2Y^2 + 2X^2Y + Y^4 + 7Y + Y^2$ is irreducible in $\mathbb{Q}(Y)[X]$.

(iv) Over a field k and for any subset S of the polynomial ring $k[X_1, \ldots, X_n]$, the set $\mathcal{Z}(S) = \{a \in k^n \mid g(a) = 0 \ \forall \ g \in S\}$ can be expressed as an intersection of finitely many sets of the form $\mathcal{Z}(f) = \{a \in k^n \mid f(a) = 0\}$. [4 + 4 + 4 = 12]

3. (i) Let H, K be subgroups of a group G.

(a) Find the stabiliser of the element $\overline{xK}$ of G/K under the action of G on G/K given by $(g, \overline{xK}) \to \overline{gxK}$ and the stabiliser of $\overline{K}$ under the induced action of H on G/K.

(b) Deduce that $[H : H \cap K] \leq [G : K]$.

(ii) Prove that no group of order 24 is simple.

(iii) Find an element g in A_4 such that $g(12)(34)g^{-1} = (13)(24)$. [(3+2)+5+2=12]

4. (i) Let G be a group with identity e. Prove that if $x^2 = e$ for every x in G then G is an Abelian group isomorphic to a direct sum of copies of $\mathbb{Z}/2\mathbb{Z}$.

(ii) Deduce that any group with at least three elements must have a non-trivial automorphism. [6+6=12]

5. (i) Prove that any nonzero commutative ring R has a maximal ideal.

(ii) Deduce that any two bases of a finitely generated free module over a commutative ring have the same cardinality.

(iii) Give an example of a nonzero module over a PID which does not have any maximal proper submodule. [5+3+4=12]

6. Let k be a field, I a nonzero proper ideal of the polynomial ring $k[X]$ and $A = k[X]/I$. Prove that:

(i) Every element in A is either a unit or a zero-divisor.

(ii) A has only finitely many maximal ideals. [6+6=12]

7. Let R be a commutative ring.

(i) Prove that if M is a finitely generated R-module with $\mathrm{Ann}_R M = (0)$, then R is isomorphic to a submodule of M^n for some n.

(ii) Deduce that if there exists a Noetherian R-module M with $\mathrm{Ann}_R M = (0)$, then R must be a Noetherian ring. [8+4=12]

8. (i) Define the tensor product of two modules over a commutative ring. (Proof of existence and uniqueness is not required.)

(ii) If $\{e_1, e_2\}$ forms a basis for the vector space V over a field k, then show that the element $e_1 \otimes e_1 + e_2 \otimes e_2$ of $V \otimes_k V$ cannot be expressed as $v \otimes w$ for any $v, w \in V$.

(iii) Prove that if M and N are Noetherian R-modules, then so is $M \otimes_R N$. [3+3+6=12]

9. Answer ANY TWO of the following questions:

(i) Compute the number of distinguishable ways in which the edges of a square can be painted with 6 colours, assuming that only one colour can be used in a single edge and the same colour can be used in different edges.

(ii) Demonstrate (with proof) a chain of four distinct prime ideals $P_1 \subset P_2 \subset P_3 \subset P_4$ in the ring $\mathbb{C}[X, Y, Z, T]/(XY - ZT)$.

(iii) Construct a short exact sequence of R-modules $0 \longrightarrow M_1 \longrightarrow M \longrightarrow M_2 \longrightarrow 0$ such that $M \cong M_1 \oplus M_2$ but the sequence is not a split exact sequence. [6+6=12]

Indian Statistical Institute
First Semester Examinations: 2016-17

Course Name: M. Math, 1st year
Subject Name : Linear Algebra
Maximum Marks: 40, Duration: Two and a half hours
Date: 28.11.2016, 2.30 PM

Answer any four questions. In case you use any matrix corresponding to a linear transformation in your solutions, please indicate the basis w.r.t which the matrix is defined. If you do not specify the basis, you will not get any marks. Marks will be deducted for indirect, incomplete, unnecessarily long and imprecise answers.

1. **a.** Prove that an R-module M is decomposable if and only if the ring of endomorphims $\mathrm{End}_R(M)$ has a non trivial idempotent.

   b. Let R be a Principal Ideal Domain and p a prime element in R.

   i. Prove that $R/(p^n)$ is an indecomposable module over R.

   ii. Deduce the structure of a finitely generated indecomposable module over R.
   3 + 4 + 3 = 10

2. **a.** Classify all possible matrices upto similarity whose minimal polynomial is $(x-1)^5$ and characteristic polynomial is $(x-1)^6$.

   b. Let T be a linear transformation from a finite dimensional vector space V to V. Prove that if the characteristic polynomial of T is irreducible, then every vector of V is a cyclic vector. **5 + 5 = 10**

3. **a.** Let N be an $n \times n$ complex matrix such that the $N^n = 0$ while $N^{n-1} \neq 0$. Prove that N is similar to N^t.

P.T.O

b. Using part a. and the Jordan canonical form, prove that every complex matrix is similar to its transpose. **3 + 7 = 10**

4. **a.** Prove that if two complex matrices A and B of the same size are similar, then the corresponding $k[x]$ modules are isomorphic.

   b. Use part a. to prove that if two idempotent complex matrices of the same size are similar, then they have the same rank. **5 + 5 = 10**

5. Let A be an $n \times n$ complex matrix such that $A^* = -A$. Show that

   a. $\det(e^A) = e^{\mathrm{Tr}(A)}$.

   b. Prove that e^A is a unitary. (Hint: Observe that iA is self adjoint) **5 + 5 = 10**

INDIAN STATISTICAL INSTITUTE

Semestral Examination: **Back Paper**
First semester

M. Math First year, 2016

Measure Theoretic Probability

Date: 26/12/16

Total Marks: 100

Duration: 3 hours

- Please try to explain every step.
- Keep all electronic devices in your bags.
- Anybody caught using unfair means will immediately get 0.

(1) Let $X_1, X_2, \cdots$ be independent $N(0,1)$ random variables. Prove that

$$\limsup_n \left(\frac{X_n}{\sqrt{2\log n}} \right) = 1 \text{ almost surely.}$$

[10]

(2) Let f be an integrable function on a measure space $(E, \mathcal{E}, \mu)$. Suppose that, for some π-system $\mathcal{A}$ containing E and generating $\mathcal{E}$, we have $\int_A f d\mu = 0$ for all $A \in \mathcal{A}$. Show that $f = 0$ almost everywhere. [10]

(3) (a) Let X be a non-negative integer valued random variable. Show that $\mathsf{E}[X] = \sum_{n=1}^{\infty} P(X \geq n)$.

(b) Show that, if $\mathsf{E}[X] = \infty$ and $X_1, X_2, \cdots$ is a sequence of independent random variables with distribution same as X, then, almost surely, $\limsup_n \frac{X_n}{n} = \infty$.

[5+10=15]

(4) If $P(A_n) \to 0$ and $\sum_{n=1}^{\infty} P(A_n \cap A_{n+1}^c) < \infty$ then show that $P(A_n \ i.o.) = 0$. [8]

(5) Suppose that $X_n \xrightarrow{d} X$.

(a) Give a counter example to the following statement: If B is a Borel subset of $\mathbb{R}$, and $P(X_n \in B) = 1$ for all n, then $P(X \in B) = 1$.

(b) Show that, if B is a closed subset of $\mathbb{R}$, then the above statement is true.

[4+8=12]

(6) Let $(X_n)_{n\geq 1}$ be an identically distributed sequence in L^2. Show that $nP(|X_1| > \epsilon\sqrt{n}) \to 0$ as $n \to \infty$, for all $\epsilon > 0$. Deduce that $n^{-\frac{1}{2}} \max_{1\leq k\leq n} |X_k| \to 0$ in probability. [5+5=10]

(7) State whether the following are true or false with appropriate reasons.

(a) Let μ and ν be probability measures on $\mathbb{R}$. Then there exists a Borel measurable $T : \mathbb{R} \to \mathbb{R}$ such that $\mu \circ T^{-1} = \nu$.

(b) Suppose $0 < q < p$. Then $X_n \to X$ in L^p implies that $X_n \to X$ in L^q.

(c) There does not exist independent and identically distributed random variables X and Y such that $X - Y$ is distributed as $U[-1,1]$. ($U[-1,1]$ denotes the uniform distribution on the interval $[-1,1]$.)

(d) If X_n's are independent and identically distributed with $E[|X_1|] < \infty$, then $\frac{X_n}{n} \xrightarrow{P} 0$.

[6×4=24]

(8) Let $(X_i)_{i\geq 1}$ be independent and identically distributed random variables with mean zero and finite variance. Let $S_n = X_1 + \cdots + X_n$. Show that $(\frac{S_n}{\sqrt{n}})_{n\geq 1}$ is tight. [4]

(9) Let $X : \Omega_1 \to \mathbb{R}$ be a random variable. If $X(\omega) = X(\omega')$ for some $\omega, \omega' \in \Omega_1$, show that there is no set $A \in \sigma(X)$ such that $\omega \in A$ and $\omega' \notin A$. If $Y : \Omega_1 \to \mathbb{R}$ is measurable with respect to $\sigma(X)$ on Ω_1, then show that Y is a function of X. [3+4=7]

INDIAN STATISTICAL INSTITUTE
Backpaper Examination: 2016–17 (First Semester)

M. MATH. I YEAR
Algebra I

Date: 27.12.2016 Maximum Marks: 100 Duration: 3 Hours

1. State whether the following statements are TRUE or FALSE with brief justification.

   (i) A_4 does not have a subgroup of index 2.

   (ii) If K is a normal subgroup of the group G and H is a normal subgroup of K, then H is a normal subgroup of G.

   (iii) If d is a nonzero element of a UFD R, then dR is contained in only finitely many principal ideals of R.

   (iv) Any two minimal generating sets of a finitely generated free module over a commutative ring have the same cardinality.

   (v) $\mathbb{C}[X]/(X^2) \otimes_{\mathbb{C}} \mathbb{C}[X]/(X^3)$ is a vector space over $\mathbb{C}$ of dimension 6. [$5 \times 4 = 20$]

2. (i) Let G be a finite group which acts transitively on X. Let N be a normal subgroup of G. Show that all the orbits of the induced action of N on X have the same size.

   (ii) Find the number of distinguishable ways in which the edges of an equilateral triangle can be painted with 6 colours, assuming that only one colour can be used in a single edge and the same colour can be used in different edges. [8+8=16]

3. Let $R = \mathbb{Q}[X]/(X^3 - X)$.

   (i) Prove that there exists an isomorphism ϕ from R to the product ring $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$.

   (ii) Which element of $\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q}$ is given by $\phi(\overline{X})$?

   (iii) For which element e of R is $\phi(e) = (1, 0, 0)$?

   (iv) Describe all prime and maximal ideals of R. Does R contain any nontrivial idempotent? [4+4+4+6=18]

4. (i) Define an Euclidean function and an Euclidean domain.

   (ii) Prove that every Euclidean domain is a PID.

   (iii) Let $R = \mathbb{Z}[i]$, $u = -3 + 11i$ and $v = 8 - i$. Find d, a, b in R such that $(u, v)R = dR$ and $d = au + bv$.

   (iv) Prove that the polynomial $X^5 + 9X^3 + 15X + 6$ is irreducible in $\mathbb{Q}[X]$. [3+8+8+5=24]

5. Prove that if R is a Noetherian ring then so is $R[X]$. [12]

6. (i) Let f be an endomorphism of a Noetherian module M. Prove that there exists a positive integer n such that $\mathrm{Ker}(f^n) \cap \mathrm{Im}(f^n) = (0)$.

   (ii) Deduce that any surjective endomorphism of a Noetherian module is an automorphism. [6+4=10]

Indian Statistical Institute, Kolkata
Backpaper examination: First semester 2016 -'17
M. Math I year

Subject	:	**Topology I**
Time	:	3 hours
Maximum score	:	100

28·12·2016

Attempt all the problems. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answerscript. Points will be deducted for missing or incomplete arguments.

(1) (a) Let $p : X \to Y$ be a continuous map. Show that if there is a continuous map $f : Y \to X$ such that $p \circ f$ equals the identity map of Y , then p is a quotient map.
(b) If $A \subset X$, then a retraction of X onto A is a continuous map $r : X \to A$ such that $r(a) = a$ for each $a \in A$. Show that a retraction is a quotient map.

[20 marks]

(2) Let Y denote the subset of $\mathbb{R}^{\mathbb{N}}$, consisting of sequences that are eventually zero. Find the closure of Y with respect to the (i) product and (ii) box topologies on $\mathbb{R}^{\mathbb{N}}$.

[15 marks]

(3) (i) Show that $\mathcal{G} := \{(a, b] : a, b \in \mathbb{R}\}$ forms a basis for a topology on $\mathbb{R}$.
(ii) Let $\mathcal{T}$ be the topology on $\mathbb{R}$ generated by $\mathcal{G}$. Show that $\mathcal{T}$ is separable and first countable.
(iii) Prove that for **any** basis $\mathcal{B}$ for $\mathcal{T}$, there exists an injective map from $\mathbb{R}$ to $\mathcal{B}$.
(iv) Is $(\mathbb{R}, \mathcal{T})$ metrizable?
(v) Is it possible to compare the topologies $\mathcal{T}$ and the usual one coming from the standard metric on $\mathbb{R}$.

[25 marks]

(4) Show that the fundamental group of the standard torus $S^1 \times S^1$ is $\mathbb{Z} \times \mathbb{Z}$.

[15 marks]

(5) Let $p : E \to B$ be a covering map.
(a) If B is Hausdorff, regular, completely regular, or locally compact Hausdorff, then so is E.
(b) If B is compact and $p^{-1}(b)$ is finite for each $b \in B$, then E is compact.

[25 marks]

(6) Show that $[0, 1]^{\mathbb{N}}$ is not locally compact with respect to the uniform topology.

[15 marks]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination

First semester

M. Math - First year 2016-2017

Analysis of Several Variables

Date: 29/12/2016 Maximum Marks: 100

Duration: 3 hours

Answer all questions.

You must state clearly any result you use.

(1) Prove that $A \mapsto e^A$ is a continuous map on the space of all $n \times n$ matrices, where

$$e^A = I_n + A + \frac{1}{2!}A^2 + \cdots + \frac{1}{n!}A^n + \ldots.$$

8

(2) Find the derivative of the map $f : M_n(\mathbb{R}) \times M_n(\mathbb{R}) \to \mathbb{R}$ defined by $f(X, Y) = Trace(XY)$ for $X, Y \in M_n(\mathbb{R})$. Give a complete proof. 8

(3) Consider the function $g : \mathbb{R}^2 \to \mathbb{R}$ defined as follows:

$$g(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Prove that g is continuous but not differentiable. 5+5

(4) Prove that the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\phi(x, y) = (x + y^2 + 2x^3y + x^6, y + x^3)$$

is a diffeomorphism. 10

(5) Let $SL(2, \mathbb{R})$ be the set of all real 2×2 matrices of determinant 1. Show that the minimum Euclidean distance of a matrix $A \in SL(2, R)$ from the null matrix is $\sqrt{2}$. 10

P.T.O

(6) Consider the 1-form $\eta = \frac{x\,dy - y\,dx}{x^2+y^2}$ on $\mathbb{R}^2 \setminus (0,0)$.

(a) Prove that η is a closed form.

(b) Evaluate the integral $\int_\gamma \eta$, where γ is the parametrized curve given by $\gamma(t) = (\cos t, \sin t)$, $t \in [0, 2\pi]$.

(c) Can η be an exact form? Justify your answer.

3+8+3

(7) Define exterior differential operator $d : \Omega^p(U) \to \Omega^{p+1}(U)$, $p \geq 0$, where U is an open subset of $\mathbb{R}^n$. Prove the following:

(a) $d \circ d = 0$.

(b) $d(f^*g) = f^*(dg)$ for every smooth real valued function g on V and a smooth function $f : U \to V$. 5+5

(8) If f is a real valued continuous function on $\mathbb{R}$ then show that

$$\int_0^x \left\{ \int_0^v \left\{ \int_0^u f(t)\,dt \right\} du \right\} dv = \frac{1}{2} \int_0^x (x-t)^2 f(t)\,dt$$

8

(9) Let $f : \mathbb{R}^n \to \mathbb{R}$ be a compactly supported continuous function. Define $h : \mathbb{R}^n \to \mathbb{R}$ by

$$h(y) = \int_{\mathbb{R}^n} f(x)\langle x, y\rangle dx, \;\; y \in \mathbb{R}^n,$$

where $\langle x, y\rangle$ denotes the inner product of the vectors x and y.

Prove that if f is invariant under the action of $O(n)$ (that is, $f(Ax) = f(x)$ for all $A \in O(n)$) then so is h. 8

(10) Consider the singular 2-cube $T : [0,1] \times [0,1] \to \mathbb{R}^3$ defined as follows:

$$T(t,z) = (\cos 2\pi t, \sin 2\pi t, z).$$

Determine the oriented boundary ∂T as a singular 1-chain. Also describe it geometrically.

Show that for any closed 1-form ω on $\mathbb{R}^3$

$$\int_{\gamma_1} \omega = \int_{\gamma_2} . \omega,$$

where $\gamma_i(t) = (\cos 2\pi t, \sin 2\pi t. i)$, $i = 0, 1$. 7+7

Indian Statistical Institute
Back paper Examinations: 2016-17

Course Name: M. Math, 1st year
Subject Name : Linear Algebra
Maximum Marks: 100, Duration: Two and a half hours

Answer all questions. 30·12·2016

1. Suppose V is a finite dimensional vector space such that $V = \oplus_{i=1}^{k} V_i$ for some subspaces V_i. Moreover, suppose $T : V \to V$ is a linear map which is of the form $\oplus_{i=1}^{k} T_i$ with respect to the above decomposition. Compute the characteristic polynomial and the minimal polynomial of T in terms of those of the T_i's. **10**

2. **a.** Let A be a 4×4 matrix whose minimal polynomial is $x(x-2)(x+2)$ and whose rank is 2. What is the characteristic polynomial of A?

   b. Classify upto similarity all matrices whose characteristic polynomial is $(x-1)(x-2)^2$ and minimal polynomial is $(x-1)(x-2)$. **10 + 10 = 20**

3. Let T be a linear transformation from a vector space V to V. Prove that if every vector of V is a cyclic vector, then the characteristic polynomial of T is irreducible. **10**

4. Prove that the product of two positive linear operators on a finite dimensional inner product space is positive if they commute. **10**

5. Let V be a finite dimensional inner product space and T a linear map from V to V.

   a. Prove that there exists a unitary operator U on V and a non negative operator N on V such that $T = UN$.

   b. Prove that the operator N defined in **a.** is unique.

P.T.O

c. Prove that if T is invertible, then the operator U defined in **a.** is also unique.
10 + 10 + 10 = 30

6. Prove that a linear map on a complex vector space is a sum of a diagonalizable map and a nilpotent map. **10**

7. Prove that a finitely generated module over a Principal Ideal Domain is a direct sum of a free module of finite rank and a torsion module. **10**

INDIAN STATISTICAL INSTITUTE
MID-SEMESTRAL EXAMINATION: 2016-2017
M.S. (Q.E.) I, II Years and M. Math. I Year
Game Theory II

Date: 20.02.2017 Maximum Marks: 100 Time: 3 hours

1 (a). When do you say that a pay-off vector is reasonable? Interpret your definition.

(b) Demonstrate the relationship between the core and reasonable set of a game?

© Prove or disprove the following: The converse of 1(b) is true as well. (2+2+6+8)

2. (a) Clearly provide an argument in favor of a stable set as a solution concept. Define a stable set of a general n-person game. Prove a sufficient condition for uniqueness of a stable set of a game. (You must define and prove here all necessary preliminaries that you require for proving this condition.)

(b) Prove or disprove the following: The only stable set of the 3-person majority game is {(0.5, 0.5,0), (0.5,0,0.5), (0,0.5,0.5)}.

(2+2+6+ 6)

3. (a)Define a market game rigorously by giving necessary preliminaries. Why should such a game exist? Provide precise arguments.

(b)Explain the role of the market.

©. Show that the core of a market game is non-empty. (Clearly state any result that you use here.)

(d) Use the same result that you employed in Question No. 3© for demonstrating non-emptiness of the core of a market game to establish that the core of the non-liability game related to the chemical firms- laundry game is empty.

(2+2+2+9+4)

4. In a 3-person buyer-seller game, the seller owning the good thinks that it is worthless to him. Of the two buyers, the first thinks that it is worth 100 and the other thinks that it is worth 50. Determine the characteristic function and the core of the game. (7+10)

5. (a) Show that the concept 'strategic equivalence' is an equivalence relation.

P.T.O

(b) An essential game has a unique representation in terms of a particular type of game. Identify the 'type' and demonstrate your claim. (6+8)

6. Do you agree or disagree with the followings? :

(a) The core of a game is a non-convex set.

(b) Superadditivity property of a game remains invariant under strategic equivalence. (5+7)

7. Give examples of non-negative valued strict subadditive and strict superadditive functions defined on the non-negative orthant of the real line. Establish your arguments indicating that the two different functions you consider satisfy respective properties. (2+2)

Indian Statistical Institute

Mid-semester Examination 2017

M.Math 1st year/M.Stat 1st year

Course name: **Algebra/Abstract Algebra**

Date: **20 February 2017**

Maximum marks: **40**

Duration: **2 hours**

Note: There are 4 questions with total marks 45. Answer as many questions as you like. The maximum you can score is 40. This is closed notes/books exam.

1. Let H be a proper subgroup of G of index $i(H)$.

(a) Suppose H is a normal subgroup. Show that G is solvable if and only if both H and G/H are solvable. (4 marks)

(b) Suppose $|G| | i(H)!$, then show that G is not simple. (4 marks)

(c) Let

$$D_n = \{a^i b^j : 0 \leq i < n,\ 0 \leq j < 2,\ o(a) = n,\ o(b) = 2,\ ba = a^{-1}b\}$$

be the dihedral group. Show that D_n is solvable for all n. (5 marks)

2. A complex number is said to be *algebraic* if it is a solution of a polynomial equation with integer coefficients. Let $\mathbb{A}$ be the set of all algebraic numbers.

(a) Show that $\mathbb{A}$ is a field. (4 marks)

(b) Show that $\mathbb{A} = \mathbb{Q}^a$. (5 marks)

3. Let $f(x) = x^6 + x^3 + 1$. Suppose K is the splitting field of f in $\mathbb{C}$.

(a) Is f irreducible over $\mathbb{Q}$? Give reasons. (3 marks)

(b) Calculate the degree $[K : \mathbb{Q}]$. (4 marks)

(c) Give an example of a field on which f is not separable. (3 marks)

4. Let K be a normal extension of k. Let G be the group of automorphisms of K over k. Set

$$K^G = \{x \in K : gx = x\ \forall g \in G\}.$$

Let K_0 be the maximal separable subextension of k in K.

(a) Show that K^G is a purely inseparable extension of k, and K is a separable extension of K^G. (3+3 marks)

(b) Show that $K^G K_0 = K$ and $K^G \cap K_0 = k$. (3 marks)

(c) Show that any finite extension of $\mathbb{F}_{p^r}$ is separable. (4 marks)

INDIAN STATISTICAL INSTITUTE, KOLKATA
MIDTERM EXAMINATION: SECOND SEMESTER 2016 - '17
M.MATH I YEAR

21.2.17

Subject	:	**Functional Analysis**
Time	:	2 hours 30 minutes
Maximum score	:	40

Attempt all the problems. Please use a new page to answer each question, making sure that the question number in the margin can be read, even after stapling. If you attempt the same problem several times, please strike out all the attempts except the final one before submitting your answer script. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answer script. Points will be deducted for missing or incomplete arguments.

(1) Let X and Y be normed linear spaces and $T : X \to Y$ be a linear map. Define

$$\|x\|_1 = \|x\| + \|T(x)\|, \forall x \in X.$$

(i) Show that $\|\cdot\|_1$ is a norm on X.
(ii) Show that $\|\cdot\|_1$ is equivalent to $\|\cdot\|$ if and only if T is continuous.

[3+4 = 7 marks]

(2) Suppose X and Y are Banach spaces and $T : X \to Y$ is a linear map such that there exists a linear map $S : Y^* \to X^*$ satisfying $S(\varphi) = \varphi \circ T$ for all $\varphi \in Y^*$. Show that T is continuous.

[7 marks]

(3) (i) Prove that the map $C([0,1]) \ni f \mapsto [f] \in L^\infty([0,1], m)$, where m is the Lebesgue measure, has a closed range.
(ii) Given a Borel subset A of $[0,1]$, prove that there exists a sequence of continuous functions f_n such that $f_n \to \chi_A$ in the w^*- topology of $L^\infty([0,1])$. Conclude that $C([0,1])$ is dense in $L^\infty([0,1])$ in w^*- topology.

[4+6+4=14 marks]

(4) Is it possible to have a bounded linear surjection $T : l^2(\mathbb{N}) \to l^1(\mathbb{N})$? Justify your answer.

[7 marks]

(5) Suppose X and Y are Banach spaces and $T : X \to Y$ is a bounded linear map whose range has finite codimension in Y (that is, $\frac{Y}{T(X)}$ is finite dimensional). Prove that $T(X)$ is closed.

[10 marks]

INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination : 2016-17
M. Math.-I Year
Topology-II

Date : 22. 02. 2017 | Maximum Score : 50 | Time : 3:00 Hours

Any result that you use should be stated clearly. All spaces are assumed to be Hausdorff. For any integer n, $n \geq 1$, S^n denotes the unit sphere in $\mathbb{R}^{n+1}$. You may assume that $\pi_1(S^1, (1,0)) \cong \mathbb{Z}$.

(1) (a) For any two based spaces (X, x_0) and (Y, y_0), prove that

$$\pi_1(X \times Y, (x_0, y_0) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0).$$

(b) Compute the fundamental group of $X = \mathbb{R}^2 - \{(0,0)\}$ by proving that X is homotopically equivalent to S^1.

(c) Let $f : S^2 \to S^2$ be a continuous function such that $f(x) \neq x$ for all $x \in S^2$. Prove that f is homotopic to the the antipodal map

$$a : S^2 \to S^2,\ (x, y, z) \stackrel{a}{\mapsto} (-x, -y, -z).$$

[8 + 6 + 6 = 20]

(2) (a) Define the notion of a covering space of a path connected, locally path connected space. Prove that any covering projection $p : E \longrightarrow B$ is an open map.

(b) Define the notion of a free action and a properly discontinuous action of a group on a space.

(c) Prove that a free action of a finite group on a space is properly discontinuous.

(d) Construct a space whose fundamental group is $\mathbb{Z}/3\mathbb{Z}$.

[4 + 4 + 4 + 4 = 16]

(3) (a) What is Deck transformation of a covering space $p : E \longrightarrow B$?

(b) Let $b_0 \in B$. Let $\mathcal{D}(p)$ be the Deck transformation group of p. Let $F_{b_0} = p^{-1}(b_0)$. Prove that F_{b_0} is a left $\mathcal{D}(p)$-space by defining suitable actions.

(c) Compute the isotropy subgroup at any point $e \in F_{b_0}$ with respect to the left $\mathcal{D}(p)$-action on F_{b_0}.

(d) Determine the Deck transformation group of the covering space

$$p : \mathbb{R}^2 \to S^1 \times S^1,\ (x, y) \stackrel{p}{\mapsto} (\exp^{2\pi i x}, \exp^{2\pi i x}).$$

[2 + 3 + 3 + 4 = 12]

(4) Justify the following statements.

(a) Let $E = S^1 \times \mathbb{R}P^2$ and $B = S^1 \times S^1 \times S^1$. Then E can be viewed as a covering space of B.

(b) Let G be a finite group acting freely on a space X and let $Y = X/G$ be the quotient space. Let $f : Y \to S^1$ be any continuous function. Then f must be homotopic to a constant map.

[6 + 6 = 12]

Indian Statistical Institute
Mid Semestral Examination: 2016-17

Course Name: M. Math, 1st year
Subject Name : Differential Geometry I
Date: 23.02.2017, Maximum Marks: 40, Duration: Two and a half hours

Answer all questions

1. A non empty subset S of $\mathbb{R}^{n+k}$ is called an n-surface in $\mathbb{R}^{n+k}$ if there exists an open set U in $\mathbb{R}^{n+k}$ and a C^∞ function $f : U \to \mathbb{R}^k$ such that $S = f^{-1}(c)$ for some c in $\mathbb{R}^k$ and $\text{Rank}Df(p) = k$ for all p in S.

   a. Show that $\mathbb{R}^n$ is an n-surface in $\mathbb{R}^{n+k}$.

   b. Prove that the 2-torus $\mathbb{T}^2$ is a 2-surface in $\mathbb{R}^4$.

   c. Compute the tangent space at the identity matrix I_n to the surface $SL(n, \mathbb{R}) \subsetneq M_n(\mathbb{R})$. You can use the formula $D(\det)(I_n)(X) = \text{Tr}(X)$. **2 + 4 + 8 = 14**

2. **a.** Show that there exists a unit normal vector field on an n-surface in $\mathbb{R}^{n+1}$.

   b. If S is a connected n-surface in $\mathbb{R}^{n+1}$, prove that there exist exactly two unit normal vector field on S. **3 + 7 = 10**

3. **a.** State and prove the inverse mapping theorem for surfaces in $\mathbb{R}^{n+k}$.

   b. Show that if S and $\widetilde{S}$ are diffeomorphic n and m surfaces in $\mathbb{R}^{n+k}$, then $n = m$. **10 + 6 = 16**

Indian Statistical Institute

Mid-Semestral Examination: 2016-2017

Programme: Master of Mathematics

Course: **Complex Analysis**

Maximum Marks: 80

Duration: 3 Hours Date: 24 February, 2017

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1. (a) State Cauchy's integral formula for derivatives. (b) Deduce Liouville's Theorem which says that an entire bounded function f must be constant. (c) If we relax the condition on f and just assume that f is holomorphic on $\mathbb{C} \setminus \{0\}$ and bounded on it, will the same conclusion hold? Justify your answer. [3+5+8=16 marks]

2. (a) Explain what you understand by a continuous branch of logarithm on an open subset of $\mathbb{C} \setminus \{0\}$. (b) Show that such a branch of logarithm must be holomorphic. (c) Is there a continuous branch of logarithm on the set $\mathbb{C} \setminus \overline{D(0,1)}$, where $\overline{D(0,1)}$ is the closed unit disc? Justify. [3+5+6=14 marks]

3. Explain why the order of a zero of a holomorphic function must be finite unless the function is identically zero. [10 marks]

4. Describe with explanation all entire functions f that satisfy the relation

$$(Ref(z))^2 - (Imf(z))^2 = 1$$

for every $z \in \mathbb{C}$. [8 marks]

5. Evaluate $\frac{1}{2\pi i} \int_{|z|=1} z^{-2} e^z \mathrm{d}z$. [6 marks]

6. Suppose $f(z) = e^{z^2+z}$. Find Max $\{|f(z)| : |z| \leq 1\}$. Explain your answer. [10 marks]

7. Suppose $f : [0,1] \longrightarrow \mathbb{C}$ is a continuous function. Show that $\left| \int_0^1 f(t)\mathrm{d}t \right| \leq \int_0^1 |f(t)|\mathrm{d}t$. [10 marks]

8. Suppose f is defined by

$$f(z) = \int_0^1 (\cos t)(\cos tz)\mathrm{d}t$$

for every $z \in \mathbb{C}$. Show that f is entire. [10 marks]

INDIAN STATISTICAL INSTITUTE
SEMESTRAL EXAMINATION, 2016-2017
M.S. (Q.E.) I, II Years and M. Math. I Year
Game Theory II

Date: 24.4.17 Maximum Marks: 100 Time: 3 hours

Note 1: Answer Parts (A) and (B) in separate answer scripts. Clearly explain the symbols you use and state all the assumptions you need for any derivation. Marks will be deducted substantially for any mistake you make in definitions and statements of assumptions, whenever you need them.

Note 2: The paper carries 110 marks. You may attempt any part of any question. The maximum you can score is 100.

A

1. Let $h = \{h_1, h_2, h_3, h_4\}$ be a set of four house owners whose preferences on the set of houses $H = \{H_1, H_2, H_3, h_4\}$ are given in the following table:

Table I : House Matching

h_1	h_2	h_3	h_4
H_3	H_1	H_1	H_2
H_2	H_2	H_4	H_3
H_1	H_4	H_2	H_1
H_4	H_3	H_3	H_4

Clearly establish how the house owners can be made better off through exchange of houses. (8)

2. Is the core of a bankruptcy game non-empty? Demonstrate your claim rigorously. (10)

3. Show that the Shapley value of a game is invariant under strategic equivalence. (5)

P.T.O

4. Define a voting game. Establish a necessary and sufficient condition for non-emptiness of the core of such a game in terms of a blocker.

(6)

5. Consider the problem of allocating costs for providing some service to a set of individuals. Assume that the following conditions hold: (a) all non-users of the service do not pay for it but all users should be charged equally; (b) the total cost of using the service is the sum of capital and operating costs, and (c) the service provider will recover the entire cost from the customers. Clearly demonstrate that there is a unique solution to this cost recovery game.

(10)

6. An firm I owns a factory and each member of a set L of laborers owns only his own labor skills. Laborers can produce nothing on their own and members of any non-empty subset S of $s = |S|$ laborers can produce exactly s units of output if they work in the factory. Formulate a coalition form game that models this situation. Justify your formulation.

(6)

7. State and prove the Bondareva-Shapley theorem by defining all necessary concepts.

(12)

8. Let $N = \{A_1, A_2, A_3\}$ be a set of 3 firms producing a homogenous output whose price function is $10 - x_{A_1} - x_{A_2} - x_{A_3}$, x_{A_i} being the output of firm A_i. The maximum output a firm can produce is 3. The cost function of firm A_i is $(1 + x_{A_i})$. The worth of any non-empty coalition $S \in 2^N$ is defined as

$$v(S) = \max_{x_{A_i}, A_i \in S} \min_{x_{A_j}, A_j \notin S} \sum_{A_i \in S} x_{A_i}\left(10 - x_{A_1} - x_{A_2} - x_{A_3}\right) - \left(\sum_{A_i \in S}\left(1 + x_{A_i}\right)\right).$$

Determine the numerical value of worth of each non-empty coalition. Also identify the set of core elements and determine its Shapley value.

(26)

p.T.O

9. Show that a solution to the two-person bargaining problem satisfying the four Nash axioms is the Nash product.

(7)

B

1. Suppose a weighted majority game has 5 players with weights 7, 3, 8, 2 and 1, given that the quota is 14. Using an efficient method determine the number of winning coalitions in the game and the number of swings for the player having weight 2.

(4+4 = 8)

2. Describe the "men propose" variant of the Gale-Shapley algorithm. In this algorithm, is it possible that before termination there arises a round where a man does not have any woman to propose to?

(8 + 4 = 12)

Indian Statistical Institute

Final Examination: 2nd Semester, 2016-2017

Programme: Master of Mathematics

Course: **Complex Analysis**

Maximum Marks: 100

Duration: 3 Hours

Date: April 24, 2017

Notation: $\mathbb{C}$ denotes the complex plane, $\mathbb{D}$ denotes the open unit disc $\{z \in \mathbb{C} : |z| < 1\}$, $\hat{\mathbb{C}}$ denotes the Riemann sphere.

1. **(a)** Suppose $\Omega \subset \hat{\mathbb{C}}$ is open and $f : \Omega \longrightarrow \hat{\mathbb{C}}$ is a map. What does it mean to say 'f is holomorphic in Ω'? Discuss all the cases.
(b) If $f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ is holomorphic and $f(\infty) = \infty$, what can you conclude about f? Prove your assertion.
(c) Determine all entire functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ satisfying the conditions $f(0) = 0, f(1) = 1$, and $|f'(z)| > 0$ for all $z \in \mathbb{C}$. [6+5+9=20 marks]

2. Suppose $\Omega \subset \mathbb{C}$ is open and $S \subset \Omega$ is a subset having no limit point in Ω. Suppose Γ is a cycle in $\Omega \backslash S$ that is homologous to zero.
(a) Show that the winding number $n(\Gamma, s) = 0$ for all but finitely many $s \in S$.
(b) Find all the poles and the corresponding residues of the function $(1 - e^z)^{-1}$.
(c) Evaluate
$$\int_0^{\pi} \frac{1}{2 - sin^2\theta} d\theta,$$
using the Residue Theorem justifying all the steps. [8+4+12=24 marks]

3. **(a)** Suppose U is a proper, open, simply connected subset of $\mathbb{C}$. Show that there is an injective holomorphic map from U to $\mathbb{D}$.
(b) Suppose $\mathscr{F}$ is the family of all functions on $\mathbb{D}$ defined by power series of the form
$$f(z) = \sum_{n \geq 1} a_n z^n,$$
where $a_n \in \mathbb{C}$ and $|a_n| \leq 1$ for every $n \geq 1$. Is $\mathscr{F}$ a normal family of holomorphic functions on $\mathbb{D}$? Prove your assertion. [12+8=20 marks]

4. Suppose f is a holomorphic function in a neighbourhood of some point $z_0 \in \mathbb{C}$.
(a) Show that f is injective near z_0 if and only if $f'(z_0) \neq 0$.
(b) Show that if f as above is injective in a neighbourhood of z_0 then the inverse of f is a holomorphic function in a neighbourhood of $f(z_0)$.
(c) Determine all automorphisms of the Riemann sphere $\hat{\mathbb{C}}$.
(d) Give an explicit example of a conformal map from $\Omega = \mathbb{C} \setminus [0, \infty)$ onto the upper half plane $\mathbb{H} = \{z \in \mathbb{C} : Im(z) > 0\}$ and prove that the map is conformal. [6+2+12+7=27 marks]

5. Does there exist an entire function f such that $f(n) = \sqrt{n} > 0$ for every integer $n \geq 1$? Prove your assertion. [10 marks]

Indian Statistical Institute

Final Examination 2017
M.Math 1st year/M.Stat 1st year

Course name: **Algebra/Abstract Algebra**
Date: **26 April 2017**
Maximum marks: **60**
Duration: **3 hours**

Note: There are 6 questions with total marks 65. Answer as many questions as you like. The maximum you can score is 60. This is closed notes/books exam.

1. Let $L|_k$ be a simple finite field extension. Prove that the number of subfields F of L containing k is at most 2^{n-1} where $n = [L : k]$. (10 marks)

2. Let $\alpha = \cos(\pi/6) + i\sin(\pi/6)$.
(i) Compute $[\mathbb{Q}(\alpha) : \mathbb{Q}]$ and determine the minimal polynomial of α over $\mathbb{Q}$. (7 marks)
(ii) Prove that $\mathbb{Q}(\alpha)|_{\mathbb{Q}}$ is a Galois extension and determine its Galois group. (8 marks)

3. Show that if N is an integer ≥ 2 and p is prime, then the polynomial $f(X) = X^5 - NpX + p$ cannot be solved by radicals over $\mathbb{Q}$. (10 marks)

4. Suppose that the Galois group of an irreducible and separable polynomial $f(X)$ is Abelian. Let E be a splitting field of $f(X)$ over k and let $\alpha_1, \dots, \alpha_n$ be the roots of $f(X)$ in E. Show that $E = k(\alpha_i)$ for each i, $1 \leq i \leq n$, and $[E : k] = \deg f$. (10 marks)

5. Let m and n be positive integers such that $\gcd(m, n) = 1$. Show that $\mathbb{Q}(\sqrt[m]{2}, \sqrt[n]{3}) = \mathbb{Q}(\sqrt[m]{2}\sqrt[n]{3})$. (8 marks)

6. Compute the Galois group of the polynomial $(X^2-1)(X^2-2)\dots(X^2-8) \in \mathbb{Q}[X]$. (12 marks)

Indian Statistical Institute
Second Semester Examinations: 2016-17

Course Name: M. Math, 1st year
Subject Name : Differential Geometry I
Maximum Marks: 50, Duration: Two and a half hours
Date: 28.4.2017, 2.30 PM

Marks will be deducted for indirect, incomplete, unnecessary long and imprecise answers. Throughout the question paper, S will denote an oriented n-surface without boundary unless mentioned otherwise. Consultation of OWN notes are permitted.

Question number 4 is compulsory. Answer any two questions among the rest

1. **a.** The mean curvature at a point p in S is defined to be the sum of eigenvalues of the Weingarten L map at p. For a unit vector v in $\mathbb{R}^{n+1}$, the normal curvature at p is defined to be the quantity $k_p(v) = \langle L_p(v),\ v\rangle$. Show that the mean curvature at p is equal to

$$\frac{1}{n}\sum_{i=1}^{n} k_p(v_i)$$

where $\{v_1, v_2, ..., v_n\}$ is ANY orthonormal basis for T_pS.

b. Let (U, ϕ) be a chart around p which is consistent with the orientation on S. Let L^{ϕ} denote the Weingarten map along ϕ. Prove that the mean curvature of S at p coincides with the mean curvature along ϕ at $\phi(p)$.

c. Let $S = f^{-1}(c)$ be an n-surface in $\mathbb{R}^{n+1}$, oriented by $\frac{\nabla f}{||\nabla f||}$. Suppose that $p \in S$ is such that $\frac{\nabla f}{||\nabla f||} = e_{n+1}$, where for $1 \leq i \leq n+1$, e_i denotes the i-th canonical basis element of $\mathbb{R}^{n+1}$. Show that the matrix for the Weingarten map L_p w.r.t the basis $\{e_1, e_2, ..., e_n\}$ of T_pS is $(-\frac{1}{||\nabla f(p)||}\frac{\partial^2 f}{\partial x_i \partial x_j}(p))$. **5 + 10 + 5 = 20**

2. **a.** Let U_1 and U_2 be open subsets of $\mathbb{R}^n$. Let $\phi : U_1 \to \mathbb{R}^{n+k}$ be a parametrized n-surface and $\psi = \phi \circ h$ be a reparametrization of ϕ where $h : U_2 \to U_1$ is a smooth map. Prove that $\text{Vol}(\phi(U_1)) = \text{Vol}(\psi(U_2))$.

P.T.O

b. Let $U = \{(\theta, \phi) \in \mathbb{R}^2 : -\pi < \theta < \pi, 0 < \phi < \pi\}$. Let $\phi : U \to \mathbb{R}^3$ be defined by

$$\phi(\theta, \phi) = (\cos\theta \ \sin\phi, \sin\theta \ \sin\phi, \cos\phi).$$

Using the orientation vector field along ϕ, compute $\text{Vol}(\phi)$.

c. Let ω be a compactly supported closed 5-form in $\mathbb{R}^5$. Prove that $\int_{\mathbb{R}^n} \omega = 0$. **5 + 5 + 10 = 20**

3. **a.** Consider the cylinder $x_1^2 + x_2^2 = 1$ in $\mathbb{R}^3$ with the normal given by $N(x_1, x_2, x_3) = (x_1, x_2, 0)$. Compute all its geodesics.

   b. Let $p \in S$ and $v \in T_p(S)$. Let $\alpha : I \to S$ be a maximal geodesic in S with initial velocity v. Let c be a real number. Show that the maximal geodesic β in S with initial velocity cv is given by the formula $\beta(t) = \alpha(ct)$.

   c. Let X, Y, Z be smooth tangent vector fields on S and $D_Z(X)$ denote the covariant derivative of X along Z. Prove that

$$D_Z \langle X, Y \rangle = \langle D_Z(X), \ Y \rangle + \langle X, D_Z(Y) \rangle .$$

   5 + 5 + 10 = 20

4. **a.** Let X and Y be two smooth tangent vector fields on S. Prove that the Lie bracket of X and Y, defined by

$$[X, Y](p) = \nabla_{X(p)} Y - \nabla_{Y(p)}(X)$$

   is again a tangent vector field to S. **5**

   b. Let S be an n-surface with boundary ∂S in $\mathbb{R}^{n+1}$. Let $p \in \partial S$. Prove that there is exactly one outward pointing unit vector in $T_p S$ which is normal to the boundary. **5**

INDIAN STATISTICAL INSTITUTE, KOLKATA
FINAL EXAMINATION: SECOND SEMESTER 2016 - '17
M.MATH I YEAR

Subject	:	**Functional Analysis**
Time	:	3 hours
Maximum score	:	60

Attempt all the problems. Please use a new page to answer each question, making sure that the question number in the margin can be read, even after stapling. If you attempt the same problem several times, please strike out all the attempts except the final one before submitting your answer script. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answer script. Points will be deducted for missing or incomplete arguments.

(1) Show that the usual $\|\cdot\|_\infty$ on $C[a,b]$ does not come from an inner product.

[7 marks]

(2) Let $\{A_n\}$ be a sequence of bounded operators in a Hilbert space $\mathcal{H}$ such that $\{\langle A_n(x), y\rangle\}$ is a Cauchy sequence in $\mathbb{C}$ for all $x, y \in \mathcal{H}$. Show that there exists an operator A such that A_n converges to A in the weak operator topology.

[10 marks]

(3) Let $\{e_n\}$ be an orthonormal basis in a Hilbert space $\mathcal{H}$. Prove that any orthonormal set $\{f_n\}$ that satisfies $\sum_{n=1}^{\infty} \|e_n - f_n\|^2 < 1$ is an orthonormal basis.
(Hint: If x is orthogonal to $\{f_n\}$, show that $\sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 < \|x\|^2$, violating Parseval's identity.)

[8 marks]

(4) Let $\mathcal{H}$ be a Hilbert space.
(i) If U is a unitary operator on $\mathcal{H}$ prove that the spectrum of U lies on the unit circle.
(ii) A self adjoint operator T on $\mathcal{H}$ is said to be positive if $\langle A(x), x\rangle \geq 0$ for all $x \in \mathcal{H}$. Show that the spectrum of T is contained in the positive real axis.
(iii) If T is a normal operator on $\mathcal{H}$ then prove that the spectral radius of T equals $\|T\|$.

[5+5+5=15 marks]

(5) Consider bounded linear maps S and $T : L^2[0,\infty) \to L^2[0,\infty)$ be defined by

$$(Sf)(x) = f(x+1) \quad \text{and} \quad (Tf)(x) = f(x) + f(x+2)$$

(i) Show that $\sigma(S) = \{z \in \mathbb{C} : |z| \leq 1\}$ (Hint: For $|\lambda| < 1$, consider functions $f : [0,\infty) \to \mathbb{C}$ of the form $f(x+n) = \lambda^n f(x)$ for all $x \in [0,1), n \in \mathbb{N}$)

(ii) Find $\sigma(T)$ (You may use the spectral mapping theorem for polynomial functions.)

[(8+7)=15 marks]

(6) Let $\mathcal{H}_0, \mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces and $\mathcal{A}, \mathcal{B}$ be (norm-) bounded subsets of $\mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ respectively. Consider the map

$$\mathcal{A} \times \mathcal{B} \ni (S,T) \mapsto S \circ T \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_2).$$

Is this map continuous if all the relevant spaces are endowed with
(i) weak operator topology
(ii) strong operator topology?
Justify your answer.

[5+5=10 marks]

INDIAN STATISTICAL INSTITUTE
Semestral Examination : 2016-2017
M. Math. - II Year
Topology-II

Date : 05. 05. 2017 Maximum Score : 50 Time : 3:00 Hours

Any result that you use should be stated clearly.

(1) (a) State Van-Kampen Theorem.
(b) Use Van-Kampen theorem to compute the fundamental group of $S^1 \vee T^2$.
(c) Determine $H_1(S^1 \vee T^2; \mathbb{Z})$ with proper justification.

[2 + 6 + 4 = 12]

(2) (a) State Eilenberg-Steenrod axioms for a homology theory.
(b) Use axioms to prove that the relative singular homology groups with integer coefficients of the pairs (D^n, S^{n-1}) and (S^n, D^n_+) are isomorphic.
(c) Prove the Dimension axiom.

[6 + 6 + 6 = 18]

(3) (a) Define degree of a map $f : S^n \to S^n$.
(b) Prove that the degree of the antipodal map $a : S^n \to S^n$ is $(-1)^{n+1}$.

[2 + 8 = 10]

(4) (a) Define the notion of a CW complex.
(b) Let $X = S^2 \vee S^4$ be the wedge of spheres of dimension 2 and dimension 4. Describe a CW-complex structure of X and compute its cellular homology groups with coefficient $\mathbb{Z}$.

[5 + 5 = 10]

(5) (a) State 'Universal coefficient Theorem' for homology.
(b) Compute $H_2(T^2; \mathbb{Z}_2)$.

[2 + 4 = 6]

INDIAN STATISTICAL INSTITUTE, KOLKATA
BACKPAPER EXAMINATION: SECOND SEMESTER 2016 - '17
M.MATH I YEAR

Subject	:	**Functional Analysis**
Time	:	3 hours
Maximum score	:	100

Date : 10/7/201

Attempt all the problems. Please use a new page to answer each question, making sure that the question number in the margin can be read, even after stapling. If you attempt the same problem several times, please strike out all the attempts except the final one before submitting your answer script. Justify every step in order to get full credit of your answers. All arguments should be clearly mentioned on the answer script. Points will be deducted for missing or incomplete arguments.

(1) Define $T : C([0,1]) \to C([0,1])$ by $(Tf)(x) = x^2 f(x)$ for all $f \in C([0,1])$ and $x \in [0,1]$. Show that T is a bounded linear operator. Find $\|T\|$ and prove that $\|I + T\| = 1 + \|T\|$.

[10 marks]

(2) Suppose X and Y are Banach spaces and $T : X \to Y$ is a linear map such that there exists a linear map $S : Y^* \to X^*$ satisfying $S(\varphi) = \varphi \circ T$ for all $\varphi \in Y^*$. Show that T is continuous.

[10 marks]

(3) Let X be a normed linear space and $\varphi : X \to \mathbb{C}$ be a linear map. Prove that φ is bounded if and only if $\ker \varphi$ is closed.

[10 marks]

(4) Consider the following subspaces of c_0 :

$$M_1 = \{\{x_n\}_{n\in\mathbb{N}} : x_1 = 0\} \text{ and } M_2 = \{\{x_n\}_{n\in\mathbb{N}} : x_1 = x_2 = 0\}$$

Prove that M_1 is isometrically isomorphic to M_2 but the corresponding quotient spaces c_0/M_1 and c_0/M_2 are not isomorphic.

[10 marks]

(5) Let X and Y be Banach spaces, $T : X \to Y$ be a bounded linear operator and $\{x_n\}_{n\in\mathbb{N}}$ be a sequence in X converging weakly to $x \in X$. Show that:
(a) $\{\|x_n\|\}_{n\in\mathbb{N}}$ is bounded.
(b) $\{Tx_n\}_{n\in\mathbb{N}}$ converges to Tx weakly.
(c) if T is compact, $\{Tx_n\}_{n\in\mathbb{N}}$ converges to Tx in norm.

[15 marks]

P.T.O

(6) A sesquilinear form B is said to be *symmetric* if $B(x, y) = \overline{B(y, x)}$ for all x and y, *positive* if $B(x, x) \geq 0$ for all x. Show that a positive, symmetric sesquilinear form satisfies the Cauchy-Schwarz inequality

$$|B(x, y)|^2 \leq B(x, x)B(y, y).$$

[10 marks]

(7) Let $\mathcal{H}$ be a Hilbert space with orthonormal basis $\{e_n\}_{n=1}^{\infty}$. Define $T : \mathcal{H} \to \mathcal{H}$ by

$$T(x) = \sum_{n=1}^{\infty} \frac{1}{n+1} \langle x, e_{n+1} \rangle e_n$$

for $x \in \mathcal{H}$. Show that T is a compact operator and find T^*.

[15 marks]

(8) Consider the operator $T : l^2(\mathbb{N}) \to l^2(\mathbb{N})$ defined by

$$l^2(\mathbb{N}) \ni (x_1, x_2, x_3, \ldots) \overset{T}{\mapsto} (x_2, x_4, x_6, \ldots) \in l^2(\mathbb{N}).$$

Find (a) $\|T\|$, (b) $\sigma_p(T)$, (c) $\sigma(T)$ and (d) T^*.

[5+5+5+5=20 marks]

INDIAN STATISTICAL INSTITUTE
Semestral Examination (Back Paper) : 2016-2017
M. Math. - I Year
Topology-II

Date : 12.7.17 Maximum Score : Time : 3:00 Hours

Any result that you use should be stated clearly.

(1) (a) State Van-Kampen Theorem.
(b) Use Van-Kampen theorem to compute the fundamental group of $S^1 \vee S^1$. Hence compute its first homology group with proper justification.

[2 + 8 = 10]

(2) (a) Define singular homology groups of a space with integer coefficients.
(b) Compute $H_0(X;\mathbb{Z})$, where X is a path connected space.
(c) Prove that $H_k(D^n, S^{n-1};\mathbb{Z}) \cong \tilde{H}_{k-1}(S^{n-1};\mathbb{Z})$ for all $k \geq 1$ and $n \geq 0$.

[8 + 6 + 6 = 20]

(3) (a) Define the notion of a CW complex.
(b) Prove that for a CW complex X $H_n(X^n, X^{n-1};\mathbb{Z})$ is a free abelian group generated by its n-cells, where X^n denotes the n-skeleton of X.
(c) Describe the cellular chain complex of a CW complex.
(d) Describe a CW complex structure of the complex projective space $\mathbb{C}P^n$ and hence compute $H_*(\mathbb{C}P^n;\mathbb{Z})$.

[6 + 10 +10 + 10 = 36]

(4) (a) Prove that $\tilde{H}_k(S^n;\mathbb{Z})$ is $\mathbb{Z}$ if $k = n$ and zero otherwise.
(b) Define degree of a map $f : S^n \to S^n$.
(c) Prove that the degree of the antipodal map $a : S^n \to S^n$ is $(-1)^{n+1}$.

[12 + 2 + 12 = 26]

(5) (a) State Universal Coefficients Theorem.
(b) Compute homology groups of $\mathbb{C}P^n$ with $\mathbb{Z}_2$ coefficients.

[2 + 6 = 8]