Unambiguous Quantum State Discrimination – A new approach

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by

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To the uncertainity that rules certainly

Declaration

I hereby declare that the dissertation report entitled **"Unambiguous Quantum State Discrimination – A new approach"** submitted to Indian Statistical Institute, Kolkata, is a bonafide record of work carried out in partial fulfilment for the award of the degree of **Master of Technology in Computer Science**. The work has been carried out under the guidance of **Guruprasad Kar**, Professor & Head, PAMU, Indian Statistical Institute, Kolkata.

I further declare that this work is original, composed by myself. The work contained herein is my own except where stated otherwise by reference or acknowledgement, and that this work has not been submitted to any other institution for award of any other degree or professional qualification.

Place : Kolkata

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CERTIFICATE

This is to certify that the dissertation entitled **"Unambiguos Quantum State Discrimination – A new approach"** submitted by **Aryasomayajula V S D Bharadwaj** to Indian Statistical Institute, Kolkata, in partial fulfilment for the award of the degree of **Master of Technology in Computer Science** is a bonafide record of work carried out by him under my supervision and guidance. The dissertation has fulfilled all the requirements as per the regulations of this institute and, in my opinion, has reached the standard needed for submission.

Dr. Guruprasad Kar

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Abstract

One of the consequences of No-Cloning[1] theorem is that we cannot identify a given state drawn from a set of non-orthogonal states with certainity using a single copy. The Unambiguous Quantum State Discrimination(UQSD) deals with number of copies required so as to identify such a state drawn from a set of non-orthogonal states though not certainly but with some probability. The UQSD between linearly independent states has been solved by A.Chefles[5] and later he solved UQSD between linearly dependent states[6].

We have taken a few examples and shown that for these examples the number of copies required for UQSD between linearly dependent states is far less than the one obtained by A.Chefles and have shown that if the set of states can be partitioned into sets of linearly independent states then from each partition a probable candidate for the state whose identity is under question can be obtained. It has been seen that this method works atleast as good as the one proposed by Anthony Chefles. Also, we deduced the condition on the size of partitions when our method would outperform the method given by A.Chefles.

Keywords: *Unambiguous,State Discrimination, Linearly independent, Linearly dependent.*

Contents

List of Figures

List of Algorithms

Chapter 1

Introduction

1.1 The No-Cloning Theorem

The quantum No-Cloning[1] theorem tells us – "There doesn't exist a universal cloning machine" – one which can make copies of a given arbitrary state. It also tells us that if at all there exists a cloning machine that can clone any given state from a given set of states then these set of states are mutually orthogonal.

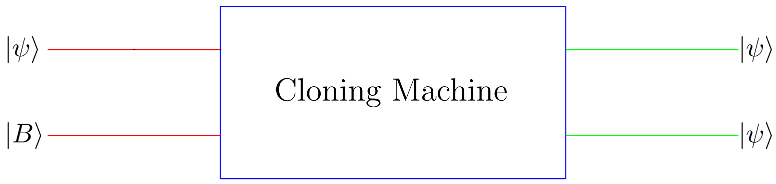

Figure 1.1: A Cloning Machine

Proof. * Let us assume a cloning machine as described in above figure exists then it corresponds to a unitary operator U_{CM} that takes two distinct states $|\psi\rangle$ and $|\phi\rangle$ along with a blank state $|B\rangle$ each time such that:

$$U_{CM}|\psi\rangle|B\rangle = |\psi\rangle|\psi\rangle \tag{1.1}$$

$$U_{CM}|\phi\rangle|B\rangle = |\phi\rangle|\phi\rangle \tag{1.2}$$

*Even though the proof presented here deals with just pure states, the actual proof is a generalized one.

Taking inner product between (1.1) and (1.2) we have $\langle\phi|\psi\rangle = \langle\phi|\psi\rangle^2$

$\Rightarrow \langle\phi|\psi\rangle = 0$ which means existence of such a unitary is possible for mutually orthogonal states only. □

So, even though there cannot exist a universal cloning machine but there exists a specialized cloning machine, the speciality being the set of states which this machine can clone are mutually orthogonal to one other.

1.2 Consequences of No-Cloning Theorem

The consequences of No-Cloning theorem are profound. Few of them are as follows:

- Non-Orthogonal states and Perfect Distingushability

  The immediate concequence of No Cloning Theorem is the limitation on the perfect distinguishability of non-orthogonal states. That is, given a state from a set of non-orthogonal states one can't make a perfect copy of it.

- Error Correction

  The classical error correction techniques cannot be applied in quantum computation as copying an arbitrary qubit is not possible.

- In Quantum Information Processing

  In quantum cryptography, this is the back bone in generating secret random key[2].

The No-Cloning theorem supports many other theorems like No-Communication theorem[3][4] etc.,

1.3 Unambiguous Quantum State Discrimination

As it was discussed in the earlier section, No-Cloning theorem puts a restriction on perfect distinguishability of non-orthogonal states. If Alice and Bob follow the BB84[2] protocol and are sharing a random secret key and Eve wants to eavesdrop on their communication, she has to conclude about the state that is being sent by Alice to Bob but with less disturbance to the system and at the same time achieving a high fidelity.

Even though we know that – a single copy of unknown given state from a set of non-orthogonal states cannot be identified perfectly, an interesting question that now arises is – "How many copies of a unknown state (taken from a set of non-orthogonal states) are required inorder to identify it perfectly?". This problem is dubbed as Unambiguous Quantum State Discrimination.

UNAMBIGUOUS QUANTUM STATE DISCRIMINATION(UQSD) can be stated as

"Number of copies of an unknown state $|\psi\rangle_i$ *taken from a set of* N *distinct non-orthogonal states* $\{|\psi\rangle_j\}_{j=1}^{N}$ *required to perfectly identify it.*[†] *"*

The next chapter deals with the solution to the UQSD problem given in the past.

[†]The discussion from now onwards is only for pure states

Chapter 2

Solution to UQSD

The problem of UQSD has been solved by Anthony Chefles in two stages. First the UQSD problem was solved for linearly independent states in 1998[5] and later in 2001[6], A.Chefles again showed how UQSD between linearly dependent states case can be converted to linearly independent case thus solving the UQSD problem.

2.1 UQSD between Linearly Independent States

Let us assume the N states under consideration of UQSD be $\{|\psi\rangle_i\}_{i=1}^{N}$. It is shown[5] that we can construct $N+1$ POVM operators $\{M_i\}_{i=1}^{N+1}$ where for each $|\psi\rangle_i$ there is corresponding M_i formed by taking outer product of some $|\phi\rangle_i$, that is $M_i = |\phi\rangle_{ii}\langle\phi|$, which is perpendicular to the sub-space spanned by the rest of the states $\{|\psi\rangle_1, |\psi\rangle_2,...,|\psi\rangle_{i-1}, |\psi\rangle_{i+1}, |\psi\rangle_{i+2}, ..., |\psi\rangle_N\}$. The M_{N+1} operator is choosen so that $\Sigma_{i=1}^{N+1} M_i = I$.

Now, when the given unknown state is measured and if the measurement results in one of the $\{M_i\}_{i=1}^{N}$, say M_j then we conclude $|\psi\rangle_j$ is the state given but if we get M_{N+1} then the measurement result is inconclusive and we repeat the experiment again.

The following algorithm describes the above discussion:

Algorithm 1 UQSD between Linearly Independent States

1: $S = \{|\psi\rangle_i\}_{i=1}^N$
2: **procedure** Create POVM Operators(S)
3: Set $O = \{\}$
4: **for** each $|\psi\rangle_j \in S$ **do**
5: Find the set of orthogonal states for the sub-space spanned by the rest.
6: Remove the components of all these orthogonal states from $|\psi\rangle_j$
7: Normalize the reminder of $|\psi\rangle_j$. This is $|\phi\rangle_j$.
8: Add $M_j = \alpha_j |\phi\rangle_{jj}\langle\phi|$ to O, α_j is a positive real constant.
9: **end for**
10: Now construct $M_{N+1} = I - \Sigma_{i=1}^N M_i$.
11: Add M_{N+1} to O
12: Determine α_i's so as to minimize the expectation of inconclusive result, M_{N+1}
13: **return** O
14: **end procedure**
15: Now perform measurement on given $|\psi\rangle_j$ using the POVM operator set O.
16: **if** the measure result is M_j and $j \neq N+1$ **then**
17: $|\psi\rangle_j$ is the given state
18: else repeat Step-15
19: **end if**

2.2 UQSD between Linearly Dependent States

Let us assume the N states under consideration of UQSD be $\{|\psi\rangle_i\}_{i=1}^N$ and they span a dimension D $(< N)$. It was shown in [5] that only linearly independent states can be unambiguously discriminated. So, to apply a POVM approach described in the previous section we need to find a way to make these states linearly independent.

A.Chefles showed[6] that if the N states $\{|\psi\rangle_i\}_{i=1}^N$ that span a dimension D are linearly dependent then the states $\{|\psi\rangle_i^{\otimes N-D+1}\}_{i=1}^N$ are linearly independent. So, we construct POVM operators for $\{|\psi\rangle_i^{\otimes N-D+1}\}_{i=1}^N$ and demand for $N - D + 1$ copies of unknown state say $|\psi\rangle_j$ and perform measurement on $|\psi\rangle_j^{\otimes N-D+1}$ and conclude about the unknown state when the measurement result is not inconclusive.

Algorithm 2 UQSD between Linearly Dependent States

1: $S = \{|\psi\rangle_i^{\otimes N-D+1}\}_{i=1}^N$
2: Create POVM Operators(S)
3: Now perform measurement on $|\psi\rangle_j^{\otimes N-D+1}$ using the POVM operator set O.
4: **if** the measure result is M_j and $j \neq N+1$ **then**
5: $|\psi\rangle_j$ is the given state
6: else repeat Step-3
7: **end if**

2.3 Related Works

The various recent works related to Quantum State Discrimination are:

- Minimum Error State Discrimination

  Here[7] inconclusive results are not allowed and error in decision about the identity of state should be minimum.

- Operator Discrimination Problem

  Here, instead of state discrimination, looking at the output state we have to identify the operator from a set of operators which has acted upon the input state.[8] [9]

Various other works on the mixed states and optimal distinguishability[10] are done.

2.4 Our Contribution

The UQSD between linearly independent states has been extensively studied and has been used in varied applications. The linearly dependent case was tackled just by converting the given set of states into linearly independent states. This doesn't take into account the arrangement of given set of states and identifies the states using exactly $N - D + 1$ copies. We have proposed a new approach for UQSD between linearly dependent states that takes the given set of states into account and identifies the state under question in atmost $N - D + 1$ copies.

Chapter 3

UQSD – A new approach

3.1 Overview

We have seen that the problem of UQSD between linearly independent states has been elegantly solved by the use of POVM measurement. Given just a single copy, one can tell with certain probability what the given state is. Also, optimization of this probability given the apriori probabilities of the set of states has been done.

When it comes to the UQSD between linearly dependent states the solution offered by Anthony Chefles[6] is very simple.If we take $N - D + 1$ copies of the states then the resulting states are linearly independent and thus we can employ POVM based measurement technique to identify $|X\rangle$ unambiguously. It is a big jump from one copy in the case of linearly independent states to $N - D + 1$ copies in the case of linearly dependent states.

The question that comes right away is – "What if the given set of states under consideration have in them several different sets of linearly independent states. Can't we employ some technique to use just a single copy for each of these sets and conclude something?" – We found that the this infact could be done.

3.2 A better partitioning of given set of states

Our approach of UQSD between linearly dependent states greatly relies on partitioning the given set of states in groups where each group is linearly independent.

Trivial Partition: A trivial partitioning scheme is to pair up the states to form (atmost) $\lfloor \frac{N}{2} \rfloor + 1$ partitions in total.

Algorithm 3 Trivial Partitioning

```
S = {|ψ⟩_i}_{i=1}^N
procedure TRIVIAL PARTITION(S, Part)
    while S is not empty do
        if |S| = 1 then
            Add S to Part
            return Part
        end if
        Make a partition P of D and add two states drawn from S to it.
        Add P to Part
        Set S = S - P
    end while
end procedure
return Part
```

Example: A trivial Partitioning:

Let the given set of states be:

$\{|0\rangle, |1\rangle, |2\rangle, \frac{|0\rangle+|1\rangle}{\sqrt{2}}, \frac{|1\rangle+|2\rangle}{\sqrt{2}}, \frac{|0\rangle+|2\rangle}{\sqrt{2}}, \frac{|0\rangle-|1\rangle}{\sqrt{2}}, \frac{|1\rangle-|2\rangle}{\sqrt{2}}\}$

Here we see that the set of states span a dimension of $D = 3$. and $N = 8$.The pratition is as follows:

$$P_1 = \{|0\rangle, |1\rangle\} \tag{3.1}$$

$$P_2 = \{|2\rangle, \frac{|0\rangle + |1\rangle}{\sqrt{2}}\} \tag{3.2}$$

$$P_3 = \{\frac{|1\rangle + |2\rangle}{\sqrt{2}}, \frac{|0\rangle + |2\rangle}{\sqrt{2}}\} \tag{3.3}$$

$$P_4 = \{\frac{|0\rangle - |1\rangle}{\sqrt{2}}, \frac{|1\rangle - |2\rangle}{\sqrt{2}}\} \tag{3.4}$$

It will be seen later that trivial partitions cannot improve the number of copies required for the UQSD. A better partitioning scheme is discussed next.

Before going into the technique let us prove something.

Lemma 3.1. *If the dimension spanned by a set of states is D then the dimension D_1 spanned by the set of states after removing any D linearly independent states from the original set cannot be greater than D*

Proof. If $D_1 > D$ then the dimension of the original set would have been D_1 by choosing these D_1 linearly independent states at the first place which contradicts the fact that the original set of states span a dimension D Thus, $D_1 \leq D$ □

Now, coming to the scheme, first make D linearly independent states as the first partition. Now, find the dimension spanned by the remaning states. Let us call it D_1. Now if $D_1 > 2$ then remove D_1 linearly independent states in the reminder states and make them into a new partition and continue the same process until $D_i = 2$ or we are left with only one or no state in our hand. When $D_i = 2$ then by Lemma 3.1 we know only size-2 partitions can be made thus we pair up those remaining set of states.

Algorithm 4 Better partitioning

$S = \{|\psi\rangle_i\}_{i=1}^{N}$
procedure PARTITION(S, Part)
    while S is not empty **do**
        Set D to the dimension of the space spanned by S
        if $D = 2$ **then**
            P = TRIVIAL PARTITION(S)
            Add P to Part
            return Part
        end if
        Make a partition P of D linearly independent states from S
        Add P to Part
        Set $S = S - P$
    end while
end procedure
return Part

Example: A better partioning

For the same states taken in previous example a better partitioning schemes results in following partitions:

$$P_1 = \{|0\rangle, |1\rangle, |2\rangle\} \tag{3.5}$$

$$P_2 = \{\frac{|0\rangle + |1\rangle}{\sqrt{2}}, \frac{|1\rangle + |2\rangle}{\sqrt{2}}, \frac{|0\rangle + |2\rangle}{\sqrt{2}}\} \tag{3.6}$$

$$P_3 = \{\frac{|0\rangle - |1\rangle}{\sqrt{2}}, \frac{|1\rangle - |2\rangle}{\sqrt{2}}\} \tag{3.7}$$

3.3 Motivating Example

Now let us first take an example to see that we can do better than $N - D + 1$ copies.

Example

Consider $\{E_i\}_{i=1}^{k}$ represent a collection of set of eigen states of k full rank operators that span the same hilbert space H_D of dimension D and also assume that all the states in this collection are distinct.Now given a unknown state $|X\rangle$ from this collection, we would like to identify $|X\rangle$

We have number of states in the collection, $N = kD$. Also, we can partition these N states into k sets each of size D where each set is nothing but one of the E_i and thus are linearly independent infact are orthogonal to one other with in the set.

We identify $|X\rangle$ in a two step process. The unknown state $|X\rangle$ must have come from one of these $k-$partitions. First we try to pickout one state from each partition which could possibly be our $|X\rangle$ and in the second step identify which among these possible candidates is actually our $|X\rangle$.

To accomplish step 1, i.e., to get possible candidates from each partition, we ask for k copies of the unknown state $|X\rangle$. We perform a measurement on each copy with k different measurement sets where the measurement sets are drawn from the states corresponding to each partition.

When measured with the measurement operators set drawn from the partition to which $|X\rangle$ actually belongs will result in outcome $|X\rangle$ where as in the remaining partitions the outcome results in some $|X_{probable}\rangle$ which is not orthogonal to $|X\rangle$ in that corresponding partition.

Thus after the first step we have k distinct states among which one is our unknown state $|X\rangle$. These k states shall span atleast a dimension of 2. Hence, to identify $|X\rangle$ we need $k-1$ copies it. In total we need $2k-1$ copies but we have $k = \frac{N}{D}$. Thus total number of copies required is $\frac{2N}{D} - 1$. The number of copies required actually is $N - D + 1$ if the property satisfied by these states is not utilized.

3.4 Remarks on the result of motivation example

We need the condition on D such that $\frac{2N}{D} - 1 \leq N - D + 1$

$$\Rightarrow 2N - D \leq ND - D^2 + D \tag{3.8}$$

$$\Rightarrow D^2 - D(N+2) + 2N \leq 0 \tag{3.9}$$

which gives the solution for D as $2 \leq D \leq N$ where equality holds in the edges i.e., when $D = 2$ and $D = N$. Thus for any $2 < D < N$ the above method is effective.
Now consider the savings S in number of copies:

$$S = (N - D + 1) - (\frac{2N}{D} - 1) \tag{3.10}$$

Differentiating S w.r.t D and setting to 0 we get $D = \sqrt{2N}$ which maximizes the number of savings. and corresponding savings,S_{max} is:

$$S_{max} = (\sqrt{N} - \sqrt{2})^2 \tag{3.11}$$

So, savings are higher as D is near to $\sqrt{2N}$.

3.5 Generalization

The results of the previous section and the result that when $D = \sqrt{2N}$ the savings are of the order of N motivates us. This section will generalize the technique used in our motivation example.

Consider a set of distinct states $\{|\psi\rangle_i\}_{i=1}^N$ that span a dimension D in Hilbert space H_D. Now consider a subset of linearly independent states (of size $\leq D$) in that set. Let the subset be $\{|\phi\rangle_i\}_{i=1}^k$ ($k \leq D$). We construct the $k+1$ POVM operators $\{M_i\}_{i=1}^{k+1}$ where each M_i corresponds to the state $|\phi_i\rangle$ and M_{k+1} corresponds to the inconclusive outcome. In case of $k < D$, to these $k+1$ operators we also append the measurement operators $\{O_i\}_{i=1}^{D-k}$ that form a measurement set of the orthogonal space to the space spanned by the $\{|\phi\rangle_i\}_{i=1}^k$ in H_D. Thus $\{M_i\}_{i=1}^{k+1}$ and $\{O_i\}_{i=1}^{D-k}$ together will form the complete set of measurement operators of H_D

Theorem 3.2. *Atmost a single probable candidate for $|X\rangle$ can be obtained from this subset using a single copy provided the results are not inconclusive.*

Proof. We ask for a single copy of $|X\rangle$ and perform measurement using this measurement set. Here, there are three possible outcomes.

1) The measurement results in M_{k+1}

2) The measurement yields one of the $\{O_i\}_{i=1}^{D-k}$ as outcome

3) The measurement collapses on one of $\{M_i\}_{i=1}^k$ say M_j (which corresponds to $|\phi_j\rangle$)

In case 1, the measurement results are inconclusive and thus we have to repeat the experiment again. In case 2, $|X\rangle$ has a component in it that is orthogonal to the space spanned by $\{|\phi_i\rangle\}_k$ thus we can safely conclude the given subset doesn't contain $|X\rangle$. In case 3, if $|X\rangle$ is actually present in this subset then $|\phi_j\rangle$ is our $|X\rangle$ but if $|X\rangle$ is not present in the given subset then $|\phi_j\rangle$ is just a state that is not orthogonal to it. The case 3 presents some ambiguity on whether or not $|\phi_j\rangle$ is $|X\rangle$. Thus, we call $|\phi_j\rangle$ as $|X_{probable}\rangle$.

□

Algorithm 5 Probable Candidate

```
1: S = {|φ⟩_i}_{i=1}^k
2: procedure PROBABLE CANDIDATE(S, |X⟩)
3:     M = CREATE POVM OPERATORS(S)
4:     if k < D then
5:         O = Orthogonal measurement set of remaining space.
6:     end if
7:     Perform measurement on |X⟩
8:     if Result = M_{k+1} then
9:         goto Step-7
10:    end if
11:    if Result ∈ O then
12:        return Null
13:    end if
14:    return |X_probable⟩ = State corresponding to the result
15: end procedure
```

Another Example:

Definition 3.1. *A set of N distinct states that span a dimension D are said to be d-spatially arranged if they can be arranged into $\lfloor \frac{N}{d} \rfloor + 1$ partitions of which $\lfloor \frac{N}{d} \rfloor$ are exactly of size $d(\leq D)$ and are linearly independent and the last partition is of size atmost $d - 1$*

To answer UQSD in the case of a d-spatially arranged states, using the two step process described in motivation example and applying result of generalization section during the first step of that process, we get number of copies of $|X\rangle$ required as $2\lfloor \frac{N}{d} \rfloor + P - 1$ where P is the size of the last partition. One can see our motivation example is a special case of $d-$spatial arrangement with $d = D$ and $P = 0$

3.6 Final Result

Now we are all set to propose our new approach.

Definition 3.2. *Let us define secondary span(s) of a given set of N distinct states that span a dimension of D as the dimension of the space spanned by remainder of the $(N - D)$states after removing a set of D linearly independent states from the given set of states.*

It is easy to see that the secondary span s is atleast 2 and atmost D whenever $N - D > 1$. Let us denote the secondary span of the given set of the N states as s_1

the secondary span of the remainder of the $N - D$ states using which s_1 is deduced be s_2 and call it ternary span. In this fashion a general $n-$ary span can be defined. From lemma 3.1 it follows that $s_i \geq s_{i+1}$

Now, let us consider the problem of UQSD of N distinct states that span a dimension $D < N$ and unknown state to be identified, $|X\rangle$ are given.

s_k denotes the $(k+1)$-ary span of given states and suppose i such spans are identified such that $s_{i+1} > 1$. Now we make the given set of states into $(i+2)$ partitions. The first partition P_1 contains the D linearly independent states. A partition P_j in second through $i+1$ partitions contains the linearly independent states that defined the $(j+1)$-ary span s_j and the last partition P_{i+2} contains the rest of the states.

The first $i+1$ partitions are linearly independent. So, we ask for $i+1$ copies of the unknown state $|X\rangle$ using which we can get $i+1$ probable candidates for $|X\rangle$.

Effectively, after this step, we are dropping atleast $D+(\sum_{j=1}^{i} s_j)-(i+1)$ states. Now, we have $N-(D+(\sum_{j=1}^{i} s_j)-(i+1))$ states that span atleast a dimension 2 (as we have $s_{i+1} > 1$) from which $|X\rangle$ can be identified by using $N-(D+(\sum_{j=1}^{i} s_j)-(i+1))-1$ copies of it.
Thus total number of copies required is

$$N - (D + (\sum_{j=1}^{i} s_j) - (i+1)) - 1 + (i+1) \tag{3.12}$$

$$= N - D - (\sum_{j=1}^{i} s_j) + 2i + 1 \tag{3.13}$$

For this method to work better, we need the condition:

$$N - D - (\sum_{j=1}^{i} s_j) + 2i + 1 \leq N - D + 1 \tag{3.14}$$

$$\Rightarrow \sum_{j=1}^{i} s_j \geq 2i \tag{3.15}$$

The minimum value that any s_j can take is 2. Thus the inequaity always holds. Hence, the method suggested performs atleast as good as the one described by A.Chefles[6].

We note that any $s_j = 2$ in the above inequality actually does nothing to the inequality. Thus if we have atleast a single $s_j \geq 3$ then we can have some savings. So, a trivial partitioning as mentioned in the beginning of the chapter works worse when $D > 2$. Also, one should note that the method in which partitions are made using secondary spans is the same the "better partitioning scheme" discussed earlier.

Chapter 4

Conclusion And Future Work

Here, in our project we proposed a new approach for Unambiguous Quantum State Discrimination between Linearly Dependent States. It is seen that given a set of N distinct linearly dependent states that span a dimension D and if a number of subset of states are linearly independent (apart from the trivial subset of size D) then we can improve upon the bound if atleast one of these subsets spans a dimension greater than 2.

We put forward the following questions:

1. If there is no secondary span that is greater than 2 is it even then possible to do better?
2. Can we apply this technique to improve upon the probability of UQSD between linearly independent states?
3. Does there exist a better partition method than the approach presented here?

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