RECOVERY OF INTERBLOCK INFORMATION

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SUMMARY. The problem of recovery of inter-block information for a general incomplete block design is examined in this paper. The ratio ρ of the inter-block variance to the intra-block variance plays a key role. Under the so-called Normal set up, the usual estimator of ρ is biased; expressions for the bias and variance are derived. Several alternative estimators of ρ having desirable properties are examined. A computational procedure for obtaining the maximum likelihood estimate is given. If a certain type of estimator of ρ is used, the estimators of treatment effects are proved to be unbiased. An expression is derived for the increase in the variance of the estimate of a treatment effect due to fluctuations of sampling in ρ.

1. Introduction

Since some incomplete block designs have low efficiency factors, Yates suggested the use of information available from inter-block comparisons to increase the precision of estimates of treatment effects. He called the process recovery of inter-block information and showed how this is to be done for a cubic lattice design (1939) and a balanced incomplete block (BIB) design (1940). Nair (1944) gave the method for partially balanced incomplete block designs and finally Rao (1947, 1956) adopted the method for any incomplete block design.

The process consists in applying the method of weighted least squares to intra-block contrasts and inter-block contrasts of observations, for the purpose of estimating the treatment effects, weights being inversely proportional to the variances of these contrasts. The ratio ρ of the inter-block variance to the intra-block variance plays a key role and since this is usually unknown, the ratio of estimates of these variances obtained from an analysis of variance of the data is substituted. The properties of estimates so obtained have not so far been investigated in detail. Recently, Graybill and Weeks (1959) have proved that under the so-called normal model, estimates of treatment effects so obtained, are unbiased in the case of BIB designs and they have also obtained (1961) a minimal set of sufficient statistics.

The problem of recovery of inter-block information in the case of a general incomplete block design is examined in detail in this paper. The usual estimator of the variance ratio ρ obtained from the analysis of variance is found to be biased and a simple correction is obtained for this bias. An expression for the variance of this estimator is derived. Some alternative estimators of ρ having desirable properties and based on quadratic estimators of inter-block and intra-block variances are also proposed. The method of maximum likelihood is shown to give rise to a somewhat complicated equation for estimation, and a numerical procedure for solving the equation by iteration is presented.

It is shown further that if a certain type of estimator of the variance ratio is used, the final estimators of treatment effects turn out to be unbiased. It is shown that there is an increase in the sampling error of the treatment effects due to the sampling fluctuation in the variance ratio, and an expression for this increment is derived.

Consider an experiment in which v treatments are applied on bk experimental units or plots, themselves divided into b blocks of k plots each. Only one treatment is applied on every plot, the actual allocation being done in the following manner.

First we consider a design, that is an arrangement of v symbols (one corresponding to each treatment) in b rows, each having k cells. The arrangement is characterised by the numbers m_{jiu}, $j = 1, 2, \ldots, v$; $i = 1, 2, \ldots, b$; $u = 1, 2, \ldots, k$ where $m_{jiu} = 1$ or 0 according as the j-th symbol (treatment) occurs on the u-th cell of the i-th row or not.

Next, the blocks are numbered $1, 2, \ldots, b$ at random and the plots in a block are numbered $1, 2, \ldots, k$ again at random and independently for different blocks. The u-th plot in the i-th block then receives the treatment corresponding to the symbol which occurs in the u-th cell of the i-th row of the design. Let y_{iu} denote the yield of this plot. Under the assumption that the yield from any plot is the sum of two components, one due to the plot and the other due to the treatment, we have

$$E(y_{iu}) = \mu + \sum_{j=1}^{v} \theta_j m_{jiu} \qquad \ldots \quad (1.1)$$

$$\operatorname{cov}(y_{iu}, y_{i'u'}) = \begin{cases} \left(1-\frac{1}{k}\right)\sigma_0^2 + \frac{1}{k}\left(1-\frac{1}{b}\right)\sigma_1^2 & \text{if } i = i',\ u = u' \\ -\frac{1}{k}\sigma_0^2 + \frac{1}{k}\left(1-\frac{1}{b}\right)\sigma_1^2 & \text{if } i = i',\ u \neq u' \\ -\frac{1}{bk}\sigma_1^2 & \text{if } i \neq i'. \end{cases} \qquad \ldots \quad (1.2)$$

Here θ_j is the effect of the j-th treatment $\sum_{j=1}^{v} \theta_j = 0$, μ is the overall mean of plot effects, and σ_0^2 and σ_1^2 are respectively the mean squares (of plot effects) within and between the blocks. From Section 3 onwards, we shall make the further assumption that the joint distribution of the random variables y_{iu}'s is multivariate normal, with first and second order moments given by (1.1) and (1.2). We shall write

$$\rho = \sigma_1^2/\sigma_0^2. \qquad \ldots \quad (1.3)$$

The problem is to estimate the parameters θ_j's and σ_0^2 and σ_1^2.

Let $\sum_{u=1}^{k} m_{jiu} = n_{ji}$, the number of times the j-th treatment occurs on plots in the i-th block. Thus $n_{ji} = 1$ or 0 and $\sum_{j=1}^{v} n_{ji} = k$, $\sum_{i=1}^{b} n_{ji} = r$. The $v \times b$ matrix $N = ((n_{ji}))$ is called the *incidence matrix*.

We shall denote by $E_{m\times n}$ a matrix of the form $m \times n$, each element of which is unity. The matrices

$$C = rI - \frac{1}{k} NN' \quad \text{and} \quad C_1 = \frac{1}{k} NN' - \frac{r^2}{bk} E_{vv} \qquad \ldots \quad (1.4)$$

play important roles in the analysis. We shall assume that the matrix C is of rank $(v-1)$: this is equivalent to the assumption that the experimental design is *connected*.

A linear function $\sum_{i,u} a_{iu} y_{iu}$ is said to be a *contrast* if $\sum_{i,u} a_{iu} = 0$. A contrast is said to belong to blocks, or simply called an *inter-block contrast* if $a_{i1} = a_{i2} = \cdots = a_{ik}$ holds for all i. A contrast is said to be an *intra-block contrast* if $\sum_{u} a_{iu} = 0$ holds for all i. A linear function $\sum_{i,u} a_{iu} y_{iu}$ is said to be *normalised* if $\sum_{i,u} a_{iu}^2 = 1$. Two linear functions $\sum_{i,u} a_{iu} y_{iu}$ and $\sum_{i,u} b_{iu} y_{iu}$ are said to be *orthogonal* if $\sum_{i,u} a_{iu} b_{iu} = 0$. It is easy to see that any inter-block contrast and any intra-block contrast are mutually orthogonal. The rank of the vector-space generated by all inter-block contrasts is $(b-1)$ and of that generated by all intra-block contrasts is $b(k-1)$.

Let B_i denote the total yield for the i-th block and T_j that for the j-th treatment and let G be the grand total, so that

$$B_i = \sum_{u} y_{iu},\ T_j = \sum_{i,u} y_{iu} n_{jiu} \quad \text{and} \quad G = \sum_{i,u} y_{iu}. \qquad \ldots \quad (1.5)$$

We shall use the row-vectors $B = (B_1, B_2, \ldots, B_b)$, and $T = (T_1, T_2, \ldots, T_v)$. The adjusted yields for the treatments are defined as

$$Q = T - \frac{1}{k} BN'. \qquad \ldots \quad (1.6)$$

Let, further

$$Q_1 = \frac{1}{k} BN' - \frac{rG}{bk} E_{1v}. \qquad \ldots \quad (1.7)$$

It can be seen that the elements of Q are intra-block contrasts and those of Q_1 are inter-block contrasts.

It is known (see, for example, Rao, 1947) that minimum variance unbiased linear estimates of the treatment effects $\theta = (\theta_1, \theta_2, \ldots, \theta_v)$, based on intra-block contrasts only, are obtained from the equations

$$\theta C = Q. \qquad \ldots \quad (1.8)$$

We shall write θ^* for the solution of these equations. If the ratio $\rho = \sigma_1^2/\sigma_0^2$ is known, both intra-block and inter-block contrasts can be used together, and minimum variance linear unbiased estimates in this case are obtained from the equation

$$\theta\left(C + \frac{1}{\rho} C_1\right) = Q + \frac{1}{\rho} Q_1. \qquad \ldots \quad (1.9)$$

The solution of these equations will be denoted by $\bar{\theta}(\rho)$. When ρ is not known, an estimate ρ^* for ρ is substituted in (1.9) and $\bar{\theta}(\rho^*)$ is taken as an estimate for θ.

For estimating ρ, the following procedure is generally recommended. (See Yates, 1939, 1940 or, for a general treatment, Rao, 1947). First the following table of analysis of variance is prepared :

TABLE ANALYSIS OF VARIANCE

source	d.f.	s.s.
blocks (unadjusted)	$b-1$	$SS_B^* = \frac{1}{k}BB'-G^2/bk$
treatments (adjusted)	$v-1$	$SS_{tr} = Q\theta^{*\prime}$
error	$e_0 = bk-b-v+1$	$SS_E = SS_T-SS_B^*-SS_{tr}$
total	$bk-1$	$SS_T = \sum_{i,u} y_{iu}^2-G^2/bk$

The adjusted sum of squares due to blocks is then computed as

$$SS_B = SS_B^*+SS_{tr}-\left(\frac{1}{r}TT'-G^2/bk\right). \qquad \ldots \quad (1.10)$$

Then s_0^2 and s_1^2 defined by

$$s_0^2 = SS_E/e_0, \quad v(r-1)s_1^2 = kSS_B-(v-k)s_0^2 \qquad \ldots \quad (1.11)$$

provide unbiased estimators of σ_0^2 and σ_1^2 respectively; and as an estimate of ρ one takes

$$R = s_1^2/s_0^2. \qquad \ldots \quad (1.12)$$

If the blocks are formed so as to achieve homogeneity within blocks, we expect $\rho \geqslant 1$; but depending on fluctuations of sampling, R may not satisfy this inequality. For this reason, a modified estimate R' (which we shall call the *truncated* form of R) given by

$$R' = \begin{cases} 1 & \text{if} \quad R \leqslant 1 \\ R & \text{if} \quad R \geqslant 1 \end{cases} \qquad \ldots \quad (1.13)$$

has at times been recommended.

2. Canonical reduction

The assumption that the rank of C is $(v-1)$ implies that there is exactly one latent root of the matrix NN' which is equal to rk, and all other latent roots are strictly smaller than rk. Let ξ_s, $s = 1, 2, \ldots, q$ be a set of orthonormal latent vectors of NN', corresponding to the q positive latent roots ϕ_s, all smaller than rk. Let ξ_s, $s = q+1$, $q+2, \ldots, v-1$ be a set of $(v-1)-q$ orthonormal $1 \times v$ vectors, each orthogonal to $\xi_1, \xi_2, \ldots, \xi_q$ and also to E_{1v}. We then define $(v-1)$ intra-block contrasts x_{0s}; $s = 1, 2, \ldots, v-1$ as follows :

$$x_{0s} = \begin{cases} k^{\frac{1}{2}}(rk-\phi_s)^{-\frac{1}{2}}Q\xi_s' & \text{for} \quad s = 1, 2, \ldots, q \\ r^{-\frac{1}{2}}Q\xi_s' & \text{for} \quad s = q+1, q+2, \ldots, v-1. \end{cases} \qquad \ldots \quad (2.1)$$

Since the rank of the vector-space generated by all intra-block contrasts is $b(k-1)$, we can find $e_0 = b(k-1)-(v-1)$ mutually orthogonal normalised intra-block contrasts, call them z_{0s}, $s = 1, 2, \ldots, e_0$, each orthogonal to $x_{01}, x_{02}, \ldots, x_{0,v-1}$.

Next, we define q inter-block contrasts

$$x_{1s} = (k\phi_s)^{-\frac{1}{2}} BN'\xi'_{s}; \; s = 1, 2, ..., q. \qquad ... \quad (2.2)$$

Since the rank of the vector-space generated by inter-block contrasts is $(b-1)$, we can find $e_1 = (b-1)-q$ mutually orthogonal normalised inter-block contrasts; call them z_{1s}; $s = 1, 2, ..., e_1$, each orthogonal to $x_{11}, x_{12}, ..., x_{1q}$. Finally, let

$$G^* = (bk)^{-\frac{1}{2}} G. \qquad .... \quad (2.3)$$

By straightforward algebra, one can easily verify the following results.

2.1. The linear transformation from y_{iu}'s to G^*, $x_{0s}(s = 1, 2, ..., v-1)$, $x_{1s}(s = 1, 2, ..., q)$, z_{0s} $(s = 1, 2, ..., e_0)$ and z_{1s} $(s = 1, 2, ..., e_1)$ is normalised orthogonal.

2.2. The transformed variables are all mutually uncorrelated and their expectations and variances are :

$$E(x_{0s}) = a_{0s}\tau_s \quad \text{where} \quad a_{0s} = \begin{cases} (r-\phi_s/k)^{\frac{1}{2}} & \text{for} \quad s = 1, 2, ..., q \\ r^{\frac{1}{2}} & \text{for} \quad s = q+1, q+2, ..., v-1, \end{cases} \qquad ... \quad (2.4)$$

$$E(x_{1s}) = a_{1s}\tau_s \quad \text{where} \quad a_{1s} = (\phi_s/k)^{\frac{1}{2}} \quad \text{for} \quad s = 1, 2, ..., q, \qquad ... \quad (2.5)$$

where

$$\tau_s = \theta\xi'_s; \quad s = 1, 2, ..., \; v-1. \qquad ... \quad (2.6)$$

$$E(z_{0s}) = 0, \quad \text{for} \quad s = 1, 2, ..., e_0, \qquad E(z_{1s}) = 0 \quad \text{for } s = 1, 2, ..., e_0$$

$$E(G^*) = (bk)^{\frac{1}{2}}\mu \qquad ... \quad (2.7)$$

$$V(x_{0s}) = \sigma_0^2, \quad \text{for} \quad s = 1, 2, ..., v-1, \quad V(z_{0s}) = \sigma_0^2, \quad \text{for } s = 1, 2, ..., e_0 \qquad ... \quad (2.8)$$

$$V(x_{1s}) = \sigma_1^2, \quad \text{for} \quad s = 1, 2, ..., q, \qquad V(z_{1s}) = \sigma_1^2, \quad \text{for } s = 1, 2, ..., e_1$$

$$V(G^*) = 0. \qquad ... \quad (2.9)$$

2.3. The equations (1.8) are equivalent to $\tau_s = t_s$ where

$$t_s = x_{0s}/a_{0s} \quad \text{for } s = 1, 2, ..., v-1. \qquad ... \quad (2.10)$$

2.4. The equations (1.9) are equivalent to $\tau_s = \bar{t}_s(\rho)$ where

$$\bar{t}_s(\rho) = \begin{cases} (\rho a_{0s}x_{0s} + a_{1s}x_{1s})/(\rho a_{0s}^2 + a_{1s}^2), & \text{for } s = 1, 2, ..., q \\ x_{0s}/a_{0s} & \text{for } s = q+1, q+2, ..., v-1. \end{cases} \qquad ... \quad (2.11)$$

2.5. The error sum of squares in the Table may be expressed as

$$SS_E = \sum_{s=1}^{e_0} z_{0s}^2 = S_0, \quad \text{say}. \qquad ... \quad (2.12)$$

2.6. The adjusted sum of squares due to blocks defined by (1.10) may be expressed as

$$SS_B = S_1 + \sum_{s=1}^{q} \phi_s t_s^2/rk \qquad ... \quad (2.13)$$

where $$S_1 = \sum_{s=1}^{q_1} z_{1s}^2 \qquad \dots \quad (2.14)$$

and $$z_s = x_{0s} - a_{0s}x_{1s}/a_{1s} \quad \text{for} \quad s = 1, 2, \dots, q. \qquad \dots \quad (2.15)$$

2.7. Let

$$w_{iu}(\theta) = y_{iu} - \sum_j m_{jiu}\theta_j \qquad \dots \quad (2.16)$$

$$B_i(\theta) = \sum_u w_{iu}(\theta). \qquad \dots \quad (2.17)$$

Then $$S_1 + \sum_{s=1}^{q} (x_{1s} - a_{1s}\tau_s)^2 = \frac{1}{k}\sum_i B_i^2(\theta) - \frac{G^2}{bk} \qquad \dots \quad (2.18)$$

and $$S_0 + \sum_{s=1}^{v-1} (x_{0s} - a_{0s}\tau_s)^2 = \sum_{i,u} w_{iu}^2(\theta) - \frac{1}{k}\sum_i B_i^2(\theta)$$

$$= \sum_{i,u} y_{iu}^2 - 2\sum_j \theta_j T_j + r\sum_j{}' \theta_j^2 - \frac{1}{k}\sum_i B_i^2(\theta). \quad \dots \quad (2.19)$$

2.8. If the joint distribution of y_{iu}'s is in addition multivariate normal, a minimal set of sufficient statistics for the parameters θ, σ_0^2, and σ_1^2 is provided by x_{0s}, $(s = 1, \dots, v-1)$, x_{1s} $(s = 1, 2, \dots, q)$, S_0 and S_1. If ρ is given, $l_s(\rho)$ $(s = 1, 2, \dots, v-1)$ and V are complete sufficient, where

$$V = S_0 + \frac{S_1}{\rho} + \sum_{s=1}^{q} \frac{z_s^2}{1 + \rho a_{0s}^2/a_{1s}^2}. \qquad \dots \quad (2.20)$$

2.9. When ρ is known $l_s(\rho)$ as defined by (2.11) is the unbiased minimum variance estimator of τ_s and its variance is given by

$$V\, l_s(\rho) = \begin{cases} \rho\sigma_0^2/(\rho a_{0s}^2 + a_{1s}^2) & \text{for } s = 1, 2, \dots, q \\ \sigma_0^2/a_{0s}^2 & \text{for } s = q+1, \dots, v-1. \end{cases} \qquad \dots \quad (2.21)$$

3. Maximum likelihood estimates

Under the assumption that the joint distribution of the random variables y_{iu}'s is multivariate normal with first and second order moments given by (1.1) and (1.2) it follows that the likelihood function L is given by

$$\log_e L = \text{const} - \tfrac{1}{2}\Big[(b-1)\log_e \sigma_1^2 + b(k-1)\log_e \sigma_0^2 + \frac{1}{\sigma_1^2}\Big\{\sum_{s=1}^{q}(x_{1s} - a_{1s}\tau_s)^2 + S_1\Big\}$$

$$+ \frac{1}{\sigma_0^2}\Big\{\sum_{s=1}^{v-1}(x_{0s} - a_{0s}\tau_s)^2 + S_0\Big\}\Big] \qquad \dots \quad (3.1)$$

where S_0 and S_1 are defined by (2.12) and (2.14) respectively. In all subsequent sections of this paper, we shall assume the joint distribution to be multivariate normal.

The likelihood equations, obtained by equating to zero the partial derivatives of $\log_e L$ with respect to the parameters, turn out to be

$$\tau_s = l_s(\rho) \qquad \text{for } s = 1, 2, \ldots, v-1 \qquad \ldots \quad (3.2)$$

where $l_s(\rho)$ is defined by (2.11) and

$$b(k-1)\sigma_0^2 = S_0 + \sum_{s=1}^{v-1} (x_{0s} - a_{0s}\tau_s)^2 \qquad \ldots \quad (3.3)$$

$$(b-1)\sigma_1^2 = S_1 + \sum_{s=1}^{q} (x_{1s} - a_{1s}\tau_s)^2. \qquad \ldots \quad (3.4)$$

The diagonal elements of the information matrix are

$$I(\tau_s, \tau_s) = \begin{cases} a_{0s}^2\sigma_0^{-2} + a_{1s}^2\sigma_1^{-2}, & \text{for } s = 1, 2, \ldots, q \\ a_{0s}^2\sigma_0^{-2}, & \text{for } s = q+1, q+2, \ldots, v-1 \end{cases} \qquad \ldots \quad (3.5)$$

$$I(\sigma_0^2, \sigma_0^2) = \tfrac{1}{2}b(k-1)\sigma_0^{-4} \qquad \ldots \quad (3.6)$$

$$I(\sigma_1^2, \sigma_1^2) = \tfrac{1}{2}(b-1)\sigma_1^{-4} \qquad \ldots \quad (3.7)$$

and all non-diagonal elements vanish.

We thus see that the maximum likelihood estimate of τ_s is $\hat{\tau}_s = l_s(\hat{\rho})$ where $\hat{\rho}$ is the maximum likelihood estimate of ρ. To compute $\hat{\rho}$ we note that it can be expressed as

$$\hat{\rho} = \frac{b(k-1)[S_1 + \sum_{s=1}^{q} (x_{1s} - a_{1s}\hat{\tau}_s)^2]}{(b-1)\left[S_0 + \sum_{s=1}^{v-1} (x_{0s} - a_{0s}\hat{\tau}_s)^2\right]}. \qquad \ldots \quad (3.8)$$

We therefore use an iterative procedure. Starting with some suitable approximation for $\hat{\tau}_s$, we obtain a first approximation for $\hat{\rho}$ using (3.8). This value of $\hat{\rho}$ is used to obtain improved values for $\hat{\tau}_s$, which, in turn, when used in (3.8) provides a second better approximation for $\hat{\rho}$. This iterative procedure is continued till one gets stable values for $\hat{\tau}_s$'s and $\hat{\rho}$.

In actual computation, we do not work with the transformed canonical variables, but make use of the result 2.7 in Section 2. The iteration formula then is

$$\rho^{(n)} = \frac{b(k-1)\left[\frac{1}{k}\sum_i B_i^2(\theta^{(n)}) - \frac{G^2}{bk}\right]}{(b-1)\left[\sum_{i,u} y_{iu}^2 - 2\sum_j \theta_j^{(n)} T_j + r\sum_j [\theta_j^{(n)}]^2 - \frac{1}{k}\sum_i B_i^2(\theta^{(n)})\right]} \qquad \ldots \quad (3.9)$$

where $\theta^{(n)} = [\theta_1^{(n)}, \theta_2^{(n)}, \ldots, \theta_v^{(n)}]$ is the n-th approximation for θ, obtained by solving the equations

$$\theta\left(C + \frac{1}{\rho^{(n-1)}} C_1\right) = Q + \frac{1}{\rho^{(n-1)}} Q_1. \qquad \ldots \quad (3.10)$$

As a first approximation for θ we may take its intra-block estimate.

The asymptotic variance of $\hat{\rho}$ obtained from the information matrix is :

$$V(\hat{\rho}) \doteq \frac{2k}{b(k-1)}\rho^2 \qquad \dots \quad (3.11)$$

The right hand side of (3.11) serves as a lower bound for the variance of any unbiased estimator of ρ.

4. Quadratic estimators σ_0^2 and σ_1^2

Since the maximum likelihood estimates are somewhat difficult to compute, we may restrict ourselves to quadratic estimators for σ_0^2 and σ_1^2. We notice that the transformed variables z_{0s} $(s = 1, 2, \dots, e_0)$, $z_{1s}(s = 1, 2, \dots, e_1)$ and $z_s(s = 1, 2, \dots, q)$ defined by (2.15) each have expectation zero and they are mutually uncorrelated. The variances of z_{0s} and z_{1s}'s are given by (2.8) and (2.9) and the variance of z_s is

$$V(z_s) = \sigma_0^2 + c_s\sigma_1^2 \qquad \dots \quad (4.1)$$

where

$$c_s = a_{0s}^2/a_{1s}^2 = (rk-\phi_s)/\phi_s. \qquad \dots \quad (4.2)$$

Obviously, we need consider only quadratic forms of the diagonal type

$$Q = b_0S_0 + b_1S_1 + \sum_{s=1}^{q} a_s z_s^2 \qquad \dots \quad (4.3)$$

where b_0, b_1 and a_s $(s = 1, 2, \dots, q)$ are the coefficients to be determined. The expectation of Q is

$$E(Q) = \left(b_0e_0 + \sum_{s=1}^{q} a_s\right)\sigma_0^2 + \left(b_1e_1 + \sum_{s=1}^{q} a_sc_s\right)\sigma_1^2. \qquad \dots \quad (4.4)$$

The variance of Q, under the assumption that the y_{iu}'s follow a joint normal distribution, is

$$V(Q) = 2\left[\left(b_0^2e_0 + \sum_{s=1}^{q} a_s^2\right) + 2\rho\sum_{s=1}^{q} a_s^2c_s + \rho^2\left(b_1^2e_1 + \sum_{s=1}^{q} a_s^2c_s^2\right)\right]\sigma_0^4. \qquad \dots \quad (4.5)$$

It is therefore possible to choose b_0, b_1 and a_s $(s = 1, 2, \dots, q)$ so as to make Q an unbiased estimator of σ_0^2 (or of σ_1^2) with a variance which is minimum for a given value of $\rho = \sigma_1^2/\sigma_0^2$. This gives, for estimating σ_0^2

$$b_0 = (e_1 + B_1)/\Delta, \; b_1 = -B_0/\Delta \qquad \dots \quad (4.6)$$

and for estimating σ_1^2

$$b_0 = -A_1/\Delta, \qquad b_1 = (e_0 + A_0)/\Delta \qquad \dots \quad (4.7)$$

and in either case

$$a_s = (b_0 + \rho^2 b_1 c_s)/(1+\rho c_s)^2 \qquad \dots \quad (4.8)$$

where

$$A_0 = \sum_{s=1}^{q} (1+\rho c_s)^{-2}, \quad A_1 = \rho^2 \sum_{s=1}^{q} c_s(1+\rho c_s)^{-2} \qquad \dots \quad (4.9)$$

$$B_0 = \sum_{s=1}^{q} c_s(1+\rho c_s)^{-2}, \; B_1 = \rho^2 \sum_{s=1}^{q} c_s^2(1+\rho c_s)^{-2}$$

and

$$\Delta = (e_0+A_0)(e_1+B_1)-B_0A_1.$$

The case where ρ is large is of special interest. In such a case the term involving ρ^2 in $V(Q)$ would be dominant, and we may like to minimise this term. The optimum unbiased estimates of σ_0^2 and σ_1^2 in this sense are given by

$$v_0 = S_0/e_0 \qquad \dots \quad (4.10)$$

$$v_1 = \frac{S_0}{e_0(b-1)} \sum_{s=1}^{q} \frac{1}{c_s} + \frac{1}{(b-1)} \left[S_1 + \sum_{s=1}^{q} \frac{z_s^2}{c_s} \right] \qquad \dots \quad (4.11)$$

respectively. In actual computation we make use of the fact that

$$S_1 + \sum_{s=1}^{q} \frac{z_s^2}{c_s} = \frac{1}{k} \sum_i B_i^2(\theta^*) - \frac{G^2}{bk} \qquad \dots \quad (4.12)$$

where $B_i(\theta)$ is defined by (2.17) and θ^* is the intra-block estimate of θ obtained from (1.8). This estimate was suggested by Shah (1962) from intuitive considerations. The variance of v_1 is

$$V(v_1) = \frac{2\sigma_0^4}{(b-1)^2} [(b-1)\rho^2 + 2(\alpha_{-1}-q)\rho + \alpha_{-2} - 2\alpha_{-1} + q + (\alpha_{-1}-q)^2/e_0] \quad \dots \quad (4.13)$$

where

$$\alpha_t = \sum_{s=1}^{q} \left(1 - \frac{\phi_s}{rk} \right)^t. \qquad \dots \quad (4.14)$$

We may compare this with the customary estimate s_1^2 of σ_1^2 as defined by (1.11). The variance of this estimator is

$$V(s_1^2) = \frac{2k^2\sigma_0^4}{v^2(r-1)^2} \Big[(e_1+\alpha_2)\rho^2 + 2(\alpha_1-\alpha_2)\rho$$

$$+ 1 - 2\alpha_1 + \alpha_2 + \left(\frac{v}{k} - 1 \right)^2 \Big/ e_0 \Big]. \qquad \dots \quad (4.15)$$

5. Unbiased estimators of ρ

As a convenient unbiased estimator of ρ we may consider a statistic of the form

$$P = \frac{aS_1 + \Sigma b_s z_s^2}{S_0} + c \qquad \dots \quad (5.1)$$

where a, $b_s (s = 1, 2, \dots, q)$ and c are constants to be suitably determined. Since for $e_0 > 2$

$$E(P) = \frac{ae_1\rho + \Sigma b_s(1+\rho c_s)}{e_0-2} + c,$$

to make P an unbiased estimator of ρ, we must have

$$\frac{\Sigma b_i}{e_0-2}+c = 0,$$

and
$$ae_1+\Sigma b_i c_i = e_0-2. \quad \ldots \quad (5.2)$$

If $e_0 > 4$ the variance of such an unbiased estimator turns out to be

$$V(P) = A_0 + A_1\rho + A_2\rho^2, \quad \ldots \quad (5.3)$$

where
$$A_0 = 3\,\Sigma\, b_i^2+2ae_1\,\Sigma\, b_i - \left(\frac{\Sigma\, b_i}{e_0-2}\right)^2$$

$$A_1 = \frac{6\,\Sigma\, b_i^2 c_i+2ae_1\,\Sigma\, b_i c_i}{(e_0-2)(e_0-4)} - \frac{2\Sigma\, b_i}{e_0-2}$$

$$A_2 = \frac{a^2 e_1(e_1+2)+3\Sigma\, b_i^2 c_i^2}{(e_0-2)(e_0-4)}-1.$$

If we like to minimise A_2 the coefficient of ρ^2 in (5.3), we have to take

$$a = \frac{3(e_0-2)}{3e_1+(e_1+2)q}, \; b_i = \frac{(e_1+2)a}{3c_i}. \quad \ldots \quad (5.4)$$

It can be seen that R given by (1.12) is *not* an unbiased estimator of ρ, but a simple correction can be applied to it to make it unbiased. We thus get

$$\left(1-\frac{2}{e_0}\right) R - \frac{2(v-k)}{e_0 v(r-1)} \quad \ldots \quad (5.5)$$

as an unbiased estimator of ρ.

Similarly, if we start with v_0 and v_1 defined by (4.10) and (4.11) as estimators of σ_0^2 and σ_1^2 respectively, we get, as another unbiased estimator of ρ :

$$\left(1-\frac{2}{e_0}\right)\frac{v_1}{v_0} - \frac{2(v-1)}{e_0(b-1)}\left(\frac{1}{E}-1\right), \quad \ldots \quad (5.6)$$

where E is the efficiency-factor of the design (Kempthorne, 1956; Roy, 1958).

$$E = \frac{(v-1)}{rk} \Big/ \left\{\frac{v-1-q}{rk}+\sum_{i=1}^{q}\frac{1}{rk-\phi_i}\right\}. \quad \ldots \quad (5.7)$$

Since with positive probability these unbiased estimators of ρ may turn out to be less than unity, we may use their truncated forms instead, as indicated by (1.13). Let x be any unbiased estimator of ρ and x' its truncated form defined by $x' = 1$ if $x \leqslant 1$, and $x' = x$, otherwise. Then, even though x' is generally a biased estimator for ρ, it can be easily seen that its mean square error can never exceed that for x,

$$E(x'-\rho)^2 \leqslant E(x-\rho)^2.$$

6. Some properties of combined intra and inter-block estimators of treatment effects

As a corollary to result 2.8, we conclude that when ρ is given, unbiased estimators of treatment effects with minimum variance are obtained from the equations $\tau_s = l_s(\rho)$, $s = 1, 2, \ldots, v-1$, where the right-hand side is given by (2.11). In this section we shall investigate the properties of the estimators of treatment effects obtained by substituting some estimator ρ^* for ρ in the above expression. For typographical simplicity, we shall write

$$l_s = l_s(\rho), \qquad l_s^* = l_s(\rho^*).$$

Let
$$w_s = \frac{(\rho^*-\rho)z_s}{1+\rho^* c_s}, \; s = 1, 2, \ldots, q. \quad \ldots \quad (6.1)$$

We then have the following :

Lemma 6.1 : *If ρ^* satisfies the conditions*

$$E(w_s) = 0, \; V(w_s) < \infty \quad \ldots \quad (6.2)$$

for all values of ρ, then $E(l_s^*) = \tau_s$

and
$$V(l_s^*) = V(l_s) + \frac{c_s^2}{a_{1s}^2(1+\rho c_s)^2} V(w_s). \quad \ldots \quad (6.3)$$

To prove this, we note that

$$l_s^* = l_s + \frac{c_s}{a_{1s}(1+\rho c_s)} w_s. \quad \ldots \quad (6.4)$$

Also, when ρ is given, l_s is the unbiased minimum variance estimator of and by the conditions of the lemma w_s is a zero-function. By Stein's theorem (1950) l_s and w_s are uncorrelated. Hence the lemma.

Let P be any statistic of the form (5.1) and let ρ^* be defined as

$$\rho^* = \begin{cases} P & \text{if } P \geqslant 1 \\ 1 & \text{otherwise.} \end{cases} \quad \ldots \quad (6.5)$$

It can then be shown that ρ^* so defined satisfies conditions (6.2). That $E(w_s^2)$ is finite can be easily checked. To show that $E(w_s) = 0$, we note that ρ^* is an even function of z_{0s}, $(s=1, 2, \ldots, e_0)$, z_{1s} $(s = 1, 2, \ldots, e_1)$ and z_s $(s = 1, 2, \ldots, q)$ and consequently, w_s is an odd function of these variables. Since the z's are mutually independent random variables each having a normal distribution with mean zero, the result follows. This is merely an extension of Graybill and Weeks (1959) argument for balanced designs to the case of general incomplete block designs. A similar argument gives

$$E(w_s w_{s'}) = 0$$

for $s \neq s' = 1, 2, \ldots, q$. Since l_s and $l_{s'}$ are independent, it follows that l_s^* and $l_{s'}^*$ are uncorrelated for $s \neq s' = 1, 2, \ldots, v-1$.

Now, any treatment contrast τ can be expressed as $\tau = \Sigma\, l_s \tau_s$ where l_s $(s = 1, 2, \ldots, v-1)$ are some constants. The minimum variance unbiased estimator of τ when ρ is known is $t = \Sigma\, l_s t_s$. When ρ is not known, for a combined inter and intra-block estimator of τ, one takes $t^* = \Sigma\, l_s t_s^*$ by substituting a suitable estimator ρ^* for ρ. If ρ^* satisfies the conditions of Lemma 6.1, we get the following :

Theorem 6.1. *The estimator t^* is unbiased for τ, and its variance is given by*

$$V(t^*) = V(t) + \sum_{s=1}^{t} \frac{c_s^2 l_s^2 V(w_s)}{a_{1s}^2 (1+\rho c_s)^2}$$

the second term being the additional variance due to the sampling fluctuation in ρ^*.

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Paper received : December, 1961.