A Generalization of Reissner-Nordström Solution. II.

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Summary. — This is a continuation of the work on a generalization of Reissner-Nordström solution, namely the interior solution of a charged (material) sphere. Following TOLMAN [2] we may easily obtain two special solutions of the field-equations, when the energy-tensor consists of two parts, namely the electromagnetic and the material parts. The second solution, obtained by assuming $e^{\nu}(\nu'/2r) = \text{const}$ is discussed here. We further obtain another solution based on the very restrictive assumption $\varrho = -p$. We also show the validity of Birkhoff's theorem for the case of a charged sphere.

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1. – In order to obtain the second solution mentioned in the first part of our note [1] we start from the simplified form of field-equations as obtained by TOLMAN [2] namely the set of equations

$$8\pi(T_1^1 - T_2^2)\frac{2}{r} = \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\exp[-\lambda]-1}{r^2}\right) + \frac{\mathrm{d}}{\mathrm{d}r}\left(\exp[-\lambda]\frac{\nu'}{2r}\right) + \exp[-(\lambda+\nu)]\frac{\mathrm{d}}{\mathrm{d}r}\left(e^{\nu}\frac{\nu'}{2r}\right), \quad (T_1^1 = T_2^2), \tag{1}$$

$$8\pi p = \exp[-\lambda]\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda\,, \tag{2}$$

$$8\pi\varrho = \exp[-\lambda]\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda\,. \tag{3}$$

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[1] R. L. BRAHMACHARY: *Nuovo Cimento*, **4**, 1216 (1956).

[2] R. C. TOLMAN: *Phys. Rev.*, **55**, 364 (1939).

In case of our charged sphere, we now obtain the following three equations

$$(4)\qquad 8\pi(t_1^1 - t_2^2)\frac{2}{r} = \frac{d}{dr}\left(\frac{\exp[-\lambda]-1}{r^2}\right) + \frac{d}{dr}\left(\exp[-r]\frac{\nu'}{2r}\right) + \\ + \exp[-(\lambda+\nu)]\frac{d}{dr}\left(e^\nu\frac{\nu'}{2r}\right), \qquad 8\pi t_1^1 = 4\pi\varepsilon r^2 ,$$

$$(5)\qquad 8\pi p - 8\pi t_1^1 = \exp[-\lambda]\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} + \Lambda ,$$

$$(6)\qquad 8\pi\varrho + 8\pi t_4^4 = \exp[-\lambda]\left(\frac{\lambda'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} - \Lambda .$$

Let us assume

$$e^\nu\frac{\nu'}{2r} = B^2 = \text{const} ,$$

$$\nu' = \frac{2B^2r}{B^2r^2 + D} .$$

Equation (4) now reduces to

$$\frac{\exp[-\lambda]-1}{r^2} + \exp[-\lambda]\frac{\nu'}{2r} = 8\pi\varepsilon r^2 + C ,$$

where c is a constant of integration, whence we obtain

$$e^\lambda = \frac{1 + (2r^2B^2/D)}{(1 + (r^2B^2/D))(1 + Cr^2 + 8\pi\varepsilon r^4)} .$$

Putting these values of λ, ν in equations (5), (6) we obtain the values of p and ϱ. Thus our complete solution is

$$e^\nu = B^2r^2 + D ,$$

$$e^\lambda = \frac{1 + (2B^2r^2/D)}{(1 + (r^2B^2/D))(1 + Cr^2 + 8\pi\varepsilon r^4)} .$$

The expressions for p and ϱ are naturally very complicated but at $r = 0$, they reduce to

$$8\pi p = \frac{4B^2}{D} + \frac{B^2C}{D} + C + \Lambda ,$$

$$8\pi\varrho = -3C + \frac{6B^2}{D} - \frac{B^2C}{D} - \Lambda .$$

2. – A further solution can be obtained under very restrictive physical conditions. The equations (4) and (5) can be combined to yield

$$\frac{\mathrm{d}p}{\mathrm{d}r} = \frac{-(\varrho + p)\nu'}{2} + 3\varepsilon r\,, \tag{7}$$

we have now to solve equations (5), (6), (7). Integrating equation (6), we obtain

$$\exp[-\lambda] = 1 - \frac{r^2}{R^2} - \frac{4\pi\varepsilon r^4}{5} - 8\pi\int \varrho r^2\,\mathrm{d}r\,.$$

If we now assume the very restricting condition $\varrho + p = 0$ which may be physically realized only in case of a « neutron-fluid » sphere, equation (7) reduces to

$$p = \tfrac{3}{2}\varepsilon r^2 + c$$

and hence we obtain

$$\varrho = -\tfrac{3}{2}\varepsilon r^2 - c\,.$$

Thus

$$\exp[-\lambda] = 1 - \frac{r^2}{R^2} - \frac{4}{5}\pi\varepsilon r^4 + \frac{12}{5}\varepsilon r^5 + \frac{8}{3}Cr^3 - 8\pi D\,.$$

As any value of ν', will satisfy (7) we can substitute p, $e^{-\lambda}$ in equation (5) and obtain the value of

$$\nu' = \frac{-\alpha r^5 - \beta r^4 - \gamma r^3 - \delta r^2 - (2 + 8\pi D)}{\alpha r^6 - (\beta r^5/4) - \gamma r^4 - (1/R^2)r^3 + r(1 - 8\pi D)}\,.$$

For very small values of r, say the case of a nucleus, we may neglect the higher powers of p and simplify ν'.

External solution: outside the charged sphere we have again the Reissner-Nordström solution if $p = 0$, $\varrho = 0$. The well known method of boundary conditions and equations of fit can be employed to find the value of constants, such as e^ν in the first part of our note, B, etc.

It may be mentioned that Birkhoff's theorem is valid in case of a charged sphere. Outside the sphere,

$$\mathcal{T}_1^1 = (T_1^1 + t_1^1) = t_1^1\,, \qquad \mathcal{T}_2^2 = t_2^2\,, \quad \text{etc.}$$

Thus,

$$T_1^1,\ T_2^2,\ T_3^3 = 0$$

$$t_1^1,\ t_2^2,\ t_3^3 \neq 0$$

and further,

$$T^1_4,\ t^1_4,\ T^4_1,\ t^4_1 = 0\ .$$

We have however the general equations

$$8\pi\mathcal{T}^1_4 = -\exp[-\lambda]\,\frac{\dot{\lambda}}{r}\,,$$

$$8\pi\mathcal{T}^4_1 = \quad \exp[-\nu]\,\frac{\dot{\lambda}}{r}\,.$$

As $\mathcal{T}^1_4 = (T^1_4 + t^1_4)$ and $\mathcal{T}^4_1 = (T^4_1,\ t^4_1) = 0$.
We have

$$\dot{\lambda} = 0\ .$$

This condition leads to the validity of Birkhoff's theorem.

RIASSUNTO (*)

L'articolo è la continuazione del lavoro sulla generalizzazione della soluzione di Reissner-Nordström, cioè la soluzione interna di una sfera (materiale) carica. Secondo Tolman, quando il tensore dell'energia consiste di due parti, l'elettromagnetica e la materiale, possiamo facilmente ottenere due soluzioni particolari delle equazioni del campo. Qui discutiamo la seconda soluzione ottenuta assumendo $e^{\nu}(\nu'/2r) = \text{cost}$. Otteniamo inoltre un'altra soluzione basata sull'ipotesi molto restrittiva $\varrho = -p$. Dimostriamo anche la validità del teorema di Birkhoff per il caso di una sfera carica.

(*) *Traduzione a cura della Redazione.*