Some contributions to free probability and random matrices

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In memory of my grandfather who is always loved,
never forgotten and forever missed.

Om Namah Śivāya

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Chapter 1

Introduction

1.1 Prologue

The concept of independence in classical probability theory is indispensable. Suppose $(\Omega, \mathcal{F}, P)$ be a probability space where Ω is a nonempty set with a σ-algebra $\mathcal{F}$ and P is a probability on Ω. Two (real-valued) random variables X and Y defined on Ω are said to be (classically) independent if

$$P(X^{-1}(A) \cap Y^{-1}(B)) = P(X^{-1}(A))P(Y^{-1}(B)),$$

for Borel subsets A and B of $\mathbb{R}$. In this case, the random variables commute i.e. $XY = YX$. However, one can use the algebra of random variables and their expectations alone to define the concept of independence. The algebra of random variables is commutative and hence the natural question arises if there are different notions of independence when one considers non-commutative algebras, such as the algebra of matrices. Voiculescu introduced the concept of free independence in operator algebra (which is non-commutative) and it became an integral part of random matrix theory and its later developments. This more general formalism of independence at an algebraic level encompasses classical probability, spectral theory, random matrix theory and quantum mechanics. A more detailed description can be found in [85]. In unital algebras, it turns out that classical and free independence are the only "universal" notions of algebraic independence. When unitality of the algebras is not assumed, another universal notion is well-known, which is Boolean independence (see [20] for details).

The following thesis contributes to the study of three interconnected fields, namely Boolean independence, free probability and random matrix theory. The motivation of the studied problems came from both classical and free probability. A brief outline of the thesis is as follows.

Subexponentiality in classical probability relates to sums of independent and identically distributed random variables with a heavy tailed distribution. It amounts to the tail of the sum being asymptotically equivalent to that of the largest summand. It is also known as the principle of one large jump and has wide applications in sums and maximum of heavy tailed distributions. The extension of the classical notion of subexponentiality to the setting of free probability was done in the work of Hazra and Maulik [59]. The extension of the notion of classical and free subexponentiality to the Boolean probability theory is done in this thesis. It is shown that the probability distribution functions with regularly varying tails play an important role in Boolean probability theory alongside classical and free probability theory.

In classical probability, it is well-known that under subexponentiality, the Lévy measure of a subordinator is tail equivalent to the infinitely divisible distribution. Under the additional assumption of heavy tails, such results can be extended to the free probability setting and it is proved that free regular infinitely divisible probability measures (analogue of a subordinator) and their Lévy measures are tail equivalent. This result turns out to have an application in random matrix theory and one can determine the tail behaviour of limiting spectral measures of a certain random matrices. In the search for a general result on asymptotic freeness in random matrices, shorter proof of asymptotic freeness in a complex Gaussian setting is provided.

The adjacency and Laplacian matrices of inhomogeneous Erdős–Rényi random graph turn out to be examples of random matrices where non-trivial free multiplicative and additive measures arise as limiting spectral distribution. The limiting behaviour of the eigenvalue spectrum was derived in [37] and the fluctuations of the extreme eigenvalues and eigenvectors are investigated in this thesis. Since the limiting spectral distribution of an adjacency matrix need not always be semicircle law, the existing methods might not be directly applicable. Hence linear algebraic techniques are developed to study the extreme eigenvalues of the adjacency matrix of an inhomogeneous Erdős–Rényi random graph.

In the following, we describe the backgrounds and frameworks for different problems studied in this thesis along with the contributions. We start with introducing some notations, followed by the operator theoretic and analytic set up of non-commutative probability theory.

1.2 Notations

We attach some preliminary notations for this chapter which is consistent throughout the thesis. The rest of the notations are introduced in consequent chapters accordingly. The natural numbers, real numbers and the non-negative real numbers are denoted by $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{R}^+$ respectively. The complex plane is marked by $\mathbb{C}$ and denote the upper half plane as $\mathbb{C}^+ := \{z \in \mathbb{C} : \Im z > 0\}$ and $\mathbb{C}^- := -\mathbb{C}^+$. The set of all probability measures on $\mathbb{R}$ and $\mathbb{R}^+$ are represented by $\mathcal{M}$ and $\mathcal{M}_+$ respectively. We write $a_n \ll b_n$ for two sequence of numbers $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ if $\left|\frac{a_n}{b_n}\right| \to 0$ as $n \to \infty$ and $a_n \sim b_n$ if $\left|\frac{a_n}{b_n}\right| \to 1$ as $n \to \infty$.

Let $\mu \in \mathcal{M}_+$ and $\alpha \in \mathbb{R}_+$. The quantity $\mu(y, \infty)$ is considered to be the *tail* of μ for large values of y. We call μ to be *regularly varying* probability measure with tail index $-\alpha$ if

$$\lim_{y\to\infty} \frac{\mu(ty, \infty)}{\mu(y, \infty)} = t^{-\alpha} \,,$$

for any real number $t > 0$. In literature, heavy tailed distributions denotes the class of distributions which do not have the moment generating function to be finite. Note that regularly varying probability measures form a large subclass of heavy tailed distributions.

1.3 Non-commutative probability and convolutions

A complex normed algebra $(\mathcal{A}, \|\cdot\|)$ is a *Banach algebra* if the norm $\|\cdot\|$ induces a complete metric. A Banach algebra is a C^**-algebra* if it possesses an involution $a \mapsto a^*$ that satisfies $\|aa^*\| = \|a\|^2$. A quadruple $(\mathcal{A}, \|\cdot\|, *, \phi)$ is called a C^**-probability space* if $(\mathcal{A}, \|\cdot\|, *)$ is a C^*-algebra and ϕ is a linear functional such that $\phi(\mathbf{1}) = 1$. Let $B(H)$ be the set of all bounded linear maps on a Hilbert space H. A C^*-algebra $\mathcal{A} \subseteq B(H)$ is said to be a *von-Neumann algebra (or* W^**-algebra)* if it is closed with respect to the weak operator topology. The pair $(\mathcal{A}, \phi)$ is called a W^**-probability space* if $\mathcal{A}$ is a W^*-algebra and ϕ is a linear functional which can be written as $\phi(a) = \langle a\zeta, \zeta\rangle$ for some unit vector $\zeta \in H$. Let $(\mathcal{A}, \phi)$ be a W^*-probability space and let T be a self-adjoint operator affiliated with $\mathcal{A}$ i.e. there exists a complex number z in the spectrum of T such that $(T - z)^{-1} \in \mathcal{A}$. Then the *law of* T is the unique probability measure μ_T on $\mathbb{R}$ such that

$$\phi(u(T)) = \int u(\lambda) d\mu_T(\lambda) \,,$$

for any bounded measurable function u on spectrum of T.

A *non-commutative probability space* is a pair $(\mathcal{A}, \phi)$ where $\mathcal{A}$ is a unital algebra (i.e. there exists a unit $\mathbf{1} \in \mathcal{A}$) over $\mathbb{C}$ and ϕ is a linear functional on $\mathcal{A}$ such that $\phi(\mathbf{1}) = 1$. For example consider $\mathcal{A} = M_N(\mathbb{C})$, the set of all $N \times N$ matrices with complex entries and $\phi = \frac{1}{N}Tr$ where Tr denotes the trace of a matrix. The elements of $\mathcal{A}$ are called *non-commutative random variables* and ϕ (the counterpart of expectation) is called a state. In non-commutative probability there are several notions of independence. Tensor, Boolean, free, monotone, anti-monotone are some of them. It was shown in [83] that the only three universal notions of independence are tensor, Boolean and free.

Definition 1.3.1. ***Tensor independence****: (Definition 5.1 of [71]) Unital sub-algebras $A_1, \cdots, A_L$ of a non-commutative probability space $(\mathcal{A}, \phi)$ are called tensor independent (the analogous notion of classical independence) if A_r's commute and*

$$\phi(a_1 \cdots a_L) = \phi(a_1) \cdots \phi(a_L)$$

whenever $a_i \in A_i$ for all $i \in \{1, 2, \ldots, L\}$.

Definition 1.3.2. ***Boolean independence****: (Definition 4.1 of [54]) Sub-algebras $A_1, \cdots, A_L$ of $(\mathcal{A}, \phi)$ (non-unital in general) are said to be Boolean independent (with respect to a unit vector ζ where $\phi(a) = \langle \zeta, a\zeta \rangle$) whenever for any $n \geqslant 1$,*

$$\phi(a_1 \cdots a_n) = \phi(a_1) \cdots \phi(a_n)$$

whenever $a_i \in A_{k(i)}$ and $k(i) \neq k(i+1)$ for all $i \in \{1, 2, \ldots, n-1\}$.

Definition 1.3.3. ***Free independence****: (Definition 5.3 of [71]) Unital sub-algebras $A_1, \cdots, A_L$ of $(\mathcal{A}, \phi)$ are called freely independent if for any $n \geqslant 1$,*

$$\phi(a_1 \cdots a_n) = 0$$

whenever $\phi(a_r) = 0$ for all $1 \leqslant r \leqslant n$ and any two consecutive a_i come from different sub-algebras.

Two random variables a and b from $\mathcal{A}$ are said to be tensor/Boolean/free independent if the algebras generated by them are tensor/Boolean/free independent respectively.

Let $\{(\mathcal{A}_n, \phi_n)\}_{n \geq 1}$ be sequence of non-commutative probability spaces. Let I be an index set and consider for each $i \in I$ and for each $n \in \mathbb{N}$ random variables $a_{i,n} \in \mathcal{A}_n$. The sequence

$(a_{i,n})_{i\in I}$ is said to be *asymptotically free* as $n \to \infty$, whenever for some positive integer k, $i(1), i(2), \ldots, i(k) \in I$ with $i(1) \neq i(2) \neq \cdots \neq i(k)$ and polynomials p_j $(j = 1, 2, \ldots, k)$ such that

$$\lim_{n\to\infty} \phi_n \left(p_j \left(a_{i(j),n}\right)\right) = 0$$

for all $j = 1, 2, \ldots, k$, then the asymptotic alternating moments vanish, i.e.,

$$\lim_{n\to\infty} \phi_n \left(p_1 \left(a_{i(1),n}\right) p_2 \left(a_{i(2),n}\right) \cdots p_k \left(a_{i(k),n}\right)\right) = 0\,.$$

The seminal works of Voiculescu ([92]) connects tensor independence with free independence. It was shown in the said work that two independent Gaussian Wigner matrices are asymptotically free. Suppose we have, two self-adjoint elements X and Y in a non-commutative probability space $(\mathcal{A}, \phi)$ with laws μ_X and μ_Y respectively. If X and Y are Boolean (or free independent) then the law of $X+Y$ and XY is denoted by $\mu_X \uplus \mu_Y$ (or $\mu_X \boxplus \mu_Y$) and $\mu_X \bowtie \mu_Y$ (or $\mu_X \boxtimes \mu_Y$) respectively. So for two independent operators, we have got two measures and we can get the convoluted measures by just adding or multiplying the operators. We also need the reverse direction, in the sense that for any two probability measures, do there exist a pair of Boolean independent and a pair of free independent random variables in some W^*-probability space such that the convolutions are well defined? The answer is yes.

The notion of freeness was extended to this context by Bercovici and Voiculescu [28]. The self-adjoint operators $\{X_i : 1 \leq i \leq k\}$ affiliated with a von Neumann algebra $\mathcal{A}$ are called freely independent, or simply free, if and only if the algebras generated by the operators, $\{f(X_i) : f \text{ bounded measurable}\}_{1\leq i\leq k}$ are free. In the free case, let $\mu_1, \mu_2, \cdots, \mu_n$ be probability measures on $\mathbb{R}$. Then by Proposition 5.3.34 of [3], there exists a W^*-probability space $(\mathcal{A}, \phi)$ with ϕ a normal, faithful, tracial state and, self-adjoint and free operators $\{X_i\}_{1\leq i\leq n}$ which are affiliated with $\mathcal{A}$, having laws $\{\mu_i\}_{1\leq i\leq n}$ respectively.

In the Boolean case the construction of the required space is provided in [54]. For a measure space $(\Omega, \mathcal{F}, P)$, call $L^2(\Omega, P)_0$ to be the orthogonal complement of the constant function, i.e.

$$L^2(\Omega, P)_0 = \left\{ g \in L^2(\Omega, P) : \int_\Omega g dP = 0 \right\}.$$

Let $\mu_1, \mu_2 \in \mathcal{M}$. Then there exists two self-adjoint operators X and Y, Boolean independent (with respect to the column vector $\zeta = (1, 0, 0)'$) on the space $\mathbb{C} \oplus L^2(\Omega, \mu_1)_0 \oplus L^2(\Omega, \mu_2)_0$ having laws μ_1 and μ_2 respectively.

This was the operator theoretic way of understanding the convolutions. It turns out that definitions of convolutions can be given analytically through properties of various transforms. Classically moment generating functions (whenever exists), characteristic functions determine measures uniquely. In non-commutative probability, there are several transforms which do the job of describing a probability measure. Suppose $\mu \in \mathcal{M}$, then its Cauchy transform $G_\mu : \mathbb{C}^+ \to \mathbb{C}^-$, is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t} d\mu(t), \quad z \in \mathbb{C}^+.$$

The Cauchy transform characterizes a probability measure uniquely.

We now present the definition of free additive convolution analytically. We use the notations

$$\Gamma_\alpha := \{z = x + iy \in \mathbb{C}^+ : x < \alpha y\} \text{ and } \Gamma_{\alpha,\beta} := \{z \in \Gamma_\alpha : |z| > \beta\}$$

for positive real numbers α and β. The region $\Gamma_{\alpha,\beta}$ is known as a Stolz angle at ∞. Define the Voiculescu transform ϕ_μ of a probability measure $\mu \in \mathcal{M}$ by

$$\phi_\mu(z) = (1/G_\mu)^{-1}(z) - z$$

for all z in an appropriate Stolz angle at ∞ where the inverse of $1/G_\mu$ exists. Then, for two probability measures μ, ν both in $\mathcal{M}$, their free additive convolution $\mu \boxplus \nu$ is characterized by the identity

$$\phi_{\mu \boxplus \nu}(z) = \phi_\mu(z) + \phi_\nu(z),$$

in the common domain of definition of ϕ_μ and ϕ_ν (see Corollary 5.5 and Corollary 5.8 of [93] for details).

The Boolean additive convolution is similarly determined by the transform K_μ (also called the energy transform) which is defined for any $z \in \mathbb{C}^+$ by the formula:

$$K_\mu(z) = z - \frac{1}{G_\mu(z)}.$$

For two probability measures μ, ν both in $\mathcal{M}$, the additive Boolean convolution $\mu \uplus \nu$ is determined by

$$K_{\mu \uplus \nu}(z) = K_\mu(z) + K_\nu(z), \text{ for } z \in \mathbb{C}^+$$

and $\mu \uplus \nu$ is again a probability measure. On the other hand the Boolean multiplicative convolution is defined for a smaller class of probability measures. Suppose μ and ν both are in $\mathcal{M}_+$ and also assume that the first moment of either μ or ν is finite, then the Boolean multiplicative convolution $\mu \circlearrowleft \nu$ is governed by the identity

$$K_{\mu \circlearrowleft \nu}(z) = K_\mu(z) K_\nu(z)\,,$$

for any $z \in \mathbb{C} \setminus \mathbb{R}$. In this thesis, motivated from the previous works on classical and free convolutions of probability measures, we have studied the behaviour of regularly varying measures under the framework of these two important Boolean convolutions. Before coming to the contributions in this topic, we introduce some more definitions that are needed to describe our results.

A probability measure $\mu \in \mathcal{M}_+$, with $\mu\,(y,\infty) > 0$ for all $y \geq 0$, is said to be *Boolean-subexponential* if for all $n \in \mathbb{N}$,

$$\underbrace{(\mu \uplus \cdots \uplus \mu)}_{n\ times}(y,\infty) \sim n\mu\,(y,\infty) \text{ as } y \to \infty\,.$$

In the above equation, if we replace Boolean convolution by classical or free convolution then we get the definition of classical or free subexponentiality respectively.

A probability measure $\mu \in \mathcal{M}$ is said to be free infinitely divisible (or $\boxplus$-infinitely divisible) if for every $n \geq 1$ there exists measures μ_n such that

$$\underbrace{(\mu_n \boxplus \cdots \boxplus \mu_n)}_{n\ times} \stackrel{d}{=} \mu\,.$$

The above is equivalent to say that $\phi_\mu(z) = n\phi_{\mu_n}(z)$. We can define the probability measures μ_t (see Theorem 2.5 of [17]) indexed by real numbers $t > 1$ by the formula

$$\phi_\mu(z) = t\phi_{\mu_t}(z)\,.$$

The set $\mathcal{M}_\boxplus$ of free infinitely divisible probability measures can be characterized by the property that: $\mu \in \mathcal{M}_\boxplus$ if and only if one can define the $\boxplus$-convolution powers $\mu^{\boxplus t}$, for any $t \in (0,\infty)$. In the same spirit as above, one can consider the parallel concept of infinite divisibility with respect to Boolean additive convolution. But here the situation turns out to be much simpler. Indeed, we have that every $\mu \in \mathcal{M}$ is infinitely divisible with respect to $\uplus$ (i.e., we can say $\mathcal{M} = \mathcal{M}_\uplus$). The bijections between Boolean infinitely divisible distribution, free infinitely divisible and classical ones were first studied by Bercovici et al. [29]. This observation is later generalized in [18]. They introduced a class of functions $\mathbf{B}_t : \mathcal{M} \to \mathcal{M}$ for all $t \geq 0$, given by

$$\mathbf{B}_t(\mu) = (\mu^{\boxplus(1+t)})^{\uplus\frac{1}{1+t}} \qquad \mu \in \mathcal{M}\,.$$

It can be shown that if $\mu \in \mathcal{M}_+$ then $\mathbf{B}_t(\mu) \in \mathcal{M}_+$ too. An important observation is that for $t = 1$, the map $\mathbf{B}_1$ coincides with the Bercovici-Pata bijection between $\mathcal{M}$ and $\mathcal{M}_{\boxplus}$. Moreover, it turns out that $(\mathbf{B}_t(\mu))_{t\geqslant 1}$ is $\boxplus$-infinitely divisible for every probability measure μ. For integer indices the map essentially tells us that

$$\mathbf{B}_n(\mu)^{\uplus(1+n)} := \underbrace{\mathbf{B}_n(\mu) \uplus \mathbf{B}_n(\mu) \uplus \cdots \uplus \mathbf{B}_n(\mu)}_{(1+n \ \ times)} = \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{(1+n \ \ times)} =: \mu^{\boxplus(1+n)}.$$

The relationship with free Brownian motion and complex Burgers equation makes it an extremely important object of study. The map is further studied in [6], [7] and [88]. It is briefly sketched in the following, about the outcomes that we have been able to explore.

1.3.1 Contributions

Suppose μ and ν are two probability measures supported on $[0, \infty)$ with regularly varying tails of indices $-\alpha$ and $-\beta$ respectively (α and β non-negative). Then what can be said about the tail behaviour of $\mu \uplus \nu$ and $\mu \Cup \nu$, the additive and multiplicative Boolean convolutions of μ and ν respectively? The chapter 2 of the thesis provides an answer to this question and is based on the work [40].

In the case of classical additive convolution, it is well known that if μ is regularly varying of index $-\alpha$, then μ is (classical) subexponential that is $\mu^{*n}(x, \infty) \sim n\mu(x, \infty)$ as $x \to \infty$ for all $N \geqslant 1$. The case of free additive convolution was studied by Hazra and Maulik [59] and it is related to the free extreme value theory of [11]. We extend this result to the Boolean additive convolution in Theorem 2.2.2.

In an influential work of Breiman ([32]), he showed the following: If μ and ν are positively supported measures, μ is regularly varying of index $-\alpha$, $\alpha > 0$ and ν is such that $\int_0^\infty y^{\alpha+\epsilon} d\nu(y) < \infty$ (for some $\epsilon > 0$) then

$$\mu \circledast \nu(x, \infty) \sim \int_0^\infty y^\alpha d\nu(y)\mu(x, \infty) \text{ as } x \to \infty\,, \tag{1.3.1}$$

where $\mu \circledast \nu$ denotes the classical multiplicative convolution. A similar result can be obtained when ν is a regularly varying measure (see [64]). In case of Boolean convolution, the behaviour turns out to be much similar for multiplicative convolution and in that case again the heavier tail wins. We derive the explicit description in Theorem 2.2.7. The constants appearing though change from the classical case.

As an application for the above results we determine the behaviour of the Belinschi-Nica map which is a one parameter family of maps $\{\mathbf{B}_t\}_{t\geqslant 0}$ on the set of probability measures. We study the case when μ is a heavy tail distribution and show that μ is regularly varying of index $-\alpha$ if and only if $\mathbf{B}_t(\mu)$ is regularly varying $-\alpha$ for $t \geqslant 0$. In particular, it shows that the support of $\mathbf{B}_t(\mu)$ will be unbounded whenever μ has such regularly varying tails.

The Boolean extreme value theory has recently been explored in [91] in parallel to the study of free extreme value theory ([19]). We show that in the subexponential case, the tail behaviour of Boolean, free and classical extremes are asymptotically equivalent. It is known that the classical subexponential random variables satisfy *the principle of one large jump*, that is, if $\{X_i\}$ are i.i.d. subexponential random variables, then for all $N \geq 1$,

$$P(\sum_{i=1}^{n} X_i > x) \sim nP(X_1 > x) \sim P(\max_{1\leq i\leq n} X_i > x) \text{ as } x \to \infty\,.$$

The free max convolution, denoted by $\boxtimes$, was introduced in [19] and the analogous result for the free one large jump principle was obtained in [59]. We have shown that Boolean subexponential distributions also follow the principle of one large jump and combining all the results of the classical, free and Boolean instances, it can be further concluded that all the tails of classical, free and Boolean max convolutions are asymptotically equivalent for regularly varying distributions.

In proving these results we exploit the relationship of regular variation with different transforms and their Laurent series expansions. Recently our results have been revisited and extended in [87].

1.4 Free infinitely divisible distributions and random matrix limits

It is well known that the classical infinitely divisible distributions have a *Lévy-Khintchine representation* given in terms of the cumulant generating function (logarithm of the characteristic function). A similar criteria for probability measure μ on $\mathbb{R}$ to be $\boxplus$-infinitely divisible (i.e. free infinitely divisible) is the existence of a finite measure σ on $\mathbb{R}$ and a real constant γ, such that

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+zt}{z-t} d\sigma(t), \qquad z \in \mathbb{C}^+.$$

One of the other important transforms in free probability theory, is the cumulant transform defined by $C_\mu(z) = z\phi_\mu(1/z)$. With an exact analogy to the classical infinitely divisible distribution, one has that μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if the free cumulant transform has the following *Lévy-Khintchine representation*:

$$C_\mu^{\boxplus}(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-zt} - 1 - tz\mathbf{1}_{[-1,1]}(t) \Big) d\nu(t), z \in \mathbb{C}^-, \tag{1.4.1}$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ν is called the *Lévy measure* on $\mathbb{R}$. These two representations are linked to each other by the relations ([16, Proposition 4.6]):

$$d\sigma(t) = a\delta_0(dt) + \frac{t^2}{1+t^2} d\nu(t),$$
$$\gamma = \eta - \int_{\mathbb{R}} t \left(\mathbb{1}_{[-1,1]}(t) - \frac{1}{1+t^2} \right) d\nu(t).$$

The free characteristic triplet (η, a, ν) of a probability measure μ is unique.

For a free infinitely divisible probability measure μ on $\mathbb{R}$ where the Lévy measure ν satisfies $\int_{\mathbb{R}} \min(1, |t|)\, d\nu(t) < \infty$ and $a = 0$, the Lévy-Khintchine representation (1.4.1) reduces to

$$C_\mu^{\boxplus}(z) = \eta' z + \int_{\mathbb{R}} \left(\frac{1}{1-zt} - 1 \right) d\nu(t), \quad z \in \mathbb{C}^-, \tag{1.4.2}$$

where $\eta' \in \mathbb{R}$. The measure μ is called a *free regular infinitely divisible distribution* (or *regular $\boxplus$-infinitely divisible measure*) if

$$\eta' \geq 0 \text{ and } \nu((-\infty, 0]) = 0\,.$$

The most typical example is compound free Poisson distributions. If the drift term η' is zero and the Lévy measure ν is $\lambda\rho$ for some constant $\lambda > 0$ and a probability measure ρ on $\mathbb{R}$, then we call μ a compound free Poisson distribution with rate λ and jump distribution ρ. To clarify these parameters, one denotes $\mu = \pi(\lambda, \rho)$. Interestingly, compound free Poisson distributions can be expressed in terms of free multiplicative convolutions. So, we first describe the later analytically followed by an overview of random matrix theory in connection to our study and explain the relation at the end of this section.

We have already seen that free additive convolution can be defined analytically with the help of Voiculescu transform. Similarly, the free multiplicative convolution can defined through the S-transform. For $\mu \in \mathcal{M}_+$, define

$$\Psi_\mu(z) = \int_0^\infty \frac{zt}{1-zt} d\mu(t),$$

for $z \in \mathbb{C} \setminus \mathbb{R}_+$. The S-transform $S_\mu : \Psi_\mu(i\mathbb{C}^+) \to i\mathbb{C}^+$ of μ is formulated as

$$S_\mu(z) = \frac{1+z}{z} \Psi_\mu^{-1}(z),$$

where the function Ψ_μ is univalent in the left half plane $i\mathbb{C}^+$ and therefore invertible on its image. Now suppose μ, ν both in $\mathcal{M}_+$. Then their free multiplicative convolution is derived via the identity

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z).$$

The free multiplicative convolution of two measures, one from $\mathcal{M}$ and symmetric and, the other from $\mathcal{M}_+$, was later defined by Arizmendi and Pérez-Abreu. In Theorem 6 of [10], it was shown that Ψ_μ univalent on the domains

$$D = \{z \in \mathbb{C}^+ : |\Re z| < \Im z\} \text{ and}$$
$$\tilde{D} = \{z \in \mathbb{C}^- : |\Re z| < |\Im z|\}.$$

Consequently, Ψ_μ becomes invertible on both $\Psi_\mu(D)$ and $\Psi_\mu(\tilde{D})$. Define

$$\tilde{S}_\mu(z) = \frac{1+z}{z} \Psi_\mu^{-1}(z)$$

for any $z \in \Psi_\mu(\tilde{D})$. Now if $\nu \in \mathcal{M}_+$ and μ is symmetric, then the free multiplicative convolution $\mu \boxtimes \nu$ is characterized by

$$S_{\mu \boxtimes \nu}(z) = S_\mu(z) S_\nu(z) \text{ and } \tilde{S}_{\mu \boxtimes \nu}(z) = \tilde{S}_\mu(z) S_\nu(z)$$

for all z in the common domain containing the interval $(0, \epsilon)$ for sufficiently small $\epsilon > 0$ (see Theorem 7 in [10]).

The convoluted measures arise naturally as limiting spectral distributions of sums and products of asymptotically free random matrices. There are examples where free additive convoluted measures and compound free Poisson distributions arise naturally as random matrix limits. Let $\lambda_1, \lambda_2, \cdots, \lambda_N$ be the eigenvalues of an $N \times N$ random matrix A_N. The *empirical spectral distribution* (in short *ESD*) is the random measure which puts mass $\frac{1}{N}$ to all the N many eigenvalues (counted with multiplicity), i.e. for any Borel set $B \subseteq \mathbb{C}$,

$$\text{ESD}(A_N)(B) = \frac{1}{N}\sum_{i=1}^{N} \delta_{\lambda_i}(B).$$

The randomness due to the random eigenvalues is still present in ESD. The *expected empirical spectral distribution* (in short *EESD*) is the non random probability measure defined by taking expectation of ESD, formulated as:

$$\text{EESD}(A_N)(B) = \frac{1}{N}\sum_{i=1}^{N} P\left(\lambda_i \in B\right).$$

If ESD(A_N) or EESD(A_N) converge to some non random measure σ in distribution, then σ is called the *Limiting spectral distribution* (in short LSD) of A_N. The *Wigner matrix* is a random matrix $A = (A_{i,j})_{i,j \leq N}$ where

(i) $\{A_{i,j},\ i < j\}$ are independent, identically distributed (real or complex valued),

(ii) $\{A_{i,i},\ i \leq N\}$ are independent, identically distributed real random variables (possibly from a different distribution),

(iii) $A_{i,j} = \bar{A}_{j,i}$ for all i and j,

(iv) $E[A_{1,2}] = 0$, $E[|A_{1,2}|^2] = 1$, $E[A_{1,1}] = 0$ and $E[A_{1,1}^2] < \infty$.

Let $\{A_N\}_{N=1}^{\infty}$ be a sequence of Wigner matrices, and for each N denote $X_N = \frac{1}{\sqrt{N}} A_N$. Then ESD$(X_N)$ converges weakly, in probability to the standard semicircle distribution (w),

$$w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\, \mathbb{1}_{(|x| \leq 2)}\,.$$

The convergence in probability to the semicircle distribution can be updated to almost sure convergence. Suppose D_N be a sequence of deterministic matrices such that ESD$(D_N) \to \nu$ weakly in probability. Then under some conditions on the distribution of X_N and D_N, the sequence of matrices X_N and D_N can be shown to be asymptotically free and hence ESD$(X_N + D_N) \to w \boxplus \nu$ weakly in probability.

The *Lévy distance* between two distribution functions F_1 and F_2 is defined by

$$d(F_1, F_2) = \inf\{\epsilon > 0 : F_1(x-\epsilon) - \epsilon \leq F_2(x) \leq F_1(x+\epsilon) + \epsilon, \text{ for all } x \in \mathbb{R}\}.$$

Let A be a random matrix with empirical spectral distribution ESD(A). The *Stieltjes transform* of A (or of ESD(A)) is given by

$$\mathcal{S}(z) = \int \frac{1}{x-z} d\left(\text{ESD}(A)\right)(x) = \frac{1}{N}\text{Tr}(A - zI_N)^{-1}$$

for $z \in \mathbb{C}^+$ and I_N is the identity matrix of order N. It is well-known that for distributions $\{F_n\}_{n\geq 1}$, F with Stieltjes transforms $\{\mathcal{S}_n\}_{n\geq 1}$ and $\mathcal{S}$ respectively, $\lim_{n\to\infty} d(F_n, F) = 0$ if and only if for all $z \in \mathbb{C}^+$, $\mathcal{S}_n(z) \to \mathcal{S}(z)$. Let $\mathcal{S}_\mu$ and G_μ denotes the Stieltjes transform and the Cauchy transform of a probability measure μ respectively, then it follows from the definition that $\mathcal{S}_\mu(z) = -G_\mu(z)$. Let N and p be two positive integers and consider the $N \times p$ matrix

$$A_{N,p} = (\mathcal{X}_{i,j})_{1\leq i\leq N, 1\leq j\leq p},$$

where $\mathcal{X}_{i,j}$'s are real valued random variables. Define the symmetric $p \times p$ matrix B_N by

$$B_N = \frac{1}{N} A'_{N,p} A_{N,p}.$$

The matrix B_N usually recognized as the sample covariance matrix associated with the process $(\mathcal{X}_{i,j})_{i,j\in\mathbb{Z}}$. It is also known as Gram matrix. Now consider N independent copies $(\mathcal{X}_{i,j})_{j\in\mathbb{Z}}$, $i \in \{1, 2, \ldots, N\}$ of a stationary sequence $(\mathcal{X}_i)_{i\in\mathbb{Z}}$ of real valued square integrable random variables with mean zero and satisfying some regularity conditions (see [70]). Assume $p/N \to c \in (0, \infty)$. Then there exists a non random probability measure μ such that ESD(B_N) converges almost surely to μ under the Lévy distance. Also the Stieltjes transform $\mathcal{S}_\mu$ of μ is uniquely determined by the equation

$$z = -\frac{1}{\hat{\mathcal{S}}_\mu} + \frac{c}{2\pi}\int_{-\pi}^{\pi} \frac{d\lambda}{\hat{\mathcal{S}}_\mu + (2\pi f(\lambda))^{-1}}, \qquad z \in \mathbb{C}^+,$$

where $\hat{\mathcal{S}}_\mu := c\mathcal{S}_\mu - (1-c)/z$ and $f(\cdot)$ is the spectral density of $(\mathcal{X}_k)_{k\in\mathbb{Z}}$. If we assume $\mathcal{X}_{i,j}$'s are independent and identically distributed with spectral density $f(\lambda) = \frac{E(\mathcal{X}_{1,1}^2)}{2\pi}$, then the Stieltjes transform represents the well known Marchenko-Pastur law (m_c) having density,

$$m_c(x) = \frac{1}{2\pi c\sigma^2 x}\sqrt{\left(\sigma^2(1-\sqrt{c})^2 - x\right)\left(x - \sigma^2(1+\sqrt{c})^2\right)}\,\mathbb{1}_{\left(\sigma^2(1-\sqrt{c})^2 \le x \le (1+\sqrt{c})^2\sigma^2\right)}$$

and a point mass $(1 - \frac{1}{c})$ at the origin if $c > 1$. In general, the limiting distribution μ can be described as the free multiplicative convolution of m_1 and $2\pi f(U)$, where U is uniformly distributed on $[-\pi, \pi]$. An important observation is that μ is compactly supported if and only if f has compact support.

We are now in a position to describe, how the compound free Poisson distributions can be written in terms of free multiplicative convolution. The standard Marchenko-Pastur law $m(= m_1)$ is a compound free Poisson distribution with rate 1 and jump distribution δ_1 whereas the distribution $\pi(1, \rho)$ coincides with $m \boxtimes \rho$, the free multiplicative convolution of m and ρ. The study of these distributions is itself very interesting in free probability and random matrix theory, because the compound free Poisson distribution $m \boxtimes \rho$ occurs not only as an LSD in the above model but also as LSD of a certain product of random matrices. It is an important question to ask whether the tail behaviour of the limit law can be understood by knowing the tail behaviour of any of its components.

Measures of the form $w \boxtimes \rho$ are also of huge interests in random matrix theory alongside the compound free Poisson distributions of the form $m \boxtimes \rho$. These two types of measures are connected by the following relation. Let $\nu^2 \in \mathcal{M}$ be the probability measure induced by the map $t \to t^2$ for a symmetric probability measure $\nu \in \mathcal{M}$. Then it can be shown that $w^2 = m$ (see Remark 3.6.2 for proof of this fact). Therefore, using Lemma 8 of [10], it can be shown that for any $\rho \in \mathcal{M}$

$$(w \boxtimes \rho)^2 = m \boxtimes \rho \boxtimes \rho.$$

From the above observation, it is not hard to see that the tail behaviour of $w \boxtimes \rho$ can be predicted form the study of the tail of $m \boxtimes \rho \boxtimes \rho$.

1.4.1 Contributions

In classical probability theory, an infinitely divisible probability measure μ also enjoys a Lévy-Khintchine representation in terms of its Lévy measure ν. In [47], it was shown that for a positively supported classically infinitely divisible probability measure (a subordinator) μ, the tails of μ and its Lévy measure ν are asymptotically equivalent if and only if any one of μ

or ν is subexponential. In analogy to the classical case, it is natural to pose whether free subexponentiality characterizes the tail equivalence of a free infinitely divisible probability measure and its free Lévy measure. Unfortunately, the result cannot be extended to the bigger class of free infinitely divisible probability measures. Since according to [9], the correct analogue of the positively supported classically infinitely divisible probability measures are the free regular probability measures, we provide a partial answer in Corollary 3.3.4 by showing the tail equivalence of a free regular probability measure and its free Lévy measure in the presence of regular variation. While in Section 3.5, the same is discussed when the measures are Boolean infinitely divisible. Note that regularly varying measures are the most important subclass of classical, free and Boolean subexponential distributions. This chapter is based on [38].

Besides, the connection of these results with the classical case is not a mere coincidence. From the famous result of Bercovici and Pata ([29]), it is known that classical and free infinitely divisible laws are in a one-to-one correspondence. It is shown in Corollary 3.4.7 that in the regularly varying set-up, the classical infinitely divisible law and its image under the Bercovici-Pata bijection are tail equivalent. The free multiplicative convolution of a measure with the semicircle law also appears naturally as limits of many random matrix models. It is shown in Corollary 3.4.4 that the tail behaviour turns out to be different from the one involving the multiplicative convolution with Marchenko-Pastur law.

1.5 Random matrices originating from random graphs

A most natural model for real life phenomena like the internet, spreading of a disease, collaboration and citations in research, social relations, and other complex networks, is random graphs. Random graphs can be understood by studying their adjacency or Laplacian matrices. Some information about the graphs is contained in the spectrum of those random matrices. The spectrum of the adjacency matrix is associated with the chromatic number and the independence number of the graph. The spectrum of Laplacian matrices is connected with the mixing time of random walks, the neighbourhood expansion, the Cheeger constant, the isoperimetric inequalities etc. One of the interesting random graphs studied in recent days is the configuration model. The motivation of this model can be understood from the following example. Suppose in Facebook wall posts the vertices of the graph are Facebook users and each edge between the user i and the user j represents one common post that is tagged to both of the users i and j. This gives rise to an undirected random diagram. Since the users can have more than one common tagged posts on their wall, the network allows multiple edges between pairs of vertices. Also, the graph may have some self-loops because of the possibility of self tagging.

Let $[N] = \{1, 2, \ldots, N\}$ be the set of vertices associated with a degree sequence $\{d_{i,N}\}_{i=1}^N$. Put $d_{i,N}$ many half edges to each vertex $i \in [N]$ and draw an uniform matching of all the half edges. In this way we can get a random multigraph $G_N = ([N], E_N)$ (called the *Configuration model*) where E_N is the random edge set obtained by the uniform adding of the half edges. The adjacency matrix A_{G_N} of the graph G_N has the (i,j)-th entry as the number of edges between the vertices i and j. If $d_{i,N} = d_N$ for all vertices i, that is, it is the random d_N-regular graph, with $1 \ll d_N \ll N$, then the ESD of $\frac{1}{\sqrt{d_N}} A_{G_N}$ converges weakly in probability to the standard semicircle law (w) defined above. Further, if we assume that the cardinality of the edge set, given by $|E_N| = \frac{1}{2} \sum_{i=1}^{N} d_{i,N}$ satisfies

$$N \ll |E_N| \ll N^2 \text{ as } N \to \infty,$$

and the normalized degrees $\hat{d}_{i,N} = \frac{d_{i,N}}{\omega_N}$ satisfy that

$$\{\hat{d}_{U_N,N}\} \text{ is uniformly integrable with } E\left[\left(\hat{d}_{U_N,N}\right)^2\right] \ll \sqrt{\frac{N}{\omega_N}},$$

where $\omega_N = (2 + \mathrm{o}(1)) \frac{|E_N|}{N}$ and U_N is uniformly chosen from $[N]$, then in [45], it was shown that the ESD of $\frac{1}{\sqrt{\omega_N}} A_{G_N}$ converges weakly in probability to $w \boxtimes \mu$, where ESD of the matrix $\operatorname{diag}\left(\hat{d}_{1,N}, \hat{d}_{2,N}, \ldots, \hat{d}_{N,N}\right)$ converges weakly to the measure μ. Due to the obstacle of non negligible dependence structure between the edges in the configuration model, Dembo and Lubetzky [45] studied the spectrum of G_N via sequences of approximations by removing some dependencies at each level of estimation, until finally arriving to a more tractable **inhomogeneous Erdős-Rényi** random graph.

From the above discussion it is evident that Erdős-Rényi random graphs are one of the most important random graph models and turns out to be very effective in the process of understanding the configuration model. Recalling that $[N] = \{1, 2, \ldots, N\}$ is the vertex set, put an edge between vertices i and j independently with probability $p_{i,j}^N$ for all $i \geq j$. The resulting undirected graph is called an *inhomogeneous Erdős-Rényi random graph*. The construction of the graph indicates that there may be self-loops but there is no multiple edge. The term inhomogeneous is for the variation of the connection probabilities across pairs of edges. One can consider,

$$p_{i,j}^N = \varepsilon_N f\left(\frac{i}{N}, \frac{j}{N}\right), \tag{1.5.1}$$

where $\varepsilon_N \geq 0$ and f is some function from $[0,1] \times [0,1]$ to $\mathbb{R}^+$. This is a generalization of the model

$$p_{i,j}^N = \frac{1}{N} f\left(\frac{i}{N}, \frac{j}{N}\right),$$

introduced in [31]. When $f \equiv 1$, is a constant function, the inhomogeneity is absent (i.e. $p_{i,j}^N = \varepsilon_N$) and the graph is called a (homogeneous) *Erdős-Rényi random graph*.

We may consider the adjacency matrix, A_N of an inhomogeneous Erdős-Rényi random graph. The study of the spectrum of A_N can be divided into two parts depending on the asymptotic behaviour of the average degree of the graph, namely the sparse regime and the non-sparse regime. If the expected degree or the average degree goes to some $c \in [0, \infty)$ then that is called the sparse regime, on the other hand, $c = \infty$ is called the non-sparse regime.

This thesis focuses on the eigenvalues outside the bulk of the ESD of adjacency matrix in the non sparse regime which is equivalent to say that $N\varepsilon_N \to \infty$ as $N \to \infty$. For the limiting spectral distribution, $\sqrt{N\varepsilon_N}$ turns out to be the correct scaling in the non sparse regime. It was shown in Theorem 1.1 of [37] that the ESD of $(N\varepsilon_N)^{-1/2}A_N$ converges weakly in probability to a non random, compactly supported probability measure. When $f \equiv 1$, the limiting spectral distribution is known to be the semicircle law. If $f(x, y) = r(x)r(y)$, for some bounded Riemann integrable function r, then the LSD is of the form

$$w \boxtimes \rho\,,$$

where ρ is the law of $r(U)$ and U is an uniform random variable on $[0, 1]$ and recall that w is the semicircle law. For inhomogeneous Erdős–Rényi random graphs the convergence of the empirical spectral distribution of both adjacency and Laplacian matrices were described in [37]. To have a better understanding, let us see some simulations.

First, we choose $f(x, y) = 2\sqrt{xy}$ i.e. f is a product of two same unit norm functions $r(t) = \sqrt{2t}$ in $L^2[0, 1]$ and fix the values of N and ε_N to be 1000 and 0.45 respectively. Then, in the Figure 1.1 the eigenvalues of the adjacency matrix are plotted under the scaling $\sqrt{N\varepsilon_N}$ and hence it is clear that the ESD converges to a compactly supported distribution, which is known to be $w \boxtimes \mu_r$ where μ_r denotes the law of $r(U)$ with U being an uniform random variable on $[0, 1]$.

One interesting aspect in Figure 1.1 is the one isolated eigenvalue outside the bulk (at a point close to $\sqrt{N\varepsilon_N} = 21.21$) of the eigenvalues of $(N\varepsilon_N)^{-1/2}A_N$.

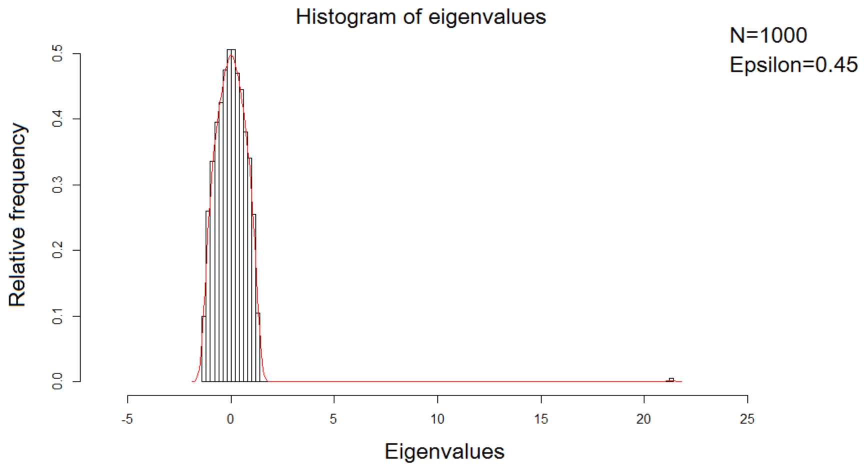

FIGURE 1.1: LSD is $w \boxtimes \mu_r$ with μ_r being the law of $r(U)$

We consider the largest eigenvalue in the above picture. When a properly scaled and centered (determined by a result in Chapter 4) rightmost eigenvalue is iterated a large number of times (40000 times, to be specific), a Gaussian curve is obtained, as shown in Figure 1.2.

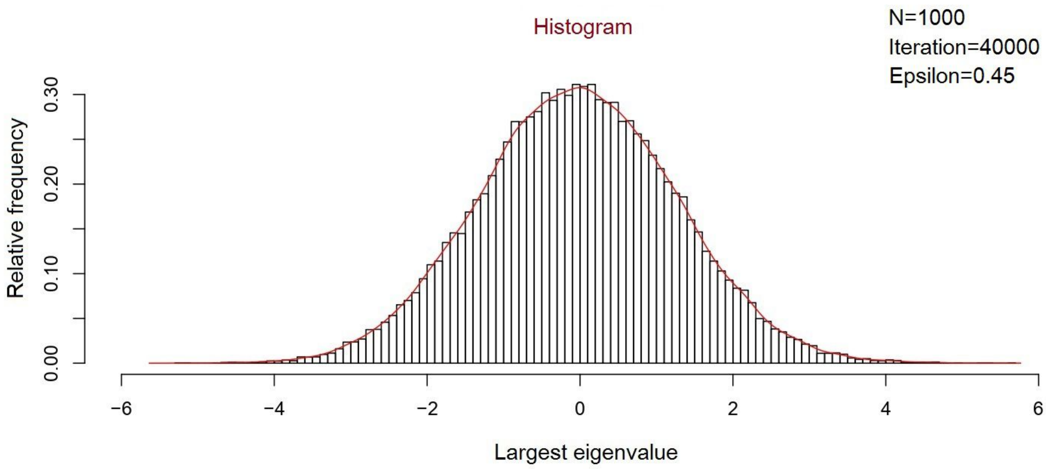

FIGURE 1.2: Gaussian nature of largest eigenvalue

Consider now another example. Let

$$f(x,y) = \mathbb{1}\left(x < \frac{1}{2}\right)\mathbb{1}\left(y < \frac{1}{2}\right) + 2\,\mathbb{1}\left(x > \frac{1}{2}\right)\mathbb{1}\left(y > \frac{1}{2}\right).$$

The inhomogeneous Erdős-Rényi graph corresponding to this function represents a stochastic block model with two communities. We again plot the histogram of eigenvalues of the adjacency matrix in Figure 1.3 In this scenario, the following picture shows that, there is exactly two eigenvalues outside the bulk of the ESD of $\frac{1}{\sqrt{N\varepsilon_N}}A_N$ (to be more precise, close to the points $\frac{1}{2}\sqrt{N\varepsilon_N} = 7.91$ and $\sqrt{N\varepsilon_N} = 15.81$). The simulation is performed when N is chosen to be 1000 and $\varepsilon_N = 0.25$.

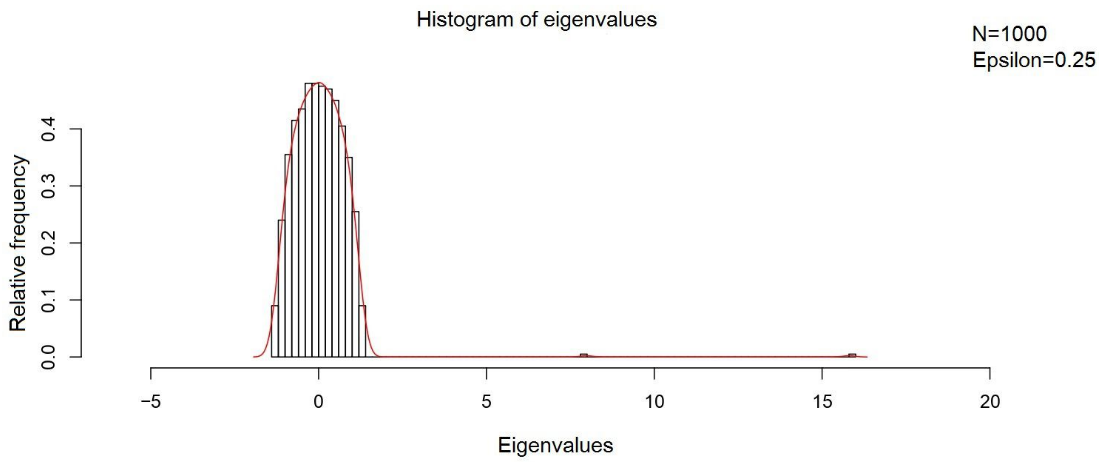

FIGURE 1.3: Two eigenvalues outside of the bulk

In the Chapters 4 and 5 of this thesis we explore the behaviour of the largest eigenvalue. We show that the eigenvalues outside the bulk of the spectrum have a Gaussian fluctuation.

The study of limiting behaviour of the largest eigenvalue, second largest eigenvalue, corresponding eigenvectors, spectral distribution of symmetric random matrices have a broad literature. The adjacency and Laplacian matrices of several random graphs are of interest to the physics and mathematics communities. For details see [44], [43] and [42].

The investigation of the largest eigenvalue of symmetric random matrices started with the pioneering works of Füredi and Komlós. In [55], they showed that after suitable scaling and centering the largest eigenvalue of a symmetric random matrix with independent entries having a strictly positive mean, converges in distribution to the Gaussian random variable under some moment assumptions on the matrix entries. A similar type of result for random (non-symmetric) matrices with independent identically distributed entries is obtained in [81]. The asymptotic behaviour of the largest eigenvalue of the adjacency matrix of sparse random graphs is studied in [65] and [50]. In [1], the authors showed that the absolute values of all the eigenvalues of a symmetric random matrix are concentrated around their respective medians with high probability.

In case of homogeneous Erdős-Rényi random graphs, one of eigenvalues of the adjacency matrix in the scaling $\frac{1}{\sqrt{N\varepsilon_N}}$ escapes to infinity as $N \to \infty$ at a rate of $\sqrt{N\varepsilon_N}$ (similar to the case of $f(x,y) = 2\sqrt{xy}$). That particular eigenvalue is the largest eigenvalue ($\lambda_1(A_N)$) in the homogeneous case. In [49], it was shown that

$$(\varepsilon_N(1-\varepsilon_N))^{-1/2} \left(\lambda_1(A_N) - \mathrm{E}[\lambda_1(A_N)]\right) \Rightarrow N(0,2).$$

The above result was shown under the assumption that

$$(\log N)^{\xi} \ll N\varepsilon_N$$

for some $\xi > 8$. In more recent works, Benaych-Georges et al. [25] and a companion article [24], have pointed out that for the Erdős–Rényi graph $G(N, d/N)$, the smallest and second largest eigenvalues of the adjacency matrix converge to the edges of the support of the asymptotic eigenvalue distribution in the regime $d \gg \log N$ while in the complementary regime $d \ll \log N$, those extreme eigenvalues are at a first order distance from the nonzero eigenvalues of the expectation matrix. The Tracy-Widom limit for the rescaled extremal eigenvalues of sparse random matrices (in particular, for adjacency matrices of Erdős–Rényi random graphs) was exhibited in [61] and [62] while the same in the non sparse regime is dealt in [48].

1.5.1 Contributions

We are interested in the outlier eigenvalues of the adjacency matrix of an inhomogeneous Erdős–Rényi random graph having edge probabilities as in (1.5.1). Motivated by the so-called stochastic block model, the function f considered here has the following form:

$$f(x,y) = \sum_{i=1}^{k} \theta_i r_i(x) r_i(y)\,,$$

where $r_1, \ldots, r_k$ are orthonormal in $L^2[0,1]$, and $\theta_1, \ldots, \theta_k > 0$. The above set-up allows to have k many eigenvalues outside the bulk of the spectrum. We investigate the behaviour of the largest eigenvalue and corresponding eigenvector in Chapters 4 and 5 under the above set up.

At first we consider the case when $f(x, y) = r(x)r(y)$, i.e. f is a product of two bounded, Riemann integrable functions. This particular case is called the rank one case and is discussed in Chapter 4. The proof techniques used in the homogeneous case by Erdős et al. [49], can be extended to this case. It is shown in Theorem 4.2.4, that the largest eigenvalue of the adjacency matrix of an inhomogeneous Erdős–Rényi random graph after suitable scaling and centering converges in distribution to the normal distribution with zero mean and some finite variance under the assumption that $(\log N)^{\xi} \ll N\varepsilon_N$ for some $\xi > 8$. In Theorem 4.2.5, it is shown that the normalized eigenvector corresponding to the largest eigenvalue is asymptotically parallel to a deterministic vector. One of the important examples of this model is the Chung-Lu graph (introduced by Chung and Lu [41]) and the results, for the same, are discussed in Section 4.5.

In Chapter 5, we consider the case,

$$f(x, y) = \sum_{i=1}^{k} \theta_i r_i(x) r_i(y) \,, (x, y) \in [0, 1] \times [0, 1],$$

where $\theta_1 \geq \theta_2 \geq \ldots \geq \theta_k > 0$ are some constants, the set $\{r_1, \ldots, r_k\}$ is an orthonormal set in $L^2[0, 1]$, and each r_i is a bounded Riemann integrable function on $[0, 1]$. An important class of examples in this frame is the stochastic block model. In this case the ideas behind the proofs are more complicated and the methods developed in [49] do not extend directly. Theorem 5.2.3 describes the joint convergence of the isolated eigenvalues, which will be defined later, among the top k eigenvalues of the adjacency matrix. We show under the assumption that eigenvalues are distinct, the outliers follow asymptotically Gaussian vector. Let

$$e_i = \begin{bmatrix} N^{-1/2} r_i(1/N) \\ N^{-1/2} r_i(2/N) \\ \vdots \\ N^{-1/2} r_i(1) \end{bmatrix} , 1 \leq i \leq k \,. \tag{1.5.2}$$

Then the behaviour of the eigenvectors corresponding to the outlier eigenvalues are also interesting. It is shown that the asymptotic behaviour of the normalized eigenvector (v_i) corresponding to the i-th largest eigenvalue $(1 \leq i \leq k)$ of A_N is asymptotically aligned with e_i and it is asymptotically orthogonal to e_j for $j \neq i$. The fluctuations of $e_j' v_i$ are studied under the additional condition $N^{-2/3} \ll \varepsilon_N \ll 1$. It is shown in Theorem 5.2.8 that $e_j' v_i$ converges to a normal distribution after a proper scaling and centering. An article [39] is based on the results of this chapter.

Chapter 2

Boolean convolutions and regular variation

2.1 Introduction

The main aim of this chapter is to study the Boolean convolution and its properties when the measures belong to the class of heavy tailed random variables.

The additive Boolean convolution of two probability measures μ and ν on the real line (denoted by $\mu \uplus \nu$) was introduced in [84] and the multiplicative Boolean convolution of two probability measures μ and ν (denoted by $\mu \boxtimes \nu$) was introduced in [26] where μ and ν are both defined on the non-negative part of the real line. Later Franz introduced the concept of Boolean independence and defined Boolean convolutions using operator theory in [54], which is similar to the approach of Bercovici and Voiculescu for the free convolutions in [28]. The definitions of Boolean convolutions using Boolean independence also agree with the former definitions.

In this chapter we are interested in a certain class of measures having power law tail behaviour. A measure is called regularly varying of index $-\alpha$, for some $\alpha > 0$, if $\mu(x, \infty) \sim x^{-\alpha} L(x)$ for some slowly varying function $L(x)$ (see explicit definition in next section). Such measures form a large class containing important distributions like Pareto and Fréchet and classical stable laws. The class of distribution functions with regularly varying tail index $-\alpha$, $\alpha \geq 0$ delivers significant applications in finance, insurance, weather, Internet traffic modelling and many other fields. In this chapter, we want to realise what happens with the Boolean convolutions of the probability measures which have regularly varying tails. In particular we want to address the following question:

Question 2.1.1. *Suppose μ and ν are two probability measures supported on $[0,\infty)$ with regularly varying tails of indices $-\alpha$ and $-\beta$ respectively (α and β non-negative). Then what can be said about the tail behaviour of $\mu \uplus \nu$ and $\mu \bowtie \nu$?*

When one considers the case of classical additive convolution, the answer is well known and the principle of one large jump gives that the heavier tail dominates. In fact it is well known, if μ is regularly varying of index $-\alpha$, then μ is (classical) subexponential in the sense that $\mu^{*n}(x,\infty) \sim n\mu(x,\infty)$ as $x \to \infty$ for all $n \geq 1$. For a contemporary review on subexponential distributions and their applications we refer to [53, 56, 64]. The case of free additive convolution was studied by Hazra and Maulik [59] and it is related to the free extreme value theory of Arous and Voiculescu [11]. One of the main aim of this chapter is to extend this result to the Boolean additive convolution.

The case of multiplicative convolution turns out to be more interesting and challenging. In classical independence, the role of Breiman's theorem is very crucial ([32]). A similar result can be obtained when ν is a regularly varying measure (see [64]). The result in the case of free multiplicative is still unknown to the best of our knowledge. We provide an example in Subsection 2.2.3 to show that the behaviour is much different from the classical case. In the Boolean convolution, the behaviour turns out to be much similar for multiplicative convolution and in that case again the heavier tail wins. We derive the explicit description in Theorem 2.2.7. The constants appearing though change from the classical case.

Boolean independence and convolutions are not studied as extensively as free independence. The Boolean convolutions of probability measures are also used in studying quantum stochastic calculus, see [21]. The Boolean Brownian motion and Poisson processes are investigated using Boolean convolutions to study the Fock space in [78], [57]. We can also observe the connection between Appell polynomials and Boolean theory in [4]. The Boolean stable laws and their relationship with free and classical stable laws were studied in recent works (see [5–8]). In a more recent study of classification of easy quantum groups, it was shown that the non-commutative analogue of de Finetti's theorem (quantum exchangeability) holds true and the notions of free independence, classical independence and half independence arise in this context (see [14]). The relation between Boolean independence and de Finetti's theorem was recently studied by Liu

[68]. Recently it has been established that in some random matrices, the asymptotic Boolean independence can arise ([58, 67, 69]).

As an application for the above results we determine the behaviour of the Belinschi-Nica map which is a one parameter family of maps $\{\mathbf{B}_t\}_{t\geq 0}$ on set of probability measures and was introduced by Belinschi and Nica [18] (see precise definition in next section). It is well known that classical infinitely divisible distributions are in bijection with the free infinitely divisible distributions. Here the map $\mathbf{B}_1$ turns out to be a bijection from Boolean to free infinitely divisible distributions. In fact it turns out that $(\mathbf{B}_t(\mu))_{t\geq 1}$ is $\boxplus$-infinitely divisible for every probability measure μ. The relationship with free Brownian motion and complex Burgers equation makes it an extremely important object of study. The map was further studied in [6], [7]. In this chapter we study the case when μ is a heavy tail distribution and show that μ is regularly varying of index $-\alpha$ if and only if $\mathbf{B}_t(\mu)$ is regularly varying $-\alpha$ for $t \geq 0$. In particular, it shows that the support of $\mathbf{B}_t(\mu)$ will be unbounded whenever μ has such regularly varying tails. The Boolean extreme value theory was recently explored in [91] in parallel to the study of free extreme value theory ([19]). We show that in the subexponential case, the tail behaviour of Boolean, free and classical extremes are asymptotically equivalent. It is known that the classical subexponential random variables satisfy *the principle of one large jump*, that is, if $\{X_i\}$ are i.i.d. subexponential random variables, then for all $n \geq 1$,

$$P\left(\sum_{i=1}^{n} X_i > x\right) \sim nP(X_1 > x) \sim P\left(\max_{1\leq i\leq n} X_i > x\right) \text{ as } x \to \infty.$$

The free max convolution, denoted by $\boxed{\vee}$, was introduced in [19] and the analogous result for the free one large jump principle was obtained in [59]. In this chapter we show that Boolean subexponential distributions follow the principle of one large jump also.

The main techniques involved in the proof of the above results is to study the transforms and their Taylor series expansion. In particular we show the remainder terms of the respective transforms carries information about the regular variation and also it is preserved under certain operations such as taking a reciprocal. These results can be independent in their own interest and can be used to study various properties of the transforms involved in free and Boolean independence. Such ideas were first explored in the works of Bercovici et al. [29] to show the bijection between free infinitely distributions with the classical counter parts. Other works relating the remainder terms of Cauchy and R-transforms were studied in [22, 27, 59].

Outline of the chapter: In the next section we develop the set-up and state our main results precisely. To prove the results we use various transforms and their relationship with the tail of the measures. In Section 2.2 we introduce some of the transforms used, an interesting property of the Belinschi-Nica map $\{\mathbf{B}_t\}_{t\geq 0}$ in Theorem 2.2.6 and the principle of one large jump in Proposition 2.2.4. In Section 2.3 we state the relationship between the tail of a regularly varying probability measure the remainder of $1/B$-transform followed by defining the remainder terms. In Section 2.4 we use these relations to provide proof of the results for the additive Boolean convolutions. Section 2.5 contains the proof of Theorem 2.2.7 about the multiplicative Boolean convolution. Finally in section 2.6 we prove the technical results which are presented in Section 2.3.

2.2 Preliminaries and main results

A real valued measurable function f defined on non-negative real line is called *regularly varying* (at infinity) with index α if for every $t > 0$, $\frac{f(tx)}{f(x)} \to t^{\alpha}$ as $x \to \infty$. If $\alpha = 0$, then f is said to be a slowly varying function (at infinity). Regular variation with index α at zero is defined analogously. In fact, f is regularly varying at zero of index α, if the function $x \mapsto f(\frac{1}{x})$ is regularly varying at infinity of index $-\alpha$. *Unless otherwise mentioned, the regular variation of a function will be considered at infinity. For regular variation at zero, we shall explicitly mention so.* A distribution function F on $[0, \infty)$ has regularly varying tail of index $-\alpha$ if $\overline{F}(x) = 1 - F(x)$ is regularly varying of index $-\alpha$. Since $\overline{F}(x) \to 0$ as $x \to \infty$, we must necessarily have $\alpha \geq 0$. A probability measure on $[0, \infty)$ with regularly varying tail is defined through its distribution function. Equivalently, a measure μ is said to have a regularly varying tail of index $-\alpha$, if $\mu(x, \infty)$ is regularly varying of index $-\alpha$ as a function of x.

We shall write $f(z) \approx g(z)$, $f(z) = \mathrm{o}(g(z))$ and $f(z) = \mathrm{O}(g(z))$ as $z \to 0$ n.t. to mean that $f(z)/g(z)$ converges to a non-zero limit, $f(z)/g(z) \to 0$ and $f(z)/g(z)$ stays bounded as $z \to 0$ n.t. respectively. If the non-zero limit is 1 in the first case, we write $f(z) \sim g(z)$ as $z \to 0$ n.t. For $f(z) = \mathrm{o}(g(z))$ as $z \to 0$ n.t., we shall also use the notations $f(z) \ll g(z)$ and $g(z) \gg f(z)$ as $z \to 0$ n.t.

Recall $\mathcal{M}$ and $\mathcal{M}_+$ are the set of probability measures supported on $\mathbb{R}$ and $\mathbb{R}_+$ respectively. By $\mathcal{M}_p$ we mean the set of probability measures on $[0, \infty)$ whose p-th moment is finite and do not have the $(p + 1)$-th moment.

2.2.1 Additive Boolean convolution

We recall that for a probability measure $\mu \in \mathcal{M}$, its Cauchy transform is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t} d\mu(t), \quad z \in \mathbb{C}^+.$$

Note that G_μ maps $\mathbb{C}^+$ to $\mathbb{C}^-$. The Boolean additive convolution is determined by the transform K_μ which is defined as

$$K_\mu(z) = z - \frac{1}{G_\mu(z)}, \text{ for } z \in \mathbb{C}^+. \tag{2.2.1}$$

For two probability measures μ and ν, the additive Boolean convolution $\mu \uplus \nu$ is determined by

$$K_{\mu\uplus\nu}(z) = K_\mu(z) + K_\nu(z), \text{ for } z \in \mathbb{C}^+ \tag{2.2.2}$$

and $\mu \uplus \nu$ is again a probability measure.

Our first result describes the behaviour of additive Boolean convolution under the regularly varying measures. Suppose $\{X_i\}_{i\geq 1}$ be independent (classically) and identically distributed non-negative regularly varying random variables of index $-\alpha$, $\alpha \geq 0$ and denote $S_n = X_1 + X_2 + \cdots + X_n$. Then it is known that

$$P(S_n > x) \sim nP(X_1 > x) \text{ as } x \to \infty. \tag{2.2.3}$$

The proof of the above fact can be found in [51]. If a sequence of random variables follows (2.2.3) then they are called subexponential. In the case of free additive convolution, the parallel result was shown in [59], which states:

$$\mu^{\boxplus n}(y,\infty) = \underbrace{(\mu \boxplus \cdots \boxplus \mu)}_{n\ times}(y,\infty) \sim n\mu(y,\infty) \text{ as } y \to \infty,$$

when μ has regularly varying tail of index $-\alpha$, $\alpha \geq 0$. We show that result can be extended to Boolean additive convolution also. To state the result we first introduce the definition of Boolean subexponentiality:

Definition 2.2.1. *A probability measure μ on $[0,\infty)$, with $\mu(y,\infty) > 0$ for all $y \geq 0$, is said to be Boolean-subexponential if for all $n \in \mathbb{N}$,*

$$\mu^{\uplus n}(y,\infty) = \underbrace{(\mu \uplus \cdots \uplus \mu)}_{n \ times}(y,\infty) \sim n\mu(y,\infty) \ \text{as } y \to \infty.$$

Our first result shows that analogue of the classical and free case is also valid in Boolean set-up.

Theorem 2.2.2. *If μ is regularly varying of index $-\alpha$, $\alpha \geq 0$, then μ is Boolean-subexponential.*

The proof uses the relation between μ and G_μ developed in [59] and also extensions to the transforms K_μ.

Applications of Boolean subexponentiality

In this subsection we see two important applications of Boolean subexponentiality. The notions of independence give rise to corresponding extreme value theory. We first show that subexponentiality in all the three notions are asymptotically equivalent. In the second application we show how the Belinschi-Nica map related to free infinitely divisible indicator behaves for a regularly varying measure.

Applications to Boolean extremes

The very immediate upshot of the definition of Boolean subexponentiality is the *principle of one large jump* which gives us the asymptotic relation between the sum and maximum of a finite collection of i.i.d. probability distributions. The extreme value theory in Boolean independence was recently explored in [91]. We briefly recall the definition of Boolean max convolution from [91].

Definition 2.2.3. *Let F_1, F_2 be two distributions on $[0,\infty)$. Their Boolean max convolution is defined by,*

$$(F_1 \cup\!\!\!\!\vee\, F_2)(t) = F_1(t) \mathbin{\underline{\wedge\!\!\!\vee}} F_2(t)$$

where the operation $\mathbin{\underline{\wedge\!\!\!\vee}}$ is defined as

$$(x \mathbin{\underline{\wedge\!\!\!\vee}} y)^{-1} - 1 = (x^{-1} - 1) + (y^{-1} - 1) \text{ for all } x, y \in [0,1].$$

Let D_+ be the set of all probability distributions on $[0, \infty)$. Then D_+ forms semigroup with respect to both the classical max convolution " $\cdot$ " and the boolean max convolution " $\cup\!\!\!\!\vee$ ". Further it is proved there that the map $X : (D_+, \cdot) \to (D_+, \cup\!\!\!\!\vee)$, given by,

$$X(F)(t) = \exp\left(1 - \frac{1}{F(t)}\right) \text{ for all } t \in [0, \infty),\ F \in D_+ \tag{2.2.4}$$

is an isomorphism while the inverse map is

$$X^{-1}(F)(t) = (1 - \log(F))^{-1}(t) = \frac{1}{1 - \log(F(t))} \text{ for all } t \in [0, \infty)\ F \in D_+. \tag{2.2.5}$$

The above isomorphism is obtained by observing an interesting isomorphism between the two semigroups $([0, 1], \cup\!\!\!\!\vee)$ and $([0, 1], .)$ where " . " is the usual multiplication of real numbers. Here we give an affirmative answer for the one large jump principle in the Boolean case and combining all the results of the classical, free and Boolean instances we can further say that all the tails of classical, free and Boolean max convolutions are asymptotically equivalent for the class of regularly varying distributions. We shall use the notations $F^{\cup\!\!\!\!\vee n}$ and $F^{\uplus n}$ for the distributions $\underbrace{F \cup\!\!\!\!\vee \cdots \cup\!\!\!\!\vee F}_{n\ times}$ and $\underbrace{F \uplus \cdots \uplus F}_{n\ times}$ respectively.

Proposition 2.2.4. *The principle of one large jump holds true for Boolean-subexponential distributions, namely, if F is Boolean-subexponential then for every $n \geq 1$,*

$$\overline{F^{\uplus n}}(y) \sim \overline{F^{\cup\!\!\!\!\vee n}}(y) \text{ as } y \to \infty.$$

Moreover if F is regularly varying with index $-\alpha$, $\alpha \geq 0$, then for all $n \geq 1$,

$$\overline{F^{\cup\!\!\!\!\vee n}}(y) \sim \overline{F^{\Box\!\!\!\!\vee n}}(y) \sim \overline{F^{n}}(y) \text{ as } y \to \infty \tag{2.2.6}$$

where F^n arises out of the classical max convolution of classical independent random variables $Z_1, \cdots, Z_n$ having identical distribution F.

Remark 2.2.5. *The proof of the above result is done only for positive integers $n \geq 1$. In fact the result holds for any real number $n > 0$ by similar calculations. See [87] for complete details.*

Application to the Belinschi-Nica map

Before going to the multiplicative Boolean convolution we want to show an application of the above result to the Belinschi-Nica map. Let us recall that $\boxplus$ denotes the free additive convolution of measures. Let us consider the map $\mathbf{B}_t : \mathcal{M} \to \mathcal{M}$ for all $t \geq 0$, given by

$$\mathbf{B}_t(\mu) = (\mu^{\boxplus(1+t)})^{\uplus \frac{1}{1+t}} \qquad \mu \in \mathcal{M}. \tag{2.2.7}$$

This map was introduced in [18] noting that every probability measure on $\mathbb{R}$ is infinitely divisible with respect to additive Boolean convolution. It was also shown there that if $\mu \in \mathcal{M}_+$ then $\mathbf{B}_t(\mu) \in \mathcal{M}_+$. When $t = 1$, the map $\mathbf{B}_1$ coincides with the Bercovici-Pata bijection between $\mathcal{M}$ and the class of all free infinitely divisible probability measures supported on $\mathbb{R}$. Here for better understanding we can consider the maps $\mathbf{B}_n : \mathcal{M}_+ \to \mathcal{M}_+$ for non-negative integers n and from the definition (2.2.7), we have

$$\mathbf{B}_n(\mu)^{\uplus(1+n)} := \underbrace{\mathbf{B}_n(\mu) \uplus \mathbf{B}_n(\mu) \uplus \cdots \uplus \mathbf{B}_n(\mu)}_{(1+n \ \ times)} = \underbrace{\mu \boxplus \mu \boxplus \cdots \boxplus \mu}_{(1+n \ \ times)} =: \mu^{\boxplus(1+n)}. \tag{2.2.8}$$

Theorem 2.2.6. *The following are equivalent for a probability measure $\mu \in \mathcal{M}_+$.*

(i) *μ is regularly varying with tail index $-\alpha$.*

(ii) *$\boldsymbol{B}_t(\mu)$ is regularly varying with tail index $-\alpha$, for $t \geq 0$.*

Furthermore, if any of the above holds, we also have as $y \to \infty$,

$$\mu(y, \infty) \sim \boldsymbol{B}_t(\mu)(y, \infty).$$

An interesting connection with complex Burgers equation was established in [18] using the following function

$$h(t, z) = F_{\mathbf{B}_t(\mu)}(z) - z, \, \forall t > 0, \ \forall \, z \in \mathbb{C}^+,$$

where F_ν is the reciprocal of the Cauchy transform. Note that it can also be written as $h(t, z) = -K_{\mathbf{B}_t}(z)$. It was shown that $h(t, z)$ satisfies the following complex Burgers equation

$$\frac{\partial h}{\partial t}(t, z) = h(t, z)\frac{\partial h}{\partial z}(t, z).$$

The complex Burgers equation (also known as the free analogue of heat equation) arises naturally due to the connections with free Brownian motion (see [93]). In the following section while proving Theorem 2.2.2 we shall study the remainder term in the K transform of a measure μ and hence from the above result one can easily derive the asymptotic behaviour of the remainder term of $h(t, z)$ (taking the Taylor series expansion in z) when μ has a regularly varying tail. Note that in the power series expansion of K, the coefficients, which are also known as Boolean cumulants can be directly computed using the moments recursively. We do not write the details of such applications but it would be clear from the derivations later.

2.2.2 Multiplicative Boolean convolution

Now we define the multiplicative Boolean convolution of two probability measures defined on $\mathbb{R}_+$. Instead of using the energy transform (as defined in Chapter 1) we follow [5] as the calculations follow steadily. For $\mu \in \mathcal{M}_+$ the function

$$\Psi_\mu(z) = \int_{\mathbb{R}} \frac{zt}{1-zt} d\mu(t), \qquad z \in \mathbb{C} \setminus \mathbb{R}_+$$

is univalent in the left-plane $i\mathbb{C}_+$ and $\Psi_\mu(i\mathbb{C}_+)$ is a region contained in the circle with diameter $(\mu(0) - 1, 0)$. It is well known that,

$$\Psi_\mu\left(z^{-1}\right) = zG_\mu(z) - 1. \tag{2.2.9}$$

The η-transform of μ, denoted by $\eta_\mu : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C} \setminus \mathbb{R}_+$, is defined by the formula:

$$\eta_\mu(z) = \frac{\Psi_\mu(z)}{1 + \Psi_\mu(z)}. \tag{2.2.10}$$

It is clear that μ is determined uniquely from the function η_μ. For $\mu \in \mathcal{M}_+$ it is known that $\eta_\mu((-\infty, 0)) \subset (-\infty, 0)$, $0 = \eta_\mu(0^-) = \lim_{x \to 0, x<0} \eta_\mu(x)$, $\eta_\mu(\overline{z}) = \overline{\eta_\mu(z)}$ for $z \in \mathbb{C} \setminus \mathbb{R}_+$. Also $\pi > \arg(\eta_\mu(z)) \geq \arg(z)$, for $z \in \mathbb{C}^+$.

The analytic function

$$B_\mu(z) = \frac{z}{\eta_\mu(z)} \tag{2.2.11}$$

is well defined in the region $z \in \mathbb{C} \setminus \mathbb{R}_+$. Now for $\mu, \nu \in \mathcal{M}_+$, their *multiplicative Boolean convolution* $\mu \uplus\!\!\!\times \nu$ is defined as the unique probability measure in $\mathcal{M}_+$ that satisfies

$$B_{\mu \times\!\!\!\!\uplus \nu}(z) = B_\mu(z) B_\nu(z) \text{ for } z \in \mathbb{C} \setminus \mathbb{R}_+. \tag{2.2.12}$$

Note that for $\mu, \nu \in \mathcal{M}_+$ which satisfies

(a) $\arg(\eta_\mu(z)) + \arg(\eta_\nu(z)) - \arg(z) < \pi$ for $z \in \mathbb{C}^+ \cup (-\infty, 0)$, and

(b) **at least one of the first moments of one of the measure μ or ν exists finitely**,

then $\mu \circledast \nu \in \mathcal{M}_+$ is well-defined.

In this chapter whenever we write the probability measure $\mu \circledast \nu$, it is assumed that the first moment $m(\nu)$ of ν must exist due to the definition of multiplicative Boolean convolution. The mean exists means it is strictly positive since the measures are supported on the positive half of the real line. When two measures shall have the same regularly varying tail, we will assume that there exists some $c \in (0, \infty)$ such that $\nu(x, \infty) \sim c\mu(x, \infty)$ as $x \to \infty$. The case where this asymptotics of tail sums fails is explained in Remark 2.5.3. Now here is the main result for multiplicative Boolean convolution:

Theorem 2.2.7. *Let μ, $\nu \in \mathcal{M}_+$. If μ is regularly varying of tail index $-\alpha$ and ν is regularly varying of tail index $-\beta$ where $\alpha \leq \beta$ and $\mu \circledast \nu \in \mathcal{M}_+$ then $\mu \circledast \nu$ is also regularly varying with tail index $-\alpha$, furthermore,*

$$\mu \circledast \nu\,(y, \infty) \sim m\,(\nu)\,\mu\,(y, \infty) \;\; \text{if } \alpha < \beta,$$

$$\mu \circledast \nu\,(y, \infty) \sim (1 + c)m\,(\nu)\,\mu\,(y, \infty)\,, \; \text{if } \alpha = \beta,$$

where $m(\nu)$ is the mean of ν.

Note that the result differs from the classical Breiman's result (1.3.1) in terms of the constants which appear in the tail equivalence relation. In classical case, the α-th moment of ν appears and in multiplicative Boolean the first moment appears only.

2.2.3 Open questions

We list some of the open questions in this subsection before going to the technicalities of the proof.

(i) Suppose μ and ν are in $\mathcal{M}_+$ and have regularly varying tails of index $-\alpha$ and $-\beta$ respectively. Then what is the tail behaviour of $\mu \boxtimes \nu$? From a result from [29, Proposition A4.3] it follows that if μ is $\boxplus$ stable of index $1/(1+s)$ and ν is of index $1/(1+t)$ then $\mu \boxtimes \nu$ is $\boxplus$ stable of index $1/(1+s+t)$. This already shows that the classical Breiman's theorem is not true in free set-up and hence it would be interesting to know what kind of behaviour the $\mu \boxtimes \nu$ distribution inherits.

(ii) In a recent work [63], it was shown that if one takes the inverse problem of Breiman's theorem, that is, if one knows that $\mu \circledast \nu$ has a regularly varying tail of index $-\alpha$ then under some necessary and sufficient conditions on ν, μ also has regularly varying tail of same index. It would be interesting to explore if such inverse problems can be answered in the free or Boolean set-up.

(iii) Following [18] we recall the definition of $\boxplus$-divisibility indicator $\phi(\mu)$ of μ given by

$$\phi(\mu) = \sup\{t \in [0, \infty) : \mu \in \mathbf{B}_t(\mathcal{M})\} \in [0, \infty].$$

The Cauchy distribution μ_{ca} (which has regularly varying tail of index -1) is fixed by the map B_1 (which is in fact the Boolean to free Bercovici-Pata bijection). Therefore by definition of $\mathbf{B}_1$ we have $\mu_{ca}^{\uplus 2} = \mu_{ca}^{\boxplus 2}$ (moreover $\mu_{ca}^{\uplus t} = \mu_{ca}^{\boxplus t}$ for all $t \geq 0$) and this along with the formula $\phi(\mathbf{B}_t(\mu)) = \phi(\mu) + t$ allows us to conclude that $\phi(\mu_{ca}) = \infty$ as observed by Belinschi and Nica. It would be interesting to understand if one can take $\boxplus$-infinitely divisible distributions with regularly varying tails and see if $\phi(\mu) = \infty$ in such cases also.

The rest of this chapter is devoted to proofs of the above results. We first develop the Tauberian type results for different transforms and then apply them to prove the results.

2.3 Regular variation of the remainder terms of Ψ, η and B transform

In this section we define the remainder term of B-transforms and shall see how regular variation is linked to it. These relations will be used to prove the main results. Note that although the B transform is used to define the multiplicative Boolean convolution, it can be used in the analysis of the additive transform by its relation to K transform in the following way. From the relations (2.2.1), (2.2.9), (2.2.10) and (2.2.11) it follows that,

$$K_\mu\left(z^{-1}\right) = \frac{1}{B_\mu\left(z\right)}. \tag{2.3.1}$$

For $\mu \in \mathcal{M}_p$, following Theorem 1.5 of [22], we have the following Laurent series like expansion

$$G_\mu(1/z) = \sum_{i=1}^{p+1} m_{i-1}(\mu) z^i + \mathrm{o}(z^{(p+1)}), \qquad z \to 0 \text{ n.t.}, \ p \geq 0.$$

Therefore using (2.2.9), we get

$$\Psi_\mu(z) = \sum_{i=1}^{p} m_i(\mu) z^i + \mathrm{o}(z^p) \text{ as } z \to 0 \text{ n.t.}, \qquad p \geq 1.$$

The remainder term $r_{G_\mu}(z)$ of the Cauchy transform was defined in [59], given by:

$$r_{G_\mu}(z) := z^{p+1}(G_\mu(z) - \sum_{i=1}^{p+1} m_{i-1}(\mu) z^{-i}). \tag{2.3.2}$$

Using (2.2.9) and the above expressions we define the remainder term $r_{\Psi_\mu}(z)$ of the Ψ-transform.

$$r_{\Psi_\mu}(z) := \begin{cases} z^{-p}(\Psi_\mu(z) - \sum_{i=1}^{p} m_i(\mu) z^i), & \text{if } p \geq 1 \\ \Psi_\mu(z), & \text{if } p = 0. \end{cases} \tag{2.3.3}$$

Using (2.2.9), (2.3.2) and (2.3.3), we get

$$r_{G_\mu}(z) = r_{\Psi_\mu}(z^{-1}). \tag{2.3.4}$$

Therefore we have from (2.3.4),

$$\Re r_{G_\mu}(z) = \Re r_{\Psi_\mu}(z^{-1}) \text{ and } \Im r_{G_\mu}(z) = \Im r_{\Psi_\mu}(z^{-1}). \tag{2.3.5}$$

Thus the remainder terms of η and B transforms can be defined analogously. Also from (2.2.11) and the fact that z lies either in the upper half or the lower half of the complex plane, we have,

$$\frac{1}{B_\mu}(z) := \frac{1}{B_\mu(z)} = \frac{\eta_\mu(z)}{z}.$$

Using $\Psi_\mu(z)$ has no constant term in its Taylor series expansion, we have for $p \geq 1, \mu \in \mathcal{M}_p$,

$$r_{\frac{1}{B_\mu}}(z) = r_{\eta_\mu}(z).$$

Equating the real and imaginary part for the above identity we get

$$\Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right)=\Re r_{\eta_\mu}\left(-iy^{-1}\right) \text{ and } \Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right)=\Im r_{\eta_\mu}\left(-iy^{-1}\right). \tag{2.3.6}$$

When $p=0$, that is, $\mu\in\mathcal{M}_0$, we write

$$r_{\frac{1}{B_\mu}}(z)=\frac{1}{B_\mu}(z) \text{ and } \frac{1}{B_\mu}\left(-iy^{-1}\right)=iy\eta_\mu\left(-iy^{-1}\right).$$

We will later see $1/B_\mu(z)$ in this case goes to infinity as $z\to 0$ non-tangentially but still we want to say it is a remainder term as it helps in keeping analogy with the other cases notationally.

Let $\mu\in\mathcal{M}_+$ and is regularly varying of tail index $-\alpha$. Then there exists a non-negative integer p such that $\mu\in\mathcal{M}_p$. We split this into five cases as follows: (i) p is a positive integer and $\alpha\in(p,p+1)$; (ii) p is a positive integer and $\alpha=p$; (iii) $p=0$ and $\alpha\in[0,1)$; (iv) $p=0$ and $\alpha=1$; (v) p is a natural number and $\alpha=p+1$, giving rise to the following five theorems.

In the following we compare the tails of μ and the behaviour of the remainder term of the $1/B_\mu$. The proof of these five theorems are deferred to the last section and depends crucially on ideas developed in [59].

We first consider the case where p is a positive integer and $\alpha\in(p,p+1)$.

Theorem 2.3.1. *Let μ be in $\mathcal{M}_p$, $p\geq 1$ and $p<\alpha<p+1$. The following are equivalent:*

(i) $\mu(y,\infty)$ *is regularly varying of index $-\alpha$.*

(ii) $\Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right)$ *is regularly varying of index $-(\alpha-p)$ and*

$$\Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right)\approx\Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \qquad \text{as } y\to\infty.$$

If any of the above statements holds, we also have, as $z\to 0$ n.t.,

$$z\ll r_{\frac{1}{B_\mu}(z)};$$

and as $y\to\infty$,

$$\Im r_{\frac{1}{B_\mu}(-iy^{-1})}\sim-\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)}y^p\mu(y,\infty)\gg y^{-1} \tag{2.3.7}$$

and

$$\Re r_{\frac{1}{B_\mu}(-iy^{-1})} \sim -\frac{\pi\left(p+2-\alpha\right)/2}{\sin\left(\pi\left(\alpha-p\right)/2\right)} y^p \mu\left(y,\infty\right) \gg y^{-1}. \tag{2.3.8}$$

Next we consider the case where p is a positive integer and $\alpha = p$. In this case although (2.3.7) holds but the final asymptotic of (2.3.8) need not be true.

Theorem 2.3.2. *Let μ be in $\mathcal{M}_p, p \geq 1$ and $\alpha = p$. The following are equivalent:*

(i) *$\mu\left(y,\infty\right)$ is regularly varying of index $-p$.*

(ii) *$\Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right)$ is slowly varying and*

$$\Re r_{\frac{1}{B_\mu}(-iy^{-1})} \gg y^{-1} \qquad \text{as } y \to \infty. \tag{2.3.9}$$

If any of the above statements holds, we also have, as $z \to 0$ n.t.,

$$z \ll r_{\frac{1}{B_\mu}(z)};$$

and as $y \to \infty$ we have,

$$\Im r_{\frac{1}{B_\mu}(-iy^{-1})} \sim -\frac{\pi}{2} y^p \mu\left(y,\infty\right) \gg y^{-1}. \tag{2.3.10}$$

In the third case we consider $\alpha \in [0,1)$.

Theorem 2.3.3. *Let μ be in $\mathcal{M}_0$ and $0 \leq \alpha < 1$. The following are equivalent:*

(i) *$\mu\left(y,\infty\right)$ is regularly varying of index $-\alpha$.*

(ii) *$\Im \frac{1}{B_\mu}\left(-iy^{-1}\right)$ is regularly varying of index $-(\alpha-1)$ and*

$$\Re \frac{1}{B_\mu}\left(-iy^{-1}\right) \approx \Im \frac{1}{B_\mu}\left(-iy^{-1}\right) \qquad \text{as } y \to \infty$$

If any of the above statements holds, we also have, as $z \to 0$ n.t.,

$$z \ll z\frac{1}{B_\mu}(z);$$

and as $y \to \infty$ we have,

$$-y^{-1}\Re\frac{1}{B_\mu}(-iy^{-1}) \sim -\frac{\pi(1-\alpha)/2}{\cos(\pi\alpha/2)}\mu(y,\infty) \gg y^{-1} \tag{2.3.11}$$

and

$$y^{-1}\Im\frac{1}{B_\mu}(-iy^{-1}) \sim -d_\alpha\mu(y,\infty) \gg y^{-1} \tag{2.3.12}$$

where

$$d_\alpha = \begin{cases} \frac{\pi(2-\alpha)/2}{\sin(\pi\alpha/2)}, & \text{when } \alpha > 0, \\ 1, & \text{when } \alpha = 0. \end{cases}$$

In the fourth case we consider $\alpha = 1$ and $p = 0$.

Theorem 2.3.4. *Let μ be in $\mathcal{M}_0$ and $\alpha = 1, r \in (0, 1/2)$. The following are equivalent:*

(i) *$\mu(y,\infty)$ is regularly varying of index -1.*

(ii) *$\Im\frac{1}{B_\mu}\left(-iy^{-1}\right)$ is slowly regularly varying and*

$$y^{-1} \ll -y^{-1}\Re\frac{1}{B_\mu}\left(-iy^{-1}\right) \ll y^{-(1-r/2)}. \tag{2.3.13}$$

If any of the above statements holds, we also have, as $z \to 0$ n.t.,

$$z \ll z\frac{1}{B_\mu}(z);$$

and as $y \to \infty$ we have,

$$y^{-(1+r/2)} \ll y^{-1}\Im\frac{1}{B_\mu}\left(-iy^{-1}\right) \sim -\frac{\pi}{2}\mu(y,\infty) \ll y^{-1+r/2} \tag{2.3.14}$$

Finally, we consider the case where $p \geq 1$ and $\alpha = p + 1$.

Theorem 2.3.5. *Let μ be in $\mathcal{M}_p$, $p \geq 1$ and $\alpha = p + 1, r \in (0, 1/2)$. The following are equivalent:*

(i) *$\mu(y,\infty)$ is regularly varying of index $-(p+1)$.*

(ii) *$\Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right)$ is regularly varying of index -1 and*

$$y^{-1} \ll \Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \ll y^{-(1-r/2)}. \tag{2.3.15}$$

If any of the above statements holds, we also have, as $z \to 0$ n.t.,

$$z \ll r_{\frac{1}{B_\mu}}(z); \tag{2.3.16}$$

As $y \to \infty$ we have,

$$y^{-(1+r/2)} \ll \Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \sim -\frac{\pi}{2} y^p \mu(y,\infty) \ll y^{-(1-r/2)}. \tag{2.3.17}$$

The proof of these theorems are deferred to Section 2.6.

2.4 Additive Boolean subexponentiality

To prove the Theorem 2.2.2 we need the following lemma.

Lemma 2.4.1. *Suppose μ and ν are two probability measures in $[0,\infty)$ with regularly varying tails of index $-\alpha$ and suppose $\nu(y,\infty) \sim c\mu(y,\infty)$ for some $c > 0$. Then*

$$\mu \uplus \nu(y,\infty) \sim (1+c)\,\mu(y,\infty) \text{ as } y \to \infty.$$

Proof. Depending on where the index $\alpha \geq 0$ lies, the proof can be split into five cases as described in Section 2.3. We shall present the proof for the case when $p \geq 1$ with $\mu \in \mathcal{M}_p$ and $\alpha \in (p, p+1)$. We shall use Theorem 2.3.1 to derive this case. The other cases can be dealt in exactly similar fashion using the four other results stated Section 2.3. Using the relation (2.3.1) we define the remainder term of the K-transform in the following obvious way:

$$r_{K_\mu}\left(\frac{1}{z}\right) = r_{\frac{1}{B_\mu}}(z). \tag{2.4.1}$$

For $\alpha \in (p, p+1)$ using Theorem (2.3.1) and taking imaginary and real parts of (2.4.1), we have

$$\Im r_{K_\mu}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu(y,\infty) \text{ and}$$
$$\Re r_{K_\mu}(iy) \sim -\frac{\pi(p+2-\alpha)/2}{\sin(\pi(\alpha-p)/2)} y^p \mu(y,\infty) \tag{2.4.2}$$

respectively.

An analogous equation for measure ν can be derived with μ being replaced by ν in (2.4.2). Now the equation (2.2.2) and the definition of the remainder term gives, $r_{K_{\mu\uplus\nu}}(z) = r_{K_\mu}(z) + r_{K_\nu}(z)$. Therefore

$$\Im r_{K_{\mu\uplus\nu}}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)}(1+c)\,y^p\mu(y,\infty) \text{ as } y\to\infty \text{ and} \tag{2.4.3}$$
$$\Re r_{K_{\mu\uplus\nu}}(iy) \sim -\frac{\pi(p+2-\alpha)/2}{\sin(\pi(\alpha-p)/2)}(1+c)\,y^p\mu(y,\infty) \text{ as } y\to\infty,$$

which are regularly varying of index $-(\alpha-p)$ with $\Im r_{K_{\mu\uplus\nu}}(iy) \approx \Re r_{K_{\mu\uplus\nu}}(iy)$ and we conclude $\mu\uplus\nu \in \mathcal{M}_p$ by looking at the remainder term of $K_{\mu\uplus\nu}$. Now again using Theorem 2.3.1 for the measure $\mu\uplus\nu$, we have

$$\Im r_{K_{\mu\uplus\nu}}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)}y^p\mu\uplus\nu(y,\infty) \text{ as } y\to\infty. \tag{2.4.4}$$

Combining (2.4.3) and (2.4.4) the result follows. □

The proof of Theorem 2.2.2 is immediate using induction which we briefly indicate below.

Proof of theorem 2.2.2. Let μ be regularly varying of tail index $-\alpha$ and supported on $[0,\infty)$. We prove

$$\mu^{\uplus n}(y,\infty) \sim n\mu(y,\infty) \text{ as } y\to\infty. \tag{2.4.5}$$

by induction on n. For $n=2$, (2.4.5) follows from the Lemma 2.4.1 with both the measures taken to be μ and $c=1$. To prove (2.4.5) for $n=m+1$ assuming $n=m$ we take $c=m$ and $\nu=\mu^{\uplus n}$ in Lemma 2.4.1. □

2.4.1 Proof of Proposition 2.2.4 and Theorem 2.2.6

Proof of Proposition 2.2.4. Using (2.2.4) and (2.2.5) we have for any $y\in[0,\infty)$,

$$\begin{aligned} F^{\cup\!\!\!\!\vee n}(y) &= X^{-1}\big((X(F(y))^n\big) \\ &= X^{-1}\left(\exp\left(n-\frac{n}{F(y)}\right)\right) \\ &= \frac{1}{1-\log\left(\exp\left(n-\frac{n}{F(y)}\right)\right)} \end{aligned}$$

$$= \frac{F(y)}{n-(n-1)F(y)}.$$

Thus for any $y \geq 0$,

$$\overline{F^{\cup\mkern-13mu\vee n}}(y) = 1 - F^{\cup\mkern-13mu\vee n}(y) = 1 - \frac{F(y)}{n-(n-1)F(y)} = \frac{n\overline{F}(y)}{1+(n-1)\overline{F}(y)}.$$

Now noting the fact that $\overline{F}(y) \to 0$ as $y \to \infty$, we have as $y \to \infty$

$$\overline{F^{\cup\mkern-13mu\vee n}}(y) \sim n\overline{F}(y) \sim \overline{F^{\uplus n}}(y). \tag{2.4.6}$$

The last asymptotic follows from the definition of Boolean-subexponentiality.

The asymptic (2.2.6) follows by combining (2.4.6), Proposition 1.1 of [59](in particular for any $n \in \mathbb{N}$, $\overline{F^{\boxed{\vee} n}}(y) \sim n\overline{F}(y)$ as $y \to \infty$), Lemma 3.8 of [64] and the fact that regularly varying distributions are classical, free and Boolean subexponential. □

Proof of Theorem 2.2.6. The proof is obvious for $t = 0$. Initially we shall prove the result for integer points $t = n$, $n \in \mathbb{N}$. After that we shall extend the proof for any non-negative real number t. We start with letting μ to be regularly varying of tail index $-\alpha$. Again to keep the exposition simple we derive the result when $\alpha \in (p, p+1)$ for some $p \in \mathbb{N}$. In the other cases the result follows by similar argument using corresponding asymptotic relations.

We have from (2.2.8),

$$K_{\mathbf{B}_n(\mu)^{\uplus(1+n)}}(iy) = K_{\mu^{\boxplus(1+n)}}(iy).$$

Therefore using (2.2.2) we can write the above expression as

$$(1+n)K_{\mathbf{B}_n(\mu)}(iy) = K_{\mu^{\boxplus(1+n)}}(iy). \tag{2.4.7}$$

Since μ has regularly varying tail of index $-\alpha$, from Theorem 1.1 of [59] we can conclude that μ is free subexponential, i.e.

$$\mu^{\boxplus(1+n)}(y,\infty) \sim (1+n)\mu(y,\infty) \text{ as } y \to \infty, \tag{2.4.8}$$

which shows that the probability measure $\mu^{\boxplus(1+n)} \in \mathcal{M}_+$ is also regularly varying of index $-\alpha$ at infinity. Therefore, using the relation between the remainder term of K and B transforms (see

(2.4.1)) and Theorem 2.3.1 we get as $y \to \infty$,

$$\Im r_{K_{\mu^{\boxplus(1+n)}}}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu^{\boxplus(1+n)}(y,\infty), \tag{2.4.9}$$

which is regularly varying of index $-(\alpha - p)$. Now taking imaginary parts of the remainder terms on both sides of (2.4.7) and using (2.4.9), we get

$$\begin{aligned}(1+n)\Im r_{K_{\mathbf{B}_n(\mu)}}(iy) &= \Im r_{K_{\mu^{\boxplus(1+n)}}}(iy) \\ &\sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu^{\boxplus(1+n)}(y,\infty) \\ &\overset{(2.4.8)}{\sim} -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)}(1+n) y^p \mu(y,\infty),\end{aligned}$$

Therefore

$$\Im r_{K_{\mathbf{B}_n(\mu)}}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu(y,\infty), \tag{2.4.10}$$

which is again regularly varying of index $-(\alpha-p)$. Similar calculations by taking the real part in place of imaginary part gives $\Re r_{K_{\mathbf{B}_n(\mu)}}(iy)$ is regularly varying of index $-(\alpha - p)$, in particular we shall then have $\Im r_{K_{\mathbf{B}_n(\mu)}}(iy) \approx \Re r_{K_{\mathbf{B}_n(\mu)}}(iy)$. Now the definition of both additive Boolean convolution and the map $\mathbf{B}$ allows us to deduce that $\mathbf{B}_n(\mu) \in \mathcal{M}_p$ if and only if $\mu \in \mathcal{M}_p$. Thus applying (2.3.7) we get $\mathbf{B}_n(\mu)$ is regularly varying of index $-\alpha$ and

$$\Im r_{K_{\mathbf{B}_n(\mu)}}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mathbf{B}_n(\mu)(y,\infty). \tag{2.4.11}$$

Hence from (2.4.10) and (2.4.11) we have as $y \to \infty$,

$$\mu(y,\infty) \sim \mathbf{B}_n(\mu)(y,\infty).$$

Conversely, suppose that $\mathbf{B}_n(\mu)$ is regularly varying of index $-\alpha$. Here also we further suppose that $\alpha \in (p, p+1)$ with $p \geq 0$. When $\alpha = p$ or $\alpha = p+1$ the conclusion can be made by using similar arguments and corresponding asymptotic relationships from [59]. Now using Theorem 2.2.2 we have, $\mathbf{B}_n(\mu)$ is Boolean-subexponential, i.e.;

$$\mathbf{B}_n(\mu)^{\uplus(1+n)}(y,\infty) \sim (1+n)\mathbf{B}_n(\mu)(y,\infty), \tag{2.4.12}$$

which also shows that $\mathbf{B}_n(\mu)^{\uplus(1+n)}$ is regularly varying of tail index $-\alpha$. Let us recall the

Voiculescu transform of a measure μ. It is known from [28] that $F_\mu = 1/G_\mu$ has a left inverse F_μ^{-1} (defined on a suitable domain) and $\phi_\mu(z) = F_\mu^{-1}(z) - z$. For probability measures μ and ν one has $\phi_{\mu\boxplus\nu}(z) = \phi_\mu(z) + \phi_\nu(z)$ on an appropriate domain. The asymptotics of remainder of ϕ_μ were derived in [59]. We can write from (2.2.8),

$$\Im r_{\phi_{\mu^{\boxplus(1+n)}}}(iy) = \Im r_{\phi_{\mathbf{B}_n(\mu)^{\uplus(1+n)}}}(iy), \tag{2.4.13}$$

where $r_{\phi_\mu}(z)$ is the remainder term of the Voiculescu transform of μ. Now using the fact that $\mathbf{B}_n(\mu)^{\uplus(1+n)}$ is regularly varying of tail index $-\alpha$, we get by applying Theorem 2.1 of [59] $\Im r_{\phi_{\mathbf{B}_n(\mu)^{\uplus(1+n)}}}(iy)$ is regularly varying of index $-(\alpha - p)$ and

$$\begin{aligned}\Im r_{\phi_{\mathbf{B}_n(\mu)^{\uplus(1+n)}}}(iy) &\sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mathbf{B}_n(\mu)^{\uplus(1+n)}(y,\infty), \\ &\overset{(2.4.12)}{\sim} -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p (1+n)\mathbf{B}_n(\mu)(y,\infty).\end{aligned} \tag{2.4.14}$$

Now combining (2.4.13), (2.4.14) and using the fact that $r_{\phi_{\mu\boxplus\nu}}(z) = r_{\phi_\mu}(z) + r_{\phi_\nu}(z)$, we have

$$\Im r_{\phi_\mu}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mathbf{B}_n(\mu)(y,\infty). \tag{2.4.15}$$

This shows that $\Im r_{\phi_\mu}(iy)$ is regularly varying of index $-(\alpha - p)$ and again applying Theorem 2.1 of [59], we get

$$\Im r_{\phi_\mu}(iy) \sim -\frac{\pi(p+1-\alpha)/2}{\cos(\pi(\alpha-p)/2)} y^p \mu(y,\infty). \tag{2.4.16}$$

Combining (2.4.15) and (2.4.16) it follows μ is regularly varying of tail index $-\alpha$ and as $y \to \infty$,

$$\mu(y,\infty) \sim \mathbf{B}_n(\mu)(y,\infty).$$

Therefore we are done for the integer case. Further we recall the definitions of $\mu^{\boxplus t}$ and $\mu^{\uplus t}$ form [18] (See also [84] and [17] for more details):

For any $t \geq 1$ and $\mu \in \mathcal{M}_+$ there exists $\mu^{\boxplus t} \in \mathcal{M}_+$ satisfying $\phi_{\mu^{\boxplus t}}(z) = t\phi_\mu(z)$ and thus

$$r_{\phi_{\mu^{\boxplus t}}(z)} = t \cdot r_{\phi_\mu}(z) \tag{2.4.17}$$

on a truncated angular domain (e.g. z with $1/z \in \Delta_{\kappa,\delta}$ for some positive κ, δ) where they are well defined. Also for any $t \geq 0$ and $\mu \in \mathcal{M}_+$ there exists $\mu^{\uplus t} \in \mathcal{M}_+$ such that $K_{\mu^{\uplus t}}(z) = tK_\mu(z)$

on the upper half plane. So

$$r_{K_{\mu^{\uplus t}(z)}} = t \cdot r_{K_\mu}(z). \tag{2.4.18}$$

Now successive use of (2.4.17) and Theorem $2.1 - 2.4$ of [59] according to suitable cases, the fact $\mu \in \mathcal{M}_p$ implies $\mu^{\boxplus t} \in \mathcal{M}_p$ which follows from Theorem 1.5 of [22] and the definition of K transform, we can say that any probability measure μ with regularly varying tail of index $-\alpha$, $\alpha \geq 0$ is more than free subexponential, i.e.,

1) If μ is regularly varying of tail index $-\alpha$ then $\mu^{\boxplus t}$ is also so for all $t \geq 1$. In particular, as $y \to \infty$, we have $\mu^{\boxplus t}(y, \infty) \sim t\mu(y, \infty)$.

Also using (2.4.18), Theorem 2.3.1-2.3.5 for respective cases and the fact that $\mu^{\uplus t} \in \mathcal{M}_p$ whenever $\mu \in \mathcal{M}_p$, we shall be able to conclude that

2) If μ is regularly varying of tail index $-\alpha$ then $\mu^{\uplus t}$ is also so for all $t \geq 0$. In particular, as $y \to \infty$, we have $\mu^{\uplus t}(y, \infty) \sim t\mu(y, \infty)$.

Now the above facts and similar calculations done in the above proof for the integer case gives us the result for any non-negative real number t. □

2.5 Some results on $\uplus$ and the proof of Theorem 2.2.7

In this short section we will prove Theorem 2.2.7 using the relation of B-transform with the multiplicative Boolean convolution. We begin by observing that when μ has p moments and ν has q moments, then the Boolean multiplicative convolution of μ and ν has exactly p moments if $p \leq q$.

Lemma 2.5.1. *Suppose $p \leq q$, $\mu \in \mathcal{M}_p$ and $\nu \in \mathcal{M}_q$, then $\mu \uplus \nu \in \mathcal{M}_p$.*

Proof. Note that when μ has infinite mean the result is obvious. Now suppose μ and ν both have finite mean then from the definition of the remainder terms we can write the B-transforms of μ and ν in the following way

$$f_1(z) := \frac{1}{m(\mu)B_\mu}(z) = 1 + c_1 z + \cdots + c_{p-1} z^{p-1} + z^{p-1} r_{f_1}(z)$$

where $c_i, i = 1, 2, \dots, p-1$ are real constants and

$$f_2(z) := \frac{1}{m(\nu)B_\nu}(z) = 1 + d_1 z + \cdots + d_{p-1}z^{p-1} + \cdots + d_{q-1}z^{q-1} + z^{q-1}r_{f_2}(z)$$

where $d_j, j = 1, 2, \dots q-1$ are also real constants. Taking the product of $f_1(z)$ and $f_2(z)$ we see that $\frac{1}{B}$-transform of $\mu \boxtimes \nu$ has a Taylor series expansion of order $p-1$ by (2.2.12). Now it is easy to see that $\mu \in \mathcal{M}_p$ is equivalent to $\frac{1}{B_\mu}(z)$ having a Taylor series expansion of order $p-1$ (see Theorem 1.5 of [22]). Therefore we see that $\mu \boxtimes \nu \in \mathcal{M}_p$. $\square$

Following equation (2.2.12) and using the results in Section 2.3 we shall derive the relation between the real or imaginary parts of the remainder terms of the product of two B-transforms. The theorem is split into several cases depending on the existence of integer moments of the two measures involved.

Theorem 2.5.2. *Suppose $\alpha \le \beta$ and let μ and ν be regularly varying with indices $-\alpha$ and $-\beta$ respectively. So there exists a non-negative integer p such that $\alpha \in [p, p+1]$ and $\mu \in \mathcal{M}_p$. We also suppose that ν has finite first moment.*[1] *Then we have the following:*

(i) *Suppose $\alpha < \beta$.*

(a) If $1 \le p < \alpha < p+1$ or $\alpha \in (0,1)$, then as $y \to \infty$

$$\Im r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \sim m(\nu)\,\Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \text{ and}$$
$$\Re r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \sim m(\nu)\,\Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right).$$

(b) If $p \ge 1$, $\alpha = p$, then as $y \to \infty$

$$\Im r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \sim m(\nu)\,\Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \text{ and}$$
$$\Re r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \gg y^{-1}.$$

(c) If $p = 0$, $\alpha = 1$, $r \in (0, 1/2)$, then as $y \to \infty$

$$\Im r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \sim m(\nu)\,\Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \text{ and}$$
$$y^{-1} \ll -y^{-1}\Re r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \ll y^{-(1-r/2)}.$$

[1] this is needed to define multiplicative Boolean convolution

(d) *If* $p \geq 1$, $\alpha = p+1$, $r \in (0, 1/2)$, *then as* $y \to \infty$

$$\Re r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \sim m(\nu)\, \Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \text{ and}$$
$$y^{-1} \ll \Im r_{\frac{1}{B_\mu B_\nu}}\left(-iy^{-1}\right) \ll y^{-(1-r/2)}.$$

(ii) *Suppose* $\alpha = \beta$ *and there exists some* $c \in (0, \infty)$ *such that* $\nu(x, \infty) \sim c\mu(x, \infty)$. *Then* ((i)a)*(with only* $p \geq 1$*),* ((i)b) *and* ((i)d) *holds with* $m(\nu)$ *is replaced by* $(1+c)m(\nu)$ *at each places.*

We shall provide a detailed proof of this result in Section 2.6. In the following, Theorem 2.2.7 for multiplicative Boolean convolution is proved using the above result.

Proof of Theorem 2.2.7. Suppose $\mu \in \mathcal{M}_p$, $1 \leq p < \alpha < p+1$ and $\nu \in \mathcal{M}_q$, $q \geq p$ with $\alpha < \beta$ then using (2.2.12) and ((i)a) of Theorem 2.5.2 we have as $y \to \infty$,

$$\Im r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right) \sim m(\nu)\, \Im r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right) \text{ and}$$
$$\Re r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right) \sim m(\nu)\, \Re r_{\frac{1}{B_\mu}}\left(-iy^{-1}\right).$$

Therefore using (2.3.7), (2.3.8) and above asymptotics, we get

$$\Im r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right) \sim -\frac{\pi\,(p+1-\alpha)\,/2}{\cos\left(\pi\,(\alpha-p)\,/2\right)} y^p m(\nu)\, \mu(y, \infty) \text{ and} \tag{2.5.1}$$
$$\Re r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right) \sim -\frac{\pi\,(p+2-\alpha)\,/2}{\sin\left(\pi\,(\alpha-p)\,/2\right)} y^p m(\nu)\, \mu(y, \infty) \text{ as } y \to \infty.$$

So $\Im r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right) \approx \Re r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right)$ and both are regularly varying $-(\alpha - p)$. By Lemma 2.5.1, we have $\mu \uplus \nu \in \mathcal{M}_p$ and therefore by applying the reverse implication of Theorem 2.3.1 we get $\mu \uplus \nu$ is regularly varying of index $-\alpha$ and again using the asymptotic equivalence (2.3.7) of Theorem 2.3.1 for the measure $\mu \uplus \nu$ we have,

$$\Im r_{\frac{1}{B_{\mu\uplus\nu}}}\left(-iy^{-1}\right) \sim -\frac{\pi\,(p+1-\alpha)\,/2}{\cos\left(\pi\,(\alpha-p)\,/2\right)} y^p \mu \uplus \nu\,(y, \infty)\,. \tag{2.5.2}$$

Hence from (2.5.1) and (2.5.2) we get $\mu \uplus \nu\,(y, \infty) \sim m(\nu)\, \mu(y, \infty)$. The other cases can be dealt similarly after employing Theorem 2.5.2 and the remaining four theorems in Section 2.3. We skip the details. □

Remark 2.5.3. *As we have mentioned in the beginning of Theorem 2.2.7, we have not dealt with the case when μ and ν both have the same regularly varying tail index but they are not tail balanced which can happen only in the case when $p \geq 1$, μ is in $\mathcal{M}_p$ and ν is in $\mathcal{M}_{p+1}$ but they are both regularly varying of tail index $-(p+1)$. In this case similar calculations will show that $\mu \uplus \nu\,(y,\infty) \sim m\,(\nu)\,\mu\,(y,\infty)$, i.e., the constant in the left of the asymptotic is 1 instead of the form $1+c$.*

2.6 Proofs of Theorem 2.3.1 to 2.3.5 and Theorem 2.5.2

To keep this chapter self contained we recall Theorem $2.1-2.4$ from [59]. The results there gave the relation between μ and the remainder of Cauchy transform. We use the equation (2.3.5) to rewrite them in terms of remainder of Ψ_μ:

Whenever p is a positive integer and $\alpha \in (p, p+1)$.

Theorem 2.6.1. *Let μ be in $\mathcal{M}_p, p \geq 1$ and $p < \alpha < p+1$. The following are equivalent:*

(i) $\mu\,(y,\infty)$ *is regularly varying of index* $-\alpha$.

(ii) $\Im r_{\Psi_\mu}\left(-iy^{-1}\right)$ *is regularly varying of index* $-(\alpha - p)$ *and*

$$\Re r_{\Psi_\mu}\left(-iy^{-1}\right) \approx \Im r_{\Psi_\mu}\left(-iy^{-1}\right).$$

If any of the above statements holds, we also have, as $z \to 0$ n.t.,

$$z \ll r_{\Psi_\mu}(z);$$

as $y \to \infty$

$$\Im r_{\Psi_\mu}\left(-iy^{-1}\right) \sim -\frac{\pi\,(p+1-\alpha)\,/2}{\cos\left(\pi\,(\alpha-p)\,/2\right)} y^p \mu\,(y,\infty) \gg y^{-1}$$

and

$$\Re r_{\Psi_\mu}\left(-iy^{-1}\right) \sim -\frac{\pi\,(p+2-\alpha)\,/2}{\sin\left(\pi\,(\alpha-p)\,/2\right)} y^p \mu\,(y,\infty) \gg y^{-1}.$$

When p is a positive integer and $\alpha = p$.

Theorem 2.6.2. *Let μ be in $\mathcal{M}_p, p \geq 1$ and $\alpha = p$. The following are equivalent:*

(i) $\mu(y,\infty)$ *is regularly varying of index* $-p$.

(ii) $\Im r_{\Psi_\mu}\left(-iy^{-1}\right)$ *is slowly varying and*

$$\Re r_{\Psi_\mu}\left(-iy^{-1}\right) \gg y^{-1}.$$

If any of the above statements holds, we also have, as $z \to 0$ *n.t.,*

$$z \ll r_{\Psi_\mu}(z);$$

as $y \to \infty$

$$\Im r_{\Psi_\mu}\left(-iy^{-1}\right) \sim -\frac{\pi}{2} y^p \mu(y,\infty) \gg y^{-1} \tag{2.6.1}$$

If we consider $\alpha \in [0,1)$, then

Theorem 2.6.3. *Let* μ *be in* $\mathcal{M}_0$ *and* $0 \le \alpha < 1$. *The following are equivalent:*

(i) $\mu(y,\infty)$ *is regularly varying of index* $-\alpha$.

(ii) $\Im\Psi_\mu\left(-iy^{-1}\right)$ *is regularly varying of index* $-\alpha$ *and*

$$\Re\Psi_\mu\left(-iy^{-1}\right) \approx \Im\Psi_\mu\left(-iy^{-1}\right).$$

If any of the above statements holds, we also have, as $z \to 0$ *n.t.,*

$$z \ll \Psi_\mu(z);$$

as $y \to \infty$

$$\Im\Psi_\mu\left(-iy^{-1}\right) \sim -\frac{\pi(1-\alpha)/2}{\cos(\pi\alpha/2)} \mu(y,\infty) \gg y^{-1}$$

and

$$\Re\Psi_\mu\left(-iy^{-1}\right) \sim -d_\alpha \mu(y,\infty) \gg y^{-1}$$

where d_α *is as in* (2.3.3).

Finally, when $p \ge 0$ and $\alpha = p+1$.

Theorem 2.6.4. *Let μ be in $\mathcal{M}_p, p \geq 1$ and $\alpha = p+1, r \in (0,1/2)$. The following are equivalent:*

(i) $\mu(y,\infty)$ *is regularly varying of index* $-(p+1)$.

(ii) $\Re r_{\Psi_\mu}\left(-iy^{-1}\right)$ *is regularly varying of index* -1 *and*

$$y^{-1} \ll \Im r_{\Psi_\mu}\left(-iy^{-1}\right) \ll y^{-(1-r/2)}.$$

If any of the above statements holds, we also have, as $z \to 0$ *n.t.,*

$$z \ll r_{\Psi_\mu}(z);$$

as $y \to \infty$

$$y^{-(1+r/2)} \ll \Re r_{\Psi_\mu}\left(-iy^{-1}\right) \sim -\frac{\pi}{2} y^p \mu(y,\infty) \ll y^{-(1-r/2)}.$$

To study the relation between the remainder terms of Ψ and η transforms we consider the following classes of functions which contains Ψ_μ depending on regular variation of μ. We shall show that the classes are closed under certain operations. Let $\mathcal{H}$ denote the set of analytic functions A having a domain $\mathcal{D}_A$ such that for all positive κ, there exists $\delta > 0$ with $\Delta_{\kappa,\delta} \subset \mathcal{D}_A$.

Definition 2.6.5. *Let $Z_{1,p}$ denote the set of all $A \in \mathcal{H}$ which satisfies the following conditions:*

(i) *For* $p \geq 0$, *A has Taylor series expansion with real coefficients of the form*

$$A(z) = \sum_{j=1}^{p} a_j z^j + z^p r_A(z)$$

where $a_1, \cdots, a_p$ are real numbers and for $p = 0$ we interpret the term in the sum as absent.

(ii) $z \ll r_A(z) \ll 1$ *as* $z \to 0$ *n.t.*

(iii) $\Re r_A(-iy^{-1}) \approx \Im r_A(-iy^{-1})$ *as* $y \to \infty$.

Let $Z_{2,p}$ be the same as $Z_{1,p}$ with (R(i)) and (R(ii)) but (R(iii)) is replaced by

(i) $y^{-(1+r/2)} \ll \Re r_A(-iy^{-1}) \ll y^{-(1-r/2)}$ *and* $y^{-1} \ll \Im r_A(-iy^{-1}) \ll y^{-(1-r/2)}$ *for any* $r \in (0,1/2)$.

Let $Z_{3,p}$ be the same as $Z_{1,p}$ with (R(i)) for $p \geq 1$ and same (R(ii)) but (R(iii)) is replaced by

(i) $\Re r_A(-iy^{-1}) \gg y^{-1}$ *and* $\Im r_A(-iy^{-1}) \gg y^{-1}$.

Remark 2.6.6. *Suppose that $\mu(y,\infty)$ is regularly varying $-\alpha$ and $\mu \in \mathcal{M}_p$ with $\alpha \in [p, p+1]$. Then note the following:*

(i) *If $p \geq 1$, $p < \alpha < p+1$ or $p = 0$, $0 \leq \alpha < 1$, then $\Psi_\mu(z) \in Z_{1,p}$ (follows from Theorem 2.6.1 and Theorem 2.6.3).*

(ii) *If $p \geq 0$, $\alpha = p+1$, then $\Psi_\mu(z) \in Z_{2,p}$ (follows from Theorem 2.6.4).*

(iii) *If $p \geq 1$, $\alpha = p$, then $\Psi_\mu(z) \in Z_{3,p}$ (follows from Theorem 2.6.2).*

Hence the proposition 2.6.7, given below, allows us to conclude that $\Psi_\mu(z) \in Z_{i,p}$ if and only if $\eta_\mu(z) \in Z_{i,p}$ for any fixed $i \in \{1,2,3\}$ and $p \in \{0,1,2,\cdots\}$.

Proposition 2.6.7. *For any fixed $i \in \{1,2,3\}$ and $p \geq 0$ (excluding $Z_{3,0}$ as this set is not defined), if $A(z) \in Z_{i,p}$ then $B(z) = A(z)(1 \pm A(z))^{-1} \in Z_{i,p}$. Furthermore, we have*

(i) $r_B(z) \sim r_A(z)$, *as $z \to 0$ n.t.;*

(ii) $\Re r_B(-iy^{-1}) \sim \Re r_A(-iy^{-1})$ *as $y \to \infty$ and*

(iii) $\Im r_B(-iy^{-1}) \sim \Im r_A(-iy^{-1})$ *as $y \to \infty$.*

Proof. We shall divide this proof into some cases because depending on i and p the calculations are different. We shall only show for $B(z) = A(z)(1 + A(z))^{-1}$. Exactly same calculation will prove the result for $B(z) = A(z)(1 - A(z))^{-1}$.

(i) Suppose $A \in Z_{1,0}$. Then $r_A(z) = A(z)$. Therefore $r_B(z) = B(z)$. This shows that (R(i)) is satisfied.

Now,

$$
\begin{aligned}
B(z) &= A(z)(1 + A(z))^{-1} &\quad (2.6.2)\\
&= A(z) + \mathrm{o}(|A(z)|) \text{ as } z \to 0 \text{ n.t.} &\quad (2.6.3)
\end{aligned}
$$

Therefore, we have $B(z) \sim A(z)$ as $z \to 0$ n.t. Therefore (R(ii)) is satisfied.

From (2.6.3) we have,

$$\Re B(-iy^{-1}) = \Re A(-iy^{-1}) + \mathrm{o}(|A(-iy^{-1})|),$$
$$\Im B(-iy^{-1}) = \Im A(-iy^{-1}) + \mathrm{o}(|A(-iy^{-1})|).$$

Now to show the equivalence of the real parts and imaginary parts, it is enough to show that

$$\frac{|A(-iy^{-1})|}{\Re A(-iy^{-1})} \text{ and } \frac{|A(-iy^{-1})|}{\Im A(-iy^{-1})}$$

remains bounded as $y \to \infty$. We shall show the first part only as the second one follows by the same arguments:

$$\begin{aligned}\left(\left|\frac{A(-iy^{-1})}{\Re A(-iy^{-1})}\right|\right)^2 &= \frac{(\Re A(-iy^{-1}))^2 + (\Im A(-iy^{-1}))^2}{(\Re A(-iy^{-1}))^2} \\ &= 1 + \left(\frac{\Im A(-iy^{-1})}{\Re A(-iy^{-1})}\right)^2,\end{aligned}$$

which goes to a constant as $y \to \infty$ by the fact that A satisfies (R(iii)). Therefore (R(iii)) is satisfied for $B(z)$ and the asymptotics in the statement also remain true.

(ii) Suppose $A \in Z_{1,p}$, $p \geq 1$. We note that $|A(z)| \to 0$ as $z \to 0$ n.t. Thus we have the following series expansion using equation (2.2.10) for $B(z)$ near zero:

$$B(z) = \sum_{i=1}^{p}(-1^{i+1})(A(z))^i + \mathrm{O}\left((A(z))^{p+1}\right).$$

Using (R(i)) and (R(ii)) we get

$$\frac{(A(z))^{p+1}}{z^p r_A(z)} = \left(\frac{A(z)}{z}\right)^{p+1} \frac{z}{r_A(z)} \to 0$$

as $z \to 0$. Hence

$$B(z) = \sum_{i=1}^{p}((-1^{i+1})(\sum_{j=1}^{p} m_j z^j + z^p r_A(z))^j) + \mathrm{o}\left(z^p r_A(z)\right). \tag{2.6.4}$$

We expand the term in the right-hand side of (2.6.6). As $z \ll r_A(z)$, all powers of z with indices greater than p can be absorbed in the last term on the right-hand side. Then collect up to p-th power of z to form a polynomial $P(z)$ of degree at most p with real coefficients without the constant term. Finally we consider the terms containing some powers of $r_A(z)$

which will contain terms of the form $z^{l_1}(z^p r_A(z))^{l_2}$ for integers $l_1 \geq 0$ and $l_2 \geq 1$ with leading term $z^p r_A(z)$ and the remaining terms can be absorbed in the last term in the right-hand side. Thus we get,

$$B(z) = P(z) + z^p r_A(z) + \mathrm{o}(z^p r_A(z)).$$

Therefore (R(i)) is satisfied. Now by uniqueness of the Taylor series expansion, we have

$$r_B(z) = r_A(z) + \mathrm{o}(r_A(z)).$$

Therefore $r_B(z) \sim r_A(z)$. So (R(ii)) is satisfied. Thus

$$\Re r_B(z) = \Re r_A(z) + \mathrm{o}(|r_A(z)|),$$
$$\Im r_B(z) = \Im r_A(z) + \mathrm{o}(|r_A(z)|).$$

Now from (R(iii)) and same calculations like in the first case we get that $B(z)$ is satisfying (R(iii)).

(iii) Suppose $A \in Z_{2,0}$. Here we only need to show (R3′) as (R(i)) and (R(ii)) have been already shown in case 1. From (2.6.2),

$$B(z) = A(z) + \mathrm{O}\left(|A(z)|^2\right).$$

Consequently,

$$\Re B(z) = \Re A(z) + \mathrm{O}\left(|A(z)|^2\right),$$
$$\Im B(z) = \Im A(z) + \mathrm{O}\left(|A(z)|^2\right).$$

It is enough to show that $\frac{|A(-iy^{-1})|^2}{\Re A(-iy^{-1})}$ and $\frac{|A(-iy^{-1})|^2}{\Im A(-iy^{-1})}$ both goes to zero as $y \to \infty$. For that

$$\begin{aligned}\frac{|A(-iy^{-1})|^2}{\Re A(-iy^{-1})} &= \Re A\left(-iy^{-1}\right) + \frac{\left(\Im A\left(-iy^{-1}\right)\right)^2}{\Re A\left(-iy^{-1}\right)} \\ &= \Re A\left(-iy^{-1}\right) + \left(\frac{\Im A\left(-iy^{-1}\right)}{y^{-(1-r/2)}}\right)^2 \frac{y^{-(1+r/2)}}{\Re A\left(-iy^{-1}\right)} y^{-1+3r/2},\end{aligned}$$

which goes to zero as $y \to \infty$ using (R3′) for $A(z)$. For the other terms similarly note that

$$\begin{aligned}\frac{|A\left(-iy^{-1}\right)|^2}{\Im A\left(-iy^{-1}\right)} &= \Im A\left(-iy^{-1}\right) + \frac{\left(\Re A\left(-iy^{-1}\right)\right)^2}{\Im A\left(-iy^{-1}\right)} \\ &= \Im A\left(-iy^{-1}\right) + \left(\frac{\Re A\left(-iy^{-1}\right)}{y^{-(1-r/2)}}\right)^2 \frac{y^{-1}}{\Im A\left(-iy^{-1}\right)} y^{-1+r}\end{aligned}$$

also goes to zero as $y \to \infty$ using (R3′) for $A(z)$. Thus (R3′) is obviously satisfied by $B(z)$.

(iv) Suppose $A \in Z_{2,p}$, $p \geq 1$. Here we need to show only (R3′). From (2.6.4) we can write using a similar argument given in case (2),

$$B(z) = P(z) + z^p r_A(z) + c_1 z^{p+1} + \mathrm{O}(z^{p+1} r_A(z)).$$

Therefore,

$$r_B\left(z\right) = r_A(z) + c_1 z + \mathrm{O}(z r_A(z)). \tag{2.6.5}$$

So, $r_B\left(z\right) \sim r_A\left(z\right)$ and evaluating (2.6.5) at the point $z = -iy^{-1}$,

$$r_B\left(-iy^{-1}\right) = r_A(-iy^{-1}) + c_1(-iy^{-1}) + \mathrm{O}(z r_A(-iy^{-1}))$$

and after taking the real parts on both sides, we have

$$\Re r_B\left(-iy^{-1}\right) = \Re r_A\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}|r_A\left(-iy^{-1}\right)|\right).$$

Now,

$$\left|\frac{y^{-1}|r_A\left(-iy^{-1}\right)|}{\Re r_A\left(-iy^{-1}\right)}\right|^2 = \frac{1}{y^2} + \frac{1}{y^2}\left(\frac{\Im r_A\left(-iy^{-1}\right)}{\Re r_A\left(-iy^{-1}\right)}\right)^2$$

and

$$y^{-1}\frac{\Im r_A\left(-iy^{-1}\right)}{\Re r_A\left(-iy^{-1}\right)} = \frac{\Im r_A\left(-iy^{-1}\right)}{y^{-(1-r/2)}} \frac{y^{-(1+r/2)}}{\Re r_A\left(-iy^{-1}\right)} y^{-1+r}$$

goes to zero as $y \to \infty$ by (R3′). As a consequence we conclude that $\Re r_B\left(-iy^{-1}\right) \sim \Re r_A\left(-iy^{-1}\right)$.

For the imaginary part asymptotic we write from (2.6.5),

$$r_B(z) = r_A(z) + \mathrm{O}(|z|). \tag{2.6.6}$$

Now putting $z = -iy^{-1}$ and taking imaginary parts on both sides we get

$$\Im r_B\left(-iy^{-1}\right) = \Im r_A(-iy^{-1}) + \mathrm{O}(|y^{-1}|)$$

and note that $y\Im r_A\left(-iy^{-1}\right) \to \infty$ as $z \to \infty$ by (R3′).

Hence $\Im r_B\left(-iy^{-1}\right) \sim \Im r_A(-iy^{-1})$. So we are done in this case.

(v) Suppose $A \in Z_{3,p}$, $p \geq 1$. Here (R(i)) and (R(ii)) is shown in case (2). Now we write the following using (2.6.6),

$$\Re r_B\left(-iy^{-1}\right) = \Re r_A\left(-iy^{-1}\right) + \mathrm{O}\left(|y^{-1}|\right),$$
$$\Im r_B\left(-iy^{-1}\right) = \Im r_A\left(-iy^{-1}\right) + \mathrm{O}\left(|y^{-1}|\right).$$

Now $y\Re r_A\left(-iy^{-1}\right)$ and $y\Im r_A\left(-iy^{-1}\right)$ both go to infinity as $y \to \infty$ by (R3″). This shows that (R3″) is also satisfied by $B(z)$ in this case.

□

We give the proofs of the main theorems stated in Theorem 2.3.1–Theorem 2.3.5. We shall only prove Theorem 2.3.1 and the rest of the theorems will follow by similar arguments.

Proof of Theorem 2.3.1. Combining Theorem 2.6.1, Proposition 2.6.7 and the definition of B-transform coming out of η-transform, we get Theorem2.3.1 because the asymptotic relationship of Ψ-transform follows from Theorem 2.6.1, the relationship between Ψ and η transforms follows from the Proposition 2.6.7 (see also Definition 2.6.5 and Remark 2.6.6) and finally the correspondence between η and B transforms is ensured by equations stated in (2.3.6). □

Proof of Theorem 2.5.2. Recall that we have assumed $\mu \in \mathcal{M}_p$ is regularly varying tail index $-\alpha$, $\alpha \geq 0$ with $\alpha \in [p, p+1]$. Since ν is regularly varying with tail index $-\beta$ with $\beta \geq \alpha$ and ν has finite first moment, there exists a positive integer $q \geq 1$ with $q \geq p$ and $\nu \in \mathcal{M}_q$. The proof of this theorem is split up into different cases depending on p, q and α, β. It contains a number of subcases because the asymptotic relations and the Taylor series like expansions differ with respect to the position of α and β in the lattice of non-negative integers.

Case ((i)a) Recall that in this case we have assumed either $1 \leq p < \alpha < p+1$ or $0 = p \leq \alpha < p+1 = 1$.

Subcase (i) Let $\mu \in \mathcal{M}_0, \nu \in \mathcal{M}_q$, $2 \leq q$ and $0 \leq \alpha < 1$ and $q \leq \beta \leq q+1$. Define

$$f_1(z) := \frac{1}{B_\mu}(z) \text{ and}$$
$$f_2(z) := \frac{1}{m(\nu)B_\nu}(z) = 1 + d_1 z + \cdots + d_{q-1}z^{q-1} + z^{q-1}r_{f_2}(z),$$

where d_j, $j = 1, 2, \ldots, q-1$ are real coefficients. Therefore

$$f_1(z) f_2(z) = f_1(z) + d_1 z f_1(z) + \cdots + d_{q-1}z^{q-1}f_1(z) + z^{q-1}f_1(z) r_{f_2}(z)$$

and using Lemma 2.5.1

$$r_{f_1 f_2}(z) = f_1(z) f_2(z) = f_1(z) + \mathrm{O}(|z f_1(z)|). \tag{2.6.7}$$

Taking the imaginary parts of (2.6.7), we get

$$\Im r_{f_1 f_2}(-iy^{-1}) = \Im f_1\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}|f_1\left(-iy^{-1}\right)|\right).$$

Observe,

$$\begin{aligned} y^{-1}\frac{|f_1\left(-iy^{-1}\right)|}{\Im f_1\left(-iy^{-1}\right)} &= \left(\frac{\left(\Re f_1\left(-iy^{-1}\right)\right)^2 + \left(\Im f_1\left(-iy^{-1}\right)\right)^2}{y^2\left(\Im f_1\left(-iy^{-1}\right)\right)^2}\right)^{1/2} \\ &= \left(\frac{1}{y^2} + \frac{1}{y^2}\left(\frac{\Re f_1\left(-iy^{-1}\right)}{\Im f_1\left(-iy^{-1}\right)}\right)^2\right)^{1/2} \to 0 \text{ as } y \to \infty, \end{aligned}$$

using (2.3.11) and (2.3.12). Hence $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im f_1\left(-iy^{-1}\right)$ as $y \to \infty$. Using same arguments for real part we also have $\Re r_{f_1 f_2}(-iy^{-1}) \sim \Re f_1\left(-iy^{-1}\right)$.

Subcase (ii) Let $\mu \in \mathcal{M}_0, \nu \in \mathcal{M}_q$, $q = 1$ and $0 \leq \alpha < 1$ and $q \leq \beta \leq q+1$. The case is similar to $q \geq 2$. Here the equation (2.6.7) gets replaced by

$$r_{f_1 f_2}(-iy^{-1}) = f_1\left(-iy^{-1}\right) + f_1\left(-iy^{-1}\right) r_{f_2}\left(-iy^{-1}\right).$$

Comparing the real part and the imaginary part we have $\Re r_{f_1 f_2}(-iy^{-1}) \sim \Re f_1\left(-iy^{-1}\right)$ and $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im f_1\left(-iy^{-1}\right)$ respectively using $\frac{\Im f_1(-iy^{-1})}{\Im f_1(-iy^{-1})}, \frac{\Re f_1(-iy^{-1})}{\Re f_1(-iy^{-1})}, \frac{\Re f_1(-iy^{-1})}{\Im f_1(-iy^{-1})}$ $\sim$ a nonzero constant and $|r_{f_2}\left(-iy^{-1}\right)| \to 0$ both as $y \to \infty$. So the $p = 0$ case is done.

Subcase (iii) Here suppose $\mu, \nu \in \mathcal{M}_1$ and $1 < \alpha < \beta < 2$. Let $f_1(z) = \frac{1}{m(\mu)B_\mu}(z)$ and $f_2(z) = \frac{1}{m(\nu)B_\nu}(z)$. Then

$$f_i(z) = 1 + r_{f_i}(z) \text{ for } i = 1, 2$$

Therefore,

$$r_{f_1 f_2}(z) = r_{f_1}(z) + r_{f_2}(z) + r_{f_1}(z)\, r_{f_2}(z).$$

So,

$$\begin{aligned} r_{f_1 f_2}(-iy^{-1}) &= r_{f_1}\left(-iy^{-1}\right) + r_{f_2}\left(-iy^{-1}\right) + r_{f_1}\left(-iy^{-1}\right) r_{f_2}\left(-iy^{-1}\right) \\ &= r_{f_1}\left(-iy^{-1}\right) + \mathrm{O}\left(|r_{f_2}\left(-iy^{-1}\right)|\right). \end{aligned} \tag{2.6.8}$$

Thus taking the real and imaginary parts

$$\Im r_{f_1 f_2}(-iy^{-1}) = \Im r_{f_1}\left(-iy^{-1}\right) + \mathrm{O}\left(|r_{f_2}\left(-iy^{-1}\right)|\right)$$

and

$$\Re r_{f_1 f_2}(-iy^{-1}) = \Re r_{f_1}\left(-iy^{-1}\right) + \mathrm{O}\left(|r_{f_2}\left(-iy^{-1}\right)|\right).$$

Using (2.3.7) and (2.3.8),

$$\left(\frac{|r_{f_2}\left(-iy^{-1}\right)|}{\Im r_{f_1}\left(-iy^{-1}\right)}\right)^2 = \frac{\left(\Re r_{f_2}\left(-iy^{-1}\right)\right)^2 + \left(\Im r_{f_2}\left(-iy^{-1}\right)\right)^2}{\left(\Im r_{f_1}\left(-iy^{-1}\right)\right)^2} \sim \frac{y^{1-\beta} l_1(y)}{y^{1-\alpha} l_2(y)} \underset{y\to\infty}{\to} 0,$$

where $l_k(y)$, $k = 1, 2$ are slowly varying functions. Therefore we get $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im r_{f_1}\left(-iy^{-1}\right)$. Similarly taking the real parts one can show $\Re r_{f_1 f_2}(-iy^{-1}) \sim \Re r_{f_1}\left(-iy^{-1}\right)$.

Subcase (iv) Let $\mu, \nu \in \mathcal{M}_1$ and $1 < \alpha < 2, \beta = 2$. The case is similar to the previous one. Using (2.6.8), we get

$$\Im r_{f_1 f_2}(z) = \Im r_{f_1}(z) + \Im r_{f_2}(z) + \Im\left(r_{f_1}(z)\, r_{f_2}(z)\right).$$

So the expression for the imaginary part becomes

$$\Im r_{f_1 f_2}(z) = \Im r_{f_1}(z) + \Im r_{f_2}(z) + \Im r_{f_1}(z)\,\Re r_{f_2}(z) + \Re r_{f_1}(z)\,\Im r_{f_2}(z)\,.$$

Using similar type of arguments with the help of (2.3.17) and (2.3.15),

we get $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im r_{f_1}\left(-iy^{-1}\right)$. The real part can be dealt similarly.

Subcase (v) Suppose $\mu, \nu \in \mathcal{M}_p, p = q \geq 2$ and $p < \alpha < \beta \leq p + 1$. Write

$$f_1(z) := \frac{1}{m(\mu)B_\mu}(z) = 1 + c_1 z + \cdots + c_{p-1}z^{p-1} + z^{p-1}r_{f_1}(z)\,.$$
$$f_2(z) := \frac{1}{m(\nu)B_\nu}(z) = 1 + d_1 z + \cdots + d_{p-1}z^{p-1} + z^{p-1}r_{f_2}(z)\,,$$

where c_i and d_i, $1 \leq i \leq p-1$ are some real constants. So,

$$f_1(z)\,f_2(z) = 1 + e_1 z + \cdots + e_{p-1}z^{p-1} + z^{p-1}\left(r_{f_1}(z) + r_{f_2}(z) + \mathrm{O}(|z|)\right),$$

where $e_i, 1 \leq i \leq p-1$ are real constants. Therefore,

$$r_{f_1 f_2}(z) = r_{f_1}(z) + r_{f_2}(z) + \mathrm{O}(|z|)\,.$$

Taking imaginary and real part on both sides we obtain

$$\Im r_{f_1 f_2}(-iy^{-1}) = \Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}\right) \text{ and} \tag{2.6.9}$$
$$\Re r_{f_1 f_2}(-iy^{-1}) = \Re r_{f_1}\left(-iy^{-1}\right) + \Re r_{f_2}\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}\right)$$

respectively. When $p < \alpha < \beta < p + 1$, we can use (2.3.7) and (2.3.8) to get

$$\frac{\Im r_{f_2}\left(-iy^{-1}\right)}{\Im r_{f_1}\left(-iy^{-1}\right)} \text{ and } \frac{\Re r_{f_2}\left(-iy^{-1}\right)}{\Re r_{f_1}\left(-iy^{-1}\right)} \to 0 \text{ as } y \to \infty.$$

Also, $y\Im r_{f_1}\left(-iy^{-1}\right), y\Re r_{f_1}\left(-iy^{-1}\right) \to \infty$ as $y \to \infty$ by (2.3.7) and (2.3.8) in the respective cases.

When $p < \alpha < p+1, \beta = p+1$ we have

$$\frac{\Im r_{f_2}\left(-iy^{-1}\right)}{\Im r_{f_1}\left(-iy^{-1}\right)} = -\frac{\Im r_{f_2}\left(-iy^{-1}\right)}{y^{-(1-r/2)}}\,\frac{1}{cy^{(p-\alpha)}l\,(y)}\,\frac{1}{y^{1-r/2}},$$

where c is a constant and $l(y)$ is a slowly varying function and r as in Theorem 2.3.5. To make this quantity tend to zero as $y \to \infty$ using (2.3.7) and (2.3.17) we need $(p - \alpha + 1 - r/2) > 0$. This can be done by a suitable choice of $r \in (0, 1/2)$ since $p + 1 - \alpha > 0$. Exactly same can be done for the real parts also.

Subcase (vi) Now suppose $\mu \in \mathcal{M}_p, \nu \in \mathcal{M}_q, 1 \leq p < q$ and $p < \alpha < p+1$ and $q \leq \beta \leq q+1$. Here we have,

$$f_1(z) := \frac{1}{m(\mu)B_\mu}(z) = 1 + c_1 z + \cdots + c_{p-1}z^{p-1} + z^{p-1}r_{f_1}(z) \tag{2.6.10}$$

and

$$f_2(z) := \frac{1}{m(\nu)B_\nu}(z) = 1 + d_1 z + \cdots + d_{p-1}z^{p-1} + \cdots + d_{q-1}z^{q-1} + z^{q-1}r_{f_2}(z), \tag{2.6.11}$$

where $c_i, 1 \leq i \leq p-1$ and $d_j, 1 \leq j \leq q-1$ are some real constants. It is easy to see using $p < q$ that

$$r_{f_1 f_2}(z) = r_{f_1}(z) + \mathrm{O}(|z|). \tag{2.6.12}$$

Observe that the asymptotics follow since $y\Im r_{f_1}\left(-iy^{-1}\right), y\Re r_{f_1}\left(-iy^{-1}\right) \to \infty$ as $y \to \infty$ (using (2.3.7) and (2.3.8) respectively).

Case ((i)b) The second part of the theorem deals with the case $p \geq 1$ and $\alpha = p$. We split again the proof into several subcases.

Subcase (i) $1 \leq p = \alpha < \beta < p+1$ or $\alpha = p, p < q \leq \beta \leq q+1$. In this case we can define f_1 and f_2 as in (2.6.10) and (2.6.11) respectively and one can get the imaginary part asymptotics using the similar calculations as in the proof of ((i)a). We only need to show $\Re r_{f_1 f_2}(-iy^{-1}) \gg y^{-1}$ in the above cases. But it is obvious since $p, q \geq 1$ and $\Im r_{f_1(-iy^{-1})}, \Re r_{f_1(-iy^{-1})} \gg y^{-1}$.

Subcase (ii) Suppose $p = q = 1$, $\alpha = p$ and $\beta = p+1$. Then we have form (2.6.8), that

$$\begin{aligned}\Im r_{f_1 f_2}(-iy^{-1}) &= \Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right) + \Im r_{f_1}\left(-iy^{-1}\right)\Re r_{f_2}\left(-iy^{-1}\right) \\ &\quad + \Re r_{f_1}\left(-iy^{-1}\right)\Im r_{f_2}\left(-iy^{-1}\right) \text{ and} \\ \Re r_{f_1 f_2}(-iy^{-1}) &= \Re r_{f_1}\left(-iy^{-1}\right) + \mathrm{O}(|r_{f_2}\left(-iy^{-1}\right)|).\end{aligned} \tag{2.6.13}$$

Now using (2.3.15), (2.3.10), we get

$$\frac{\Im r_{f_2}\left(-iy^{-1}\right)}{\Im r_{f_1}\left(-iy^{-1}\right)} = \frac{\Im r_{f_2}\left(-iy^{-1}\right)}{y^{-(1-r/2)}} \frac{1}{y^{1-r/2}} \frac{1}{l(y)} \to 0 \text{ as } y \to \infty, \tag{2.6.14}$$

while observing

$$\frac{\Im r_{f_1}\left(-iy^{-1}\right) \Re r_{f_2}\left(-iy^{-1}\right)}{\Im r_{f_1}\left(-iy^{-1}\right)} = \Re r_{f_2}\left(-iy^{-1}\right) \to 0 \text{ as } y \to \infty.$$

Therefore $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im r_{f_1}\left(-iy^{-1}\right)$. We now consider the equation (2.6.13). From (2.3.9) and (2.3.16), it follows that

$$\Re r_{f_1}\left(-iy^{-1}\right) \gg y^{-1} \text{ and } y\left|r_{f_2}\left(-iy^{-1}\right)\right| = \left|\frac{r_{f_2}\left(-iy^{-1}\right)}{-iy^{-1}}\right| \gg 1.$$

These show that $\Re r_{f_1 f_2}(-iy^{-1}) \gg y^{-1}$.

Subcase (iii) Now let $p = q \geq 2$, $\alpha = p$ and $\beta = p + 1$. For imaginary part again using the equations (2.6.9), (2.3.10) and calculations as in (2.6.14) we have $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im r_{f_1}\left(-iy^{-1}\right)$. On the other hand, the real part asymptotic follows from (2.3.9), (2.3.17).

Case ((i)c) Let $\mu \in \mathcal{M}_0, \nu \in \mathcal{M}_q, 1 \leq q$ and $\alpha = 1$ and $q \leq \beta \leq q+1$ with $\alpha \neq \beta$. In this case we have from (2.6.7) and consequently, $\Im r_{f_1 f_2}(-iy^{-1}) = \Im f_1\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}|f_1\left(-iy^{-1}\right)|\right)$.

Now,

$$\begin{aligned} y^{-1}\frac{|f_1\left(-iy^{-1}\right)|}{\Im f_1\left(-iy^{-1}\right)} &= \left(\frac{\left(\Re f_1\left(-iy^{-1}\right)\right)^2 + \left(\Im f_1\left(-iy^{-1}\right)\right)^2}{y^2\left(\Im f_1\left(-iy^{-1}\right)\right)^2}\right)^{1/2} \\ &= \left(\frac{1}{y^2} + \frac{1}{y^2}\left(\frac{\Re f_1\left(-iy^{-1}\right)}{\Im f_1\left(-iy^{-1}\right)}\right)^2\right)^{1/2} \to 0 \text{ as } y \to \infty \end{aligned}$$

because

$$\begin{aligned} y^{-1}\frac{\Re f_1\left(-iy^{-1}\right)}{\Im f_1\left(-iy^{-1}\right)} &= y^{-1}\frac{\Re f_1\left(-iy^{-1}\right)}{y^{r/2}} \frac{y^{r/2}}{\Im f_1\left(-iy^{-1}\right)} \\ &= \frac{\Re f_1\left(-iy^{-1}\right)}{y^{r/2}} \frac{1}{y^{1-r/2}\Im f_1\left(-iy^{-1}\right)} \to 0 \text{ as } y \to \infty \end{aligned}$$

with the help of (2.3.14) and (2.3.13). Therefore $\Im r_{f_1 f_2}(-iy^{-1}) \sim \Im f_1\left(-iy^{-1}\right)$.

Again from (2.6.7), $\Re r_{f_1f_2}(-iy^{-1}) = \Re f_1\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}|f_1\left(-iy^{-1}\right)|\right)$. Now if we are able to show that $\Re r_{f_1f_2}(-iy^{-1}) \sim \Re f_1\left(-iy^{-1}\right)$, then we are done. For that we proceed exactly as in the case of imaginary part of this case. It is enough to show that $y^{-1}\frac{\Im f_1(-iy^{-1})}{\Re f_1(-iy^{-1})} \to 0$ as $y \to \infty$ to get the required result. Note in this case $\Re f_1\left(-iy^{-1}\right) \to \infty$ and using (2.3.14) and (2.3.13), we get

$$y^{-1}\frac{\Im f_1\left(-iy^{-1}\right)}{\Re f_1\left(-iy^{-1}\right)} = y^{-1+r/2}\frac{\Im f_1\left(-iy^{-1}\right)}{y^{r/2}}\frac{1}{\Re f_1\left(-iy^{-1}\right)} \to 0 \text{ as } y \to \infty.$$

Case ((i)d) Suppose $\mu \in \mathcal{M}_p, \nu \in \mathcal{M}_q, 1 \le p < q$ and $\alpha = p+1$ and $q \le \beta \le q+1$ with $\alpha \neq \beta$. In this case we have $\Im r_{f_1f_2}(-iy^{-1}) = \Im r_{f_1}\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}\right)$ from (2.6.12). $y\Im r_{f_1}\left(-iy^{-1}\right) \to \infty$ as $y \to \infty$ by (2.3.15). Therefore $\Im r_{f_1f_2}(-iy^{-1}) \sim \Im r_{f_1}\left(-iy^{-1}\right)$.

For the real part calculations we recall (2.6.10), (2.6.11) and write the remainder term of $f_1f_2(z)$ in the following way:

$$r_{f_1f_2}(z) = r_{f_1}\left(z\right) + dz + zr_{f_1}\left(z\right) + zr_{f_2}\left(z\right) + \mathrm{O}\left(|z^2|\right),$$

where d is some real constant. We note that the term $zr_{f_2}\left(z\right)$ may or may not occur in the above expression depending on the value of $q-p=1$ or >1. Therefore

$$\Re r_{f_1f_2}(-iy^{-1}) = \Re r_{f_1}\left(-iy^{-1}\right) - \frac{\Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right)}{y} + \mathrm{O}\left(y^{-2}\right).$$

Using (2.3.16), $y^2\Re r_{f_1}\left(-iy^{-1}\right) \to \infty$ as $y \to \infty$. For the term in the middle after dividing by $\Re r_{f_1}(-iy^{-1})$ we observe the following:

When $q \le \beta < q+1$ the numerator is regularly varying with tail index between $(-1, 0]$ while the denominator is slowly varying and this allows us to conclude that the term

$$\frac{\Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right)}{y\Re r_{f_1}\left(-iy^{-1}\right)} \to 0 \text{ as } y \to \infty.$$

Also when $\beta = q+1$ we can write

$$\frac{\Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right)}{y\Re r_{f_1}\left(-iy^{-1}\right)} = \frac{1}{y^{1-r}}\frac{\Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right)}{y^{-(1-r/2)}}\frac{y^{-(1+r/2)}}{\Re r_{f_1}\left(-iy^{-1}\right)}$$

$\to 0$ as $y \to \infty$, using (2.3.17) and (2.3.15). Thus $\Re r_{f_1f_2}(-iy^{-1}) \sim \Re r_{f_1}\left(-iy^{-1}\right)$. □

Proof of Theorem 2.5.2 ((ii)). In this case we assume $\alpha = \beta$, $\mu, \nu \in \mathcal{M}_p$ and $\nu(x,\infty) \sim c\mu(x,\infty)$ for some $c \in (0,\infty)$. The methods are similar to the previous one, so we shall briefly sketch these proofs.

Case (i) Suppose $p \geq 1$ and $p < \alpha = \beta < p+1$.

Subcase (i) First suppose $p = 1$. Then from (2.6.8), we have

$$r_{f_1 f_2}(-iy^{-1}) = r_{f_1}\left(-iy^{-1}\right) + r_{f_2}\left(-iy^{-1}\right) + \mathrm{o}\left(\left|r_{f_1}\left(-iy^{-1}\right)\right|\right).$$

Taking imaginary parts on both sides we get

$$\Im r_{f_1 f_2}(-iy^{-1}) = \Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right) + \mathrm{o}\left(\left|r_{f_1}\left(-iy^{-1}\right)\right|\right).$$

Using (2.3.7), (2.3.8) and the tail equivalence condition we derive

$$\frac{\Im r_{f_2}\left(-iy^{-1}\right)}{\Im r_{f_1}\left(-iy^{-1}\right)} \to c \text{ and } \left(\frac{\left|r_{f_1}\left(-iy^{-1}\right)\right|}{\Im r_{f_1}\left(-iy^{-1}\right)}\right) \to (1+c^2)^{1/2} \text{ as } y \to \infty.$$

Therefore we have $\Im r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Im r_{f_1}\left(-iy^{-1}\right)$. Exactly same calculations taking the real part into consideration gives $\Re r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Re r_{f_1}\left(-iy^{-1}\right)$.

Subcase (ii) Suppose $p \geq 2$. Then we have the equation (2.6.9), given by

$$\Im r_{f_1 f_2}(-iy^{-1}) = \Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}\right).$$

Subsequently using (2.3.7) we get as $y \to \infty$

$$\frac{\Im r_{f_2}\left(-iy^{-1}\right)}{\Im r_{f_1}\left(-iy^{-1}\right)} \to c \text{ and } y\Im r_{f_1}\left(-iy^{-1}\right) \to \infty.$$

Thus $\Im r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Im r_{f_1}\left(-iy^{-1}\right)$ and exactly same calculations with real parts give $\Re r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Re r_{f_1}\left(-iy^{-1}\right)$.

Case (ii) When $\alpha = p$, similar calculations like above provides $\Im r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Im r_{f_1}\left(-iy^{-1}\right)$ since we have the same imaginary part asymptotics in this case also. The calculations done in the proof of ((i)b) assures us that $\Re r_{f_1 f_2}(-iy^{-1}) \gg y^{-1}$.

Case (iii). Here we suppose $p \geq 1$, $\alpha = \beta = p+1$.

Subcase (i) First consider $p = 1$. From (2.6.8), we have,

$$\Re r_{f_1 f_2}(-iy^{-1}) = \Re r_{f_1}\left(-iy^{-1}\right) + \Re r_{f_2}\left(-iy^{-1}\right) + \Re(r_{f_1}\left(-iy^{-1}\right) r_{f_2}\left(-iy^{-1}\right)).$$

From the identity (2.3.17), it appears that,

$$\frac{\Re r_{f_2}\left(-iy^{-1}\right)}{\Re r_{f_1}\left(-iy^{-1}\right)} \to c \text{ as } y \to \infty.$$

Also it follows that,

$$\frac{\Re r_{f_1}\left(-iy^{-1}\right) \Re r_{f_2}\left(-iy^{-1}\right)}{\Re r_{f_1}\left(-iy^{-1}\right)} = \Re r_{f_2}\left(-iy^{-1}\right) \to 0 \text{ as } y \to \infty.$$

By applications of (2.3.17) and (2.3.15) we observe that,

$$\frac{\Im r_{f_1}\left(-iy^{-1}\right) \Im r_{f_2}\left(-iy^{-1}\right)}{\Re r_{f_1}\left(-iy^{-1}\right)} = \frac{\Im r_{f_1}\left(-iy^{-1}\right)}{y^{-(1-r/2)}} \frac{\Im r_{f_2}\left(-iy^{-1}\right)}{y^{-(1-r/2)}} \frac{y^{-(1+r/2)}}{\Re r_{f_1}\left(-iy^{-1}\right)} \frac{1}{y^{1-3r/2}}$$

$\to 0$ as $y \to \infty$. Thus $\Re r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Re r_{f_1}\left(-iy^{-1}\right)$. Exactly same calculation taking the imaginary part gives us $\Im r_{f_1 f_2}(-iy^{-1}) \sim (1+c)\Im r_{f_1}\left(-iy^{-1}\right)$. Therefore we are done when $p = 1$.

Subcase (ii) Suppose $p \geq 2$. The real part can be dealt as in the proof of case ((i)d). For the imaginary part note that from (2.6.9) we have,

$$\Im r_{f_1 f_2}(-iy^{-1}) = \Im r_{f_1}\left(-iy^{-1}\right) + \Im r_{f_2}\left(-iy^{-1}\right) + \mathrm{O}\left(y^{-1}\right).$$

Now as $y \to \infty$, $y^{-(1-r/2)} \gg \Im r_{f_1}\left(-iy^{-1}\right), \Im r_{f_2}\left(-iy^{-1}\right) \gg y^{-1}$ by (2.3.15). Therefore we have $\Im r_{f_1 f_2}(-iy^{-1}) \gg y^{-1}$. Noting the fact that $y^{-1} \ll y^{-(1-r/2)}$ as $y \to \infty$, it can be concluded that $\Im r_{f_1 f_2}(-iy^{-1}) \ll y^{-(1-r/2)}$, $r \in (0, 1/2)$. □

Chapter 3

Regular variation and free regular infinitely divisible laws

3.1 Introduction

In the previous chapter we understood the tail behaviour of measures under Boolean convolutions while in this chapter we shall focus on the free multiplicative convolution. The motivation comes from random matrix theory as well as classical probability theory.

The limiting spectral distribution (LSD) of product of two or more random matrices is important in the field of random matrix theory. It arises naturally, for example, in study of multivariate F-matrix (the product of mutually independent sample covariance matrix and the inverse of another sample covariance matrix). The limiting spectral distributions of F-matrices were studied in [95], [12]. In addition, products of random matrices arise in study of high dimensional time-series, for example, see [74], [75]. For a history of the product of random matrices the reader is referred to [13].

The existence of a non-random LSD of the product of a sample covariance matrix and a non-negative definite Hermitian matrix, which are mutually independent, was given explicitly in terms of the Stieltjes transform in [82]. A stronger result in this direction is obtained using the moment method and truncation arguments in [13], by replacing the non-negative definite assumption by a Lindeberg type one, on the entries of the Hermitian matrices. When one considers Wishart matrices, a more explicit description of the LSD can be given in terms of free probability; see [70], [36] etc.

It is well known in random matrix theory that the Marchenko-Pastur law (also called the free Poisson distribution) turns out to be the limiting spectral distribution of a sequence of Wishart random matrices$(W_N)_{N\geq 1}$. Suppose for each $N \geq 1$, Y_N is an $N \times N$ independent random Hermitian matrix with LSD ρ. It is shown in [36] that the expected empirical distribution of $W_N Y_N$ (equivalently, $W_N^{1/2} Y_N W_N^{1/2}$) converges to $m \boxtimes \rho$ as $N \to \infty$ where $\boxtimes$ denotes the free multiplicative convolution. It is not difficult to see that ρ is compactly supported if and only if so is $m \boxtimes \rho$. Therefore it is natural to ask whether there is any relation between the tail behaviour of $m \boxtimes \rho$ and ρ? In this chapter, an affirmative answer is given to that question when ρ has a power law tail decay. Thus, based on the LSD of Y_N, one can describe the tail behaviour of that of $W_N Y_N$. In general, it is very hard to write down an explicit formula for the limit distribution.

It is noteworthy that the probability measures of the form $m \boxtimes \rho$ are free regular probability measures (see [9]) which form a special subclass of free infinitely divisible distributions (also called the $\boxplus$-infinitely divisible distributions, see [28]). The free cumulant transform of a free regular probability measure can be described through a Lévy-Khintchine representation. Interestingly, it turns out that ρ is the Lévy measure of $m \boxtimes \rho$. Therefore it is natural to wonder whether there is any relation between the tail behaviour of a free regular probability measure and its Lévy measure.

In classical probability theory, a classically infinitely divisible probability measure μ also enjoys a Lévy-Khintchine representation in terms of its Lévy measure ν. In [47], it was shown that for a positively supported classically infinitely divisible probability measure (a subordinator) μ, the tails of μ and its Lévy measure ν are asymptotically equivalent if and only if any one of μ or ν is subexponential. Later the result was extended in the works of Pakes [73] and Watanabe [96] to classify the subexponentiality of classically infinitely divisible distributions on $\mathbb{R}$. Recently, in [97], the subexponential densities of absolutely continuous infinitely divisible distributions on the half line is characterized under some additional assumptions. In analogy to the classical case, it is natural to pose whether free subexponentiality characterizes the tail equivalence of a free infinitely divisible probability measure and its free Lévy measure. But unfortunately the result can not be extended to the bigger class of free infinitely divisible probability measures. Since according to Arizmendi et al. [9], the correct analogue of the positively supported classically infinitely divisible probability measures are the free regular probability measures, in this chapter, we provide a partial answer in Theorem 3.3.1 by showing the tail equivalence of a free regular probability measure and its free Lévy measure in presence of regular variation. Note that regularly varying measures are the most important subclass of both free and classical subexponential distributions ([59]).

As an application of this result, the exact tail behaviour of the free multiplicative convolution of Marchenko-Pastur law with another regularly varying measure is derived in Corollary 3.4.1. Besides, the connection of these results with the classical case is not a mere coincidence. From the famous result of Bercovici and Pata ([29]), it is known that classical and free infinitely divisible laws are in a one-to-one correspondence. It is shown in Corollary 3.4.7 that in the regularly varying set-up, the classical infinitely divisible law and its image under the Bercovici-Pata bijection are tail equivalent. The free multiplicative convolution of a measure with Wigner's semicircle law also appears naturally as limits of many random matrix models. It is shown in Corollary 3.4.4 that the tail behaviour turns out to be different from the one involving the Marchenko-Pastur law.

In Section 3.2 the basic notations and transforms used in free probability are introduced. Subsequently, the main results and their proofs are in Section 3.3. Section 3.4 collects some corollaries arising out of the main results. The relation between the tails of a probability measure and its Boolean Lévy measure is briefly sketched in Section 3.5. An independent proof of our result in case of free Poisson distributions (i.e. Corollary 3.4.1) is given in Section 3.6. The proofs depend heavily on relations between the transforms and regular variation. To keep this chapter self contained, the main result of [59] is quoted in the Appendix.

3.2 Preliminaries and Main results

3.2.1 Notations and basic definitions:

Recall the notions of regular variation and the tail of a probability measure from the begining of Section 2.2. A distribution F on $[0, \infty)$ is called (classical) subexponential if $\overline{F^{(n)}}(x) \sim \overline{F}(x)$ as $x \to \infty$, for all $n \geq 0$. Here $F^{(n)}$ denotes the n-th (classical) convolution of F. It is apparent that both regular variation and subexponentiality of a probability measure μ is defined through its distribution function F.

For a complex number z, $\Re z$ and $\Im z$ denote its real and imaginary parts, respectively. Given positive numbers η, δ and M, let us define the following cone:

$$\Gamma_\eta = \{z \in \mathbb{C}^+ : |\Re z| < \eta \Im z\} \text{ and } \Gamma_{\eta,M} = \{z \in \Gamma_\eta : |z| > M\}.$$

Then we shall say that $f(z) \to l$ as z goes to ∞ non-tangentially, abbreviated by "n.t.", if for any $\epsilon > 0$ and $\eta > 0$, there exists $M \equiv M(\eta, \epsilon) > 0$, such that $|f(z) - l| < \epsilon$, whenever $z \in \Gamma_{\eta,M}$. This is same as saying that the convergence in $\mathbb{C}^+$ is uniform in each cone Γ_η. The boundedness can be defined analogously.

We use the notations "$f(z) \approx g(z)$", "$f(z) = \mathrm{o}(g(z))$" and "$f(z) = \mathrm{O}(g(z))$ as $z \to \infty$ n.t." in an analogous way as defined in the previous chapter, to mean, respectively, that "$f(z)/g(z)$ converges to a non-zero finite limit", "$f(z)/g(z) \to 0$" and "$f(z)/g(z)$ stays bounded as $z \to \infty$ n.t." If the limit is 1 in the first case, we write $f(z) \sim g(z)$ as $z \to \infty$ n.t. For $f(z) = \mathrm{o}(g(z))$ as $z \to \infty$ n.t., we shall also use the notations $f(z) \ll g(z)$ and $g(z) \gg f(z)$ as $z \to \infty$ n.t.

Recall the sets $\mathcal{M}$, $\mathcal{M}_+$ and $\mathcal{M}_p$ to be the same as defined before. The set $\mathcal{M}_{p,\alpha}$ will contain all probability measures in $\mathcal{M}_p$ with regularly varying tail index $-\alpha$ such that $p \le \alpha \le p+1$.

Despite of repetitiveness, it is not unpleasant to recall the Cauchy transform. For a probability measure $\mu \in \mathcal{M}$, its Cauchy transform is defined as

$$G_\mu(z) = \int_{-\infty}^{\infty} \frac{1}{z-t} d\mu(t), \quad z \in \mathbb{C}^+.$$

Note that G_μ maps $\mathbb{C}^+$ to $\mathbb{C}^-$. Set $F_\mu = \frac{1}{G_\mu}$, which maps $\mathbb{C}^+$ to $\mathbb{C}^+$.

The free cumulant transform (C_μ) and the Voiculescu transform (ϕ_μ) of a probability measure μ are defined as

$$C_\mu(z) = z\phi_\mu\left(\frac{1}{z}\right) = zF_\mu^{-1}\left(\frac{1}{z}\right) - 1,$$

for z in a domain $D_\mu = \{z \in \mathbb{C}^- : 1/z \in \Gamma_{\eta,M}\}$ where F_μ^{-1} is defined; The free additive convolution of two probability measures μ_1, μ_2 on $\mathbb{R}$ is defined as the probability measure $\mu_1 \boxplus \mu_2$ on $\mathbb{R}$ such that $\phi_{\mu_1 \boxplus \mu_2}(z) = \phi_{\mu_1}(z) + \phi_{\mu_2}(z)$ or equivalently $C_{\mu_1 \boxplus \mu_2}(z) = C_{\mu_1}(z) + C_{\mu_2}(z)$ for $z \in D_{\mu_1} \cap D_{\mu_2}$. It turns out that $\mu_1 \boxplus \mu_2$ is the distribution of the sum $X_1 + X_2$ of two free random variables X_1 and X_2 having distributions μ_1 and μ_2 respectively. On the other hand, the free multiplicative operation $\boxtimes$ on $\mathcal{M}$ is defined as follows (see [28]). Let μ_1, μ_2 be probability measures on $\mathbb{R}$, with $\mu_1 \in \mathcal{M}_+$ and let X_1, X_2 be free random variables such that $\mu_{X_i} = \mu_i$. Since μ_1 is supported on $\mathbb{R}^+$, X_1 is a positive self-adjoint operator and $\mu_{X_1^{1/2}}$ is uniquely determined by μ_1. Hence the distribution $\mu_{X_1^{1/2} X_2 X_1^{1/2}}$ of the self-adjoint operator $X_1^{1/2} X_2 X_1^{1/2}$ is determined by μ_1 and μ_2. This measure is called the free multiplicative convolution of μ_1 and μ_2 and it is denoted by $\mu_1 \boxtimes \mu_2$. This operation on $\mathcal{M}_+$ is associative and commutative.

We recall (Theorem 1.3 and Theorem 1.5 of [22]) the remainder terms in Laurent series expansion of Cauchy and Voiculescu transforms for probability measures μ with finite p moments and summarize the following expressions from [59]:

$$r_{G_\mu}(z) = z^{p+1}\left(G_\mu(z) - \sum_{j=1}^{p+1} m_{j-1}(\mu)z^{-j}\right) \tag{3.2.1}$$

and

$$r_{\phi_\mu}(z) = z^{p-1}\left(\phi_\mu(z) - \sum_{j=0}^{p-1} \kappa_{j+1}(\mu)z^{-j}\right), \tag{3.2.2}$$

where $\{m_j(\mu) : j \le p\}$ and $\{\kappa_j(\mu) : j \le p\}$ denotes the moment and free cumulant sequences of the probability measure μ, respectively.

3.2.2 Classical infinite divisibility and known results

A probability measure μ is called classically infinitely divisible, if for every $n \in \mathbb{N}$, there exists a probability measure μ_n such that $\mu = \mu_n * \mu_n * \cdots * \mu_n (n \text{ times})$, where $*$ is the classical convolution of probability measures. A detailed description about classical infinite divisibility can be found in [80]. It is well known that a probability measure μ on $\mathbb{R}$ is classically infinitely divisible if and only if its classical cumulant transform $C^*_\mu(w) := \log \int_{\mathbb{R}} e^{iwx} d\mu(x)$ has the following Lévy-Khintchine representation (see [80] or [16])

$$C^*_\mu(w) = i\eta w - \frac{1}{2}aw^2 + \int_{\mathbb{R}} \left(e^{iwt} - 1 - iwt\mathbf{1}_{[-1,1]}(t)\right) d\nu(t), \quad w \in \mathbb{R}, \tag{3.2.3}$$

where $\eta \in \mathbb{R}$, $a \ge 0$ and ν is a Lévy measure on $\mathbb{R}$, that is, $\int_{\mathbb{R}} \min\left(1, t^2\right) d\nu(t) < \infty$ and $\nu(\{0\}) = 0$. If this representation exists, the triplet (η, a, ν) is called the classical characteristic triplet of μ and the triplet is unique.

Another form of $C^*_\mu(w)$ is given by

$$C^*_\mu(w) = i\gamma w + \int_{\mathbb{R}} \left(e^{iwt} - 1 - \frac{iwt}{1+t^2}\right) \frac{1+t^2}{t^2} d\sigma(t), \quad w \in \mathbb{R},$$

where γ is a real constant and σ is a finite measure on $\mathbb{R}$.

One has the following relationships between the two representations (see equations (2.3) below definition 2.1 in [16]):

$$\begin{aligned} a &= \sigma(\{0\}), \\ d\nu(t) &= \frac{1+t^2}{t^2}\mathbf{1}_{\mathbb{R}\setminus\{0\}}d\sigma(t), \qquad (3.2.4) \\ \eta &= \gamma + \int_{\mathbb{R}} t\left(\mathbf{1}_{[-1,1]}(t) - \frac{1}{1+t^2}\right)d\nu(t). \qquad (3.2.5) \end{aligned}$$

In general, when one does not have the Brownian component, it is easier to consider the Laplace transform (if exists) of the measure, for example, in the case of compound Poisson. In this situation let us recall the classical result which studies the tail equivalence of Lévy measure and the infinitely divisible distribution. In [47] it was shown subexponentiality is a property which makes an infinitely divisible measure and its Lévy measure tail equivalent.

Theorem 3.2.1 ([47])**.** *Let μ be a classical infinitely divisible probability measure on $[0,\infty)$. Suppose μ has the Lévy-Khintchine representation of the form,*

$$f(s) = \int_{0-}^{\infty} e^{-st}d\mu(t) = exp\left\{-as - \int_0^{\infty}(1-e^{-st})d\nu(t)\right\}$$

where ν is a Lévy measure satisfying $\int_0^\infty \min\{1,t\}d\nu(t) < \infty$. Then the following statements are equivalent:

(a) μ is subexponential,

(b) $\overline{\nu}$ is subexponential,

(c) $\mu(x,\infty) \sim \nu(x,\infty)$ as $x \to \infty$.

Here, the probability measure $\overline{\nu}$, supported on the interval $(1,\infty)$ is defined by

$$\overline{\nu}(1,x) = \nu(1,x]/\nu(1,\infty)\,.$$

The remarkable feature of this result is that tail equivalence gives subexponentiality. In Section 3.3 we will address the partial extension of this result in the free setting.

3.2.3 Free infinite divisibility and free regular probability measures

Free infinitely divisible probability measures are defined in analogy with classical infinitely divisible probability measures. Infinitely divisible measures can also be described in terms of a representation through Voiculescu and free cumulant transforms. A probability measure μ is called *free infinitely divisible*, if for every $n \in \mathbb{N}$, there exists a probability measure μ_n such that $\mu = \mu_n \boxplus \mu_n \boxplus \cdots \boxplus \mu_n (n$ times) holds. Also a probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible i.e. free infinitely divisible if and only if there exists a finite measure σ on $\mathbb{R}$ and a real constant γ, such that

$$\phi_\mu (z) = \gamma + \int_{\mathbb{R}} \frac{1+zt}{z-t} d\sigma (t) , \qquad z \in \mathbb{C}^+ \tag{3.2.6}$$

A probability measure μ on $\mathbb{R}$ is $\boxplus$-infinitely divisible if and only if the free cumulant transform has the representation:

$$C_\mu^{\boxplus}(z) = \eta z + az^2 + \int_{\mathbb{R}} \Big(\frac{1}{1-zt} - 1 - tz\mathbf{1}_{[-1,1]}(t) \Big) d\nu(t), z \in \mathbb{C}^-, \tag{3.2.7}$$

where $\eta \in \mathbb{R}$, $a \geq 0$ and ν is called the Lévy measure on $\mathbb{R}$. In the expressions (3.2.6) and (3.2.7), similar to the equations (3.2.4) and (3.2.5) holds true (see Proposition 4.16 of [16]). The free characteristic triplet (η, a, ν) of a probability measure μ is unique.

For a free infinitely divisible probability measure μ on $\mathbb{R}$ where the Lévy measure (Definition 2.1 in [16]) ν satisfies $\int_{\mathbb{R}} \min(1, |t|) \, d\nu(t) < \infty$ and $a = 0$ the Lévy-Khintchine representation (3.2.7) reduces to

$$C_\mu^{\boxplus} (z) = \eta' z + \int_{\mathbb{R}} \left(\frac{1}{1-zt} - 1 \right) d\nu (t) , \quad z \in \mathbb{C}^-, \tag{3.2.8}$$

where $\eta' \in \mathbb{R}$. The measure μ is called a **free regular infinitely divisible distribution** (or **regular $\boxplus$-infinitely divisible measure**) if $\eta' \geq 0$ and $\nu((-\infty, 0]) = 0$.

The most typical example is compound free Poisson distributions. If the drift term η' is zero and the Lévy measure ν is $\lambda\rho$ for some constant $\lambda > 0$ and a probability measure ρ on $\mathbb{R}$, then we call μ a compound free Poisson distribution with rate λ and jump distribution ρ. To clarify these parameters, we denote $\mu = \pi(\lambda, \rho)$.

Example 3.2.2 ([9], Remark 8)**.**

(i) *The Marchenko-Pastur law m is a compound free Poisson with rate 1 and jump distribution δ_1.*

(ii) *The compound free Poisson $\pi(1,\rho)$ coincides with the free multiplication $m \boxtimes \rho$.*

We shall use both the Voiculescu transform and the cumulant transform to state our theorems. The notations $\mu_{\boxplus,V}^{\gamma,\sigma}$ or $\mu_{\boxplus,C}^{\eta',0,\nu}$ shall occur whenever we write the Voiculescu transform or the cumulant transform of a free regular probability measure μ respectively. The indices V and C are used to distinguish between the occurrence in Voiculescu transform or in the cumulant transform. The use of γ, σ, η' and ν in the indices are clear from (3.2.9) and (3.2.10) while in $\mu_{\boxplus,C}^{\eta',0,\nu}$, the index 0 is to indicate the non existence of the Gaussian part in the representation of the cumulant transform. Let $\mu_{\boxplus,V}^{\gamma,\sigma} = \mu_{\boxplus,C}^{\eta',0,\nu}$ be a free regular infinitely divisible probability measure. Then its Voiculescu and cumulant transforms have the representations:

$$\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}(z) = \gamma + \int_{\mathbb{R}+} \frac{1+tz}{z-t} d\sigma(t), \tag{3.2.9}$$

$$C_{\mu_{\boxplus,C}^{\eta',0,\nu}}(z) = \eta' z + \int_{\mathbb{R}+} \left(\frac{1}{1-zt} - 1 \right) d\nu(t) \tag{3.2.10}$$

respectively following (3.2.6) and (3.2.8). In the above representation the pair (γ, σ) is related to (η', ν) in the following way:

$$\begin{aligned} d\sigma(t) &= \frac{t^2}{1+t^2} d\nu(t), \\ \gamma &= \eta' + \int_{\mathbb{R}+} \frac{t}{1+t^2} d\nu(t), \qquad \eta' \geq 0. \end{aligned} \tag{3.2.11}$$

The proof of Theorem 3.3.1 demands the finiteness of the measure σ appearing the Voiculescu transform while the Lévy measure may not be a finite measure for a free regular infinitely divisible measure.

3.3 Main results and their proofs

Now we are ready to state the main results of this chapter while keeping in mind all the notations defined above. The following theorem gives us the tail equivalence between a free regular probability measure and the finite measure σ occurring in the Voiculescu transform.

Theorem 3.3.1. *Suppose that $\mu_{\boxplus,V}^{\gamma,\sigma}$ is free regular infinitely divisible measure. Then the following statements are equivalent:*

(i) $\mu_{\boxplus,V}^{\gamma,\sigma}$ *has regularly varying tail of index* $-\alpha$.

(ii) σ *has regularly varying tail of index* $-\alpha$.

If either of the above holds, then $\mu_{\boxplus,V}^{\gamma,\sigma}(x,\infty) \sim \sigma(x,\infty)$ *as* $x \to \infty$.

Remark 3.3.2. *It follows from Theorem* 4.2 *of [9] that for a free regular infinitely divisible probability measure* $\mu_{\boxplus,V}^{\gamma,\sigma}$*, both* $\mu_{\boxplus,V}^{\gamma,\sigma}$ *and* σ *are concentrated on* $[0,\infty)$.

We fix the notations $m_{-1}(\sigma) = \gamma$, $m_0(\sigma) = \sigma(\mathbb{R}^+)$ and $\overline{\sigma}$ for the probability measure $\sigma/m_0(\sigma)$. Recall the remainder terms of the Cauchy and Voiculescu transforms as defined in (3.2.1) and (3.2.2) respectively. To prove Theorem 3.3.1, we first state and prove the following Lemma.

Lemma 3.3.3. *Let* $\mu_{\boxplus,V}^{\gamma,\sigma}$ *be a regular* $\boxplus$*-infinitely divisible probability measure.*

(i) *Voiculescu transform of* $\mu_{\boxplus,V}^{\gamma,\sigma}$ *and Cauchy transform of* $\overline{\sigma}$ *are related by*

$$\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}(z) = m_{-1}(\sigma) - m_0(\sigma)\, z + \left(1+z^2\right) m_0(\sigma)\, G_{\overline{\sigma}}(z)\,. \tag{3.3.1}$$

(ii) *In this case,* $\mu_{\boxplus,V}^{\gamma,\sigma}$ *and* σ *have same number of moments.*

(iii) *If both* $\mu_{\boxplus,V}^{\gamma,\sigma}$ *and* σ *have* p *moments, then the* p *cumulants of* $\mu_{\boxplus,V}^{\gamma,\sigma}$ *and* p *moments of* σ *satisfy the relation*

$$\kappa_p\left(\mu_{\boxplus,V}^{\gamma,\sigma}\right) = m_{p-2}(\sigma) + m_p(\sigma) \tag{3.3.2}$$

and the remainder terms of $\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}$ *and* $G_{\overline{\sigma}}$ *satisfy*

$$r_{\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}}(z) = m_{p-1}(\sigma)\, z^{-1} + m_p(\sigma)\, z^{-2} + \left(1+z^{-2}\right) m_0(\sigma)\, r_{G_{\overline{\sigma}}}(z)\,. \tag{3.3.3}$$

Proof. (i) Using (3.2.9), we have

$$\begin{aligned}
\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}(z) &= \gamma + m_0(\sigma) \int_0^\infty \frac{1+tz}{z-t}\, d\overline{\sigma}(t) \\
&= \gamma + m_0(\sigma)\, G_{\overline{\sigma}}(z) + m_0(\sigma)\, z \int_0^\infty \frac{t}{z-t}\, d\overline{\sigma}(t) \\
&= m_{-1}(\sigma) + m_0(\sigma)\, G_{\overline{\sigma}}(z) - m_0(\sigma)\, z + m_0(\sigma)\, z^2 G_{\overline{\sigma}}(z)\,.
\end{aligned}$$

(ii) Here it is easy to conclude that for a non trivial free regular probability measure $\mu_{\boxplus,V}^{\gamma,\sigma}$, one must have $\gamma > 0$. Now we shall prove the existence of a p moment of σ is equivalent to the existence of a p moment of $\mu_{\boxplus,V}^{\gamma,\sigma}$. We shall follow the proof of Proposition 2.3 of [22].

First suppose σ admits a moment of order p. For all positive integer n, let us define the positive finite measure σ_n on $[0,\infty)$ by $\sigma_n(A) = \sigma\big(A \cap [0,n)\big)$. By dominated convergence theorem σ_n converges weakly to σ. Thus by Theorem 3.8 of [15] we have $\mu_{\boxplus,V}^{\gamma,\sigma_n}$ converges weakly to $\mu_{\boxplus,V}^{\gamma,\sigma}$. Therefore,

$$\int_0^\infty t^p d\mu_{\boxplus,V}^{\gamma,\sigma}(t) \le \liminf_n \int_0^\infty t^p d\mu_{\boxplus,V}^{\gamma,\sigma_n}(t).$$

The range of the integral is $\mathbb{R}^+$ instead of $\mathbb{R}$ because $\mu_{\boxplus,V}^{\gamma,\sigma_n}$ is again a free regular measure (since $\mu_{\boxplus,V}^{\gamma,\sigma}$ is so) and the Remark 3.3.2 gives $\mu_{\boxplus,V}^{\gamma,\sigma_n}$ is concentrated on $[0,\infty)$. Thus,

$$\int_0^\infty t^p d\mu_{\boxplus,V}^{\gamma,\sigma}(t) \le \liminf_n m_p(\mu_{\boxplus,V}^{\gamma,\sigma_n}).$$

To show that $\mu_{\boxplus,V}^{\gamma,\sigma}$ has p^{th} moment finite, it is enough to show that $\{m_p(\mu_{\boxplus,V}^{\gamma,\sigma_n})\}_n$ is bounded. By the equation (2.1) of [22], we have the q^{th} free cumulant $\kappa_q(\mu_{\boxplus,V}^{\gamma,\sigma_n}) = m_{q-2}(\sigma_n)+m_q(\sigma_n)$ since σ_n's are compactly supported (with the convention that $m_{-1}(\sigma_n) = \gamma$). So, for all n,

$$\begin{aligned} m_p(\mu_{\boxplus,V}^{\gamma,\sigma}) &= \sum_{\pi \in NC(p)} \prod_{V \in \pi} \kappa_{|V|}(\mu_{\boxplus,V}^{\gamma,\sigma_n}) \\ &= \sum_{\pi \in NC(p)} \prod_{V \in \pi} \big(m_{q-2}(\sigma_n) + m_q(\sigma_n)\big) \\ &\le \sum_{\pi \in NC(p)} \prod_{V \in \pi} \big(m_{q-2}(\sigma) + m_q(\sigma)\big) < \infty, \end{aligned}$$

where $NC(p)$ is the set of all non crossing partitions of $\{1, 2, \dots, n\}$ and $|V|$ is the number of elements in the block V of π.

Next suppose $\mu_{\boxplus,V}^{\gamma,\sigma}$ admits a moment of order p. Then by Theorem 1.3 of [22], $\phi_{\mu_{\boxplus,V}^{\gamma/n,\sigma/n}}$ admits a Laurent series expansion of order $p+1$. Thus for all positive integer n, we have $\phi_{\mu_{\boxplus,V}^{\gamma/n,\sigma/n}}(z) = \frac{1}{n}\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}(z)$. Now support of $\mu_{\boxplus,V}^{\gamma/n,\sigma/n}(z)$ is contained in $[0,\infty)$ (as $\mu_{\boxplus,V}^{\gamma/n,\sigma/n}(z)$ is a free regular measure with σ_n has support on $[0,\infty)$) and the uniqueness of the Laurent series expansion allows us to conclude that $\mu_{\boxplus,V}^{\gamma/n,\sigma/n}(z)$ has a Laurent series expansion of order $p+1$. Moreover we have $\kappa_i(\mu_{\boxplus,V}^{\gamma/n,\sigma/n}) = \frac{1}{n}\kappa_i(\mu_{\boxplus,V}^{\gamma,\sigma})$ for all

$i \in \{1, 2, \ldots, p\}$. From Theorem $5.10(iii)$ of [28], we conclude that

$$d\sigma(t) = \lim_{n\to\infty} \frac{nt^2}{1+t^2} d\mu_{\boxplus,V}^{\gamma/n,\sigma/n}(t).$$

Therefore we have,

$$\begin{aligned}\int_0^\infty t^p d\sigma(t) &\leq \liminf_n \int_0^\infty \frac{t^p nt^2}{1+t^2} d\mu_{\boxplus,V}^{\gamma/n,\sigma/n}(t) \\ &\leq \liminf_n \int_0^\infty nt^p d\mu_{\boxplus,V}^{\gamma/n,\sigma/n}(t) \\ &= \liminf_n nm_p(\mu_{\boxplus,V}^{\gamma/n,\sigma/n}) \\ &= \liminf_n \sum_{\pi\in NC(p)} n \prod_{V\in\pi} \kappa_{|V|}(\mu_{\boxplus,V}^{\gamma/n,\sigma/n}) \\ &= \liminf_n \sum_{\pi\in NC(p)} n^{1-\#\pi} \prod_{V\in\pi} \kappa_{|V|}(\mu_{\boxplus,V}^{\gamma,\sigma}) < \infty,\end{aligned}$$

where in the third line we have used $\mu_{\boxplus,V}^{\gamma/n,\sigma/n}((-\infty, 0)) = 0$ since for all n, $\mu_{\boxplus,V}^{\gamma/n,\sigma/n}$ is free regular and σ/n has support on $[0,\infty)$ and in the last line $\#\pi$ indicated the number of blocks in the partition π.

(iii) If both $\mu_{\boxplus,V}^{\gamma,\sigma}$ and σ have p moments finite, considering Laurent series expansion of $G_{\overline{\sigma}}$ in (3.3.1) and the fact that $m_j(\sigma) = m_0(\sigma) m_j(\overline{\sigma})$ for $0 \leq j \leq p$, we have

$$\begin{aligned}\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}(z) =& m_{-1}(\sigma) - m_0(\sigma)z + (1+z^2)\sum_{j=1}^{p+1} m_{j-1}(\sigma)z^{-j} \\ &+ (1+z^2)z^{-(p+1)} m_0(\sigma) r_{G_{\overline{\sigma}}}(z) \\ =& \sum_{j=1}^{p} (m_{j-2}(\sigma) + m_j(\sigma))z^{-(j-1)} + z^{-(p-1)}\big(m_{p-1}(\sigma)z^{-1} \\ &+ m_p(\sigma)z^{-2} + (1+z^{-2})m_0(\sigma)r_{G_{\overline{\sigma}}}(z)\big).\end{aligned}$$

Since $r_{G_\sigma}(z) = \mathrm{o}(1)$ as $z \to \infty$ n.t., we have

$$m_{p-1}(\sigma) z^{-1} + m_p(\sigma) z^{-2} + \left(1+z^{-2}\right) m_0(\sigma) r_{G_{\overline{\sigma}}}(z) = \mathrm{o}(1)$$

as $z \to \infty$ n.t. Thus, by uniqueness of Laurent series expansion (which is equivalent to the uniqueness of Taylor series expansion given in Lemma A.1 of [22]), we obtain (3.3.2) as well as (3.3.3).

□

It can be shown, using the expansions of Voiculescu and Cauchy transforms, that, if $\mu_{\boxplus,V}^{\gamma,\sigma}$ is a compactly supported probability measure, then σ is also compactly supported and their cumulants and moments are related exactly by the formula stated in (3.3.2). Further note that $m_{p-2}(\sigma)+m_p(\sigma)$ is also the classical cumulant of a classical infinitely divisible distribution.

Proof of Theorem 3.3.1. First assume that $\mu_{\boxplus,V}^{\gamma,\sigma}$ is regularly varying with tail index $-\alpha$ for some $\alpha \geq 0$. Then there exists a unique nonnegative integer p such that $\alpha \in [p,p+1]$ and the measure $\mu_{\boxplus,V}^{\gamma,\sigma} \in \mathcal{M}_{p,\alpha}$. Also, by Lemma 3.3.3((ii)), we have $\overline{\sigma} \in \mathcal{M}_p$ as well. Furthermore evaluating (3.3.3) at $z = iy$ and equating the real and the imaginary parts respectively, we have,

$$\left(1-y^{-2}\right) m_0(\sigma)\, \Re r_{G_{\overline{\sigma}}}(iy) - m_p(\sigma)\, y^{-2} = \Re r_{\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}}(iy) \tag{3.3.4}$$

and

$$\left(1-y^{-2}\right) m_0(\sigma)\, \Im r_{G_{\overline{\sigma}}}(iy) - m_{p-1}(\sigma)\, y^{-1} = \Im r_{\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}}(iy)\,. \tag{3.3.5}$$

Now if $\alpha \in [p,p+1)$, using Theorem 3.7.1, we have from (3.3.5), as $y \to \infty$,

$$\begin{aligned}\left(1-y^{-2}\right) m_0(\sigma)\, \Im r_{G_{\overline{\sigma}}}(iy) - m_{p-1}(\sigma)\, y^{-1} &= \Im r_{\phi_{\mu_{\boxplus,V}^{\gamma,\sigma}}}(iy)\\ &\sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu_{\boxplus,V}^{\gamma,\sigma}(y,\infty)\,,\end{aligned}$$

which is regularly varying of index $-(\alpha-p)$ with $\alpha - p < 1$. Thus, as $y \to \infty$,

$$\begin{aligned} m_0(\sigma)\, \Im r_{G_{\overline{\sigma}}}(iy) &\sim \left(1-y^{-2}\right) m_0(\sigma)\, \Im r_{G_{\overline{\sigma}}}(iy)\\ &\sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu_{\boxplus,V}^{\gamma,\sigma}(y,\infty)\end{aligned}$$

and is also regularly varying of index $-(\alpha-p)$. Now by Theorem 3.7.1, $\overline{\sigma}$ and hence σ has regularly varying tail of index $-\alpha$ and

$$\Im r_{G_{\overline{\sigma}}}(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \overline{\sigma}(y,\infty) \text{ as } y \to \infty.$$

Putting two asymptotic equivalences together, we get $\mu_{\boxplus,V}^{\gamma,\sigma}(y,\infty) \sim m_0(\sigma)\overline{\sigma}(y,\infty) = \sigma(y,\infty)$ as same argument works for the case $\alpha = p+1$ with the help of Theorem 3.7.2 and equation (3.3.4).

To get the converse statement, we shall start with σ to be regularly varying with index $-\alpha$. Thus $\sigma \in \mathcal{M}_{p,\alpha}$ for some integer $p \geq 0$. Lemma 3.3.3((ii)) gives $\mu_{\boxplus,V}^{\gamma,\sigma} \in \mathcal{M}_p$ also, and we get the equations (3.3.4) and (3.3.5). Arguing exactly the same way like above we shall be able to conclude that $\mu_{\boxplus,V}^{\gamma,\sigma}$ is regularly varying with tail index $-\alpha$. □

Noting the relations between the measures appearing in the Lévy-Khintchine representations of the Voiculescu and cumulant transform of a free regular measure, the following corollary is immediate and this will also be very important to link our result with the classical one in Corollary 3.4.7.

Corollary 3.3.4. *Suppose $\mu_{\boxplus,C}^{\eta',0,\nu}$ is a free regular infinitely divisible measure. Then the following are equivalent:*

(i) *$\mu_{\boxplus,C}^{\eta',0,\nu}$ has regularly varying tail of index $-\alpha$.*

(ii) *ν has regularly varying tail of index $-\alpha$.*

If either of the above holds, then $\mu_{\boxplus,C}^{\eta',0,\nu}(x,\infty) \sim \nu(x,\infty)$ as $x \to \infty$.

Remark 3.3.5. *Note that in Corollary 3.3.4, the measure ν may not be a finite measure. Since ν being a Lévy measure of a free regular probability measure we have $\nu(1,\infty) < \infty$ and therefore there is no ambiguity in talking about its tail behaviour.*

Proof of Corollary 3.3.4. First we observe the following with the notations σ, ν and a be as in (3.2.11). Suppose $a = 0$. Then from (3.2.11), taking integral from x to infinity on both sides we get,

$$\begin{aligned}\nu(x,\infty) &= \sigma(x,\infty) + \int_x^\infty \frac{1}{t^2} d\sigma(t) \\ &\leq (1+\frac{1}{x^2})\sigma(x,\infty). \text{ since } x < t.\end{aligned}$$

Therefore,

$$\sigma(x,\infty) \leq \nu(x,\infty) \leq (1+\frac{1}{x^2})\sigma(x,\infty).$$

Taking limit as $x \to \infty$ we get

$$\sigma(x, \infty) \sim \nu(x, \infty) \text{ as } x \to \infty. \tag{3.3.6}$$

Now Corollary 3.3.4 is immediate from Theorem 3.3.1 and the equation (3.3.6) as

$$\mu_{\boxplus, C}^{\eta', 0, \nu}(x, \infty) = \mu_{\boxplus, V}^{\gamma, \sigma}(x, \infty) \overset{Theorem\ 3.3.1}{\sim} \sigma(x, \infty) \overset{(3.3.6)}{\sim} \nu(x, \infty).$$

□

3.4 Some corollaries

As an application of our main result we study the compound free Poisson distribution which turn out to be the free analogue of the classical compound Poisson distribution. Recall, that if G is a proper distribution on $[0, \infty)$ and $\lambda > 0$ then the (classical) compound Poission distribution is defined as

$$F(x) = e^{-\lambda} \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} G^{(n)}(x)$$

where $G^{(0)}$ is Dirac mass at 0 and $G^{(n)}$ is the n-th classical convolution of G. It was shown in Theorem 3 of [47] that F is subexponential if and only if G is subexponential and this is also equivalent to $\overline{F}(x) \sim \lambda \overline{G}(x)$ as $x \to \infty$. We show that a partial analogue of this result is true in the free setting when one restricts to regularly varying measures.

The representation of a compound free Poisson distribution $\mu = \pi(1, \rho)$ as $\mu = m \boxtimes \rho$ makes it an interesting object to study further as they arise as limits of empirical distribution of random matrices.

As a corollary of Corollary 3.3.4, we get the following:

Corollary 3.4.1. *Let ρ be a positively supported probability measure. Then for the compound free Poisson distribution $\mu = \pi(1, \rho)$ which coincides with the free multiplication $m \boxtimes \rho$, the following are equivalent.*

(i) *The tail of μ is regularly varying with index $-\alpha$.*

(ii) *The tail of ρ is regularly varying with index $-\alpha$.*

If any of the above holds, then $\mu(y,\infty) \sim \rho(y,\infty)$ *as* $y \to \infty$.

Proof of Corollary 3.4.1. This result follows directly from Corollary 3.3.4 while noticing from the example 3.2.2 that ρ is the Lévy measure of the compound free Poisson distribution $\mu = m \boxtimes \rho$. □

An independent proof of Corollary 3.4.1 is given in Section 3.6. Now we describe two situations where the above results can be applied. The first one is for random matrices while the other one is for the free stable laws.

Example 3.4.2. *As mentioned in the introduction of this chapter,* $m \boxtimes \rho$ *often occurs as a limiting spectral distribution. For example consider for all* $N \geq 1$, $W_N = \frac{1}{M_N} X_N^* X_N$ *where* X_N *is a complex Gaussian random matrix with i.i.d. entries and the sequence* $\{M_N\}_{N\geq 1}$ *is such that* $\lim_{N\to\infty} N/M_N = \lambda \in (0,\infty)$. *Also take* Y_N*, for all* $N \geq 1$ *to be random complex Hermitian matrices independent of the entries of* X_N*. Suppose there exists a non random probability measure* ρ *on* $\mathbb{R}$ *such that empirical spectral distribution of* Y_N *converges to* ρ *weakly in probability. In this framework, when* $\lambda = 1$*, the expected empirical spectral distribution of* $W_N Y_N$ *converges to* $m \boxtimes \rho$ *weakly as* $N \to \infty$ *(see Theorem* 2.3 *of [36]). Therefore if we take* ρ *to be regularly varying with tail index* $-\alpha$, $\alpha \geq 0$*, we are able to conclude that the tail of the limiting spectral distribution of* $W_N Y_N$ *is same as that of* ρ *using Corollary 3.4.1.* □

Example 3.4.3. *Following [29] we define two probability measures* μ *and* ν *to be equivalent (denote as* $\mu \sim \nu$*) if* $\mu(S) = \nu(aS + b)$ *for every Borel set* $S \subseteq \mathbb{R}$*, for some* $a \in \mathbb{R}^+$ *and* $b \in \mathbb{R}$*. A measure* μ *(excluding point mass measures) is said to be* $\boxplus$*-stable if for every* $\nu_1, \nu_2 \in \mathcal{M}$ *such that* $\nu_1 \sim \mu \sim \nu_2$*, it follows that* $\nu_1 \boxplus \nu_2 \sim \mu$*. Associated with every* $\boxplus$*-stable measure* μ *there is a number* $\alpha \in (0,2]$ *such that the measure* $\mu \boxplus \mu$ *is a translate of the measure* $D_{1/2^\alpha}\mu$ *where* $D_a\mu(S) = \mu(aS)$*. The number* α *is called the stability index of* μ*. The probability measure* $\mu^{(2)}$ *will be the image of* μ *under the map* $t \to t^2$ *on* $\mathbb{R}$.

We give a proper example where Corollary 3.4.1 follows directly. From the appendix of [29] we get that the Voiculescu transform of a $\boxplus$*-stable probability measure with stability index* $\alpha \in (0,1)$ *is of the form*

$$\phi(z) = -e^{i\alpha\rho\pi} z^{-\alpha+1}$$

where ρ *is called the asymmetry coefficient. Now using Theorem 3.7.1 of Appendix we can conclude that the* $\boxplus$*-stable probability measures in* $\mathcal{M}_0$ *with stability index* $\alpha \in (0,1)$ *are exactly regularly varying probability measures with tail index* $-\alpha$.

Let μ_α be a regularly varying symmetric free α-stable law with $0 < \alpha < 2$. Then $\mu_\alpha^{(2)} = \rho_{\frac{\alpha}{2}} \boxtimes m$, where $\rho_{\frac{\alpha}{2}}$ is a free positive $\frac{\alpha}{2}$ stable law.

The above statement can be verified by the following arguments. First from Corollary 21 of [76], observe that the positive $\frac{\alpha}{2}$-stable law $\mu_\alpha^{(2)}$ enjoys the relation

$$\mu_\alpha^{(2)} = (\rho_\beta \boxtimes \rho_\beta) \boxtimes m,$$

where ρ_β is a free positive $2\alpha/(2+\alpha)$ stable law. Applying Proposition 13 of [10], it follows that $\rho_\beta \boxtimes \rho_\beta = \rho_{\frac{\alpha}{2}}$. Hence $\mu_\alpha^{(2)} = \rho_{\frac{\alpha}{2}} \boxtimes m$. Observe $\mu_\alpha^{(2)}$ and $\rho_{\frac{\alpha}{2}}$ are in $\mathcal{M}_0$ implies that both have regularly varying tail of index $-\frac{\alpha}{2}$. The Corollary 3.4.1 can be seen as generalizing this behaviour to a much more general class of probability measures. $\square$

It is a pertinent question that whether the conclusion involving Marchenko-Pastur law can be replaced by the standard Wigner's semicircle law, w. The measures of the form $w \boxtimes \rho$ for some $\rho \in \mathcal{M}_+$ has appeared as the limiting spectral distributions of random matrices (see [2, 35, 37]), free type W distributions (see [76]) and in several other places.

The first observation in this regard is that in general one cannot say that $w \boxtimes \rho$ is free infinitely divisible for some $\rho \in \mathcal{M}_+$. In fact if one considers the measure w_+ having density

$$f_{w_+}(x) = \frac{1}{2\pi}\sqrt{4 - (x-2)^2}\mathbf{1}_{[0,4]}(x),$$

then it was shown in Corollary 3.5 of [79] that $w \boxtimes w_+$ is not a free infinitely divisible measure. The obstacle comes from the fact that w_+ is not free regular. So a valid question in this regard is whether $w \boxtimes \rho$ is regularly varying if ρ is free regular with regularly varying tail? We give a partial answer when the measure $\rho \boxtimes \rho$ is regularly varying of index $-\alpha$, $\alpha \geq 0$. Such a particular case can arise in free stable laws and goes back to the works of Bercovici, Pata and Biane (in particular see Proposition A4.3 of [29]), which states if ρ_α and ρ_β are free stable laws of index $\alpha, \beta \in (0,1)$ respectively, then $\rho_\alpha \boxtimes \rho_\beta$ is free stable law of index $\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}$ and hence regularly varying of index $-\frac{\alpha\beta}{\alpha+\beta-\alpha\beta}$ (as discussed in the second paragraph of Example 3.4.3). So combining these observations we have the following corollary where we use the definition of regularly varying measures supported on $\mathbb{R}$ instead of $\mathbb{R}^+$. Since we restrict ourselves to symmetric probability measures we don't go into the details of the definition of regular variation of such tail balanced measures.

Corollary 3.4.4. *Let $\rho \in \mathcal{M}_+$ and w be the standard Wigner measure. Then the following are equivalent.*

(i) $\rho_0 = \rho \boxtimes \rho$ *is free regular infinitely divisible, regularly varying probability measure with tail index* $-\alpha$, $\alpha \geq 0$.

(ii) $\mu = w \boxtimes \rho$ *is free infinitely divisible, regularly varying with tail index* $-\frac{\alpha}{2}$.

Proof. Assume ((i)). Theorem 22 of [76] says that for $\rho \in \mathcal{M}_+$ and w be the standard Wigner measure, then $\rho_0 = \rho \boxtimes \rho$ is a free regular infinitely divisible probability measure if and only if $\mu = w \boxtimes \rho$ is a symmetric free infinitely divisible probability measure. Now from Lemma 8 of [10] we get

$$\mu^2 = w^2 \boxtimes \rho \boxtimes \rho = m \boxtimes \rho \boxtimes \rho = m \boxtimes \rho_0.$$

(See Remark 3.6.2 for a proof of the fact that $w^2 = m$.) Since ρ_0 is regularly varying with tail index $-\alpha$ we have from Corollary 3.4.1 that μ^2 is also regularly varying with tail index $-\alpha$. Thus by using the transform $x \mapsto \sqrt{x}$ we get that the symmetric measure μ is regularly varying with tail index $-\frac{\alpha}{2}$. Thus we have shown ((ii)).

The arguments given above can be reversed to show that ((ii)) implies ((i)). □

The following is also an immediate consequence of the above discussion and Corollary 3.4.4.

Corollary 3.4.5. *Let $\alpha \in (0,1)$ and the measure $\rho_{\frac{2\alpha}{\alpha+1}}$ is free stable of index $\frac{2\alpha}{\alpha+1}$. Then $w \boxtimes \rho_{\frac{2\alpha}{\alpha+1}}$ is regularly varying with tail index $-\frac{\alpha}{2}$.*

Now we relate our result for the free regular probability measures (Corollary 3.3.4) and the famous classical result (stated in Theorem 3.2.1) via the notion of Bercovici-Pata bijection.

Definition 3.4.6 ([29]). *The **Bercovici-Pata bijection** between the set of classical infinitely divisible probability measures $I(*)$ and the set of free infinitely divisible probability measures $I(\boxplus)$ is the mapping $\Lambda : I(*) \to I(\boxplus)$ that sends the measure μ in $I(*)$ with classical characteristic triplet (η, a, ν) (see equation (3.2.3)) to the measure $\Lambda(\mu)$ in $I(\boxplus)$ with free characteristic triplet (η, a, ν) (see equation (3.2.7)).*

Corollary 3.4.7. *Suppose $\alpha \geq 0$, $\eta' > 0$ and $\nu \in \mathcal{M}_+$ satisfies $\int_{\mathbb{R}_+} min(1,t)\, d\nu(t) < \infty$. Then the classical infinitely divisible probability measure $\mu_*^{\eta',0,\nu}$ has regularly varying tail of* *index $-\alpha$ if and only if the free regular infinitely divisible probability measure $\mu_{\boxplus,C}^{\eta',0,\nu}$, the image of $\mu_*^{\eta',0,\nu}$ under Bercovici-Pata bijection, has regularly varying tail of index $-\alpha$. In either case,*

$$\mu_*^{\eta',0,\nu}(x,\infty) \sim \mu_{\boxplus,C}^{\eta',0,\nu}(x,\infty) \text{ as } x \to \infty.$$

Proof of Corollary 3.4.7. Suppose the classical infinitely divisible probability measure $\mu_*^{\eta',0,\nu}$ has regularly varying tail, then by Theorem 3.2.1 we have the measure ν in the Laplace transform has the same regularly varying tail. Now both measures in the Laplace transform and the Fourier transform is same ν by Remark 21.6 of [80], since the measure ν satisfies the conditions $a = 0$ in (3.2.3), $\int_{-\infty}^{0} d\nu(t) = 0$, $\int_0^1 t d\nu(t) < \infty$ and $\eta > 0$. Then the relation (3.3.6) assures that σ has also the same regular variation and finally applying corollary 3.3.4 we can conclude that $\mu_{\boxplus,C}^{\eta',0,\nu}$ has the same regular variation like $\mu_*^{\eta',0,\nu}$. The arguments can also be reversed. □

3.5 Regular variation and Boolean infinitely divisible laws

Recall that the Boolean to free Bercovici-Pata bijection (B_1) was introduced in [29] (the definition of B_t for any $t \geq 0$ is given in (2.2.7)). The map B_1 establishes an one to one correspondence between the set of Boolean infinitely divisible probability measures on $\mathbb{R}$ to the set of all free infinitely divisible probability measures on $\mathbb{R}$ such that $B_1(\mu \uplus \nu) = B_1(\mu) \boxplus B_1(\nu)$. Also B_1 is a homeomorphism with respect to weak convergence.

According to Speicher and Woroudi [84], any probability measure on $\mathbb{R}$ is infinitely divisible with respect to Boolean convolution and therefore enjoys a Lévy-Khintchine representation of the form:

$$z - F_\mu(z) = \gamma_\mu + \int_{\mathbb{R}} \frac{1+tz}{z-t} d\rho_\mu(t)$$

where $\gamma_\mu \in \mathbb{R}$ and the Boolean Lévy measure ρ_μ of μ is a finite non-negative measure.

Theorem 3.5.1. *For any positively supported probability measure μ and $\alpha \geq 0$, the following are equivalent:*

(i) *μ is regularly varying with tail index $-\alpha$,*

(ii) *ρ_μ is regularly varying with tail index $-\alpha$.*

If any of the above holds then $\mu(y,\infty) \sim \rho_\mu(y,\infty)$ as $y \to \infty$.

Proof. First suppose μ be a positively supported probability measure with regularly varying with tail index $-\alpha$. By definition of Boolean to free Bercovici-Pata bijection we have $z - F_\mu(z) = \phi_{B_{1_\mu}}(z)$. Hence using Theorem 2.2.6 we get $B_1(\mu)$ is also regularly varying with tail index $-\alpha$. Now we know that a probability measure ν is free regular if and only if $B_1^{-1}(\nu)$ is supported on the non-negative part of the real line. This shows that $B_1(\mu)$ is free regular. Thus applying Corollary 3.3.4, we get that $B_1(\mu)$ and ρ_μ both have the same regularly varying tail index since ρ_μ is the free Lévy measure of $B_1(\mu)$. We finish the proof by observing that the arguments can be reversed to get the converse implication. □

It is very natural to note that a similar result like Corollary 3.4.7 can be obtained in an obvious way to connect the classical, free and Boolean cases which is not further described.

3.6 Another proof of Corollary 3.4.1

In this section, we provide the proof of Corollary 3.4.1 without using the Theorem 3.3.1. We have, $\mu = m \boxtimes \rho$ where m is the Marchenko-Pastur law. Interestingly the cumulant transform of μ becomes equal to the Ψ-transform of ρ, the Lévy measure of μ, i.e.,

$$C_\mu(z) = \int_0^\infty \frac{zt}{1-zt} d\rho(t) = \Psi_\rho(z), \qquad z \in \mathbb{C}^-. \tag{3.6.1}$$

In addition, it follows that for any $z \in \mathbb{C}^+$,

$$C_\mu\left(\frac{1}{z}\right) = \int_0^\infty \frac{t}{z-t} d\rho(t) = \frac{\phi_\sigma(z)}{z}.$$

Hence,

$$\begin{aligned} \phi_\mu(z) &= \int_0^\infty \frac{zt}{z-t} d\rho(t) \\ &= -z \int_0^\infty \frac{z-t-z}{z-t} d\rho(t) \\ &= -z + z^2 G_\rho(z), \qquad z \in \mathbb{C}^-. \end{aligned} \tag{3.6.2}$$

Now recall that for a probability measure σ on $\mathbb{R}$ which has all moments finite up to order p we have the following Taylor expansions from Theorem 1.3 and Theorem 1.5 of [22]:

$$R_\sigma(z) = \sum_{i=0}^{p-1} k_{i+1}(\sigma) z^i + \mathrm{o}\left(z^{p-1}\right). \tag{3.6.3}$$

Using (3.6.3) and the relation $C_\sigma(z) = zR_\sigma(z)$, we have,

$$C_\sigma(z) = \sum_{i=1}^{p} k_i(\sigma) z^i + \mathrm{o}(z^p). \tag{3.6.4}$$

Also it is easy to see that,

$$\Psi_\sigma(z) = \sum_{i=1}^{p} m_i(\sigma) z^i + \mathrm{o}(z^p). \tag{3.6.5}$$

Where $m_i(\sigma), k_i(\sigma)$ are the ith moment and ith cumulant of σ respectively. Therefore $C_{\sigma_1}(z) = \Psi_{\sigma_2}(z)$ for two probability measures σ_1 and σ_2 where σ_2 has moments finite up to order p if and only if σ_1 has finite cumulants up to order p and

$$k_i(\sigma_1) = m_i(\sigma_1), \qquad \forall\ 1 \le i \le p, \tag{3.6.6}$$

by comparing the first p terms of the Taylor expansions of (3.6.4) and (3.6.5). Hence the equation (3.6.1) allows us to conclude the relation (3.6.6) is true for $\sigma_1 = \mu$ and $\sigma_2 = \rho$.

Lemma 3.6.1. *Suppose $\mu = m \boxtimes \rho$ and ρ is a probability measure such that the map $y \mapsto \rho(y, \infty)$ is regularly varying with tail index $-\alpha$. Then,*

$$r_{\phi_\mu}(z) = r_{G_\rho}(z). \tag{3.6.7}$$

Proof. There exists $p \in \mathbb{N}$ such that $p \le \alpha \le p+1$ and $\mu \in \mathcal{M}_p$ since the map $y \mapsto \rho(y, \infty)$ is regularly varying with tail index $-\alpha$. Therefore the remainder term of the Voiculescu transform of μ can be written in the following way using (3.2.2):

$$\begin{aligned} r_{\phi_\mu}(z) &= z^{p-1}\left(\phi_\mu(z) - \sum_{j=0}^{p-1} k_{j+1}(\sigma) z^{-j}\right) \\ &= z^{p-1}\left(z^2 G_\rho(z) - z - \sum_{j=0}^{p-1} m_{j+1}(\sigma) z^{-j}\right), \end{aligned}$$

where in the second line we have used (3.6.2) and the moment cumulant relationship (3.6.6). Relabelling the index in the sum, we get,

$$\begin{aligned} r_{\phi_\mu}(z) &= z^{p+1}\left(G_\rho(z) - \frac{1}{z} - \sum_{j=2}^{p+1} m_{j-1}(\sigma)\, z^{-j}\right) \\ &= z^{p+1}\left(G_\rho(z) - \sum_{j=1}^{p+1} m_{j-1}(\sigma)\, z^{-j}\right). \end{aligned}$$

Now the lemma follows from (3.2.1). □

Proof of Corollary 3.4.1. First we suppose $p \le \alpha < p+1$ and ρ is regularly varying of index $-\alpha$. Then we have from [59],

as $y \to \infty$,

$$\Im r_{G_\rho}(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \rho(y,\infty) \tag{3.6.8}$$

using (3.6.7) and the above asymptotic equivalence (3.6.8) we get,

$$\Im r_{\phi_\mu}(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \rho(y,\infty) \tag{3.6.9}$$

Therefore $\Im r_{\phi_\mu}(iy)$ is regularly varying of index $-(\alpha-p)$ and consequently μ is regularly varying of index $-\alpha$ by Theorem 2.1 of [59]. Thus as $y \to \infty$,

$$\Im r_{\phi_\mu}(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty) \tag{3.6.10}$$

Putting the last two asymptotic equivalences (3.6.9) and (3.6.10) together, we get

$$\mu(y,\infty) \sim \rho(y,\infty)$$

as $n \to \infty$.

The arguments can also be reversed to give the converse statement in this case.

Now suppose $\alpha = p+1$ and ρ is regularly varying of index $-\alpha$ with ρ has has moments finite up to order p only. Then as $y \to \infty$, $\Re r_{\phi_\mu}(iy) = \Re r_{\phi_\rho}(iy) \sim -\frac{\pi}{2} y^p \rho(y,\infty)$.

Again by similar arguments like the previous case we shall have $\mu(y,\infty) \sim \rho(y,\infty)$ in this case also and the arguments can be reversed too. □

Remark 3.6.2. *The standard Marchenko-Pastur law m which has the density $f_m(x) = \frac{1}{2\pi x}\sqrt{x(4-x)}\mathbf{1}_{[0,4]}(\mathbf{x})$ on $[0,\infty)$ is the image under the mapping $x \mapsto x^2$ of the standard Wigner semi-circle law w having the density $f_w(x) = \dfrac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(\mathbf{x})$ on $\mathbb{R}$.*

Proof. The semi-circle distribution and the Marchenko-Pastur distribution have the S-transforms $S_w(z)$ and $S_m(z)$ given by:

$$S_w(z) = \frac{1}{\sqrt{z}}, \tag{3.6.11}$$

and

$$S_m(z) = \frac{1}{z+1}. \tag{3.6.12}$$

Now from Theorem 6 of [10] we have,

$$\begin{aligned} S_{w^2}(z) &= \frac{z}{z+1}S_w^2(z) \\ &\overset{(3.6.11)}{=} \frac{z}{z+1} \times \frac{1}{z} \\ &= \frac{1}{z+1} \\ &\overset{(3.6.12)}{=} S_m(z). \end{aligned}$$

Since w has positive densities on both positive and negative parts of $\mathbb{R}$, one also needs to check the $\tilde{S}$-transform (see page 11 for definition of $\tilde{S}$-transform). But a similar calculation will provide us that $\tilde{S}_{w^2}(z) = S_m(z)$ both in suitable domains for z. □

3.7 Appendix

In the above proofs we have used some important results from [59]. We recall these results here to help the reader. The following two theorems are written in a more compact form which are in particular Theorems $2.1 - 2.4$ in [59].

Theorem 3.7.1. *Let p be a nonnegative integer and μ be a probability measure in the class $\mathcal{M}_p$ and $\alpha \in [p, p+1)$. The following statements are equivalent:*

(i) $y \mapsto \mu(y,\infty)$ *is regularly varying of index* $-\alpha$.

(ii) $y \mapsto \Im r_G(iy)$ *is regularly varying of index* $-(\alpha - p)$.

(iii) $y \mapsto \Im r_\phi(iy)$ *is regularly varying of index* $-(\alpha - p)$, $\Re r_\phi(iy) \gg y^{-1}$ *as* $y \to \infty$ *and* $r_\phi(z) \gg z^{-1}$ *as* $z \to \infty$ *n.t.*

If any of the above statements holds, we also have, as $z \to \infty$ *n.t.,* $r_G(z) \sim r_\phi(z) \gg z^{-1}$*; as* $y \to \infty$,

$$\Im r_\phi(iy) \sim \Im r_G(iy) \sim -\frac{\frac{\pi(p+1-\alpha)}{2}}{\cos\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty) \gg \frac{1}{y} \text{ and } \Re r_\phi(iy) \sim \Re r_G(iy) \gg \frac{1}{y}.$$

If $\alpha > p$ *and any of the statements* (i)-(iii) *holds, we further have, as* $y \to \infty$,

$$\Re r_\phi(iy) \sim \Re r_G(iy) \sim -\frac{\frac{\pi(p+2-\alpha)}{2}}{\sin\frac{\pi(\alpha-p)}{2}} y^p \mu(y,\infty).$$

If $\alpha = p = 0$ *and any of the statements* (i)-(iii) *holds, we further have, as* $y \to \infty$,

$$\Re r_\phi(iy) \sim \Re r_G(iy) \sim -\mu(y,\infty).$$

Theorem 3.7.2. *Let* p *be a nonnegative integer and* μ *be a probability measure in the class* $\mathcal{M}_p$*. Let* $\beta \in (0, 1/2)$ *and* $\alpha = p + 1$*. The following statements are equivalent:*

(i) $y \mapsto \mu(y,\infty)$ *is regularly varying of index* $-(p+1)$.

(ii) $y \mapsto \Re r_G(iy)$ *is regularly varying of index* -1.

(iii) $y \mapsto \Re r_\phi(iy)$ *is regularly varying of index* -1, $y^{-1} \ll \Im r_\phi(iy) \ll y^{-(1-\beta/2)}$ *as* $y \to \infty$ *and* $z^{-1} \ll r_\phi(z) \ll z^{-\beta}$ *as* $z \to \infty$ *n.t.*

If any of the above statements holds, we also have, as $z \to \infty$ *n.t.,* $z^{-1} \ll r_G(z) \sim r_\phi(z) \ll z^{-\beta}$*; as* $y \to \infty$,

$$y^{-(1+\beta/2)} \ll \Re r_\phi(iy) \sim \Re r_G(iy) \sim -\frac{\pi}{2} y^p \mu(y,\infty) \ll y^{-(1-\beta/2)}$$

and

$$y^{-1} \ll \Im r_\phi(iy) \sim \Im r_G(iy) \ll y^{-(1-\beta/2)}.$$

Chapter 4

Largest eigenvalue in the rank one case

4.1 Introduction

The final two chapters of this thesis consider the generalization of one of the most studied random graph, namely the Erdős–Rényi random graph (ERRG). It is a graph on N vertices where an edge is present independently with probability ε_N. Given a graph on N vertices, say, $\{1, \ldots, N\}$, let A_N denote the adjacency matrix of the graph, whose (i, j)-th entry is 1 if there is an edge between vertices i and j, and 0 otherwise. Important statistics of the graph are the eigenvalues and eigenvectors of A_N which encode crucial information about the graph. The adjacency matrix of the ERRG is a symmetric matrix with diagonal entries zero, and the entries above the diagonal are independent and identically distributed Bernoulli random variables with parameter ε_N. We consider an inhomogeneous extension of the ERRG where the presence of an edge between vertices i and j is given by a Bernoulli random variable with parameter $p_{i,j}$ and these $\{p_{i,j} : 1 \le i < j \le N\}$ need not be same. When $p_{i,j}$ are same for all vertices i and j it shall be referred as (homogeneous) ERRG.

The mathematical foundations of inhomogeneous ERRG where the connection probabilities $p_{i,j}$ come from a discretization of a symmetric, non-negative function f on $[0, 1]^2$ was initiated in [31]. The said article considered edge probabilities given by

$$p_{i,j} = \frac{1}{N} f\left(\frac{i}{N}, \frac{j}{N}\right). \tag{4.1.1}$$

In that case the average degree is bounded and the phase transition picture on the largest cluster size was studied in the same article. See [30, 89] for further results on inhomogeneous ERRG.

We consider a similar set-up but the average degree is unbounded and study the properties of eigenvalues of the adjacency matrix. The connection probabilities are given by

$$p_{i,j} = \varepsilon_N f\left(\frac{i}{N}, \frac{j}{N}\right)$$

with the assumption that

$$N\varepsilon_N \to \infty. \tag{4.1.2}$$

Let $\lambda_1(A_N) \geq \ldots \geq \lambda_N(A_N)$ be the eigenvalues of A_N. It was shown in [37] (see also [98] for a graphon approach) that the empirical distribution of the centered adjacency matrix converges, after scaling with $\sqrt{N\varepsilon_N}$, to a compactly supported measure μ_f. When $f \equiv 1$, the limiting law μ_f turns out to be the semicircle law (w). Note that $f \equiv 1$ corresponds to the (homogeneous) ERRG (see [46, 86] also for the homogeneous case). It was shown in [37] that when f is of the form $f(x, y) = r(x)r(y)$ the limit distribution of the bulk becomes $w \boxtimes \mu$, where μ is the law of $r(U)$ where U is Uniform random variable on $[0, 1]$.

Quantitative estimates on the largest eigenvalue of the homogeneous case (when $N\varepsilon_N \gg (\log N)^4$) were studied in [55, 94] and it follows from their work that the smallest and second largest eigenvalue converges to the edge of the support of semicircular law. The results were improved recently in [24] and the condition on sparsity can be extended to the case $N\varepsilon_N \gg \log N$ (which is also the connectivity threshold for the graph). It was shown that inhomogeneous ERRG also has similar behaviour. The largest eigenvalue of inhomogeneous ERRG when $N\varepsilon_N \ll \log N$ was treated in [25].

It is well known that in the classical case of a (standard) Wigner matrix, the largest eigenvalue converges to the Tracy-Widom law. We note that there is a different scaling between the edge and bulk of the spectrum in ERRG. As pointed out before that the bulk scales at $(N\varepsilon_N)^{-1/2}$ and the largest eigenvalue has the scaling $(N\varepsilon_N)^{-1}$. Letting

$$W_N = A_N - \mathrm{E}(A_N)\,, \tag{4.1.3}$$

where $\mathrm{E}(A_N)$ is the entrywise expectation of A_N, it is easy to see that

$$A_N = \varepsilon_N\, \mathbb{1}\, \mathbb{1}' + W_N\,,$$

where $\mathbb{1}$ is the $N \times 1$ vector with each entry 1. Since the empirical spectral distribution of $(N\varepsilon_N)^{-1/2} W_N$ converges to semi-circle law, the largest eigenvalue of the same converges to 2 almost surely. As $\mathrm{E}[A_N]$ is a rank-one matrix, it turns out that the largest eigenvalue of A_N scales like $N\varepsilon_N$, which is different from the bulk scaling.

To derive the fluctuations one needs to study in details what happens to the rank-one perturbations of a Wigner matrix. When W_N is a symmetric random matrix with independent and identically distributed entries and the perturbation comes from a rank-one matrix then the fluctuation of the largest eigenvalue depends on the form of the deformation matrix (see [33, 34, 52]). For example, when

$$M_N = \frac{W_N}{\sqrt{N}} + P_N$$

where $P_N = \theta\, \mathbb{1}\, \mathbb{1}'$ then $\lambda_1(M_N)$ has a Gaussian fluctuation. If P_N is a diagonal matrix with single non-zero entry, the fluctuations depend on the distribution of the entries of W_N. The rank-one case was extended to the finite rank case in the works of Benaych-Georges et al. [23], Pizzo et al. [77]. We do not go into further discussion of the results there as they crucially use the fact that bulk behaviour (after scaling) in the limit is semicircular law, which is not generally the case here.

The scaling limit of the maximum eigenvalues of an ERRG turns out to be interesting. Recall that the fluctuations of the maximum eigenvalue ($\lambda_1(A_N)$) in the homogeneous case were studied in [49]. It was proved that

$$(\varepsilon_N(1-\varepsilon_N))^{-1/2} \left(\lambda_1(A_N) - \mathrm{E}[\lambda_1(A_N)]\right) \Rightarrow N(0,2).$$

The above result was shown under the assumption that

$$(\log N)^{\xi} \ll N\varepsilon_N \tag{4.1.4}$$

for some $\xi > 8$, which is a stronger assumption than (4.1.2). Further, Under the assumption that $N^{\xi} \ll N\varepsilon_N$ for some $\xi \in (2/3, 1]$, it was proved in Theorem 2.7 of [48] that the second largest eigenvalue ($\lambda_2(A_N)$) of the (homogeneous) ERRG after an appropriate centering scaling converge in distribution to the Tracy-Widom law. The conditions were recently improved in [66]. The properties of the largest eigenvector in the homogeneous case was studied in [48, 66, 86].

The adjacency matrix of the inhomogeneous ERRG does not fall directly into the purview of the above results, since W_N, as in (4.1.3), is a symmetric matrix, with independent entries above the diagonal, but the entries have a variance profile, which also depends on the size of the graph. The inhomogeneity does not allow the use of local laws suitable for semicircle law in an obvious way. The last two chapters of this thesis aims at extending the results obtained in [49] for the case that f is a constant to the case that f is a non-negative, symmetric, bounded, Riemann integrable function on $[0,1]^2$ which induces an integral operator of finite rank k, under the assumption that (4.1.4) holds.

In this chapter we consider the rank one case i.e., in the definition of edge probabilities in (4.1.1) take $f(x,y)$ to be a product of two bounded and Riemann integrable functions (details are in (4.2.5)). The introduction to the integral operator arising out of f is not needed in this chapter while in the finite rank case, considered in the next chapter, we use the integral operator. A crucial observation in the rank one case is that there is exactly one eigenvalue of the adjacency matrix A_N, outside the bulk which escapes to infinity. Inspired by the calculations done in [49], here we shall show in Theorem 4.2.4 that the largest eigenvalue of the adjacency matrix of an inhomogeneous Erdős–Rényi random graph after suitable scaling and centering converges in distribution to the normal distribution with zero mean and some finite variance.

In the next section, we describe the model followed by the main theorems of this chapter. Section 4.3 states some necessary estimates, describes the master equation (4.3.12) for the largest eigenvalue of the adjacency matrix and the proof of first order asymptotics of the same. The proofs of the main theorems are in Section 4.4. The results for the well known Chung-Lu graph is illustrated in Section 4.5. Finally, Section 4.6 provides the proofs of the technical lemmas stated in Section 4.3.

4.2 Set up and main results

Let $f\colon [0,1]\times[0,1]\to[0,\infty)$ be a bounded Riemann integrable function, satisfying

$$f(x,y)=f(y,x) \qquad \forall\, x,y\in[0,1]\,. \tag{4.2.1}$$

Since f is bounded on a compact domain, there exists a positive real constant M such that

$$M:=\sup_{x,y\in[0,1]} f(x,y)\,. \tag{4.2.2}$$

A sequence of positive real numbers $(\varepsilon_N : N \geq 1)$ is fixed that satisfies

$$\lim_{N\to\infty} \varepsilon_N = \varepsilon_\infty \,, \qquad \text{and} \tag{4.2.3}$$

$$\frac{(\log N)^\xi}{N} \ll \varepsilon_N \ll 1 \text{ as } N \to \infty \text{ for some } \xi > 8. \tag{4.2.4}$$

In the next chapter, we shall assume only the existence of ε_∞, which is enough to conclude the results. For the sake of simplicity in the expressions we assume $\varepsilon_\infty = 0$ in this chapter although that need not be the case. Consider the random graph $\mathbb{G}_N$ on vertices $\{1, \ldots, N\}$ where, for each (i, j) with $1 \leq i \leq j \leq N$, an edge is present between vertices i and j with probability $\varepsilon_N f(\frac{i}{N}, \frac{j}{N})$, independently of other pairs of vertices. In particular, $\mathbb{G}_N$ is an undirected graph with possible self loops but without multiple edges. Boundedness of f ensures that $\varepsilon_N f(\frac{i}{N}, \frac{j}{N}) \leq 1$ for all $1 \leq i \leq j \leq N$ when N is large enough.

The adjacency matrix of $\mathbb{G}_N$ is denoted by A_N. Clearly, A_N is a symmetric random matrix whose upper triangular entries including the diagonal are independent Bernoulli random variables, i.e.,

$$A_N(i,j) \triangleq \text{BER}\left(\varepsilon_N f\left(\frac{i}{N}, \frac{j}{N}\right)\right), \qquad 1 \leq i \leq j \leq N \,.$$

As mentioned in the introduction of this chapter, we consider the case when f has a multiplicative structure. Let r be bounded and Riemann integrable on $[0, 1]$ and

$$f(x, y) = r(x)r(y) \text{ for all } x, y \in [0, 1]. \tag{4.2.5}$$

The following definition will be used both in this and the next chapter.

Definition 4.2.1. *A sequence of events E_N occurs* with high probability*, abbreviated as w.h.p., if*

$$P(E_N^c) = O\left(e^{-(\log N)^\eta}\right) \,,$$

for some $\eta > 1$. For random variables Y_N, Z_N,

$$Y_N = O_{hp}(Z_N) \,,$$

means there exists a deterministic finite constant C such that

$$|Y_N| \leq C|Z_N| \text{ w.h.p.} \,,$$

and

$$Y_N = o_{hp}(Z_N)\,,$$

means that for all $\delta > 0$*,*

$$|Y_N| \le \delta |Z_N| \text{ w.h.p.}$$

We shall say

$$Y_N = O_p(Z_N)\,,$$

to mean that

$$\lim_{x\to\infty} \sup_{N\ge 1} P(|Y_N| > x|Z_N|) = 0\,,$$

and

$$Y_N = o_p(Z_N)\,,$$

to mean that for all $\delta > 0$*,*

$$\lim_{N\to\infty} P(|Y_N| > \delta |Z_N|) = 0\,.$$

The reader may note that if $Z_N \neq 0$ a.s., then "$Y_N = O_p(Z_N)$" and "$Y_N = o_p(Z_N)$" are equivalent to "$(Z_N^{-1}Y_N : N \ge 1)$ is stochastically tight" and "$Z_N^{-1}Y_N \xrightarrow{P} 0$", respectively. Besides, "$Y_N = O_{hp}(Z_N)$" is a much stronger statement than $Y_N = O_p(Z_N)$, and so is "$Y_N = o_{hp}(Z_N)$" than "$Y_N = o_p(Z_N)$".

In the rest of this chapter and the next chapter, the subscript 'N' is dropped from notations like A_N, W_N, ε_N, e_N etc. and the ones that will be introduced. We state the main theorems of this chapter under under the above set up i.e. A is the adjacency matrix of the Inhomogeneous ERRG $\mathbb{G}_N$ such that (4.2.3), (4.2.4) and (4.2.5) are satisfied along with $\varepsilon_\infty = 0$. Let $\mu_{\max}$ be the largest eigenvalue of A and define

$$e = \begin{bmatrix} N^{-1/2} r(1/N) \\ N^{-1/2} r(2/N) \\ \vdots \\ N^{-1/2} r(1) \end{bmatrix}.$$

The following theorem gives us the first order asymptotics of $\mu_{\max}$.

Theorem 4.2.2. *We have,*

$$\mu_{\max} - N\varepsilon e' e = \mathrm{O}_{hp}\left(\sqrt{N\varepsilon}\right). \tag{4.2.6}$$

The proof of the above theorem is provided in the next section. The following theorem tells us the behaviour of $\mathrm{E}\mu_{\max}$.

Theorem 4.2.3. *We have,*

$$\mathrm{E}\mu_{\max} = \frac{\beta_2}{(e'e)^2} + N\varepsilon e'e + \mathrm{O}\left(\varepsilon + (N\varepsilon)^{-1/2}\right) \tag{4.2.7}$$

where

$$\beta_2 = \int_0^1 r^3(x)dx \int_0^1 r(x)dx.$$

Our next result tries to address the limiting behaviour of the largest eigenvalue of the adjacency matrix under the product structure of f.

Theorem 4.2.4. *We have,*

$$\varepsilon^{-1/2}\left(\mu_{\max} - \mathrm{E}[\mu_{\max}]\right) \Rightarrow N(0, \sigma^2) \tag{4.2.8}$$

where $N(0,\, \sigma^2)$ is a normal random variable with mean zero and variance

$$\sigma^2 := 2\left(\frac{\int_0^1 r^3(x)dx}{\int_0^1 r^2(x)dx}\right)^2. \tag{4.2.9}$$

The following theorem essentially states that the eigenvector corresponding to the largest eigenvalue $\mu_{\max}$ is asymptotically aligned with the vector e.

Theorem 4.2.5. *Let v be the normalized eigenvector corresponding to the largest eigenvalue $\mu_{\max}$ of A. Then,*

$$e'v - 1 = \mathrm{O}_{hp}\left((N\varepsilon)^{-1/2}\right). \tag{4.2.10}$$

The proofs of the above theorems are in the next two sections.

4.3 Some estimates and the first order asymptotics for $\mu_{\max}$

Consider $\mathrm{E}[A]$ to be the matrix having entry wise expected values of A. Then we shall be able to write,

$$\mathrm{E}[A] := N\varepsilon ee'.$$

Observe that $e'e := \frac{1}{N}\sum_{i=0}^{N} r^2(i/N)$ converges to $\int_0^1 r^2(x)dx$ as $N \to \infty$. Here without loss of generality we assume that integral is non-zero. We will always choose N large enough such that $\varepsilon r(\frac{i}{N})r(\frac{j}{N}) < 1$. We generally refer W (as defined in (4.1.3)) to be the Wigner-type matrix because it is a symmetric matrix with centered, independent upper triangular entries. The boundedness of f allows us to conclude that the entries of W are pointwise bounded. The following bound on the moments of W turns out to be useful later. We have,

$$\mathrm{E}[|W(i,j)|^n] \leq C^n \varepsilon, \tag{4.3.1}$$

for some constant $C > 0$ independent of n and N. The following lemma gives a bound on the largest eigenvalue of W. The result is similar to [49, Lemma 4.3]. We provide a proof of it in the later section as the entries of W are no longer identically distributed. The result immediately follows from the work of Vu [94]. The result also does not depend on the specific form of f.

Lemma 4.3.1. *Let $\|\cdot\|$ denotes the spectral norm of a matrix and suppose the conditions* (4.2.1) *and* (4.2.4) *are satisfied. Then the event*

$$\|W\| \leq 2\sqrt{MN\varepsilon} + C_1(N\varepsilon)^{1/4}(\log N)^{\xi/4}$$

occurs w.h.p. for some constant $C_1 > 0$ and M is as in (4.2.2).

We record a few estimates that will subsequently be used in the proofs of both this and the next chapter. Since their proofs are routine, they are being postponed to Section 4.6. The notations e_1 and e_2, introduced in the next lemma and used in the subsequent lemmas, should not be confused with e_j defined in (5.2.2) (in the next chapter). Continuing to suppress 'N' in the subscript, let

$$L = \lfloor \log N \rfloor , \tag{4.3.2}$$

where $\lfloor x \rfloor$ is the largest integer less than or equal to x. It is noteworthy to mention that the following lemmas also do no depend on the rank of f. The following lemma gives an estimate on $\mathrm{E}[e'W^n e]$.

Lemma 4.3.2. *There exists $0 < C_1 < \infty$ such that if e_1 and e_2 are $N \times 1$ vectors with each entry in $[-1/\sqrt{N}, 1/\sqrt{N}]$, then*

$$\left|\mathrm{E}\left(e_1' W^n e_2\right)\right| \leq (C_1 N\varepsilon)^{n/2}, 2 \leq n \leq L.$$

If we consider the equation (4.3.12), then we notice that bounds on terms like $e'W^n e$ are needed. The next lemma shows that for n not large enough, the value concentrates around the mean. The lemmas are stated in a slightly more general setting because they will be used in Chapter 5.

Lemma 4.3.3. *There exists $\eta_1 > 1$ such that for e_1, e_2 as in Lemma 4.3.2, it holds that*

$$\max_{2\le n\le L} P\left(\left|e_1'W^n e_2 - \mathrm{E}\left(e_1'W^n e_2\right)\right| > N^{(n-1)/2}\varepsilon^{n/2}(\log N)^{n\xi/4}\right)$$
$$= \mathrm{O}\left(e^{-(\log N)^{\eta_1}}\right), \tag{4.3.3}$$

where ξ is as in (4.1.4). In addition,

$$e_1'We_2 = o_{hp}\left(N\varepsilon\right). \tag{4.3.4}$$

Lemma 4.3.4. *If e_1, e_2 are as in Lemma 4.3.2, then*

$$\mathrm{Var}\left(e_1'We_2\right) = O(\varepsilon),$$

and

$$\mathrm{E}\left(e_1'W^3 e_2\right) = O(N\varepsilon). \tag{4.3.5}$$

Now we proceed towards the proof of Theorem 4.2.2. Note that the Assumption (4.2.4) and the Lemma 4.3.1 allows us to conclude that

$$\|W\| \le (2\sqrt{M} + C_1)\sqrt{N\varepsilon}. \tag{4.3.6}$$

w.h.p. for all large N. From (4.1.3) we have $|\mu_{\max} - N\varepsilon e'e| \le \|W\|$ since $\mathrm{E}[A]$ is a rank one matrix with largest eigenvalue $N\varepsilon e'e$. Therefore using (4.3.6), we have,

$$|\mu_{\max} - N\varepsilon e'e| \le (2\sqrt{M} + C_1)\sqrt{N\varepsilon} \tag{4.3.7}$$

w.h.p. An appeal to the equation (4.2.4) yields from (4.3.7), that

$$\mu_{\max} \ge \frac{N\varepsilon e'e}{2} \tag{4.3.8}$$

happens w.h.p. for all large N. Using (4.3.6) and (4.3.8) we get that there exists an $\varepsilon_0 > 0$ such that

$$\frac{\|W\|}{\mu_{\max}} \le \frac{2(2\sqrt{M} + C_1)}{\sqrt{N\varepsilon e'e}} \le 1 - \varepsilon_0, \tag{4.3.9}$$

w.h.p. for all large N. For further calculations in this subsection, we work on the high probability event where the above holds. Let v be the eigenvector corresponding to the largest eigenvalue $\mu_{\max}$ of A, that is,

$$Av = \mu_{\max} v.$$

Then using (4.1.3) we have $(\mu_{\max} I - W)v = N\varepsilon ee'v = N\varepsilon(e'v)e$, where I is the $N \times N$ identity matrix. Now it follows from (4.3.9), that $\mu_{\max}$ is not an eigenvalue of W, then the matrix $(\mu_{\max} I - W)$ is invertible and so,

$$v = N\varepsilon e'v(\mu_{\max} I - W)^{-1} e. \tag{4.3.10}$$

Premultiplying the above equation by e', we get,

$$1 = N\varepsilon e'(\mu_{\max} I - W)^{-1} e, \tag{4.3.11}$$

where we have used $e'v \neq 0$ (since $\mu_{\max}$ is not an eigenvalue of W w.h.p.). Notice that the equation (4.3.11) can be expressed as

$$\begin{aligned} \mu_{\max} &= N\varepsilon e' \left(I - \frac{W}{\mu_{\max}} \right)^{-1} e \\ &= N\varepsilon \sum_{n=0}^{\infty} e' \left(\frac{W}{\mu_{\max}} \right)^{n} e \end{aligned} \tag{4.3.12}$$

where the series is valid due to (4.3.9). The equation (4.3.7) suggests that

$$\frac{\mu_{\max}}{\sqrt{N\varepsilon e'e}} \xrightarrow{P} 1.$$

The above probability convergence is not enough to conclude Theorem 4.2.2 about the first order asymptotics of $\mu_{\max}$. We are going to prove the theorem with the help of (4.3.12).

Proof of Theorem 4.2.2. Let us consider the Laurent series expansion of $\mu_{\max}$ from (4.3.12), in the following way,

$$\mu_{\max} = N\varepsilon e'e + N\varepsilon \frac{e'We}{\mu_{\max}} + N\varepsilon \sum_{n=2}^{\infty} \frac{e'W^n e}{\mu_{\max}^n}, \tag{4.3.13}$$

which holds w.h.p. due to Lemma 4.3.1. The term $e'We$ in the numerator in the second term on the right hand side is $\mathrm{O}_{hp}(\sqrt{N\varepsilon})$ by an application of Lemma 4.3.1. Hence using (4.3.8) we have the second term of $\mathrm{O}_{hp}(\sqrt{N\varepsilon})$. The final term in (4.3.13) can be bounded in the following way:

$$N\varepsilon \sum_{n=2}^{\infty} \frac{e'W^n e}{\mu_{\max}^n} \le N\varepsilon \sum_{n=2}^{\infty} \left(\frac{\|W\|}{\mu_{\max}}\right)^{n/2} .$$

Now using (4.3.9) we get that the right hand side of the above equation is $\mathrm{O}_{hp}(1)$ w.h.p. This gives us the equation (4.2.6). □

4.4 Proof of main Theorems

The section will be devoted to the proofs of our main results. In the Proposition 4.4.1 we provide the main representation of $\mu_{\max}$ (with the help of (4.3.12)) which will later be used to derive the distributional convergence stated in Theorem 4.2.4. The following Proposition describes a more detailed description of $\mu_{\max}$ and $\mathrm{E}[\mu_{\max}]$.

Proposition 4.4.1. *Let the assumptions of Theorem 4.2.4 hold. Then w.h.p.,*

$$\mu_{\max} = \mu_0 + \frac{e'We}{e'e} + \Xi \tag{4.4.1}$$

where $\mu_0 = \mu_0(N)$ is a deterministic real number, Ξ satisfies

$$\mathrm{E}[|\Xi|] = \mathrm{O}\left(\frac{(\log N)^{\frac{\xi}{2}}}{\sqrt{N}}\right) \tag{4.4.2}$$

and

$$\Xi = \mathrm{O}_{hp}\left(\frac{(\log N)^{\frac{\xi}{2}}}{\sqrt{N}}\right) . \tag{4.4.3}$$

Using the above Proposition we derive Theorem 4.2.4. Later in the present section, we deduce the proof of the proposition 4.4.1.

Proof of Theorem 4.2.4. First observe that Proposition 4.4.1 allows us to write

$$\frac{\mu_{\max} - \mathrm{E}\mu_{\max}}{\sqrt{\varepsilon}} = \frac{e'We}{\sqrt{\varepsilon}e'e} + \mathrm{O}_{hp}\left(\frac{(\log N)^{\frac{\xi}{2}}}{\sqrt{N\varepsilon}}\right) . \tag{4.4.4}$$

Note that using the assumption $N\varepsilon \gg (\log N)^{\xi}$ we can see that the last factor goes to zero in probability. Hence to show the distributional convergence stated in (4.2.8), it is enough to show that

$$\frac{e'We}{\sqrt{\varepsilon e'e}} \Rightarrow N\left(0,\sigma^2\right)\,, \tag{4.4.5}$$

where σ^2 is as defined in (4.2.9). Observe that

$$\begin{aligned} e'We &= \frac{1}{N}\sum_{i,j=1}^{N} W(i,j)r\left(\frac{i}{N}\right)r\left(\frac{j}{N}\right) \\ &= \frac{2}{N}\sum_{i\le j} W(i,j)r\left(\frac{i}{N}\right)r\left(\frac{j}{N}\right) - \frac{1}{N}\sum_{1\le i\le N} W(i,i)r^2\left(\frac{i}{N}\right). \end{aligned} \tag{4.4.6}$$

Let us consider the first term in the right hand side of (4.4.6) and set $k = 1+2+\cdots+N$. Define

$$\begin{aligned} \sigma_k^2 &:= \sum_{i\le j} \mathrm{Var}\left(\frac{2}{N}W(i,j)r\left(\frac{i}{N}\right)r\left(\frac{j}{N}\right)\right) \\ &= \varepsilon\sum_{i\le j}\frac{4}{N^2}r^3\left(\frac{i}{N}\right)r^3\left(\frac{j}{N}\right)\left(1-\varepsilon r\left(\frac{i}{N}\right)r\left(\frac{j}{N}\right)\right). \end{aligned}$$

Therefore $\sigma_k = \mathrm{O}(\sqrt{\varepsilon})$. It is easy to see that the conditions of the Lyapunov's Central limit theorem are satisfied and hence

$$\frac{2}{N\sigma_k}\sum_{i\le j} W(i,j)r\left(\frac{i}{N}\right)r\left(\frac{j}{N}\right) \Rightarrow N(0,1).$$

In particular

$$\frac{e'We}{\sigma_k} \Rightarrow N(0,1),$$

since the second term in the right hand side of the equation (4.4.6) after dividing by σ_k converges to zero in probability. The distributional convergence in (4.4.5) follows from the following observation

$$\lim_{N\to\infty}\frac{\sigma_k}{\sqrt{\varepsilon e'e}} = \sqrt{2}\frac{\int_0^1 r^3(x)dx}{\int_0^1 r^2(x)dx}.$$

□

Now we write down the proof of Proposition 4.4.1.

Proof of Proposition 4.4.1. We consider the high probability event A_0 where Theorem 4.2.2, Lemma 4.3.1 and Lemma 4.3.3 hold. Therefore on A_0 we have

$$\frac{\|W\|}{\mu_{\max}} \le \frac{C_3}{(\log N)^{\xi/2}} \tag{4.4.7}$$

for some constant $C_3 > 0$, using condition (4.2.4) and the equation (4.3.9) on A_0. Note that the expansion of $\mu_{\max}$ up to second order (or up to any finite order) i.e. of the form (4.3.13) is not enough to conclude this proposition. So, we write the equation (4.3.12) as

$$\mu_{\max} = N\varepsilon \sum_{n=0}^{L} \frac{e'W^n e}{\mu_{\max}^n} + N\varepsilon \sum_{n>L} \frac{e'W^n e}{\mu_{\max}^n}, \tag{4.4.8}$$

where L is as in (4.3.2). Using (4.4.7), the second term in the right hand side of the above expression can be bounded above by a constant multiple of

$$\sum_{n>L} \frac{\|W\|^{n-1}}{\mu_{\max}^{n-1}} \le \left(\frac{1}{(\log N)^{\xi/2}} \right)^L, \tag{4.4.9}$$

and the final expression is $\mathrm{O}\left((\log N)^{-\log N}\right)$. We rewrite the first term in (4.4.8) as follows

$$\begin{aligned} N\varepsilon \sum_{n=0}^{L} \frac{e'W^n e}{\mu_{\max}^n} &= N\varepsilon \sum_{n=0}^{L} \frac{\mathrm{E}[e'W^n e]}{\mu_{\max}^n} \\ &\quad + N\varepsilon \sum_{n=2}^{L} \frac{e'W^n e - \mathrm{E}[e'W^n e]}{\mu_{\max}^n} + \frac{N\varepsilon}{\mu_{\max}} e'We. \end{aligned} \tag{4.4.10}$$

Let us consider the second term in the above equation. Using Lemma 4.3.3 and the equation (4.3.8) we have

$$\begin{aligned} \left| N\varepsilon \sum_{n=2}^{L} \frac{e'W^n e - \mathrm{E}[e'W^n e]}{\mu_{\max}^n} \right| &\le N\varepsilon \sum_{n=2}^{L} \frac{2^n N^{(n-1)/2} \varepsilon^{n/2} (\log N)^{n\xi/4}}{(N\varepsilon e'e)^n} \\ &\le C_4 \frac{N\varepsilon}{\sqrt{N}} \sum_{n=2}^{\infty} \frac{2^n (\log N)^{n\xi/4}}{(N\varepsilon)^{n/2}} \\ &= \mathrm{O}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}} \right), \end{aligned}$$

where C_4 is a constant, that has been used to bound $e'e$. Absorbing the bound of (4.4.9) in $\mathrm{O}\left((\log N)^{\xi/2}/\sqrt{N}\right)$ we can reduce (4.4.10) to the following:

$$\mu_{\max} = \frac{N\varepsilon}{\mu_{\max}} e'We + \sqrt{N\varepsilon}\sum_{n=0}^{L} \frac{\mathrm{E}[e'W^n e]}{\mu_{\max}^n} + \mathrm{O}_{hp}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right). \tag{4.4.11}$$

Let us consider the following polynomial equation in μ_0, given by

$$\mu_0 = N\varepsilon \sum_{n=0}^{L} \frac{\mathrm{E}[e'W^n e]}{\mu_0^n}. \tag{4.4.12}$$

Since the factors $\mathrm{E}[e'W^n e]$ are non-negative for large N (see Lemma 4.6.1), the equation (4.4.12) has a unique positive solution for μ_0. Now any positive real solution which satisfies the above equation must be of the form $\mu_0 = N\varepsilon + \mathrm{O}(1)$. Let $\zeta = \mu_{\max} - \mu_0$. Then the expression of μ_0 and Theorem 4.2.2 allows us to conclude that

$$\zeta = \mathrm{O}_{hp}\left(\sqrt{N\varepsilon}\right)$$

on A_0. Hence using (4.4.11) and (4.4.12) we have

$$\zeta = \frac{N\varepsilon e'We}{\mu_{\max}} + N\varepsilon \sum_{n=2}^{L}\left(\frac{\mathrm{E}[e'W^n e]}{\mu_{\max}^n} - \frac{\mathrm{E}[e'W^n e]}{\mu_0^n}\right) + \mathrm{O}_{hp}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right). \tag{4.4.13}$$

Now for the first term, using the first order asymptotics of $\mu_{\max}$ from Theorem 4.2.2, Lemma 4.3.3 with $n = 1$ and the fact that $\mathrm{E}[e'We] = 0$, we have,

$$\begin{aligned}\frac{N\varepsilon e'We}{\mu_{\max}} &= \frac{e'We}{e'e}\left(1 + \mathrm{O}(N\varepsilon)^{-1/2}\right)\\ &= \frac{e'We}{e'e} + \mathrm{o}_{hp}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right).\end{aligned}$$

Using $\mu_{\max} = \zeta + \mu_0$, the second term in (4.4.13) can be expressed as

$$\begin{aligned}N\varepsilon \sum_{n=2}^{L} \mathrm{E}[e'W^n e]\left(\frac{1}{\mu_{\max}^n} - \frac{1}{\mu_0^n}\right) =& N\varepsilon \sum_{n=2}^{L} \frac{\mathrm{E}[e'W^n e]}{\mu_0^n}\left[\left(1 + \frac{\zeta}{\mu_0}\right)^{-n} - 1\right]\\ =& \sum_{r=1}^{\infty} d_r \zeta^r,\end{aligned}$$

where

$$d_r = (-1)^r N\varepsilon \sum_{n=2}^{L} \frac{2^{n+r} n(n+1)\cdots(n+r-1)\mathrm{E}[e'W^n e]}{r!\mu_0^{n+r}}.$$

Using Lemma 4.3.2 it follows that

$$d_r = \mathrm{O}\left((N\varepsilon)^{-r}\right)$$

Thus we get,

$$\begin{aligned}\zeta - \sum_{r=1}^{\infty} d_r \zeta^r &= \zeta(1 - \sum_{r=1}^{\infty} d_r \zeta^{r-1}) \\ &= \frac{e'We}{e'e} + \mathrm{O}_{hp}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right).\end{aligned}$$

Taking the factor $(1 - \sum_{r=1}^{\infty} d_r \zeta^{r-1})$ on the right hand side followed by applications of Lemma 4.3.3 and the expression $d_r \zeta^{r-1} = \mathrm{O}\left((N\varepsilon)^{-(r+1)/2)}\right)$ for any $r \geq 1$, we have,

$$\begin{aligned}\zeta &= \mu_{\max} - \mu_0 \\ &= \frac{e'We}{e'e} + \mathrm{O}_{hp}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right).\end{aligned}$$

Thus we have shown equation (4.4.1) with

$$\Xi = \mu_{\max} - \mu_0 - \frac{e'We}{e'e}.$$

where Ξ satisfies (4.4.3) w.h.p. Further using

$$\mathrm{E}[|\Xi|] \leq \left(\mathrm{E}[|\Xi|^2]\mathrm{E}[\mathbf{1}_{A_0^c}]\right)^{\frac{1}{2}} + \mathrm{O}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right),$$

where $\mathbf{1}_{A_0}$ is the indicator function of the event A_0 and we have used Cauchy-Schwartz inequality in the first part on the second line. Note that

$$\begin{aligned}\mathrm{E}[\mu_{\max}^2] &\leq \mathrm{E}[\mathrm{Tr}\, A^2] \\ &= \sum_{i,j=1}^{N} \mathrm{E}\left[(A(i,j))^2\right],\end{aligned}$$

which is bounded by N^2 because $A(i,j)$ is either 1 or 0 for any $i, j \in \{1, 2, \ldots, N\}$. Now

$$\begin{aligned}\mathrm{E}[|\Xi|^2] &= \mathrm{E}[|\mu_{\max} - \mu_0 - e'We|^2] \\ &\leq \mathrm{E}[|\mu_{\max}|^2] + \mathrm{E}[|\mu_0|^2] + \mathrm{E}[|e'We|^2]\end{aligned}$$

$$+ 2\left(\mathrm{E}[|\mu_{\max}|^2]\right)^{\frac{1}{2}}\left(|e'We|^2\right)^{\frac{1}{2}} + 2\left(\mathrm{E}[|\mu_{\max}|^2]\right)^{\frac{1}{2}}\mu_0 + 2\left(\mathrm{E}[|e'We|^2]\right)^{\frac{1}{2}}\mu_0$$
$$\le N^{C_5}$$

for some $C_5 > 0$ since $\mu_0 \le N$ for all large N and, for some constant $C_6 > 0$, $\mathrm{E}[|e'We|^2] \le C_6 N^4$, since $W(i,j) \le 2$ for any $i, j \in \{1, 2, \ldots, N\}$. Therefore,

$$\mathrm{E}[|\Xi|] \le N^{\frac{C_5}{2}} e^{-\frac{\nu}{2}(\log N)^{\xi/4-1}} + \mathrm{O}\left(\frac{(\log N)^{\xi/2}}{\sqrt{N}}\right).$$

Finally, the required bound for $\mathrm{E}[|\Xi|]$ follows from the fact that

$$N^{\frac{C_5}{2}} e^{-\frac{\nu}{2}(\log N)^{\frac{\xi}{4}-1}} \ll \frac{(\log N)^{\frac{\xi/}{2}}}{\sqrt{N}}.$$

□

Proof of Theorem 4.2.5. Let v be the normalized eigenvector described in (4.3.10). Denoting $K = N\varepsilon e'v$, we get,

$$\begin{aligned} v &= \frac{K}{\mu_{\max}}\left(I - \frac{W}{\mu_{\max}}\right)^{-1} e \\ &= \frac{K}{\mu_{\max}}\sum_{n=0}^{\infty}\left(\frac{W}{\mu_{\max}}\right)^n e\,. \end{aligned} \tag{4.4.14}$$

Now v being normalized, the expression (4.4.14) allows us to write,

$$\begin{aligned} 1 &= v'v \\ &= \frac{K^2}{\mu_{\max}^2}\sum_{m,n=0}^{\infty}\mu_{\max}^{-m-n}(W^n e)'W^m e. \end{aligned}$$

Therefore, we have,

$$K^{-2} = \frac{1}{\mu_{\max}^2}\left(e'e + \mathrm{O}_{hp}(N\varepsilon)^{-1/2}\right),$$

by using Lemma 4.3.1 and (4.3.9). Now putting $K = N\varepsilon e'v$ in the above equation, we get,

$$\begin{aligned} (e'v)^2 &= \left(\frac{\mu_{\max}}{N\varepsilon}\right)^2\left(e'e + \mathrm{O}_{hp}(N\varepsilon)^{-1/2}\right)^{-1} \\ &= \left(\frac{\mu_{\max}}{N\varepsilon e'e}\right)^2\left(1 + \mathrm{O}_{hp}(N\varepsilon)^{-1/2}\right). \end{aligned}$$

An appeal to Theorem 4.2.2 establishes the equation (4.2.10) and that completes the proof. □

Proof of Theorem 4.2.3. A formal moment calculation will give us the fact that $e'W^3e = \mathrm{O}_{hp}(N\varepsilon)$. Now using (4.3.12) and the just said fact, we have the expression for $\mu_{\max}$ as

$$\begin{aligned}\mu_{\max} &= N\varepsilon \sum_{n=0}^{\infty} e' \left(\frac{W}{\mu_{\max}}\right)^n e \\ &= N\varepsilon e'e + \frac{N\varepsilon}{\mu_{\max}} e'We + \frac{N\varepsilon}{\mu_{\max}^2} e'W^2e + \mathrm{O}_{hp}\left((N\varepsilon)^{-1}\right).\end{aligned}$$

Let us again denote this high probability event by A_0 (with an abuse of notation) and the following computations are are on this event. Iterating the expression for $\mu_{\max}$ in the right hand side of the above equation we get,

$$\begin{aligned}\mu &= N\varepsilon e'e \\ &+ N\varepsilon e'We\left(N\varepsilon e'e + \frac{N\varepsilon}{\mu_{\max}} e'We + \frac{N\varepsilon}{\mu_{\max}^2} e'W^2e + \mathrm{O}\left((N\varepsilon)^{-1}\right)\right)^{-1} \\ &+ N\varepsilon e'W^2e\left(N\varepsilon e'e + \frac{N\varepsilon}{\mu_{\max}} e'We + \frac{N\varepsilon}{\mu_{\max}^2} e'W^2e + \mathrm{O}\left((N\varepsilon)^{-1}\right)\right)^{-2} \\ &+ \mathrm{O}\left((N\varepsilon)^{-1}\right), \\ &= N\varepsilon e'e + \frac{e'We}{e'e}\left(1 - \frac{e'We}{\mu_{\max} e'e} - \mathrm{O}\left((N\varepsilon)^{-1}\right)\right) \\ &+ \frac{e'W^2e}{N\varepsilon (e'e)^2}\left(1 - \frac{2e'We}{\mu_{\max} e'e} - \mathrm{O}\left((N\varepsilon)^{-1}\right)\right) + \mathrm{O}\left((N\varepsilon)^{-1}\right), \\ &= N\varepsilon e'e + \frac{e'We}{e'e} - \frac{(e'We)^2}{\mu_{\max}(e'e)^2} + \frac{e'W^2e}{N\varepsilon(e'e)^2} + \mathrm{O}\left((N\varepsilon)^{-1/2}\right).\end{aligned}$$

We plug in the asymptotics of $\mu_{\max}$ again in above equation to get,

$$\begin{aligned}\mu_{\max} &= N\varepsilon e'e + \frac{e'We}{e'e} + \frac{e'W^2e}{N\varepsilon(e'e)^2} \\ &- \frac{(e'We)^2}{(e'e)^2}\left(N\varepsilon e'e + \frac{N\varepsilon}{\mu_{\max}} e'We + \frac{N\varepsilon}{\mu_{\max}^2} e'W^2e + \mathrm{O}\left((N\varepsilon)^{-1}\right)\right)^{-1} \\ &+ \mathrm{O}\left((N\varepsilon)^{-1/2}\right) \\ &= N\varepsilon e'e + \frac{e'We}{e'e} + \frac{e'W^2e}{N\varepsilon(e'e)^2} - \frac{(e'We)^2}{N\varepsilon(e'e)^3} + \mathrm{O}\left((N\varepsilon)^{-1/2}\right).\end{aligned}$$

Now we write $\mu_{\max} = \mu_{\max}\mathbf{1}_{A_0} + \mu_{\max}\mathbf{1}_{A_0^c}$. We take expectation on both sides and get the following from the above expression of $\mu_{\max}$,

$$\mathrm{E}\mu_{\max} = N\varepsilon e'e + \frac{\mathrm{E}\left(e'W^2e\right)}{N\varepsilon(e'e)^2} - \frac{\mathrm{E}\left(e'We\right)^2}{N\varepsilon(e'e)^3} + \mathrm{O}\left((N\varepsilon)^{-1/2}\right).$$

Finally we get the equation (4.2.7) from the above equation by observing that $e'e = \mathrm{O}(1)$, $\mathrm{E}[e'W^2e] = N\varepsilon(\beta_2 + \mathrm{O}(\varepsilon))$ and $|\mathrm{E}[e'We]^2| = \varepsilon(\beta_2 + \mathrm{O}(\varepsilon))$. $\square$

4.5 An example

The following random graph was introduced in [41]. For $N \geq 1$, let $(d_{N_i}\colon\ 1 \leq i \leq N)$ be a sequence of positive real numbers. Abbreviate

$$m = \max_{1\leq i\leq N} d_{N_i}\,, \qquad \sigma = \sum_{i=1}^{N} d_{N_i}\,.$$

Assume for some $\xi > 8$,

$$(\log N)^{\xi} \ll m \ll N \tag{4.5.1}$$

and

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} \delta_{d_{N_i}/m} = \eta \quad \text{weakly} \tag{4.5.2}$$

for some measure η on $\mathbb{R}$. Suppose η has finite, non-zero second moment. Consider an inhomogeneous Erdős–Rényi graph on N vertices where an edge exists between i and j with probability $d_{N_i}d_{N_j}/\sigma$, for $1 \leq i \leq j \leq N$, which is called a **Chung-Lu graph**. Let A and $\mu_{\max}$ denote the adjacency matrix and the largest eigenvalue of A respectively. It was shown in [37] that the ESD of $(\sqrt{N\varepsilon})^{-1}A$ converges to $w \boxtimes \eta$ weakly in probability. The following theorem describes the behavior of $\mu_{\max}$.

Theorem 4.5.1. *Under the hypotheses mentioned above,*

$$\frac{\sqrt{\sigma}}{m}\left(\mu_{\max} - \mathrm{E}[\mu_{\max}]\right) \Rightarrow N(0, \sigma^2)$$

where $N(0,\, \sigma^2)$ is a normal random variable with mean zero and variance

$$\sigma^2 := 2\left(\frac{\int_0^1 x^3 d\eta(x)}{\int_0^1 x^2 d\eta(x)}\right)^2.$$

Proof. Observe that the edge probabilities can be written as

$$\frac{d_{N_i} d_{N_j}}{\sigma} = \frac{m^2}{\sigma} \frac{d_{N_i}}{m} \frac{d_{N_j}}{m}$$

and the assumption (4.5.2) ensures that for any $n \geq 0$,

$$\frac{1}{N} \sum_{i=1}^{N} \left(\frac{d_{N_i}}{m} \right)^n \to \int_0^1 x^n d(\eta)x. \tag{4.5.3}$$

The above expression with $n = 1$ gives, $\frac{\sigma}{Nm} = \int_0^1 x d\eta(x) + \mathrm{o}(1)$. Let $\varepsilon = \frac{m^2}{\sigma}$, then

$$\varepsilon = \frac{m^2}{Nm \left(\int_0^1 x d\eta(x) + \mathrm{o}(1) \right)} \ll 1,$$

by the assumption (4.5.1). Also,

$$N\varepsilon = \frac{Nm^2}{\sigma} = \mathrm{O}(m) \gg (\log N)^{\xi}.$$

It is noteworthy to mention that while in the proof of Theorem 4.2.4, we have only used the values of the function r at the rational points in $[0, 1]$. Here d_{N_i}/m can be thought as the values of the function r at the point i/N, although it may not be true that such a function r can be defined in the whole of $[0, 1]$. Define e in the following way:

$$e = \begin{bmatrix} N^{-1/2} d_{N_1}/m \\ N^{-1/2} d_{N_2}/m \\ \vdots \\ N^{-1/2} d_N/m \end{bmatrix}.$$

The theorem follows from the observation that the arguments given in the proof of Theorem 4.2.4 can be imitated using (4.5.3). □

4.6 Proof of the technical lemmas

Proof of Lemma 4.3.1. Note that for any even integer n

$$\mathrm{E}(\|W\|^n) \leq \mathrm{E}(\mathrm{Tr}(W^n)). \tag{4.6.1}$$

Using $\mathrm{E}(W(i,j)^2) \le \varepsilon M$ (from (4.3.1)) and condition (4.2.4) it is immediate that conditions of Theorem 1.4 of [94] are satisfied. We shall use the following estimate from the proof of that result. It follows from [94, Section 4]

$$\mathrm{E}(\mathrm{Tr}(W^n)) \le C_7 N (2\sqrt{MN\varepsilon})^n \tag{4.6.2}$$

where C_7 is some positive constant and there exists a constant $a > 0$ such that k can be chosen as

$$n = \sqrt{2}a(\varepsilon M)^{1/4} N^{1/4}.$$

Using (4.6.1), (4.6.2) and $(1-x)^n \le e^{-nx}$ for $n, x > 0$,

$$\begin{aligned} &P\left(\|W\| \ge 2\sqrt{MN\varepsilon} + C_1(N\varepsilon)^{1/4}(\log N)^{\xi/4}\right) \\ &= C_7 N \left(1 - \frac{C_1(N\varepsilon)^{1/4}(\log N)^{\xi/4}}{2\sqrt{MN\varepsilon} + C_1(N\varepsilon)^{1/4}(\log N)^{\xi/4}}\right)^n \\ &\le C_7 N \exp\left(-\frac{nC_1(N\varepsilon)^{1/4}(\log N)^{\xi/4}}{2\sqrt{MN\varepsilon} + C_1(N\varepsilon)^{1/4}(\log N)^{\xi/4}}\right). \end{aligned} \tag{4.6.3}$$

Now plugging in the value of n in the bound (4.6.3) and using

$$2\sqrt{M} + C_1(N\varepsilon)^{-1/4}(\log N)^{\xi/4} \le 2\sqrt{M} + C_1$$

we have

$$(4.6.3) \le C_7 N \exp\left(-\frac{C_1 a M^{1/4}\sqrt{2}(\log N)^{\xi/4}}{2\sqrt{M} + C_1}\right) \le e^{-C_8(\log N)^{\xi/4}}$$

for some constant $C_8 > 0$ and N large enough. This proves the lemma. □

Proof of Lemma 4.3.2. Let A_0 be the event where Lemma 4.3.1 holds, that is, $\|W\| \le C_9\sqrt{N\varepsilon}$ for some constant C_9. Since the entries of e_1 and e_2 are in $[-1/\sqrt{N}, 1/\sqrt{N}]$ so $\|e_i\| \le 1$ for $i = 1, 2$. Hence on the high probability event it holds that

$$\left|\mathrm{E}\left(e_1' W^n e_2 \mathbf{1}_{A_0}\right)\right| \le (C_9 N\varepsilon)^{n/2}.$$

We show that the above expectation on the low probability event A_0^c is negligible. For that first observe

$$|\mathrm{E}[(e_1' W^n e_2)^2]| \le N^{nC_{10}}$$

for some constant $0 < C_{10} < \infty$. Thus using Lemma 4.3.1 one has

$$\begin{aligned}\left|\mathrm{E}\left(e_1' W^n e_2 \mathbf{1}_{A_0^c}\right)\right| &\le \left|\mathrm{E}\left[(e_1' W^n e_2)^2\right]^{1/2}\right| P(A_0^c)^{1/2} \\ &\le \exp\left(nC_{10}\log N - 2^{-1}C_2(\log N)^{\xi/4}\right)\end{aligned}$$

Since $n \le \log N$ and $\xi > 8$ the result follows. □

Proof of Lemma 4.3.3. The proof is similar to the proof of Lemma 6.5 of [49]. The exponent in the exponential decay is crucial, so the proof is briefly sketched. Observe that

$$\begin{aligned}& e_1' W^n e_2 - \mathrm{E}\left(e_1' W^n e_2\right) \\ &= \sum_{i\in\{1,\ldots,N\}^{n+1}} e_1(i_1)e_2(i_{n+1})\left(\prod_{l=1}^n W(i_l, i_{l+1}) - \mathrm{E}\left[\prod_{l=1}^n W(i_l, i_{l+1})\right]\right) \end{aligned} \tag{4.6.4}$$

To use the independence, one can split the matrix W as $W' + W''$ where the upper triangular matrix W' has entries $W'(i,j) = W(i,j)\mathbf{1}(i \leqslant j)$ and the lower triangular matrix W'' with entries $W''(i,j) = W(i,j)\mathbf{1}(i > j)$. Therefore the above quantity under the sum breaks into 2^n terms each having similar properties. Denote one such term as

$$L_n = \sum_{i\in\{1,\ldots,N\}^{n+1}} e_1(i_1)e_2(i_{n+1})\left(\prod_{l=1}^n W'(i_l, i_{l+1}) - \mathrm{E}\left[\prod_{l=1}^n W'(i_l, i_{l+1})\right]\right).$$

Using the fact that each entry of e_1 and e_2 are bounded by $1/\sqrt{N}$, it follows by imitating the proof of Lemma 6.5 of [49] that

$$\mathrm{E}[|L_n|^p] \le \frac{(C_{11}np)^{np}\,(N\varepsilon)^{np/2}}{N^{p/2}},$$

where p is an even integer and C_{11} is a positive constant, independent of n and p. Rest of the $2^n - 1$ terms arising in (4.6.4) have the same bound and hence

$$\begin{aligned}& P\left(\left|e_1' W^n e_2 - \mathrm{E}\left(e_1' W^n e_2\right)\right| > N^{(n-1)/2}\varepsilon^{n/2}(\log N)^{n\xi/4}\right) \\ &\le \frac{(2C_{11}np)^{np}\,(N\varepsilon)^{np/2}}{N^{p/2}N^{p(n-1)/2}\varepsilon^{pn/2}(\log N)^{pn\xi/4}} = \frac{(2C_{11}np)^{np}}{(\log N)^{pn\xi/4}}.\end{aligned}$$

Choose $\eta \in (1,\, \xi/4)$ and consider

$$p = \frac{(\log N)^\eta}{2C_{11}n},$$

(with N large enough to make p an even integer) to get

$$P\left(\left|e_1'W^ne_2 - \mathrm{E}\left(e_1'W^ne_2\right)\right| > N^{(n-1)/2}\varepsilon^{n/2}(\log N)^{n\xi/4}\right)$$
$$\leq \exp\left(-\frac{1}{2C_{11}}(\log N)^{\eta}(\frac{\xi}{4}-\eta)\log\log N\right).$$

Note that $n \leq L$, ensures that $p > 1$. Since the bound is uniform over all $2 \leq n \leq L$, the first bound (4.3.3) follows.

For (4.3.4) one can use Hoeffding's inequality [60, Theorem 2] as follows.

Define

$$\widetilde{A}(k,l) = A(k,l)e_1(k)e_2(l), \quad 1 \leq k \leq l \leq N.$$

Since $A(k,l)$ are Bernoulli random variables, so one has $\{\widetilde{A}(k,l) : 1 \leq k \leq l \leq N\}$ are independent random variables taking values in $[-1/N, 1/N]$ and hence by Hoeffding's inequality we have, for any $\delta > 0$,

$$P\left(\left|\sum_{1\leq k\leq l\leq N}\widetilde{A}(k,l) - E\left(\sum_{1\leq k\leq l\leq N}\widetilde{A}(k,l)\right)\right| > \delta N\varepsilon\right)$$
$$\leq 2\exp\left(-\delta^2(N\varepsilon)^2\right) \leq 2\exp\left(-\delta^2(\log N)^{2\xi}\right).$$

Dealing with the case $k > l$ similarly, the desired bound on $e_1'We_2$ follows. □

Proof of Lemma 4.3.4. Follows by a simple moment calculation. □

Lemma 4.6.1. *For any $n \geq 0$, $\mathrm{E}[e'W^ne] \geq 0$ for any $N \geq N_0$, where $N_0 \in \mathbb{N}$ does not depend on n.*

Proof. We have

$$\mathrm{E}[e'W^ne] = \frac{1}{N}\sum_{i_1,\ldots,i_{n+1}=1}^{N}\mathrm{E}\left(\prod_{t=1}^{n}W(i_t,i_{t+1})\right)r\left(\frac{i_1}{N}\right)r\left(\frac{i_{n+1}}{N}\right).$$

On the right hand side of the above identity, any term under the expectation in the sum is of the form

$$\prod_{t=1}^{n'}\mathrm{E}\left((W(i_t,i_{t+1}))^{l_t}\right)$$

for some $n' \leq n$, $l_t \geq 1$ with the independent random variables $\{W(i_t, i_{t+1})\}_{t=1}^{n'}$ and $\sum_{t=1}^{n'} l_t = n$. Thus if we can show that for any $l_t \geq 1$, $\mathrm{E}\left((W(i_t, i_{t+1}))^{l_t}\right) \geq 0$ the lemma follows because range of the function r is $[0, M]$. Now from (4.1.3) using the notation $q := \varepsilon r\left(\frac{i_t}{N}\right) r\left(\frac{i_{t+1}}{N}\right)$, we get

$$\begin{aligned}
\mathrm{E}\left((W(i_t, i_{t+1}))^{l_t}\right) &= (A(i_t, i_{t+1}) - q)^{l_t} \\
&= (1-q)^{l_t} q + (-q)^{l_t}(1-q) \\
&= (1-q)q[(1-q)^{l_t - 1} + (-q)^{l_t - 1}] \geq 0
\end{aligned}$$

wherever $q \leq \frac{1}{2}$. □

Chapter 5

Largest eigenvalue in the finite rank case

5.1 Introduction

Consider an inhomogeneous Erdős-Rényi random graph on vertices $\{1, 2, \ldots, N\}$ with connection probability between the vertices i and j, given by

$$p_{i,j} = \varepsilon f\left(\frac{i}{N}, \frac{j}{N}\right).$$

Here f is of the form

$$f(x,y) = \sum_{i=1}^{k} \theta_i r_i(x) r_i(y)\,, \text{ for all } (x,y) \in [0,1] \times [0,1], \tag{5.1.1}$$

for some $k \geq 1$. In Chapter 4, we have studied the behaviour of the largest eigenvalue of the adjacency matrix when $k = 1$. The case $k \geq 2$ turns out to be substantially difficult than the case $k = 1$ for the following reason. If $k = 1$, that is,

$$\mathrm{E}(A) = uu'\,,$$

for some $N \times 1$ deterministic column vector u, then with high probability it holds that

$$u' \left(\lambda I - W\right)^{-1} u = 1\,,$$

where λ is the largest eigenvalue of A (see (4.3.10)). The above equation facilitates the asymptotic study of λ. However, when $k \geq 2$, the above equation takes a complicated form. The observation which provides a way out of this is that λ is also an eigenvalue of a $k \times k$ matrix with high probability; the same is recorded in Lemma 5.4.2 of Section 5.4. Besides, working with the eigenvalues of a $k \times k$ matrix needs more linear algebraic work when $k \geq 2$. For example, the proof of Lemma 5.4.9, which is one of the major steps in the proof of a main result, becomes a tautology when $k = 1$.

The following results are obtained in the current chapter. The function f, defined in (5.1.1), induces an integral operator of finite rank k, under the assumption that (4.1.4) holds. If the largest eigenvalue of the integral operator has multiplicity 1, then the largest eigenvalue of the adjacency matrix has a Gaussian fluctuation. More generally, it is shown that the eigenvalues which correspond to isolated eigenvalues, which will be defined later, of the induced integral operator jointly converge to a multivariate Gaussian law. Under the assumption that the function f is Lipschitz continuous, the leading order term in the expansion of the expected value of the isolated eigenvalues is obtained. Furthermore, under an additional assumption, the inner product of the eigenvector with the discretized eigenfunction of the integral operator corresponding to the other eigenvalues is shown to have a Gaussian fluctuation. Some important examples of such f include the rank-one case, and the stochastic block models. Whether the fluctuation of the $(k+1)$-th eigenvalue after appropriate centring and scaling is asymptotically Tracy-Widom, remains an open question.

The mathematical set-up and the main results of the chapter are stated in Section 5.2. Theorem 5.2.3 shows that of the k largest eigenvalues, the isolated ones, centred by their mean and appropriately scaled, converge to a multivariate normal distribution. Theorem 5.2.6 studies the first and second order of the expectation of the top k isolated eigenvalues. Theorems 5.2.7 and 5.2.8 study the behaviour of the eigenvectors corresponding to the top k isolated eigenvalues. Section 5.3 illustrates the example of stochastic block models. The proofs of the main results are in Section 5.4 and some essential lemmas are proved in Section 5.5.

5.2 The set-up and the results

Recall that f is a bounded Riemann integrable function from $[0,1]\times[0,1]$ to $[0,\infty)$ which is symmetric, that is,

$$f(x,y)=f(y,x)\,,0\le x,y\le 1\,.$$

The integral operator I_f with kernel f is defined from $L^2[0,1]$ to itself by

$$\big(I_f(g)\big)(x)=\int_0^1 f(x,y)g(y)\,dy\,,\;0\le x\le 1\,.$$

Besides the above, we assume that I_f is a non-negative definite operator and the range of I_f has a finite dimension.

Under the above assumptions I_f turns out to be a compact self-adjoint operator. Let $\theta_1\ge\cdots\ge\theta_k$, where k is the dimension of the range of I_f, denote the non-zero eigenvalues of I_f, with corresponding eigenfunctions $r_1,\ldots,r_k$. Spectral theory implies that $r_1,\ldots,r_k$ are orthonormal. These functions are Riemann integrable by assumption; see Lemma 5.5.1 in Section 5.5. Thus, for $g\in L^2[0,1]$ it holds that

$$I_f(g)=\sum_{i=1}^k\theta_i\langle\, r_i,g\rangle_{L^2[0,1]}r_i.$$

Note that this gives

$$\int_0^1\left(\sum_{i=1}^k\theta_i r_i(x)r_i(y)g(y)\right)\,dy=\int_0^1 f(x,y)g(y)\,dy\;\text{ for almost all } x\in[0,1].$$

Since g is an arbitrary function in $L^2[0,1]$ this immediately gives

$$f(x,y)=\sum_{i=1}^k\theta_i r_i(x)r_i(y)\,,\;\text{for almost all } (x,y)\in[0,1]\times[0,1]. \tag{5.2.1}$$

Since the functions on both sides of the above equation are Riemann integrable, the corresponding Riemann sums are approximately equal, and hence there is no loss of generality in assuming that the above equality holds for every x and y.

Let $(\varepsilon \equiv \varepsilon_N : N \geq 1)$ be a real sequence satisfying

$$0 < \varepsilon \leq \left[\sup_{0 \leq x,y \leq 1} f(x,y) \right]^{-1} , N \geq 1 \, .$$

Here and elsewhere, the subscript 'N' is suppressed in the notation. We assume that (4.1.4) holds for some $\xi > 8$ and ε_∞ is same as in (4.2.3). It's worth emphasizing that we do not assume that ε necessarily goes to zero, although that may be the case.

We recall that $\mathbb{G}_N$ is an inhomogeneous Erdős-Rényi graph where an edge is placed between vertices i and j with probability $\varepsilon f(i/N, j/N)$, for $i \leq j$, the choice being made independently for each pair in $\{(i,j) : 1 \leq i \leq j \leq N\}$. Note that we allow self-loops. Let A be the adjacency matrix of $\mathbb{G}_N$. In other words, A is an $N \times N$ symmetric matrix, where $\{A(i,j) : 1 \leq i \leq j \leq N\}$ is a collection of independent random variable, and

$$A(i,j) \sim \text{Bernoulli} \left(\varepsilon f \left(\frac{i}{N}, \frac{j}{N} \right) \right) , \, 1 \leq i \leq j \leq N \, .$$

A few more notations are needed for stating the main results. For a moment, set $\theta_0 = \infty$ and $\theta_{k+1} = -\infty$, and define the set of indices i for which θ_i is isolated as follows:

$$\mathcal{I} = \{1 \leq i \leq k : \theta_{i-1} > \theta_i > \theta_{i+1}\} \, .$$

For an $N \times N$ real symmetric matrix M, $\lambda_1(M) \geq \ldots \geq \lambda_N(M)$ denote its eigenvalues. Finally, after recalling the Definition 4.2.1 about the high probability events, the notations o_{hp}, O_{hp} and, O_p, we state the main results of this chapter. The first result complements Theorem 4.2.2 of the rank one case to rank k $(< \infty)$ case and is about the first order behaviour of $\lambda_i(A)$.

Theorem 5.2.1. *For every* $1 \leq i \leq k$*,*

$$\lambda_i(A) = N\varepsilon\theta_i \left(1 + o_{hp}(1)\right) \, .$$

An immediate consequence of the above is that for all $1 \le i \le k$, $\lambda_i(A)$ is non-zero w.h.p. and hence dividing by the same is allowed, as done in the next result. Define

$$e_i = \begin{bmatrix} N^{-1/2} r_i(1/N) \\ N^{-1/2} r_i(2/N) \\ \vdots \\ N^{-1/2} r_i(1) \end{bmatrix}, \; 1 \le i \le k\,. \tag{5.2.2}$$

The second main result studies the asymptotic behaviour of $\lambda_i(A)$, for $i \in \mathcal{I}$, after appropriate centering and scaling.

Theorem 5.2.2. *For every $i \in \mathcal{I}$, as $N \to \infty$,*

$$\lambda_i(A) = \mathrm{E}\left(\lambda_i(A)\right) + \frac{N\theta_i\varepsilon}{\lambda_i(A)} e_i' W e_i + o_p(\sqrt{\varepsilon})\,,$$

where W is as defined in (4.1.3).

The next result generalizes Theorem 4.2.4, and is a corollary of the previous two theorems.

Theorem 5.2.3. *Assuming (4.1.4), (4.2.3) and (5.2.1), if $\mathcal{I}$ is a non-empty set, then as $N \to \infty$,*

$$\left(\varepsilon^{-1/2}\left(\lambda_i(A) - \mathrm{E}[\lambda_i(A)]\right) : i \in \mathcal{I}\right) \Rightarrow \left(G_i : i \in \mathcal{I}\right)\,, \tag{5.2.3}$$

where the right hand side is a multivariate normal random vector in $\mathbb{R}^{|\mathcal{I}|}$, with mean zero and

$$\mathrm{Cov}(G_i, G_j) = 2\int_0^1\int_0^1 r_i(x) r_i(y) r_j(x) r_j(y) f(x,y)\left[1 - \varepsilon_\infty f(x,y)\right]\, dx\, dy\,,$$

for all $i, j \in \mathcal{I}$.

It may be checked that the Lindeberg-Lévy central limit theorem implies that as $N \to \infty$,

$$\left(\varepsilon^{-1/2} e_i' W e_i : i \in \mathcal{I}\right) \Rightarrow \left(G_i : i \in \mathcal{I}\right), \tag{5.2.4}$$

where the right hand side is as in Theorem 5.2.3. Therefore, the latter would follow from Theorems 5.2.1 and 5.2.2.

Remark 5.2.4. *If $f > 0$ a.e. on $[0,1]\times[0,1]$, then the Krein-Rutman theorem (see Lemma 5.5.2) implies that $1 \in \mathcal{I}$, and that $r_1 > 0$ a.e. Thus, in this case, if $\varepsilon_\infty = 0$, then*

$$\mathrm{Var}(G_1) = 2\int_0^1\int_0^1 r_1(x)^2 r_1(y)^2 f(x,y)\,dx\,dy > 0\,.$$

Remark 5.2.5. *That the claim of Theorem 5.2.3 may not hold if $i \notin \mathcal{I}$ is evident from the following example. Suppose that $\varepsilon_\infty = 0$ and*

$$f(x,y) = \mathbb{1}\left(x \vee y < \frac{1}{2}\right) + \mathbb{1}\left(x \wedge y > \frac{1}{2}\right)\,, 0 \le x, y \le 1\,.$$

Then, Theorem 5.2.3 itself implies that there exists $\beta_N \in \mathbb{R}$ such that

$$\varepsilon^{-1/2}\left(\lambda_1(A) - \beta\right) \Rightarrow G_1 \vee G_2\,,$$

where G_1 and G_2 are i.i.d. from normal with mean 0 and variance 2, and hence there doesn't exist a centering and a scaling by which $\lambda_1(A)$ converges weakly to a non-degenerate normal distribution.

For the remaining results in this section, f will be assumed to be a Lipschitz function. The next main result of the chapter studies asymptotics of $\mathrm{E}(\lambda_i(A))$ for $i \in \mathcal{I}$.

Theorem 5.2.6. *Assume that f is Lipschitz continuous, that is, there exists $K < \infty$ such that*

$$|f(x,y) - f(x',y')| \le K\left(|x-x'| + |y-y'|\right)\,. \tag{5.2.5}$$

Then, for all $i \in \mathcal{I}$,

$$\mathrm{E}\left[\lambda_i(A)\right] = \lambda_i(B) + O\left(\sqrt{\varepsilon} + (N\varepsilon)^{-1}\right)\,,$$

where B is a $k \times k$ symmetric deterministic matrix, depending on N, defined by

$$B(j,l) = \sqrt{\theta_j\theta_l}N\varepsilon e_j' e_l + \theta_i^{-2}\sqrt{\theta_j\theta_l}(N\varepsilon)^{-1}\mathrm{E}\left(e_j' W^2 e_l\right)\,, 1 \le j, l \le k\,,$$

and e_j and W are as defined in (5.2.2) and (4.1.3), respectively.

The next result studies the asymptotic behaviour of the normalized eigenvector corresponding to $\lambda_i(A)$, again for isolated vertices i. It is shown that the same is asymptotically aligned with e_i, and hence it is asymptotically orthogonal to e_j. Upper bounds on rates of convergence are obtained.

Theorem 5.2.7. *As in Theorem 5.2.6, let f be a Lipschitz continuous function. Then, for a fixed $i \in \mathcal{I}$,*

$$\lim_{N\to\infty} P\left(\lambda_i(A) \text{ is an eigenvalue of multiplicity } 1\right) = 1\,. \tag{5.2.6}$$

If v is the eigenvector, with L^2-norm 1, of A corresponding to $\lambda_i(A)$, then

$$e_i'v = 1 + O_p\left((N\varepsilon)^{-1}\right)\,, \tag{5.2.7}$$

that is, $N\varepsilon(1 - e_i'v)$ is stochastically tight. When $k \geq 2$, it holds that

$$e_j'v = O_p\left((N\varepsilon)^{-1}\right)\,,\, j \in \{1,\ldots,k\}\setminus\{i\}\,. \tag{5.2.8}$$

The last main result of this chapter studies finer fluctuations of (5.2.8) under an additional condition.

Theorem 5.2.8. *Continue assuming f to be Lipschitz continuous, and let $k \geq 2$ and $i \in \mathcal{I}$. Furthermore, assume that*

$$N^{-2/3} \ll \varepsilon \ll 1\,. \tag{5.2.9}$$

If v is as in Theorem 5.2.7, then, for all $j \in \{1,\ldots,k\}\setminus\{i\}$,

$$e_j'v = \frac{1}{\theta_i - \theta_j}\left[\theta_i \frac{1}{\lambda_i(A)} e_i'We_j + (N\varepsilon)^{-2}\frac{1}{\theta_i}\mathrm{E}\left(e_i'W^2e_j\right)\right] + o_p\left(\frac{1}{N\sqrt{\varepsilon}}\right)\,.$$

Remark 5.2.9. *An immediate consequence of Theorem 5.2.8 is that under (5.2.9), there exists a deterministic sequence $(z_N : N \geq 1)$ given by*

$$z = \frac{1}{(N\varepsilon)^2\theta_i(\theta_i - \theta_j)}\mathrm{E}\left(e_i'W^2e_j\right)\,,$$

such that as $N \to \infty$,

$$N\sqrt{\varepsilon}\left(e_j'v - z\right)$$

converges weakly to a normal distribution with mean zero, for all $i \in \mathcal{I}$ and $j \in \{1,\ldots,k\}\setminus\{i\}$. Furthermore, the convergence holds jointly for all i and j (satisfying the above), and with (5.2.3), to a multivariate normal distribution in $\mathbb{R}^{k|\mathcal{I}|}$ with mean zero, whose covariance matrix is not hard to calculate.

5.3 An example

The stochastic block model

An important example is the stochastic block model, defined as follows. Suppose that

$$f(x,y)=\sum_{i,j=1}^{k} p(i,j)\,\mathbb{1}_{B_i}(x)\,\mathbb{1}_{B_j}(y)\,,0\le x,y\le 1\,,$$

where p is a $k\times k$ symmetric positive definite matrix, and $B_1,\ldots,B_k$ are disjoint Borel subsets of $[0,1]$ whose boundaries are sets of measure zero, that is, their indicators are Riemann integrable. We show below how to compute the eigenvalues and eigenfunctions of I_f, the integral operator associated with f.

Let β_i denote the Lebesgue measure of B_i, which we assume without loss of generality to be strictly positive. Rewrite

$$f(x,y)=\sum_{i,j=1}^{k}\tilde{p}(i,j)s_i(x)s_j(y)\,,$$

where

$$\tilde{p}(i,j)=p(i,j)\sqrt{\beta_i\beta_j}\,,1\le i,j\le k\,,$$

and

$$s_i=\beta_i^{-1/2}\,\mathbb{1}_{B_i}\,,1\le i\le k\,.$$

Thus, $\{s_1,\ldots,s_k\}$ is an orthonormal set in $L^2[0,1]$. Let

$$\tilde{p}=U'DU\,,$$

be a spectral decomposition of $\tilde{p}$, where U is a $k\times k$ orthogonal matrix, and

$$D=\mathrm{Diag}(\theta_1,\ldots,\theta_k)\,,$$

for some $\theta_1\ge\ldots\ge\theta_k>0$.

Define functions $r_1, \ldots, r_k$ by

$$\begin{bmatrix} r_1(x) \\ \vdots \\ r_k(x) \end{bmatrix} = U \begin{bmatrix} s_1(x) \\ \vdots \\ s_k(x) \end{bmatrix}, \; x \in [0,1] \, .$$

It is easy to see that $r_1, \ldots, r_k$ are orthonormal in $L^2[0,1]$, and for $0 \le x, y \le 1$,

$$\begin{aligned} f(x,y) &= [s_1(x) \, \ldots \, s_k(x)] \, \tilde{p} \, [s_1(x) \, \ldots \, s_k(x)]' \\ &= [r_1(x) \, \ldots \, r_k(x)] \, U \tilde{p} U' \, [r_1(x) \, \ldots \, r_k(x)]' \\ &= \sum_{i=1}^{k} \theta_i r_i(x) r_i(y) \, . \end{aligned}$$

Thus, $\theta_1, \ldots, \theta_k$ are the eigenvalues of I_f, and $r_1, \ldots, r_k$ are the corresponding eigenfunctions.

5.4 Proof of the main results

This section is devoted to the proof of the main results. We shall be using the estimates stated in Section 4.3 of the previous chapter. At this point, it should be clarified that in this section, e_j will always be as defined in (5.2.2). We start with showing that Theorem 5.2.1 is a corollary of Lemma 4.3.1.

Proof of Theorem 5.2.1. For a fixed $i \in \{1, \ldots, k\}$, it follows that

$$|\lambda_i(A) - \lambda_i\left(\mathrm{E}(A)\right)| \le \|W\| = O_{hp}\left((N\varepsilon)^{1/2}\right) ,$$

by Lemma 4.3.1. In order to complete the proof, it suffices to show that

$$\lim_{N \to \infty} (N\varepsilon)^{-1} \lambda_i\left(\mathrm{E}(A)\right) = \theta_i \, ,$$

which however follows from the observation that (5.2.1) implies that

$$\mathrm{E}(A) = N\varepsilon \sum_{j=1}^{k} \theta_j e_j e_j' \, . \tag{5.4.1}$$

This completes the proof. □

Proceeding towards the proof of Theorem 5.2.2, let us fix $i \in \mathcal{I}$, once and for all, denote

$$\mu = \lambda_i(A)\,,$$

and let V be a $k \times k$ matrix, depending on N which is suppressed in the notation, defined by

$$V(j,l) = \begin{cases} N\varepsilon\sqrt{\theta_j\theta_l}\, e_j' \left(I - \frac{1}{\mu}W\right)^{-1} e_l, & \text{if } \|W\| < \mu\,, \\ 0, & \text{else}\,, \end{cases}$$

for all $1 \le j, l \le k$. It should be noted that if $\|W\| < \mu$, then $I - W/\mu$ is invertible. The first step towards Theorem 5.2.2 is to show that $V/N\varepsilon$ converges to $\mathrm{Diag}(\theta_1, \ldots, \theta_k)$, that is, the $k \times k$ diagonal matrix with diagonal entries $\theta_1, \ldots, \theta_k$, w.h.p.

Lemma 5.4.1. *As $N \to \infty$,*

$$V(j,l) = N\varepsilon\theta_j \left(\mathbb{1}(j = l) + o_{hp}(1)\right)\,, 1 \le j, l \le k\,.$$

Proof. For fixed $1 \le j, l \le k$, writing

$$\left(I - \frac{1}{\mu}W\right)^{-1} = I + O_{hp}\left(\mu^{-1}\|W\|\right)\,,$$

we get that

$$V(j,l) = N\varepsilon\sqrt{\theta_j\theta_l}\left(e_j' e_l + \frac{1}{\mu}O_{hp}(\|W\|)\right)\,.$$

Since

$$\lim_{N\to\infty} e_j' e_l = \mathbb{1}(j = l)\,, \tag{5.4.2}$$

and

$$\|W\| = o_{hp}(\mu)$$

by Lemma 4.3.1 and Theorem 5.2.1, the proof follows. □

The next step, which is one of the main steps in the proof of Theorem 5.2.2, shows that the i-th eigenvalues of A and V are equal w.h.p.

Lemma 5.4.2. *With high probability,*

$$\mu = \lambda_i(V)\,.$$

The proof of the above lemma is based on the following fact which is a direct consequence of the Gershgorin circle theorem; see Theorem 1.6, pg 8 of [90].

Fact 1. *Suppose that U is an $n \times n$ real symmetric matrix. Define*

$$R_l = \sum_{1 \le j \le n,\, j \ne l} |U(j,l)|\,,\ 1 \le l \le n\,.$$

If for some $1 \le m \le n$ it holds that

$$U(m,m) + R_m < U(l,l) - R_l\,,\ \textit{for all}\ 1 \le l \le m-1\,, \tag{5.4.3}$$

and

$$U(m,m) - R_m > U(l,l) + R_l\,,\ \textit{for all}\ m+1 \le l \le n\,, \tag{5.4.4}$$

then

$$\{\lambda_1(U), \ldots, \lambda_n(U)\} \setminus \left(\bigcup_{1 \le l \le k,\, l \ne m} [U(l,l) - R_l, U(l,l) + R_l] \right) = \{\lambda_m(U)\}\,.$$

Remark 5.4.3. *The assumptions (5.4.3) and (5.4.4) of Fact 1 mean that the Gershgorin disk containing the m-th largest eigenvalue is disjoint from any other Gershgorin disk.*

Proof of Lemma 5.4.2. The first step is to show that

$$\mu \in \{\lambda_1(V), \ldots, \lambda_k(V)\}\ \text{w.h.p.} \tag{5.4.5}$$

To that end, fix $N \ge 1$ and a sample point for which $\|W\| < \mu$. The following calculations are done for that fixed sample point.

Let v be an eigenvector of A, with norm 1, corresponding to $\lambda_i(A)$. That is,

$$\mu v = Av = Wv + N\varepsilon \sum_{l=1}^{k} \theta_l (e_l' v) e_l\,, \tag{5.4.6}$$

by (5.4.1). Since $\mu > \|W\|$, $\mu I - W$ is invertible, and hence

$$v = N\varepsilon \sum_{l=1}^{k} \theta_l (e_l' v)\, (\mu I - W)^{-1} e_l\,. \tag{5.4.7}$$

Fixing $j \in \{1, \ldots, k\}$ and premultiplying the above by $\sqrt{\theta_j}\mu e_j'$ yields

$$\mu\sqrt{\theta_j}(e_j'v) = N\varepsilon \sum_{l=1}^{k} \sqrt{\theta_j}\theta_l(e_l'v)e_j' \left(I - \frac{1}{\mu}W\right)^{-1} e_l = \sum_{l=1}^{k} V(j,l)\sqrt{\theta_l}(e_l'v)\,.$$

As the above holds for all $1 \le j \le k$, this means that if

$$u = \left[\sqrt{\theta_1}(e_1'v)\ldots\sqrt{\theta_k}(e_k'v)\right]', \tag{5.4.8}$$

then

$$Vu = \mu u\,. \tag{5.4.9}$$

Recalling that in the above calculations a sample point is fixed such that $\|W\| < \mu$, what we have shown, in other words, is that a vector u satisfying the above exists w.h.p.

In order to complete the proof of (5.4.5), it suffices to show that u is a non-null vector w.h.p. To that end, premultiply (5.4.6) by v' to obtain that

$$\mu = v'Wv + N\varepsilon\|u\|^2\,.$$

Dividing both sides by $N\varepsilon$ and using Lemma 4.3.1 implies that

$$\|u\|^2 = \theta_i + o_{hp}(1)\,.$$

Thus, (5.4.5) follows.

Lemma 5.4.1 shows that for all $l \in \{1, \ldots, i-1\}$,

$$\begin{aligned}&\left[V(i,i) + \sum_{1\le j\le k,\, j\ne i} |V(i,j)|\right] - \left[V(l,l) - \sum_{1\le j\le k,\, j\ne l} |V(l,j)|\right]\\ =&N\varepsilon\left(\theta_i - \theta_l\right)\left(1 + o_{hp}(1)\right),\end{aligned}$$

as $N \to \infty$. Since $i \in \mathcal{I}$, $\theta_i - \theta_l < 0$, and hence

$$V(i,i) + \sum_{1\le j\le k,\, j\ne i} |V(i,j)| < V(l,l) - \sum_{1\le j\le k,\, j\ne l} |V(l,j)| \text{ w.h.p.}$$

A similar calculation shows that for $l \in \{i+1, \ldots, k\}$,

$$V(i,i) - \sum_{1 \le j \le k,\, j \ne i} |V(i,j)| > V(l,l) + \sum_{1 \le j \le k,\, j \ne l} |V(l,j)| \text{ w.h.p.}$$

In view of (5.4.5) and Fact 1, the proof would follow once it can be shown that for all $l \in \{1, \ldots, k\} \setminus \{i\}$,

$$|\mu - V(l,l)| > \sum_{1 \le j \le k,\, j \ne l} |V(l,j)| \text{ w.h.p.}$$

This follows, once again, by dividing both sides by $N\varepsilon$ and using Theorem 5.2.1 and Lemma 5.4.1. This completes the proof. □

The next step is to write

$$\left(I - \frac{1}{\mu}W\right)^{-1} = \sum_{n=0}^{\infty} \mu^{-n} W^n \,, \tag{5.4.10}$$

which is possible because $\|W\| < \mu$. Denote

$$Z_{j,l,n} = e_j' W^n e_l \,, 1 \le j, l \le k \,, n \ge 0 \,,$$

and for $n \ge 0$, let Y_n be a $k \times k$ matrix with

$$Y_n(j,l) = \sqrt{\theta_j \theta_l} N\varepsilon Z_{j,l,n} \,, 1 \le j, l \le k \,.$$

The following bounds will be used several times.

Lemma 5.4.4. *It holds that*

$$\mathrm{E}\left(\|Y_1\|\right) = O\left(N\varepsilon^{3/2}\right) \,,$$

and

$$\|Y_1\| = o_{hp}\left((N\varepsilon)^2\right) \,.$$

Proof. Lemma 4.3.4 implies that

$$\mathrm{Var}\left(Z_{j,l,1}\right) = O(\varepsilon) \,, 1 \le j, l \le k \,.$$

Hence,

$$\begin{aligned}\mathrm{E}\|Y_1\| =&O\left(N\varepsilon\sum_{j,l=1}^{k}\mathrm{E}|Z_{j,l,1}|\right)\\ =&O\left(N\varepsilon\sum_{j,l=1}^{k}\sqrt{\mathrm{Var}(Z_{j,l,1})}\right)\\ =&O\left(N\varepsilon^{3/2}\right),\end{aligned}$$

the equality in the second line using the fact that $Z_{j,l,1}$ has mean 0. This proves the first claim. The second claim follows from (4.3.4) of Lemma 4.3.3. □

The next step is to truncate the infinite sum in (5.4.10) to level L, where $L = [\log N]$ as defined before.

Lemma 5.4.5. *It holds that*

$$\mu = \lambda_i\left(\sum_{n=0}^{L}\mu^{-n}Y_n\right) + o_{hp}\left(\sqrt{\varepsilon}\right).$$

Proof. From the definition of V, it is immediate that for $1 \le j,l \le k$,

$$V(j,l) = N\varepsilon\sqrt{\theta_j\theta_l}\sum_{n=0}^{\infty}\mu^{-n}e_j'W^ne_l\ \mathbb{1}(\|W\|<\mu),$$

and hence

$$V = \mathbb{1}(\|W\|<\mu)\sum_{n=0}^{\infty}\mu^{-n}Y_n\,.$$

For the sake of notational simplicity, let us suppress $\mathbb{1}(\|W\|<\mu)$. Therefore, with the implicit understanding that the sum is set as zero if $\|W\|\ge\mu$, for the proof it suffices to check that

$$\left\|\sum_{n=L+1}^{\infty}\mu^{-n}Y_n\right\| = o_{hp}(\sqrt{\varepsilon})\,. \tag{5.4.11}$$

To that end, Theorem 5.2.1 and Lemma 4.3.1 imply that

$$\begin{aligned}\left\|\sum_{n=L+1}^{\infty}\mu^{-n}Y_n\right\| \le& \sum_{n=L+1}^{\infty}|\mu|^{-n}\|Y_n\|\\ =&O_{hp}\left((N\varepsilon)^{-(L-1)/2}\right).\end{aligned}$$

In order to prove (5.4.11), it suffices to show that as $N \to \infty$,

$$-\log \varepsilon = o\left((L-1)\log(N\varepsilon)\right) . \tag{5.4.12}$$

To that end, recall (4.1.4) to argue that

$$N^{-1} = o(\varepsilon) \tag{5.4.13}$$

and

$$\log\log N = O(\log(N\varepsilon)) . \tag{5.4.14}$$

By (5.4.13), it follows that

$$\begin{aligned} -\log \varepsilon =& O\left(\log N\right) \\ =& o\left(\log N \log\log N\right) \\ =& o\left((L-1)\log(N\varepsilon)\right) , \end{aligned}$$

the last line using (5.4.14). Therefore, (5.4.12) follows, which ensures (5.4.11), which in turn completes the proof. □

In the next step, Y_n is replaced by its expectation for $n \geq 2$.

Lemma 5.4.6. *It holds that*

$$\mu = \lambda_i \left(Y_0 + \mu^{-1} Y_1 + \sum_{n=2}^{L} \mu^{-n} \mathrm{E}(Y_n) \right) + o_{hp}\left(\sqrt{\varepsilon}\right) .$$

Proof. In view of Theorem 5.2.1 and Lemma 5.4.5, all that has to be checked is

$$\sum_{n=2}^{L} (N\varepsilon)^{-n} \| Y_n - \mathrm{E}(Y_n) \| = o_{hp}(\sqrt{\varepsilon}) . \tag{5.4.15}$$

For that, invoke Lemma 4.3.3 to claim that

$$\begin{aligned} \max_{2 \leq n \leq L,\, 1 \leq j,l \leq k} & P\left(|Z_{j,l,n} - \mathrm{E}(Z_{j,l,n})| > N^{(n-1)/2} \varepsilon^{n/2} (\log N)^{n\xi/4} \right) \\ &= O\left(e^{-(\log N)^{\eta_1}} \right) , \end{aligned} \tag{5.4.16}$$

where ξ is as in (4.1.4).

Our next claim is that there exists $C_2 > 0$ such that for N large,

$$\bigcap_{2\le n\le L, 1\le j,l\le k} \left[|Z_{j,l,n} - \mathrm{E}(Z_{j,l,n})| \le N^{(n-1)/2}\varepsilon^{n/2}(\log N)^{n\xi/4}\right] \tag{5.4.17}$$

$$\subset \left[\sum_{n=2}^{L}(N\varepsilon)^{-n}\|Y_n - \mathrm{E}(Y_n)\| \le C_2\sqrt{\varepsilon}\left((N\varepsilon)^{-1}(\log N)^{\xi}\right)^{1/2}\right].$$

To see this, suppose that the event on the left hand side holds. Then, for fixed $1 \le j, l \le k$, and large N,

$$\begin{aligned}
&\sum_{n=2}^{L}(N\varepsilon)^{-n}\left\|Y_n(j,l) - \mathrm{E}\left[Y_n(j,l)\right]\right\| \\
\le&\theta_1 N\varepsilon\sum_{n=2}^{L}(N\varepsilon)^{-n}\left|Z_{j,l,n} - \mathrm{E}\left(Z_{j,l,n}\right)\right| \\
\le&\theta_1\sum_{n=2}^{\infty}(N\varepsilon)^{-(n-1)}N^{(n-1)/2}\varepsilon^{n/2}(\log N)^{n\xi/4} \\
=&\left[1-(N\varepsilon)^{-1/2}(\log N)^{\xi/4}\right]^{-1}\theta_1\sqrt{\varepsilon}(N\varepsilon)^{-1/2}(\log N)^{\xi/2}.
\end{aligned}$$

Thus, (5.4.17) holds for some $C_2 > 0$.

Combining (5.4.16) and (5.4.17), it follows that

$$\begin{aligned}
&P\left(\sum_{n=2}^{L}(N\varepsilon)^{-n}\|Y_n - \mathrm{E}(Y_n)\| > C_2\sqrt{\varepsilon}\left((N\varepsilon)^{-1}(\log N)^{\xi}\right)^{1/2}\right) \\
=&O\left(\log N e^{-(\log N)^{\eta_1}}\right) \\
=&o\left(e^{-(\log N)^{(1+\eta_1)/2}}\right).
\end{aligned}$$

This, with the help of (4.1.4), establishes (5.4.15) from which the proof follows. □

The goal of the next two lemmas is replacing μ by a deterministic quantity in

$$\sum_{n=2}^{L}\mu^{-n}\mathrm{E}(Y_n).$$

Lemma 5.4.7. *For N large, the deterministic equation*

$$x = \lambda_i\left(\sum_{n=0}^{L}x^{-n}\mathrm{E}(Y_n)\right), \; x > 0, \tag{5.4.18}$$

has a solution $\tilde{\mu}$ such that

$$0 < \liminf_{N\to\infty} (N\varepsilon)^{-1}\tilde{\mu} \le \limsup_{N\to\infty} (N\varepsilon)^{-1}\tilde{\mu} < \infty \,. \tag{5.4.19}$$

Proof. Define a function

$$h : (0,\infty) \to \mathbb{R}\,,$$

by

$$h(x) = \lambda_i \left(\sum_{n=0}^{L} x^{-n} \mathrm{E}(Y_n) \right) .$$

Our first claim is that for any fixed $x > 0$,

$$\lim_{N\to\infty} (N\varepsilon)^{-1} h\left(xN\varepsilon\right) = \theta_i \,. \tag{5.4.20}$$

To that end, observe that since $\mathrm{E}(Y_1) = 0$,

$$h\left(xN\varepsilon\right) = \lambda_i \left(\mathrm{E}(Y_0) + \sum_{n=2}^{L} (xN\varepsilon)^{-n} \mathrm{E}(Y_n) \right) .$$

Recalling that

$$Y_0(j,l) = N\varepsilon\sqrt{\theta_j\theta_l}\, e_j' e_l \,, 1 \le j,l \le k\,,$$

it follows by (5.4.2) that

$$\lim_{N\to\infty} (N\varepsilon)^{-1} \mathrm{E}(Y_0) = \mathrm{Diag}(\theta_1,\ldots,\theta_k)\,. \tag{5.4.21}$$

Lemma 4.3.2 implies that

$$\mathrm{E}(Z_{j,l,n}) \le \left(O(N\varepsilon)\right)^{n/2} ,$$

uniformly for $2 \le n \le L$, and hence there exists $0 < C_3 < \infty$ with

$$\|\mathrm{E}(Y_n)\| \le (C_3 N\varepsilon)^{n/2+1} \,,\, 2 \le n \le L\,. \tag{5.4.22}$$

Therefore,

$$\left\| \sum_{n=2}^{L} (xN\varepsilon)^{-n} \mathrm{E}(Y_n) \right\| \le \sum_{n=2}^{\infty} (xN\varepsilon)^{-n} (C_3 N\varepsilon)^{n/2+1} \to C_3^2 x^{-2} \,,$$

as $N \to \infty$. With the help of (5.4.21), this implies that

$$\lim_{N\to\infty} (N\varepsilon)^{-1} \left(\sum_{n=0}^{L} (xN\varepsilon)^{-n} \mathrm{E}(Y_n) \right) = \mathrm{Diag}(\theta_1, \ldots, \theta_k) \,,$$

and hence (5.4.20) follows. It follows that for a fixed $0 < \delta < \theta_i$.

$$\lim_{N\to\infty} (N\varepsilon)^{-1} \left[N\varepsilon(\theta_i + \delta) - h\left((\theta_i + \delta)N\varepsilon\right) \right] = \delta \,,$$

and thus, for large N,

$$N\varepsilon(\theta_i + \delta) > h\left((\theta_i + \delta)N\varepsilon\right) \,.$$

Similarly, again for large N,

$$N\varepsilon(\theta_i - \delta) < h\left((\theta_i - \delta)N\varepsilon\right) \,.$$

Hence, for N large, (5.4.18) has a solution $\tilde{\mu}$ in $[(N\varepsilon)(\theta_i - \delta), (N\varepsilon)(\theta_i + \delta)]$, which trivially satisfies (5.4.19). Hence the proof. □

Lemma 5.4.8. *If $\tilde{\mu}$ is as in Lemma 5.4.7, then*

$$\mu - \tilde{\mu} = O_{hp}\left((N\varepsilon)^{-1} \|Y_1\| + \sqrt{\varepsilon} \right) \,.$$

Proof. Lemmas 5.4.6 and 5.4.7 imply that

$$\begin{aligned}
&|\mu - \tilde{\mu}| \\
=& \left| \lambda_i \left(Y_0 + \mu^{-1} Y_1 + \sum_{n=2}^{L} \mu^{-n} \mathrm{E}(Y_n) \right) - \lambda_i \left(\sum_{n=0}^{L} \tilde{\mu}^{-n} \mathrm{E}(Y_n) \right) \right| + o_{hp}(\sqrt{\varepsilon}) \\
\le& \|\mu^{-1} Y_1\| + |\mu - \tilde{\mu}| \sum_{n=2}^{L} \mu^{-n} \tilde{\mu}^{-n} \|E(Y_n)\| \sum_{j=0}^{n-1} \mu^j \tilde{\mu}^{n-1-j} + o_{hp}(\sqrt{\varepsilon}) \\
=& |\mu - \tilde{\mu}| \sum_{n=2}^{L} \mu^{-n} \tilde{\mu}^{-n} \|E(Y_n)\| \sum_{j=0}^{n-1} \mu^j \tilde{\mu}^{n-1-j} + O_{hp}\left((N\varepsilon)^{-1} \|Y_1\| + \sqrt{\varepsilon} \right) \,.
\end{aligned}$$

Thus,

$$|\mu - \tilde{\mu}| \left[1 - \sum_{n=2}^{L} \mu^{-n} \tilde{\mu}^{-n} \|E(Y_n)\| \sum_{j=0}^{n-1} \mu^j \tilde{\mu}^{n-1-j} \right] \le O_{hp}\left((N\varepsilon)^{-1} \|Y_1\| + \sqrt{\varepsilon} \right) \,. \quad (5.4.23)$$

Equations (5.4.19) and (5.4.22) imply that

$$\left|\sum_{n=2}^{L}\mu^{-n}\tilde{\mu}^{-n}\|E(Y_n)\|\sum_{j=0}^{n-1}\mu^j\tilde{\mu}^{n-1-j}\right| = O_{hp}\left(\sum_{n=2}^{\infty}n(N\varepsilon)^{-(n+1)}(C_3N\varepsilon)^{n/2+1}\right)$$
$$= O_{hp}\left((N\varepsilon)^{-1}\right)$$
$$= o_{hp}(1),\ N\to\infty\,. \tag{5.4.24}$$

This completes the proof with the help of (5.4.23). □

The next lemma is arguably the most important step in the proof of Theorem 5.2.2, the other major step being Lemma 5.4.2.

Lemma 5.4.9. *There exists a deterministic $\bar{\mu}$, which depends on N, such that*

$$\mu = \bar{\mu} + \mu^{-1}Y_1(i,i) + o_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right)\,.$$

Proof. Define a $k\times k$ deterministic matrix

$$X = \sum_{n=0}^{L}\tilde{\mu}^{-n}\mathrm{E}(Y_n)\,,$$

which, as usual, depends on N. Lemma 5.4.8 and (5.4.24) imply that

$$\left\|X - \sum_{n=0}^{L}\mu^{-n}\mathrm{E}(Y_n)\right\| \le |\mu-\tilde{\mu}|\sum_{n=2}^{L}\mu^{-n}\tilde{\mu}^{-n}\|\mathrm{E}(Y_n)\|\sum_{j=0}^{n-1}\mu^j\tilde{\mu}^{n-1-j}$$
$$= o_{hp}\left(|\mu-\tilde{\mu}|\right)$$
$$= o_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right)\,.$$

By Lemma 5.4.6 it follows that

$$\mu = \lambda_i\left(\mu^{-1}Y_1 + X\right) + o_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right)\,. \tag{5.4.25}$$

Let

$$H = X + \mu^{-1}Y_1 - \left(X(i,i) + \mu^{-1}Y_1(i,i)\right)I\,,$$

$$M = X - X(i,i)I\,,$$

and

$$\bar{\mu} = \lambda_i(X)\,.$$

Clearly,

$$\begin{aligned}\lambda_i\left(\mu^{-1}Y_1 + X\right) =& X(i,i) + \mu^{-1}Y_1(i,i) + \lambda_i(H) \\ =& \bar{\mu} - \lambda_i(M) + \mu^{-1}Y_1(i,i) + \lambda_i(H)\,.\end{aligned}$$

Thus, the proof would follow with the aid of (5.4.25) if it can be shown that

$$\lambda_i(H) - \lambda_i(M) = o_{hp}\left((N\varepsilon)^{-1}\|Y_1\|\right)\,. \tag{5.4.26}$$

If $k = 1$, then $i = 1$ and hence $H = M = 0$. Thus, the above is a tautology in that case. Therefore, assume without loss of generality that $k \geq 2$.

Proceeding towards proving (5.4.26) when $k \geq 2$, set

$$U_1 = (N\varepsilon)^{-1}M\,, \tag{5.4.27}$$

and

$$U_2 = (N\varepsilon)^{-1}H\,. \tag{5.4.28}$$

The main idea in the proof of (5.4.26) is to observe that the eigenvector of U_1 corresponding to $\lambda_i(U_1)$ is same as that of M corresponding to $\lambda_i(M)$, and likewise for U_2 and X. Hence, the first step is to use this to get a bound on the differences between the eigenvectors in terms of $\|U_1 - U_2\|$.

An important observation that will be used later is that

$$\|U_1 - U_2\| = O_{hp}\left((N\varepsilon)^{-2}\|Y_1\|\right)\,. \tag{5.4.29}$$

The second claim of Lemma 5.4.4 implies that the right hand side above is $o_{hp}(1)$. The same implies that for $m = 1, 2$ and $1 \leq j, l \leq k$,

$$U_m(j,l) = (\theta_j - \theta_i)\,\mathbb{1}(j = l) + o_{hp}(1)\,, N \to \infty\,. \tag{5.4.30}$$

In other words, as $N \to \infty$, U_1 and U_2 converge to $\mathrm{Diag}(\theta_1 - \theta_i, \ldots, \theta_k - \theta_i)$ w.h.p. Therefore,

$$\lambda_i(U_m) = o_{hp}(1)\,, m = 1, 2\,. \tag{5.4.31}$$

Let $\tilde{U}_m$, for $m = 1, 2$, be the $(k-1) \times (k-1)$ matrix (recall that $k \geq 2$) obtained by deleting the i-th row and the i-th column of U_m, and let $\tilde{u}_m$ be the $(k-1) \times 1$ vector obtained from the i-th column of U_m by deleting its i-th entry. It is worth recording, for possible future use, that

$$\|\tilde{u}_m\| = o_{hp}(1)\,, m = 1, 2\,, \tag{5.4.32}$$

which follows from (5.4.30), and that

$$\|\tilde{u}_1 - \tilde{u}_2\| = O_{hp}\left((N\varepsilon)^{-2}\|Y_1\|\right)\,, \tag{5.4.33}$$

follows from (5.4.29).

Equations (5.4.30) and (5.4.31) imply that $\tilde{U}_m - \lambda_i(U_m)I_{k-1}$ converges w.h.p. to

$$\mathrm{Diag}(\theta_1 - \theta_i, \ldots, \theta_{i-1} - \theta_i, \theta_{i+1} - \theta_i, \theta_k - \theta_i)\,.$$

Since $i \in \mathcal{I}$, the above matrix is invertible. Fix $\delta > 0$ such that every matrix in the closed δ-neighborhood B_δ, in the sense of operator norm, of the above matrix is invertible. Let

$$C_4 = \sup_{E \in B_\delta} \|E^{-1}\|\,. \tag{5.4.34}$$

Then, $C_4 < \infty$. Besides, there exists $C_5 < \infty$ satisfying

$$\left\|E_1^{-1} - E_2^{-1}\right\| \leq C_5\|E_1 - E_2\|\,, E_1, E_2 \in B_\delta\,. \tag{5.4.35}$$

Fix $N \geq 1$ and a sample point such that $\tilde{U}_m - \lambda_i(U_m)I_{k-1}$ belongs to B_δ. Then, it is invertible. Define a $(k-1) \times 1$ vector

$$\tilde{v}_m = -\left[\tilde{U}_m - \lambda_i(U_m)I_{k-1}\right]^{-1}\tilde{u}_m\,, m = 1, 2\,,$$

and a $k \times 1$ vector

$$v_m = [\tilde{v}_m(1), \ldots, \tilde{v}_m(i-1), 1, \tilde{v}_m(i), \ldots, \tilde{v}_m(k-1)]', m = 1, 2.$$

It is immediate that

$$\|\tilde{v}_m\| \le C_4 \|\tilde{u}_m\|, m = 1, 2. \tag{5.4.36}$$

Our next claim is that

$$U_m v_m = \lambda_i(U_m) v_m, m = 1, 2. \tag{5.4.37}$$

This claim is equivalent to

$$[U_m - \lambda_i(U_m) I_k] v_m = 0. \tag{5.4.38}$$

Let $\bar{U}_m$ be the $(k-1) \times k$ matrix obtained by deleting the i-th row of $U_m - \lambda_i(U_m) I_k$. Since the latter matrix is singular, and $\tilde{U}_m - \lambda_i(U_m) I_{k-1}$ is invertible, it follows that the i-th row of $U_m - \lambda_i(U_m) I_k$ lies in the row space of $\bar{U}_m$. In other words, the row spaces of $U_m - \lambda_i(U_m) I_k$ and $\bar{U}_m$ are the same, and so do their null spaces. Thus, (5.4.38) is equivalent to

$$\bar{U}_m v_m = 0.$$

To see the above, observe that the i-th column of $\bar{U}_m$ is $\tilde{u}_m$, and hence we can partition

$$\bar{U}_m = \left[\bar{U}_{m1}\, \tilde{u}_m\, \bar{U}_{m2}\right],$$

where $\bar{U}_{m1}$ and $\bar{U}_{m2}$ are of order $(k-1) \times (i-1)$ and $(k-1) \times (k-i)$, respectively. Furthermore,

$$\left[\bar{U}_{m1}\, \bar{U}_{m2}\right] = \tilde{U}_m - \lambda_i(U_m) I_{k-1}.$$

Therefore,

$$\bar{U}_m v_m = \tilde{u}_m + \left[\bar{U}_{m1}\, \bar{U}_{m2}\right] \tilde{v}_m = \tilde{u}_m + \left(\tilde{U}_m - \lambda_i(U_m) I_{k-1}\right) \tilde{v}_m = 0.$$

Hence, (5.4.38) follows, which proves (5.4.37).

Next, we note

$$\|v_1 - v_2\|$$

$$
\begin{aligned}
&=\|\tilde{v}_1-\tilde{v}_2\| \\
&\le \left\|\left(\tilde{U}_1-\lambda_i(U_1)I_{k-1}\right)^{-1}\right\| \|\tilde{u}_1-\tilde{u}_2\| \\
&\quad + \left\|\left(\tilde{U}_1-\lambda_i(U_1)I_{k-1}\right)^{-1}-\left(\tilde{U}_2-\lambda_i(U_2)I_{k-1}\right)^{-1}\right\| \|\tilde{u}_2\| \\
&\le C_4\|\tilde{u}_1-\tilde{u}_2\|+C_5\left\|\left(\tilde{U}_1-\lambda_i(U_1)I_{k-1}\right)-\left(\tilde{U}_2-\lambda_i(U_2)I_{k-1}\right)\right\| \|\tilde{u}_2\|\,,
\end{aligned}
$$

C_4 and C_5 being as in (5.4.34) and (5.4.35), respectively. Recalling that the above calculation was done on an event of high probability, what we have proven, with the help of (5.4.29) and (5.4.33), is that

$$
\|v_1-v_2\| = O_{hp}\left((N\varepsilon)^{-2}\|Y_1\|\right) .
$$

Furthermore, (5.4.32) and (5.4.36) imply that

$$
\|\tilde{v}_m\| = o_{hp}(1)\,.
$$

Finally, noting that

$$
U_m(i,i)=0\,, m=1,2\,,
$$

and that

$$
v_m(i)=1\,, m=1,2\,,
$$

it follows that

$$
\begin{aligned}
&|\lambda_i(U_1)-\lambda_i(U_2)| \\
=&\left|\sum_{1\le j\le k,\, j\ne i} U_1(i,j)v_1(j)-\sum_{1\le j\le k,\, j\ne i} U_2(i,j)v_2(j)\right| \\
\le&\sum_{1\le j\le k,\, j\ne i}|U_1(i,j)||v_1(j)-v_2(j)|+\sum_{1\le j\le k,\, j\ne i}|U_1(i,j)-U_2(i,j)|v_2(j)| \\
=&O_{hp}\left(\|\tilde{u}_1\|\|v_1-v_2\|+\|U_1-U_2\|\|\tilde{v}_2\|\right) \\
=&o_{hp}\left((N\varepsilon)^{-2}\|Y_1\|\right) .
\end{aligned}
$$

Recalling (5.4.27) and (5.4.28), (5.4.26) follows, which completes the proof in conjunction with (5.4.25). □

Now, we are in a position to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. Recalling that

$$Y_1(i,i) = \theta_i N\varepsilon\, e_i' W e_i\,,$$

it suffices to show that

$$\mu - \mathrm{E}(\mu) = \mu^{-1} Y_1(i,i) + o_p(\sqrt{\varepsilon})\,. \tag{5.4.39}$$

Lemma 5.4.9 implies that

$$\begin{aligned} \mu - \bar{\mu} =& \mu^{-1} Y_1(i,i) + o_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right) \\ =& O_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right)\,, \end{aligned} \tag{5.4.40}$$

a consequence of which, combined with Lemma 5.4.4, is that

$$\lim_{N\to\infty} (N\varepsilon)^{-1}\bar{\mu} = \theta_i\,. \tag{5.4.41}$$

Thus,

$$\begin{aligned} \left|\frac{1}{\bar{\mu}} Y_1(i,i) - \frac{1}{\mu} Y_1(i,i)\right| =& O_{hp}\left((N\varepsilon)^{-2}|\mu - \bar{\mu}|\|Y_1\|\right) \\ =& o_{hp}\left(|\mu - \bar{\mu}|\right) \\ =& o_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right) \\ =& o_p(\sqrt{\varepsilon})\,, \end{aligned} \tag{5.4.42}$$

Lemma 5.4.4 implying the second line, the third line following from (5.4.40) and the fact that

$$\|Y_1\| = O_p\left(N\varepsilon^{3/2}\right)\,, \tag{5.4.43}$$

which is also a consequence of the former lemma, being used for the last line. Using Lemma 5.4.9 once again, we get that

$$\mu = \bar{\mu} + \frac{1}{\bar{\mu}} Y_1(i,i) + o_{hp}\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right)\,. \tag{5.4.44}$$

Let

$$R = \mu - \bar{\mu} - \frac{1}{\bar{\mu}} Y_1(i,i)\,.$$

Clearly,

$$\mathrm{E}(R) = \mathrm{E}(\mu) - \bar{\mu}\,,$$

and (5.4.44) implies that for $\delta > 0$ there exists $\eta > 1$ with

$$\mathrm{E}|R| \le \delta\left(\sqrt{\varepsilon} + (N\varepsilon)^{-1}\mathrm{E}\|Y_1\|\right) + \mathrm{E}^{1/2}\left(\mu - \bar{\mu} - \frac{1}{\bar{\mu}}Y_1(i,i)\right)^2 O\left(e^{-(\log N)^\eta}\right)\,.$$

Lemma 5.4.4 implies that

$$\mathrm{E}|R| \le o(\sqrt{\varepsilon}) + \mathrm{E}^{1/2}\left(\mu - \bar{\mu} - \frac{1}{\bar{\mu}}Y_1(i,i)\right)^2 O\left(e^{-(\log N)^\eta}\right)\,.$$

Next, (5.4.41) and that $|\mu| \le N^2$ a.s. imply that

$$\begin{aligned}\mathrm{E}^{1/2}\left(\mu - \bar{\mu} - \frac{1}{\bar{\mu}}Y_1(i,i)\right)^2 =& O\left(N^2\right)\\ =& o\left(\varepsilon^{1/2}N^3\right)\\ =& o\left(\varepsilon^{1/2}e^{(\log N)^\eta}\right)\,.\end{aligned}$$

Thus,

$$\mathrm{E}|R| = o(\sqrt{\varepsilon})\,,$$

and hence

$$\mathrm{E}(\mu) = \bar{\mu} + o(\sqrt{\varepsilon})\,.$$

This, in view of (5.4.44), implies that

$$\begin{aligned}\mu =& \mathrm{E}(\mu) + \frac{1}{\bar{\mu}}Y_1(i,i) + o_p\left((N\varepsilon)^{-1}\|Y_1\| + \sqrt{\varepsilon}\right)\\ =& \mathrm{E}(\mu) + \frac{1}{\bar{\mu}}Y_1(i,i) + o_p\left(\sqrt{\varepsilon}\right)\,,\end{aligned}$$

the second line following from (5.4.43). This establishes (5.4.39) with the help of (5.4.42), and hence the proof. □

Theorems 5.2.1 and 5.2.2 establish Theorem 5.2.3 with the help of (5.2.4). Now we shall proceed toward proving Theorem 5.2.6. For the rest of this section, (5.2.5) will be assumed, that is,

f is Lipschitz continuous. As a consequence, the functions $r_1, \ldots, r_k$, which are eigenfunctions of the integral operator I_f, are also Lipschitz.

The following lemma essentially proves Theorem 5.2.6.

Lemma 5.4.10. *If f is a Lipschitz function, then*

$$\mu = \lambda_i \left(Y_0 + (N\varepsilon\theta_i)^{-2} \mathrm{E}(Y_2) \right) + O_p \left(\sqrt{\varepsilon} + (N\varepsilon)^{-1} \right) .$$

Proof. Lemma 5.4.6 implies that

$$\mu = \lambda_i \left(\sum_{n=0}^{3} \mu^{-n} \mathrm{E}(Y_n) \right) + O_p \left(\mu^{-1} \|Y_1\| + \sum_{n=4}^{L} \mu^{-n} \|\mathrm{E}(Y_n)\| \right) + o_p(\sqrt{\varepsilon}) .$$

Equation (5.4.43) implies that

$$\mu = \lambda_i \left(\sum_{n=0}^{3} \mu^{-n} \mathrm{E}(Y_n) \right) + O_p \left(\sqrt{\varepsilon} + \sum_{n=4}^{L} \mu^{-n} \|\mathrm{E}(Y_n)\| \right) .$$

From (5.4.22), it follows that

$$\sum_{n=4}^{L} \mu^{-n} \|\mathrm{E}(Y_n)\| = O_p \left((N\varepsilon)^{-1} \right) ,$$

and hence

$$\mu = \lambda_i \left(\sum_{n=0}^{3} \mu^{-n} \mathrm{E}(Y_n) \right) + O_p \left(\sqrt{\varepsilon} + (N\varepsilon)^{-1} \right) . \tag{5.4.45}$$

Lemma 4.3.4, in particular (4.3.5) therein, implies that

$$\|\mathrm{E}(Y_3)\| = O \left((N\varepsilon)^2 \right) ,$$

and hence

$$\mu^{-3} \|\mathrm{E}(Y_3)\| = O_p \left((N\varepsilon)^{-1} \right) .$$

This, in conjunction with (5.4.45), implies that

$$\mu = \lambda_i \left(Y_0 + \mu^{-2} \mathrm{E}(Y_2) \right) + O_p \left(\sqrt{\varepsilon} + (N\varepsilon)^{-1} \right) . \tag{5.4.46}$$

An immediate consequence of the above and (5.4.22) is that

$$\mu = \lambda_i(Y_0) + O_p(1)\,. \tag{5.4.47}$$

Applying Fact 1 as in the proof of Lemma 5.4.2, it can be shown that

$$|\lambda_i(Y_0) - Y_0(i,i)| \le \sum_{1\le j\le k,\, j\ne i} |Y_0(i,j)|\,. \tag{5.4.48}$$

Since r_i and r_j are Lipschitz functions, it holds that

$$e_i' e_j = \mathbb{1}(i=j) + O\left(N^{-1}\right)\,.$$

Hence, it follows that

$$Y_0(i,i) = N\varepsilon\left(\theta_i + O(N^{-1})\right) = N\varepsilon\theta_i + O(\varepsilon)\,,$$

and similarly,

$$Y_0(i,j) = O(\varepsilon)\,, j \ne i\,.$$

Combining these findings with (5.4.48) yields that

$$\lambda_i(Y_0) = N\varepsilon\theta_i + O(\varepsilon)\,. \tag{5.4.49}$$

Equations (5.4.47) and (5.4.49) together imply that

$$\mu = N\varepsilon\theta_i + O_p(1)\,. \tag{5.4.50}$$

Therefore,

$$\begin{aligned}
&\left\|\mu^{-2}\mathrm{E}(Y_2) - (N\varepsilon\theta_i)^{-2}\mathrm{E}(Y_2)\right\| \\
=&O_p\left((N\varepsilon)^{-3}\|\mathrm{E}(Y_2)\|\right) \\
=&O_p\left((N\varepsilon)^{-1}\right)\,.
\end{aligned}$$

This in conjunction with (5.4.46) completes the proof. □

Theorem 5.2.6 is a simple corollary of the above lemma, as shown below.

Proof of Theorem 5.2.6. A consequence of Theorem 5.2.2 is that

$$\mu - \mathrm{E}(\mu) = O_p(\sqrt{\varepsilon})\,.$$

The claim of Lemma 5.4.10 is equivalent to

$$\lambda_i(B) - \mu = O_p\left(\sqrt{\varepsilon} + (N\varepsilon)^{-1}\right)\,.$$

The proof follows by adding the two equations, and noting that B is a deterministic matrix. □

Next we proceed towards the proof of Theorem 5.2.7, for which the following lemma will be useful.

Lemma 5.4.11. *If f is Lipschitz continuous, then as $N \to \infty$,*

$$e_j'\left(I - \mu^{-1}W\right)^{-n} e_l = \mathbb{1}(j = l) + O_p\left((N\varepsilon)^{-1}\right)\,, 1 \le j, l \le k\,, n = 1, 2\,.$$

Proof. For a fixed $n = 1, 2$, expand

$$\left(I - \mu^{-1}W\right)^{-n} = I + n\mu^{-1}W + O_p\left(\mu^{-2}\|W\|^2\right)\,.$$

The proof can be completed by proceeding along similar lines as in the proof of Lemma 5.4.10. □

Now we are in a position to prove Theorem 5.2.7.

Proof of Theorem 5.2.7. Theorem 5.2.1 implies that (5.2.6) holds for any $i \in \mathcal{I}$. Fix such an i, denote

$$\mu = \lambda_i(A)\,,$$

and let v be the eigenvector of A, having norm 1, corresponding to μ, which is uniquely defined with probability close to 1.

Fix $k \ge 2$, and $j \in \{1, \ldots, k\} \setminus \{i\}$. Premultiplying (5.4.7) by e_j' yields that

$$e_j'v = N\varepsilon \sum_{l=1}^{k} \theta_l (e_l'v) e_j' \left(\mu I - W\right)^{-1} e_l\,, \text{ w.h.p.}$$

Therefore,

$$\begin{aligned} & e_j'v\left(1-\theta_j\frac{N\varepsilon}{\mu}e_j'\left(I-\mu^{-1}W\right)^{-1}e_j\right) \\ =&\frac{N\varepsilon}{\mu}\sum_{1\le l\le k,\, l\neq j}\theta_l(e_l'v)e_j'\left(I-\mu^{-1}W\right)^{-1}e_l\,,\ \text{w.h.p.} \end{aligned}$$

Lemma 5.4.11 implies that as $N\to\infty$,

$$1-\theta_j\frac{N\varepsilon}{\mu}e_j'\left(I-\mu^{-1}W\right)^{-1}e_j\xrightarrow{P}1-\frac{\theta_j}{\theta_i}\neq 0\,.$$

Therefore,

$$\begin{aligned} e_j'v=&O_p\left(\frac{N\varepsilon}{\mu}\sum_{1\le l\le k,\, l\neq j}\theta_l(e_l'v)e_j'\left(I-\mu^{-1}W\right)^{-1}e_l\right) \\ =&O_p\left(\sum_{1\le l\le k,\, l\neq j}\left|e_j'\left(I-\mu^{-1}W\right)^{-1}e_l\right|\right) \\ =&O_p\left((N\varepsilon)^{-1}\right)\,, \end{aligned}$$

the last line being another consequence of Lemma 5.4.11. Thus, (5.2.8) holds.

Using (5.4.7) once again, we get that

$$1=(N\varepsilon)^2\sum_{l,m=1}^{k}\theta_l\theta_m(e_l'v)(e_m'v)e_l'\left(\mu I-W\right)^{-2}e_m\,,$$

that is,

$$\begin{aligned} & \theta_i^2(e_i'v)^2e_i'\left(I-\mu^{-1}W\right)^{-2}e_i \\ =&(N\varepsilon)^{-2}\mu^2-\sum_{(l,m)\in\{1,\ldots,k\}^2\setminus\{(i,i)\}}\theta_l\theta_m(e_l'v)(e_m'v)e_l'\left(I-\mu^{-1}W\right)^{-2}e_m\,. \end{aligned}$$

Using Lemma 5.4.11 once again, it follows that

$$e_i'\left(I-\mu^{-1}W\right)^{-2}e_i=1+O_p\left((N\varepsilon)^{-1}\right)\,.$$

Thus, (5.2.7) would follow once it's shown that

$$(N\varepsilon)^{-2}\mu^2=\theta_i^2+O_p\left((N\varepsilon)^{-1}\right)\,, \tag{5.4.51}$$

and that for all $(l,m) \in \{1,\ldots,k\}^2 \setminus \{(i,i)\}$,

$$(e_l'v)(e_m'v)e_l'\left(I-\mu^{-1}W\right)^{-2}e_m = O_p\left((N\varepsilon)^{-1}\right). \tag{5.4.52}$$

Equation (5.4.51) is a trivial consequence of (5.4.50). For (5.4.52), assuming without loss of generality that $l \neq i$, (5.2.8) implies that

$$\begin{aligned}\left|(e_l'v)(e_m'v)e_l'\left(I-\mu^{-1}W\right)^{-2}e_m\right| &= \left|(e_m'v)e_l'\left(I-\mu^{-1}W\right)^{-2}e_m\right| O_p\left((N\varepsilon)^{-1}\right)\\ &\le \left|e_l'\left(I-\mu^{-1}W\right)^{-2}e_m\right| O_p\left((N\varepsilon)^{-1}\right)\\ &= O_p\left((N\varepsilon)^{-1}\right),\end{aligned}$$

the last line following from Lemma 5.4.11. Thus, (5.4.52) follows, which in conjunction with (5.4.51) establishes (5.2.7). This completes the proof. □

Finally, Theorem 5.2.8 is proved.

Proof of Theorem 5.2.8. Fix $i \in \mathcal{I}$. Recall (5.4.8) and (5.4.9), and let u be as defined in the former. Let $\tilde{u}$ be the column vector obtained by deleting the i-th entry of u, $\tilde{V}_i$ be the column vector obtained by deleting the i-th entry of the i-th column of V, and $\tilde{V}$ be the $(k-1)\times(k-1)$ matrix obtained by deleting the i-th row and i-th column of V. Then, (5.4.9) implies that

$$\mu\tilde{u} = \tilde{V}\tilde{u} + u(i)\tilde{V}_i\,, \text{ w.h.p.} \tag{5.4.53}$$

Lemma 5.4.1 implies that

$$\left\|I_k - \mu^{-1}V - \text{Diag}\left(1-\frac{\theta_1}{\theta_i},\ldots,1-\frac{\theta_k}{\theta_i}\right)\right\| = o_{hp}(1)\,,$$

and hence $I_{k-1} - \mu^{-1}\tilde{V}$ is non-singular w.h.p. Thus, (5.4.53) implies that

$$\tilde{u} = u(i)\mu^{-1}\left(I_{k-1}-\mu^{-1}\tilde{V}\right)^{-1}\tilde{V}_i\,, \text{ w.h.p.} \tag{5.4.54}$$

The next step is to show that

$$\left\|\mu^{-1}V - \text{Diag}\left(\frac{\theta_1}{\theta_i},\ldots,\frac{\theta_k}{\theta_i}\right)\right\| = o_p\left(\sqrt{\varepsilon}\right). \tag{5.4.55}$$

To see this, use the fact that f is Lipschitz to write for a fixed $1 \le j, l \le k$,

$$
\begin{aligned}
V(j,l) =& N\varepsilon\sqrt{\theta_j\theta_l}\left(e_j' e_l + \mu^{-1} e_j' W e_l + O_p\left(\mu^{-2}\|W\|^2\right)\right) \\
=& N\varepsilon\sqrt{\theta_j\theta_l}\left(e_j' e_l + O_p\left((N\varepsilon)^{-1}\right)\right) \\
=& N\varepsilon\theta_j\left(\mathbb{1}(j=l) + O_p\left((N\varepsilon)^{-1}\right)\right) \\
=& N\varepsilon\theta_j\left(\mathbb{1}(j=l) + o_p\left(\sqrt{\varepsilon}\right)\right) ,
\end{aligned} \tag{5.4.56}
$$

the last line following from the fact that

$$
(N\varepsilon)^{-1} = o\left(\sqrt{\varepsilon}\right) , \tag{5.4.57}
$$

which is a consequence of (5.2.9). This along with (5.4.50) implies that

$$
(N\varepsilon\theta_i)^{-1}\mu = 1 + o_p\left(\sqrt{\varepsilon}\right) . \tag{5.4.58}
$$

Combining this with (5.4.56) yields that

$$
\mu^{-1}V(j,l) = \theta_i^{-1}\theta_j\,\mathbb{1}(j=l) + o_p\left(\sqrt{\varepsilon}\right) .
$$

Thus, (5.4.55) follows, an immediate consequence of which is that

$$
\left\|\left(I_{k-1} - \mu^{-1}\tilde{V}\right)^{-1} - \tilde{D}\right\| = o_p\left(\sqrt{\varepsilon}\right) , \tag{5.4.59}
$$

where

$$
\tilde{D} = \left[\mathrm{Diag}\left(1 - \frac{\theta_1}{\theta_i}, \ldots, 1 - \frac{\theta_{i-1}}{\theta_i}, 1 - \frac{\theta_{i+1}}{\theta_i}, \ldots, 1 - \frac{\theta_k}{\theta_i}\right)\right]^{-1} .
$$

Next, fix $j \in \{1, \ldots, k\} \setminus \{i\}$. By similar arguments as above, it follows that

$$
\begin{aligned}
V(i,j) =& N\varepsilon\sqrt{\theta_i\theta_j}\left(\sum_{n=0}^{3}\mu^{-n} e_i' W^n e_j + O_p\left(\mu^{-4}\|W\|^4\right)\right) \\
=& N\varepsilon\sqrt{\theta_i\theta_j}\sum_{n=0}^{3}\mu^{-n} e_i' W^n e_j + O_p\left((N\varepsilon)^{-1}\right) \\
=& N\varepsilon\sqrt{\theta_i\theta_j}\sum_{n=1}^{2}\mu^{-n} e_i' W^n e_j + o_p\left(\sqrt{\varepsilon}\right) ,
\end{aligned}
$$

using (5.4.57) once again, because

$$N\varepsilon e_i' e_j = O(\varepsilon) = o\left(\sqrt{\varepsilon}\right) ,$$

and

$$N\varepsilon\mu^{-3} e_i' W^3 e_j = O_p\left((N\varepsilon)^{-2}\mathrm{E}(e_i' W^3 e_j)\right) = o_p\left(\sqrt{\varepsilon}\right) ,$$

by (4.3.5). Thus,

$$\begin{aligned} V(i,j) - N\varepsilon\sqrt{\theta_i\theta_j}\mu^{-1}e_i'We_j =& N\varepsilon\sqrt{\theta_i\theta_j}\mu^{-2}e_i'W^2e_j + o_p\left(\sqrt{\varepsilon}\right) \\ =& N\varepsilon\sqrt{\theta_i\theta_j}\mu^{-2}\mathrm{E}\left(e_i'W^2e_j\right) + o_p\left(\sqrt{\varepsilon}\right) \\ =& (N\varepsilon)^{-1}\theta_j^{1/2}\theta_i^{-3/2}\mathrm{E}\left(e_i'W^2e_j\right) + o_p\left(\sqrt{\varepsilon}\right) , \end{aligned}$$

the second line following from Lemma 4.3.3, and the last line from (5.4.57), (5.4.58) and Lemma 4.3.2. In particular,

$$V(i,j) = O_p(1) .$$

The above in conjunction with (5.4.59) implies that

$$\begin{aligned} &\left[\left(I_{k-1} - \mu^{-1}\tilde{V}\right)^{-1}\tilde{V}_i\right](j) \\ =& \left(1 - \frac{\theta_j}{\theta_i}\right)^{-1}\sqrt{\theta_i\theta_j}\left[(N\varepsilon)^{-1}\theta_i^{-2}\mathrm{E}\left(e_i'W^2e_j\right) + N\varepsilon\mu^{-1}e_i'We_j\right] + o_p(\sqrt{\varepsilon}) . \end{aligned}$$

In light of (5.4.54), the above means that

$$\begin{aligned} &e_j'v \\ =&(e_i'v)\mu^{-1}\left(1 - \frac{\theta_j}{\theta_i}\right)^{-1}\left[(N\varepsilon)^{-1}\theta_i^{-1}\mathrm{E}\left(e_i'W^2e_j\right) + N\varepsilon\theta_i\mu^{-1}e_i'We_j + o_p(\sqrt{\varepsilon})\right] \\ =&\mu^{-1}\left(1 - \frac{\theta_j}{\theta_i}\right)^{-1}\left[(N\varepsilon)^{-1}\theta_i^{-1}\mathrm{E}\left(e_i'W^2e_j\right) + N\varepsilon\theta_i\mu^{-1}e_i'We_j + o_p(\sqrt{\varepsilon})\right] , \end{aligned}$$

the last line following from (5.2.7) and (5.4.57). Using (5.4.58) once again yields that

$$N\varepsilon(e_j'v) = \frac{1}{\theta_i - \theta_j}\left[(N\varepsilon)^{-1}\theta_i^{-1}\mathrm{E}\left(e_i'W^2e_j\right) + N\varepsilon\theta_i\mu^{-1}e_i'We_j\right] + o_p(\sqrt{\varepsilon}) .$$

This completes the proof. □

5.5 Appendix

Lemma 5.5.1. *The eigenfunctions* $\{r_i : 1 \leq i \leq k\}$ *of the operator* I_f *are Riemann integrable.*

Proof. Let $D_f \subset [0,1] \times [0,1]$ be the set of discontinuity points f. Since f is Riemann integrable, the Lebesgue measure of D_f is 0. Let

$$D_f^x = \{y \in [0,\, 1] :\, (x, y) \in D_f\},\, x \in [0,\, 1]\,.$$

If λ is the one dimensional Lebesgue measure, then Fubini's theorem implies that

$$E = \{x \in [0, 1] : \lambda(D_f^x) = 0\}$$

has full measure. Fix an $x \in E$ and consider $x_n \to x$ and observe that

$$f(x_n, y) \to f(x, y) \text{ for all } y \notin D_f^x.$$

Fix $1 \leq i \leq k$ and let θ_i be the eigenvalue with corresponding eigenfunction r_i, that is,

$$r_i(x) = \frac{1}{\theta_i} \int_0^1 f(x, y) r_i(y)\, dy. \tag{5.5.1}$$

Using f is bounded and $r \in L^2[0, 1]$, dominated convergence theorem implies

$$r_i(x_n) = \frac{1}{\theta_i} \int_{(D_f^x)^c} f(x_n, y) r_i(y)\, dy \to \frac{1}{\theta_i} \int_0^1 f(x, y) r_i(y)\, dy = r_i(x)$$

and hence r is continuous at x. So the discontinuity points of r_i form a subset of E^c which has Lebesgue measure 0. Further, (5.5.1) shows that r_i is bounded and hence Riemann integrability follows. □

The following result is a version of the Perron-Frobenius theorem in the infinite dimensional setting (also known as the Krein-Rutman theorem). Since our integral operator is positive, self-adjoint and finite dimensional so the proof in this setting is much simpler and can be derived following the work of Ninio [72]. In what follows, we use for $f, g \in L^2[0, 1]$, the inner product

$$\langle f,\, g \rangle = \int_0^1 f(x) g(x) dx.$$

Lemma 5.5.2. *Suppose $f > 0$ a.e. on $[0,1] \times [0,1]$. Then largest eigenvalue θ_1 of T_f is positive and the corresponding eigenfunction r_1 can be chosen such that $r_1(x) > 0$ for almost every $x \in [0,1]$. Further, $\theta_1 > \theta_2$.*

Proof. First observe that

$$\begin{aligned} 0 < \theta_1 = \langle r_1,\, \theta_1 r_1 \rangle = \langle r_1,\, I_f(r_1) \rangle = |\langle r_1,\, I_f(r_1) \rangle| \\ \leq \langle u_1, I_f(u_1) \rangle \leq \theta_1 \end{aligned}$$

where $u_1(x) = |r_1|(x)$ and the last inequality follows from the Rayleigh-Ritz formulation of the largest eigenvalue. Hence note that the string of inequalities is actually an equality, that is,

$$\langle r_1,\, I_f(r_1) \rangle = \langle u_1, I_f(u_1) \rangle.$$

Breaking $r_1 = r_1^+ - r_1^-$ implies either $r_1^+ = 0$ or $r_1^- = 0$ almost everywhere. Without loss of generality assume that $r_1 \geq 0$ almost everywhere. Using

$$\theta_1 r_1(x) = \int_0^1 f(x,y) r_1(y)\, dy$$

Note that if $r_1(x)$ is zero for some x then due to the positivity assumption on f, $r_1(y) = 0$ for almost every $y \in [0,1]$ which is a contradiction. Hence we have that $r_1(x) > 0$ almost every $x \in [0,1]$.

For the final claim, without loss of generality assume that $\int_0^1 r_1(x)\, dx \geq 0$. If $\theta_1 = \theta_2$, then the previous argument would give us $r_2(x) > 0$ and this will contradict the orthogonality of r_1 and r_2. $\square$

Bibliography

[1] N. Alon, M. Krivelevich, and V. H. Vu. On the concentration of eigenvalues of random symmetric matrices. *Israel J. Math.*, 131:259–267, 2002. ISSN 0021-2172. doi: 10.1007/BF02785860. URL https://doi.org/10.1007/BF02785860.

[2] G. Anderson and O. Zeitouni. A law of large numbers for finite-range dependent random matrices. *Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences*, 61(8):1118–1154, 2008.

[3] G. W. Anderson, A. Guionnet, and O. Zeitouni. *An introduction to random matrices*, volume 118 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010. ISBN 978-0-521-19452-5.

[4] M. Anshelevich. Appell polynomials and their relatives II. Boolean theory. *Indiana Univ. Math. J.*, 58(2):929–968, 2009. ISSN 0022-2518. doi: 10.1512/iumj.2009.58.3523. URL http://dx.doi.org/10.1512/iumj.2009.58.3523.

[5] M. Anshelevich, J.-C. Wang, and P. Zhong. Local limit theorems for multiplicative free convolutions. *J. Funct. Anal.*, 267(9):3469–3499, 2014. ISSN 0022-1236. doi: 10.1016/j.jfa.2014.08.015. URL http://dx.doi.org/10.1016/j.jfa.2014.08.015.

[6] O. Arizmendi and T. Hasebe. Semigroups related to additive and multiplicative, free and Boolean convolutions. *Studia Math.*, 215(2):157–185, 2013. ISSN 0039-3223. doi: 10.4064/sm215-2-5. URL http://dx.doi.org/10.4064/sm215-2-5.

[7] O. Arizmendi and T. Hasebe. Classical and free infinite divisibility for Boolean stable laws. *Proc. Amer. Math. Soc.*, 142(5):1621–1632, 2014. ISSN 0002-9939. URL https://doi.org/10.1090/S0002-9939-2014-12111-3.

[8] O. Arizmendi and T. Hasebe. Classical scale mixtures of Boolean stable laws. *Transactions of the American Mathematical Society*, 368(7):4873–4905, 2016.

[9] O. Arizmendi, T. Hasebe, and N. Sakuma. On the law of free subordinators. *ALEA Lat. Am. J. Probab. Math. Stat.*, 10(1):271–291, 2013. ISSN 1980-0436.

[10] O. Arizmendi E. and V. Pérez-Abreu. The S-transform of symmetric probability measures with unbounded supports. *Proc. Amer. Math. Soc.*, 137(9):3057–3066, 2009. ISSN 0002-9939. doi: 10.1090/S0002-9939-09-09841-4. URL http://dx.doi.org/10.1090/S0002-9939-09-09841-4.

[11] G. B. Arous and D. V. Voiculescu. Free extreme values. *The Annals of Probability*, pages 2037–2059, 2006.

[12] Z. D. Bai, Y. Q. Yin, and P. R. Krishnaiah. On limiting empirical distribution function of the eigenvalues of a multivariate F matrix. *Teor. Veroyatnost. i Primenen.*, 32(3):537–548, 1987. ISSN 0040-361X.

[13] Z. D. Bai, B. Miao, and B. Jin. On limit theorem for the eigenvalues of product of two random matrices. *J. Multivariate Anal.*, 98(1):76–101, 2007. ISSN 0047-259X. URL https://doi.org/10.1016/j.jmva.2006.06.001.

[14] T. Banica, S. Curran, and R. Speicher. De Finetti theorems for easy quantum groups. *The Annals of Probability*, pages 401–435, 2012.

[15] O. E. Barndorff-Nielsen and S. Thorbjørnsen. Self-decomposability and Lévy processes in free probability. *Bernoulli*, 8(3):323–366, 2002. ISSN 1350-7265.

[16] O. E. Barndorff-Nielsen, U. Franz, R. Gohm, B. Kümmerer, and S. Thorbjørnsen. *Quantum independent increment processes. II*, volume 1866 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2006. ISBN 978-3-540-24407-3; 3-540-24407-7. Structure of quantum Lévy processes, classical probability, and physics. Lectures from the School “Quantum Independent Increment Processes: Structure and Applications to Physics” held in Greifswald, March 9–22, 2003, Edited by Michael Schüermann and Franz.

[17] S. T. Belinschi and H. Bercovici. Atoms and regularity for measures in a partially defined free convolution semigroup. *Math. Z.*, 248(4):665–674, 2004. ISSN 0025-5874. URL https://doi.org/10.1007/s00209-004-0671-y.

[18] S. T. Belinschi and A. Nica. On a remarkable semigroup of homomorphisms with respect to free multiplicative convolution. *Indiana Univ. Math. J.*, 57(4):1679–1713, 2008. ISSN

0022-2518. doi: 10.1512/iumj.2008.57.3285. URL http://dx.doi.org/10.1512/iumj.2008.57.3285.

[19] G. Ben Arous and D. V. Voiculescu. Free extreme values. *Ann. Probab.*, 34(5):2037–2059, 2006. ISSN 0091-1798. URL https://doi.org/10.1214/009117906000000016.

[20] A. Ben Ghorbal and M. Schürmann. Non-commutative notions of stochastic independence. *Math. Proc. Cambridge Philos. Soc.*, 133(3):531–561, 2002. ISSN 0305-0041. doi: 10.1017/S0305004102006072. URL https://doi.org/10.1017/S0305004102006072.

[21] A. Ben Ghorbal and M. Schürmann. Quantum stochastic calculus on Boolean Fock space. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 7(4):631–650, 2004. ISSN 0219-0257. doi: 10.1142/S0219025704001815. URL http://dx.doi.org/10.1142/S0219025704001815.

[22] F. Benaych-Georges. Taylor expansions of R-transforms: application to supports and moments. *Indiana Univ. Math. J.*, 55(2):465–481, 2006. ISSN 0022-2518. doi: 10.1512/iumj.2006.55.2691. URL http://dx.doi.org/10.1512/iumj.2006.55.2691.

[23] F. Benaych-Georges, A. Guionnet, and M. Maida. Fluctuations of the extreme eigenvalues of finite rank deformations of random matrices. *Electronic Journal of Probability*, 16:1621–1662, 2011.

[24] F. Benaych-Georges, C. Bordenave, and A. Knowles. Spectral radii of sparse random matrices. *To appear in Ann. Inst. Henri Poincaré Probab. Stat.*, 2017. URL https://arxiv.org/pdf/1704.02945.pdf.

[25] F. Benaych-Georges, C. Bordenave, and A. Knowles. Largest eigenvalues of sparse inhomogeneous Erdős–Rényi graphs. *The Annals of Probability*, 47(3):1653–1676, 2019.

[26] H. Bercovici. On Boolean convolutions. In *Operator theory 20*, volume 6 of *Theta Ser. Adv. Math.*, pages 7–13. Theta, Bucharest, 2006.

[27] H. Bercovici and V. Pata. Functions of regular variation and freely stable laws. *Annali di Matematica Pura ed Applicata*, 178(1):245–269, 2000.

[28] H. Bercovici and D. Voiculescu. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, 42(3):733–773, 1993. ISSN 0022-2518. doi: 10.1512/iumj.1993.42.42033. URL http://dx.doi.org/10.1512/iumj.1993.42.42033.

[29] H. Bercovici, V. Pata, and P. Biane. Stable laws and domains of attraction in free probability theory. *Annals of Mathematics*, 149:1023–1060, 1999.

[30] S. Bhamidi, R. van der Hofstad, J. van Leeuwaarden, et al. Scaling limits for critical inhomogeneous random graphs with finite third moments. *Electronic Journal of Probability*, 15:1682–1702, 2010.

[31] B. Bollobás, S. Janson, and O. Riordan. The phase transition in inhomogeneous random graphs. *Random Structures & Algorithms*, 31(1):3–122, 2007.

[32] L. Breiman. On some limit theorems similar to the arc-sin law. *Theory of Probability & Its Applications*, 10(2):323–331, 1965.

[33] M. Capitaine, C. Donati-Martin, D. Féral, et al. The largest eigenvalues of finite rank deformation of large Wigner matrices: convergence and nonuniversality of the fluctuations. *The Annals of Probability*, 37(1):1–47, 2009.

[34] M. Capitaine, C. Donati-Martin, and D. Féral. Central limit theorems for eigenvalues of deformations of Wigner matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 48(1):107–133, 2012. ISSN 0246-0203. doi: 10.1214/10-AIHP410. URL https://doi.org/10.1214/10-AIHP410.

[35] A. Chakrabarty, R. S. Hazra, and D. Sarkar. From random matrices to long range dependence. *Random Matrices Theory Appl.*, 5(2):1650008, 52, 2016. ISSN 2010-3263. doi: 10.1142/S2010326316500088. URL https://doi.org/10.1142/S2010326316500088.

[36] A. Chakrabarty, S. Chakraborty, and R. S. Hazra. A note on the folklore of free independence. *arXiv preprint arXiv:1802.00952*, 2018.

[37] A. Chakrabarty, R. S. Hazra, F. den Hollander, and M. Sfragara. Spectra of adjacency and Laplacian matrices of inhomogeneous Erdős-Rényi random graphs. *To appear in Random matrices: Theory and applications*, 2018. URL https://arxiv.org/pdf/1807.10112.pdf.

[38] A. Chakrabarty, S. Chakraborty, and R. S. Hazra. Regular variation and free regular infinitely divisible laws. *Statist. Probab. Lett.*, 156:108607, 11, 2020. ISSN 0167-7152. doi: 10.1016/j.spl.2019.108607. URL https://doi.org/10.1016/j.spl.2019.108607.

[39] A. Chakrabarty, S. Chakraborty, and R. S. Hazra. Eigenvalues outside the bulk of inhomogeneous Erdős–Rényi random graphs. *To appear in the Journal of Statistical Physics*, 2020. URL https://doi.org/10.1007/s10955-020-02644-7.

[40] S. Chakraborty and R. S. Hazra. Boolean convolutions and regular variation. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15(2):961–991, 2018. doi: 10.30757/alea.v15-36. URL https://doi.org/10.30757/alea.v15-36.

[41] F. Chung and L. Lu. The average distances in random graphs with given expected degrees. *Proc. Natl. Acad. Sci. USA*, 99(25):15879–15882, 2002. ISSN 0027-8424. doi: 10.1073/pnas.252631999. URL https://doi.org/10.1073/pnas.252631999.

[42] F. Chung and L. Lu. *Complex graphs and networks*, volume 107 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2006. ISBN 978-0-8218-3657-6; 0-8218-3657-9. doi: 10.1090/cbms/107. URL https://doi.org/10.1090/cbms/107.

[43] F. R. K. Chung. *Spectral graph theory*, volume 92 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997. ISBN 0-8218-0315-8.

[44] D. M. Cvetković, M. Doob, and H. Sachs. *Spectra of graphs*, volume 87 of *Pure and Applied Mathematics*. Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1980. ISBN 0-12-195150-2. Theory and application.

[45] A. Dembo and E. Lubetzky. Empirical spectral distributions of sparse random graphs. *arXiv preprint arXiv:1610.05186*, 2016.

[46] X. Ding, T. Jiang, et al. Spectral distributions of adjacency and Laplacian matrices of random graphs. *Annals of Applied Probability*, 20(6):2086–2117, 2010.

[47] P. Embrechts, C. M. Goldie, and N. Veraverbeke. Subexponentiality and infinite divisibility. *Z. Wahrsch. Verw. Gebiete*, 49(3):335–347, 1979. ISSN 0044-3719. doi: 10.1007/BF00535504. URL http://dx.doi.org/10.1007/BF00535504.

[48] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. Spectral statistics of Erdős–Rényi graphs II: Eigenvalue spacing and the extreme eigenvalues. *Communications in Mathematical Physics*, 314(3):587–640, Sep 2012. ISSN 1432-0916. doi: 10.1007/s00220-012-1527-7. URL https://doi.org/10.1007/s00220-012-1527-7.

[49] L. Erdős, A. Knowles, H.-T. Yau, and J. Yin. Spectral statistics of Erdős–Rényi graphs I: Local semicircle law. *Ann. Probab.*, 41(3B):2279–2375, 2013. ISSN 0091-1798. doi: 10.1214/11-AOP734. URL https://doi.org/10.1214/11-AOP734.

[50] U. Feige and E. Ofek. Spectral techniques applied to sparse random graphs. *Random Structures Algorithms*, 27(2):251–275, 2005. ISSN 1042-9832. doi: 10.1002/rsa.20089. URL https://doi.org/10.1002/rsa.20089.

[51] W. Feller. *An introduction to probability theory and its applications. Vol. II.* Second edition. John Wiley & Sons, Inc., New York-London-Sydney, 1971.

[52] D. Féral and S. Péché. The largest eigenvalue of rank one deformation of large Wigner matrices. *Communications in mathematical physics*, 272(1):185–228, 2007.

[53] S. Foss, D. Korshunov, S. Zachary, et al. *An introduction to heavy-tailed and subexponential distributions*, volume 6. Springer, 2011.

[54] U. Franz. Monotone and Boolean convolutions for non-compactly supported probability measures. *Indiana Univ. Math. J.*, 58(3):1151–1185, 2009. ISSN 0022-2518. doi: 10.1512/iumj.2009.58.3578. URL http://dx.doi.org/10.1512/iumj.2009.58.3578.

[55] Z. Füredi and J. Komlós. The eigenvalues of random symmetric matrices. *Combinatorica*, 1(3):233–241, 1981. ISSN 0209-9683. doi: 10.1007/BF02579329. URL https://doi.org/10.1007/BF02579329.

[56] C. M. Goldie and C. Klüppelberg. Subexponential distributions. *A practical guide to heavy tails: statistical techniques and applications*, pages 435–459, 1998.

[57] T. Hamdi. Monotone and Boolean unitary Brownian motions. *Infin. Dimens. Anal. Quant. Probab. Relat. Top.*, 18(2):1550012, 19, 2015. ISSN 0219-0257. doi: 10.1142/S0219025715500125. URL http://dx.doi.org/10.1142/S02190257155001.

[58] Z. Hao and M. Popa. A combinatorial result on asymptotic independence relations for random matrices with non-commutative entries. *J. Operator Theory*, 80(1):47–76, 2018. ISSN 0379-4024. doi: 10.7900/jot.2017may21.2170. URL https://doi.org/10.7900/jot.2017may21.2170.

[59] R. S. Hazra and K. Maulik. Free subexponentiality. *Ann. Probab.*, 41(2):961–988, 2013. ISSN 0091-1798. doi: 10.1214/11-AOP706. URL http://dx.doi.org/10.1214/11-AOP706.

[60] W. Hoeffding. Probability inequalities for sums of bounded random variables. *Journal of the American Statistical Association*, 58(301):13–30, 1963.

[61] J. Y. Hwang, J. O. Lee, and K. Schnelli. Local law and Tracy-Widom limit for sparse sample covariance matrices. *Ann. Appl. Probab.*, 29(5):3006–3036, 2019. ISSN 1050-5164. doi: 10.1214/19-AAP1472. URL https://doi.org/10.1214/19-AAP1472.

[62] J. Y. Hwang, J. O. Lee, and W. Yang. Local law and Tracy–Widom limit for sparse stochastic block models. *arXiv preprint arXiv:1903.02179*, 2019.

[63] M. Jacobsen, T. Mikosch, J. Rosiński, and G. Samorodnitsky. Inverse problems for regular variation of linear filters, a cancellation property for σ-finite measures and identification of stable laws. *The Annals of Applied Probability*, pages 210–242, 2009.

[64] A. H. Jessen and T. Mikosch. Regularly varying functions. *Publ. Inst. Math. (Beograd) (N.S.)*, 80(94):171–192, 2006. ISSN 0350-1302. doi: 10.2298/PIM0694171J. URL http://dx.doi.org/10.2298/PIM0694171J.

[65] M. Krivelevich and B. Sudakov. The largest eigenvalue of sparse random graphs. *Combin. Probab. Comput.*, 12(1):61–72, 2003. ISSN 0963-5483. doi: 10.1017/S0963548302005424. URL https://doi.org/10.1017/S0963548302005424.

[66] J. O. Lee and K. Schnelli. Local law and Tracy-Widom limit for sparse random matrices. *Probab. Theory Related Fields*, 171(1-2):543–616, 2018. ISSN 0178-8051. doi: 10.1007/s00440-017-0787-8. URL https://doi.org/10.1007/s00440-017-0787-8.

[67] R. Lenczewski. Asymptotic properties of random matrices and pseudomatrices. *Advances in Mathematics*, 228(4):2403–2440, 2011.

[68] W. Liu. A noncommutative de Finetti theorem for Boolean independence. *Journal of Functional Analysis*, 269(7):1950–1994, 2015.

[69] C. Male. Traffics distributions and independence: the permutation invariant matrices and the notions of independence. *To appear in Memors of the AMS*, 2011. URL https://arxiv.org/pdf/1111.4662.pdf.

[70] F. Merlevède and M. Peligrad. On the empirical spectral distribution for matrices with long memory and independent rows. *Stochastic Process. Appl.*, 126(9):2734–2760, 2016. ISSN 0304-4149. URL https://doi.org/10.1016/j.spa.2016.02.016.

[71] A. Nica and R. Speicher. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2006. ISBN 978-0-521-85852-6; 0-521-85852-6. doi: 10.1017/CBO9780511735127. URL http://dx.doi.org/10.1017/CBO9780511735127.

[72] F. Ninio. A simple proof of the Perron–Frobenius theorem for positive symmetric matrices. *Journal of Physics A: Mathematical and General*, 9(8):1281, 1976.

[73] A. G. Pakes. Convolution equivalence and infinite divisibility: corrections and corollaries. *J. Appl. Probab.*, 44(2):295–305, 2007. ISSN 0021-9002. doi: 10.1239/jap/1183667402. URL https://doi.org/10.1239/jap/1183667402.

[74] G. Pan. Strong convergence of the empirical distribution of eigenvalues of sample covariance matrices with a perturbation matrix. *J. Multivariate Anal.*, 101(6):1330–1338, 2010. ISSN 0047-259X. URL https://doi.org/10.1016/j.jmva.2010.02.001.

[75] G. Pan and J. Gao. Asymptotic theory for sample covariance matrix under cross-sectional dependence. *Preprint*, 2012.

[76] V. Pérez-Abreu and N. Sakuma. Free infinite divisibility of free multiplicative mixtures of the Wigner distribution. *J. Theoret. Probab.*, 25(1):100–121, 2012. ISSN 0894-9840. URL https://doi.org/10.1007/s10959-010-0288-5.

[77] A. Pizzo, D. Renfrew, and A. Soshnikov. On finite rank deformations of Wigner matrices. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(1):64–94, 2013. ISSN 0246-0203. doi: 10.1214/11-AIHP459. URL https://doi.org/10.1214/11-AIHP459.

[78] N. Privault. Quantum stochastic calculus for the uniform measure and Boolean convolution. In *Séminaire de Probabilités, XXXV*, volume 1755 of *Lecture Notes in Math.*, pages 28–47. Springer, Berlin, 2001. doi: 10.1007/978-3-540-44671-2_2. URL http://dx.doi.org/10.1007/978-3-540-44671-2_2.

[79] N. Sakuma. On free regular infinitely divisible distributions. In *Spectra of random operators and related topics*, RIMS Kôkyûroku Bessatsu, B27, pages 115–121. Res. Inst. Math. Sci. (RIMS), Kyoto, 2011.

[80] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013. ISBN 978-1-107-65649-9. Translated from the 1990 Japanese original, Revised edition of the 1999 English translation.

[81] J. W. Silverstein. The spectral radii and norms of large-dimensional non-central random matrices. *Comm. Statist. Stochastic Models*, 10(3):525–532, 1994. ISSN 0882-0287. doi: 10.1080/15326349408807308. URL https://doi.org/10.1080/15326349408807308.

[82] J. W. Silverstein. Strong convergence of the empirical distribution of eigenvalues of large-dimensional random matrices. *J. Multivariate Anal.*, 55(2):331–339, 1995. ISSN 0047-259X. URL https://doi.org/10.1006/jmva.1995.1083.

[83] R. Speicher. On universal products. In *The Fields Institute Communications*. Citeseer, 1997.

[84] R. Speicher and R. Woroudi. Boolean convolution. In *Free probability theory (Waterloo, ON, 1995)*, volume 12 of *Fields Inst. Commun.*, pages 267–279. Amer. Math. Soc., Providence, RI, 1997.

[85] T. Tao. 254A, Notes 5: Free probability, 2010. URL https://terrytao.wordpress.com/2010/02/10/245a-notes-5-free-probability/#more-3466.

[86] L. V. Tran, V. H. Vu, and K. Wang. Sparse random graphs: eigenvalues and eigenvectors. *Random Structures Algorithms*, 42(1):110–134, 2013. ISSN 1042-9832. doi: 10.1002/rsa.20406. URL https://doi.org/10.1002/rsa.20406.

[87] Y. Ueda. Limit theorems for classical, freely and Boolean max-infinitely divisible distributions. *arXiv preprint arXiv:1907.11996*, 2019.

[88] Y. Ueda. Max-convolution semigroups and extreme values in limit theorems for the free multiplicative convolution. *arXiv preprint arxiv:2003.05382*, 2020. URL https://arxiv.org/pdf/2003.05382.pdf.

[89] R. van der Hofstad. Critical behavior in inhomogeneous random graphs. *Random Structures & Algorithms*, 42(4):480–508, 2013.

[90] R. S. Varga. *Geršgorin and his circles*, volume 36 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 2004. ISBN 3-540-21100-4. doi: 10.1007/978-3-642-17798-9. URL https://doi.org/10.1007/978-3-642-17798-9.

[91] J. G. Vargas and D. V. Voiculescu. Boolean extremes and dagum distributions. *To appear in Indiana University Mathematics Journal*, 2017. URL https://arxiv.org/pdf/1711.06227.pdf.

[92] D. Voiculescu. Limit laws for random matrices and free products. *Invent. Math.*, 104 (1):201–220, 1991. ISSN 0020-9910. doi: 10.1007/BF01245072. URL https://doi.org/10.1007/BF01245072.

[93] D. Voiculescu. The analogues of entropy and of Fisher's information measure in free probability theory, I. *Communications in mathematical physics*, 155(1):71–92, 1993.

[94] V. H. Vu. Spectral norm of random matrices. *Combinatorica*, 27(6):721–736, 2007. ISSN 0209-9683. doi: 10.1007/s00493-007-2190-z. URL https://doi.org/10.1007/s00493-007-2190-z.

[95] K. W. Wachter. The limiting empirical measure of multiple discriminant ratios. *Ann. Statist.*, 8(5):937–957, 1980. ISSN 0090-5364. URL http://links.jstor.org/sici?sici=0090-5364(198009)8:5<937:TLEMOM>2.0.CO;2-Y&origin=MSN.

[96] T. Watanabe. Convolution equivalence and distributions of random sums. *Probab. Theory Related Fields*, 142(3-4):367–397, 2008. ISSN 0178-8051. doi: 10.1007/s00440-007-0109-7. URL https://doi.org/10.1007/s00440-007-0109-7.

[97] T. Watanabe. Subexponential densities of infinitely divisible distributions on the half line. *arXiv preprint arXiv:1912.08366*, 2019.

[98] Y. Zhu. Graphon approach to limiting spectral distributions of Wigner–type matrices. *To appear in Random Structures & Algorithms*, 2018. URL https://arxiv.org/pdf/1806.11246.pdf.

List of Publications

(i) A. Chakrabarty, S. Chakraborty, and R. S. Hazra. Eigenvalues outside the bulk of inhomogeneous Erdős–Rényi random graphs. *To appear in the Journal of Statistical Physics*, 2020. URL https://doi.org/10.1007/s10955-020-02644-7.

(ii) A. Chakrabarty, S. Chakraborty, and R. S. Hazra. Regular variation and free regular infinitely divisible laws. *Statist. Probab. Lett.*, 156:108607, 11, 2020. ISSN 0167-7152. URL https://doi.org/10.1016/j.spl.2019.108607.

(iii) S. Chakraborty and R. S. Hazra. Boolean convolutions and regular variation. *ALEA Lat. Am. J. Probab. Math. Stat.*, 15(2):961–991, 2018. URL https://doi.org/10.30757/alea.v15-36.