Quantum Symmetries in Noncommutative Geometry

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A thesis submitted to the Indian Statistical Institute
in partial fulfillment of the requirements
for the award of the degree of
Doctor of Philosophy.

March, 2020

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Dedicated to the memory of Alexander Grothendieck

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Acknowledgments

To Kolkata Metro: where many of the ideas took form, where I indulged in my other mathematical interests orthogonal to this thesis and which provided a beautiful atmosphere for solitary thinking, a big thanks.

To a certain fellow commuter: for a wonderful March, 2019, and for making me aware of some functions of my cingulate cortex, thank you!

To ISI, Kolkata: for allowing me to enroll as a graduate student and for the full financial support throughout, thank you.

To the staffs of SMU office: thank you for actively helping us graduate students with the mundane tasks.

To Paramita di and Urmi di: thanks a lot for helping me with the desktops and Paramita di, in particular, for bearing with me (us) for a good three years.

To all my teachers at ISI, Kolkata: for the wonderful course-work and the pressure-free atmosphere that helped us learn and most importantly enjoy the process, I am indebted to you all.

To Jaydeb da: for giving a chance to be at OTOA and caring for my future endeavors, thanks a lot.

To the anonymous referee: for her/his many suggestions for improvements and corrections, thank you.

To Ré, Karen and Andrey: for inviting me to the conference, taking such good care of me, for countless discussions over email and the still pending paper (Ré), for helping me out with the food and drinks each time (Karen), and for translating "can I have some water please" every instance (Andrey), many many thanks!

To Arnab da: for so much fun and discussions at NISER and Presidency and while hunting for a coffee mug at M.G. road, thank you.

To Soumalya da: for inviting me to IISER, Kolkata, for the discussions, and for our joint future endeavors, a lot of thanks.

To Indranil da: for the invitation to TIFR, Mumbai, for teaching me all I know about complex geometry, and for the paper, I cannot thank you enough.

To Sutanu da: for all the fun, discussions, math, arguments, Asterix, bad jokes, coffee, several visits, conferences, advice, and the still unfinished paper, many many many thanks!

To Sukrit, Biltu, Gopal, Nurul, Sumit, Aparajita, Priyanka: for a wonderful experience, thanks.

To Mainak and Jayanta: for all the discussions and for excusing me for my bad jokes, thank you!

To Sourjya and Asfaq: for taking in my many outbursts, sharing little secrets and of course math, thanks a lot.

To Sugato da and Aritra: the space is really low for you guys!

To Jyotishman da: I shall not "list" the ways you helped me; in the fear of leaving many things out. You were always there, a silent presence, and I am yet to come up with something more than a thank you.

To Debashish da: throughout, I acted like a whining child, not content with what I had. But you understood me, supported me, letting me take each step on my own and helped me stand again when I tumbled down. I am fortunate to have you as my supervisor. I cannot express your influence on me but I should say that what we are working on at the moment, were "not of my immediate interests" once!

To my family: to all my relatives, a big thank you for supporting me all the way. Ma, Baba and Didi, you suffered the most from my instability and ingratitude. You know what you are to me.

And finally, to all my cats, dogs, all the living and non-living things inseparably entwined with me: you are part of me and I cannot tell you, even though I wish to, how you make each and every day special.

Introduction

This thesis concerns itself with the study of quantum symmetries within the realm of noncommutative geometry. We capture these symmetries in two levels of generality, namely, Hopf algebras [**Swe69**] (or compact quantum groups, [**Wor98**]) in the context of noncommutative differential geometry a la Connes [**Con94**] and Hopf algebroids [**Böh09**] in the context of noncommutative Kähler geometry a la Ó Buachalla [**B́17**]. We now briefly explain these terms.

Noncommutative differential geometry. The necessity for noncommutativity was first realized by Heisenberg in his formulation of Matrix mechanics [**Wc19**]. But the justification that "classical" corresponds to commutativity and "quantum" to noncommutativity came after a few years with the work of Gelfand and Naimark. The Gelfand-Naimark theorem says that there is an anti-equivalence between the category of (locally) compact Hausdorff spaces and (proper, vanishing at infinity) continuous maps and the category of (not necessarily) unital C^*-algebras and $*$-homomorphisms. This means that the entire topological information of a locally compact Hausdorff space is encoded in the commutative C^*-algebra of continuous functions vanishing at infinity. On the algebraic side, Hilbert's Nullstellensatz establishes, ignoring nilpotents, an equivalence between commutative algebras and affine varieties. These two theorems led to the development of modern algebraic geometry. But it also gives rise to the question if one can view a possibly noncommutative (C^*-) algebra as the algebra of (continuous) regular "functions on some noncommutative (topological space) variety".

In classical Riemannian geometry on spin manifolds, the Dirac operator on the Hilbert space of square integrable sections of the spinor bundle contains a lot of geometric information. The metric, the volume form, the dimension of the manifold can all be captured from the Dirac operator, for instance. Motivated by this, Alain Connes developed his noncommutative differential geometry with the central object as the spectral triple which is a triple (A, H, D) consisting of a separable Hilbert space H, a $*$-subalgebra A of $B(H)$ and a self-adjoint, typically unbounded, operator D satisfying some natural conditions. In a certain sense, this noncommutativity may be seen as a manifestation of the singular behavior of the spaces involved. For instance, the naive spaces attached to quotients of group actions or the leaf space of a foliation are highly pathological. The Connes approach to these spaces begins with attaching a noncommutative algebra - the "algebra of functions on the noncommutative space", eventually constructing a spectral triple on that algebra and extracting geometric information, which was inaccessible by usual geometric methods, hitherto.

Hopf algebras. It is hard to undermine the role of linear algebraic groups in algebraic geometry or compact Lie groups in differential geometry. For instance, according to Klein or later Cartan, these groups (with some extra data, which we ignore for now) govern "geometries", nowadays known as Klein geometry or Cartan geometry, respectively. When we pass from the group G to the function algebra $\mathcal{O}(G)$, consisting of regular functions in the case of algebraic groups and continuous functions in the Lie case, depending on the context, we see that the group multiplication $G \times G \to G$ transforms into a map $\mathcal{O}(G) \to \mathcal{O}(G \times G)$. If we use an appropriate notion of tensor product, also depending on the context, and identify $\mathcal{O}(G \times G)$ with $\mathcal{O}(G) \otimes \mathcal{O}(G)$, then what we get is a "comultiplication" $\Delta : \mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$. Moreover, the inversion gives an involution $S : \mathcal{O}(G) \to \mathcal{O}(G)$. The algebra $\mathcal{O}(G)$ together with Δ and S is the basic example of a Hopf algebra. Using either the Gelfand-Naimark theorem or Hilbert's Nullstellensatz (again ignoring nilpotents, which we can by a theorem of Cartier), we see that commutative Hopf algebras are precisely the function algebras of compact Lie groups or linear algebraic groups. Dually, enveloping algebras of Lie algebras exhaust essentially all of the "cocommutative" Hopf algebras, by Cartier-Gabriel-Kostant theorem.

Independently developed by Woronowicz and Drinfeld, (compact) quantum groups provide natural examples of noncommutative spaces, the "function algebras" of which are noncommutative and noncocommutative Hopf algebras. These are obtained by deforming a Lie group (and get a compact quantum group) or dually, the Lie algebra (and get the quantized universal enveloping algebras). Hopf algebras found application in low-dimensional topology, quantum field theory and knot theory. Drinfeld used quantum groups to provide systematic solutions to the quantum Yang-Baxter equation, which were known only in scattered examples, previously.

Quantum isometry groups. As soon as the basic theory of quantum groups was settled, the question of viewing them as symmetries or equivalently their actions on noncommutative spaces, emerged. Following suggestions of Connes, Wang [**Wan98**] defined and proved the existence of quantum automorphism groups on finite dimensional C^*-algebras. Since then, many interesting examples of such quantum groups, particularly the quantum permutation groups of finite sets and finite graphs, were extensively studied [**Ban05a**, **Bic03**]. It is thus natural to see what happens if one replaces the finite structures by classical or noncommutative manifolds. Motivated by this Goswami [**Gos09**] formulated and studied the quantum analogues of the group of Riemannian isometries, calling it the quantum isometry group. It was defined more or less along the classical line, characterizing isometries using the Laplacian, which is available from the spectral triple defining the noncommutative manifold.

However, as observed by Bhowmick and Goswami in [**BG09**], the verification of the required properties of the Laplacian is not always easy and it is more desirable to define the quantum isometry group in terms of the Dirac operator directly. This dream was fulfilled in [**BG09**] and a number of computations were done, thus successfully deploying quantum groups to Riemannian aspects of noncommutative differential geometry.

Generalized symmetries. Since a large part of this thesis is titled under this name, a few words are in order. We borrowed this term from [**KP11**]. There are many places in geometry where the usual notion of symmetry that is captured by a group, is not sufficient to deal with the pathology at hand. Singular spaces like orbifolds, foliations, pathological group actions come to mind. A need for a generalized symmetry is apparent, which can be successfully encoded using Lie groupoids and pseudogroups, facts that can be traced back to Lie himself and Cartan. Groupoids, which are a joint generalization of spaces and groups, were systematically deployed in differential geometry by Ehresmann and provide a symmetry concept finding applications in foliation theory [**MM03**].

A very natural question to ask is what should the generalized symmetries in noncommutative geometry correspond to. An infinitesimal version of Lie groupoids is provided by that of Lie algebroids, culminating in a beautiful theory of duality. As expounded upon above, these two dual object should give rise to some sort of symmetry object in noncommutative geometry, and it is not hard to guess that a generalization of the concept of Hopf algebras is required. Apart from this geometric motivation, an extension of Hopf algebra concepts surprisingly came from the work of Connes and Moscovici on index theory of transverse elliptic operators [**CM98**]. There are other examples coming from topology and geometry, all the more emphasizing the need for a noncommutative generalization of the concept of Hopf algebras.

Hopf algebroids. Many of the needs described above have been handled by allowing for a not necessarily commutative ring A replacing the commutative ground ring k (in this thesis, $\mathbb{C}$) of a Hopf algebra. Now a Hopf algebra is a bialgebra over k together with an antipode. So a notion of a Hopf algebroid should involve some kind of a generalized bialgebra over A together with an antipode. Such a generalized bialgebra is referred to as a bialgebroid. One can guess that such a structure should consist of a "coalgebra structure over A" and an "algebra structure over A". It becomes a bit technical to make sense of this but can be done nevertheless with the notion of Takeuchi product. That the notion is the "right" one can be seen from the fact the module category over the bialgebroid is monoidal, a fact familiar from the usual bialgebra setup.

It is also satisfying to note the arrow-reversal in the definition of a bialgebroid. We recall that a groupoid consists of two spaces, the base space and the total space, with two maps - the source and the target and a partially defined multiplication. On the other hand, a bialgebroid, roughly consists of a base ring (A) and a total ring, two maps again called source and target, with a comultiplication taking values in the Takeuchi product, which is a smaller subspace sitting inside the full tensor product of the total ring with itself over A, resembling the partial multiplication. This would suggest that the antipode should be an endomorphism of the total ring. But this straightforward generalization is not possible for many reasons. Approaches begin to differ from this point on and there are many competing definitions of what should an antipode, and hence a Hopf algebroid, consist of. See for example [**Lu96**, **Xu01**]. In spite of this Hopf algebroids have been used in many instances. In [**Mrč99**, **Mrč07**], noncommutative Hopf algebroids over commutative base have been used to study principal bundles with groupoid symmetry. Motivated by problems in cyclic cohomology, the authors of [**KR04**] introduced a notion of para-Hopf algebroid.

An alternative, more symmetric and natural, definition of Hopf algebroids was given in [**BS04**]. The crucial idea is to put two distinct bialgebroid structures on a single total ring, the so-called left bialgebroid and a right bialgebroid. An antipode is then seen as some sort of an intertwiner between these two different structures. It is this definition that we make use of in this thesis, although we do not use both structures simultaneously, except for a few places. The need for the introduction of Hopf algebroids in this thesis will be addressed after a few moments.

Noncommutative complex geometry. Classical complex geometry, a beautiful subject in its own right, also stands at the threshold of two different ways of studying geometry: differential geometry and algebraic geometry. Amenable to tools and methods belonging to these two different camps of doing geometry, it provides a bridge between these two areas, via Serre's GAGA [**Ser55**]. A complex manifold is a smooth manifold with a complex coordinate system. Interestingly, every smooth complex variety is a complex manifold. Kodaira's embedding theorem characterizes compact complex manifolds that may be embedded in $\mathbb{C}P^N$. Chow's theorem says that every compact complex submanifold of $\mathbb{C}P^N$ is a smooth complex projective variety. This harmony and Serre's GAGA show that one can go back and forth between these two areas, enriching both.

As was hinted above, one can develop noncommutative algebraic geometry, replacing the commutative algebra of regular functions by a possibly noncommutative one. This has been investigated and developed by many authors, the most prominent one being noncommutative projective algebraic geometry. Classical geometry tells us to expect a link between this camp and noncommutative differential geometry, hence the dream of noncommutative complex geometry - the missing link. The first robust framework and many connections to noncommutative projective algebraic geometry were given in [**BPS13**], approaching the subject using Woronowicz's differential calculus setup. Taking lessons from classical geometry again and noting that the Dolbeault complex provides one of the most important elliptic complexes, one cannot but suspect that a spectral triple a la Connes is lurking behind somewhere. That this is the case was shown by Ó Buachalla who introduced a beautiful framework of noncommutative Kähler geometry on quantum homogeneous space [**B́17**]. The dominating example is the quantum projective space and it is shown that one can go as far as proving a version of Hodge decomposition theorem and Kähler identities.

Both the frameworks in [**BPS13**] and [**B́17**] recover classical complex and Kähler geometry, respectively. But the question arises about how can one capture using these frameworks the singular spaces. There is a body of theory developed for these spaces, the so called transverse Kähler foliation and Kähler orbifolds. The frameworks mentioned above don't suffice because of the presence of singularities. This points to the appearance of groupoids or Hopf algebroids in this setup.

With the main players of this thesis out of the way, we now give an outline of each chapter and mention the principal results obtained in this thesis.

Chapter 1 consists of the bare minimum of preliminaries needed for this thesis. We stress that some of the chapters have their own preliminaries sections. This chapter is intended to make the thesis essentially self contained. Section 1.1 recalls basic Hopf algebra theory, introduces quantum groups and some of the main examples. The theory of compact quantum groups a la Woronowicz is introduced in Section 1.2. We will use some unbounded operator theory in later chapters, the required background for which is also recalled here. Section 1.3 recalls Riemannian and spin geometry so as to put the next section in context. Complex geometry is also introduced in a way that is well-suited to our purposes. The final Section 1.4 of this chapter introduces spectral triples a la Connes and quantum isometry groups a la Goswami and Bhowmick-Goswami. We have presented the material without going into all the details, mentioning only the results needed in this thesis.

With Chapter 2, we begin the investigation of quantum symmetry in noncommutative geometry. This chapter focuses on the Riemannian aspects of noncommutative geometry. One of the many ways of obtaining genuine noncommutative spaces, be they quantum groups or spectral triples, is by deforming the algebra associated with a classical space. The Podleś sphere and Woronowicz's special unitary group are prime examples of these. Both of these spaces have a structure of noncommutative manifolds and associated quantum isometry groups have been computed by the pioneers themselves. The odd dimensional spheres introduced by Vaksman-Soibelmann provide another class of noncommutative space and we compute the quantum isometry group of this space. The main novelty lies in the fusion of purely analytic techniques with algebraic ones. Section 2.2 provides further background and recalls the necessary prerequisites. The first part of Section 2.3, under some assumption, computes the quantum symmetry of the odd sphere in an algebraic way. The second part justifies the algebraic assumption made in the previous part and identifies the quantum isometry group of the odd sphere with the quantum unitary group, using analytic tools. This generalizes the result obtained by Bhowmick-Goswami for $SU_q(2)$.

Chapter 3 recalls the framework of noncommutative complex and Kähler geometry as developed in [**BPS13**] and [**B́17**], respectively. We present the material in a way so as to pave the way for the presentation of the next chapter. We begin by reviewing Woronowicz's differential calculus. After that, complex and Kähler structures are introduced and some of the noncommutative analogues of standard classical theorems are presented. Finally, we present the main example, that of quantum projective space and the associated Heckenberger-Kolb calculus.

We return to the investigation of quantum symmetries in noncommutative geometry with Chapter 4, now in the context of noncommutative complex geometry, as described in the last chapter. We exploit the viewpoint advocated by Haefliger, that singular spaces are regular spaces with a generalized symmetry object. For instance, a foliation can be thought of as the geometry represented by a complete transversal together with the holonomy pseudogroup. These data can be succinctly summarized by the (reduced) holonomy groupoid, which is one of the most important examples of Lie groupoids. Now a transverse Kähler foliation is a foliation where the complete transversal admits a Kähler structure, equivariant under the action of the holonomy pseudogroup. This promptly hints towards a framework of noncommutative complex geometry equivariant under some generalized symmetry, i.e., that of a Hopf algebroid. In Section 4.2 we present all the requisite definitions regarding Hopf algebroids, groupoids and foliation. We recall how the Connes convolution algebra of a groupoid naturally provides examples of Hopf algebroids. We discuss modules over Hopf algebroids, exemplify them using vector bundles over Lie groupoids. We introduced $*$-structures, absent from the literature hitherto, and a not-so-straightforward formalism to deal with the associated structures. Section 4.3 introduces Hopf algebroid equivariant Kähler structures. We start by defining equivariant differential calculus. This is quite a bit technical, as can be seen from the classical case itself. As groupoids generalize both spaces and groups at the same time, its action should commute with the exterior derivative and satisfy Leibniz rule simultaneously. We present this case as an example and move onto defining complex and Kähler structures. We present in Section 4.4 two of the most important theorems in Kähler geometry, namely Hodge decomposition and Formality for Kähler manifolds, in our setting. Next in Section 4.5 we construct a genuine noncommutative example of a Hopf algebroid and show it fits into our framework, namely that of Connes-Moscovici Hopf algebroid, part of which already existed in the literature. We end the chapter with Section 4.6 which discusses further directions for future research and some comments on earlier work.

The last Chapter 5 deals with finite spaces. These are the spaces that are thought to be testing ground for any noncommutative theory. The first part deals with the authors' first foray into the noncommutative world and classifies complex structures on a three-point space. The final Section 5.3 of this thesis initiates a program for universal action of Hopf algebroids analogous to that of Wang's in the setting of compact quantum groups. We obtain such a bialgebroid for a finite space. Finally, we build a bialgebroid which makes the universal one forms on the finite space equivariant.

We end with a list of references used in the process of obtaining the results that constitute this thesis.

CHAPTER 1

Preliminaries

In this chapter, we gather the background material for this thesis. The contents of the first three sections are well-known, but we give a rough outline so as to be self-contained. References will be given along the way. The fourth and final section is relatively new and we state only results that we are going to use.

1.1. Algebraic preliminaries

We presume the reader's familiarity with quantum groups but nevertheless, we start from the very basics. More details can be found in the references [**Swe69**, **KS97**, **CP95**, **Kas95**]. All algebras appearing in this thesis will be over $\mathbb{C}$.

1.1.1. Quantum groups. Let us start by recalling the following

Definition 1.1.1. *An algebra over $\mathbb{C}$ is a triple (A, m, u) with A a $\mathbb{C}$-vector space, $m : A \otimes A \to A$ a linear map called the multiplication, $u : \mathbb{C} \to A$ a linear map called the unit, and such that*

$$m(\mathrm{id} \otimes m) = m(m \otimes \mathrm{id}), \qquad m(u \otimes \mathrm{id}) = m(\mathrm{id} \otimes u) = \mathrm{id}\,. \tag{1.1.1}$$

The advantage of putting the definition in this way is that one can then dualize.

Definition 1.1.2. *A coalgebra over $\mathbb{C}$ is a triple (C, Δ, ε) with C a $\mathbb{C}$-vector space, Δ a linear map called the comultiplication, $\varepsilon : C \to \mathbb{C}$ a linear map called the counit, and such that*

$$(\Delta \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes \Delta)\Delta, \qquad (\mathrm{id} \otimes \varepsilon)\Delta = (\varepsilon \otimes \mathrm{id})\Delta = \mathrm{id}\,. \tag{1.1.2}$$

We have identified $A \otimes \mathbb{C}$ $(C \otimes \mathbb{C})$ and $\mathbb{C} \otimes A$ $(\mathbb{C} \otimes C)$ with A $(C$, respectively$)$. As an example, consider a set S and construct the vector space $\mathbb{C}S$ with basis S. Define $\Delta : \mathbb{C}S \to \mathbb{C}S \otimes \mathbb{C}S$ by $s \to s \otimes s$ and $\varepsilon : \mathbb{C}S \to \mathbb{C}$ by $s \to 1$. Then $(\mathbb{C}S, \Delta, \varepsilon)$ is a coalgebra which is called the group-like coalgebra. We will see other examples shortly.

Sweedler notation. Let (C, Δ, ε) be a coalgebra. Let $c \in C$ have $\Delta(c) = \sum_i c_{1i} \otimes c_{2i}$ with $c_{ji} \in C$. We indicate such an expression in the form

$$\Delta(c) = c_{(1)} \otimes c_{(2)}, \tag{1.1.3}$$

suppressing the sigma $\sum$ and the index i. Thus coassociativity of Δ reads

$$c_{(1)} \otimes (c_{(2)})_{(1)} \otimes (c_{(2)})_{(2)} = (c_{(1)})_{(1)} \otimes (c_{(1)})_{(2)} \otimes c_{(2)}. \tag{1.1.4}$$

So we write $\Delta_2(c) = (\Delta \otimes \mathrm{id})\Delta(c) = (\mathrm{id} \otimes \Delta)\Delta(c)$ in the form

$$\Delta_2(c) = c_{(1)} \otimes c_{(2)} \otimes c_{(3)}. \tag{1.1.5}$$

Defining inductively, $\Delta_1 = \Delta$, $\Delta_{n+1} : C \to C^{\otimes(n+2)}$, $\Delta_{n+1} = (\Delta \otimes \mathrm{id})\Delta_n$, it follows that there is no ambiguity in writing

$$\Delta_n(c) = c_{(1)} \otimes \ldots \otimes c_{(n+1)}. \tag{1.1.6}$$

Definition 1.1.3. *Let $(C, \Delta_C, \varepsilon_C)$ and $(D, \Delta_D, \varepsilon_D)$ be coalgebras. A linear map $f : C \to D$ is called a coalgebra morphism if $(f \otimes f)\Delta_C = \Delta_D f$ and $\varepsilon_D f = \varepsilon_C$. In Sweedler notation, this can be written as*

$$f(c)_{(1)} \otimes f(c)_{(2)} = f(c_{(1)}) \otimes f(c_{(2)}), \qquad \varepsilon_D(f(c)) = \varepsilon_C(c). \tag{1.1.7}$$

Let (C, Δ, ε) be a coalgebra. A subspace $V \subset C$ is a right coideal if $\Delta(V) \subset V \otimes C$, a left coideal if $\Delta(V) \subset C \otimes V$ and a (two-sided) coideal if $\Delta(V) \subset V \otimes C + C \otimes V$, $\varepsilon(V) = 0$. If I is a coideal then C/I is a coalgebra in a canonical way. Moreover, for a coalgebra morphism $f : C \to D$, $\ker f$ is a coideal and $\operatorname{im} f$ is a coalgebra and $C/\ker f$ is canonically isomorphic to $\operatorname{im} f$.

Given an algebra A, a left A-module consists of a space N and a linear map $\psi : A \otimes N \to N$ such that

$$\psi(u \otimes \mathrm{id}) = \mathrm{id}, \qquad \psi(m \otimes \mathrm{id}) = \psi(\mathrm{id} \otimes \psi). \tag{1.1.8}$$

Dually, if C is a coalgebra, a right comodule consists of a space M and a linear map $\omega : M \to M \otimes C$ such that

$$(\mathrm{id} \otimes \varepsilon)\omega = \mathrm{id}, \qquad (\omega \otimes \mathrm{id})\omega = (\mathrm{id} \otimes \Delta)\omega. \tag{1.1.9}$$

For example, the coalgebra C itself is a right comodule over itself via the comultiplication Δ. Other than that any right coideal V of C is a right comodule over C.

Given (M, ω_M), (N, ω_N) right comodules over C, $f : M \to N$ is a comodule morphism if it satisfies

$$\omega_N f = (f \otimes \mathrm{id})\omega_M. \tag{1.1.10}$$

All the concepts like ker, im carry over.

Sweedler notation. The Sweedler notation for coalgebras can be extended to the setting of comodules. For $m \in M$, we indicate $\omega(m)$ by

$$\omega(m) = m_{(0)} \otimes m_{(1)}. \tag{1.1.11}$$

We also inductively define

$$m_{(0)} \otimes \ldots \otimes m_{(n)} = (\omega \otimes \mathrm{id})(m_{(0)} \otimes \ldots \otimes m_{(n-1)}). \tag{1.1.12}$$

Note that the index 0 indicates the comodule tensorand and positive indices indicate the coalgebra tensorands.

Let C and D be coalgebras. We define $\Delta_{C\otimes D} : C \otimes D \to C \otimes D \otimes C \otimes D$ by $\Delta_{C\otimes D} = (\mathrm{id} \otimes P \otimes \mathrm{id})(\Delta_C \otimes \Delta_D)$, where $P : C \otimes D \to D \otimes C$ is the permutation of the two factors: $P(c \otimes d) = d \otimes c$. Then $(C \otimes D, \Delta_{C\otimes D})$ is a coalgebra called the tensor product of C and D. Explicitly, $\Delta(c \otimes d) = c_{(1)} \otimes d_{(1)} \otimes c_{(2)} \otimes d_{(2)}$ and $\varepsilon(c \otimes d) = \varepsilon(c)\varepsilon(d)$.

Now suppose that (H, m, u) is an algebra and (H, Δ, ε) is a coalgebra. Thus $H \otimes H$ is an algebra as well as a coalgebra. One has the following proposition.

Proposition 1.1.4. *The following are equivalent:*

i) m and u are coalgebra morphisms;
ii) Δ and ε are algebra morphisms.

Definition 1.1.5. *A pentuple $(H, m, u, \Delta, \varepsilon)$ satisfying one of the conditions of the above proposition is called a bialgebra.*

Before defining what a Hopf algebra is, let us recall the convolution product. Let C be a coalgebra and A an algebra. We put an algebra structure on $\mathrm{Hom}_{\mathbb{C}}(C, A)$, called the convolution algebra as follows:

$$f * g = m(f \otimes g)\Delta \tag{1.1.13}$$

for $f, g \in \mathrm{Hom}_{\mathbb{C}}(C, A)$. Explicitly, $(f * g)(c) = f(c_{(1)})g(c_{(2)})$. The identity for this operation is $u\varepsilon$.

Now suppose $(H, m, u, \Delta, \varepsilon)$ is a bialgebra. We write for the underlying coalgebra H^C and for the algebra H^A. Then $\mathrm{Hom}_{\mathbb{C}}(H^C, H^A)$ is an algebra under the convolution product. We note that the identity operator $\mathrm{id} : H \to H$ is an element of $\mathrm{Hom}_{\mathbb{C}}(H^C, H^A)$.

A convolution inverse $S \in \mathrm{Hom}_{\mathbb{C}}(H^C, H^A)$ of $1 : H \to H$ is called an antipode of the bialgebra H. Explicitly, $S(h_{(1)})h_{(2)} = h_{(1)}S(h_{(2)}) = \varepsilon(h)1$ for all $h \in H$. Note that by definition, an antipode if exists, is unique.

Definition 1.1.6. *A bialgebra with an antipode is called a Hopf algebra.*

We observe that one can form tensor products of (co)modules over a Hopf algebra H, thus providing a monoidal category structure on the category of (co)modules. Explicitly, given modules V and W over H, $V \otimes W$ carries the following H-module structure:

$$h \cdot (v \otimes w) = (h_{(1)} \cdot v) \otimes (h_{(2)} \cdot w), \tag{1.1.14}$$

for $v \in V$, $w \in W$ and $h \in H$. Dually, given comodules V and W, $V \otimes W$ carries the following comodule structure:

$$(v \otimes w)_{(0)} \otimes (v \otimes w)_{(1)} = v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}, \tag{1.1.15}$$

where $v \in V$, $w \in W$. We use Sweedler notation in the above formula.

(Co)semisimplicity. Here, we recall the definitions of semi-and cosemisimplicity.

Definition 1.1.7. *Let H be a Hopf algebra. A left integral in H is an element $t \in H$ such that $ht = \varepsilon(h)t$, for all $h \in H$. Similarly, one can define right integrals.*

Denoting the spaces of left and right integrals by $\int_H^l$ and $\int_H^r$, respectively, we say H is unimodular if $\int_H^l = \int_H^r$. For example, for a finite group G, the element $t = \sum_{g \in G} g$ generates the spaces of left and right integrals. The following is the analogue of Maschke's theorem in representation theory of finite groups.

Theorem 1.1.8. *Let H be any finite dimensional Hopf algebra. Then H is semisimple as an algebra if and only if $\varepsilon(\int_H^l) \neq 0$ if and only if $\varepsilon(\int_H^r) \neq 0$.*

Now we come to the dual picture.

Definition 1.1.9. *We call a coalgebra C simple if it has no proper subcoalgebras. C is said to be cosemisimple if it is direct sum of simple subcoalgebras.*

Definition 1.1.10. *Let H be a Hopf algebra. A Haar functional on H is a linear form $\mathbf{h} : A \to \mathbb{C}$ such that $\mathbf{h}(1) = 1$, and for all a in H*

$$(\mathrm{id} \otimes \mathbf{h})\Delta(a) = \mathbf{h}(a)1, \qquad (\mathbf{h} \otimes \mathrm{id})\Delta(a) = \mathbf{h}(a)1. \tag{1.1.16}$$

The following is the dual Maschke theorem.

Theorem 1.1.11. *Let H be any Hopf algebra, not necessarily finite dimensional. Then H is cosemisimple as a coalgebra if and only if there exists a Haar functional $\mathbf{h}$ on H.*

We now introduce $*$-structures.

Definition 1.1.12. *A $*$-algebra A is an algebra A endowed with a map $a \mapsto a^*$, called an involution, such that*

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad (ab)^* = b^* a^*, \quad (a^*)^* = a, \tag{1.1.17}$$

where $\alpha, \beta \in \mathbb{C}$ and $a, b \in A$.

It follows from these conditions that $1^* = 1$. Dually,

Definition 1.1.13. *A $*$-coalgebra C is a coalgebra C endowed with an involution $a \mapsto a^*$ such that*

$$(\alpha a + \beta b)^* = \overline{\alpha} a^* + \overline{\beta} b^*, \quad \Delta(a^*) = \Delta(a)^*, \quad (a^*)^* = a, \tag{1.1.18}$$

$\alpha, \beta \in \mathbb{C}$ and $a, b \in C$ and we endow $C \otimes C$ with the involution $(a \otimes b)^* = a^* \otimes b^*$.

In Sweedler notation, the condition $\Delta(a^*) = \Delta(a)^*$ reads $(a^*)_{(1)} \otimes (a^*)_{(2)} = (a_{(1)})^* \otimes (a_{(2)})^*$. It also follows from these conditions that $\varepsilon(a^*) = \overline{\varepsilon(a)}$.

Now a $*$-bialgebra A is a bialgebra A endowed with an involution $*$ such that $(A, *)$ is a $*$-algebra as well as a $*$-coalgebra.

Definition 1.1.14. *A Hopf algebra which is a $*$-bialgebra is called a Hopf $*$-algebra.*

Observe that the last definition does not contain a requirement of the compatibility of $*$ with the antipode S. This is a consequence of the definition: for $a \in A$,

$$S(S(a^*)^*) = a. \tag{1.1.19}$$

In particular, S becomes invertible.

Definition 1.1.15. *A compact quantum group (CQG) algebra is a cosemisimple Hopf $*$-algebra H with Haar functional $\mathbf{h}$ such that $\mathbf{h}(a^*a) > 0$, for all $a \neq 0$.*

It is known that H is a CQG algebra if and only if it is isomorphic to the dense Hopf $*$-algebra $\mathcal{S}$ of a compact quantum group S, in the sense of the next section. This and other examples will be described in the next section.

Let us end with an example. So consider a $\mathbb{C}$-Lie algebra, $\mathfrak{g}$ and the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. Define Δ and ε on the generators as follows: $\Delta(x) = x \otimes 1 + 1 \otimes x$ and $\varepsilon(x) = 1$ for $x \in \mathfrak{g}$. We extend Δ and ε to all of $U(\mathfrak{g})$ by the universality of $U(\mathfrak{g})$. One can show that endowed with these structures $U(\mathfrak{g})$ becomes a Hopf algebra. Observe that it is cocommutative. By a theorem of Cartier, more or less, these are the cocommutative Hopf algebras. Dually, let G be an algebraic group and $\mathcal{O}(G)$ be the algebra of regular functions. Dualizing the group multiplication, we get the comultiplication Δ of $\mathcal{O}(G)$. The counit is obtained by dualizing the unit map of the group G. Thus we get a commutative Hopf algebra $\mathcal{O}(G)$. Again by a theorem of Cartier, these are more or less all the commutative Hopf algebras. We "quantize" them next.

1.1.2. Quantized universal enveloping algebras. We briefly describe the quantized universal enveloping algebras. Although we don't need them in the later parts, these provide the first systematic way of producing noncommutative, noncocommutative Hopf algebras. We start by describing the quantum $\mathfrak{sl}_2$. We fix a complex number q such that $q \neq 0$ and $q^2 \neq 1$.

Definition 1.1.16. *Let $U_q(\mathfrak{sl}_2)$ be the algebra over $\mathbb{C}$ with generators E, F, K and K^{-1} and defining relations*

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \tag{1.1.20}$$

and

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}}. \tag{1.1.21}$$

As we observed above that $U(\mathfrak{sl}_2)$ carries a Hopf algebra structure, $U_q(\mathfrak{sl}_2)$ likewise is a Hopf algebra.

Proposition 1.1.17. *There is a unique Hopf algebra structure on $U_q(\mathfrak{sl}_2)$ such that*

$$\Delta(E) = E \otimes K + 1 \otimes E, \quad \Delta(F) = F \otimes 1 + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \tag{1.1.22}$$

$$\varepsilon(K) = 1, \quad \varepsilon(E) = \varepsilon(F) = 0. \tag{1.1.23}$$

Furthermore,

$$S(K) = K^{-1}, \quad S(E) = -EK^{-1}, \quad S(F) = -KF. \tag{1.1.24}$$

Definition 1.1.18. *The algebra $U_q(\mathfrak{sl}_2)$ endowed with the above Hopf algebra structure is called the quantum algebra $U_q(\mathfrak{sl}_2)$.*

Observe that as an algebra $U_q(\mathfrak{sl}_2)$ is noncommutative and as a coalgebra it is noncocommutative. This construction can be generalized to produce deformation of any semisimple Lie algebra. We briefly describe this construction.

Let $\mathfrak{g}$ be a finite dimensional semisimple Lie algebra with rank l. Let the Cartan matrix be (a_{ij}) and $d_i = \frac{(\alpha_i, \alpha_i)}{2}$, where α_i are the simple roots. Let q be a fixed complex number and let $q_i = q^{d_i}$. Suppose that $q_i^2 \neq 1$ for $i = 1, \dots, l$.

Definition 1.1.19. *Let $U_q(\mathfrak{g})$ be the algebra over $\mathbb{C}$ with $4l$ generators E_i, F_i, K_i and K_i^{-1}, $1 \leq i \leq l$ and defining relations*

$$K_iK_j = K_jK_i, \qquad K_iK_i^{-1} = K_i^{-1}K_i = 1, \tag{1.1.25}$$

$$K_iE_jK_i^{-1} = q^{a_{ij}}E_j, \qquad K_iF_jK_j^{-1} = q^{-a_{ij}}F_j, \tag{1.1.26}$$

$$[E_i, F_j] = \delta_{ij}\frac{K_i - K_i^{-1}}{q_i - q_i^{-1}}, \tag{1.1.27}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0, \qquad i \neq j, \tag{1.1.28}$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \begin{bmatrix} 1-a_{ij} \\ r \end{bmatrix}_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0, \qquad i \neq j, \tag{1.1.29}$$

where

$$\begin{bmatrix} n \\ r \end{bmatrix}_q = \frac{[n]_q!}{[r]_q![n-r]_q!}, \qquad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{1.1.30}$$

Proposition 1.1.20. *There is a unique Hopf algebra structure on $U_q(\mathfrak{g})$ such that*

$$\Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i, \tag{1.1.31}$$

$$\varepsilon(K_i) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0. \tag{1.1.32}$$

Furthermore,

$$S(K_i) = K_i^{-1}, \quad S(E_i) = -E_iK_i^{-1}, \quad S(F_i) = -K_iF_i. \tag{1.1.33}$$

Definition 1.1.21. *The algebra $U_q(\mathfrak{g})$ endowed with the above Hopf algebra structure is called the Drinfeld-Jimbo quantize (or quantum) universal enveloping algebra associated to the Lie algebra $\mathfrak{g}$ and the complex number q.*

Observe also that $U_q(\mathfrak{g})$ is noncommutative and noncocommutative. These algebras also provide systematic examples of what are called quasitriangular Hopf algebras but since we don't need them, we don't get into these considerations. We will meet more examples in the next chapter.

1.2. Analytic preliminaries

Here, we summarize the analytic prerequisites used in this thesis. Although we assume some familiarity with the theory of C^*-algebras and compact quantum groups, we go over these in a manner sufficient for our purpose. References for the first two subsections are [**Dav96**, **JT91**, **Lan95**, **Mur90**]

1.2.1. C^*-algebras and Hilbert C^*-modules. Let A be an algebra over $\mathbb{C}$. A norm $\|\cdot\|$ on A is said to be submultiplicative if

$$\|ab\| \leq \|a\|\|b\| \tag{1.2.1}$$

for $a, b \in A$. The pair $(A, \|\cdot\|)$ is called a normed algebra. If the algebra is unital and $\|1\| = 1$ then A is a unital normed algebra.

A complete normed algebra is called a Banach algebra. Unital Banach algebras are defined in the obvious way.

Example 1.2.1. *The set $C_b(\Omega)$ of all bounded complex-valued functions on a topological space Ω is a unital Banach algebra.*

Example 1.2.2. *Recall that a function f from a locally compact Hausdorff space to $\mathbb{C}$ is said to vanish at infinity if for each $\varepsilon > 0$ the set $\{\omega \in \Omega \mid |f(\omega)| \geq \varepsilon\}$ is compact. The set of such functions, $C_0(\Omega)$ is a Banach algebra.*

EXAMPLE 1.2.3. *The set $B(X)$ of all bounded linear maps from X to X with X a Banach space, is a Banach algebra. The norm is the operator norm:*

$$\|u\| = \sup_{x \neq 0} \frac{\|u(x)\|}{\|x\|}. \tag{1.2.2}$$

The spectrum of an element $a \in A$ (assumed to be unital) is the set $\{\lambda \in \mathbb{C} \mid \lambda - a \text{ is not invertible}\}$, which is denoted by $\sigma(a)$.

THEOREM 1.2.4. *If a is an element of a unital Banach algebra then $\sigma(a)$ is non-empty.*

A character on an abelian algebra A is a non-zero homomorphism $\tau : A \to \mathbb{C}$. The set of all characters is denoted by $\Omega(A)$. If A is an abelian Banach algebra then it can be shown that $\Omega(A)$ is contained in the closed unit ball of A^*. $\Omega(A)$, endowed with the relative weak* topology is called the spectrum of A.

Suppose that A is an abelian Banach algebra whose spectrum is non-empty. For $a \in A$, we define the function

$$\hat{a} : \Omega(A) \to \mathbb{C}, \qquad \tau \mapsto \tau(a). \tag{1.2.3}$$

It can be shown that these functions are continuous and in fact vanish at infinity, i.e., $\hat{a} \in C_0(\Omega(A))$ for each $a \in A$. $\hat{a}$ is called the Gelfand transform of a.

THEOREM 1.2.5. *Suppose that A is an abelian Banach algebra and that $\Omega(A)$ is non-empty. Then the map*

$$A \to C_0(\Omega(A)), \qquad a \mapsto \hat{a} \tag{1.2.4}$$

is a norm-decreasing homomorphism. Moreover, $\sigma(a) = \hat{a}(\Omega(A))$.

A Banach $*$-algebra is a $*$-algebra together with a complete submultiplicative norm $\|\cdot\|$ such that $\|a^*\| = \|a\|$ for each $a \in A$. A C^*-algebra is a Banach $*$-algebra such that the "C^*-property" holds:

$$\|a^* a\| = \|a\|^2 \tag{1.2.5}$$

for $a \in A$. $C_b(\Omega)$ and $C_0(\Omega)$ for appropriate Ω are C^*-algebras, the involution is defined as $f \mapsto \overline{f}$. If H is a Hilbert space then $B(H)$ is a C^*-algebra, the involution being the operator adjoint. We will see that $B(H)$ is the archetypal example of a C^*-algebra.

THEOREM 1.2.6. *If A is a non-zero abelian C^*-algebra then the Gelfand representation*

$$A \to C_0(\Omega(A)), \qquad a \mapsto \hat{a} \tag{1.2.6}$$

is an isometric $*$-isomorphism.

An element a in a C^*-algebra is said to be normal if $aa^* = a^*a$.

THEOREM 1.2.7. *Let a be a normal element of a C^*-algebra A and suppose that z is the inclusion map of $\sigma(a)$ in $\mathbb{C}$. Then there is a unique unital $*$-homomorphism $\phi : C(\sigma(a)) \to A$ such that $\phi(z) = a$. Moreover, ϕ is isometric and $\mathrm{im}(\phi)$ is the C^*-subalgebra of A generated by 1 and a.*

The unique unital $*$-homomorphism ϕ is called the functional calculus at $a \in A$. We write $f(a)$ for $\phi(f)$.

THEOREM 1.2.8. *Let a be a normal element of a unital C^*-algebra A, and let $f \in C(\sigma(a))$. Then*

$$\sigma(f(a)) = f(\sigma(a)). \tag{1.2.7}$$

Moreover, if $g \in C(\sigma(f(a)))$, then

$$(gf)(a) = g(f(a)), \tag{1.2.8}$$

where gf is the composition

$$\sigma(a) \xrightarrow{\quad f \quad} f(\sigma(a)) = \sigma(f(a)) \xrightarrow{\quad g \quad} \mathbb{C} \tag{1.2.9}$$

An element a of a C^*-algebra A is said to be positive if a is hermitian and $\sigma(a) \subset \mathbb{R}^+$.

THEOREM 1.2.9. *If a is an arbitrary element of a C^*-algebra A then a^*a is positive.*

A linear map $\phi : A \to B$ between the C^*-algebras A and B is said to be positive if $\phi(A^+) \subset B^+$, where A^+, respectively B^+, denote the set of all positive elements of A, respectively B.

EXAMPLE 1.2.10. *Let A be the C^* algebra $C(S^1)$ and $\int$ be integration with respect to the normalized arc length measure:*

$$\int : C(S^1) \to \mathbb{C}, \qquad f \mapsto \int f dm. \tag{1.2.10}$$

Then $\int$ is positive.

EXAMPLE 1.2.11. *Let tr denote the usual trace on $M_n(\mathbb{C})$. Then it is positive.*

Observe that if τ is a positive linear functional on a C^*-algebra A then

$$A \times A \to \mathbb{C}, \qquad (a, b) \mapsto \tau(a^*b) \tag{1.2.11}$$

is a sesquilinear form on A.

A representation of a C^*-algebra A consists of a Hilbert space H and a $*$-homomorphism $\phi : A \to B(H)$. It is faithful if ϕ is injective.

Now given a positive linear functional τ on a C^*-algebra A, one can construct a representation (H_τ, ϕ_τ) of A, called the Gelfand-Naimark-Segal representation associated to τ. If A is non-zero then its universal representation is the direct sum of (H_τ, ϕ_τ) where τ ranges over all states, i.e., positive linear functionals of norm one.

THEOREM 1.2.12. *If A is a C^*-algebra then it has a faithful representation. In particular, the universal representation is faithful.*

The minimal tensor product. We will be denoting topological tensor products by $\hat{\otimes}$ to distinguish it from the algebraic one, which is denoted by $\otimes$.

THEOREM 1.2.13. *If H and K are Hilbert spaces then there is unique inner product $\langle \cdot, \cdot \rangle$ on $H \otimes K$, the algebraic tensor product of H and K, such that*

$$\langle x \otimes y, x' \otimes y' \rangle = \langle x, x' \rangle \langle y, y' \rangle, \tag{1.2.12}$$

where $x, x' \in H$ and $y, y' \in K$.

The Hilbert space completion of $H \otimes K$ with respect to the inner product as above is denoted by $H \hat{\otimes} K$ and called the Hilbert space tensor product of H and K. Observe that

$$\|x \otimes y\| = \|x\| \|y\|. \tag{1.2.13}$$

PROPOSITION 1.2.14. *Let H and K be Hilbert spaces and suppose that $u \in B(H)$ and $v \in B(K)$. Then there is a unique operator $u \hat{\otimes} v \in B(H \hat{\otimes} K)$ such that $(u \hat{\otimes} v)(x \otimes y) = u(x) \otimes v(y)$ for $x \in H$ and $y \in K$. Moreover, $\|u \hat{\otimes} v\| = \|u\| \|v\|$.*

We now describe a C^*-norm on the $*$-algebra $A \otimes B$, given that A and B are C^*-algebras. But before that, observe, that if $\phi : A \to C$ and $\psi : B \to C$ are $*$-homomorphisms, where C is another C^*-algebra, with commuting images, then there is a unique $*$-homomorphism $\pi : A \otimes B \to C$ such that $\pi(a \otimes b) = \phi(a)\psi(b)$.

PROPOSITION 1.2.15. *Let A and B be C^*-algebras and suppose that (H, ϕ) and (K, ψ) are representations of A and B respectively. Then there is a unique $*$-homomorphism $\pi : A \otimes B \to B(H \hat{\otimes} K)$ such that*

$$\pi(a \otimes b) = \phi(a) \otimes \psi(b). \tag{1.2.14}$$

Moreover, if ϕ and ψ are injective then so is π.

Again, we will denote π by $\phi \otimes \psi$. Now we take (H, ϕ) and (K, ψ) to be the respective universal representations of A and B. By the above Proposition, there is a unique $*$-homomorphism $\pi : A \otimes B \to B(H \hat{\otimes} K)$ such that $\pi(a \otimes b) = \phi(a) \otimes \psi(b)$. The function

$$\| \cdot \| : A \otimes B \to \mathbb{C}, \qquad c \mapsto \|\pi(c)\| \tag{1.2.15}$$

is a C^*-norm on $A \otimes B$, called the minimal C^*-norm. The C^*-completion of $A \otimes B$ with respect to this norm is said to be the minimal tensor product of the C^*-algebras A and B and denoted $A \hat{\otimes} B$. We end this section by observing that $\|a \otimes b\| = \|a\|\|b\|$.

Hilbert C^*-modules. Let A be a C^*-algebra with norm $\| \cdot \|$. A pre-Hilbert A-module is a right A-module E together with a map $\langle \cdot, \cdot \rangle : E \times E \to A$ which is linear in the second variable and satisfies the following:

i) $\langle x, ya \rangle = \langle x, y \rangle a$;
ii) $\langle x, y \rangle^* = \langle y, x \rangle$;
iii) $\langle x, x \rangle \geq 0$;
iv) $x \neq 0$ implies $\langle x, x \rangle \neq 0$;

for all $x, y \in E$ and $a \in A$. We have the following lemma.

Lemma 1.2.16. *Let E be a pre-Hilbert A-module and for $x \in E$, define $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. Then E is a normed vector space and the following inequalities hold:*

i) $\|xa\| \leq \|x\|\|a\|$;
ii) $\|\langle x, y \rangle\| \leq \|x\|\|y\|$;

for $x, y \in E$ and $a \in A$.

We define a Hilbert A-module to be a pre-Hilbert A-module which is complete in the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$.

Example 1.2.17. *A Hilbert $\mathbb{C}$-module is just a Hilbert space.*

Example 1.2.18. *Let A be a C^*-algebra. Then A is a Hilbert A-module in itself. The inner product is defined as $\langle a, b \rangle = a^* b$.*

We will meet many more examples of Hilbert modules later on. Now let E_1 and E_2 be a pair of Hilbert A-modules and we define the space $L(E_1, E_2)$ as the set of all maps $T : E_1 \to E_2$ for which there exists another map $T^* : E_2 \to E_1$ such that $\langle T_1 x, y \rangle = \langle x, T_2 y \rangle$ for all $x \in E_1$ and $y \in E_2$. T^* is obviously called the adjoint of T. Its existence implies that T is A-linear and that T is bounded. It is a fact that $L(E)$ is a C^*-algebra, the norm being the operator norm.

For each pair of elements $x, y \in E$, we define $\Theta_{x,y} : E \to E$ by $\Theta_{x,y}(z) = x\langle y, z \rangle$, $z \in E$. One checks that $\Theta_{x,y} \in L(E)$ and that $\Theta^*_{x,y} = \Theta_{y,x}$. The closed linear span of the set $\{\Theta_{x,y} \mid x, y \in E\}$ is denoted by $K(E)$. When $A = \mathbb{C}$, so that Hilbert A-modules are Hilbert spaces, $K(E)$ is just the set of compact operators. The space $K(A)$ is canonically isomorphic to A itself and the space $L(A)$ is called the multiplier algebra of A, often denoted $M(A)$. If A is unital then $M(A)$ coincides with A.

1.2.2. Compact quantum groups. References for this subsection are [**DK94**, **MVD98**, **Wor87**, **Wor88**, **Wor98**]. Let us start with the following

Definition 1.2.19. *Let A be a C^*-algebra. A unital $*$-homomorphism $\Delta : A \to A \hat{\otimes} A$ is called coassociative if*

$$(\mathrm{id} \,\hat{\otimes} \Delta)\Delta = (\Delta \hat{\otimes} \,\mathrm{id})\Delta. \tag{1.2.16}$$

In analogy with the algebraic case, Δ is called a comultiplication. Here, we use the minimal tensor product of C^*-algebras. $(\mathrm{id} \,\hat{\otimes} \Delta)$ and $(\Delta \hat{\otimes} \,\mathrm{id})$ are the continuous extensions of the obvious maps on the algebraic tensor product.

With this in hand, we have

DEFINITION 1.2.20. *A compact quantum group (CQG, for short)* $\mathbb{G}$ *is a pair* $(C(\mathbb{G}), \Delta)$*, where* $C(\mathbb{G})$ *is a unital* C^**-algebra and* $\Delta : C(\mathbb{G}) \to C(\mathbb{G})\hat{\otimes}C(\mathbb{G})$ *is a comultiplication, such that* $\Delta(C(\mathbb{G}))(C(\mathbb{G})\hat{\otimes}1)$ *and* $\Delta(C(\mathbb{G}))(1\hat{\otimes}C(\mathbb{G}))$ *are dense in* $C(\mathbb{G})\hat{\otimes}C(\mathbb{G})$*.*

Let G be a compact topological group and $C(G)$ be the function algebra. The multiplication of G induces a unital $*$-homomorphism $\Delta : C(G) \to C(G\times G)$. Identifying $C(G\times G)$ with $C(G)\hat{\otimes}C(G)$, this gives rise to a comultiplication on $C(G)$. One can show then that $(C(G), \Delta)$ satisfies the coassociativity and the density property as above in the Definition. Thus $(C(G), \Delta)$ is a compact quantum group. This is the only compact quantum group with the C^*-algebra being commutative. More precisely, if $\mathbb{G} = (C(\mathbb{G}), \Delta)$ is a compact quantum group with $C(\mathbb{G})$ commutative then the character space inherits a group structure and thus $(C(\mathbb{G}), \Delta)$ is of the form $(C(G), \Delta)$, the G being $\Omega(C(\mathbb{G}))$. We will meet more examples later on.

DEFINITION 1.2.21. *A morphism from a compact quantum group* $\mathbb{G}$ *to another compact quantum group* $\mathbb{G}'$ *is given by a unital* C^**-homomorphism* $\Phi : C(\mathbb{G}') \to C(\mathbb{G})$ *of the underlying* C^**-algebras such that*

$$(\Phi \otimes \Phi)\Delta_{\mathbb{G}'} = \Delta_{\mathbb{G}}\Phi. \tag{1.2.17}$$

DEFINITION 1.2.22. *A Woronowicz* C^**-subalgebra of a compact quantum group* $\mathbb{G}$ *is a pair* $(\mathbb{G}', \Phi)$*, where* $\mathbb{G}'$ *is a compact quantum group and* $\Phi : C(\mathbb{G}') \to C(\mathbb{G})$ *is an injective morphism.*

The existence of Haar measure for compact groups extends to compact quantum groups also.

DEFINITION 1.2.23. *Let* $\mathbb{G} = (C(\mathbb{G}), \Delta)$ *be a compact quantum group. A state* $\mathbf{h}$ *on* $C(\mathbb{G})$ *is called a Haar state on* $\mathbb{G}$ *if* $(\mathbf{h}\hat{\otimes}\,\mathrm{id})\Delta(a) = (\mathrm{id}\,\hat{\otimes}\mathbf{h})\Delta(a) = \mathbf{h}(a)1$ *for all* $a \in C(\mathbb{G})$*.*

We have

THEOREM 1.2.24. *There exists a unique Haar state* $\mathbf{h}$ *on* $\mathbb{G}$ *for any compact quantum group* $\mathbb{G} = (C(\mathbb{G}), \Delta)$*.*

Representation theory. We now turn to defining what a representation of a compact quantum group means. Let H be a Hilbert space and $K(H)$ be the C^*-algebra of all compact operators on H. We consider the multiplier algebra $M(K(H)\hat{\otimes}C(\mathbb{G}))$ which embeds in two different ways into $M(K(H)\hat{\otimes}C(\mathbb{G})\hat{\otimes}C(\mathbb{G}))$. The first one is given by extending $x \mapsto x\hat{\otimes}1$ from $K(H)\hat{\otimes}C(\mathbb{G}) \to K(H)\hat{\otimes}C(\mathbb{G})\hat{\otimes}C(\mathbb{G})$. For the second, we send $x\hat{\otimes}a$ to $x\hat{\otimes}1\hat{\otimes}a$ as a map from $K(H)\hat{\otimes}C(\mathbb{G})$ to $K(H)\hat{\otimes}C(\mathbb{G})\hat{\otimes}C(\mathbb{G})$. One uses the "leg numbering notation" to denote the image of $v \in M(K(H)\hat{\otimes}C(\mathbb{G}))$ by $v_{(12)}$ and $v_{(13)}$ under the two embeddings, respectively.

DEFINITION 1.2.25. *Let* $\mathbb{G} = (C(\mathbb{G}), \Delta)$ *be a compact quantum group. A representation of* $\mathbb{G}$ *consists of a Hilbert space* H *and an element* $v \in M(K(H)\hat{\otimes}C(\mathbb{G}))$ *such that*

$$(\mathrm{id}\,\hat{\otimes}\Delta)v = v_{(12)}v_{(13)}. \tag{1.2.18}$$

If v is unitary as an element of the C^*-algebra $M(K(H)\hat{\otimes}C(\mathbb{G}))$ then the representation is called unitary. Note that a representation can also be interpreted as an element of $L(H\hat{\otimes}C(\mathbb{G}))$, the C^*-algebra of adjointable operators on the Hilbert $C(\mathbb{G})$-module $H\hat{\otimes}C(\mathbb{G})$. We use the identification $K(H)\hat{\otimes}C(\mathbb{G}) \cong K(H\hat{\otimes}C(\mathbb{G}))$.

Let us discuss the compact group case briefly. Let u be a unitary representation of G on H, i.e., a continuous homomorphism of G into $B(H)$, $B(H)$ taken with strong operator topology. One can show that u can be viewed as an element of $M(K(H)\hat{\otimes}C(G))$, using the "strict topology" on $B(H)$ which is the multiplier algebra of $K(H)$. In the same way, elements of $M(K(H)\hat{\otimes}C(G)\hat{\otimes}C(G))$ are viewed as strictly continuous $B(H)$-valued functions on $G \times G$. Then $u_{(12)}$ ($u_{(13)}$) written in terms "elements" becomes $u_{(12)}(p, q) = u(p)$ ($u_{(13)}(p, q) = u(q)$, respectively). Using the definition of comultiplication Δ, one can see that $(\mathrm{id}\,\hat{\otimes}\Delta)u = u_{(12)}u_{(13)}$ is nothing but $u(pq) = u(p)u(q)$!

The Haar state can be used to construct the regular representation as in the compact group case. Let $\mathcal{H}$ be the GNS space associated to $\mathbf{h}$ and ξ_0 be the cyclic vector. Here, $\mathbf{h}$ is the Haar state of the compact quantum group $\mathbb{G}$. Let K be another faithful and non-degenerate representation of $C(\mathbb{G})$.

PROPOSITION 1.2.26. *There is a unitary operator u on $\mathcal{H}\hat{\otimes}K$ defined by $u(a\xi_0\hat{\otimes}\eta) = \Delta(a)(\xi_0\hat{\otimes}\eta)$ for $a \in C(\mathbb{G})$ and $\eta \in K$.*

One can show that u is a multiplier in $M(K(\mathcal{H})\hat{\otimes}C(\mathbb{G}))$ and satisfies $(\mathrm{id}\,\hat{\otimes}\Delta)u = u_{(12)}u_{(13)}$. Thus u is a unitary representation of the compact quantum group $\mathbb{G}$, called the regular representation. For a compact group G, one does indeed get the right regular representation of G.

After defining the notion of regular representation, we briefly discuss an exact analogue of Peter-Weyl theorem.

DEFINITION 1.2.27. *Let v be a representation of a compact quantum group $\mathbb{G}$ on a Hilbert space H. A closed subspace H_1 of H is said to be invariant if $(e\hat{\otimes}1)v(e\hat{\otimes}1) = v(e\hat{\otimes}1)$, where e is the projection onto the subspace H_1. The representation v is said to be irreducible if the only invariant subspaces of H are $\{0\}$ and H itself.*

Like in the case of compact groups, one can show that if v is a unitary representation of the compact quantum group $\mathbb{G}$ on H and H_1 is an invariant subspace then the orthogonal complement $H_1^{\perp}$ of H_1 is also invariant.

Now,

DEFINITION 1.2.28. *Let v and w be two representations of $\mathbb{G}$ on H_1 and H_2 respectively. An intertwiner between v and w is a bounded linear operator $x \in B(H_1, H_2)$ such that $(x\hat{\otimes}1)v = w(x\hat{\otimes}1)$.*

Then it can be shown that any non-degenerate finite dimensional representation is equivalent to a unitary representation. Any unitary representation decomposes into irreducible ones.

THEOREM 1.2.29. *Let v be a unitary representation of the compact quantum group $\mathbb{G}$ on the Hilbert space H. Then there is a set $\{e_\alpha \mid \alpha \in I\}$ of mutually orthogonal finite dimensional projections with sum 1 and satisfying*

$$(e_\alpha\hat{\otimes}1)v = v(e_\alpha\hat{\otimes}1), \tag{1.2.19}$$

and $v(e_\alpha\hat{\otimes}1)$, considered as an element in $B(e_\alpha H)\hat{\otimes}C(\mathbb{G})$, is a finite dimensional unitary representation of $\mathbb{G}$.

If v is a representation on a finite dimensional Hilbert space H, we write the matrix units in $B(H)$ by (e_{pq}), so that $v = \sum e_{pq} \otimes v_{pq}$. We define $\overline{v} = \sum e_{pq} \otimes v_{pq}^*$. Then $\overline{v}$ is still a representation and is called the conjugate representation of v. If v is irreducible then one can show that $\overline{v}$ is also irreducible. Moreover, if v is unitary then $\overline{v}$ is equivalent to a unitary representation.

If A_0 denotes the subspace spanned by the matrix elements of finite dimensional unitary representations then it can be shown that A_0 is a dense $*$-subalgebra of $C(\mathbb{G})$. Moreover, Δ maps A_0 inside the algebraic tensor product $A_0 \otimes A_0$ and (A_0, Δ) becomes a Hopf $*$-algebra (Definition 1.1.14). In order to describe the counit and the antipode, one takes a complete set $\{u^\alpha \mid \alpha \in I\}$ of mutually inequivalent, irreducible unitary representations. It can be shown that the elements $\{u^\alpha_{pq} \mid \alpha \in I, 1 \le p, q \le n(\alpha)\}$ form a basis for A_0. Then the counit and the antipode are given by

$$\varepsilon(u^\alpha_{pq}) = \delta_{pq}, \qquad S(u^\alpha_{pq}) = (u^\alpha_{pq})^*. \tag{1.2.20}$$

Alternative approach. In the last paragraph, we mentioned that for each compact quantum group $\mathbb{G}$ there is a dense Hopf $*$-algebra $(A_0, \Delta|_{A_0})$. It can be shown that if one starts with a Hopf $*$-algebra A_0 which is spanned by the coefficients of its finite dimensional irreducible unitary representations then Haar state exists. Such an algebra is called a compact quantum group algebra (Definition 1.1.15). One can moreover show that these algebras admit C^*-completion which produces compact quantum groups. A special case is that when the algebra A_0 is generated by the matrix elements of a distinguished irreducible unitary representation called the fundamental representation. We call such an algebra a compact quantum matrix algebra.

1.2.3. Examples of compact quantum groups. In this section we describe some of the main examples of compact quantum groups. But before that we define action of compact quantum groups on C^*-algebras.

Action of compact quantum group on C^*-algebras. One says that the compact quantum group $\mathbb{G}$ acts on a unital C^*-algebra B if there is a unital C^*-homomorphism $\alpha : B \to B\hat{\otimes}C(\mathbb{G})$ such the following hold:

i) $(\alpha\hat{\otimes}\,\mathrm{id})\alpha = (\mathrm{id}\,\hat{\otimes}\Delta)\alpha$;
ii) the linear span of $\alpha(B)(\mathrm{id}\,\hat{\otimes}C(\mathbb{G}))$ is norm-dense in $B\hat{\otimes}C(\mathbb{G})$.

It is well-known [**Pod87**] that the last condition is equivalent to the existence of a norm-dense, unital $*$-subalgebra B_0 of B such that $\alpha(B_0)$ is mapped inside the algebraic tensor product $B_0 \otimes A_0$, A_0 being the dense Hopf $*$-algebra inside $C(\mathbb{G})$, and on B_0, $(\mathrm{id}\otimes\varepsilon)\alpha = \mathrm{id}$.

Definition 1.2.30. *Suppose that the compact quantum group $\mathbb{G}$ acts by α on the C^*-algebra B. We say that α is faithful if there is no proper C^*-subalgebra A' of $C(\mathbb{G})$ such that the following hold:*

i) $(A', \Delta|_{A'})$ *is itself a compact quantum group;*
ii) the inclusion $j : A' \to C(\mathbb{G})$ *satisfies* $\Delta j = (j\hat{\otimes}j)\Delta$;
iii) α *is an action of* $(A', \Delta|_{A'})$.

Now suppose that the unital C^*-algebra B carries an action of the compact quantum group $\mathbb{G}$. A continuous linear functional ϕ on B is said to be invariant under the action α if $(\phi\hat{\otimes}\,\mathrm{id})\alpha(b) = \phi(b)1$ for all $b \in B$. With these in hand, we now move onto examples.

The Wang algebras. Fix an $n \times n$ invertible, positive matrix $Q = (Q_{ij})$. Let $A_{u,n}(Q)$ be the universal C^*-algebra (for the notion of a universal C^*-algebra, see [**Dav96**]) generated by $\{u_{ij} \mid i, j = 1, \cdot, n\}$ subject to the following relations:

$$uu^* = u^*u = I_n, \qquad u^tQ\overline{u}Q^{-1} = Q\overline{u}Q^{-1}u^t = I_n; \tag{1.2.21}$$

where $u = (u_{ij})$, $u^t = (u_{ji})$, $\overline{u} = (u^*_{ij})$ and $u^* = \overline{u}^t$. It can be shown that $A_{u,n}(Q)$ is a compact quantum group with comultiplication Δ given on the generators by

$$\Delta(u_{ij}) = \sum_k u_{ik}\hat{\otimes}u_{kj}. \tag{1.2.22}$$

Furthermore, $A_{u,n}(Q)$ is the universal object in the category of compact quantum groups which act on the finite dimensional C^*-algebra $M_n(\mathbb{C})$ such the functional $M_n(\mathbb{C}) \to \mathbb{C}$, $x \mapsto Tr(Q^tx)$ is invariant under the action, see [**Wan98**].

The quantum permutation group. Let $C(S_n^+)$ be the universal C^*-algebra generated by a_{ij} $i, j = 1, \dots, n$ subject to the following conditions:

$$a_{ij}^2 = a_{ij} = a_{ij}^*, \quad \sum_i a_{ij} = 1, \quad \sum_j a_{ij} = 1, \quad i, j = 1, \dots, n. \tag{1.2.23}$$

It can be shown that $C(S_n^+)$ is the underlying C^*-algebra of a compact quantum group S_n^+, called the quantum permutation group, with comultiplication given on generators by $\Delta(a_{ij}) = \sum_k a_{ik}\hat{\otimes}a_{kj}$. Furthermore, if a_{ij} are assumed to commute with each other then we indeed get back the permutation group S_n on n letters. Let X_n be the set $\{1, \dots, n\}$. Then $C(X_n)$ has the following presentation $C(X_n) = C^*\{e_i \mid e_i^2 = e_i = e_i^*, \sum_k e_i = k, i = 1, \dots, n\}$. It can be shown that $C(S_n^+)$ is the universal object in the category of compact quantum groups acting on the C^*-algebra $C(X_n)$, see [**Wan98**].

The compact quantum group $U_q(2)$. We now introduce the compact quantum group $U_q(2)$. A more general version is given in the next chapter. We refer to [**KS97**] for more details. As a unital C^*-algebra, $C(U_q(2))$ is generated by four elements u_1^1, u_2^1, u_1^2 and u_2^2, satisfying

$$u_1^1u_2^1 = qu_2^1u_1^1, \quad u_1^1u_1^2 = qu_1^2u_1^1, \quad u_2^1u_2^2 = qu_2^2u_2^1, \quad u_1^2u_2^2 = qu_2^2u_1^2, \tag{1.2.24}$$

$$u_2^1u_1^2 = u_1^2u_2^1, \quad u_1^1u_2^2 - u_2^2u_1^1 = (q - q^{-1})u_2^1u_1^2 \tag{1.2.25}$$

and the condition that the matrix $u = \begin{pmatrix} u_1^1 & u_2^1 \\ u_1^2 & u_2^2 \end{pmatrix}$ is unitary. Thus the matrix u becomes the fundamental unitary for $U_q(2)$. The CQG structure is given by

$$\Delta(u_j^i) = \sum_k u_k^i \hat{\otimes} u_j^k. \tag{1.2.26}$$

The compact quantum group $SU_q(2)$. We end by describing one of the most studied compact quantum groups, a more general version will appear in the next chapter, see [**Wor89**] for more details. Let q belong to $[-1, 1]$, $q \neq 0$. The C^*-algebra $C(SU_q(2))$ is defined as the universal C^*-algebra generated by α and γ satisfying

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1, \quad \gamma\gamma^* = \gamma^*\gamma, \quad q\gamma\alpha = \alpha\gamma, \quad q\gamma^*\alpha = \alpha\gamma^*. \tag{1.2.27}$$

The fundamental representation of $SU_q(2)$ is given by $\begin{pmatrix} \alpha & -q\gamma^* \\ \gamma & \alpha^* \end{pmatrix}$. There is a coproduct Δ on $C(SU_q(2))$ given by

$$\Delta(\alpha) = \alpha\hat{\otimes}\alpha - q\gamma^*\hat{\otimes}\gamma, \quad \Delta(\gamma) = \gamma\hat{\otimes}\alpha + \alpha^*\hat{\otimes}\gamma, \tag{1.2.28}$$

which makes it into a compact quantum group.

1.2.4. Review of unbounded operators. Here we recall some facts concerning unbounded operators, that we will need later on. References are [**RS72**, **Sch12**]. We start with the following

DEFINITION 1.2.31. *An operator on a Hilbert space H is a linear map T from its domain $D(T)$, a linear subspace of H, into H. We will always assume that the domain $D(T)$ is dense and call T a densely defined unbounded operator on H.*

EXAMPLE 1.2.32. *Let $H = L^2(\mathbb{R})$ and $D(T)$ be the set of all functions $\phi \in L^2(\mathbb{R})$ such that $\int_{\mathbb{R}} x^2|\phi(x)|^2dx < \infty$. For $\phi \in D(T)$, define $(T\phi)(x) = x\phi(x)$. Then T is unbounded. To see this, we choose ϕ with support near plus or minus infinity so that we can make $\|T\phi\|$ as large as we want while keeping $\|\phi\| = 1$.*

The graph of the linear transformation T, denoted $\Gamma(T)$ is the set $\{(\phi, T\phi) \mid \phi \in D(T)\}$. It is a subset of $H \times H$ which is a Hilbert space with inner product $\langle(\phi_1, \psi_1), (\phi_2, \psi_2)\rangle = \langle\phi_1, \phi_2\rangle + \langle\psi_1, \psi_2\rangle$. T is said to be closed if $\Gamma(T)$ is a closed subspace of $H \times H$. Given two operators T, T_1 on H, T_1 is said to be an extension of T if $\Gamma(T_1) \supseteq \Gamma(T)$. An operator T is said to be closable if T has a closed extension. The smallest closed extension of such T is called the closure of the operator, denoted $\overline{T}$.

DEFINITION 1.2.33. *Let T be a densely defined linear operator on a Hilbert space H. Let $D(T^*)$ be the set of all $\phi \in H$ for which there is an $\eta \in H$ such that*

$$\langle T\psi, \phi\rangle = \langle\psi, \eta\rangle, \tag{1.2.29}$$

for all $\psi \in D(T)$. For each such ϕ, define $T^*\phi = \eta$. T^* is called the adjoint of T.

We have the following theorem.

THEOREM 1.2.34. *Let T be a densely defined operator on a Hilbert space H. Then the following hold:*

i) T^* is closed;
ii) T is closable if and only if $D(T^*)$ is dense in which case $\overline{T} = T^{**}$;
iii) If T is closable then $(\overline{T})^* = T^*$.

Let T be a closed operator on a Hilbert space H. A complex number λ is in the resolvent set, $\rho(T)$, if $\lambda I - T$ is a bijection of $D(T)$ onto H with a bounded inverse. If $\lambda \in \rho(T)$, $R_\lambda(T) = (\lambda I - T)^{-1}$ is called the resolvent of T at λ. Similarly, the definition of spectrum is the same as they are for bounded operators.

A densely defined operator T on a Hilbert space H is called symmetric (or Hermitian) if $T \subseteq T^*$ and self-adjoint if $T = T^*$. A symmetric operator T is called essentially self-adjoint if its closure $\overline{T}$ is self-adjoint. If T is closed, a subset $D \subseteq D(T)$ is called a core for T if $\overline{T|_D} = T$. The following is the basic criterion for self-adjointness.

THEOREM 1.2.35. *Let T be a symmetric operator on a Hilbert space H. Then the following are equivalent.*

i) *T is self-adjoint;*
ii) *T is closed and* $\ker(T^* \pm i) = \{0\}$;
iii) $\operatorname{ran}(T \pm i) = H$.

This has an important corollary.

COROLLARY 1.2.36. *Let T be a symmetric operator on a Hilbert space H. Then the following are equivalent.*

i) *T is essentially self-adjoint;*
ii) $\ker(T^* \pm i) = \{0\}$;
iii) $\operatorname{ran}(T \pm i)$ *are dense.*

The spectral theorem. One can extend the spectral theorem for bounded self-adjoint operators to unbounded self-adjoint operators.

PROPOSITION 1.2.37. *Let (M, μ) be a measure space with μ a finite measure. Suppose that f is a measurable, real-valued function on M that is finite a.e. $[\mu]$. Then the operator $T_f : \phi \mapsto f\phi$ on $L^2(M, \mu)$ with domain $D(T_f) = \{\phi \mid f\phi \in L^2(M, \mu)\}$ is self-adjoint and $\sigma(T_f)$ is the essential range of f.*

We now state one form of the spectral theorem.

THEOREM 1.2.38. *Let A be a self-adjoint operator on a separable Hilbert space H with domain $D(A)$. Then there is a measure space (M, μ) with μ a finite measure, a unitary operator $U : H \to L^2(M, \mu)$ and a real-valued function f on M which is finite a.e.$[\mu]$, such that*

i) *$\psi \in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot) \in L^2(M, \mu)$;*
ii) *if $\phi \in U(D(A))$ then $(UAU^{-1}\phi)(m) = f(m)\phi(m)$.*

The functional calculus form of the spectral theorem is as follows.

THEOREM 1.2.39. *Let A be a self-adjoint operator on H. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions on $\mathbb{R}$ into $B(H)$ such that*

i) *$\hat{\phi}$ is a $*$-homomorphism;*
ii) *$\hat{\phi}$ is norm continuous, i.e., $\|\hat{\phi}\|_{B(H)} \leq \|h\|_\infty$;*
iii) *let $h_n(x)$ be a sequence of bounded Borel functions with $h_n(x) \to x$ as $n \to \infty$ for each x and $|h_n(x)| \leq |x|$ for all x and n. Then, for any $\psi \in D(A)$, $\hat{\phi}(h_n)(\psi) \to A\psi$ as $n \to \infty$;*
iv) *if $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded then $\hat{\phi}(h_n) \to p\hat{h}i(h)$ strongly;*
v) *if $A\psi = \lambda\psi$ then $\hat{\phi}(h)\psi = h(\lambda)\psi$;*
vi) *if $h \geq 0$ then $\hat{\phi}(h) \geq 0$.*

In the case where A is bounded, one can define e^{itA} by the power series which converges in norm; one does not need the spectral theorem. But when A is unbounded, the power series for e^{itA} may not be defined, for $\phi \in D(A)$ might not be in $D(A^n)$ for some n. One uses the spectral theorem to define e^{itA}. To describe e^{itA} more explicitly, we use the projection-valued measure form of the spectral theorem. Let P_Ω be the operator $\chi_\Omega(A)$ where χ_Ω is the characteristic function of the measurable set $\Omega \subseteq \mathbb{R}$. This family of operators $\{P_\Omega\}$ enjoys the following properties:

i) each P_Ω is an orthogonal projection;
ii) $P_\emptyset = 0$ and $P_\mathbb{R} = I$;
iii) if $\Omega = \cup_n \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$ whenever $n \neq m$ then $P_\Omega = s\text{-}\lim_{N\to\infty} P_{\Omega_N}$;
iv) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Such a family is called a projection-valued measure. Given $\phi \in H$, $\langle \phi, P_\Omega \phi \rangle$ is a well-defined Borel measure on $\mathbb{R}$ which is denoted by $d\langle \phi, P_\lambda \phi \rangle$. For a bounded Borel function g, we define $g(A)$ by

$$\langle \phi, g(A)\phi \rangle = \int_{-\infty}^{\infty} g(\lambda) d\langle \phi, P_\lambda \phi \rangle. \tag{1.2.30}$$

It can be shown that $g(A)$ so defined satisfies the properties in Theorem 5.8. Now, for unbounded complex-valued g, let

$$D_g = \{\phi \mid \int_{-\infty}^{\infty} |g(\lambda)|^2 d\langle \phi, P_\lambda \phi \rangle < \infty\}. \tag{1.2.31}$$

Then D_g is dense in H and $g(A)$ is defined on D_g by

$$\langle \phi, g(A)\phi \rangle = \int_{-\infty}^{\infty} g(\lambda) d\langle \phi, P_\lambda \phi \rangle, \tag{1.2.32}$$

which is written symbolically as $g(A) = \int g(\lambda) dP_\lambda$. For $\phi, \psi \in D(A)$,

$$\langle \phi, A\psi \rangle = \int_{-\infty}^{\infty} \lambda d\langle \phi, P_\lambda \psi \rangle. \tag{1.2.33}$$

If g is real-valued then $g(A)$ is self-adjoint on D_g. Putting all these together,

THEOREM 1.2.40. *There is a one-one correspondence between self-adjoint operators A and projection valued measures $\{P_\Omega\}$ on H, the correspondence being given by*

$$A = \int_{-\infty}^{\infty} \lambda dP_\lambda. \tag{1.2.34}$$

THEOREM 1.2.41. *Let A be a self-adjoint operator and define $U(t) = e^{itA}$. Then*

i) for each $t \in \mathbb{R}$, $U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $s, t \in \mathbb{R}$;
ii) if $\phi \in H$ and $t \to t_0$, then $U(t)\phi \to U(t_0)\phi$;
iii) for $\psi \in D(A)$, $\frac{U(t)\psi - \psi}{t} \to iA\psi$ as $t \to 0$;
iv) if $\lim_{t\to 0} \frac{U(t)\psi - \psi}{t}$ exists, then $\psi \in D(A)$.

An operator-valued function $U(t)$ satisfying i) and ii) above is said to be a strongly continuous one-parameter unitary group. We end with a definition.

DEFINITION 1.2.42. *Let A be an operator on a Hilbert space H. The set $C^\infty(A) = \cap_{n=1}^{\infty} D(A^n)$ is called the C^∞-vectors for A. A vector $\phi \in C^\infty(A)$ is called an analytic vector for A if*

$$\sum_{n=0}^{\infty} \frac{\|A^n \phi\|}{n!} t^n < \infty \tag{1.2.35}$$

for some $t > 0$.

1.3. Geometric preliminaries

Here we collect some geometric notions to be used in this thesis.

1.3.1. Hodge theory on differentiable manifolds. We assume the reader is familiar with basic manifold theory but nevertheless we start from the basics. References are [**MS74**, **War83**, **dC92**, **BT82**].

Differentiable manifolds and vector bundles. Let $U \subset \mathbb{R}^n$ be an open set and $f : U \to \mathbb{R}$ be a function. One says that f is differentiable of class C^k on U, for k a nonnegative integer, if the partial derivatives $\frac{\partial^\alpha f}{\partial r^\alpha}$ exist and are continuous on U with $k = \sum_i \alpha_i$. Here r_i are the standard global coordinates of $\mathbb{R}^n$. If $f : U \to \mathbb{R}^m$ then f is C^k if each $f_i = r_i f$ is. f is C^∞ if f is C^k for each $k \geq 0$.

DEFINITION 1.3.1. *A locally Euclidean space M of dimension n is a Hausdorff topological space M for which each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$. If ϕ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of $\mathbb{R}^n$, ϕ is called a coordinate map, the functions $x_i = r_i \phi$ are called the coordinate functions, and the pair $(U, \phi) = (U, x_1, \dots, x_n)$ is called a coordinate system.*

DEFINITION 1.3.2. *A differentiable structure $\mathcal{F}$ of class C^k ($1 \leq k \leq \infty$) on a locally Euclidean space M is a collection of coordinate systems $\{(U_\alpha, \phi_\alpha) \mid \alpha \in A\}$ satisfying the following:*
i) $\bigcup_{\alpha \in A} U_\alpha = M$;
ii) $\phi_\alpha \phi_\beta^{-1}$ *is C^k for all $\alpha, \beta \in A$; these are called the transition functions;*
iii) The collection $\mathcal{F}$ is maximal with respect to ii); Explicitly, if (U, ϕ) is a coordinate system such that $\phi \phi_\alpha^{-1}$ and $\phi_\alpha \phi^{-1}$ are C^k for all $\alpha \in A$, then $(U, \phi) \in \mathcal{F}$.

One can show that if $\mathcal{F}_0$ is any collection of coordinate systems satisfying i) and ii) then there is a unique differentiable structure $\mathcal{F}$ containing it.

DEFINITION 1.3.3. *An n-dimensional differentiable manifold of class C^k is a pair $(M, \mathcal{F})$ consisting of an n-dimensional, second countable, locally Euclidean space M together with a differentiable structure $\mathcal{F}$ of class C^k.*

One usually writes only M for a differentiable manifold and we will only consider such M of class C^∞.

EXAMPLE 1.3.4. *The standard differentiable structure on Euclidean space $\mathbb{R}^n$ is obtained by taking $\mathcal{F}$ to be the maximal collection containing $(\mathbb{R}^n, \mathrm{id})$.*

EXAMPLE 1.3.5. *Let V be a finite dimensional real vector space and fix a basis $\{e_1, \dots, e_n\}$. The elements of the dual basis $\{r_1, \dots, r_n\}$ form the coordinates of a global coordinate system and uniquely determines a differentiable structure. Thus the complex n-space $\mathbb{C}^n$ is a $2n$-dimensional real manifold.*

EXAMPLE 1.3.6. *An open subset U of a differentiable manifold M admits a differentiable structure by restricting that of M to U.*

EXAMPLE 1.3.7. *Let S^n be the n-sphere inside $\mathbb{R}^{n+1}$ and let $N = (0, \dots, 0, 1)$, $S = (0, \dots, 0, -1)$. The standard differentiable structure on S^n is obtained by taking the maximal collection containing $(S^n - N, p_N)$ and $(S^n - S, p_S)$, where p_N and p_S are the stereographic projections from N and S, respectively.*

For $U \subset M$ open, an $f : U \to \mathbb{R}$ is a C^∞ function on U if $f\phi^{-1}$ is C^∞ for each coordinate map ϕ on M. A continuous $\psi : M \to N$ is differentiable of class C^∞ if $g\psi$ is C^∞ for all C^∞ functions g defined on open sets of N. A bijective C^∞ map with C^∞ inverse is called a diffeomorphism. We recall that given a point $p \in M$, two functions f and g defined on some open sets of M containing p are said to have the same germ at p if they agree on some neighborhood of p. This defines an equivalence relation and we let $\mathbf{F}_p$ denote the set of equivalence classes of such germs. This is an algebra in a canonical way.

One defines a tangent vector v at the point $p \in M$ to be a linear derivation of the algebra $\mathbf{F}_p$. The space of such tangent vectors is called the tangent space of M at p and is denoted by T_pM. This becomes a real vector space in a canonical way of dimension $\dim M = n$. A basis for T_pM maybe constructed as follows: we choose a coordinate system $(U, x_1, \dots, x_n)$ around p and for each $i = 1, \dots, n$, define the tangent vector $\frac{\partial}{\partial x_i}\Big|_p$ by

$$(\frac{\partial}{\partial x_i}\Big|_p) f = \frac{f\phi^{-1}}{\partial r_i}\Big|_{\phi(p)} \tag{1.3.1}$$

for each C^∞ functions defined on a neighborhood of p. Then one can show that these tangent vectors are linearly independent and span T_pM providing a basis.

Now let $\psi : M \to N$ be C^∞ and let $p \in M$. The differential of ψ at p is the linear map

$$d\psi : T_pM \to T_{\psi(p)}N \tag{1.3.2}$$

defined as follows. Recall T_pM is the collection of all linear derivations of the algebra $\mathbf{F}_p$. Pick $v \in T_pM$. We define $d\psi(v)$ on a germ g at $\psi(p)$ by setting

$$d\psi(v)(g) = v(g\psi). \tag{1.3.3}$$

With this definition, $d\psi$ becomes a linear map from T_pM to $T_{\psi(p)}N$ that also satisfies the chain rule:

$$d(\phi\psi)_p = d\phi_{\psi(p)}d\psi_p \tag{1.3.4}$$

for $\psi : M \to N$ and $\phi : N \to X$ C^∞ maps. All the tangent spaces T_pM glue to the tangent bundle $TM = \bigcup_{p\in M} T_pM$ which is an example of a smooth real vector bundle.

Definition 1.3.8. *Let E and M be differentiable manifolds. A C^∞ map $\pi : E \to M$ is called a (real) vector bundle of rank r if the following conditions are satisfied.*

i) $E_p = \pi^{-1}(p)$, for $p \in M$ is a real vector space of dimension r and is called the fiber at p;
ii) for every $p \in M$, there is a neighborhood U of p and a diffeomorphism

$$\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r \tag{1.3.5}$$

such that $\phi(E_p) \subset \{p\} \times \mathbb{R}^r$ and ϕ restricted to E_p is an $\mathbb{R}$-vector space isomorphism between E_p and $\{p\} \times \mathbb{R}^r$.

The pair (U, ϕ) is called a local trivialization of the vector bundle $\pi : E \to M$, with total space E and base space M. Given two local trivializations (U_α, ϕ_α) and (U_β, ϕ_β), the map

$$\phi_\alpha\phi_\beta^{-1} : (U_\alpha \cap U_\beta) \times \mathbb{R}^r \to (U_\alpha \cap U_\beta) \times \mathbb{R}^r \tag{1.3.6}$$

induces

$$g_{\alpha\beta} : (U_\alpha \cap U_\beta) \to GL(r, \mathbb{R}) \tag{1.3.7}$$

given by

$$g_{\alpha\beta}(m) = \phi_\alpha|_{E_m}(\phi_\beta|_{E_m})^{-1} : \mathbb{R}^r \to \mathbb{R}^r. \tag{1.3.8}$$

These $g_{\alpha\beta}$ are called the transition functions (or the cocycles) of the vector bundle $\pi : E \to M$ and satisfy the cocycle property

$$g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} = \mathrm{id}_r \quad \text{on} \quad U_\alpha \cap U_\beta \cap U_\gamma, \tag{1.3.9}$$

where the product is pointwise matrix product. One can moreover recover the vector bundle from these cocycle data.

As mentioned above, the tangent bundle is the disjoint union of the various tangent spaces

$$TM = \bigcup_p T_pM \tag{1.3.10}$$

and the projection is the one that collapses whole of T_pM to p. For the local trivializations one considers the coordinate systems (U_α, ϕ_α) and notes that for each $p \in U_\alpha$, T_pU_α is canonically identified with $\mathbb{R}^n$. Thus TU_α is canonically identified with $U_\alpha \times \mathbb{R}^n$, giving a trivialization for the tangent bundle.

The algebraic operations for vector spaces to produce new ones from given ones can be carried through for vector bundles also. Essentially, one does the constructions fiberwise and then patches them up with the help of the transition functions. For example, the tensor product construction is carried through as follows: given bundles E and F,

$$E \otimes F = \bigcup_m E_m \otimes F_m, \tag{1.3.11}$$

and the transition functions are given by $g^{E\otimes F}_{\alpha\beta} = g^E_{\alpha\beta} \otimes g^F_{\alpha\beta}$. More generally, one can form new vector bundles using continuous functors on the category of finite dimensional vector spaces. The most important examples in our case will be the Λ^k-the exterior product, S^k-the symmetric product and the dual $(-)^*$. We apply the dual functor to the tangent bundle and obtain the cotangent bundle T^*M. Applying the exterior and symmetric product functors to the cotangent bundle we get, respectively, the k-th exterior bundle $\bigwedge^k T^*M$, the k-th symmetric bundle S^kT^*M.

In order to define metric and differential forms, one needs the fundamental notion of a section of a vector bundle.

Definition 1.3.9. *A C^∞ section of a vector bundle $\pi : E \to M$ is a C^∞ map $s : M \to E$ such that*

$$\pi s = \mathrm{id}_M, \tag{1.3.12}$$

i.e., s maps a point p in the base M to a vector $s(p) \in E_p$, the fiber over p. We denote the set of all sections of a vector bundle $\pi : E \to M$ by $\Gamma(M, E)$, or sometimes simply by $\Gamma(E)$.

Since each fiber E_p is a vector space, one can define addition of two sections pointwise. Similarly, one can define scalar multiplication. What is more fundamental is that one can multiply a C^∞ function and a section pointwise giving $\Gamma(E)$ a $C^\infty(M)$-module structure.

A section X of the tangent bundle TM is called a smooth vetor field on M, a section of the exterior bundle Λ^kT^*M is called a differential k-form on M. One usually denotes $\Gamma(\Lambda^kT^*M)$ by $\Omega^k(M)$.

One may even consider local sections. Let U be an open set in M. A C^∞ map $s : U \to E$ is a section over U of the vector bundle E if πs is the identity on U. A collection of sections $s_1, \ldots, s_r$ over an open set U in M is a frame on U if for every point p in U, $s_1(p), \ldots, s_r(p)$ forms a basis of the vector space $E_p = \pi^{-1}(p)$.

Recall that on a coordinate system (U, ϕ), $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$ forms a trivialization (or a local frame over U) of TU. Let $(dx_1, \ldots, dx_n)$ be the pointwise-dual trivialization of T^*U. Then any k-form can be written locally, i.e., on a such a coordinate system as

$$\omega = \sum_{|I|=p}{}' f_I dx_I, \tag{1.3.13}$$

where f_I are local C^∞ functions, $I = (i_1, \ldots, i_k)$, $|I|$ = number of indices, $dx_I = dx_{i_1} \wedge \ldots \wedge dx_{i_p}$ and $\sum'$ signifies that the sum is taken over strictly increasing indices. Using this local description, one defines the exterior product

$$\Omega^k(M) \times \Omega^l(M) \to \Omega^{k+l}(M), \quad (\omega, \eta) \mapsto \omega \wedge \eta, \tag{1.3.14}$$

the exterior differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ which satisfies $d^2 = 0$ together with the Leibniz rule

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta, \tag{1.3.15}$$

and a differentiable $f : M \to N$, the pull-back map

$$f^* : \Omega^k(N) \to \Omega^k(M) \tag{1.3.16}$$

Definition 1.3.10. *The de Rham cohomology of a differentiable manifold M is defined as*

$$H^k(M, \mathbb{R}) = \frac{\ker(d : \Omega^k(M) \to \Omega^{k+1}(M))}{\mathrm{im}(d : \Omega^{k-1}(M) \to \Omega^k(M))}. \tag{1.3.17}$$

If M is compact and oriented, i.e., Λ^nT^*M is trivial and a trivializing section (a global nowhere vanishing smooth n-form) has been chosen up to scaling by positive functions, then "integration" yields a linear map

$$\int_M : H^n(M, \mathbb{R}) \to \mathbb{R}, \quad [\omega] \mapsto \int_M \omega. \tag{1.3.18}$$

Riemannian manifolds and Hodge theory. We start with a definition.

DEFINITION 1.3.11. *A Riemannian manifold is a manifold endowed with a positive definite symmetric bilinear form g on each T_pM for $p \in M$ such that for each coordinate system $(U, x_1, \ldots, x_n)$, the functions $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$ are smooth.*

One can endow $\Lambda^k T^*M$ with natural inner product using the metric $\langle , \rangle$. If M is orientable, then one has a unique n-form, the volume form $\mathrm{vol} = \mathrm{vol}_{(M,g)}$, which is of norm one and positively oriented at every point. The Gram-Schmidt process applied fibre-wise produces local orthonormal frames for all these bundles. We shall usually denote a local orthonormal frame of T^*M by $(e_1, \ldots, e_n)$.

Let (M, g) be an oriented Riemannian manifold of dimension n. Using the metric and the orientation one introduces the Hodge $\star$-operator

$$\star : \Omega^k(M) \to \Omega^{n-k}(M). \tag{1.3.19}$$

The form $\star 1$ is the volume form $\mathrm{vol}_{(M,g)}$. The adjoint d^* of the exterior differential d is given by

$$d^* := (-1)^{n(k+1)+1} \star d \star : \Omega^k(M) \to \Omega^{k-1}(M), \tag{1.3.20}$$

and the Laplace operator is

$$\Delta := d^*d + dd^*. \tag{1.3.21}$$

Since d^* has degree -1, Δ has degree 0, i.e., Δ induces endomorphism of each $\Omega^k(M)$. Since the metric g induces natural inner product on any fibre $\Lambda^k T_p^*M$ for all $p \in M$, one can introduce an inner product on the space of global k-forms whenever M is compact.

DEFINITION 1.3.12. *If (M, g) is a compact oriented Riemannian manifold then for $\omega, \eta \in \Omega^k(M)$ one defines*

$$\langle \omega, \eta \rangle = \int_M g(\omega, \eta) \,\mathrm{vol}_{(M,g)} = \int_M \omega \wedge \star\eta. \tag{1.3.22}$$

Using this inner product, one can prove

LEMMA 1.3.13. *If M is compact, then*

$$\langle d\omega, \eta \rangle = \langle \omega, d^*\eta \rangle \quad \text{ and } \langle \Delta\omega, \eta \rangle = \langle \omega, \Delta\eta \rangle, \tag{1.3.23}$$

i.e., d^* is the adjoint of the operator d and Δ is self-adjoint.

We now recall what harmonic forms are.

DEFINITION 1.3.14. *A form $\omega \in \Omega^k(M)$ is harmonic if $\Delta(\omega) = 0$. The space of all harmonic k-forms is denoted by $\mathcal{H}^k(M, g)$.*

On a compact oriented Riemannian manifold (M, g), one can show, that a form ω is harmonic, i.e., $\Delta(\omega) = 0$ if and only if $d\omega = d^*\omega = 0$. As a corollary, one gets that the natural map

$$\mathcal{H}^k(M, g) \to H^k(M, \mathbb{R}) \tag{1.3.24}$$

that associates a harmonic form to its cohomology class is injective. We also note that $\Delta\star = \star\Delta$, and $\star : \mathcal{H}^k(M, g) \cong \mathcal{H}^{n-k}(M, g)$. All of these together with some hard analysis culminate into the Hodge-decomposition theorem.

THEOREM 1.3.15. *Let (M, g) be a compact oriented Riemannian manifold. Then, with respect to the inner product $\langle , \rangle$ there exists an orthogonal decomposition:*

$$\Omega^k(M) = d(\Omega^{k-1}(M)) \oplus \mathcal{H}^k(M, g) \oplus d^*(\Omega^{k+1}(M)). \tag{1.3.25}$$

Moreover, the space of harmonic forms $\mathcal{H}^k(M, g)$ is finite dimensional.

As a corollary, the natural map $\mathcal{H}^k(M, g) \to H^k(M, \mathbb{R})$ is an isomorphism, i.e., every cohomology class has a unique harmonic representative.

1.3.2. Essentials of complex geometry. Here we collect some basic definitions and facts from complex geometry. References are [**Wel08**, **Huy05**, **GH94**].

Complex manifolds and holomorphic vector bundles.

DEFINITION 1.3.16. *A holomorphic atlas on a differentiable manifold is an atlas* $\{(U_i, \phi_i)\}$ *of the form* $\phi_i : U_i \cong \phi_i(U_i) \subset \mathbb{C}^n$ *such that the transition functions* $\phi_{ij} = \phi_i\phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ *are holomorphic. The pair* (U_i, ϕ_i) *is called a holomorphic chart. Two holomorphic atlases* $\{(U_i, \phi_i)\}$, $\{(U_i', \phi_i')\}$ *are called equivalent if all the maps* $\phi_i {\phi_j'}^{-1} : \phi_j'(U_i \cap U_j') \to \phi_i(U_i \cap U_j')$ *are holomorphic.*

DEFINITION 1.3.17. *A complex manifold* X *of dimension* n *is a (real) differentiable manifold of dimension* $2n$ *endowed with an equivalence class of holomorphic atlases.*

We now define holomorphic functions on a complex manifold.

DEFINITION 1.3.18. *A holomorphic function on a complex manifold* X *is a function* $f : X \to \mathbb{C}$ *such that* $f\phi_i^{-1} : \phi_i(U_i) \to \mathbb{C}$ *is holomorphic for any chart* (U_i, ϕ_i) *of a holomorphic atlas in the equivalence class of defining* X.

DEFINITION 1.3.19. *Let* X *and* Y *be two complex manifolds. A continuous map* $f : X \to Y$ *is a holomorphic map if for any holomorphic charts* (U, ϕ) *and* (U', ϕ') *of* X *and* Y*, respectively, the map* $\phi' f \phi^{-1} : \phi(f^{-1}(U') \cap U) \to \phi'(U')$ *is holomorphic. Two complex manifolds* X *and* Y *are called biholomorphic if there exists a holomorphic homeomorphism* $f : X \to Y$.

EXAMPLE 1.3.20. *The most basic complex manifold is provided by the* n*-dimensional complex space* $\mathbb{C}^n$*. The open subsets of* $\mathbb{C}^n$ *serve as the local model for arbitrary complex manifolds.*

EXAMPLE 1.3.21. *The complex projective space* $\mathbb{C}P^n$ *is a compact complex manifold.*

EXAMPLE 1.3.22. *The complex torus* $\mathbb{C}^n/\mathbb{Z}^{2n}$ *is a complex manifold, where* $\mathbb{Z}^{2n} \subset \mathbb{R}^{2n} = \mathbb{C}^n$ *is the natural inclusion.*

Next we define holomorphic vector bundles.

DEFINITION 1.3.23. *Let* X *be a complex manifold. A holomorphic vector bundle of rank* r *on* X *is a complex manifold* E *with a holomorphic map* $\pi : E \to X$ *and the structure of an* r*-dimensional complex vector space on any fiber* $E_x = \pi^{-1}(x)$ *satisfying the following condition: there exists an open covering* $X = \bigcup U_i$ *and biholomorphic maps* $\psi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^r$ *commuting with the projections to* U_i *such that the induced map* $\pi^{-1}(x) \cong \mathbb{C}^r$ *is* $\mathbb{C}$*-linear.*

Let $X = \bigcup U_i$ be an open covering by charts $\phi_i : U_i \to \phi_i(U_i) \subset \mathbb{C}^n$. By definition the Jacobian of the transition maps $\phi_{ij} = \phi_i\phi_j^{-1} : \phi_j(U_i \cap U_j) \to \phi_i(U_i \cap U_j)$ is the matrix $J(\phi_{ij})(\phi_j(z)) = \left(\frac{\partial \phi_{ij}^k}{\partial z_l}(\phi_j(z))\right)_{k,l}$.

DEFINITION 1.3.24. *The holomorphic tangent bundle of a complex manifold* X *of dimension* n *is the holomorphic vector bundle* $\mathcal{T}_X$ *on* X *of rank* n *which is given by the transition functions* $\psi_{ij}(z) = J(\phi_{ij})(\phi_j(z))$.

The holomorphic cotangent bundle Ω_X *is the dual of* $\mathcal{T}_X$*. The bundle of holomorphic* p*-forms is* $\Omega_X^p = \bigwedge^p \Omega_X$ *for* $0 \leq p \leq n$ *and* $K_X = det(\Omega_X) = \bigwedge^n \Omega_X$ *is called the canonical line bundle of* X.

Now we describe an alternative way of defining complex manifolds that leads to noncommutative complex geometry.

DEFINITION 1.3.25. *An almost complex manifold is a differentiable manifold* X *together with a vector bundle endomorphism* $J : TX \to TX$ *with* $J^2 = -\mathrm{id}$*. Here,* TX *is the real tangent bundle of the underlying real manifold.*

The endomorphism is also called the almost complex structure on the underlying differentiable manifold. If an almost complex structure exists, then the real dimension of X is even.

Proposition 1.3.26. *Any complex manifold X admits a natural almost complex structure.*

Now let X be an almost complex manifold. Then $T_{\mathbb{C}}X$ denotes the complexification of TX, i.e., $T_{\mathbb{C}}X = TX \otimes \mathbb{C}$.

Proposition 1.3.27. *i) Let X be an almost complex manifold. Then there exists a direct sum decomposition*

$$T_{\mathbb{C}}X = T^{1,0}X \oplus T^{0,1}X \tag{1.3.26}$$

of complex vector bundles on X, such that the $\mathbb{C}$-linear extension of J acts as multiplication by i on $T^{1,0}X$ and by $-i$ on $T^{0,1}X$.

ii) If X is a complex manifold, then $T^{1,0}X$ is naturally isomorphic (as a complex vector bundle) to the holomorphic tangent bundle $\mathcal{T}_X$.

The bundles $T^{1,0}X$ and $T^{0,1}X$ are called the holomorphic, respectively, anti-holomorphic tangent bundles of the almost complex manifold X. One defines the complex vector bundles

$$\bigwedge_{\mathbb{C}}^{k} X = \bigwedge^{k}(T_{\mathbb{C}}X)^*, \quad \bigwedge^{p,q} X = \bigwedge^{p} T^{1,0}X \otimes_{\mathbb{C}} \bigwedge^{q} T^{0,1}X. \tag{1.3.27}$$

The space of sections are denoted by $\Omega^k_{\mathbb{C}}(X)$, and $\Omega^{p,q}(X)$, respectively. Elements in $\Omega^{p,q}(X)$ are called forms of type (p,q). We denote the projections $\Omega(X) \to \Omega^k(X)$ and $\Omega(X) \to \Omega^{p,q}(X)$ by Π^k and $\Pi^{p,q}$, respectively.

Proposition 1.3.28. *There exists a natural direct sum decomposition*

$$\bigwedge_{\mathbb{C}}^{k} X = \bigoplus_{p+q=k} \bigwedge^{p,q} X, \quad \Omega^k_{\mathbb{C}}(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X). \tag{1.3.28}$$

Moreover, $\overline{\bigwedge^{p,q} X} = \bigwedge^{q,p} X$ and $\overline{\Omega^{p,q}(X)} = \Omega^{q,p}(X)$, where the bar denotes complex conjugation.

Definition 1.3.29. *Let X be an almost complex manifold. If $d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)$ is the $\mathbb{C}$-linear extension of the exterior differential, then one defines*

$$\partial = \Pi^{p+1,q} d : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X), \tag{1.3.29}$$

and

$$\overline{\partial} = \Pi^{p,q+1} d : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X). \tag{1.3.30}$$

It can be shown that ∂ and $\overline{\partial}$ satisfy Leibniz rule. We next discuss what an almost complex structure needs in order to be induced by a complex one.

Proposition 1.3.30. *Let X be an almost complex manifold. Then the following two conditions are equivalent.*

i) $d(\omega) = \partial(w) + \overline{\partial}(\omega)$ for $\omega \in \Omega(X)$;

ii) On $\Omega^{1,0}$ one has $\Pi^{0,2} d = 0$.

Both conditions hold true if X is a complex manifold.

Definition 1.3.31. *An almost complex structure J on X is called integrable if the conditions of Proposition 1.3.30 are satisfied.*

We have another characterization of integrability of an almost complex structure.

Proposition 1.3.32. *An almost complex structure J is integrable if and only if the Lie brackets of the vector fields preserves $T^{0,1}X$, i.e., $[T^{0,1}X, T^{0,1}X] \subset T^{0,1}X$.*

As a corollary to this we have,

Corollary 1.3.33. *If I is an integrable almost complex structure then $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$. Conversely if $\overline{\partial}^2 = 0$ then J is integrable.*

The following theorem answers the question raised above.

THEOREM 1.3.34 (Newlander-Nirenberg). *Any integrable almost complex structure is induced by a complex structure.*

Thus a complex manifold and a differentiable manifold with an integrable almost complex structure are describing the same object.

Let us denote by $\Omega^{p,q}(E)$ the space of E-valued forms of type (p,q) for a complex vector bundle over a complex manifold X.

LEMMA 1.3.35. *If E is holomorphic vector bundle then there exists a natural $\mathbb{C}$-linear operator $\overline{\partial}_E : \Omega^{p,q}(E) \to \Omega^{p+1,q}(E)$ with $\overline{\partial}_E^2 = 0$ and which satisfies the Leibniz rule $\overline{\partial}_E(f \cdot \omega) = \overline{\partial}(f) \wedge \omega + f\overline{\partial}_E(\omega)$.*

The following is known as the Koszul-Malgrange theorem.

THEOREM 1.3.36. *Let E be a complex vector bundle over a complex manifold X. A holomorphic structure on E is uniquely determined by a $\mathbb{C}$-linear operator $\overline{\partial}_E : \Omega^0(E) \to \Omega^{0,1}(E)$ satisfying the Leibniz rule and the integrability condition $\overline{\partial}_E^2 = 0$.*

Kähler manifolds and Hodge theory. Let X be a complex manifold and let J be the induced almost complex structure. Recall the definition (1.3.11) of a Riemannian metric,

DEFINITION 1.3.37. *A Riemannian metric g on X is a hermitian structure on X if for any point $x \in X$ the scalar product g_x on T_xX is compatible with the almost complex structure J_x, i.e., $g_x(J_x(\cdot), J_x(\cdot)) = g_x(\cdot,\cdot)$.*

We call the form $\omega = g(J(\cdot),\cdot)$, the fundamental form associated to g.

LEMMA 1.3.38. *The fundamental form ω is real and of type $(1,1)$.*

Locally the fundamental form is of the form

$$\omega = \frac{i}{2} \sum_{i,j=1}^{n} h_{ij} dz_i \wedge \overline{dz_j}. \tag{1.3.31}$$

The complex manifold endowed with a hermitian structure g is called a hermitian manifold. We note that the hermitian structure is completely determined by the almost complex structure and the fundamental form. Indeed, $g(\cdot,\cdot) = \omega(\cdot, J(\cdot))$. One defines the following vector bundle homomorphisms on any hermitian manifold of complex dimension n:

i) The Lefschetz operator

$$L : \bigwedge^{k} X \to \bigwedge^{k+2} X, \quad \alpha \mapsto \alpha \wedge \omega, \tag{1.3.32}$$

is an operator of degree 2.

ii) The Hodge $\star$-operator (see (1.3.19))

$$\star : \bigwedge^{k} X \to \bigwedge^{2n-k} X, \tag{1.3.33}$$

is induced by the metric g and the natural orientation of the complex manifold X, of real dimension $2n$. It is a fact that the Hodge $\star$-operator maps $\Omega^{p,q}(X)$ to $\Omega^{n-q,n-p}(X)$.

iii) The dual Lefschetz operator

$$\Lambda = \star^{-1} L \star : \bigwedge^{k} X \to \bigwedge^{k-2} X, \tag{1.3.34}$$

is an operator of degree -2 and depends on the Kähler form ω and the metric g and hence on the almost complex structure J.

All three operators can be extended $\mathbb{C}$-linearly to the complexified bundles $\bigwedge_{\mathbb{C}} X$, which we again denote by the same symbols L, $\star$ and Λ.

PROPOSITION 1.3.39. *Let (X,g) be a hermitian manifold. Then there exists a direct sum decomposition of vector bundles*

$$\bigwedge^k X = \bigoplus_{i\geq 0} L^i(P^{k-2i}X), \tag{1.3.35}$$

where $P^{k-2i}X = \ker(\Lambda : \bigwedge^{k-2i} X \to \bigwedge^{k-2i-2} X)$ is the bundle of primitive forms.

The decomposition is compatible with the bidegree decomposition $\bigwedge_{\mathbb{C}}^k X = \bigoplus_{p+q=k} \bigwedge^{p,q} X$ and one has $P_{\mathbb{C}}^k X = \bigoplus_{p+q=k} P^{p,q}X$, where $P^{p,q}X = P_{\mathbb{C}}^{p+q} \cap \bigwedge^{p,q} X$. Define the operators H and I:

$$H = \sum_{k=0}^{2n}(k-n)\Pi^k, \quad \mathrm{I} = \sum_{p,q=0}^{n} i^{p-q}\Pi^{p,q}, \tag{1.3.36}$$

where Π^k and $\Pi^{p,q}$, are the natural projections $\Omega(X) \to \Omega^k(X)$ and $\Omega(X) \to \Omega^{p,q}(X)$, respectively. Using the bundle of primitive forms, the Hodge $\star$-operator can be given an explicit form, known as Weil's formula:

$$\star\,(L^j(\omega)) = (-1)^{\frac{k(k+1)}{2}} \frac{j!}{(n-k-j)!} L^{n-k-j}\,\mathrm{I}(\omega), \tag{1.3.37}$$

for $\omega \in P^k$.

Recall that on an arbitrary m-dimensional Riemannian manifold (M,g) the adjoint operator d^* (see (1.3.20)) is defined as

$$d^* = (-1)^{m(k+1)+1} \star d\star : \Omega^k(M) \to \Omega^{k+1}(M), \tag{1.3.38}$$

and the Laplace operator is given by

$$\Delta_d = dd^* + d^*d. \tag{1.3.39}$$

If the dimension of M is even, for instance, if M admits a complex structure, then $d^* = -\star d\star$. Analogously, one defines ∂^* and $\overline{\partial}^*$ as follows.

DEFINITION 1.3.40. *If (X,g) is a hermitian manifold, then*

$$\partial^* = -\star\overline{\partial}\star, \quad \overline{\partial}^* = -\star\partial\star. \tag{1.3.40}$$

∂^* takes $\Omega^{p,q}(X)$ to $\Omega^{p-1,q}(X)$ and similarly, $\overline{\partial}^*$ takes $\Omega^{p,q}(X)$ to $\Omega^{p,q-1}(X)$. Moreover, from the decomposition $d = \partial + \overline{\partial}$, it follows that $d^* = \partial^* + \overline{\partial}^*$ and $\partial^{*2} = \overline{\partial}^{*2} = 0$. For a hermitian manifold (X,g), one defines the ∂- and $\overline{\partial}$-Laplacians Δ_∂ and $\Delta_{\overline{\partial}}$ as follows:

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \quad \Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*. \tag{1.3.41}$$

DEFINITION 1.3.41. *A Kähler structure is a hermitian structure g for which the fundamental form ω is closed, i.e., $d\omega = 0$. In this case the fundamental form is called the Kähler form.*

The complex manifold endowed with the Kähler structure ω is called a Kähler manifold. Complex projective spaces are Kähler, and by a theorem which says any complex submanifold of a Kähler manifold is Kähler, any projective manifold is Kähler. Obviously, a Riemann surface is Kähler. We now note some remarkable properties of Kähler manifolds. We start with the Kähler identities.

PROPOSITION 1.3.42. *Let X be a complex manifold, endowed with a Kähler metric g. Then the following identities hold true.*

i) $[\overline{\partial}, L] = [\partial, L] = 0$ *and* $[\overline{\partial}^*, \Lambda] = [\partial^*, \Lambda] = 0$.
ii) $[\overline{\partial}^*, L] = i\partial$, $[\partial^*, L] = -i\overline{\partial}$ *and* $[\Lambda, \overline{\partial}] = -i\partial^*$, $[\Lambda, \partial] = i\overline{\partial}^*$.
iii) $\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d$ *and* Δ_d *commutes with* $\star$, L, Λ, ∂, $\overline{\partial}$, ∂^*, $\overline{\partial}^*$.

If X is a complex manifold with a hermitian structure g, we denote the hermitian extension of the Riemannian metric by $g_{\mathbb{C}}$. It naturally induces hermitian products on all form bundles. See also Definition 1.3.12

Definition 1.3.43. *Let (X, g) be a compact hermitian manifold. Then one defines a hermitian product on $\Omega_{\mathbb{C}}(X)$ by*

$$\langle \omega, \eta \rangle = \int_X g_{\mathbb{C}}(\omega, \eta) \star 1. \tag{1.3.42}$$

Proposition 1.3.44. *Let (X, g) be a compact hermitian manifold. Then the following decompositions are orthogonal with respect to $\langle , \rangle$.*

i) The degree decomposition $\Omega_{\mathbb{C}}(X) = \oplus_k \Omega^k_{\mathbb{C}}(X)$;
ii) The bidegree decomposition $\Omega^k_{\mathbb{C}}(X) = \oplus_{p+q=k} \Omega^{p,q}(X)$;
iii) The Lefschetz decomposition $\Omega^k_{\mathbb{C}}(X) = \oplus_{i \geq 0} L^i P^{k-2i}_{\mathbb{C}}(X)$

Moreover, with respect to $\langle , \rangle$, the operators ∂^*, $\overline{\partial}^*$ are formal adjoints to ∂ and $\overline{\partial}$, respectively. For a hermitian manifold, one defines a form $\omega \in \Omega^k(X)$ to be $\overline{\partial}$-harmonic if $\Delta_{\overline{\partial}}(\omega) = 0$ and the space of harmonic forms are defined as

$$\mathcal{H}^k_{\overline{\partial}}(X, g) = \{\omega \in \Omega^k_{\mathbb{C}}(X) \mid \Delta_{\overline{\partial}}(\omega) = 0\} \tag{1.3.43}$$

and

$$\mathcal{H}^{p,q}_{\overline{\partial}}(X, g) = \{\omega \in \Omega^{p,q}(X) \mid \Delta_{\overline{\partial}}(\omega) = 0\}. \tag{1.3.44}$$

Similarly one considers the case for ∂. For X compact, being $\overline{\partial}$-harmonic is equivalent to the joint vanishing of $\overline{\partial}$ and $\overline{\partial}^*$ on that form. We can now state the Hodge decomposition theorem.

Theorem 1.3.45. *Let (X, g) be a compact hermitian manifold. Then there exist two natural orthogonal decompositions*

$$\Omega^{p,q}(X) = \partial \Omega^{p-1,q}(X) \oplus \mathcal{H}^{p,q}_{\partial}(X, g) \oplus \partial^* \Omega^{p+1,q}(X), \tag{1.3.45}$$

and

$$\Omega^{p,q}(X) = \overline{\partial} \Omega^{p,q-1}(X) \oplus \mathcal{H}^{p,q}_{\overline{\partial}}(X, g) \oplus \partial^* \Omega^{p,q+1}(X). \tag{1.3.46}$$

The spaces $\mathcal{H}^{p,q}(X)$ are finite dimensional. If (X, g) is assumed to be Kähler, then $\mathcal{H}^{p,q}_{\partial}(X, g) = \mathcal{H}^{p,q}_{\overline{\partial}}(X, g)$.

1.3.3. Differential operators on manifolds. As the name suggests, here we discuss differential operators, to motivate the next section, and to exemplify the unbounded operator theory reviewed in 1.2.4. We shall meet noncommutative versions in the following chapters, so it seems useful to have the classical picture reviewed. A wonderful reference for these material is [**HR00**].

First-order differential operators. Recall the notion of a differentiable manifold and of a vector bundle from 1.3.1.

Definition 1.3.46. *Let M be a differentiable manifold and S be a smooth complex vector bundle over M. A first-order linear differential operator on S is $\mathbb{C}$-linear map*

$$D : \Gamma(M, S) \to \Gamma(M, S) \tag{1.3.47}$$

satisfying the following properties:

i) if u_1 and u_2 are two smooth sections of S which agree on an open set $U \subset M$, then Du_1 and Du_2 agree on U.
ii) if we choose a coordinate system $(U, x_1, \ldots, x_n)$ of M which also trivializes the bundle S, then D can be represented in local coordinates by a formula

$$Du = \sum_j A_j \frac{\partial u}{\partial x_j} + Bu, \tag{1.3.48}$$

where A_j and B are smooth, matrix-valued functions on U.

The functions A_j and B depend on the particular coordinate system and the trivialization. But if $\xi = \sum_j \xi_j dx_j$ is a cotangent vector at $x \in M$, then the expression

$$\sigma_D(x,\xi) = \sum_j A_j \xi_j, \tag{1.3.49}$$

interpreted as an endomorphism of the fibre S_x is independent of the choice of the coordinates. In fact, if g is any smooth function on M and if $\rho(g)$ denotes the operator on $\Gamma(M,S)$ of multiplication by g, then

$$\sigma_D(x, dg)u(x) = ([D, \rho(g)]u)(x), \tag{1.3.50}$$

for any smooth section u.

Definition 1.3.47. *The symbol of a (first-order) differetial operator $D : \Gamma(M,S) \to \Gamma(M,S)$ is the vector bundle homomorphism*

$$\sigma_D : T^*M \to \mathrm{End}(S), \tag{1.3.51}$$

given by (1.3.49).

We shall, from now on assume that M is a compact oriented Riemannian manifold (M,g) and that the bundle admits a hermitian metric $(\,,)$, i.e., smoothly varying inner products on the complex fibres (see Definition 1.3.37). We can then define an inner product on the space $\Gamma(M,S)$ by

$$\langle u, v\rangle = \int_M (u(x), v(x)) \,\mathrm{vol}_{(M,g)}, \tag{1.3.52}$$

and obatain the Hilbert space $L^2(M,S)$ by completing $\Gamma(M,S)$ with respect to the inner product (1.3.52).

Proposition 1.3.48. *Let $D : \Gamma(M,S) \to \Gamma(M,S)$ be a first-order differential operator. Then there is a unique first-order differential operator $D^\dagger : \Gamma(M,S) \to \Gamma(M,S)$, called the formal adjoint of D, such that*

$$\langle Du, v\rangle = \langle u, D^\dagger v\rangle, \tag{1.3.53}$$

for all $u, v \in \Gamma(M,S)$. The symbols of D and $D^\dagger$ are related in the following way:

$$\sigma_{D^\dagger}(x,\xi) = \sigma_D(x,\xi)^*. \tag{1.3.54}$$

We now think of D as an unbounded operator on $L^2(M,S)$ densely defined with domain $\Gamma(M,S)$, in the sense of Definition1.2.31. Recall also the notion of closability from the discussion before Definition 1.2.33.

Proposition 1.3.49. *Every first-order differential operator is closable.*

By an abuse of notation, we shall write D instead of the closure $\overline{D}$. We see that a first-order differential operator D is symmetric, i.e., $\langle Du, v\rangle = \langle u, Dv\rangle$ if and only if D equals its formal adjoint. One can show that every symmetric first-order differential operator is essentially self-adjoint, see the discussion before Theorem 1.2.35.

Finally, we note that if D is a self-adjoint first-order differential operator and $\rho(g) \in B(L^2(M,S))$ is the multiplication operator by a smooth function g, then $[D, \rho(g)]$ is also a multiplication operator on $\Gamma(M,S)$, because of (1.3.50).

Remark 1.3.50. *In particular, $[D, \rho(g)]$ extends to a bounded operator on $L^2(M,S)$.*

Dirac operators and spin manifolds. We begin by introducing some terminology regarding $\mathbb{Z}_2$-grading. We recall that a complex vector space V is said to be $\mathbb{Z}_2$-graded if there is a decomposition

$$V = V^+ \oplus V^- \tag{1.3.55}$$

of V into a direct sum of subspaces. $V^\pm$ are called the positive and negative parts, respectively. The grading operator γ_V is the operator whose ± 1-eigenspaces are $V^\pm$, respectively. A graded Hilbert space is a Hilbert space provided with a $\mathbb{Z}_2$-grading for which the positive and negative parts are closed, orthogonal subspaces. A graded vector bundle is a vector bundle whose fibres are graded vector spaces such that the transition functions respect the grading. An endomorphism of a graded vector space (Hilbert space, vector bundle) is even if it commutes with the grading operator and odd if it anti-commutes with the grading operator.

DEFINITION 1.3.51. *Let M be a Riemannian manifold and let S be a smooth, graded hermitian vector bundle over M. A Dirac operator on S is an odd, symmetric first-order differential operator on S whose symbol $\sigma_D(x,\xi)$ has the property that*

$$\sigma_D(x,\xi)^2 u = -\|\xi\|^2 u, \tag{1.3.56}$$

for every $x \in M$, $\xi \in T_x^*M$ and $u \in S_x$.

If $\sigma_D(x,\xi)$ is the symbol of a Dirac operator then it follows from the symmetry and grading of D that $\sigma_D(x,\xi)$ is a skew-adjoint, odd endomorphism of S_x for all x and ξ. The condition $\sigma_D(x,\xi)^2 u = -\|\xi\|^2 u$ implies that $\sigma_D(x,\xi)$ is an invertible endomorphism of S_x whenever $\xi \neq 0$. A differential operator with this property is called elliptic. Thus Dirac operators are elliptic.

DEFINITION 1.3.52. *Let M be a Riemannian manifold. A Dirac bundle on M is a graded hermitian vector bundle S on M together with an $\mathbb{R}$-linear morphism of vector bundles*

$$T^*M \to \mathrm{End}(S) \tag{1.3.57}$$

which associates to each cotangent vector $\xi \in T_x^*M$ a skew-adjoint, odd endomorphism $u \mapsto \xi \cdot u$ of S_x whose square is multiplication by the scalar $-\|\xi\|^2$ on S_x.

The action of a vector $\xi \in T_x^*M$ on S_x will be referred to as Clifford multiplication by ξ. The relation $\xi^2 = -\|\xi\|^2 1$ is the Clifford relation, which we shall describe in a moment.

Thus there is a one-one correspondence between Dirac bundle structures on a graded hermitian vector bundle S and equivalence classes of Dirac operators on S, two Dirac operators being equivalent if they have the same symbols. The following examples are from the last two Subsections 1.3.1 and 1.3.2.

EXAMPLE 1.3.53. *Recall from Subsection 1.3.1, that on any smooth manifold M of dimension n, one has the de Rham complex*

$$\Omega^0_{\mathbb{C}}(M) \xrightarrow{\quad d \quad} \Omega^1_{\mathbb{C}}(M) \xrightarrow{\quad d \quad} \dots \xrightarrow{\quad d \quad} \Omega^n_{\mathbb{C}}(M) \tag{1.3.58}$$

where $\Omega^k_{\mathbb{C}}(M)$ is the space of smooth, complex-valued k-forms on M and d is the exterior differential. Now suppose that M is Riemannian and let S be the complex exterior algebra bundle $\Lambda T^*_{\mathbb{C}}M$, whose sections are complex-valued differential forms on M. One grades S by dividing the differential forms on M into those of odd and even degree. Let d be the exterior differential thought of as a first-order differential operator on S and let d^* be its formal adjoint, see (1.3.20) *and Lemma 1.3.13. Then the de Rham operator $D = d + d^*$ is a Dirac operator on S. The symbol of D is given by*

$$\sigma_D(x,\xi) = E_\xi - E_{\xi^*}, \tag{1.3.59}$$

where E_ξ is the exterior multiplication by ξ.

EXAMPLE 1.3.54. *Let M be a hermitian manifold of complex dimension n, see Definition 1.3.37. We recall the operators ∂ and $\overline{\partial}$ from Definition 1.3.29 and note that they satisfy*

$$\partial^2 = \overline{\partial}^2 = 0, \tag{1.3.60}$$

see Corollary 1.3.33. Thus one has the following complex, called the Dolbeault complex

$$\Omega^{0,0}(M) \xrightarrow{\quad \overline{\partial} \quad} \Omega^{0,1}(M) \xrightarrow{\quad \overline{\partial} \quad} \dots \xrightarrow{\quad \overline{\partial} \quad} \Omega^{0,n}(M). \tag{1.3.61}$$

Let S be the exterior algebra bundle $\Lambda T^{0,1}M$ and we grade it the same way as in the previous example. Let $\overline{\partial}$ be thought of as a first-order differential operator on the bundle S and let $\overline{\partial}^*$ be its formal adjoint. Then the Dolbeault operator $\mathcal{D} = \overline{\partial} + \overline{\partial}^*$ is a Dirac operator on S and its symbol is given by

$$\sigma_{\mathcal{D}}(x, df) = E_{\overline{\partial} f} - E_{\overline{\partial} f^*}, \tag{1.3.62}$$

where E_ξ again denotes exterior multiplication by ξ.

Having described two basic examples of Dirac operators, we now move onto defining spin manifolds.

DEFINITION 1.3.55. *A p-multigraded Dirac operator is a Dirac operator D on a graded hermitian vector bundle S together with p odd endomorphisms $\varepsilon_1, \ldots, \varepsilon_p$ of S such that $\varepsilon_j D = D\varepsilon_j$ for all j and*

$$\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad \varepsilon_j\varepsilon_i + \varepsilon_i\varepsilon_j = 0, \quad (i \neq j). \tag{1.3.63}$$

Similarly, a p-multigraded Dirac bundle is a Dirac bundle S equipped with p odd endomorphisms $\varepsilon_1, \ldots, \varepsilon_p$ of S such that

$$\varepsilon_j = -\varepsilon_j^*, \quad \varepsilon_j^2 = -1, \quad \varepsilon_j\varepsilon_i + \varepsilon_i\varepsilon_j = 0, \quad (i \neq j). \tag{1.3.64}$$

and such that each ε_j commutes with every Clifford multiplication operator on every fibre S_x.

One can construct n-multigraded Dirac operators on an n-dimensional Riemannian manifold using the concept of Clifford algebras.

DEFINITION 1.3.56. *The complex Clifford algebra for $\mathbb{R}^n$ is the complex $*$-algebra $\mathbb{C}_n$ generated by elements $e_1, \ldots, e_n$ (corresponding to the standard orthonormal basis of $\mathbb{R}^n$) such that*

$$e_j = -e_j^*, \quad e_j^2 = -1 \quad (j = 1, \ldots, n) \tag{1.3.65}$$

and

$$e_ie_j + e_je_i = 0 \quad (i \neq j). \tag{1.3.66}$$

The algebra $\mathbb{C}_n$ is linearly spanned by the 2^n monomials $e_{j_1} \ldots e_{j_k}$, where $j_1 < \cdots < j_k$ and $0 \leq k \leq n$. One grades $\mathbb{C}_n$ by assigning to each monomial its degree modulo 2.

Now let M be a Riemannian manifold of dimension n. If $e_1, \ldots, e_n$ is a local orthonormal frame for T^*M over an open set $U \subset M$, then one can construct an n-graded Dirac bundle structure on the trivial bundle $U \times \mathbb{C}_n$ over U in the following way. Clifford multiplication by an element e_j of the frame is the left multiplication by the jth generator of $\mathbb{C}_n$, and the n-multigrading operators $\varepsilon_1, \ldots, \varepsilon_n$ for the bundle are right multiplication by the same generators. This is the local model of a globally defined Dirac bundle called a complex spinor bundle:

DEFINITION 1.3.57. *Let M be an n-dimensional Riemannian manifold. A complex spinor bundle on M is an n-multigraded Dirac bundle S which is locally isomorphic to the trivial bundle with fiber $\mathbb{C}_n$, the Clifford multiplication being determined by some local orthonormal frame as above.*

The real Clifford algebra $\mathbb{R}_n$ is the $\mathbb{R}$-subalgebra of $\mathbb{C}_n$ generated by the elements $e_1, \ldots, e_n$ and a real spinor bundle is a real Dirac bundle which is locally isomorphic to $U \times \mathbb{R}_n$. One can complexify a real spinor bundle and obtain a complex spinor bundle. A complex spinor bundle determines an orientation on M, therefore it can only exist on orientable manifolds. If one starts with an oriented Riemannian manifold then one only considers those complex spinor bundle for which the induced orientation agress with the orientation one started with.

For every complex spinor bundle S on a Riemannian manifold M there is a corresponding Dirac operator D. There is no canonical choice but all the choices contain the same amount of structure (K-homology class, to be precise). So one introduces an equivalence relation and calls one such class a Spin^c structure. For that we recall that for graded vector spaces V_1 and V_2 their graded tensor product $V_1 \otimes^{gr} V_2$ is the ordinary tensor product $V_1 \otimes V_2$ with the grading

$$V^+ = (V_1^+ \otimes V_2^+) \oplus (V_1^- \otimes V_2^-), \quad V^- = (V_1^+ \otimes V_2^-) \oplus (V_1^- \otimes V_2^+). \tag{1.3.67}$$

Graded tensor product of graded vector bundles are defined fibre-wise. Now there is a canonical complex spinor bundle on $\mathbb{R}$, namely, the trivial bundle $S_{\mathbb{R}} = \mathbb{R} \times \mathbb{C}_1$ for which the Dirac operator is

$$D_{\mathbb{R}} = e_1 \cdot \frac{d}{dx}. \tag{1.3.68}$$

If S is a complex spinor bundle on a Riemannian manifold M then $S \otimes^{gr} S_{\mathbb{R}}$ is a complex spinor bundle over $M \times \mathbb{R}$.

DEFINITION 1.3.58. *Let M and M' be two Riemannian manifolds with the same underlying smooth manifold, and let S and S' be complex spinor bundles on M and M', respectively. The pairs (M,S) and (M',S') are said to be concordant if there is a pair consisting of a Riemannian metric and a complex spinor bundle on $\mathbb{R}\times M$ which over some non-empty open interval of $\mathbb{R}$ agress with $(\mathbb{R}\times M, S_{\mathbb{R}}\otimes^{gr} S)$ and which over some other non-empty open interval of $\mathbb{R}$ agress with $(\mathbb{R}\times M', S_{\mathbb{R}}\otimes^{gr} S')$*

We now finally come to

DEFINITION 1.3.59. *A Spin^c-structure on a smooth manifold M is a concordance class of Riemannian metrics and complex spinor bundles on M. A Spin^c-manifold is a smooth manifold which is provided with a Spin^c-structure.*

Similarly, a Spin-structure on a smooth manifold M is a concordance class of Riemannian metics and real spinor bundles on M. A Spin-manifold is a smooth manifold which is provided with a Spin-structure. We should also add that we have followed [**HR00**] for this presentation. One can also introduce Spin-manifolds using the language of principal bundles, see [**Fri00**, **LM89**] and the wonderful [**BHMS07**] for an account of principal bundles from noncommutative geometry perspective. We end with an example.

PROPOSITION 1.3.60. *Let M be a Riemannian manifold of dimension $n = 2k$. There is a one-one correspondence between isomorphism classes of complex spinor bundles on M and isomorphism classes of Dirac bundles of rank 2^k.*

Thus we get

EXAMPLE 1.3.61. *Recall from Example 1.3.54, that the Dolbeault operator on a complex manifold of complex dimension n acts on a Dirac bundle of rank 2^n. Hence the Dolbeault operator determines a Spin^c-structure on M.*

1.4. Noncommutative geometry and quantum isometry groups

In this section, we collect the necessary background in noncommutative geometry. We also describe the theory of quantum isometry groups.

1.4.1. Spectral triples. We start by defining the basic objects of study in noncommutative geometry. References are [**GBVF01**, **Con94**, **Con85**, **Lan97**].

DEFINITION 1.4.1. *A spectral triple of compact type is a triple (A, H, D) consisting of:*

a) an associative algebra A of bounded operators on a Hilbert space H, and
b) an unbounded self-adjoint operator D on H such that
  i) for every $a \in A$, the operators $a(D \pm i)^{-1}$ are compact, and
  ii) for every $a \in A$, the operator $[D, a]$ is defined on $\mathrm{dom}(D)$ and extends to a bounded operator on H.

In b), D is self-adjoint in the sense of unbounded operator theory, see the discussion before Theorem 1.2.35. By the same theorem, $(D \pm i)$ map $\mathrm{dom}(D)$ bijectively onto H. In ii), it is assumed that each $a \in A$ maps $\mathrm{dom}(D)$ into itself. Also if the algebra A has a unit, which acts as the identity operator on the Hilbert space H, then i) is equivalent to the assertion that $(D \pm i)^{-1}$ be compact operators, which is equivalent to the assertion that there exist an orthonormal basis for H consisting of eigenvectors v_j of D, with eigenvalues λ_j converging to ∞ in absolute value.

One calls a spectral triple (A, H, D) even if the Hilbert space H is graded in the sense of the previous subsection (see the discussion before Definition 1.3.51), that the grading operator γ maps the domain of D into itself, anticommutes with D and commutes with each $a \in A$. Spectral triples without a grading operator is referred to as odd.

Given an algebra A, an odd (even) spectral triple on A is an odd (even) spectral triple $(\rho(A), H, D)$ (respectively, $(\rho(A), H, D, \gamma)$) where $\rho : A \to B(H)$ is a homomorphism.

DEFINITION 1.4.2. *Two spectral triples* $(\rho_1(A), H_1, D_1)$ *and* $(\rho_2(A), H_2, D_2)$ *are said to be unitarily equivalent if there is a unitary operator* $U : H_1 \to H_2$ *such that* $D_2 = UD_1U^*$ *and* $\rho_2(\cdot) = U\rho_1(\cdot)U^*$ *where* ρ_j, $j = 1, 2$ *are the representations of* A *on* H_j, *respectively.*

We now introduce our first examples. The remark 1.3.50 is essential in the following examples.

EXAMPLE 1.4.3. *Let* M *be a compact smooth* Spin^c*-manifold. We recall (Definition 1.3.59) that this means we are provided with a concordance class of Riemannian metrics and complex spinor bundles. Let* S *be such a complex spinor bundle with Dirac operator* D. *Then* $(C^\infty(M), L^2(S), D)$ *is a spectral triple of compact type.*

EXAMPLE 1.4.4. *Let* M *be a compact oriented Riemannian manifold. We recall from Example 1.3.53 that* $d + d^*$ *is a Dirac operator. Then* $(C^\infty(M), L^2(\Lambda T^*_{\mathbb{C}}M), d + d^*)$ *is a spectral triple of compact type.*

EXAMPLE 1.4.5. *The noncommutative* 2*-torus* A_θ *is the universal* C^**-algebra generated by two unitaries* U *and* V *satisfying* $UV = e^{2\pi i\theta}VU$ *where* $\theta \in [0, 1]$. *There are two derivations* d_1 *and* d_2 *on* A_θ *given by the rule:*

$$d_1(U) = U, \qquad d_1(V) = 0 \tag{1.4.1}$$

and

$$d_2(U) = 0, \qquad d_2(V) = V. \tag{1.4.2}$$

These are well-defined on the dense $*$*-subalgebra* A_θ^∞:

$$A_\theta^\infty = \{\sum_{m,n\in\mathbb{Z}} a_{mn}U^mV^n \mid \sup_{m,n} |m^k n^l a_{mn}| < \infty \forall k, l \in \mathbb{N}\}. \tag{1.4.3}$$

There is a unique faithful trace on A_θ^∞ *given by:*

$$\tau(\sum_{m,n} a_{mn}U^mV^n) = a_{00}. \tag{1.4.4}$$

Let $H = L^2(\tau) \oplus L^2(\tau)$, *where* $L^2(\tau)$ *denotes the GNS space of* A_θ^∞ *with respect to the state* τ. *We embed* A_θ^∞ *as a subalgebra of* $B(H)$ *by* $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. *We define* D *as*

$$D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}. \tag{1.4.5}$$

Then (A_θ^∞, H, D) *is a spectral triple of compact type.*

The space of noncommutative forms.

DEFINITION 1.4.6. *Let* A *be an arbitrary* $\mathbb{C}$*-algebra. Let* $\Omega_u^1 A$ *denote the kernel of the multiplication map* $\mu : A \otimes A \to A$. *We then define the universal differential graded algebra over* A *to be the tensor algebra*

$$\Omega_u A = \bigoplus_{n=0}^{\infty} (\underbrace{\Omega_u^1 A \otimes_A \ldots \otimes_A \Omega_u^1 A}_{n}), \tag{1.4.6}$$

endowed with the unique degree one derivation such that

$$d(a) := 1 \otimes a - a \otimes 1 \tag{1.4.7}$$

for $a \in A$.

Given a spectral triple (A, H, D) one constructs a compatible differential calculus on A by means of a suitable representation of the universal algebra $\Omega_u A$ in the algebra of bounded operators on H. The map

$$\pi_u : \Omega_u A \to B(H) \tag{1.4.8}$$

given by

$$\pi_u(a_0 da_1 \ldots da_p) := a_0[D, a_1] \ldots [D, a_p], \qquad a_j \in A \tag{1.4.9}$$

is a homomorphism of algebras. One can show

PROPOSITION 1.4.7. *Let $J_0 := \oplus_p J_0^p$ be the graded two sided ideal of $\Omega_u A$ given by*

$$J_0^p := \{\omega \in \Omega_u^p A \mid \pi_u(\omega) = 0\}. \tag{1.4.10}$$

Then $J := J_0 + dJ_0$ is graded differential ideal of $\Omega_u A$.

The elements of J are called junk forms.

DEFINITION 1.4.8. *The differential graded algebra of Connes' forms over the algebra A is defined by*

$$\Omega_D A := \Omega_u A/J \cong \pi_u(\Omega_u A)/\pi_u(J). \tag{1.4.11}$$

See Chapter 6 of [**Con85**] for many interesting examples and constructions that enable one to call a spectral triple a noncommutative manifold.

1.4.2. Quantum isometry groups. In this subsection, we briefly describe the theory of quantum isometry groups. We begin by recalling the notion of isometry for a Riemannian manifold.

DEFINITION 1.4.9. *Let (M, g) be a Riemannian manifold. A diffeomorphism $\psi : M \to M$ (i.e., ψ is a differentiable bijection with differentiable inverse) is called an isometry if*

$$g(u, v)_p = g(d\psi_p(u), d\psi_p(v))_{\psi(p)}, \quad \text{for all } p \in M, u, v \in T_pM. \tag{1.4.12}$$

Let us write $\mathrm{Iso}(M)$ for the collection of all isometries of the Riemannian manifold M. There is a natural group structure on $\mathrm{Iso}(M)$. In fact, one can prove more.

THEOREM 1.4.10. *The group $\mathrm{Iso}(M)$ of isometries of the Riemannian manifold M is a Lie group with respect to the compact-open topology. If M is compact then $\mathrm{Iso}(M)$ is compact.*

We recall that a Lie group is a smooth manifold together with a group structure such that both structures are compatible in a way.

EXAMPLE 1.4.11. *The isometry group of $\mathbb{R}^n$ with the standard Riemannian structure (making the standard basis an orthonormal frame) is the Euclidean group $E(n) = T(n) \rtimes O(n)$, where $T(n)$ is the translation group.*

EXAMPLE 1.4.12. *The isometry group of $S^n (\subset \mathbb{R}^{n+1})$ with induced Riemannian structure is $O(n+1)$.*

EXAMPLE 1.4.13. *The isometry group of the flat torus $\mathbb{T}^n = \underbrace{S^1 \times \cdots \times S^1}_{n\text{-times}}$ i.e., with the product metric, is $\mathbb{T}^n \rtimes (\mathbb{Z}_2^n \rtimes S_n)$, where S_n is the permutation group on n-letters.*

Now to define a quantum version of $\mathrm{Iso}(M)$ one needs a point-free definition of an isometry of M. For that we need to recall the Laplacian Δ from Eq. (1.3.21) and that it is an endomorphism of each $\Omega^k(M)$, so in particular of $\Omega^0(M) = C^\infty(M)$.

PROPOSITION 1.4.14. *Let M be a compact oriented Riemannian manifold. A diffeomorphism $\psi : M \to M$ is an isometry if and only if ψ commutes with Δ in the sense*

$$\Delta(f\psi) = \Delta(f)\psi, \quad \text{for all } f \in C^\infty(M). \tag{1.4.13}$$

Thus we can rephrase the definition of an isometry of M using the Laplacian Δ. To obtain a description of $\mathrm{Iso}(M)$ in terms of Δ one considers the category with pairs (G, α), where G is a compact metrizable group acting on M by the smooth and isometric action α, as objects. If (G_1, α_1) and (G_2, β_2) are two such pairs then a morphism between these is a group homomorphism $\pi : G_1 \to G_2$ such that $\beta\pi = \alpha$. Then, it is a classical result that the isometry group of M is the universal object in this category.

More generally, the isometry group of a classical compact Riemannian manifold, viewed as a compact metrizable space (forgetting the group structure), can be seen to be the universal object of a category whose object class consists of subsets (not generally subgroups) of the set of smooth isometries of the manifold. Then it can be proved that this universal compact set has a canonical group structure. Motivated by this and using ideas of Woronowicz and Sołtan, Goswami considered in [**Gos09**] a bigger category with objects as pairs (S, f), where S is a compact metrizable space and $f : S \times M \to M$ such that the map $m \mapsto f(s, m)$ from M to itself is a smooth isometry for all $s \in S$. The morphisms are defined analogously as above. One can then prove

THEOREM 1.4.15. *Let M be a compact oriented Riemannian manifold and let $C^\infty(M)_0$ be the span of eigenvectors of Δ. Then $\mathrm{Iso}(M)$ is the universal object of the category with objects as pairs $(C(Y), \alpha)$ where Y is a compact metrizable space and α is a unital C^*-homomorphism from $C(M) \to C(M) \otimes C(Y)$ satisfying the following.*

i) $\alpha(C(M))(1 \otimes C(Y))$ is dense in $C(M) \otimes C(Y)$.
ii) $\alpha_\phi := (\mathrm{id} \otimes \phi)\alpha$ maps $C^\infty(M)_0$ into itself and commutes with Δ on $C^\infty(M)_0$, for every state ϕ on $C(Y)$.

Therefore, to define the quantum isometry group, it is reasonable to consider a category of compact quantum groups which act on the manifold (or more generally on a spectral triple) in a way so as to preserve the Riemannian structure, which is precisely formulated in [**Gos09**], where it is also proven that a universal object in the category of such quantum groups does exist if one makes some natural regularity assumption on the spectral triple at hand.

There are two formulations of the quantum isometry group of a noncommutative manifold. One is based on the Laplacian, obtained from the space of forms and the other, more natural, based on the Dirac operator, already part of the spectral triple. Since we will only need the latter, we do not describe the former, referring the reader to the comprehensive book [**GB16**].

So we want to rephrase the definition of an isometry of a compact oriented Riemannian manifold M in terms of some Dirac operator. To begin with, let M be a compact oriented Riemannian manifold. We recall the triple $(C^\infty(M), L^2(\Lambda T^*_{\mathbb{C}}M), d + d^*)$ from Example 1.4.4 and define

$$\begin{aligned}\mathrm{Iso}((C^\infty(M), L^2(\Lambda T^*_{\mathbb{C}}M), d + d^*)) := &\{\phi \in Aut(C(M)) \mid \exists \text{ a unitary } U \text{ on } L^2(\Lambda T^*_{\mathbb{C}}M) \text{ such that} \\ &U(d + d^*) = (d + d^*)U \text{ and } \rho(f\phi) = U\rho(f)U^* \text{ for all } f \in C(M)\},\end{aligned}$$

where $\rho(f)$ is the multiplication operator by f. [**Par95**] proves

THEOREM 1.4.16. $\mathrm{Iso}((C^\infty(M), L^2(\Lambda T^*_{\mathbb{C}}M), d + d^*)) \cong \mathrm{Iso}(M)$.

Thus one obtains a description of $\mathrm{Iso}(M)$ in terms of the de Rham operator $d + d^*$. On the other hand, the following theorem describes an orientation-preserving isometry (i.e., an isometry ψ such that $\psi^*(\mathrm{vol}_{(M,g)}) = \mathrm{vol}_{(M,g)}$, ψ^* being the pull-back Eq. (1.3.16)) for a Spin-manifold (see Definition 1.3.59) using the Dirac operator.

THEOREM 1.4.17. *Let M be a compact Riemannian* Spin*-manifold with the induced orientation. Let S be the associated real spinor bundle and D be the Dirac operator on S. Let $\psi : M \to M$ be a smooth injective map which is an orientation-preserving isometry. Then there exists a unitary U_ψ on $L^2(S)$ commuting with D such that $U_\psi \rho(f) U_\psi^* = \rho(f\psi)$, for every $f \in C(M)$, where $\rho(f)$ denotes the multiplication operator by f on $L^2(S)$.*

Conversely, suppose that U is a unitary on $L^2(S)$ such that $UD = DU$ and the map $\alpha_U(X) = UXU^{-1}$ for X in $B(L^2(S))$ maps $C(M)$ into $L^\infty(M)$. Then there is a smooth injective orientation-preserving isometry ψ on M such that $U = U_\psi$.

It is also possible to describe a family of orientation-preserving isometries in an operator-theoretic way.

THEOREM 1.4.18. *Let M be a compact Riemannian* Spin*-manifold with the induced orientation. Let S be the associated real spinor bundle and D be the Dirac operator on S. Let X be a compact metrizable space and let $\psi : X \times M \to M$ be a map such that*

i) the map ψ_x defined by $\psi_x(m) = \psi(x, m)$ is a smooth orientation-preserving isometry and *ii)* $x \mapsto \psi_x \in C^\infty(M, M)$ *is continuous with respect to the locally convex topology on* $C^\infty(M, M)$.

Then there exists a ($C(X)$-linear) unitary U_ψ *on the Hilbert* $C(X)$*-module* $L^2(S) \otimes C(X)$ *such that for all* $x \in X$, $U_x := (\mathrm{id} \otimes ev_x)U_\psi$ *is a unitary of the form* U_{ψ_x} *on the Hilbert space* $L^2(S)$ *commuting with* D *and* $U_x \rho(f) U_x^{-1} = \rho(f\psi_x^{-1})$.

Conversely, if there exists a $C(X)$*-linear unitary* U *on* $L^2(S) \otimes C(X)$ *such that* $U_x = (\mathrm{id} \otimes ev_x)U$ *is a unitary commuting with* D *for all* x *and* $(\mathrm{id} \otimes ev_x)\alpha_U(C^\infty(M)) \subset L^\infty(M)$ *for all* $x \in X$, *then there exists a map* $\psi : X \times M \to M$ *satisfying the conditions mentioned above such that* $U = U_\psi$.

Thus one is lead to the following definition.

Definition 1.4.19. *A quantum family of orientation-preserving isometries for the compact-type spectral triple* (A, H, D) *is given by a pair* (S, u), *where* S *is a unital* C^**-algebra and* u *is a linear map from* H *to* $H \otimes S$ *such that* $\tilde{u}$ *given by* $\tilde{u}(\xi \otimes b) = u(\xi)(1 \otimes b)$, *extends to a unitary element of* $M(\mathcal{K}(H) \otimes S)$ *satisfying*

i) for every state ϕ *on* S, $u_\phi D = (D \otimes \mathrm{id})u_\phi$ *where* $u_\phi = (\mathrm{id} \otimes \phi)\tilde{u}$;

ii) $(\mathrm{id} \otimes \phi)\mathrm{ad}_u(a) \in A''$, *for all* $a \in A$ *and for all state* ϕ *on* S, *where* $\mathrm{ad}_u(x) = \tilde{u}(x \otimes 1)\tilde{u}^*$ *for* $x \in B(H)$.

In case the C^**-algebra* S *has a coproduct* Δ *such that* (S, Δ) *is a compact quantum group and* U *is a unitary representation of* (S, Δ) *on* H, *it is said that* (S, Δ) *acts by orientation-preserving isometries on the spectral triple.*

Remark 1.4.20. *Our discussion goes equally well in the presence of a grading operator.*

Now consider the category $\mathbf{Q}(A, H, D)$ with object class consisting of all quantum families of orientation-preserving isometries (S, u) of the spectral triple (A, H, D). If (S, u) and (S', u') are two such pairs then a morphism between these is a unital C^*-homomorphism $\Phi : S \to S'$ such that $(\mathrm{id} \otimes \Phi)(u) = u'$. We also consider another category $\mathbf{Q}'(A, H, D)$ whose objects are the triples (S, Δ, u), where (S, Δ) is a compact quantum group acting by orientation-preserving isometries on the spectral triple (A, H, D), and u is the corresponding unitary representation. The morphisms are homomorphisms of compact quantum groups which are also morphisms of the underlying quantum families of orientation-preserving isometries. We note that the forgetful functor $Q' \to Q$ is faithful. In general, a universal object might not exist either for $\mathbf{Q}(A, H, D)$ or $\mathbf{Q}'(A, H, D)$, see [**GB16**] for an example. But one has the following existence result.

Theorem 1.4.21. *Let* (A, H, D) *be a spectral triple of compact-type and assume that* D *has a one dimensional eigenspace spanned by a unit vector* ξ, *which is cyclic and separating for the algebra* A. *Moreover, assume that each eigenvector of* D *belongs to the dense subspace* $A\xi$ *of* H. *Then*

i) there exists a universal object $(\widetilde{S_0}, u_0)$ *in the category* $\mathbf{Q}(A, H, D)$.

ii) $\widetilde{S_0}$ *admits a comultiplication* Δ_0 *such that* $(\widetilde{S_0}, \Delta_0)$ *is a compact quantum group and* $(\widetilde{S_0}, \Delta_0, u_0)$ *is a universal object in the category* $\mathbf{Q}'(A, H, D)$.

Let S_0 be the Woronowicz C^*-subalgebra (see Definition 1.2.22) of $(\widetilde{S_0}, \Delta_0)$ generated by $\{(t_{\xi,\eta} \otimes \mathrm{id})(\mathrm{ad}_{u_0}(a)) \mid \xi, \eta \in H, a \in A\}$. It is the largest Woronowicz C^*-subalgebra of $(\widetilde{S_0}, \Delta_0)$ for which ad_{u_0} is faithful on A.

Notation. We write $\mathrm{Qiso}^+(A, H, D)$ for S_0, $\widetilde{\mathrm{Qiso}^+(A, H, D)}$ for $\widetilde{S_0}$ and refer to $\mathrm{Qiso}^+(A, H, D)$ as the quantum group of orientation-preserving isometries of (A, H, D).

Although, the above theorem ensures existence of $\mathrm{Qiso}^+(A, H, D)$, it is not fit for computations. In many of the examples, the Hilbert space H is the L^2-space of A with respect to some faithful state τ and it suffices to look for actions of compact quantum groups preserving the state τ and commuting with the Dirac operator D. So one introduces another category tailored for this situation that enables one to compute several examples, which includes the Chakraborty-Pal spectral triple on $SU_q(2)$ as well as Connes' spectal triples on group algebras coming from length functions; see [**GB16**]. In Chapter 2 (Subsection 2.3.2), we shall introduce this category and use it in our computation of the quantum group of orientation-preserving isometries of the odd sphere. We now move on to some examples.

1.4.3. Examples of quantum isometry groups. Here we discuss some examples of quantum isometry groups.

Equivariant spectral triple on $SU_q(2)$. We begin by recalling that for each n in $\{0, 1/2, 1, \ldots, \}$, there is a unique irreducible $(2n+1)$-dimensional representation of $SU_q(2)$, which we denote by T^n. Let t^n_{ij} be the ij-th matrix element of T^n. They form an orthogonal basis of $H = L^2(SU_q(2))$. Denote by e^n_{ij} the normalized t^n_{ij} so that $\{e^n_{ij} \mid n = 0, 1/2, 1, \ldots, i, j = -n, -n+1, \ldots, n\}$ is an orthonormal basis. We consider the spectral triple (A^∞, H, D) on $SU_q(2)$ constructed by Chakraborty and Pal ([**CP03**]) given as follows. Let A^∞ be the linear span of t^n_{ij}, $H = L^2(SU_q(2))$ and D be defined as:

$$D(e^n_{ij}) = \begin{cases} (2n+1)e^n_{ij} & n \neq i \\ -(2n+1)e^n_{ij}, & n = i \end{cases}.$$

Then for this spectral triple, one has

Theorem 1.4.22. *The quantum group of orientation-preserving isometries of $SU_q(2)$ is the quantum unitary group $U_q(2)$.*

It is a fact that the Dirac operator constructed above for $SU_q(2)$ has no classical counterpart, see Remark 5.1 of [**CP03**]. Nevertheless, if we view $SU(2)$ as the 3-sphere S^3, then the isometry group would be $U(2)$. Our computation in Chapter 2 conform to this, where we have identified $U_q(2)$ as the quantum group of orientation-preserving isometries of S^3_q. Thus $U_q(2)$ naturally arises as the quantum isometry group for the q-analogues of both $SU(2)$ and S^3 in the noncommutative case as well.

The noncommutative 2-torus. We recall the spectral triple on A_θ from Example 1.4.5. Then for this spectral triple

Theorem 1.4.23. *The quantum group of orientation-preserving isometries of the noncommutative 2-torus is again the CQG $C(\mathbb{T}^2)$.*

Taking the classical limit, we obtain the spectral triple (A^∞, H, D) on $\mathbb{T}^2$ given by $A^\infty = C^\infty(\mathbb{T}^2)$, $H = L^2(\mathbb{T}^2) \oplus L^2(\mathbb{T}^2)$ and $D = \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}$, where we view $C(\mathbb{T}^2)$ as the universal C^*-algebra generated by two commuting unitaries U and V, and d_1 and d_2 are derivations on A^∞ defined by:

$$d_1(U) = U, \quad d_1(V) = 0, \quad d_2(U) = 0, \quad d_2(V) = V.$$

The above theorem then yields that the quantum group of orientation-preserving isometries of the torus $\mathbb{T}^2$ is the $\mathbb{T}^2$ itself which is also the group of orientation-preserving isometries of the torus $\mathbb{T}^2$.

CHAPTER 2

Quantum symmetry of the odd sphere

2.1. Introduction

In the previous chapter 1.2.3, we presented examples of compact quantum groups arising from Wang's work, who defined quantum permutation group of finite sets and quantum automorphism group of finite dimensional matrix algebras. This was followed by flurry of work by several other mathematicians including Banica, Bichon and others [**Ban05b**, **Ban05a**, **Bic03**]. The last section of the previous chapter 1.4.3 described in some detail the work of Goswami and his collaborators (including Bhowmick, Skalski and others) who approached the problem from a geometric perspective and formulated an analogue of the Riemannian isometry groups in the framework of (compact) quantum groups acting on C^*-algebras. See also [**EGMW17**, **EW14**].

A useful procedure of producing genuine examples of 'noncommutative spaces' is to deform the coordinate algebra or some other suitable function algebra underlying a classical space. In this context, it is natural to ask the following: what is the quantum isometry group of a non-commutative space obtained by deforming a classical space? It is expected that under mild assumptions, it should be isomorphic with a deformation of the isometry group of the classical space, at least when the classical space is connected. Indeed, for a quite general class of cocycle deformation (called the Rieffel deformation), such a result has been proved by Bhowmick, Goswami and Joardar [**GJ14**]. See also [**EW16**, **EW17**].

However, no such general result has yet been achieved for the Drinfeld-Jimbo type q-deformation of semisimple Lie groups and the corresponding homogeneous spaces. The goal of the present paper is to make some progress in this direction. We have been able to prove the above result for q-deformed odd spheres, i.e., S_q^{2N-1} [**VS90**]. Classically (for $q = 1$), these are nothing but the spheres $\{(z_1, \ldots, z_N) \mid \sum_i |z_i|^2 = 1\}$ inside $\mathbb{C}^N$. The universal group that can act 'linearly', i.e., leaves the span of the complex coordinates $z_1, \ldots, z_N$ invariant and also preserves the canonical inner product on $span\{z_1, \ldots, z_n\}$ coming from the standard inner product of $\mathbb{C}^N$, is the unitary group $U(N)$.

We have proved in Subsection 2.3.1 (Theorem 2.3.7) a q-analogue of this result. More precisely, we have proved the following theorem.

THEOREM 2.1.1. *Let Q be a Hopf $*$-algebra coacting on $\mathcal{O}(S_q^{2N-1})$ by ρ making it a $*$-comodule algebra, where we have viewed $\mathcal{O}(S_q^{2N-1})$ as a $*$-coideal subalgebra of $\mathcal{O}(SU_q(N))$. Moreover, suppose that*

i) ρ leaves the subspace $V = span\{z_1, \ldots, z_N\}$ invariant, i.e., $\rho(z_i) = \sum_j z_j \otimes q_i^j$ for some $q_j^i \in Q$. We write $\mathbf{q} = (q_j^i)$ for the matrix of Q-valued coefficients;

ii) ρ preserves the inner product on V induced by the Haar functional.

Then there is a unique $*$-morphism $\Psi : \mathcal{O}(U_q(N)) \to Q$ such that $(\mathrm{id} \otimes \Psi)\rho_u = \rho$, ρ_u as in Proposition 2.3.5.

Using this, we have also identified (Theorem 2.3.19) $U_q(N)$ with the (orientation-preserving) quantum isometry group of a natural spectral triple on S_q^{2N-1} constructed in [**CP08**].

REMARK 2.1.2. *We can compare the above result with* [**EGMW17**, Theorem 1.1]. *In* [**EGMW17**], *the algebra $\mathcal{A}$ on which Hopf coactions are considered is commutative, which is replaced by a q-deformed quantized function algebra in the present article. Moreover, flexibility of choice of a non-degenerate bilinear form in* [**EGMW17**, Theorem 1.1, Condition (i)] *is gone in our case; we have the somewhat* *rigid requirement of preserving a canonical non-degenerate sesquilinear form coming from the Haar functional. In fact, a special advantage of working with a commutative algebra in* [**EGMW17**] *is that any bilinear form on $\mathcal{A}$ admits a natural extension (as a bilinear form) on $\mathcal{A} \otimes \mathcal{A}$ which is invariant under the flip map. This is no longer true for quantized function algebra, if we consider the obvious q-analogue of the flip, associated with the natural braiding.*

REMARK 2.1.3. *As pointed out by the referee, our setup is surprisingly similar to that of* [**HM98**]. *Further investigation is needed in that direction.*

2.2. Preliminaries

We collect some preliminaries here that are needed for the rest of the chapter. We introduce some well known material on coquasitriangular Hopf algebras. The main object of investigation, the Vaksman-Soibelman (also called quantum or odd) sphere, is also introduced.

2.2.1. Coquasitriangular Hopf algebras. Let us recall that H denotes a Hopf algebra with comultiplication Δ, counit ε, antipode S, unit 1 and multiplication m. We use Sweedler notation throughout, i.e., for the coproduct we write $\Delta(a) = a_{(1)} \otimes a_{(2)}$ and for a right coaction ρ, we write $\rho(x) = x_{(0)} \otimes x_{(1)}$. A general reference is [**KS97**].

DEFINITION 2.2.1. *A coquasitriangular Hopf algebra is a Hopf algebra H equipped with a linear form $\mathbf{r} : H \otimes H \to \mathbb{C}$ such that the following conditions hold:*

i) *$\mathbf{r}$ is invertible with respect to the convolution, that is, there exists another linear form $\overline{\mathbf{r}} : H \otimes H \to \mathbb{C}$ such that $\mathbf{r} * \overline{\mathbf{r}} = \overline{\mathbf{r}} * \mathbf{r} = \varepsilon \otimes \varepsilon$ on $H \otimes H$;*

ii) *$m_{H^{op}} = \mathbf{r} * m_H * \overline{\mathbf{r}}$ on $H \otimes H$;*

iii) *$\mathbf{r}(m_H \otimes \mathrm{id}) = \mathbf{r}_{(13)} * \mathbf{r}_{(23)}$ and $\mathbf{r}(\mathrm{id} \otimes m_H) = \mathbf{r}_{(13)} * \mathbf{r}_{(12)}$ on $H \otimes H \otimes H$,*

where $\mathbf{r}_{(12)}(a \otimes b \otimes c) = \mathbf{r}(a \otimes b)\varepsilon(c)$, $\mathbf{r}_{(23)}(a \otimes b \otimes c) = \varepsilon(a)\mathbf{r}(b \otimes c)$ and $\mathbf{r}_{(13)}(a \otimes b \otimes c) = \varepsilon(b)\mathbf{r}(a \otimes c)$, a, b, c in H.

REMARK 2.2.2. *A linear form $\mathbf{r}$ on $H \otimes H$ with the properties i)-iii) is called a universal r-form on H.*

Since linear forms on $H \otimes H$ correspond to bilinear forms on $H \times H$, we can consider any linear form $\mathbf{r} : H \otimes H \to \mathbb{C}$ as a bilinear form on $H \times H$ and write $\mathbf{r}(a, b) := \mathbf{r}(a \otimes b)$, $a, b \in H$. Then the above conditions i)-iii) read as

$$\mathbf{r}(a_{(1)}, b_{(1)})\overline{\mathbf{r}}(a_{(2)}, b_{(2)}) = \overline{\mathbf{r}}(a_{(1)}, b_{(1)})\mathbf{r}(a_{(2)}, b_{(2)}) = \varepsilon(a)\varepsilon(b), \tag{2.2.1}$$

$$ba = \mathbf{r}(a_{(1)}, b_{(1)})a_{(2)}b_{(2)}\overline{\mathbf{r}}(a_{(3)}, b_{(3)}), \tag{2.2.2}$$

$$\mathbf{r}(ab, c) = \mathbf{r}(a, c_{(1)})\mathbf{r}(b, c_{(2)}), \tag{2.2.3}$$

$$\mathbf{r}(a, bc) = \mathbf{r}(a_{(1)}, c)\mathbf{r}(a_{(2)}, b), \tag{2.2.4}$$

with $a, b, c \in H$.

REMARK 2.2.3. *It can be shown that $\mathbf{r}(S(a), S(b)) = \mathbf{r}(a, b)$.*

Let H be a coquasitriangular Hopf algebra with universal r-form $\mathbf{r}$. For right H-comodules V and W we define a linear mapping $\mathbf{r}_{V,W} : V \otimes W \to W \otimes V$ by

$$\mathbf{r}_{V,W}(v \otimes w) = \mathbf{r}(v_{(1)}, w_{(1)})w_{(0)} \otimes v_{(0)}, \tag{2.2.5}$$

$v \in V$, $w \in W$.

REMARK 2.2.4. *It can be shown that $\mathbf{r}_{V,W}$ is an isomorphism of the right H-comodules $V \otimes W$ and $W \otimes V$.*

The compatibility of a universal r-form and a $*$ structure is described in the following definition.

DEFINITION 2.2.5. *A universal r-form $\mathbf{r}$ of a Hopf $*$-algebra H is called real if $\mathbf{r}(a \otimes b) = \overline{\mathbf{r}(b^* \otimes a^*)}$.*

2.2.2. The quantum semigroup $M_q(N)$. Let q be a positive real number. We now introduce some of the well known deformations of classical objects.

DEFINITION 2.2.6. *The FRT bialgebra, also called the coordinate algebra of the quantum matrix space, denoted $\mathcal{O}(M_q(N))$, is the free unital $\mathbb{C}$-algebra with a set of N^2 generators $\{u^i_j \mid i, j = 1, \ldots, N\}$ and defining relations*

$$u^i_k u^j_k = q u^j_k u^i_k, \qquad u^k_i u^k_j = q u^k_j u^k_i, \qquad i < j, \tag{2.2.6}$$

$$u^i_l u^j_k = u^j_k u^i_l, \qquad i < j, \qquad k < l, \tag{2.2.7}$$

$$u^i_k u^j_l - u^j_l u^i_k = (q - q^{-1}) u^j_k u^i_l, \qquad i < j, \qquad k < l. \tag{2.2.8}$$

PROPOSITION 2.2.7. *There is a unique bialgebra structure on the algebra $\mathcal{O}(M_q(N))$ such that*

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j, \qquad and \qquad \varepsilon(u^i_j) = \delta_{ij}, \qquad i, j = 1, \ldots, N. \tag{2.2.9}$$

The above construction can be realized more conceptually as follows. Let $\hat{R} : \mathbb{C}^N \otimes \mathbb{C}^N \to \mathbb{C}^N \otimes \mathbb{C}^N$ be the linear operator whose matrix with respect to the standard basis of $\mathbb{C}^N$ is given by

$$\hat{R}^{ij}_{mn} = q^{\delta_{ij}} \delta_{in} \delta_{jm} + (q - q^{-1}) \delta_{im} \delta_{jn} \theta(j - i), \tag{2.2.10}$$

where θ is the Heaviside symbol, that is, $\theta(k) = 1$ if $k > 0$ and $\theta(k) = 0$ if $k \leq 0$. Let the inverse of $\hat{R}$ be $\hat{R}^-$. Also, let $\check{R}$ be the "dual" operator defined by $\check{R}^{ij}_{mn} = \hat{R}^{nm}_{ji}$ and $\check{R}^{-ij}_{mn} = \hat{R}^{-ij}_{mn}$.

REMARK 2.2.8. *It is known that $\hat{R}$ satisfies*

$$(\hat{R} - qI)(\hat{R} + q^{-1} I) = 0, \tag{2.2.11}$$

where I is the identity operator.

The following shows that $M_q(N)$ is universal in a sense.

PROPOSITION 2.2.9. *i) There is a linear map $\phi : \mathbb{C}^N \to \mathbb{C}^N \otimes \mathcal{O}(M_q(N))$ such that $\mathbb{C}^N$ is a right comodule of $\mathcal{O}(M_q(N))$ with coaction ϕ and $\hat{R}$ is a comodule morphism, i.e., $(\hat{R} \otimes \mathrm{id})\phi^{(2)} = \phi^{(2)} \hat{R}$, where $\phi^{(2)}$ is the induced coaction on $\mathbb{C}^N \otimes \mathbb{C}^N$ given by $\phi^{(2)}(v \otimes w) = v_0 \otimes w_0 \otimes v_1 w_1$ ($\phi(v) = v_0 \otimes v_1$);*
ii) If A is any other bialgebra and $\psi : \mathbb{C}^N \to \mathbb{C}^N \otimes A$ is a right coaction of A on $\mathbb{C}^N$ such that $\hat{R}$ is a comodule morphism (in the sense described above) then there exists a unique bialgebra morphism $\Theta : \mathcal{O}(M_q(N)) \to A$ such that $(\mathrm{id} \otimes \Theta)\phi = \psi$.

Since $\mathcal{O}(M_q(N))$ is only a bialgebra, we want to construct a Hopf algebra out of it. For that we need the following definition.

DEFINITION 2.2.10. *The quantum determinant, denoted $\mathcal{D}_q$, is the element of $\mathcal{O}(M_q(N))$ defined by*

$$\sum_{\pi \in S_N} (-q)^{\ell(\pi)} u^1_{\pi(1)} \ldots u^N_{\pi(N)}, \tag{2.2.12}$$

where S_N is the symmetric group on N letters and $\ell(\pi)$ is the number of inversions in π.

REMARK 2.2.11. *It is an important fact that $\mathcal{D}_q$ is central, nonzero and group-like in $\mathcal{O}(M_q(N))$. We recall that group-like means $\Delta(\mathcal{D}_q) = \mathcal{D}_q \otimes \mathcal{D}_q$. Applying $(\varepsilon \otimes \mathrm{id})$ on the identity $\Delta(\mathcal{D}_q) = \mathcal{D}_q \otimes \mathcal{D}_q$ yields $\mathcal{D}_q = \varepsilon(\mathcal{D}_q)\mathcal{D}_q$, hence $\varepsilon(\mathcal{D}_q) = 1$ as $\mathcal{D}_q \neq 0$.*

2.2.3. The quantum group $SU_q(N)$. The deformation of the special linear group is realized as follows.

DEFINITION 2.2.12. *The coordinate algebra of the quantum special linear group is defined to be the quotient*

$$\mathcal{O}(SL_q(N)) = \mathcal{O}(M_q(N))/\langle \mathcal{D}_q - 1\rangle$$

of the algebra $\mathcal{O}(M_q(N))$ by the two-sided ideal generated by the element $\mathcal{D}_q - 1$.

The following shows that $\mathcal{O}(SL_q(N))$ is indeed a Hopf algebra.

PROPOSITION 2.2.13. *There is a unique Hopf algebra structure on the algebra $\mathcal{O}(SL_q(N))$ with comultiplication Δ and counit ε such that*

$$\Delta(u^i_j) = \sum_k u^i_k \otimes u^k_j \qquad \text{and} \qquad \varepsilon(u^i_j) = \delta_{ij}. \tag{2.2.13}$$

The antipode S of the Hopf algebra is given by

$$S(u^i_j) = (-q)^{i-j} \sum_{\pi\in S_{N-1}} (-q)^{\ell(\pi)} u^{k_1}_{\pi(l_1)} \dots u^{k_{N-1}}_{\pi(l_{N-1})}, \tag{2.2.14}$$

where $\{k_1, \dots, k_{N-1}\} := \{1, \dots, N\} \setminus \{j\}$ and $\{l_1, \dots, l_{N-1}\} := \{1, \dots, N\} \setminus \{i\}$ as ordered sets.

The composite $\mathbb{C}^N \xrightarrow{\phi} \mathbb{C}^N \otimes \mathcal{O}(M_q(N)) \to \mathbb{C}^N \otimes \mathcal{O}(SL_q(N))$ gives the natural coaction of $\mathcal{O}(SL_q(N))$ on $\mathbb{C}^N$.

As quantum groups are understood to be quasitriangular Hopf algebras, quantum function algebras are assumed to be coquasitriangular Hopf algebras. We want to think of $\mathcal{O}(SL_q(N))$ as the quantum function algebra of $SL(N)$.

THEOREM 2.2.14. *$\mathcal{O}(SL_q(N))$ is a coquasitriangular Hopf algebra with universal r-form $\mathbf{r}_t$ uniquely determined by*

$$\mathbf{r}_t(u^i_j \otimes u^k_l) = t\hat{R}^{ki}_{jl}, \tag{2.2.15}$$

where t is the unique positive real number such that $t^N = q^{-1}$.

REMARK 2.2.15. *It can be shown that the morphism $\mathbf{r}_{\mathbb{C}^N,\mathbb{C}^N}$ induced by the universal r-form $\mathbf{r}_t$ of $\mathcal{O}(SL_q(N))$, equals $t\hat{R}$. Let us denote it by σ.*

The following resembles complex conjugation.

PROPOSITION 2.2.16. *There is a unique $*$-structure on the Hopf algebra $\mathcal{O}(SL_q(N))$ given by $(u^i_j)^* = S(u^j_i)$, making it into a Hopf $*$-algebra.*

Let us now introduce the quantum version of the real form $SU(N)$ of $SL(N)$.

DEFINITION 2.2.17. *The coordinate algebra of the quantum special unitary group $SU_q(N)$ is the Hopf $*$-algebra $\mathcal{O}(SL_q(N))$ of the above proposition.*

THEOREM 2.2.18. *The universal r-form $\mathbf{r}_t$ of $\mathcal{O}(SU_q(N))$ is real, in the sense of Definition 2.2.5.*

We recall that H is a CQG algebra if and only if it is isomorphic to the dense Hopf $*$-algebra $\mathcal{S}$ of a compact quantum group S. The following captures the compactness of the real form $SU(N)$.

THEOREM 2.2.19. *$\mathcal{O}(SU_q(N))$ is a CQG algebra.*

In fact, there is a natural C^*-norm on $\mathcal{O}(SU_q(N))$. Upon completion with respect to this norm, one gets $C(SU_q(N))$, the underlying C^*-algebra of $SU_q(N)$ which is a compact quantum group in the sense of Definition 1.2.20. Moreover, $C(SU_q(N))$ is the universal C^*-algebra generated by $\mathcal{O}(SU_q(N))$.

2.2.4. The odd sphere. We now introduce the main example to be studied. We remark that these are q-deformations of the $(2N-1)$-dimensional spheres. One could as well define q-deformations of even dimensional real spheres which are quantum homogeneous spaces of $\mathcal{O}(SO_q(n))$ for appropriate n. We do not deal with these because our proof of some of the main results crucially use the fact that $\hat{R}$ has two eigenvalues whereas in the case of $\mathcal{O}(SO_q(n))$, it has three.

DEFINITION 2.2.20. *The coordinate algebra $\mathcal{O}(S_q^{2N-1})$ of the quantum sphere is the free unital $\mathbb{C}$-algebra with a set of $2N$ generators $\{z_i, z_i^* \mid i = 1, \ldots, N\}$ and defining relations*

$$z_i z_j = q z_j z_i, \qquad z_i^* z_j^* = q^{-1} z_j^* z_i^*, \qquad i < j \tag{2.2.16}$$

$$z_i z_j^* = q z_j^* z_i, \qquad i \neq j \tag{2.2.17}$$

$$z_i z_i^* - z_i^* z_i + q^{-1}(q - q^{-1}) \sum_{k>i} z_k z_k^* = 0, \tag{2.2.18}$$

$$\sum_{i=1}^{N} z_i z_i^* = 1, \tag{2.2.19}$$

together with the $*$-structure $(z_i)^* = z_i^*$ and $(z_i^*)^* = z_i$.

There is a natural C^*-norm on $\mathcal{O}(S_q^{2N-1})$. Upon completion with respect to this norm, one gets the unital C^*-algebra $C(S_q^{2N-1})$ which is the universal C^*-algebra generated by $\mathcal{O}(S_q^{2N-1})$.

PROPOSITION 2.2.21. *Putting $z_i = u_i^1$ and $z_i^* = (u_i^1)^* = S(u_1^i)$ gives an embedding of $\mathcal{O}(S_q^{2N-1})$ into $\mathcal{O}(SU_q(N))$ making $\mathcal{O}(S_q^{2N-1})$ into a quantum homogeneous space for $\mathcal{O}(SU_q(N))$ with the coaction*

$$\Delta_R(z_i) = \sum_j z_j \otimes u_i^j, \qquad \Delta_R(z_i^*) = \sum_j z_j^* \otimes S(u_j^i). \tag{2.2.20}$$

The compact quantum group $SU_q(N)$ acts on $C(S_q^{2N-1})$ in the C^*-algebraic sense, lifting the above coaction on $\mathcal{O}(S_q^{2N-1})$. Classically, the sphere S^{2N-1} can be described as a homogeneous space for $SU(N)$. The above proposition states the quantum version of it. Moreover, one can view the sphere as a homogeneous space for $U(N)$ also. As expected, the last statement continues to hold in the quantum world too. We need the following definition.

DEFINITION 2.2.22. *The coordinate algebra of the quantum general linear group is defined to be the quotient*

$$\mathcal{O}(GL_q(N)) = \mathcal{O}(M_q(N))[t]/\langle t\mathcal{D}_q - 1\rangle$$

of the polynomial algebra $\mathcal{O}(M_q(N))[t]$ in t over $\mathcal{O}(M_q(N))$ by the two-sided ideal generated by the element $t\mathcal{D}_q - 1$.

PROPOSITION 2.2.23. *There is a unique Hopf algebra structure on the algebra $\mathcal{O}(GL_q(N))$ with comultiplication Δ and counit ε such that*

$$\Delta(u_j^i) = \sum_k u_k^i \otimes u_j^k \qquad \text{and} \qquad \varepsilon(u_j^i) = \delta_{ij}. \tag{2.2.21}$$

The antipode S of the Hopf algebra is given by

$$S(u_j^i) = (-q)^{i-j} {\mathcal{D}_q}^{-1} \sum_{\pi \in S_{N-1}} (-q)^{\ell(\pi)} u_{\pi(l_1)}^{k_1} \ldots u_{\pi(l_{N-1})}^{k_{N-1}}, \tag{2.2.22}$$

and

$$S(\mathcal{D}_q) = {\mathcal{D}_q}^{-1}, \tag{2.2.23}$$

where $\{k_1, \ldots, k_{N-1}\} := \{1, \ldots, N\} \setminus \{j\}$ and $\{l_1, \ldots, l_{N-1}\} := \{1, \ldots, N\} \setminus \{i\}$ as ordered sets.

We have the following analogue of the real form $U(N)$.

PROPOSITION 2.2.24. *There is a unique $*$-structure on the Hopf algebra $\mathcal{O}(GL_q(N))$ given by $(u_j^i)^* = S(u_i^j)$, making it into a Hopf $*$-algebra.*

DEFINITION 2.2.25. *The coordinate algebra of the quantum unitary group $U_q(N)$ is the Hopf $*$-algebra $\mathcal{O}(GL_q(N))$ of the above proposition. In this $*$-algebra, the quantum determinant $\mathcal{D}_q$ becomes a unitary element.*

In analogy with $U(N)$, the fact that $U_q(N)$ is a compact quantum group is reflected in the following theorem.

THEOREM 2.2.26. *$\mathcal{O}(U_q(N))$ is a CQG algebra.*

Thus we have the analogues of Proposition 2.2.16 and Theorem 2.2.19. Moreover, Proposition 2.2.21 remains true in this case and makes $\mathcal{O}(S_q^{2N-1})$ a quantum homogeneous space for $\mathcal{O}(U_q(N))$. There is a natural C^*-norm on $\mathcal{O}(U_q(N))$. Upon completion with respect to this norm, one gets $C(U_q(N))$, the underlying C^*-algebra of $U_q(N)$ which is a compact quantum group in the sense of Definition 1.2.20. $C(U_q(N))$ is also the universal C^*-algebra generated by $\mathcal{O}(U_q(N))$. Moreover, $U_q(N)$ acts on $C(S_q^{2N-1})$ in the C^*-algebraic sense lifting the algebraic coaction on $\mathcal{O}(S_q^{2N-1})$. We end this section with a lemma.

LEMMA 2.2.27. *Suppose Q is a Hopf $*$-algebra and $q^i_j \in Q$, $i,j = 1,\dots,N$ such that the following hold:*

i) $\Delta(q^i_j) = \sum_k q^i_k \otimes q^k_j$ *and* $\varepsilon(q^i_j) = \delta_{ij}$*;*
ii) q^i_j *satisfy the FRT relations* (2.2.6), (2.2.7) *and* (2.2.8)*;*
iii) $\mathbf{q}$ *satisfies* $\mathbf{q}\mathbf{q}^* = I_n$*, where* $\mathbf{q} = (q^i_j)$ *and* $\mathbf{q}^* = \overline{\mathbf{q}}^t$, $\overline{\mathbf{q}} = ((q^i_j)^*)$*.*

Then there is a unique $*$-morphism $\Psi : \mathcal{O}(U_q(N)) \to Q$ between these Hopf $*$-algebras such that $\Psi(u^i_j) = q^i_j$.

PROOF. Define $\Phi : \mathcal{O}(M_q(N)) \to Q$ on the generators u^i_j by $\Phi(u^i_j) = q^i_j$. Then by hypotheses i), ii), Definition 2.2.6 and Proposition 2.2.7, Φ extends to a bialgebra morphism. Using Remark 2.2.11, we conclude:

$$\Delta(\Phi(\mathcal{D}_q)) = (\Phi \otimes \Phi)\Delta(\mathcal{D}_q) = \Phi(\mathcal{D}_q) \otimes \Phi(\mathcal{D}_q) \tag{2.2.24}$$

and

$$\varepsilon(\Phi(\mathcal{D}_q)) = \varepsilon(\mathcal{D}_q) = 1. \tag{2.2.25}$$

Thus (2.2.24), (2.2.25) imply that $\Phi(\mathcal{D}_q)$ is a nonzero group-like element in Q. Moreover, applying $(S \otimes \mathrm{id})$ and $(\mathrm{id} \otimes S)$ on either side of (2.2.24), we observe $S(\Phi(\mathcal{D}_q))\Phi(\mathcal{D}_q) = \Phi(\mathcal{D}_q)S(\Phi(\mathcal{D}_q)) = \varepsilon(\Phi(\mathcal{D}_q))1 = 1$. Hence $\Phi(\mathcal{D}_q)$ is invertible with $\Phi(\mathcal{D}_q)^{-1} = S(\Phi(\mathcal{D}_q))$ We also have $q^i_j\Phi(\mathcal{D}_q) = \Phi(\mathcal{D}_q)q^i_j$, by the centrality of $\mathcal{D}_q$, (see Remark 2.2.11), hence q^i_j commute with $\Phi(\mathcal{D}_q)^{-1} = S(\Phi(\mathcal{D}_q))$.

Now define $\tilde{\Phi} : \mathcal{O}(M_q(N))[t] \to Q$ by

$$\tilde{\Phi}(u^i_j) = q^i_j, \quad \text{and} \quad \tilde{\Phi}(t) = S(\Phi(\mathcal{D}_q)).$$

This is well-defined because q^i_j commute with $S(\Phi(\mathcal{D}_q))$. Clearly, $\tilde{\Phi}$ sends the ideal generated by $t\mathcal{D}_q - 1$ to 0. Hence $\tilde{\Phi}$ descends to $\Psi : \mathcal{O}(GL_q(N)) \to Q$ which is, by Proposition 2.2.23, a bialgebra morphism.

We are left to show that Ψ is a $*$-map, with respect to the Hopf $*$-algebra structure on $\mathcal{O}(GL_q(N))$ defined in Proposition 2.2.24 i.e., the Hopf $*$-structure of $\mathcal{O}(U_q(N))$. To this end, applying $m(S \otimes \mathrm{id})$ and $m(\mathrm{id} \otimes S)$ on either side of the relation $\Delta(q^i_j) = \sum_k q^i_k \otimes q^k_j$ (hypothesis i)), it follows that the antipode S satisfies

$$S(\mathbf{q})\mathbf{q} = \mathbf{q}S(\mathbf{q}) = I_N,$$

i.e., $S(\mathbf{q}) = \mathbf{q}^{-1}$, where $S(\mathbf{q}) = (S(q^i_j))$. This, together with hypothesis iii), implies $S(\mathbf{q}) = \mathbf{q}^*$ i.e., $S(q^i_j) = (q^j_i)^*$. Since Ψ is a bialgebra morphism, it preserves the antipode and so Proposition 2.2.24 implies that Ψ is a $*$-morphism. Finally, the proof of uniqueness is straightforward, hence omitted. $\square$

2.3. Main results

We are now prepared to describe the main results obtained in [**BG19a**].

2.3.1. Quantum symmetry of the odd sphere - algebraic version. We first look for quantum symmetry of the odd sphere in a completely algebraic way. Later, we will fuse these with analytic tools to obtain the quantum isometry group of the odd sphere. We start with a basic observation.

LEMMA 2.3.1. *Let H be any cosemisimple Hopf $*$-algebra with the Haar functional $\mathbf{h}$. Then for $a, b \in H$, $\mathbf{h}(a^*_{(1)}b)a^*_{(2)} = \mathbf{h}(a^*b_{(1)})S(b_{(2)})$.*

PROOF. By definition, $\mathbf{h}(x)1 = \mathbf{h}(x_{(1)})x_{(2)}$ for $x \in H$. Applying the antipode S, we get $\mathbf{h}(x)1 = \mathbf{h}(x_{(1)})S(x_{(2)})$. Now,

$$\begin{aligned}\mathbf{h}(a^*_{(1)}b)a^*_{(2)} &= \mathbf{h}(a^*_{(1)}b_{(1)})S(a^*_{(2)}b_{(2)})a^*_{(3)}\\ &= \mathbf{h}(a^*_{(1)}b_{(1)})S(b_{(2)})S(a^*_{(2)})a^*_{(3)}\\ &= \mathbf{h}(a^*_{(1)}b_{(1)})S(b_{(2)})\varepsilon(a^*_{(2)})\\ &= \mathbf{h}(a^*_{(1)}\varepsilon(a^*_{(2)})b_{(1)})S(b_{(2)})\\ &= \mathbf{h}(a^*b_{(1)})S(b_{(2)}).\end{aligned}$$

□

The following lemma exploits the relation between the two apparently different concepts, namely coquasitriangularity and faithfulness of the Haar functional.

LEMMA 2.3.2. *Let H be a coquasitriangular CQG algebra with Haar functional $\mathbf{h}$ and real universal r-form $\mathbf{r}$. Let the induced inner product be denoted by $\langle , \rangle$, i.e., $\langle a, b\rangle = \mathbf{h}(a^*b)$. Let V be any subcomodule of H and $r_{V,V}$ be the induced morphism on $V \otimes V$. Then $r_{V,V}$ is hermitian with respect to the restricted inner product.*

PROOF. We have

$$\begin{aligned}\langle \mathbf{r}_{V,V}(v\otimes w), v'\otimes w'\rangle &= \overline{\mathbf{r}(v_{(1)}, w_{(1)})}\langle w_{(0)}\otimes v_{(0)}, v'\otimes w'\rangle\\ &= \mathbf{r}(w^*_{(1)}, v^*_{(1)})\langle w_{(0)}, v'\rangle\langle v_{(0)}, w'\rangle \qquad \text{(we use reality of } \mathbf{r})\\ &= \mathbf{r}(\mathbf{h}(w^*_{(0)}v')w^*_{(1)}, \mathbf{h}(v^*_{(0)}w')v^*_{(1)})\\ &= \mathbf{r}(\mathbf{h}(w^*v'_{(0)})S(v'_{(1)}), \mathbf{h}(v^*w'_{(0)})S(w'_{(1)})) \qquad \text{(using Lemma 2.3.1)}\\ &= \mathbf{r}(S(v'_{(1)}), S(w'_{(1)}))\langle w, v'_{(0)}\rangle\langle v, w'_{(0)}\rangle\\ &= \mathbf{r}(v'_{(1)}, w'_{(1)})\langle v\otimes w, w'_{(0)}\otimes v'_{(0)}\rangle \qquad \text{(by Remark 2.2.3)}\\ &= \langle v\otimes w, \mathbf{r}_{V,V}(v'\otimes w')\rangle.\end{aligned}$$

□

We recall a standard fact. Let V be a finite dimensional vector space with inner product $\langle , \rangle$. Let H be a Hopf $*$-algebra with a coaction $\rho : V \to V \otimes H$ on V. We say that ρ preserves the inner product if $\langle v_{(0)}, w_{(0)}\rangle v^*_{(1)}w_{(1)} = \langle v, w\rangle 1$ for all $v, w \in V$, where we use Sweedler notation, i.e., $\rho(v) = v_{(0)} \otimes v_{(1)}$ and likewise for $\rho(w)$.

PROPOSITION 2.3.3. [**KS97**, page 402] *If ρ preserves the inner product on V and W is a subcomodule of V then the orthogonal complement $W^\perp$ (with respect to $\langle , \rangle$) is also a subcomodule of V.*

For any CQG algebra, an inner product is given by $\langle a, b\rangle = h(a^*b)$.

LEMMA 2.3.4. *Let A and Q be Hopf $*$-algebras, $B \subset A$ a $*$-coideal subalgebra of A and a comodule algebra over Q with coaction $\rho : B \to B \otimes Q$ such that $\rho(b^*) = \rho(b)^*$ for all $b \in B$. Suppose that A is cosemisimple and ρ preserves the restriction of the Haar functional $\mathbf{h}$ of A to B i.e., $(\mathbf{h}\otimes\mathrm{id})\rho(b) = \mathbf{h}(b)1$ for all $b \in B$. Then ρ preserves the induced inner product on B given by $\langle a, b\rangle = \mathbf{h}(a^*b)$, in the sense described in the paragraph preceding Proposition 2.3.3.*

The proof is straightforward, hence omitted.

PROPOSITION 2.3.5. *Let* $\pi : \mathcal{O}(U_q(N)) \to \mathcal{O}(SU_q(N))$ *be the quotient homomorphism and* ρ_u, ρ_{su}, *respectively, be the corresponding coactions on* $\mathcal{O}(S_q^{2N-1})$ *so that* $(\mathrm{id} \otimes \pi)\rho_u = \rho_{su}$. *Then* ρ_u *preserves the restriction of the Haar functional* $\mathbf{h}$ *on* $\mathcal{O}(S_q^{2N-1})$.

PROOF. Let f be any linear functional on $\mathcal{O}(S_q^{2N-1})$ such that $f(1) = 1$. Let f' be defined as $(f \otimes h_u)\rho_u$, where h_u is the Haar functional of $U_q(N)$. Then f' is ρ_u invariant. By $(\mathrm{id} \otimes \pi)\rho_u = \rho_{su}$, we get that f' is ρ_{su} invariant too. It is well known that the restriction of $\mathbf{h}$ is the only functional with this property [**KS97**]. Hence, the conclusion follows. $\square$

We think the following is well known. We included it because we couldn't find it in the literature.

PROPOSITION 2.3.6. *The set* $\{z_1^{k_1} \dots z_N^{k_N} (z_{N-1}^*)^{l_{N-1}} \dots (z_1^*)^{l_1}, z_1^{k_1} \dots z_{N-1}^{k_{N-1}} (z_N^*)^{l_N} \dots (z_1^*)^{l_1} \mid k_1, \dots, k_N, l_{N-1}, \dots, l_1 \in \mathbb{N} \cup \{0\}, \quad l_N \in \mathbb{N}\}$ *is a vector space basis of* $\mathcal{O}(S_q^{2N-1})$.

PROOF. It is a simple application of Bergman's Diamond lemma [**Ber78**] $\square$

We state and prove below the main result concerning Hopf coactions on the quantum spheres satisfying suitable conditions.

THEOREM 2.3.7. *Let* Q *be a Hopf* $*$*-algebra coacting on* $\mathcal{O}(S_q^{2N-1})$ *by* ρ *making it a* $*$*-comodule algebra, where we have viewed* $\mathcal{O}(S_q^{2N-1})$ *as a* $*$*-coideal subalgebra of* $\mathcal{O}(SU_q(N))$. *Moreover, suppose that*

i) ρ *leaves the subspace* $V = span\{z_1, \dots, z_N\}$ *invariant, i.e.,* $\rho(z_i) = \sum_j z_j \otimes q_i^j$ *for some* $q_j^i \in Q$. *We write* $\mathbf{q} = (q_j^i)$ *for the matrix of* Q*-valued coefficients;*
ii) ρ *preserves the inner product on* V *induced by the Haar functional.*

Then there is a unique $*$*-morphism* $\Psi : \mathcal{O}(U_q(N)) \to Q$ *such that* $(\mathrm{id} \otimes \Psi)\rho_u = \rho$.

Before we go to the proof, we prove a lemma.

LEMMA 2.3.8. *In the notation of Theorem 2.3.7, let* μ *be the multiplication of* $\mathcal{O}(S_q^{2N-1})$ *restricted to* $V \otimes V$. *Then* $\ker(\mu) = \mathrm{im}(\hat{R} - qI)$.

PROOF. We first claim that $\ker(\mu) = span\{z_i \otimes z_j - q z_j \otimes z_i; i < j\}$. Clearly, $z_i \otimes z_j - q z_j \otimes z_i \in \ker(\mu)$. Moreover, $v = \sum_{i,j} c_{ij} z_i \otimes z_j \in V \otimes V$ $(c_{ij} \in \mathbb{C})$ is in $\ker(\mu)$ if and only if

$$0 = \sum_{ij} c_{ij} z_i z_j = \sum_{i<j} (c_{ij} + q^{-1} c_{ji}) z_i z_j + \sum_i c_{ii} z_i^2 \qquad \text{(by (2.2.16))}.$$

It follows, by the linear independence of $\{z_i z_j, z_i^2; i < j\}$ (Proposition 2.3.6), that $c_{ii} = 0$ for all i and $c_{ij} + q^{-1} c_{ji} = 0$ for all $i < j$. Hence, v reduces to the form $v = \sum_{i<j} c_{ij}(z_i \otimes z_j - q z_j \otimes z_i)$, proving the claim.

On the other hand, using the definition of $\hat{R}$ given by (2.2.10), an easy computation gives

$$(\hat{R} - qI)(z_i \otimes z_j) = \begin{cases} q^{-1}(z_i \otimes z_j - q z_j \otimes z_i) & i < j, \\ z_j \otimes z_i - q z_i \otimes z_j & i > j, \\ 0 & i = j. \end{cases} \tag{2.3.1}$$

Hence $\mathrm{im}(\hat{R} - qI) = span\{z_i \otimes z_j - q z_j \otimes z_i; i < j\} = \ker(\mu)$. $\square$

PROOF OF THEOREM 2.3.7. We start with the observation that since V is a Q-comodule, we have that $\Delta(q_j^i) = \sum_k q_k^i \otimes q_j^k$ and $\varepsilon(q_j^i) = \delta_{ij}$.

Now recall the map σ from Remark 2.2.15. By Lemma 2.3.2, σ is hermitian. Since $\sigma = t\hat{R}$ and t is real, $\hat{R}$ is also hermitian. The multiplication of $\mathcal{O}(S_q^{2N-1})$ is a Q-comodule morphism, $\mathcal{O}(S_q^{2N-1})$ being a Q-comodule algebra. Since V is assumed to be a subcomodule of $\mathcal{O}(S_q^{2N-1})$, the restriction μ of the multiplication to $V \otimes V$ is also a Q-comodule morphism. We recall that $V \otimes V$ is a Q-comodule with coaction $\rho_{V \otimes V}(v \otimes w) = v_{(0)} \otimes w_{(0)} \otimes v_{(1)} w_{(1)}$, where $\rho(v) = v_{(0)} \otimes v_{(1)}$. As μ is a Q-comodule morphism, we have $\rho\mu = (\mu \otimes \mathrm{id})\rho_{V \otimes V}$. Thus $\ker(\mu) = \mathrm{im}(\hat{R} - qI)$ (from Lemma 2.3.8 above) is also a Q-comodule whose orthogonal complement is the image of $\hat{R}+q^{-1}I$, by Remark 2.2.8. Hence, by Proposition 2.3.3, $\mathrm{im}(\hat{R}+q^{-1}I)$ is a Q-comodule. $\hat{R}$ has two eigenvalues, namely, q and $-q^{-1}$, the corresponding eigenspaces being $\mathrm{im}(\hat{R}+q^{-1}I)$ and $\mathrm{im}(\hat{R}-qI)$, respectively. Since ρ preserves both the eigenspaces, $\hat{R}$ becomes a Q-comodule morphism. By Proposition 2.2.9, $\mathbf{q}$ then satisfies the FRT relations (2.2.6), (2.2.7), (2.2.8).

By assumption, ρ preserves the relation $\sum_{i=1}^{N} z_i z_i^* = 1$. Applying ρ to both sides, comparing coefficients and using Proposition 2.3.6 we get that $\mathbf{q}\mathbf{q}^* = I_n$.

Thus, Lemma 2.2.27 yields a unique $*$-morphism $\Psi : \mathcal{O}(U_q(N)) \to Q$ such that $\Psi(u^i_j) = q^i_j$. Moreover, $(\mathrm{id}\otimes\Psi)\rho_u$ and ρ agree on the generators z_i, hence they are equal. For any other Ψ' satisfying $(\mathrm{id}\otimes\Psi')\rho_u = \rho$, we get by evaluating both sides on the generators z_i, that $\Psi'(u^i_j) = q^i_j$. Hence, by the uniqueness in Lemma 2.2.27, $\Psi' = \Psi$. ☐

REMARK 2.3.9. *The equation* (2.2.19) *can also be written as* $\sum_{i=1}^{N} q^{-2i} z_i^* z_i = q^{-2}$. *Now applying* ρ *to both sides, comparing coefficients and using Proposition 2.3.6, we see that* $\mathbf{q}$ *satisfies* $E\overline{\mathbf{q}}E^{-1}\mathbf{q}^t = \mathbf{q}^t E\overline{\mathbf{q}}E^{-1} = I_n$, *where* E *is the matrix*

$$\frac{1}{q^{n-1}[n]_q}\, diag\,(1, q^2, q^4, \ldots, q^{2(n-1)}), \qquad [n]_q := \frac{q^n - q^{-n}}{q-q^{-1}}.$$

See [**BDDD14, VDW96**].

We finally have the following theorem.

THEOREM 2.3.10. *Consider the category* $\mathcal{C}$ *consisting of Hopf* $*$*-algebras satisfying the hypotheses of Theorem 2.3.7 as objects and Hopf* $*$*-algebra morphisms intertwining the coactions as morphisms. Then* $\mathcal{O}(U_q(N))$ *is a universal object in this category.*

PROOF. By Proposition 2.3.5 and Lemma 2.3.4, $\mathcal{O}(U_q(N))$ is an object in this category. Then Theorem 2.3.7 shows that $\mathcal{O}(U_q(N))$ is universal with that property. ☐

2.3.2. Quantum symmetry of the odd sphere - analytic version. We provide an application of the main result of the previous section, namely, that of determining the quantum isometry group of the odd sphere, in the sense of [**BG09**]. We begin by describing the unitary representations of the compact quantum group $SU_q(N)$.

Irreducible unitary representations of the quantum group $SU_q(N)$ are indexed by Young tableaux $\lambda = (\lambda_1, \ldots, \lambda_N)$, where λ_i's are nonnegative integers $\lambda_1 \geq \cdots \geq \lambda_N$ [**Wor88**]. Let H_λ be the carrier Hilbert space corresponding to λ whose basis elements are parametrized by arrays of the form

$$\mathbf{r} = \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1,N-1} & r_{1N} \\ r_{21} & r_{22} & \cdots & r_{2,N-1} & \\ \vdots & \vdots & & & \\ r_{N-1,1} & r_{N-1,2} & & & \\ r_{N1} & & & & \end{bmatrix},$$

where r_{ij}'s are integers satisfying $r_{1j} = \lambda_j$ for $j = 1, \ldots, N$, $r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0$ for all i, j. Such arrays are known as Gelfand-Tsetlin (GT) tableaux (see [**CP08**] for details). For a GT tableaux $\mathbf{r}$, $\mathbf{r}_i$ will denote its ith row.

Let us denote the Haar functional of $SU_q(N)$ again by $\mathbf{h}$. Let $L^2(SU_q(N))$ denote the corresponding GNS space and $L^2(S_q^{2N-1})$ denote the closure of $C(S_q^{2N-1})$ in $L^2(SU_q(N))$. Let us quote the following result from [**CP08**].

PROPOSITION 2.3.11. *Assume* $N > 2$. *The restriction of the right regular representation of* $SU_q(N)$*to* $L^2(S_q^{2N-1})$ *decomposes as a direct sum of the irreducibles, with each copy occurring exactly once, given by the Young tableau* $\lambda_{n,k} = (n+k, k, k, \ldots, k, 0)$ *with* $n, k \in \mathbb{N}_0$.

Let the irreducible representation corresponding to the Young tableau $\lambda_{n,k} = (n+k, k, k, \ldots, k, 0)$ be denoted by $V_{n,k}$. According to our previous notation $V = V_{1,0}$, the irreducible with Young tableau $\lambda_{1,0} = (1, 0, \dots, 0)$. Moreover, observe that $V_{0,1}$, the irreducible with Young tableau $\lambda_{0,1} = (1, 1, \dots, 1, 0)$, is the conjugate corepresentation to V. In our previous notation, this is nothing but the span of $z_1^*, \dots, z_N^*$, i.e., $V_{0,1} = V^* = \{v^* \mid v \in V\}$.

Consider the space

$$W^{\otimes(n,k)} := \underbrace{V_{1,0} \otimes \cdots \otimes V_{1,0}}_{\text{n times}} \otimes \underbrace{V_{0,1} \otimes \cdots \otimes V_{0,1}}_{\text{k times}}.$$

Let us denote the image of it in the algebra under multiplication by

$$W^{\bullet(n,k)} := \underbrace{V_{1,0} \bullet \cdots \bullet V_{1,0}}_{\text{n times}} \bullet \underbrace{V_{0,1} \bullet \cdots \bullet V_{0,1}}_{\text{k times}}.$$

Let $\Lambda^{\otimes(n,k)}$ be the set of the Young tableaux for the irreducible representations occurring in the decomposition of $W^{\otimes(n,k)}$ and $\Lambda^{\bullet(n,k)}$ be the corresponding set for the decomposition of $W^{\bullet(n,k)}$. The following proposition gives a description of $W^{\bullet(n,k)}$.

PROPOSITION 2.3.12. *i) $V_{n,k}$ occurs with multiplicity exactly one in the orthogonal decomposition of $W^{\bullet(n,k)}$ into irreducibles;*

ii) If a copy of $V_{m,l}$ occurs in the irreducible decomposition of the orthogonal complement $(V_{n,k})^{\perp}$ of $V_{n,k}$ in $W^{\bullet(n,k)}$ then we must have:

$$\textit{either,}\quad l < k \ \textit{and}\ m \le n + k - l;$$
$$\textit{or,}\quad l = k \ \textit{and}\ m < n.$$

PROOF. By Proposition 2.3.11, it follows that any $\lambda \in \Lambda^{\bullet(n,k)}$ must be of the form $\lambda_{m,l} = (m+l, l, \dots, l, 0)$ and has multiplicity one. It is known (see e.g., [**CP95**, page 326, Proposition 10.1.16]) that any $\lambda \in \Lambda^{\otimes(n,k)}$ is dominated by $\lambda_{n,k} = (n + k, k, \dots, k, 0)$ which is equivalent to $l \le k$ and $m + l \le n + k$. ii) follows immediately from these remarks.

For i), let us first remark that $\mathcal{O}(SU_q(N))$, hence $\mathcal{O}(S_q^{2N-1})$ is an integral domain (see [**KS98**, page 98]). Now, if v and $\overline{v}$ are the primitive vectors for $V_{1,0}$ and $V_{0,1}$, respectively, then $v^n\overline{v}^k \in W^{\bullet(n,k)}$ is nonzero and belongs to the weight space corresponding to $\lambda_{n,k}$, which follows from the definition of weight spaces as in [**CP95**] and the fact that each K_i is group-like ([**CP95**], so that $K_i(xy) = (K_ix)(K_iy)$). ☐

The following will be useful in constructing a non-commutative structure on the sphere, see [**CP08**].

PROPOSITION 2.3.13. *Let Γ_0 be the set of all GT tableaux $\mathbf{r}^{nk}$ given by*

$$r_{ij}^{nk} = \begin{cases} n+k & \textit{if} \quad i = j = 1 \\ 0 & \textit{if} \quad i = 1, j = N \\ k & \textit{otherwise,} \end{cases}$$

for some $n, k \in \mathbb{N}$. Let Γ_0^{nk} be the set of all GT tableaux with top row $(n + k, k, k, \dots, k, 0)$. Then the family of vectors

$$\{e_{\mathbf{r}^{nk},\mathbf{s}} \mid n, k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{nk}\}$$

form a complete orthonormal basis for $L^2(S_q^{2N-1})$.

Let us now put a non-commutative structure on the quantum sphere.

THEOREM 2.3.14. *Let $A = \mathcal{O}(S_q^{2N-1})$ and H be $L^2(S_q^{2N-1})$. Take π to be the inclusion. Finally, define the operator $D : e_{\mathbf{r^{nk}},\mathbf{s}} \mapsto d(\mathbf{r}^{nk})e_{\mathbf{r^{nk}},\mathbf{s}}$ on $L^2(S_q^{2N-1})$ where the $d(\mathbf{r}^{nk})$'s are given by*

$$d(\mathbf{r}^{nk}) = \begin{cases} -k & \textit{if} \quad n = 0 \\ n+k & \textit{if} \quad n > 0. \end{cases}$$

Then (A, H, D), as constructed above, is a spectral triple of compact type on $\mathcal{O}(S_q^{2N-1})$.

We recall the following for the reader's convenience.

THEOREM 2.3.15. *Let (A, H, D) be a spectral triple of compact-type and assume that D has a one dimensional eigenspace spanned by a unit vector ξ, which is cyclic and separating for the algebra A. Moreover, assume that each eigenvector of D belongs to the dense subspace $A\xi$ of H. Then*

i) there exists a universal object $(\widetilde{S_0}, u_0)$ in the category $\mathbf{Q}(A, H, D)$.

ii) $\widetilde{S_0}$ admits a comultiplication Δ_0 such that $(\widetilde{S_0}, \Delta_0)$ is a compact quantum group and $(\widetilde{S_0}, \Delta_0, u_0)$ is a universal object in the category $\mathbf{Q}'(A, H, D)$.

Let S_0 be the Woronowicz C^*-subalgebra of $(\widetilde{S_0}, \Delta_0)$ generated by $\{(t_{\xi,\eta} \otimes \mathrm{id})(\mathrm{ad}_{u_0}(a)) \mid \xi, \eta \in H, a \in A\}$. It is the largest Woronowicz C^*-subalgebra of $(\widetilde{S_0}, \Delta_0)$ for which ad_{u_0} is faithful on A.

Notation. We write $\mathrm{Qiso}^+(A, H, D)$ for S_0, $\widetilde{\mathrm{Qiso}^+(A, H, D)}$ for $\widetilde{S_0}$ and refer to $\mathrm{Qiso}^+(A, H, D)$ as the quantum group of orientation-preserving isometries of (A, H, D).

For the spectral triple in Theorem 2.3.14, the cyclic separating vector ξ is $1_{\mathcal{O}(S_q^{2N-1})}$. The following is from [**BG09**]

DEFINITION 2.3.16. *Let (A, H, D) be the spectral triple in Theorem 2.3.14. Let $\widehat{\mathbf{Q}}(A, H, D)$ be the category with objects (S, α) where S is a compact quantum group with an action on A such that*

i) α is $\mathbf{h}$ preserving;

ii) α commutes with $\widehat{D}$, i.e., $\alpha\widehat{D} = (\widehat{D} \otimes \mathrm{id})\alpha$, where $\widehat{D}$ is the operator $A \to A$ given by $\widehat{D}(a)\xi = D(a\xi)$, ξ as in Theorem 2.3.15 which is $1_{\mathcal{O}(S_q^{2N-1})}$ in our case.

Note that the eigenspaces of $\widehat{D}$ and D are in one-one correspondence. In fact, eigenspaces of $\widehat{D}$ are of the form $\{a \in A \mid a\xi \in V_\lambda\}$, where V_λ is the finite dimensional eigenspace of D with respect to eigenvalue λ. We recall

PROPOSITION 2.3.17. *There exists a universal object S in the category $\widehat{\mathbf{Q}}(A, H, D)$ and it is isomorphic to $\mathrm{Qiso}^+(A, H, D)$.*

We conclude by describing the quantum isometry group of the sphere. This generalizes [**BG09**, Theorem 4.13, p. 2559].

LEMMA 2.3.18. *Given a compact quantum group S with an action α on A, the following are equivalent:*

i) (S, α) is an object of the category $\widehat{\mathbf{Q}}(A, H, D)$, (A, H, D) as in Theorem 2.3.14;

ii) α is linear (meaning it preserves V as in Theorem 2.3.7) and preserves $\mathbf{h}$;

iii) α preserves each irreducible $V_{n,k}$, occurring in Proposition 2.3.11.

PROOF. i) $\implies$ ii): Since α commutes with $\widehat{D}$, it preserves the eigenspaces of $\widehat{D}$, in particular V, and by definition of the category $\widehat{\mathbf{Q}}(A, H, D)$, it preserves $\mathbf{h}$.

ii) $\implies$ iii): Recall the notation $W^{\bullet(n,k)}$ from the discussion below Proposition 2.3.11. Clearly, α preserves $W^{\bullet(n,0)}$ as it is a homomorphism and preserves $V_{1,0}$. It also preserves $W^{\bullet(0,k)}$ because it is a $*$-homomorphism and preserves $V_{0,1} = V_{1,0}^*$. Hence α preserves $W^{\bullet(n,k)}$. We will use this fact to show α preserves $V_{n,k}$ for all n and k. Let $\mathbf{P}(k)$ be the statement "α preserves $V_{n,k}$ for all n". We now proceed to prove this statement for all k by induction. We break the proof in several steps.

Step1: We prove that $\mathbf{P}(0)$ holds. Thus we need to show α preserves $V_{n,0}$ for all n. We use induction on n. Clearly, α preserves $V_{0,0}$. Next, we assume that α preserves $V_{m,0}$ for all $m < n$. By Proposition 2.3.12, each irreducible contained in $(V_{n,0})^\perp$ is of the form $V_{m,0}$ with $m < n$, which is preserved by α. Hence α preserves $V_{n,0}$, by Proposition 2.3.3.

Step2: Now we assume $\mathbf{P}(l)$ holds for all $l < k$, i.e., α preserves $V_{n,l}$ for all n and for all $l < k$. We have to prove that $\mathbf{P}(k)$ holds, i.e., α preserves $V_{n,k}$ for all n. We use induction on n (with fixed k) to prove this.

Step2a: We prove that α preserves $V_{0,k}$. By Proposition 2.3.12, each irreducible occurring in $(V_{0,k})^{\perp}$ is of the form $V_{0,l}$ with $l < k$. By assumption α preserves $V_{0,l}$ for all $l < k$. Hence, by Proposition 2.3.3, α preserves $V_{0,k}$.

Step2b: Next we assume that α preserves $V_{m,k}$ for all $m < n$ and prove that it also preserves $V_{n,k}$ (k fixed). By Proposition 2.3.12, each irreducible contained in $(V_{n,k})^{\perp}$ is of the form $V_{m,l}$ with $l < k$ or $V_{m,k}$ with $m < n$. α preserves $V_{m,l}$ with $l < k$ (the main induction hypothesis that $\mathbf{P}(l)$ holds for all $l < k$) and $V_{m,k}$ with $m < n$, by the induction hypothesis at the beginning of the present step. Hence, by Proposition 2.3.3, α preserves $V_{n,k}$. This completes the induction on n, proving α preserves $V_{n,k}$ for all n.

So we have proved $\mathbf{P}(0)$ holds and $\mathbf{P}(k)$ holds assuming $\mathbf{P}(l)$ holds for all $l < k$. This completes the induction on k, thus completing the proof of ii) $\implies$ iii).

iii) $\implies$ i): Condition iii) implies that α leaves each eigenspaces of $\widehat{D}$ invariant, hence, it commutes with $\widehat{D}$. Since $\mathbf{h}(1) = 1$ and $\ker(\mathbf{h})$ is the span of all $V_{n,k}$ with $n + k \neq 0$, α preserves $\ker(\mathbf{h})$. But then, $a - \mathbf{h}(a)1 \in \ker(\mathbf{h})$, so that $\alpha(a - \mathbf{h}(a)1) = \alpha(a) - \mathbf{h}(a)(1 \otimes 1) \in \ker(\mathbf{h}) \otimes S$, implying $(\mathbf{h} \otimes \mathrm{id})(\alpha(a)) = \mathbf{h}(a)1$. Hence, α preserves $\mathbf{h}$. $\square$

THEOREM 2.3.19. *The quantum group of orientation-preserving isometries for the spectral triple in Theorem 2.3.14 is the compact quantum group $U_q(N)$.*

PROOF. By Proposition 2.3.17, there exists a universal object (S, α) in $\widehat{\mathbf{Q}}(A, H, D)$. By ii) and iii) of Lemma 2.3.18, α preserves $\mathbf{h}$ and leaves the algebra generated by $\{V_{n,k}; n, k \in \mathbb{N} \cup \{0\}\}$, i.e., $\mathcal{O}(S_q^{2N-1})$ invariant. Moreover, as each $V_{n,k}$ is finite dimensional, $\alpha(\mathcal{O}(S_q^{2N-1})) \subset \mathcal{O}(S_q^{2N-1}) \otimes \mathcal{S}$, where $\mathcal{S}$ is the dense Hopf $*$-algebra inside the compact quantum group S. In particular, by ii) of Lemma 2.3.18, the coaction of $\mathcal{S}$ on $\mathcal{O}(S_q^{2N-1})$ satisfies the hypotheses of Theorem 2.3.7, hence is an object of the category $\mathcal{C}$ as in Theorem 2.3.10. Thus we have a unique morphism $\Psi : \mathcal{O}(U_q(N)) \to \mathcal{S}$ in $\mathcal{C}$. As $C(U_q(N))$ is the universal C^*-algebra corresponding to $\mathcal{O}(U_q(N))$, Ψ extends to a C^*-algebra morphism (again denoted by Ψ) $C(U_q(N)) \to S$. It is clearly a morphism of compact quantum groups and intertwines the two (C^*-algebraic) actions, hence a morphism in $\widehat{\mathbf{Q}}(A, H, D)$.

On the other hand, it follows from Lemma 2.3.18 that $U_q(N)$ is an object in the category $\widehat{\mathbf{Q}}(A, H, D)$ as the canonical action of the compact quantum group $U_q(N)$ on $C(S_q^{2N-1})$ satisfies ii) of Lemma 2.3.18. Hence we have a unique morphism $\Theta : S \to C(U_q(N))$ in $\widehat{\mathbf{Q}}(A, H, D)$. Clearly, $\Psi\Theta$ is the identity morphism, since S is the universal object. To show that $\Theta\Psi$ is the identity morphism, we observe that it is a morphism of compact quantum groups, hence takes $\mathcal{O}(U_q(N))$ to itself. Since $\mathcal{O}(U_q(N))$ is universal in $\mathcal{C}$, $\Theta\Psi$ is identity, at least on $\mathcal{O}(U_q(N))$. We recall that $C(U_q(N))$ is the universal C^*-algebra generated by $\mathcal{O}(U_q(N))$, hence $\Theta\Psi$ lifts uniquely to $C(U_q(N))$, implying that $\Theta\Psi$ is also the identity morphism. Thus S is isomorphic to $U_q(N)$. $\square$

CHAPTER 3

Noncommutative complex geometry

In this chapter, we briefly describe the framework of noncommutative complex and Kähler geometry, developed in [**B́16**, **B́17**]. See also [**BPS13**, **KLvS11**, **KM11**, **PS03**].

3.1. Preliminaries

In this section we gather some preliminaries.

3.1.1. Quantum homogeneous space. We start by defining quantum homogeneous spaces, on which the theory is developed. All the Hopf algebras appearing will be assumed to be cosemisimple, $\mathbf{h}$ denoting the Haar functional. Let G be a Hopf algebra. For a right G-comodule V with coaction ρ, we say that an element $v \in V$ is coaction-invariant if $\rho(v) = v \otimes 1$. We denote the subspace of all coaction-invariant elements by V^G, and call it the coaction-invariant subspace of the coaction. We also use the analogous conventions for left comodules.

Definition 3.1.1. *For H a Hopf algebra, a homogeneous right H-coaction on G is a coaction of the form $(\mathrm{id} \otimes \pi)\Delta$, where $\pi : G \to H$ is a surjective Hopf algebra map. A quantum homogeneous space $M := G^H$ is the coaction-invariant subspace of such a coaction.*

In the rest of this chapter, we will always use the symbols G, H, π and M in this sense. As is easily seen, M is a subalgebra of G. Moreover, if G and H are Hopf $*$-algebras, and π is a Hopf $*$-algebra map, then M is a $*$-subalgebra of G.

The most well-known example of a quantum homogeneous space is the Podleś sphere [**Pod87**]. The odd sphere introduced in 2.2.4 is also an example of a quantum homogeneous space. In fact, it is the q-deformation of the homogeneous space $SU(N+1)/SU(N)$. We will later meet the quantum projective space which is the main example of this chapter.

3.1.2. Complexes and Double Complexes. For $(S, +)$ a commutative semigroup, an S-graded algebra is an algebra of the form $A = \bigoplus_{s \in S} A^s$, where each A^s is a linear subspace of A, and $A^s A^t \subset A^{s+t}$, for all $s, t \in S$. If $a \in A^s$, then we say that a is a homogeneous element of degree s. A homogeneous mapping of degree t on A is a linear mapping $L : A \to A$ such that if $a \in A^s$, then $L(a) \in A^{s+t}$. We say that a subspace B of A is homogeneous if it admits a decomposition $B = \oplus_{s \in S} B^s$, with $B^s \subset A^s$, for all $s \in S$.

A pair (A, d) is called a complex if A is an $\mathbb{N}_0$-graded algebra, and d is a homogeneous mapping of degree 1 such that $d^2 = 0$. A triple $(A, \partial, \overline{\partial})$ is called a double complex if A is an $\mathbb{N}_0^2$-graded algebra, ∂ is homogeneous mapping of degree $(1, 0)$, $\overline{\partial}$ is homogeneous mapping of degree $(0, 1)$, and

$$\partial^2 = \overline{\partial}^2 = 0, \quad \partial\overline{\partial} = -\overline{\partial}\partial. \tag{3.1.1}$$

Note we can associate to any double complex $(A, \partial, \overline{\partial})$ three different complexes

$$(A, d := \partial + \overline{\partial}), \quad (A, \partial), \quad (A, \overline{\partial}), \tag{3.1.2}$$

where the $\mathbb{N}_0$-grading on A is given by $A^k := \bigoplus_{a+b=k} A^{(a,b)}$.

For any complex (A, d), we call an element d-closed if it is contained in $\ker(d)$, and d-exact if it is contained in $\mathrm{im}(d)$. Moreover, the d-cohomology group of order k is the space

$$H_d^k := \frac{\ker(d : A^k \to A^{k+1})}{\mathrm{im}(d : A^{k-1} \to A^k)}. \tag{3.1.3}$$

For a double complex $(A, \partial, \overline{\partial})$ we define ∂-closed, $\overline{\partial}$-closed, ∂-exact, and $\overline{\partial}$-exact forms analogously. The ∂-cohomology group H^k_{∂}, and the $\overline{\partial}$-cohomology group $H^k_{\overline{\partial}}$, are the cohomology groups of the complexes (A, ∂) and $(A, \overline{\partial})$. Finally, we note that we have the decompositions

$$H^k_{\partial} = \bigoplus_{a+b=k} H^{(a,b)}_{\partial}, \quad H^k_{\overline{\partial}} = \bigoplus_{a+b=k} H^{(a,b)}_{\overline{\partial}}, \tag{3.1.4}$$

where $H^{(a,b)}_{\partial}$ and $H^{(a,b)}_{\overline{\partial}}$ are the a-th, and b-th, cohomology groups of the complexes $(A^{(\cdot,b)}, \partial)$ and $(A^{(a,\cdot)}, \overline{\partial})$ respectively, where the gradings are the obvious ones.

3.1.3. Differential $*$-Calculi. A complex (A, d) is called a differential graded algebra if d is a graded derivation, which is to say, if it satisfies the graded Leibniz rule

$$d(\alpha\beta) = d(\alpha)\beta + (-1)^k \alpha d(\beta), \tag{3.1.5}$$

for all $\alpha \in A^k$, $\beta \in A$. The operator d is called the differential of the differential graded algebra. The following is the noncommutative version of the local description of a form in Subsection 1.3.1.

DEFINITION 3.1.2. *A differential calculus over an algebra A is a differential graded algebra (Ω, d) such that $\Omega^0 = A$, and*

$$\Omega^k = span_{\mathbb{C}}\{a_0 da_1 \wedge \ldots \wedge da_k \mid a_0, \ldots, a_k \in A\}. \tag{3.1.6}$$

We use $\wedge$ to denote the multiplication between elements of a differential calculus when both are of order greater than 0. We call an element of a differential calculus a form. A differential map between two differential calculi $(\Omega, \delta_{\Omega})$ and (Γ, d_{Γ}), defined over the same algebra A, is a bimodule map $\phi : \Omega \to \Gamma$ such that $\phi d_{\Omega} = d_{\Gamma}\phi$.

We call a differential calculus (Ω, d) over a $*$-algebra A a differential $*$-calculus if the involution of A extends to an involutive conjugate-linear map on Ω, for which $(d\omega)^* = d\omega^*$, for all $\omega \in \Omega$, and

$$(\omega \wedge \nu)^* = (-1)^{kl} \nu^* \wedge \omega^*, \tag{3.1.7}$$

for all $\omega \in \Omega^k$, $\nu \in \Omega^l$. We say that a form $\omega \in \Omega$ is real if $\omega^* = \omega$.

A differential calculus Ω over a quantum homogeneous space M is said to be covariant if $\rho : M \to G \otimes M$ extends to a necessarily unique algebra map $\rho : \Omega \to G \otimes \Omega$ such that

$$\rho(mdn) = \rho(m)(\mathrm{id} \otimes d)\rho(n) = m_{(1)}n_{(1)} \otimes m_{(2)}dn_{(2)}, \tag{3.1.8}$$

where $m, n \in M$.

3.1.4. Orientability and Closed Integrals. We say that a differential calculus has total dimension n if $\Omega^k = 0$, for all $k > n$, and $\Omega^n \neq 0$. If in addition there exists an (A, A)-bimodule isomorphism $\mathrm{vol} : \Omega^n \simeq A$, then we say that Ω is orientable. We call a choice of such an isomorphism an orientation. If Ω is a covariant calculus over a quantum homogeneous space M and vol is a covariant morphism, then we say that Ω is covariantly orientable. Note all covariant orientations are equivalent up to scalar multiple. If Ω is a $*$-calculus over a $*$-algebra, then a $*$-orientation is an orientation which is also a $*$-map. A $*$-orientable calculus is one which admits a $*$-orientation.

When the calculus is defined over a quantum homogeneous space, we define the integral, with respect to vol, to be the map which is zero on all Ω^k, for $k < n$, and

$$\int : \Omega^n \to \mathbb{C}, \quad \omega \mapsto \mathbf{h}(\mathrm{vol}(\omega)), \tag{3.1.9}$$

where $\mathbf{h}$ is the Haar functional. We say that the integral is closed if $\int d\omega = 0$, for all $\omega \in \Omega^{n-1}$.

3.2. Noncommutative Kähler structures

In this section we introduce the framework of noncommutative complex geometry.

3.2.1. Complex structures. We begin with the definition of a noncommutative complex structure.

DEFINITION 3.2.1. *An almost complex structure for a differential $*$-calculus Ω, over a $*$-algebra A, is an $\mathbb{N}_0^2$-algebra grading $\bigoplus_{(a,b)\in\mathbb{N}_0^2}\Omega^{(a,b)}$ for Ω such that*

i) $\Omega^k = \bigoplus_{a+b=k}\Omega^{(a,b)}$, *for all* $k \in \mathbb{N}_0$,
ii) $(\Omega^{(a,b)})^* = \Omega^{(b,a)}$, *for all* $(a,b) \in \mathbb{N}_0^2$.

We call an element of $\Omega^{(a,b)}$ an (a,b)-form. Let ∂ and $\overline{\partial}$ be the unique homogeneous operators of order $(1,0)$, and $(0,1)$ respectively, defined by

$$\partial|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a+1,b)}}\, d, \quad \overline{\partial}|_{\Omega^{(a,b)}} := \mathrm{proj}_{\Omega^{(a,b+1)}}\, d, \tag{3.2.1}$$

where $\mathrm{proj}_{\Omega^{(a+1,b)}}$, and $\mathrm{proj}_{\Omega^{(a,b+1)}}$, are the projections from Ω^{a+b+1} onto $\Omega^{(a+1,b)}$, and $\Omega^{(a,b+1)}$, respectively. The proof of the following lemma carries over directly from the classical setting [**Huy05**].

LEMMA 3.2.2. *If $\bigoplus_{(a,b)\in\mathbb{N}_0^2}\Omega^{(a,b)}$ is an almost complex structure for a differential $*$-calculus Ω over an algebra A, then the following two conditions are equivalent:*

i) $d = \partial + \overline{\partial}$,
ii) the triple $\left(\bigoplus_{(a,b)\in\mathbb{N}^2}\Omega^{(a,b)}, \partial, \overline{\partial}\right)$ *is a double complex.*

With this in hand,

DEFINITION 3.2.3. *When the conditions in Lemma 3.2.2 hold for an almost complex structure, then we say that it is integrable.*

We call an integrable almost complex structure a complex structure, and the double complex $(\bigoplus_{(a,b)\in\mathbb{N}^2}\Omega^{(a,b)}, \partial, \overline{\partial})$ its Dolbeault double complex. An easy consequence of integrability is that

$$\partial(\omega^*) = (\overline{\partial}\omega)^*, \quad \overline{\partial}(\omega^*) = (\partial\omega)^*, \tag{3.2.2}$$

for all $\omega \in \Omega$.

3.2.2. Hermitian and Kähler structures. Throughout this section Ω denotes a differential $*$-calculus, over an algebra A, of total dimension $2n$. As a first step towards the definition of a hermitian form, we present a direct noncommutative generalization of the classical definition of an almost symplectic form.

DEFINITION 3.2.4. *An almost symplectic form for Ω is a central real 2-form σ such that, with respect to the Lefschetz operator*

$$L : \Omega \to \Omega, \quad \omega \mapsto \sigma \wedge \omega, \tag{3.2.3}$$

isomorphisms are given by

$$L^{n-k} : \Omega^k \to \Omega^{2n-k}, \tag{3.2.4}$$

for all $1 \leq k < n$.

Note that since σ is a central real form, L is an (A,A)-bimodule $*$-homomorphism. Moreover, if σ is an almost symplectic form for a covariant calculus over a quantum homogeneous space M, then L is a covariant morphism if and only if σ is a left G-coaction-invariant form.

DEFINITION 3.2.5. *For L the Lefschetz operator of any almost symplectic form, the space of primitive k-forms is*

$$P^k := \{\alpha \in \Omega^k \mid L^{n-k+1}(\alpha) = 0\}, \ \textit{if } k \leq n, \quad \textit{and} \quad P^k := 0, \ \textit{if } k > n. \tag{3.2.5}$$

One has

PROPOSITION 3.2.6. *For L the Lefschetz operator of any almost symplectic form, we have the A-bimodule decomposition*

$$\Omega^k \simeq \bigoplus_{j\geq 0} L^j(P^{k-2j}), \tag{3.2.6}$$

which we call the Lefschetz decomposition.

Now we present the following noncommutative generalization of the classical notion of a symplectic form [**Huy05**].

DEFINITION 3.2.7. *A symplectic form is a d-closed almost symplectic form.*

We now introduce a hermitian structure for a differential $*$-calculus, which is essentially just a symplectic form interacting with a complex structure in a natural way. In the commutative case each such form is the fundamental form of a uniquely identified hermitian metric [**Huy05**].

DEFINITION 3.2.8. *An hermitian structure for a $*$-calculus Ω is a pair $(\Omega^{(\cdot,\cdot)}, \sigma)$ where $\Omega^{(\cdot,\cdot)}$ is a complex structure and σ is an almost symplectic form, called the hermitian form, such that $\sigma \in \Omega^{(1,1)}$.*

When Ω is a covariant $*$-calculus over a quantum homogeneous space, $\Omega^{(\cdot,\cdot)}$ is a covariant complex structure, and σ is a left G-coaction-invariant form, then we say that $(\Omega^{(\cdot,\cdot)}, \sigma)$ is a covariant hermitian structure.

DEFINITION 3.2.9. *For $h \in \mathbb{R}_{>0}$, the h-Hodge map associated to a hermitian structure is the morphism uniquely defined by*

$$\star_h\left(L^j(\omega)\right) = (-1)^{\frac{k(k+1)}{2}} i^{a-b} \frac{[j]_h!}{[n-j-k]_h!} L^{n-j-k}(\omega), \quad \omega \in P^{(a,b)} \subset P^k, \tag{3.2.7}$$

where $[m]_h := h^{m-1} + h^{m-3} + \cdots + h^{-m+1}$ denotes the quantum integer corresponding to m. We call h the Hodge parameter of the Hodge map.

We have

LEMMA 3.2.10. *It holds that*

i) $\star_h^2(\omega) = (-1)^k \omega$, *for all* $\omega \in \Omega^k$,
ii) $\star_h$ *is an isomorphism,*
iii) $\star_h(\Omega^{(a,b)}) = \Omega^{(n-b,n-a)}$,
iv) $\star_h$ *is a $*$-map.*

By reversing the classical order of definition, we use the Hodge map to associate a metric to any hermitian structure. Here, Ω denotes a differential $*$-calculus of total dimension $2n$, and $(\Omega^{(\cdot,\cdot)}, \kappa)$ denotes a hermitian structure for Ω.

DEFINITION 3.2.11. *The metric associated to the hermitian structure $(\Omega^{(\cdot,\cdot)}, \kappa)$ is defined to be the map $g : \Omega \otimes_M \Omega \to M$ for which $g(\Omega^k \otimes_M \Omega^l) = 0$, for all $k \neq l$, and*

$$g(\omega \otimes \nu) = \mathrm{vol}(\omega \wedge \star_h(\nu^*)), \quad \omega, \nu \in \Omega^k. \tag{3.2.8}$$

With this definition of the metric, one has

LEMMA 3.2.12. *It holds that*

i) the $\mathbb{N}_0^2$-decomposition of Ω is orthogonal with respect to g,
ii) the Lefschetz decomposition of Ω is orthogonal with respect to g.

As a corollary to this,

COROLLARY 3.2.13. *It holds that $g(\omega \otimes_M \nu) = (g(\nu \otimes_M \omega))^*$, for all $\omega, \nu \in \Omega$.*

Now we specialize to the case where Ω is a covariant calculus over a quantum homogeneous space M, and $(\Omega^{(\cdot,\cdot)}, \kappa)$ is a covariant hermitian structure.

DEFINITION 3.2.14. *A hermitian structure is said to be positive definite if* $g(\omega \otimes_M \omega^*) > 0$ *for* $0 \neq \omega \in \Omega$.

We are now ready to introduce the inner product associated to a hermitian structure and to establish the existence of adjoints with respect to this pairing.

LEMMA 3.2.15. *For* $\star_h$ *the Hodge map of a positive definite hermitian structure, an inner product is given by*

$$\langle \cdot, \cdot \rangle : \Omega \otimes \Omega \to \mathbb{C}, \quad \omega \otimes \nu \mapsto \int \omega \wedge \star_h(\nu^*) = \mathbf{h}g(\omega \otimes_M \nu^*). \tag{3.2.9}$$

Moreover, the Peter–Weyl decomposition of Ω *is orthogonal with respect to* $\langle \cdot, \cdot \rangle$.

We continue to assume that Ω is a covariant $*$-calculus over a quantum homogeneous space M, and $(\Omega^{(\cdot,\cdot)}, \kappa)$ is a covariant hermitian structure. Moreover, $(\Omega^{(\cdot,\cdot)}, \kappa)$ is assumed to be positive definite.

LEMMA 3.2.16. *For all values of the Hodge parameter* h, *the Hodge map is unitary.*

One also has

LEMMA 3.2.17. *It holds that* $\Lambda := L^* = \star_h^{-1} L \star_h$.

As a corollary

COROLLARY 3.2.18. *It holds that* $P^k = \ker(\Lambda : \Omega^k \to \Omega^{k-2})$.

Consider now the counting operators,

$$H, K : \Omega \to \Omega, \quad H(\omega) = (k-n)\omega, \quad K(\omega) = h^{k-n}\omega, \quad \omega \in \Omega^k. \tag{3.2.10}$$

For a classical hermitian manifold the operators H, L, and Λ, define a representation of $\mathfrak{sl}_2$. We now show that in the noncommutative setting H, L, Λ, and K give a representation of the quantized enveloping algebra of $\mathfrak{sl}_2$.

PROPOSITION 3.2.19. *We have the relations*

$$[H, L]_{h^{-2}} = [2]_h LK, \quad [L, \Lambda] = H, \quad [H, \Lambda]_{h^2} = -[2]_{h^2} K\Lambda, \tag{3.2.11}$$

where $[A, B]_{h^{\pm 2}} = AB - h^{\pm 2} BA$.

COROLLARY 3.2.20. *A representation* ρ *of* $U_h(\mathfrak{sl}_2)$ *is given by*

$$\rho(E) = L, \quad \rho(K) = K, \quad \rho(F) = \Lambda. \tag{3.2.12}$$

Now we call the adjoints of d, ∂, and $\overline{\partial}$ the codifferential, holomorphic codifferential, and anti-holomorphic codifferential, respectively. Classically, these operators have expressions in terms of the Hodge operator analogous to the expression given above for the dual Lefschetz operator. The following lemma shows that this is also true in the noncommutative setting.

LEMMA 3.2.21. *It holds that*

$$d^* = - \star_h d \star_h, \quad \partial^* = - \star_h \overline{\partial} \star_h, \quad \overline{\partial}^* = - \star_h \partial \star_h . \tag{3.2.13}$$

COROLLARY 3.2.22. *For all* $\omega \in \Omega$, *it holds that*

$$d^*(\omega^*) = \big(d^*(\omega)\big)^*, \quad \partial^*(\omega^*) = \big(\overline{\partial}^*(\omega)\big)^*, \quad \overline{\partial}^*(\omega^*) = \big(\partial^*(\omega)\big)^*. \tag{3.2.14}$$

We now define the d-, ∂-, and $\overline{\partial}$-Laplacians to be, respectively,

$$\Delta_d := (d + d^*)^2, \quad \Delta_\partial := (\partial + \partial^*)^2, \quad \Delta_{\overline{\partial}} := (\overline{\partial} + \overline{\partial}^*)^2. \tag{3.2.15}$$

Moreover, we define the space of d-harmonic, ∂-harmonic, and $\overline{\partial}$-harmonic forms to be, respectively,

$$\mathcal{H}_d := \ker(\Delta_d), \quad \mathcal{H}_\partial := \ker(\Delta_\partial), \quad \mathcal{H}_{\overline{\partial}} := \ker(\Delta_{\overline{\partial}}). \tag{3.2.16}$$

When Ω is a covariant calculus over a quantum homogeneous space, $\Delta_d, \Delta_\partial$, and $\Delta_{\overline{\partial}}$, are left G-comodule maps, and so, each space of harmonic forms is a left G-comodule.

Definition 3.2.23. *A Kähler structure for a differential $*$-calculus is a hermitian structure $(\Omega^{(\cdot,\cdot)},\kappa)$ such that the hermitian form κ is d-closed. We call such a κ a Kähler form.*

Every 2-form in a $*$-calculus with total dimension 2 is obviously d-closed. Hence, just as in the classical case, with respect to any choice of complex structure, every $\kappa \in \Omega^{(1,1)}$ is a Kähler form. The following is the first set of Kähler identities.

Lemma 3.2.24. *For any Kähler structure $(\Omega^{(\cdot,\cdot)},\kappa)$, we have the following relations*

$$[\partial, L] = 0, \quad [\overline{\partial}, L] = 0, \quad [\partial^*, \Lambda] = 0, \quad [\overline{\partial}^*, \Lambda] = 0. \tag{3.2.17}$$

Now the second set of Kähler identities.

Theorem 3.2.25. *The four identities*

$$[L, \partial^*] = i\overline{\partial}, \quad [L, \overline{\partial}^*] = -i\partial, \quad [\Lambda, \partial] = i\overline{\partial}^*, \quad [\Lambda, \overline{\partial}] = -i\partial^*, \tag{3.2.18}$$

hold in both of the following cases:

i) the Hodge parameter is fixed at $h = 1$,
ii) the domain is restricted to $P^\bullet$ the space of primitive elements.

Corollary 3.2.26. *When the Hodge parameter is fixed at $h = 1$, it holds that*

$$\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0, \quad \partial^*\overline{\partial} + \overline{\partial}\partial^* = 0, \quad \Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}. \tag{3.2.19}$$

3.2.3. The Hodge decomposition and the hard Lefschetz theorem. We now come to Hodge decomposition, the principal result of the paper [**B́17**]. Here, Ω denotes a covariant $*$-calculus, of total dimension $2n$, over a quantum homogeneous space M. Moreover, $(\Omega^{(\cdot,\cdot)},\kappa)$ denotes a positive definite covariant hermitian structure such that the associated integral is closed.

Lemma 3.2.27. *It holds that*

i) $\mathcal{H}_d \simeq \ker(d) \cap \ker(d^*)$,
ii) $\mathcal{H}_\partial \simeq \ker(\partial) \cap \ker(\partial^*)$,
iii) $\mathcal{H}_{\overline{\partial}} \simeq \ker(\overline{\partial}) \cap \ker(\overline{\partial}^*)$.

Theorem 3.2.28. *The following decompositions are orthogonal with respect to $\langle\cdot,\cdot\rangle$*

i) $\Omega \simeq \mathcal{H}_d \oplus d\Omega \oplus d^*\Omega$,
ii) $\Omega \simeq \mathcal{H}_\partial \oplus \partial\Omega \oplus \partial^*\Omega$,
iii) $\Omega \simeq \mathcal{H}_{\overline{\partial}} \oplus \overline{\partial}\Omega \oplus \overline{\partial}^*\Omega$.

Corollary 3.2.29. *It holds that*

$$\ker(d) \simeq \mathcal{H}_d \oplus d\Omega, \quad \ker(\partial) \simeq \mathcal{H}_\partial \oplus d\Omega, \quad \ker(\overline{\partial}) \simeq \mathcal{H}_{\overline{\partial}} \oplus d\Omega, \tag{3.2.20}$$

and so, we have the isomorphisms

$$\mathcal{H}^k_d \to H^k_d, \quad \mathcal{H}^{(a,b)}_\partial \to H^{(a,b)}_\partial, \quad \mathcal{H}^{(a,b)}_{\overline{\partial}} \to H^{(a,b)}_{\overline{\partial}}. \tag{3.2.21}$$

Using this corollary, we show that the Hodge map and the $*$-map induce isomorphisms on the cohomology ring of Ω, and present some easy but interesting consequences.

Lemma 3.2.30. *The Hodge map $\star_h$, and the $*$-map, commute with the Laplacian Δ_d, and so, induce isomorphisms on $H^\bullet_d$.*

Proportionality of the Laplacians implies equality of harmonic forms:

$$\mathcal{H}^k_d = \bigoplus_{a+b=k} \mathcal{H}^{(a,b)}_\partial = \bigoplus_{a+b=k} \mathcal{H}^{(a,b)}_{\overline{\partial}}. \tag{3.2.22}$$

Hence, Corollary 3.2.29 implies the following decomposition of cohomology classes.

COROLLARY 3.2.31. *It holds that*

$$H_d^k \simeq \bigoplus_{a+b=k} H_{\partial}^{(a,b)} \simeq \bigoplus_{a+b=k} H_{\overline{\partial}}^{(a,b)}. \tag{3.2.23}$$

Moreover, the decomposition is independent of the choice of Kähler form.

LEMMA 3.2.32. *When the Hodge parameter is fixed at $h = 1$,*

$$[L, \Delta_d] = [\Lambda, \Delta_d] = 0. \tag{3.2.24}$$

DEFINITION 3.2.33. *For a Kähler structure, the (a,b)-primitive cohomology group is the vector space*

$$H_{prim}^{(a,b)} := \ker\Big(L^{n-(a+b)+1} : H^{(a,b)} \to H^{(n-b+1,n-a+1)}\Big). \tag{3.2.25}$$

Moreover, we denote $H_{prim}^k := \bigoplus_{a+b=k} H_{prim}^{(a,b)}$.

The following is a noncommutative generalization of the classical hard Lefschetz theorem.

THEOREM 3.2.34. *Let $(\Omega^{(\cdot,\cdot)}, \kappa)$ be a Kähler structure. Then it holds that*

i) $L^k : H_d^{n-k} \to H_d^{n+k}$ *is an isomorphism, for* $k = 0, \dots, n$,

ii) $H^k \simeq \bigoplus_{a+b=k-2i} L^i H_{prim}^{(a,b)}$.

3.3. Quantum projective space

We now come to the main example of this chapter, see [**HK06**, **B́17**, **B́16**] for more details.

3.3.1. The Heckenberger-Kolb Calculi for quantum projective space. Recall the definition of the quantum groups $U_q(N)$ and $SU_q(N)$ from Subsections 2.2.3 and 2.2.4, respectively. The quantum N-projective space is the subalgebra of coaction-invariant elements of a $\mathcal{O}(U_q(N))$-coaction on $\mathcal{O}(SU_q(N+1))$. This subalgebra is a q-deformation of the coordinate algebra of the complex manifold $SU(N+1)/U(N)$. Recall that $\mathbb{C}P^N$ is isomorphic to $SU(N+1)/U(N)$.

DEFINITION 3.3.1. *Let $\alpha_N : \mathcal{O}(SU_q(N+1)) \to \mathcal{O}(U_q(N))$ be the surjective Hopf $*$-algebra map defined by setting $\alpha_N(u_1^1) = \mathcal{D}_q^{-1}$, $\alpha_N(u_i^1) = \alpha_N(u_1^i) = 0$, for $i = 2, \dots, n+1$, and $\alpha_N(u_j^i) = u_{j-1}^{i-1}$, for $i, j = 2, \dots, n+1$. Quantum projective n-space $\mathbb{C}P_q^N$ is defined to be the quantum homogeneous space of the corresponding homogeneous coaction $(\mathrm{id} \otimes \alpha_N)\Delta$.*

We recall some details about first-order differential calculi necessary for our presentation of the Heckenberger-Kolb calculus below. A first-order differential calculus over A is a pair (Ω^1, d), where Ω^1 is an (A, A)-bimodule and $d : A \to \Omega^1$ is a linear map for which the Leibniz rule, $d(ab) = a(db) + (da)b$, for$a, b, \in A$, holds and for which $\Omega^1 = span_{\mathbb{C}}\{adb \mid a, b \in A\}$. The notions of differential map, and left-covariance when the calculus is defined over a quantum homogeneous space M, have obvious first-order analogues. The direct sum of two first-order differential calculi (Ω^1, d_Ω) and (Γ^1, d_Γ) is the first-order calculus $(\Omega^1 \oplus \Gamma^1, d_\Omega + d_\Gamma)$. Finally, we say that a left-covariant first-order calculus over M is irreducible if it does not possess any non-trivial quotients by a left-covariant M-bimodule.

We say that a differential calculus (Γ, d_Γ) extends a first-order calculus (Ω^1, d_Ω) if there exists a bimodule isomorphism $\phi : \Omega^1 \to \Gamma^1$ such that $d_\Gamma = \phi d_\Omega$. It can be shown that any first-order calculus admits an extension Ω which is maximal in the sense that there exists a unique differential map from Ω onto any other extension of Ω^1. We call this extension the maximal prolongation of the first-order calculus.

Now Heckenberger and Kolb's classification of first-order calculi over $\mathbb{C}P_q^N$ is as follows.

THEOREM 3.3.2. *There exist exactly two non-isomorphic irreducible left-covariant first-order differential calculi of finite dimension over $\mathbb{C}P_q^N$. We call the direct sum of these two calculi the Heckenberger-Kolb calculus of $\mathbb{C}P_q^N$.*

We denote these two calculi by $\Omega^{(1,0)}$ and $\Omega^{(0,1)}$, and denote their direct sum by Ω^1. The maximal prolongation of the direct sum Ω^1 is called the Heckenberger-Kolb calculus over $\mathbb{C}P_q^N$.

PROPOSITION 3.3.3. *For the Heckenberger-Kolb calculus over $\mathbb{C}P_q^N$, there is a unique covariant complex structure $\Omega^{(\cdot,\cdot)}$ such that $\Phi(\Omega^{(a,b)}) = V^{(a,b)}$, where Φ is the "Takeuchi equivalence".*

3.3.2. A Kähler structure for the Heckenberger-Kolb calculus. In this subsection we construct a covariant hermitian $(\Omega^{(\cdot,\cdot)}, \kappa)$ structure for the Heckenberger-Kolb calculus over $\mathbb{C}P_q^N$. In the classical case, it follows from the classification of covariant metrics on complex projective space that κ is equal, up to scalar multiple, to the fundamental form of the Fubini-Study metric. Throughout this subsection we will, by abuse of notation, denote $\Phi(L), \Phi(\mathrm{vol})$, and $\Phi(\star_q)$, by L, vol, and $\star_q$, respectively. One can then prove

PROPOSITION 3.3.4. *There exists a left G-coaction-invariant closed form κ such that the pair $(\Omega^{(\cdot,\cdot)}, \kappa)$ is a covariant hermitian structure for the Heckenberger-Kolb calculus over $\mathbb{C}P_q^N$.*

Moreover,

LEMMA 3.3.5. *There exists an open real interval around 1, such that when q is contained in this interval, the hermitian structure $(\Omega^{(\bullet,\bullet)}, \kappa)$ is positive definite.*

Finally we end with

THEOREM 3.3.6. *The hermitian structure $(\Omega^{(\cdot,\cdot)}, \kappa)$ for $\mathbb{C}P_q^N$ is a Kähler structure.*

CHAPTER 4

Generalized symmetry in noncommutative complex geometry

4.1. Introduction

As we saw in the last Chapter 3, the theory of noncommutative complex geometry was initiated in [**B́16**, **B́17**], although there are precursors; see [**BPS13**, **FGR97**, **KLvS11**, **PS03**]. It attempts to provide a fresh insight into various aspects of noncommutative geometry, such as the construction of spectral triples for quantum groups, by considering "complex structures". It also promises a fruitful interaction between noncommutative geometry and noncommutative projective algebraic geometry. Identifying "differential forms" as the basic objects of study, the framework of noncommutative complex geometry is developed in the setting of Woronowicz's differential calculus, see [**Wor89**]. The classical complex geometry being the obvious example, the setup in [**B́17**] takes as its motivating example the family of quantum flag manifolds. It is possible to proceed, as shown in there, as far as proving a version of the Hard Lefschetz theorem.

Singular spaces, such as the leaf space of a foliation, have been studied extensively in classical geometry as well as noncommutative geometry. These spaces provided the main impetus for the development of noncommutative geometry, see [**Con82**]. Classically, "transverse geometry" attempts to study such singular spaces using symmetry, which most of the time turns out to be a pseudogroup. This was exemplified in the beautiful paper [**Hae80**]. It led to the systematic study of spaces with pseudogroup symmetry. It is natural to ask whether one can do complex geometry over such spaces. That one can, was done in a volume of works, [**CW91**, **EKA90**], to name a few.

Now, pseudogroups and groupoids are very much noncommutative in their nature. This led to Connes' construction of the highly noncommutative groupoid C^*-algebra of the holonomy groupoid of a foliation, which was successfully applied to the questions in index theory. However, the fact that groupoids consist of symmetries is not so conspicuous in this construction. To take the symmetry into account, one is naturally led to the language of Hopf algebroids, as shown in [**Kal11**, **Mrč99**, **Mrč07**].

Thus, the study of complex geometry over such singular spaces consists of studying regular spaces with highly noncommutative symmetry, which are also generalized, in that they are not Hopf algebras.

The goal of the present chapter is to introduce Hopf algebroid symmetry in noncommutative geometry. We formulate and study a quite general framework of Hopf algebroid covariance of noncommutative complex and Kähler structures. We have been able to accommodate all the existing examples in our framework. Another notable and novel aspect of our work is a new definition of Hopf algebroid action or covariance on differential calculus which seems to work in a very general context. We present the Connes-Moscovici Hopf algebroid as one of the most interesting examples of our setup.

The material in this chapter forms the preprint [**BBG**].

4.2. Preliminaries

4.2.1. Hopf algebras over noncommutative base - Hopf algebroids. We recall the definition of Hopf algebroids from [**KP11**]. See also [**Böh09**, **BS04**]. We begin by defining a generalization of bialgebras.

Definition 4.2.1. *Let A be a $\mathbb{C}$-algebra. An (s,t)-ring over A is a $\mathbb{C}$-algebra H with homomorphisms $s : A \to H$ and $t : A^{op} \to H$ whose images commute in H.*

The functions s and t are referred to as the source and target maps respectively. An (s,t)-ring structure is equivalent to the structure of an A^e-algebra on H.

Definition 4.2.2. *Let H be an (s,t)-ring over A. The Takeuchi product is the subspace*

$$H \times_A H := \{\sum_i h_i \otimes_A h'_i \in H \otimes_A H \mid \sum_i h_i t(a) \otimes h'_i = \sum_i h_i \otimes h'_i s(a) \quad \forall a \in A\}$$

of $H \otimes_A H$, where the tensor product $\otimes_A$ is defined with respect to the following (A,A)-bimodule structure on H:

$$a_1 \cdot h \cdot a_2 := s(a_1)t(a_2)h, \qquad a_1, a_2 \in A, \quad h \in H. \tag{4.2.1}$$

This Takeuchi product becomes a unital algebra with factorwise multiplication as well as an (s,t)-ring. Before we go onto the definition of a bialgebroid, let us recall the definition of an A-coalgebra.

Definition 4.2.3. *Let A be a $\mathbb{C}$-algebra. A coalgebra over A is a triple (C, Δ, ε) with C an (A,A)-bimodule, Δ an (A,A)-bimodule morphism called the comultiplication, $\varepsilon : C \to A$ an (A,A)-bimodule morphism called the counit, and such that*

$$(\Delta \otimes_A \mathrm{id})\Delta = (\mathrm{id} \otimes_A \Delta)\Delta, \qquad (\mathrm{id} \otimes_A \varepsilon)\Delta = (\varepsilon \otimes_A \mathrm{id})\Delta = \mathrm{id}\,. \tag{4.2.2}$$

A left bialgebroid over A is then an algebra with a compatible coalgebra structure over A. More precisely,

Definition 4.2.4. *Let A_l be a $\mathbb{C}$-algebra. A left bialgebroid over A_l is an (s_l,t_l)-ring H_l equipped with the structure of an A_l-coalgebra $(\Delta_l, \varepsilon_l)$ with respect to the (A_l, A_l)-bimodule structure* (4.2.1)*, subject to the following conditions:*

i) the (left) coproduct $\Delta_l : H_l \to H_l \otimes_{A_l} H_l$ maps into the subset $H_l \times_{A_l} H_l$ and defines a morphism $\Delta_l : H_l \to H_l \times_{A_l} H_l$ of unital $\mathbb{C}$-algebras;

ii) the (left) counit has the property:

$$\varepsilon_l(hh') = \varepsilon_l(hs_l(\varepsilon_l h')) = \varepsilon_l(ht_l(\varepsilon_l h')) \qquad h, h' \in H_l. \tag{4.2.3}$$

We denote the above left bialgebroid by $(H_l, A_l, s_l, t_l, \Delta_l, \varepsilon_l)$ or simply by H_l.

Remark 4.2.5. *From* (4.2.3) *above and the fact that ε_l is an (A_l, A_l)-bimodule morphism, it follows that $\varepsilon_l(s_l(a)h) = a\varepsilon_l(h)$, $\varepsilon_l(t_l(a)h) = \varepsilon_l(h)a$, and it also follows that $\varepsilon_l(1_{H_l}) = 1_{A_l}$. So we have that $\varepsilon_l s_l = \varepsilon_l t_l = \mathrm{id}_{A_l}$.*

Lemma 4.2.6. *In a left bialgebroid, the left counit is unique.*

Proof. Indeed, if both ε_l^1 and ε_l^2 make $(H_l, A_l, s_l, t_l, \Delta_l, \varepsilon_l^1)$ and $(H_l, A_l, s_l, t_l, \Delta_l, \varepsilon_l^2)$ left bialgebroids, then we have:

$$\varepsilon_l^2(h) = \varepsilon_l^2(s_l\varepsilon_l^1(h_1)h_2) = \varepsilon_l^1(h_1)\varepsilon_l^2(h_2) = \varepsilon_l^1(t_l\varepsilon_l^2(h_2)h_1) = \varepsilon_l^1(h).$$

□

Given an (s,t)-ring H, there is another (A,A)-bimodule structure on H:

$$a_1 \cdot h \cdot a_2 = ht(a_1)s(a_2), \qquad a_1, a_2 \in A \quad h \in H. \tag{4.2.4}$$

With respect to this bimodule structure, the tensor product $\otimes_A$ is defined. Inside $H \otimes_A H$, there is the Takeuchi product:

$$H \times^A H := \{\sum_i h_i \otimes_A h'_i \in H \otimes_A H \mid \sum_i s(a)h_i \otimes h'_i = \sum_i h_i \otimes t(a)h'_i \quad \forall a \in A\}.$$

This again becomes a unital algebra with factorwise multiplication and also is an (s,t)-ring.

Definition 4.2.7. *Let A_r be a $\mathbb{C}$-algebra. A right bialgebroid over A_r is an (s_r,t_r)-ring H_r equipped with the structure of an A_r-coalgebra $(\Delta_r, \varepsilon_r)$ with respect to the (A_r, A_r)-bimodule structure* (4.2.4)*, subject to the following conditions:*

i) the (right) coproduct $\Delta_r : H_r \to H_r \otimes_{A_r} H_r$ maps into $H_r \times^{A_r} H_r$ and defines a morphism $\Delta_r : H_r \to H_r \times^{A_r} H_r$ of unital $\mathbb{C}$-algebras;

ii) the (right) counit has the property:

$$\varepsilon_r(hh') = \varepsilon_r(s_r(\varepsilon_r h)h') = \varepsilon_r(t_r(\varepsilon_r h)h') \qquad h, h' \in H_r. \tag{4.2.5}$$

We denote a right bialgebroid by $(H_r, A_r, s_r, t_r, \Delta_r, \varepsilon_r)$ or simply by H_r. Note that if $(H_l, A_l, s_l, t_l, \Delta_l, \varepsilon_l)$ is a left bialgebroid, then $(H_l^{op}, A_l, t_l, s_l, \Delta_l, \varepsilon_l)$ is a right bialgebroid.

REMARK 4.2.8. *As in Remark 4.2.5, we have $\varepsilon_r s_r = \varepsilon_r t_r = \mathrm{id}_{A_r}$. Also as above, the right counit is unique.*

Sweedler notation. We shall use Sweedler notation with subscripts $\Delta_l(h) = h_{(1)} \otimes h_{(2)}$ for left comultiplication while the right comultiplication are indicated by superscripts: $\Delta_r(h) = h^{(1)} \otimes h^{(2)}$.

We now define a Hopf algebroid as an algebra endowed with a left and a right bialgebroid structure together with an antipode "intertwining" the left bialgebroid and the right bialgebroid structures. More precisely:

DEFINITION 4.2.9. *A Hopf algebroid is given by a triple (H_l, H_r, S), where $H_l = (H_l, A_l, s_l, t_l, \Delta_l, \varepsilon_l)$ is a left A_l-bialgebroid and $H_r = (H_r, A_r, s_r, t_r, \Delta_r, \varepsilon_r)$ is a right A_r-bialgebroid on the same $\mathbb{C}$-algebra H, and $S : H \to H$ is invertible $\mathbb{C}$-linear. These structures are subject to the following four conditions:*

i) the images of s_l and t_r as well as those of t_l and s_r, coincide:

$$s_l\varepsilon_l t_r = t_r, \quad t_l\varepsilon_l s_r = s_r, \quad s_r\varepsilon_r t_l = t_l, \quad t_r\varepsilon_r s_l = s_l; \tag{4.2.6}$$

ii) mixed coassociativity holds:

$$(\Delta_l \otimes \mathrm{id}_H)\Delta_r = (\mathrm{id}_H \otimes \Delta_r)\Delta_l, \qquad (\Delta_r \otimes \mathrm{id}_H)\Delta_l = (\mathrm{id}_H \otimes \Delta_l)\Delta_r; \tag{4.2.7}$$

iii) for all $a_1 \in A_l$, $a_2 \in A_r$ and $h \in H$, we have

$$S(t_l(a_1)ht_r(a_2)) = s_r(a_2)S(h)s_l(a_1); \tag{4.2.8}$$

iv) the antipode axioms hold:

$$\mu_H(S \otimes \mathrm{id}_H)\Delta_l = s_r\varepsilon_r, \qquad \mu_H(\mathrm{id}_H \otimes S)\Delta_r = s_l\varepsilon_l. \tag{4.2.9}$$

We apply ε_r to the first two and ε_l to the second pair of identities in (4.2.6) and get that A_l and A_r are anti-isomorphic as $\mathbb{C}$-algebras:

$$\begin{aligned} \phi &:= \varepsilon_r s_l : A_l^{op} \to A_r, \qquad & \phi^{-1} &:= \varepsilon_l t_r : A_r \to A_l^{op}, \\ \theta &:= \varepsilon_r t_l : A_l \to A_r^{op}, \qquad & \theta^{-1} &:= \varepsilon_l s_r : A_r^{op} \to A_l. \end{aligned} \tag{4.2.10}$$

The antipode is anti-algebra and anti-coalgebra morphism (between different coalgebras) and satisfies the equations

$$\mathrm{flip}(S \otimes S)\Delta_l = \Delta_r S, \qquad \mathrm{flip}(S \otimes S)\Delta_r = \Delta_l S, \tag{4.2.11}$$

where flip : $H \otimes_{\mathbb{C}} H \to H \otimes_{\mathbb{C}} H$ is the flip permuting two factors of the tensor product (this becomes an (A_l, A_l)-respectively (A_r, A_r)-bimodule). Similar formulas hold for the inverse S^{-1}. The following identities will be used:

$$\begin{array}{llll} s_r\varepsilon_r s_l = Ss_l, & s_l\varepsilon_l s_r = Ss_r, & s_r\varepsilon_r t_l = S^{-1}s_l, & s_l\varepsilon_l t_r = S^{-1}s_r, \\ t_r\varepsilon_r s_l = St_l, & t_l\varepsilon_l s_r = St_r, & t_r\varepsilon_r t_l = S^{-1}t_l, & t_l\varepsilon_l t_r = S^{-1}t_r, \\ \varepsilon_r s_l\varepsilon_l = \varepsilon_r S, & \varepsilon_l s_r\varepsilon_r = \varepsilon_l S, & \varepsilon_r t_l\varepsilon_l = \varepsilon_r S^{-1}, & \varepsilon_l t_r\varepsilon_r = \varepsilon_l S^{-1}, \end{array} \tag{4.2.12}$$

and

$$\begin{array}{cc} \mu_H(S \otimes s_l\varepsilon_l)\Delta_l = S, & \mu_H(s_r\varepsilon_r \otimes S)\Delta_r = S, \\ \mu_{H^{op}}(\mathrm{id}_H \otimes S^{-1})\Delta_l = t_r\varepsilon_r, & \mu_{H^{op}}(S^{-1} \otimes \mathrm{id}_H)\Delta_r = t_l\varepsilon_l, \\ \mu_{H^{op}}(t_l\varepsilon_l \otimes S^{-1})\Delta_l = S^{-1}, & \mu_{H^{op}}(S^{-1} \otimes t_r\varepsilon_r)\Delta_r = S^{-1}. \end{array} \tag{4.2.13}$$

LEMMA 4.2.10. *In a Hopf algebroid, the antipode is unique.*

PROOF. Indeed, if both S_1 and S_2 make (H_l, H_r, S_1) and (H_l, H_r, S_2) Hopf algebroids then we have

$$S_2(h) = s_r\varepsilon_r(h^{(1)})S_2(h^{(2)}) = S_1(h^{(1)}_{(1)})h^{(1)}_{(2)}S_2(h^{(2)})$$
$$= S_1(h_{(1)})h^{(1)}_{(2)}S_2(h^{(2)}_{(2)}) = S_1(h_{(1)})s_l\varepsilon_l(h_{(2)}) = S_1(h).$$

□

Finally, note that if (H_l, H_r, S) is a Hopf algebroid, then $(H_r^{op}, H_l^{op}, S^{-1})$ is also a Hopf algebroid.

4.2.2. The main example - Étale groupoids. We now introduce our main example besides Hopf algebras. A Hopf algebra is a Hopf algebroid with $A_l = A_r = \mathbb{C}$. We follow [**MM03**]. See also [**Con94**, **Har15**, **Kal11**].

DEFINITION 4.2.11. *A groupoid G is a small category in which each arrow is invertible. More explicitly, a groupoid consists of a space of objects G_0, a space of arrows G_1 (often denoted by G itself) and five structure maps relating the two:*

i) source and target maps $s, t : G_1 \to G_0$, assigning to each arrow g its source $s(g)$ and target $t(g)$; one says that g is from $s(g)$ to $t(g)$;
ii) a partially defined composition of arrows, that is, only for those arrows g, h for which source and target match, that is $s(g) = t(h)$; in other words, a map $m : G_2 := G_1 \,{}^{s}\!\times^{t}_{G_0} G_1 \to G_1$, $(g, h) \mapsto gh$ that is associative whenever defined, producing the composite arrow going from $s(gh) = s(h)$ to $t(gh) = t(g)$;
iii) a unit map $1 : G_0 \to G_1$, $x \mapsto 1_x$, that has the property $1_{t(g)}g = g1_{s(g)} = g$;
iv) an inversion $inv : G_1 \to G_1$, $g \mapsto g^{-1}$ that produces the inverse arrow going from $s(g^{-1}) = t(g)$ to $t(g^{-1}) = s(g)$, fulfilling $g^{-1}g = 1_{s(g)}$, $gg^{-1} = 1_{t(g)}$.

These maps can be assembled into a diagram

$$G_2 \xrightarrow{m} G_1 \xrightarrow{inv} G_1 \underset{t}{\overset{s}{\rightrightarrows}} G_0 \xrightarrow{1} G_1 \tag{4.2.14}$$

An arrow may be denoted by $x \xrightarrow{g} y$ to indicate that $y = s(g)$ and $x = t(g)$.

A topological groupoid is a groupoid in which both G_1 and G_0 are topological spaces and all the structure maps are continuous. Similarly one defines smooth groupoids, where in addition s and t are required to be surjective submersions in order to ensure that $G_2 = G_1 \,{}^{s}\!\times^{t}_{G_0} G_1$ remains a manifold. A topological (or smooth) groupoid is called étale if the source map is a local homeomorphism (or local diffeomorphism); this condition implies that all structures maps are local homeomorphisms (or local diffeomorphisms, respectively). In the smooth case, this equivalently amounts to saying that $\dim G_1 = \dim G_0$. In particular, an étale groupoid has zero-dimensional source and target fibers, and hence they are discrete. We shall only be dealing with smooth étale groupoids.

We give some examples of étale groupoids below.

EXAMPLE 4.2.12.

i) The unit groupoid has a single manifold M as both its object and arrow space. All the maps are identity functions.
ii) A (discrete) group is a one-object groupoid (called the point groupoid).
iii) The translation groupoid $\Gamma \ltimes M$ of a smooth left action of a discrete group has as object space M and arrow space $\Gamma \times M$. The source is $(g, m) \mapsto m$, the target is $(g, m) \mapsto gm$ and the multiplication is $(g, m)(g', m') = (gg', m')$.
*iv) Orbifold groupoids or proper étale groupoids. We refer to [**MM03**, **Har15**] for more details.*
v) Let $(M, \mathcal{F})$ be a foliated manifold. Then the (reduced) holonomy groupoid is étale.

As the last example is one of our main motivating examples, we shall describe it in a slightly greater details. See [**CW91**, **MM03**, **CLN85**, **CM01b**]. A foliation $\mathcal{F}$ on M is given by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ modeled on a manifold N_0 ($\mathbb{R}^n$ or $\mathbb{C}^n$), i.e.,

i) $\{U_i\}$ is an open covering of M;
ii) $f_i : U_i \to N_0$ are submersions with connected fibers defining $\mathcal{F}$;
iii) g_{ij} are local diffeomorphisms of N_0 and $g_{ij} f_j = f_i$ on $U_i \cap U_j$.

The manifold $N = \sqcup f_i(U_i)$ is called the transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$, and the pseudogroup P generated by g_{ij} is called the holonomy pseudogroup on the transverse manifold. To any pseudogroup P on some manifold X we can associate an étale (effective) groupoid $\Gamma(P)$ over X as follows: for any $x, y \in X$ let

$$\Gamma(P)(x,y) = \{germ_x g \mid g \in P, x \in dom(g), g(x) = y\}. \tag{4.2.15}$$

The multiplication in $\Gamma(P)$ is given by the composition of transitions. Equipped with classical sheaf topology $\Gamma(P)_1$ becomes a smooth manifold and $\Gamma(P)$ becomes an étale groupoid. In our case, $\Gamma(P)$ is called the reduced holonomy groupoid of $(M, \mathcal{F})$ and is denoted $Hol_N(M, \mathcal{F})$ (but also we write $\Gamma(P)$ sometimes).

We now show one gets Hopf algebroids naturally from étale groupoids following [**KP11**, **Mrč07**]. Before that we introduce the following.

Fiber sum notation. Let E and F are vector bundles over two manifolds X and Y, respectively. Suppose $\phi : X \to Y$ is an étale map (i.e., a local homeomorphism) and $\alpha : E \cong \phi^* F$ an isomorphism of vector bundles. Then the push-forward (or fiber sum) of ϕ, denoted by $\phi_* : \Gamma_c(X, E) \to \Gamma_c(Y, F)$, is defined by

$$(\phi_* s)(y) = \sum_{\phi(x)=y} \alpha(s(x)), \tag{4.2.16}$$

where $x \in X, y \in Y$ and $s \in \Gamma_c(X, E)$. Here we identify the fiber $\phi^* F_z$ with $F_{\phi(z)}$ using the definition of pullback.

If G is an étale groupoid over a compact Hausdorff G_0, the space $C_c^\infty(G)$ of smooth functions on $G = G_1$ with compact support carries a Hopf algebroid structure. Although $G = G_1$ often happens to be non-Hausdorff in examples, we assume this condition in this paper since the reduced holonomy groupoid of a Riemannian foliation is always Hausdorff. We have two $C^\infty(G_0)$-actions on $C_c^\infty(G)$ by left and right multiplication with respect to which we define the four tensor products denoted by $\otimes^{ll}_{C^\infty(G_0)}$, $\otimes^{rr}_{C^\infty(G_0)}$, $\otimes^{rl}_{C^\infty(G_0)}$ and $\otimes^{lr}_{C^\infty(G_0)}$. We need the following isomorphisms

$$\begin{aligned}
\Omega_{s,t} &: C_c^\infty(G) \otimes^{rl}_{C^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G\ {}^s\!\times^t_{G_0} G) = C_c^\infty(G_2) \\
\Omega_{t,t} &: C_c^\infty(G) \otimes^{ll}_{C^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G\ {}^t\!\times^t_{G_0} G) = C_c^\infty(G_2) \\
\Omega_{s,s} &: C_c^\infty(G) \otimes^{rr}_{C^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G\ {}^s\!\times^s_{G_0} G) = C_c^\infty(G_2) \\
\Omega_{t,s} &: C_c^\infty(G) \otimes^{lr}_{C^\infty(G_0)} C_c^\infty(G) \to C_c^\infty(G\ {}^t\!\times^s_{G_0} G) = C_c^\infty(G_2)
\end{aligned} \tag{4.2.17}$$

all given by the formulas

$$\Omega_{-,-}(u \otimes^{--}_{C^\infty(G_0)} u')(g, g') = u(g)u(g'), \tag{4.2.18}$$

for $u, u' \in C_c^\infty(G)$ and (g, g') in the respective pullback $G\ {}^-\!\times^-_{G_0} G$. The maps are isomorphism, as it was shown in [**Mrč07**]. We now give the Hopf algebroid structure maps for $C_c^\infty(G)$ over $C^\infty(G_0)$:

Ring structure. On the base algebra $C^\infty(G_0)$ one has the commutative pointwise product, whereas the total algebra $C_c^\infty(G)$ is equipped with a convolution product, defined as the composition

$$* : C_c^\infty(G) \otimes^{rl}_{C^\infty(G_0)} C_c^\infty(G) \xrightarrow{\Omega_{s,t}} C_c^\infty(G_2) \xrightarrow{m_*} C_c^\infty(G). \tag{4.2.19}$$

Explicitly,

$$(u * v)(g) := *(u \otimes v) = (m_* \Omega^{s,t}(u \otimes v))(g) = \sum_{g = g_1 g_2} u(g_1)u(g_2), \tag{4.2.20}$$

which can be used in showing associativity of the product.

Source and target maps. For $f \in C^\infty(G_0)$ and $u \in C_c^\infty(G)$,

$$(f * u)(g) = f(t(g))u(g) \qquad \text{and} \qquad (u * f)(g) = u(g)f(s(g)). \tag{4.2.21}$$

It can be shown that $C^\infty(G_0)$, identified with those functions in $C_c^\infty(G)$ having support on $1_{G_0} \subset G$, is a commutative subalgebra of $C_c^\infty(G)$. We put for the (left and right bialgebroid) source and target maps

$$s_l \equiv s_r \equiv t_l \equiv t_r \equiv 1_* : C^\infty(G_0) \to C_c^\infty(G), \tag{4.2.22}$$

i.e., the injection as subalgebra given by the fiber sum of the unit map $1 : G_0 \to G$. More explicitly,

$$s_l : f \mapsto \overline{f}, \qquad \text{where} \qquad \overline{f}(g) = \begin{cases} f(x) & \text{if } g = 1_x \text{ for some } x \in G_0 \\ 0 & \text{otherwise.} \end{cases} \tag{4.2.23}$$

Left and right comultiplications. Using the isomorphism $\Omega_{-,-}$, the left and right comultiplications are given as follows:

$$\begin{aligned} \Delta_l : C_c^\infty(G) &\to C_c^\infty(G\ {}^t\!\times_{G_0}^t G) \cong C_c^\infty(G) \otimes^{ll} C_c^\infty(G), \\ (\Delta_l u)(g, g') &= \begin{cases} u(g) & \text{if } g = g', \\ 0 & \text{else,} \end{cases} \end{aligned} \tag{4.2.24a}$$

$$\begin{aligned} \Delta_r : C_c^\infty(G) &\to C_c^\infty(G\ {}^s\!\times_{G_0}^s G) \cong C_c^\infty(G) \otimes^{rr} C_c^\infty(G), \\ (\Delta_r u)(g, g') &= \begin{cases} u(g) & \text{if } g = g', \\ 0 & \text{else.} \end{cases} \end{aligned} \tag{4.2.24b}$$

Alternatively, $\Delta_l = d_*^l$ and $\Delta_r = d_*^r$ for the diagonal maps $d^l : G \to G\ {}^t\!\times_{G_0}^t G$, $g \mapsto (g, g)$ and $d^r : G \to G\ {}^s\!\times_{G_0}^s G$, $g \mapsto (g, g)$.

Left and right counits. Both left and right counits are respectively determined by the fiber sum of the target and source maps of the groupoid. For any $x \in G_0$,

$$\begin{aligned} \varepsilon_l : C_c^\infty(G) \to C^\infty(G_0), &\quad (\varepsilon_l(u))(x) = \textstyle\sum_{t(g)=x} u(g) \\ \varepsilon_r : C_c^\infty(G) \to C^\infty(G_0), &\quad (\varepsilon_r(u))(x) = \textstyle\sum_{s(g)=x} u(g). \end{aligned} \tag{4.2.25}$$

Antipode. The antipode is given by the groupoid inversion,

$$S : C_c^\infty(G) \to C_c^\infty(G), \qquad (S(u))(g) = u(g^{-1}) = (inv_* u)(g). \tag{4.2.26}$$

THEOREM 4.2.13. *With the above structure maps, $C_c^\infty(G)$ becomes a Hopf algebroid over $C^\infty(G_0)$.*

The proof is in [**KP11**]. See also [**Con82**, **Con85**, **Kor08**, **Kor09**].

4.2.3. Modules over Hopf algebroids. Let $H = (H_l, H_r, S)$ be a Hopf algebroid. A left module over H is simply a left module over the underlying $\mathbb{C}$-algebra H. We denote the structure map by $(h, m) \mapsto h \cdot m$. The left bialgebroid structure H_l induces an (A_l, A_l)-bimodule structure on each module and a monoidal structure on the category of modules. More explicitly, let M be an H-module. Then the (A_l, A_l)-bimodule structure is given by

$$a_1 \cdot m \cdot a_2 = s_l(a_1) \cdot t_l(a_2) \cdot m, \tag{4.2.27}$$

for all $a_1, a_2 \in A_l$ and $m \in M$. The left coproduct defines the monoidal structure $(M, N) \mapsto M \otimes_A N$, where $M \otimes_A N$ is equipped with the H-module structure

$$h \cdot (m \otimes n) := h_{(1)} \cdot m \otimes h_{(2)} \cdot n, \qquad h \in H, m \in M, n \in N. \tag{4.2.28}$$

The monoidal unit is given by A_l with left H-action $h \cdot a = \varepsilon_l(hs_l(a))$. Note that $\varepsilon_l(ht_l(a)) = \varepsilon_l(hs_l(\varepsilon_l(t_l(a)))) = \varepsilon_l(hs_l(a))$. Also A_l being the monoidal unit it is an algebra in the category of H-modules, i.e., it is an H-module algebra. This structure will be important for us in the examples we consider.

REMARK 4.2.14. *We state the definition of an H-module algebra explicitly. It is a $\mathbb{C}$-algebra and left H-module B such that the multiplication in B is A_l-balanced and*

i) $h \cdot 1_B = s_l \varepsilon_l(h) \cdot 1_B$;
ii) $h \cdot (bb') = (h_{(1)} \cdot b)(h_{(2)} \cdot b')$.

for $b, b' \in B$ *and* $h \in H$. *Note that* B *has a canonical* A_l*-ring structure. Its unit is the map* $A_l \to B$, $a \mapsto s_l(a) \cdot 1_B = t_l(a) \cdot 1_B$.

Similarly, one can consider right H-modules as modules over the $\mathbb{C}$-algebra H. Such modules get the structure of an (A_r, A_r)-bimodule and the category becomes monoidal using the right coproduct. The monoidal unit is A_r. We now see some examples coming from the geometry of groupoids. We follow [**Kal11**].

Definition 4.2.15. *A smooth left action of a Lie groupoid* G *on a smooth manifold* P *along a smooth map* $\pi : P \to G_0$ *is a smooth map* $\mu : G_1{}^s \times^{\pi}_{G_0} P \to P$, $(g, p) \mapsto g \cdot p$, *which satisfies the conditions* $\pi(g \cdot p) = t(g)$, $1_{\pi(p)} \cdot p = p$ *and* $g' \cdot (g \cdot p) = (g'g) \cdot p$ *for all* $g', g \in G_1$ *and* $p \in P$ *with* $s(g') = t(g)$ *and* $s(g) = \pi(p)$.

We define right actions of étale groupoids on smooth manifolds in a similar way.

Definition 4.2.16. *Let* G *be an étale groupoid, and let* E *be a smooth complex vector bundle over* G_0. *A representation of the groupoid* G *on* E *is a smooth left action* $\rho : G_1 {}^s \times^{p}_{G_0} E \to E$, *denoted by* $\rho(g, v) = g \cdot v$, *of* G *on* E *along the bundle projection* $p : E \to G_0$ *such that for any arrow* $x \xrightarrow{g} y$ *the induced map* $g_* : E_x \to E_y$, $v \mapsto g \cdot v$, *is a linear isomorphism. A section* $u : G_0 \to E$ *is called* G*-invariant if for any arrow* $x \xrightarrow{g} y$, *it holds that* $g \cdot u(x) = u(y)$.

Let us see what representations mean in the examples above.

Example 4.2.17.

i) *Representations of the unit groupoid associated to a smooth manifold correspond precisely to complex vector bundles.*
ii) *Representations of the point groupoid associated to a (discrete) group* Γ *correspond to representations of the group on finite dimensional complex vector spaces.*
iii) *Representations of the translation groupoid* $\Gamma \ltimes M$ *corresponds to* Γ*-equivariant complex vector bundles over* M.
iv) *Representations of the orbifold groupoid are the orbibundles.*
v) *Representations of the holonomy groupoid are the transversal vector bundles.*
vi) *For an étale groupoid* G *the complexified tangent bundle of* G_0 *becomes a representation of* G. *The cotangent bundle, exterior bundle all inherit this natural representation, so it makes sense to speak of vector fields, differential forms or Riemannian metrics etc. on étale groupoids (vector fields, differential forms or Riemannian metrics etc. on* G_0, *respectively, invariant under the action). Also note that the exterior derivative* d *is invariant under the* G*-action. This follows from naturality of* d *and a local argument.*

Proposition 4.2.18. *Let* E *be representation of the étale groupoid* G. *The space of smooth sections* $\Gamma^\infty(E)$ *over* G_0 *becomes a module over* $C_c^\infty(G)$ *by the formulas*

$$(a \cdot u)(x) = \sum_{t(g)=x} a(g)(g \cdot u(s(g))), \tag{4.2.29}$$

for $a \in C_c^\infty(G)$ *and* $u \in \Gamma^\infty(E)$.

The proof is in [**Kal11**]. Moreover, each module of finite type and constant rank appears in this way, giving a version of Serre-Swan theorem. See [**Con85**] for an example coming from Sobolev spaces.

4.2.4. $*$-structures and conjugate modules. We introduce $*$-structures on Hopf algebroids which will be needed in order to view them as symmetry objects. This is one of the main results of the present paper. We view the ensuing structures as the first step in defining a "compact"-type Hopf algebroid in analogy with CQG-algebras [**DK94**], though we do not go in that direction here.

Let (H_l, H_r, S) be a Hopf algebroid such that H, A_l and A_r are $*$-algebras, s_l and s_r are $*$-preserving (the involutions for H, A_r and A_l are denoted by the same symbol $*$). Assume that

$$\varepsilon_l t_r(a_1^*) = (\varepsilon_l s_r(a_1))^*, \qquad \varepsilon_r t_l(a_2^*) = (\varepsilon_r s_l(a_2))^* \tag{4.2.30}$$

hold for all $a_1 \in A_r$, $a_2 \in A_l$.

Lemma 4.2.19. *We have*

$$h^* t_l(a)^* \otimes_{A_r} h'^* = h^* \otimes_{A_r} h'^* s_l(a)^*. \tag{4.2.31}$$

Proof. We compute

$$\begin{aligned} h^* t_l(a)^* \otimes_{A_r} h'^* &= h^* s_r(\varepsilon_r(t_l(a)))^* \otimes_{A_r} h'^* \\ &= h^* s_r((\varepsilon_r t_l(a))^*) \otimes_{A_r} h'^* \\ &= h^* s_r \varepsilon_r s_l(a^*) \otimes_{A_r} h'^* \\ &= h^* \cdot \varepsilon_r s_l(a^*) \otimes_{A_r} h'^* \\ &= h^* \otimes_{A_r} \varepsilon_r s_l(a^*) \cdot h'^* \\ &= h^* \otimes_{A_r} h'^* t_r \varepsilon_r s_l(a^*) \\ &= h^* \otimes_{A_r} h'^* s_l(a^*) \\ &= h^* \otimes_{A_r} h'^* s_l(a)^*. \end{aligned} \tag{4.2.32}$$

□

Lemma 4.2.19 says that the map $(* \otimes *) : H_l \otimes_{\mathbb{C}} H_l \to H_r \otimes_{A_r} H_r$ descends to an isomorphism $(* \otimes *) : H_l \otimes_{A_l} H_l \to H_r \otimes_{A_r} H_r$. So we can make sense of

$$\Delta_r * = (* \otimes *)\Delta_l. \tag{4.2.33}$$

In Sweedler notation,

$$(h^*)^{(1)} \otimes (h^*)^{(2)} = (h_{(1)})^* \otimes (h_{(2)})^* \,. \tag{4.2.34}$$

Definition 4.2.20. *Let (H_l, H_r, S) be a Hopf algebroid such that H, A_l and A_r are $*$-algebras while s_l and s_r are $*$-preserving. Then (H_l, H_r, S) is said to be a Hopf $*$-algebroid if* (4.2.30) *and* (4.2.33) *hold.*

Some immediate corollaries of Definition 4.2.20 are:

i) $(* \otimes *) : H_l \otimes_{A_l} H_l \to H_r \otimes_{A_r} H_r$ induces an isomorphism $H_l \times_{A_l} H_l \to H_r \times^{A_r} H_r$.
ii) From (4.2.6), $t_l * = s_r \varepsilon_r t_l * = s_r * \varepsilon_r s_l = * s_r \varepsilon_r s_l = * S s_l$, with the last equality following from (4.2.12).
iii) Similarly, $t_r * = * S s_r$.

Proposition 4.2.21. *Let (H_l, H_r, S) be a Hopf $*$-algebroid. Then the counits and the antipode satisfy*

$$\varepsilon_r S^{-1} * = * \varepsilon_r, \qquad \varepsilon_l S^{-1} * = * \varepsilon_l, \qquad S * S * = \mathrm{id}_H \tag{4.2.35}$$

and A_l becomes an H-module $*$-algebra, i.e., the H-action satisfies

$$(h \cdot a)^* = S(h)^* \cdot a^* \qquad h \in H, \quad a \in A_l. \tag{4.2.36}$$

Proof. We have

$$h^* = s_l \varepsilon_l((h^*)_{(1)})(h^*)_{(2)} = s_l \varepsilon_l((h^{(1)})^*)(h^{(2)})^*,$$

so

$$h = h^{(2)}(s_l \varepsilon_l((h^{(1)})^*))^*.$$

Similarly,

$$h = h^{(1)}(t_l \varepsilon_l((h^{(2)})^*))^*.$$

Now,

$$* s_l \varepsilon_l * = s_l * \varepsilon_l * = t_r \varepsilon_r s_l * e_l * = t_r * \varepsilon_r t_l \varepsilon_l * \,.$$

Similarly,

$$*t_l\varepsilon_l* = s_r * \varepsilon_r t_l \varepsilon_l * .$$

So we conclude that $*\varepsilon_r t_l\varepsilon_l*$ satisfies the right counit axioms. Hence $\varepsilon_r = *\varepsilon_r t_l\varepsilon_l* = *\varepsilon_r S^{-1}*$. Similarly, $\varepsilon_l = *\varepsilon_l S^{-1} *$. From this we observe that $s_l\varepsilon_l* = *t_r\varepsilon_r$ and $s_r\varepsilon_r* = *t_l\varepsilon_l$. Using the above observation and proceeding exactly as before, it follows that $*S^{-1}*$ satisfies the antipode axioms. By uniqueness, we have $S = *S^{-1}*$, which implies $S * S* = \mathrm{id}_H$.

Finally,

$$\begin{aligned}
S(h)^* \cdot a^* &= \varepsilon_l(S(h)^* s_l(a^*)) \\
&= \varepsilon_l(S(h)^* s_l(a)^*) \\
&= \varepsilon_l * (s_l(a)S(h)) \\
&= \varepsilon_l * (St_l(a)S(h)) \\
&= \varepsilon_l * S(ht_l(a)) \\
&= *\varepsilon_l(ht_l(a)) \\
&= (\varepsilon_l(hs_l(a)))^* \\
&= (h \cdot a)^*.
\end{aligned}$$

□

Besides Hopf $*$-algebras, the Hopf algebroid in Theorem 4.2.13 becomes a central example of Hopf $*$-algebroids:

PROPOSITION 4.2.22. *The space $C_c^\infty(G)$ becomes a Hopf $*$-algebroid over $C^\infty(G_0)$ with $*$-structure given by*

$$u^*(g) = \overline{u(g^{-1})} \text{ for } u \in C_c^\infty(G) \text{ and } f^*(x) = \overline{f(x)} \text{ for } f \in C^\infty(G_0). \tag{4.2.37}$$

PROOF. This follows from direct computations. □

Another class of examples, which we have not mentioned above, comes from weak Hopf algebras studied in [**BNS99**]. Our $*$-structure is the same as C^*-structure mentioned in [**BNS99**]. Following this and the standard theory of CQG-algebras, leads to opening up a new direction of study, namely, (co)representation theory of Hopf $*$-algebroids and the interplay of the $*$-structure and (co)integrals.

We shall systematically use the language of conjugate modules in order to keep track of various aspects. See [**BM09**, **BPS13**].

Let (H_l, H_r, S) be a Hopf $*$-algebroid and M an H-module. We define the conjugate module $\overline{M}$ by declaring that

i) $\overline{M} = M$ as abelian group;
ii) we write $\overline{m}$ for an element $m \in M$ when we consider it as an element of $\overline{M}$;
iii) the module operation for $\overline{M}$ is $h \cdot \overline{m} = \overline{S(h)^* \cdot m}$.

Again, let B be a $*$-algebra and let E be a (B, B) bimodule. The conjugate bimodule $\overline{E}$ is defined by the following three conditions:

i) $\overline{E} = E$ as abelian group;
ii) We write $\overline{e}$ for an element $e \in E$ when we consider it as an element of $\overline{E}$;
iii) The bimodule operations for $\overline{E}$ are $b \cdot \overline{e} = \overline{e \cdot b^*}$ and $\overline{e} \cdot b = \overline{b^* \cdot e}$.

If $\theta : E \to F$ is any morphism, then we define $\overline{\theta} : \overline{E} \to \overline{F}$ by $\overline{\theta}(\overline{e}) = \overline{\theta(e)}$.

We make $\overline{B}$ an associative algebra by defining the multiplication $\overline{b}\overline{b'} := \overline{b'b}$. As an $\mathbb{R}$-algebra, $\overline{B}$ is isomorphic to B^{op} via the map $b \mapsto \overline{b}$. We make $\overline{B}$ a $\mathbb{C}$-algebra through the algebra homomorphism $\mathbb{C} \to \overline{B}$, $\lambda \mapsto \overline{\lambda^* 1_B}$. We now define $\# : B \to \overline{B}$, $b \mapsto \overline{b^*}$. Then $\#$ is an isomorphism of $\mathbb{C}$ algebras. If $\theta : B \to B'$ is a morphism then we say that θ is $*$-preserving if $\#\theta = \overline{\theta}\#$.

So we see that the conclusion in (4.2.36) that A_l is an H-module $*$ algebra, is nothing but the assertion that $\# : A_l \to \overline{A_l}$ is an H-module morphism. We also see that for an H-module M, the induced (A_l, A_l)-bimodule structure matches with the prescription above. Thus our $*$-structure naturally produces examples of "Bar categories" in the sense of [**BM09**].

LEMMA 4.2.23. *Let B be an H-module $*$-algebra, and let the invariant subalgebra B_H be defined as $B_H = \{b \in B \mid h \cdot b = s_l\varepsilon_l(h) \cdot b\}$. Then $\# : B \to \overline{B}$ induces an isomorphism $\# : B_H \to \overline{B}_H$.*

PROOF. This follows from the fact that $\#$ is an H-module morphism. □

In fact, we can say more:

PROPOSITION 4.2.24. *Let B be an H-module $*$-algebra. Then B_H is also a $*$-algebra. So that, by Lemma 4.2.23 we can identify $(\overline{B})_H = \overline{(B_H)}$ as algebras.*

PROOF. Let $b \in B_H$. We compute

$$\begin{aligned}(h \cdot b^*)^* &= S(h)^* \cdot b \\ &= s_l\varepsilon_l(S(h)^*) \cdot b \\ &= s_l(\varepsilon_l(h)^*) \cdot b \\ &= (s_l\varepsilon_l(h))^* \cdot b\end{aligned}$$

so that

$$\begin{aligned}h \cdot b^* &= ((s_l\varepsilon_l(h))^* \cdot b)^* \\ &= (S(s_l\varepsilon_l(h)^*))^* \cdot b^* \\ &= S^{-1}s_l\varepsilon_l(h) \cdot b^* \\ &= t_l\varepsilon_l(h) \cdot b^*.\end{aligned}$$

Next observe that taking $h = s_l(a)$ for $a \in A_l$ in the last equality gives $s_l(a) \cdot b^* = t_l(a) \cdot b^*$. So that for all $h \in H$ we get

$$s_l\varepsilon_l(h) \cdot b^* = t_l\varepsilon_l(h) \cdot b^*$$

which in turn implies that $b^* \in B_H$. □

4.3. Noncommutative Kähler structures

4.3.1. Differential calculi. Let $H = (H_l, H_r, S)$ be a Hopf $*$-algebroid. We start by defining a differential calculus. We follow the setup in [**B́17**], as expounded in the last chapter 3.

DEFINITION 4.3.1. *An $\mathbb{N}_0$-graded H-module is an $\mathbb{N}_0$-graded $\mathbb{C}$-vector space which is also an H-module such that the H-action preserves the $\mathbb{N}_0$-grading.*

DEFINITION 4.3.2. *An $\mathbb{N}_0$-graded H-module algebra is an $\mathbb{N}_0$-graded algebra which is also an H-module algebra such that the H-action preserves the $\mathbb{N}_0$-grading.*

DEFINITION 4.3.3. *A pair $(B,\, d)$ is called an H-covariant complex if B is an $\mathbb{N}_0$-graded H-module algebra, and d is homogeneous of degree one satisfying $d^2 = 0$, such that*

$$A_l \text{ and } H_0 \text{ generate } H \text{ as algebra,} \tag{4.3.1}$$

where

$$H_0 := \{h \in H \mid [h - s_l\varepsilon_l(h), d] = [h - t_l\varepsilon_l(h), d] = 0\}\,. \tag{4.3.2}$$

DEFINITION 4.3.4. *A triple $(B, \partial, \overline{\partial})$ is called an H-covariant double complex if B is an $\mathbb{N}_0^2$-graded H-module algebra, ∂ is homogeneous of degree $(1, 0)$, and $\overline{\partial}$ is homogeneous of degree $(0, 1)$, such that $\partial^2 = 0$, $\overline{\partial}^2 = 0$, $\partial\overline{\partial} + \overline{\partial}\partial = 0$ and they satisfy (4.3.1).*

For any H-covariant complex $(B,\, d)$, we call an element d-closed if it is contained in $\ker(d)$ and d-exact if it is contained in $im(d)$. For an H-covariant double complex $(B, \partial, \overline{\partial})$, we define ∂-closed, $\overline{\partial}$-closed, ∂-exact and $\overline{\partial}$-exact elements analogously.

DEFINITION 4.3.5. *An H-covariant complex (B, d) is called an H-covariant differential graded algebra if d satisfies the graded Leibniz rule*

$$d(bb') = d(b)b' + (-1)^k b d(b') \qquad b \in B^k, \quad b' \in B. \tag{4.3.3}$$

DEFINITION 4.3.6. *An H-covariant differential calculus over an H-module algebra B (with unit map $\imath_B$) is an H-covariant differential graded algebra (Ω, d) (with unit map $\imath_\Omega$) such that $\Omega^0 = B$, the two H-action on B coming from B itself and Ω^0 coincide, and*

$$\Omega^k = \operatorname{span}_{\mathbb{C}}\{b_0 db_1 \wedge \ldots \wedge db_k \mid b_0, \ldots, b_k \in B\}. \tag{4.3.4}$$

Notation. We use $\wedge$ to denote the multiplication between elements of a differential calculus when both are of order greater that 0. We call an element of a differential calculus a form.

Observe that the coincidence of the two H-actions on B implies that the two unit maps also coincide. Observe also that the induced (A_l, A_l)-bimodule structure on Ω coincide with the one coming from the unit map.

DEFINITION 4.3.7. *An H-covariant differential calculus (Ω, d) over an H-module $*$-algebra B is a $*$-differential calculus if the involution of B extends to a degree zero involutive conjugate linear map on Ω, for which $(d\omega)^* = d(\omega^*)$ for all $\omega \in \Omega$, and*

$$(\omega \wedge \eta)^* = (-1)^{kl} \eta^* \wedge \omega^*, \qquad \omega \in \Omega^k, \quad \eta \in \Omega^l$$

making Ω an H-module $*$-algebra.

We say that a form is real if $\omega^* = \omega$.

LEMMA 4.3.8. *For an H-covariant $*$-differential calculus (Ω, d), we have*

i) $[h - s_l\varepsilon_l(h), d] = 0 \implies [S^{-1}(h^*) - t_l\varepsilon_l(S^{-1}(h^*)), d] = 0$;
ii) $[h - t_l\varepsilon_l(h), d] = 0 \implies [S^{-1}(h^*) - s_l\varepsilon_l(S^{-1}(h^*)), d] = 0$;

for $h \in H$. Thus combining the two, we get that $h \in H_0$ if and only if $S^{-1}(h^*) \in H_0$.

PROOF. For $\omega \in \Omega$, we compute

$$\begin{aligned}
0 = ([h - s_l\varepsilon_l(h), d](\omega^*))^* &= ((h - s_l\varepsilon_l(h)) \cdot d(\omega^*) - d((h - s_l\varepsilon_l(h)) \cdot \omega^*))^* \\
&= ((h - s_l\varepsilon_l(h)) \cdot (d\omega)^*)^* - d(((h - s_l\varepsilon_l(h)) \cdot \omega^*)^*) \\
&= (S(h - s_l\varepsilon_l(h)))^* \cdot d\omega - d(S(h - s_l\varepsilon_l(h))^* \cdot \omega) \\
&= [(S(h - s_l\varepsilon_l(h)))^*, d](\omega).
\end{aligned}$$

And similarly, $0 = ([h - t_l\varepsilon_l(h), d](\omega^*))^* = [(S(h - t_l\varepsilon_l(h)))^*, d](\omega)$. Now

$$\begin{aligned}
(S(h - s_l\varepsilon_l(h)))^* = S(h)^* - S(s_l\varepsilon_l(h))^* &= S^{-1}(h^*) - S^{-1}(s_l(\varepsilon_l(h)^*)) \\
&= S^{-1}(h^*) - t_l\varepsilon_l(S^{-1}(h^*))
\end{aligned}$$

and

$$\begin{aligned}
(S(h - t_l\varepsilon_l(h)))^* = S(h)^* - S(t_l\varepsilon_l(h))^* &= S^{-1}(h^*) - s_l(\varepsilon_l(h)^*) \\
&= S^{-1}(h^*) - s_l\varepsilon_l(S^{-1}(h^*)).
\end{aligned}$$

Thus we get $S^{-1}(h^*) \in H_0$ if $h \in H_0$. The other direction follows from $(S*)^2 = \mathrm{id}$ ☐

LEMMA 4.3.9. *On $\overline{\Omega}$, defining the product as $\overline{\omega} \wedge \overline{\eta} = (-1)^{kl}\overline{\eta \wedge \omega}$ for $\omega \in \Omega^k$, $\eta \in \Omega^l$ makes $(\overline{\Omega}, \overline{d})$ an H-covariant differential graded algebra. Then an H-covariant $*$-differential calculus is an H-covariant differential calculus such that $\# : (\Omega, d) \to (\overline{\Omega}, \overline{d})$ is H-linear and a differential graded algebra homomorphism.*

PROOF. The second part follows from the discussion prior to Lemma 4.2.23. For the first part, we observe that given $\omega \in \Omega$ and $h \in H$,

$$\begin{aligned}
[h - s_l\varepsilon_l(h), \overline{d}](\overline{\omega}) &= (h - s_l\varepsilon_l(h)) \cdot \overline{d}(\overline{\omega}) - \overline{d}((h - s_l\varepsilon_l(h)) \cdot \overline{\omega}) \\
&= (h - s_l\varepsilon_l(h)) \cdot \overline{d\omega} - \overline{d}(\overline{(S(h - s_l\varepsilon_l(h)))^* \cdot \omega}) \\
&= \overline{S(h - s_l\varepsilon_l(h))^* \cdot d\omega} - \overline{d(S(h - s_l\varepsilon_l(h))^* \cdot \omega)} \\
&= \overline{[S(h - s_l\varepsilon_l(h))^*, d](\omega)}
\end{aligned}$$

and similarly, $[h - t_l\varepsilon_l(h), \overline{d}](\overline{\omega}) = \overline{[S(h - t_l\varepsilon_l(h))^*, d](\omega)}$. Now the lemma follows from Lemma 4.3.8. □

DEFINITION 4.3.10. *We define the space of invariant forms Ω_0 of Ω as*

$$\Omega_0 = \{\omega \in \Omega \mid h \cdot \omega = s_l\varepsilon_l(h) \cdot \omega = t_l\varepsilon_l(h) \cdot \omega \textit{ for all } h \in H_0\}.$$

Observe that we recover the usual definition of invariant subalgebra as in Lemma 4.2.23 if the differential d is identically 0.

PROPOSITION 4.3.11. *For the space of invariant forms we have,*

i) $(\Omega_0, d|_{\Omega_0})$ *is a differential graded algebra;*
ii) Ω_0 *is a $*$-algebra;*
iii) $d|_{\Omega_0}$ *satisfies* $d|_{\Omega_0}(\omega^*) = (d|_{\Omega_0}\omega)^*$ *for all* $\omega \in \Omega_0$*;*
iv) $\# : (\Omega_0, d|_{\Omega_0}) \to (\overline{\Omega}_0, \overline{d}|_{\overline{\Omega}_0})$ *is a differential graded algebra homomorphism.*

PROOF. i) That Ω_0 is an algebra follows from the same proof as in d identically 0 case. Moreover, that d preserves Ω_0 follows from the definition of H_0.

ii) Observe that for $h \in H_0$ and $\omega \in \Omega_0$

$$\begin{aligned}
(h \cdot \omega^*)^* = S(h)^* \cdot \omega &= S^{-1}(h^*) \cdot \omega \\
&= t_l\varepsilon_l(S^{-1}(h^*)) \cdot \omega \\
&= t_l(\varepsilon_l(h)^*) \cdot \omega \\
&= (Ss_l\varepsilon_l(h))^* \cdot \omega
\end{aligned}$$

so that

$$h \cdot \omega^* = ((Ss_l\varepsilon_l(h))^* \cdot \omega)^* = s_l\varepsilon_l(h) \cdot \omega^*.$$

Again

$$\begin{aligned}
(h \cdot \omega^*)^* = S(h)^* \cdot \omega &= S^{-1}(h^*) \cdot \omega \\
&= s_l\varepsilon_l(S^{-1}(h^*)) \cdot \omega \\
&= s_l(\varepsilon_l(h)^*) \cdot \omega \\
&= (s_l\varepsilon_l(h))^* \cdot \omega
\end{aligned}$$

so that

$$h \cdot \omega^* = ((s_l\varepsilon_l(h))^* \cdot \omega)^* = S^{-1}s_l\varepsilon_l(h) \cdot \omega^* = t_l\varepsilon_l(h) \cdot \omega^*.$$

iii) holds because d satisfies the property.

iv) Follows from ii). □

We shall denote the differential on Ω_0 only by d, assuming that it really means d is restricted to Ω_0. Now we come to our example. According to Haefliger [**Kor08**]:

DEFINITION 4.3.12. *A transverse structure on a foliated manifold $(M, \mathcal{F})$ is a structure on the transversal manifold N, invariant under the action of the holonomy pseudogroup P.*

Since the groupoid $\Gamma(P)$ is constructed out of P, it follows that P invariant structures are $\Gamma(P)$ invariant. The normal bundle $N(M, \mathcal{F})$ of the foliation $\mathcal{F}$ is isomorphic to the tangent bundle TN of N. Thus, basic forms on the foliated manifold $(M, \mathcal{F})$ are in bijective correspondence with $\Gamma(P)$-invariant forms on the transverse manifold N (see [**Kor08**]). To see what does $\Gamma(P)$ invariant forms correspond to, we introduce the following.

DEFINITION 4.3.13. *A local bisection of a Lie groupoid G is a local section $\sigma : U \to G$ of $s : G \to G_0$ defined on an open subset $U \subset G_0$ such that $t\sigma$ is an open embedding.*

If G is étale, any arrow g induces a germ of a homeomorphism $\sigma_g : (U, s(g)) \to (V, t(g))$ from a neighborhood U of $s(g)$ to a neighborhood V of $t(g)$ as follows: choosing U small enough such that a bisection σ exists and $t|_{\sigma U}$ is a homeomorphism into $V := t(\sigma U)$, we set $\sigma_g := t\sigma$. We do not distinguish between σ_g and the actual germ of this map at the point $s(g)$.

LEMMA 4.3.14. *Let G be an étale groupoid, and let E be a smooth complex vector bundle over G_0 with a G-representation. Then a section $u : G_0 \to E$ is G-invariant if and only if it is $C_c^\infty(G)$-invariant.*

PROOF. Recall that a section u of the bundle E is G invariant, if $g \cdot u(x) = u(y)$ for all arrow $x \xrightarrow{g} y$, while u is $C_c^\infty(G)$ invariant if $a \cdot u = \varepsilon_l(a)u$ for all $a \in C_c^\infty(G)$. That G-invariance implies $C_c^\infty(G)$-invariance is clear. For the converse, pick an arrow $x \xrightarrow{g} y$ and a bisection (U, σ) such that $g \in \sigma(U)$ [**MM03**]. Then choose any function $a \in C_c^\infty(G)$ with support in $\sigma(U)$ and $a(g) = 1$. Note that on a bisection $\sigma(U)$, we have $a(t|_{\sigma(U)})^{-1} = \varepsilon_l(a)$ and $a \cdot u = a(t|_{\sigma(U)})^{-1}u = \varepsilon_l(a)u$. Hence the lemma follows. □

Now take $B = C^\infty(G_0)$ and $\Omega = \Omega(G_0)$, the $\mathbb{C}$-valued smooth functions and forms on G_0, respectively.

LEMMA 4.3.15. *The differential d on G_0 satisfies*

$$d(a \cdot \omega) = d(\varepsilon_l(a)) \wedge \omega + a \cdot d(\omega) \tag{4.3.5}$$

for $a \in C_c^\infty(G)$ and $\omega \in \Omega(G_0)$. Hence $[a - \varepsilon_l(a), d] = 0$ for all $a \in C_c^\infty(G)$, thus implying $H_0 = C_c^\infty(G)$ (see (4.3.2) *for H_0).*

PROOF. As observed above in the proof of Lemma 4.3.14, on a bisection $\sigma(U)$, we have $a(t|_{\sigma(U)})^{-1} = \varepsilon_l(a)$ and $a \cdot u = a(t|_{\sigma(U)})^{-1}u = \varepsilon_l(a)u$. Now (4.3.5) follows from Leibniz rule and locality of d. The last statement follows from (4.3.5) and the fact that $s_l \equiv t_l$. □

Denote by $\Omega(G_0)^G$ the G-invariant forms. Then forms on the "orbit or leaf space" are captured as follows.

PROPOSITION 4.3.16. *The pair $(\Omega(G_0), d)$ is a $C_c^\infty(G)$-covariant differential calculus, and we have $(\Omega(G_0)^G, d) = (\Omega(G_0)_{C_c^\infty(G)}, d)$ as differential graded algebras.*

PROOF. Since G acts by local diffeomorphisms, it follows that d is G-invariant. So d descends to $\Omega(G_0)^G$. The proposition now follows from Lemma 4.3.14 and Lemma 4.3.15. □

DEFINITION 4.3.17.

i) We say that an H-covariant differential calculus (Ω, d) over an H-module algebra B has total dimension n if $\Omega^k = 0$, for all $k > n$, and $\Omega^n \neq 0$.
ii) If in addition, there exists a (B, B)-bimodule and an H-module isomorphism $\mathrm{vol} : \Omega^n \to B$*, then we say that Ω is orientable.*
iii) If Ω is a $*$-calculus over a $*$-algebra, then a $*$-orientation is an orientation which is also $*$-preserving, meaning $\overline{\mathrm{vol}}\# = \#\,\mathrm{vol}$.
iv) A $*$-orientable calculus is one which admits a $*$-orientation.
v) Let τ be a state on B, i.e., a unital linear functional $\tau : B \to \mathbb{C}$ such that $\tau(b^*b) \geq 0$. We call the functional $\tau\,\mathrm{vol}$ *the integral associated to τ and denote it by $\int_\tau$.*
vi) We say that the integral is closed if $\int_\tau(d\omega) = 0$ for all $\omega \in \Omega^{n-1}$.

DEFINITION 4.3.18. *An étale groupoid G is oriented if G_0 is oriented in the ordinary sense and G acts by orientation-preserving local diffeomorphisms.*

PROPOSITION 4.3.19. *With $B = C^\infty(G_0)$ and $\Omega = \Omega(G_0)$, orientation in the sense of Definition 4.3.17 coincide with groupoid orientation on G.*

PROOF. This follows from Proposition 4.3.16. □

LEMMA 4.3.20. *Assume that $(\Omega,\, d)$ is $*$-oriented with orientation* vol *and of total dimension $2n$. Then $(\Omega_0,\, d)$ is $*$-oriented.*

PROOF. Since vol is assumed to be H-linear, it restricts to Ω_0, which in turn shows that $\Omega_H^{2n} \neq 0$ so that it also has total dimension $2n$. The lemma now follows from Lemma 4.2.23 and Proposition 4.3.11. □

4.3.2. Complex structures. The setup below is due to [**B́17**], see Chapter 3 and we follow it closely. We shall omit the proofs of some of the results here as they are essentially given in [**B́17**].

DEFINITION 4.3.21. *An H-covariant almost complex structure for an H-covariant $*$-differential calculus $(\Omega,\, d)$ over an H-module $*$-algebra B is an $\mathbb{N}_0^2$-algebra grading $\oplus_{(k,l)\in\mathbb{N}_0^2}\Omega^{(k,l)}$ for Ω such that*

i) the H-action preserves the $\mathbb{N}_0^2$-grading;
ii) $\Omega^n = \oplus_{k+l=n}\Omega^{(k,l)}$, for all $n \in \mathbb{N}_0$;
iii) $\# : \Omega \to \overline{\Omega}$ preserves the $\mathbb{N}_0^2$-grading, where the $\mathbb{N}_0^2$-grading on $\overline{\Omega}$ is given by $\overline{\Omega}^{(k,l)} = \overline{\Omega^{(l,k)}}$.

Let ∂ and $\overline{\partial}$ be the unique homogeneous operators of order $(1,0)$ and $(0,1)$ respectively, defined by

$$\partial\,|_{\Omega^{(k,l)}} = \mathrm{proj}_{\Omega^{(k+1,l)}}\, d \qquad \overline{\partial}\,|_{\Omega^{(k,l)}} = \mathrm{proj}_{\Omega^{(k,l+1)}}\, d, \tag{4.3.6}$$

where $\mathrm{proj}_{\Omega^{(k,l+1)}}$ and $\mathrm{proj}_{\Omega^{(k,l+1)}}$ are the projections from $\Omega^{(k+l+1)}$ onto $\Omega^{(k+1,l)}$ and $\Omega^{(k,l+1)}$, respectively.

As in [**B́17**], we have:

LEMMA 4.3.22. *If $\oplus_{(k,l)\in\mathbb{N}_0^2}\Omega^{(k,l)}$ is an H-covariant almost complex structure for an H-covariant $*$-differential calculus $(\Omega,\, d)$ over an H-module $*$-algebra B, then the following two conditions are equivalent:*

i) $d = \partial + \overline{\partial}$;
ii) the triple $(\oplus_{(k,l)\in\mathbb{N}_0^2}\Omega^{(k,l)}, \partial, \overline{\partial})$ is an H-covariant double complex.

PROOF. The proof of the equivalence is in [**B́17**]. All we have to show is the H-covariant part in ii). Observe that $\mathrm{proj}_{\Omega^{(k+1,l)}}$ and $\mathrm{proj}_{\Omega^{(k,l+1)}}$ are H-linear. Then for $h \in H_0$,

$$[h - s_l\varepsilon_l(h), \partial\,|_{\Omega^{(k,l)}}] = [h - t_l\varepsilon_l(h), \partial\,|_{\Omega^{(k,l)}}] = 0\,.$$

Thus we get (4.3.1) for $\partial\,|_{\Omega^{(k,l)}}$, and similarly for $\overline{\partial}\,|_{\Omega^{(k,l)}}$, hence the covariance. □

DEFINITION 4.3.23. *When the conditions in Lemma 4.3.22 hold for an almost complex structure, then we say that the almost complex structure is integrable.*

We also call an integrable almost complex structure a complex structure and the double complex $(\oplus_{(k,l)\in\mathbb{N}_0^2}\Omega^{(k,l)}, \partial, \overline{\partial})$ its Dolbeault double complex. Note that

$$\partial(\omega^*) = (\overline{\partial}\omega)^*, \qquad \overline{\partial}(\omega^*) = (\partial\omega)^*, \qquad \omega \in \Omega, \tag{4.3.7}$$

as they follow from the integrability condition.

LEMMA 4.3.24. *Suppose that $(\Omega,\, d)$ admits an H-covariant complex structure. Then $(\Omega_0,\, d)$ admits a complex structure. We call this a transverse complex structure on B_0.*

REMARK 4.3.25. *Strictly speaking, we haven't defined what complex structure (or any other structures) means on an algebra without any equivariance. The idea is to forget the "H-covariant" part and take the rest as the corresponding definition. In the present situation, a complex structure is a bigrading that satisfies Conditions i) and ii) in Definition 4.3.21 with $d = \partial + \overline{\partial}$.*

PROOF OF LEMMA 4.3.24. Condition i) in Definition 4.3.21 implies that (Ω_0, d) admits an $\mathbb{N}_0^2$-algebra grading by $(\Omega_0)^{(k,l)} = \Omega_0^{(k,l)}$, $(k,l) \in \mathbb{N}_0^2$. Condition ii) follows automatically, while Condition iii) follows from that fact that $\#$ is H-linear. ∂ and $\overline{\partial}$ restrict to the space of invariant forms as in Proposition 4.3.11. Finally, $d = \partial + \overline{\partial}$ then follows automatically. □

As in [**CW91**], we define:

DEFINITION 4.3.26. *The foliation $\mathcal{F}$ on a foliated manifold $(M, \mathcal{F})$ is transversely holomorphic if it carries a transverse complex structure in the sense of Definition 4.3.12.*

If the foliation $\mathcal{F}$ is transversely holomorphic, the normal bundle $N(M, \mathcal{F})$ of $\mathcal{F}$ has a complex structure corresponding to the complex structure on N. Therefore any complex-valued basic k-form can be represented as a sum of the k-forms of pure type (r, s) corresponding to the decomposition of k-forms on the complex manifold N. Let $\Omega^k_{\mathbb{C}}(M, \mathcal{F})$ denote the space of complex-valued basic k-forms on the foliated manifold $(M, \mathcal{F})$, and denote by $\Omega^{(r,s)}_{\mathbb{C}}(M, \mathcal{F})$ the space of complex-valued basic forms of pure type (r, s). Then we have $\Omega^k_{\mathbb{C}}(M, \mathcal{F}) = \oplus_{r+s=k} \Omega^{(r,s)}_{\mathbb{C}}(M, \mathcal{F})$. The exterior derivative $d : \Omega^k_{\mathbb{C}}(M, \mathcal{F}) \to \Omega^{k+1}_{\mathbb{C}}(M, \mathcal{F})$ decomposes into two components $d = \partial + \overline{\partial}$, where ∂ is of bidegree $(1, 0)$ and $\overline{\partial}$ is of bidegree $(0, 1)$, i.e., $\partial : \Omega^{(r,s)} \to \Omega^{(r+1,s)}$ and $\overline{\partial} : \Omega^{(r,s)} \to \Omega^{(r,s+1)}$.

Keeping in mind Definition 4.3.26 and the case for orbifolds (see [**BBF$^+$17**]), we make

DEFINITION 4.3.27. *An étale groupoid G is holomorphic if G_0 is a complex manifold and G acts by local biholomorphic transformations.*

This fits into our framework as follows:

PROPOSITION 4.3.28. *An étale groupoid G is holomorphic if and only if $(\Omega(G_0), d)$ admits a $C_c^\infty(G)$-covariant complex structure.*

PROOF. First observe that an almost complex structure on G_0 is also given by a bundle map $J : T^*(G_0) \to T^*(G_0)$ (and its extension to the exterior algebra bundle) such that $J^2 = -\operatorname{id}_{T^*(G_0)}$. The bidegree decomposition is a consequence of this fact. Since bundle maps are sections of the HOM-bundle, G is almost complex if and only if $(\Omega(G_0), d)$ admits a $C_c^\infty(G)$-covariant almost complex structure, by Lemma 4.3.14. Since integrability is same in both sense, we have the proposition proved. □

The orbit space inherits a complex structure:

COROLLARY 4.3.29. *If G is holomorphic, then $(\Omega(G_0)^G, d)$ admits a complex structure.*

PROOF. This follows from Proposition 4.3.28 and Lemma 4.3.24. □

4.3.3. Hermitian and Kähler structures. We fix an H-covariant $*$-differential calculus (Ω, d) over an H-module $*$-algebra B of total dimension $2n$. As in [**Ḃ17**], the following is a non-commutative generalization of an almost symplectic form.

DEFINITION 4.3.30. *An almost symplectic form for Ω is a central real H-invariant 2-form σ ($h \cdot \sigma = s_l \varepsilon_l(h) \cdot \sigma$ for all $h \in H$) such that, the Lefschetz operator*

$$L : \Omega \to \Omega, \qquad \omega \mapsto \sigma \wedge \omega$$

satisfies the following condition: the maps

$$L^{n-k} : \Omega^k \to \Omega^{2n-k} \tag{4.3.8}$$

are isomorphisms for all $0 \leq k < n$.

Since σ is a central real form, L is a (B,B)-bimodule morphism and $*$-preserving ($\overline{L}\# = \# L$). Moreover, the H-invariance condition implies that L is also an H-module morphism. Indeed, we have

$$h\cdot(\sigma\wedge\omega) = h_{(1)}\cdot\sigma\wedge h_{(2)}\cdot\omega = \imath_B(\varepsilon_l(h_{(1)}))\sigma\wedge h_{(2)}\cdot\omega$$
$$= \sigma\wedge\imath_B(\varepsilon_l(h_{(1)}))(h_{(2)}\cdot\omega)) = \sigma\wedge(s_l\varepsilon_l(h_{(1)})h_{(2)})\cdot\omega = \sigma\wedge h\cdot\omega\,.$$

Definition 4.3.31. *A symplectic form is a d-closed almost symplectic form.*

Buachalla, [**Ó17**], introduced hermitian structure which is an almost symplectic form compatible with a complex structure.

Definition 4.3.32. *A hermitian structure for Ω is a pair $(\Omega^{(\cdot,\cdot)},\sigma)$, where $\Omega^{(\cdot,\cdot)}$ is an H-covariant complex structure, and σ is an almost symplectic form, called the hermitian form, such that $\sigma\in\Omega^{(1,1)}$.*

We have:

Lemma 4.3.33. *Suppose that $(\Omega^{(\cdot,\cdot)},\sigma)$ is a hermitian structure for $(\Omega,\, d)$. Then σ induces a hermitian structure on $(\Omega_0,\, d)$.*

Proof. By definition, $\sigma\in\Omega_0$. The H-linearity of L shows that σ is an almost symplectic form for $(\Omega_0,\, d)$. Finally, $\sigma\in(\Omega^{(1,1)})_0 = (\Omega_0)^{(1,1)}$, by Lemma 4.3.24. □

We say that an almost complex structure is of diamond type if $\Omega^{(a,b)} = 0$ whenever $a>n$ or $b>n$. Supposing $a>n$ and observing that the isomorphism L^{a+b-n} maps $\Omega^{(n-b,n-a)}$ onto $\Omega^{(a,b)}$, we see that the existence of a hermitian structure implies that the complex structure has to be of diamond type.

Definition 4.3.34. *The Hodge map associated to a hermitian structure is the morphism uniquely defined by*

$$\star\,(L^j(\omega)) = (-1)^{\frac{k(k+1)}{2}} i^{a-b}\frac{[j]!}{[n-j-k]!}L^{n-j-k}(\omega) \qquad \omega\in P^{(a,b)}\subset P^k. \tag{4.3.9}$$

Recall the notion of primitive forms from Definition 3.2.5. Observe that $\star$ is an H-module morphism. Hence it descends to Ω_0.

Lemma 4.3.35. *We have*

i) $\star^2(\omega) = (-1)^k\omega$ *for all* $\omega\in\Omega^k$,
ii) $\star$ *is an isomorphism,*
iii) $\star(\Omega^{(a,b)}) = \Omega^{(n-b,n-a)}$,
iv) $\star$ *is a* $*$*-preserving.*

Proof. This is Lemma 3.2.10. The proof is same as given in [**Ó17**]. One checks the statements on primitive forms and then uses Lefschetz decomposition, see Proposition 3.2.6 and Proposition 4.3.38 below. □

Given a hermitian structure $(\Omega^{(\cdot,\cdot)},\kappa)$, we first recover the hermitian metric associated to it:

Definition 4.3.36. *Define $g:\Omega\otimes_B\overline{\Omega}\to B$ by $g(\omega\otimes\overline{\eta}) = 0$ for $\omega\in\Omega^k$, $\eta\in\Omega^l$, $k\neq l$, and*

$$g(\omega\otimes\overline{\eta}) = \mathrm{vol}(\omega\wedge\star(\eta^*)) \tag{4.3.10}$$

for $\omega,\eta\in\Omega^k$.

A metric on the orbit space should be an invariant one as is showed in the following lemma.

Lemma 4.3.37. *For $\omega,\eta\in\Omega^k$ and $h\in H$, it holds that*

$$g(h_{(1)}\cdot\omega\otimes h_{(2)}\cdot\overline{\eta}) = h\cdot g(\omega\otimes\overline{\eta}), \tag{4.3.11}$$

so that g is H-covariant.

PROOF. We compute

$$\begin{aligned}
g(h_{(1)}\cdot\omega\otimes h_{(2)}\cdot\overline{\eta}) &= g(h_{(1)}\cdot\omega\otimes\overline{S(h_{(2)})^*\cdot\eta})\\
&= \mathrm{vol}(h_{(1)}\cdot\omega\wedge\star(S(h_{(2)})^*\cdot\eta)^*)\\
&= \mathrm{vol}(h_{(1)}\cdot\omega\wedge\star((S(S(h_{(2)})^*))^*)\cdot\eta^*)\\
&= \mathrm{vol}(h_{(1)}\cdot\omega\wedge\star(h_{(2)}\cdot\eta^*)\\
&= \mathrm{vol}(h_{(1)}\cdot\omega\wedge h_{(2)}\cdot\star(\eta^*))\\
&= \mathrm{vol}(h\cdot(\omega\wedge\star(\eta^*)))\\
&= h\cdot\mathrm{vol}(\omega\wedge\star(\eta^*))\\
&= h\cdot g(\omega\otimes\overline{\eta}),
\end{aligned}$$

□

PROPOSITION 4.3.38. *The following decompositions are orthogonal with respect to* $\langle,\rangle$*:*

i) The degree decomposition $\Omega=\oplus_k\Omega^k$*;*
ii) The bidegree decomposition $\Omega^k=\oplus_{(a,b)}\Omega^{(a,b)}$*;*
iii) The Lefschetz decomposition $\Omega^k=\oplus_{j\geq 0}L^j(P^{k-2j})$.

PROOF. Again the proof is same as in [**B́17**]. But we repeat the proof of iii) as it is beautiful in its own right. First we prove that the decomposition holds. Assume that the decomposition holds for some $k\leq n-2$. Consider the composition $L^{n-k}:\Omega^k\xrightarrow{L}\Omega^{k+2}\xrightarrow{L^{n-k-1}}\Omega^{2n-k}$. Since $L^{n-k}:\Omega^k\to\Omega^{2n-k}$ is an isomorphism of (B,B)-bimodules, we have the (B,B)-bimodule decomposition

$$\begin{aligned}
\Omega^{k+2} &\cong ker(L^{n-k-1}\mid_{\Omega^{k+2}})\oplus L(\Omega^k)\\
&= ker(L^{n-(k+2)+1}\mid_{\Omega^{k+2}})\oplus L(\Omega^k)\\
&= P^{k+2}\oplus L(\Omega^k)\\
&= P^{k+2}\oplus(\oplus_{j\geq 0}L^{j+1}(P^{k-2j}))\\
&= P^{k+2}\oplus(\oplus_{j\geq 1}L^{j}(P^{k+2-2j}))\\
&= \oplus_{j\geq 0}L^{j}(P^{k+2-2j}).
\end{aligned}$$

Since $\Omega^0=P^0$ and $\Omega^1=P^1$, it follows from an inductive argument that the decomposition holds for each space of forms of degree less that or equal to n. For forms of degree greater than n, we see that, for $k=0,\ldots,n$,

$$\begin{aligned}
\Omega^{2n-k} &\cong L^{n-k}(\Omega^k)\\
&\cong L^{n-k}(\oplus_{j\geq 0}L^j(P^{k-2j}))\\
&= \oplus_{j\geq n-k}L^j(P^{2n-k-2j})\\
&= \oplus_{j\geq 0}L^j(P^{2n-k-2j}),
\end{aligned}$$

where the last equality follows from the fact that, for $j=0,\ldots,n-k-1$, either $2n-k-2j>n$ and $P^{2n-k-2j}=0$ by definition, or $k+2\leq 2n-k-2j\leq n$, and so, we have $L^j(P^{2n-k-2j})=0$.

Now we prove that the decomposition is orthogonal. Given $\omega\in P^k$, $\eta\in P^l$, $g(L^i(\omega)\otimes\overline{L^j(\eta))}$ is nonzero only if $2i+k=2j+l$. Assuming $\eta\in P^{(a,b)\subset P^l}$, we have

$$\begin{aligned}
g(L^i(\omega)\otimes\overline{L^j(\eta)}) &= \mathrm{vol}(L^i(\omega)\wedge\star L^j(\eta^*))\\
&= c\,\mathrm{vol}(L^i(\omega))\wedge L^{n-j-l}(\eta^*) \qquad \text{(where c is the scalar part in the definition of $\star$)}\\
&= c\,\mathrm{vol}(L^{i+n-j-l}(\omega)\wedge\eta^*).
\end{aligned}$$

Assuming $j>i$, we have $L^{i+n-j-l}=L^{n-k+(j-i)}$, hence, for $\omega\in P^k$, $L^{n-k+(j-i)}(\omega)=0$ and the result follows. The case $j<i$ is similar. □

The above proposition implies the following.

Corollary 4.3.39. *We have $g(\omega\otimes\overline{\eta}) = g(\eta\otimes\overline{\omega})^*$ for $\omega,\eta\in\Omega$.*

Proof. By Proposition 4.3.38, it suffices to prove the result for $g(L^j(\omega)\otimes\overline{L^j(\eta)})$, for $\omega,\eta\in P^{(a,b)}\subset P^k$. We have,

$$\begin{aligned}
g(L^j(\omega)\otimes\overline{L^j(\eta)}) &= \mathrm{vol}(L^j(\omega)\wedge\star L^j(\eta^*))\\
&= (-1)^{\frac{k(k+1)}{2}} i^{b-a}\frac{[j]!}{[n-j-k]!}\,\mathrm{vol}(L^j(\omega)\wedge L^{n-k-j}(\eta^*))\\
&= ((-1)^{\frac{k(k+1)}{2}} i^{a-b}\frac{[j]!}{[n-j-k]!}(-1)^{k^2}\,\mathrm{vol}(L^j(\eta)\wedge L^{n-k-j}(\omega^*)))^*\\
&= ((-1)^{\frac{k(k+1)}{2}} i^{b-a}\frac{[j]!}{[n-j-k]!}\,\mathrm{vol}(L^j(\eta)\wedge L^{n-k-j}(\omega^*)))^*\\
&= (g(L^j(\eta)\otimes\overline{L^j(\omega)}))^*.
\end{aligned}$$

□

We recall from [**MM03**]:

Definition 4.3.40. *The foliation $\mathcal{F}$ on a foliated manifold $(M,\mathcal{F})$ is transversely Riemannian if it carries a transverse Riemannian structure in the sense of Definition 4.3.12.*

The metric on $N(M,\mathcal{F})$ is induced from a bundle-like metric on M. Recall from [**CW91**]:

Definition 4.3.41. *The foliation $\mathcal{F}$ on a foliated manifold $(M,\mathcal{F})$ is transversely hermitian if it carries a transverse hermitian structure in the sense of Definition 4.3.12.*

The operator $\star:\Lambda^k(M,\mathcal{F})\to\Lambda^{2q-k}(M,\mathcal{F})$ defined via the transverse part of the bundle-like metric of $\mathcal{F}$ extends to $\Lambda^k_{\mathbb{C}}(M,\mathcal{F})\to\Lambda^{2q-k}_{\mathbb{C}}(M,\mathcal{F})$, where q is the complex codimension of $\mathcal{F}$.

Being motivated by this, we make the following definition.

Definition 4.3.42. *An étale groupoid G is hermitian if G_0 admits a G-invariant hermitian structure.*

Again, algebraically we have the following proposition.

Proposition 4.3.43. *An étale groupoid G is hermitian if and only if $(\Omega(G_0),\,d)$ admits a $C_c^\infty(G)$-covariant hermitian structure.*

Proof. The proof of the statement that G is hermitian implies that $(\Omega(G_0),\,d)$ admits a $C_c^\infty(G)$-invariant hermitian structure is straightforward. For the converse, we recover the hermitian metric as in Definition 4.3.36, and Lemma 4.3.37 shows that it is G-invariant. Compatibility follows from Proposition 4.3.38. □

Corollary 4.3.44. *If G is hermitian, then $(\Omega(G_0)^G,\,d)$ admits a hermitian structure.*

Proof. This follows from Proposition 4.3.43 and Lemma 4.3.33. □

The hermitian structure is said to be positive definite if $g(\omega\otimes\overline{\omega})>0$ for all nonzero $\omega\in\Omega$. In that case, we define an inner product (positive definite, hermitian) on Ω by setting

$$\langle\omega,\eta\rangle = \tau g(\omega\otimes\overline{\eta}) = \int_\tau \omega\wedge\star(\eta^*) \tag{4.3.12}$$

for $\omega,\eta\in\Omega$ and a fixed faithful state τ on B. We denote the corresponding norm of ω by $\|\omega\|$. Moreover, Lemma 4.3.37 shows that g induces a metric on Ω_0 that takes values in B_0. Applying τ, we get an inner product on Ω_0 which is really the restriction of $\langle\cdot,\cdot\rangle$ to Ω_0. From now on, we assume that the hermitian structure to be positive definite.

Proposition 4.3.45. *The Hodge map $\star$ is unitary.*

PROOF. See the proof of Lemma 5.10 in [**Ḃ17**]. □

We now define the Laplacians.

DEFINITION 4.3.46.

i) *The codifferential is defined as* $d^* := - \star d \star$;
ii) *the holomorphic codifferential is defined as* $\partial^* := - \star \overline{\partial} \star$;
iii) *the anti-holomorphic codifferential is defined as* $\overline{\partial}^* = - \star \partial \star$.

Observe that for $\omega \in \Omega$,

$$d^*(\omega^*) = (d^*\omega)^*, \qquad \partial^*(\omega^*) = (\overline{\partial}^*\omega)^* \qquad \text{and} \qquad \overline{\partial}^*(\omega^*) = (\partial^*\omega)^*. \tag{4.3.13}$$

Now, it is natural to define the d-, ∂- and $\overline{\partial}$- Laplacians, respectively as

$$\Delta_d := (d + d^*)^2, \qquad \Delta_\partial := (\partial + \partial^*)^2, \qquad \Delta_{\overline{\partial}} := (\overline{\partial} + \overline{\partial}^*)^2. \tag{4.3.14}$$

PROPOSITION 4.3.47. *The operator adjoints of d, ∂ and $\overline{\partial}$ are d^*, ∂^* and $\overline{\partial}^*$, respectively.*

The following will be used later.

COROLLARY 4.3.48. *The Laplacians Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$ are symmetric.*

We have:

LEMMA 4.3.49. *The operator d^* (respectively ∂, $\overline{\partial}$) and hence Δ_d (respectively Δ_∂, $\Delta_{\overline{\partial}}$) descends to Ω_0.*

PROOF. Since $\star$ is H-linear, we have for $h \in H_0$,

$$[h - s_l\varepsilon_l(h), d^*] = [h - t_l\varepsilon_l(h), d^*] = 0\,.$$

Hence d^* descends to Ω_0. □

Given the Laplacians Δ_d, Δ_∂ and $\Delta_{\overline{\partial}}$, we define the d-harmonic, ∂-harmonic and $\overline{\partial}$-harmonic forms to be, respectively

$$\mathcal{H}_d := \ker(\Delta_d), \qquad \mathcal{H}_\partial := \ker(\Delta_\partial), \qquad \mathcal{H}_{\overline{\partial}} := \ker(\Delta_{\overline{\partial}}). \tag{4.3.15}$$

PROPOSITION 4.3.50. *We have*

i) $\Delta_d\omega = 0$ *if and only if* $d\omega = 0$ *and* $d^*\omega = 0$;
ii) $\Delta_\partial\omega = 0$ *if and only if* $\partial\omega = 0$ *and* $\partial^*\omega = 0$;
iii) $\Delta_{\overline{\partial}}\omega = 0$ *if and only if* $\overline{\partial}\omega = 0$ *and* $\overline{\partial}^*\omega = 0$.

PROOF. We only prove i), the other proofs being similar. Clearly, $\Delta_d\omega = 0$ if $d\omega = 0$ and $d^*\omega = 0$. Now

$$\langle \Delta_d\omega, \omega\rangle = \|d\omega\|^2 + \|d^*\omega\|^2.$$

Thus if $\Delta_d\omega = 0$, then the both terms on right-hand side must vanish, i.e., $d\omega = 0$ and $d^*\omega = 0$. □

According to [**Ḃ17**], Kähler structures are defined as follows.

DEFINITION 4.3.51. *A Kähler structure is a hermitian structure $(\Omega^{(\cdot,\cdot)}, \kappa)$ such that the hermitian form κ is d-closed. Such a form is called a Kähler form.*

THEOREM 4.3.52. *The following relations hold:*

$$\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0, \qquad \partial^*\overline{\partial} + \overline{\partial}\partial^* = 0, \qquad \Delta_d = 2\Delta_\partial = 2\Delta_{\overline{\partial}}. \tag{4.3.16}$$

PROOF. See the proof of Corollary 7.6 of [**Ḃ17**]. □

PROPOSITION 4.3.53. *We have*

i) $\mathcal{H}^k_\partial = \oplus_{a+b=k}\mathcal{H}^{(a,b)}_\partial$ *and* $\mathcal{H}^k_{\overline{\partial}} = \oplus_{a+b=k}\mathcal{H}^{(a,b)}_{\overline{\partial}}$, *where*

$$\mathcal{H}^{(a,b)}_\partial = \{\omega \in \Omega^{(a,b)} \mid \Delta_\partial \omega = 0\}.$$

Similarly, define $\mathcal{H}^{(a,b)}_{\overline{\partial}}$;

ii) if the hermitian structure is Kähler, then both decompositions coincide with $\mathcal{H}^k_d = \oplus_{a+b=k}\mathcal{H}^{(a,b)}_d$. *In particular,* $\mathcal{H}^k_d = \mathcal{H}^k_\partial = \mathcal{H}_{\overline{\partial}^k}$.

The proof in [**B́17**] does not use equivariance. Hence the above proposition also holds for (Ω_0, d).

PROPOSITION 4.3.54. *The Hodge map* $\star$ *and the map* $\alpha \mapsto \overline{\alpha}^*$ *commute with the Laplacian* Δ_d. *Hence, in the Kähler case, they also commute with* Δ_∂ *and* $\Delta_{\overline{\partial}}$.

LEMMA 4.3.55. *A Kähler structure* $(\Omega^{(\cdot,\cdot)}, \kappa)$ *on* (Ω, d) *induces via* κ *a Kähler structure on* (Ω_0, d).

PROOF. Since κ is automatically $d|_{\Omega_0}$-closed, the lemma follows from Lemma 4.3.33. □

Following [**CW91**], we have:

DEFINITION 4.3.56. *The foliation* $\mathcal{F}$ *on a foliated manifold* $(M, \mathcal{F})$ *is transversely Kähler if it carries a transverse Kähler structure in the sense of Definition 4.3.12.*

The Kähler form of N defines a basic $(1,1)$-form on $(M, \mathcal{F})$ which is called the transverse Kähler form of the foliation $\mathcal{F}$. Motivated by this and the case for orbifolds, we define:

DEFINITION 4.3.57. *An étale groupoid* G *is Kähler if* G_0 *admits a* G*-invariant Kähler structure.*

The following is routine:

PROPOSITION 4.3.58. *An étale groupoid* G *is Kähler if and only if* $(\Omega(G_0), d)$ *admits a* $C^\infty_c(G)$*-covariant Kähler structure.*

PROOF. This follows from Proposition 4.3.43 and Proposition 4.3.16. □

COROLLARY 4.3.59. *If* G *is Kähler, then* $(\Omega(G_0)^G, d)$ *admits a Kähler structure.*

PROOF. This follows from Proposition 4.3.58 and Lemma 4.3.55. □

4.4. Hodge theory and formality for noncommutative Kähler structures

In this section, we prove a version of the Hodge decomposition and a formality theorem.

4.4.1. The Hodge decomposition. We begin by remarking that in [**B́17**], cosemisimplicity is used to prove the theorem for quantum homogeneous spaces. What we prove below corresponds to, classically, Hodge decomposition for G_0. Ideally, one should use only the compactness for G_0 without any equivariance. This is what we do. To descend to the space of invariant forms, we need something more. More about it below (see Definition 4.4.7). Following [**War83**], we make the following definition.

DEFINITION 4.4.1. *For* $\eta \in \Omega^k$, *a weak solution to* $\Delta_d(\omega) = \eta$ *is a bounded linear functional* $l : \Omega^k \to \mathbb{C}$ *such that*

$$l(\Delta_d(\phi)) = \langle \eta, \phi \rangle, \qquad \text{for all } \phi \in \Omega^k. \tag{4.4.1}$$

The next definition is equivalent to the ellipticity of the Laplacian in the classical situation.

DEFINITION 4.4.2. *The hermitian structure is said to be d-regular if the following are satisfied:*

i) Let $\eta \in \Omega^k$, *and let* l *be a weak solution of* $\Delta_d(\omega) = \eta$. *Then there exists an element* $\omega \in \Omega^k$ *such that*

$$l(\nu) = \langle \omega, \nu \rangle$$

for every $\nu \in \Omega^k$.

ii) For a sequence $\{\eta_n\}$ *in* Ω^k *such that* $\|\eta_n\| \leq c$ *and* $\|\Delta_d(\eta_n)\| \leq c$ *for all* n *and for some constant* $c > 0$, *there exists a Cauchy subsequence of* $\{\eta_n\}$ *in* Ω^k.

A sufficient condition for regularity is provided in Theorem 4.4.18. We now show that, as in the classical situation, regularity is sufficient for the decomposition to hold.

THEOREM 4.4.3. *Assume that the hermitian structure is d-regular. Then for each k with $0 \leq k \leq 2n$, the space $\mathcal{H}_d^k$ of d-harmonic forms is finite dimensional and we have the following orthogonal direct sum decomposition of Ω^k called the Hodge decomposition:*

$$\begin{aligned} \Omega^k &= \Delta_d(\Omega^k) \oplus \mathcal{H}_d^k \\ &= (dd^* \oplus d^*d)(\Omega^k) \oplus \mathcal{H}_d^k \\ &= d(\Omega^{k-1}) \oplus d^*(\Omega^{k+1}) \oplus \mathcal{H}_d^k. \end{aligned} \tag{4.4.2}$$

PROOF. We closely follow [**War83**]. If $\mathcal{H}_d^k$ were not finite dimensional, then $\mathcal{H}_d^k$ would contain an infinite orthonormal sequence. But by condition ii) in Definition 4.4.2, this orthonormal sequence would contain a Cauchy subsequence, which is impossible. Thus $\mathcal{H}_d^k$ is finite dimensional.

Observe that it is sufficient to prove the first equality.

Let $\omega_1, \dots, \omega_l$ be an orthonormal basis of $\mathcal{H}_d^k$. Then an arbitrary form $\eta \in \Omega^k$ can uniquely be written as

$$\eta = \nu + \sum_{i=1}^{l} \langle \eta, \omega_i \rangle \omega_i, \tag{4.4.3}$$

where ν lies in $(\mathcal{H}_d^k)^\perp$. Thus we have an orthogonal direct sum decomposition

$$\Omega^k = (\mathcal{H}_d^k)^\perp \oplus \mathcal{H}_d^k. \tag{4.4.4}$$

The theorem will be proved by showing that $(\mathcal{H}_d^k)^\perp = \Delta_d(\Omega^k)$. We let P denote the projection operator of Ω^k onto $\mathcal{H}_d^k$ so that $P(\eta)$ is the harmonic part of η.

It can be shown that $\Delta_d(\Omega^k) \subset (\mathcal{H}_d^k)^\perp$. Indeed, if $\omega \in \Omega^k$ and $\eta \in \mathcal{H}_d^k$, then

$$\langle \Delta_d(\omega), \eta \rangle = \langle \omega, \Delta_d(\eta) = 0.$$

Conversely, we claim that

$$(\mathcal{H}_d^k)^\perp \subset \Delta_d(\Omega^k)\,. \tag{4.4.5}$$

In order to prove (4.4.5), we first need the following inequality.

We claim that there is a constant $c > 0$ such that

$$\|\eta\| \leq c\|\Delta_d(\eta)\| \qquad \text{for all } \eta \in (\mathcal{H}_d^k)^\perp. \tag{4.4.6}$$

Suppose the contrary. Then there exists a sequence $\eta_j \in (\mathcal{H}_d^k)^\perp$ with $\|\eta_j\| = 1$ and $\|\Delta_d(\eta_j)\| \to 0$. By condition ii) in Definition 4.4.2, a subsequence of the η_j, which for convenience we can assume to be $\{\eta_j\}$ itself, is Cauchy. Thus $\lim_{j\to\infty} \langle \eta_j, \psi \rangle$ exists for each $\psi \in \Omega^k$. We define a linear functional l on Ω^k be setting

$$l(\psi) = \lim_{j\to\infty} \langle \eta_j, \psi \rangle \qquad \text{for } \psi \in \Omega^k. \tag{4.4.7}$$

Now l is clearly bounded, and

$$l(\Delta_d(\phi)) = \lim_{j\to\infty} \langle \eta_j, \Delta_d(\phi) \rangle = \lim_{j\to\infty} \langle \Delta_d(\eta_j), \phi \rangle = 0, \tag{4.4.8}$$

so l is weak solution of $\Delta_d(\eta) = 0$. By condition i) in Definition 4.4.2, there exists $\eta \in \Omega^k$ such that $l(\psi) = \langle \eta, \psi \rangle$. Consequently, $\eta_j \to \eta$. Since $\|\eta_j\| = 1$ and $\eta_j \in (\mathcal{H}_d^k)^\perp$, it follows that $\|\eta\| = 1$ and $(\mathcal{H}_d^k)^\perp$. But $\Delta_d(\eta) = 0$, so $\eta \in \mathcal{H}_d^k$, which is a contradiction. Thus the inequality in (4.4.6) is proved.

Now we shall use (4.4.6) to prove (4.4.5). Let $\eta \in (\mathcal{H}_d^k)^\perp$. We define a linear functional l on $\Delta_d(\Omega^k)$ by setting

$$l(\Delta_d(\phi)) = \langle \eta, \phi \rangle \text{ for all } \phi \in \Omega^k. \tag{4.4.9}$$

This l is well-defined; for if $\Delta_d(\phi_1) = \Delta_d(\phi_2)$, then $\phi_1 - \phi_2 \in \mathcal{H}_d^k$, so that $\langle \eta, \phi_1 - \phi_2 \rangle = 0$. Also l is a bounded linear functional on $\Delta_d(\Omega^k)$, for let $\phi \in \Omega^k$ and let $\psi = \phi - P(\phi)$. Then using the above inequality, we obtain that

$$\begin{aligned} \mid l(\Delta_d(\phi)) \mid = \mid l(\Delta_d(\phi)) \mid = \mid \langle \eta, \psi \rangle \mid &\leq \|\eta\| \|\psi\| \\ &\leq c\|\eta\| \|\Delta_d(\psi)\| = c\|\eta\| \|\Delta_d(\phi)\|. \end{aligned} \tag{4.4.10}$$

By the Hahn-Banach theorem, l extends to a bounded linear functional on Ω^k. Thus l is a weak solution of $\Delta_d(\omega) = \eta$. By condition i) in Definition 4.4.2, there exists $\omega \in \Omega^k$ such that $\Delta_d(\omega) = \eta$. Hence (4.4.5) is proved. Consequently, we have

$$(\mathcal{H}_d^k)^{\perp} = \Delta_d(\Omega^k), \tag{4.4.11}$$

and the Hodge decomposition is proved. □

Similarly, ∂-regularity and $\overline{\partial}$-regularity lead to Hodge decompositions for Δ_∂ and $\Delta_{\overline{\partial}}$, with finite dimensional harmonic spaces $\mathcal{H}_\partial^{(a,b)}$, $\mathcal{H}_{\overline{\partial}}^{(a,b)}$, respectively. Moreover, if the hermitian structure is Kähler, then d-regularity coincide with ∂-regularity and $\overline{\partial}$-regularity. In this situation, $\mathcal{H}_\partial^{(a,b)} = \mathcal{H}_{\overline{\partial}}^{(a,b)}$.

From now on, we assume d-, ∂- and $\overline{\partial}$-regularity.

Corollary 4.4.4. *We have*

$$\ker(d) = \mathcal{H}_d \oplus d(\Omega), \quad \ker(\partial) = \mathcal{H}_\partial \oplus \partial(\Omega), \quad \ker(\overline{\partial}) = \mathcal{H}_{\overline{\partial}} \oplus \overline{\partial}(\Omega), \tag{4.4.12}$$

and

$$\mathcal{H}_d^k = H_d^k, \quad \mathcal{H}_\partial^{(a,b)} = H_\partial^{(a,b)} \quad \mathcal{H}_{\overline{\partial}}^{(a,b)} = H_{\overline{\partial}}^{(a,b)}. \tag{4.4.13}$$

where H_d^k is the k-th cohomology of $(\Omega,\, d)$, $H_\partial^{(a,b)}$ is the a-th cohomology of $(\Omega^{(\cdot,b)}, \partial)$ and $H_{\overline{\partial}}^{(a,b)}$ is the b-th cohomology of $(\Omega^{(a,\cdot)}, \overline{\partial})$.

Corollary 4.4.5. *Let $(\Omega^{(\cdot,\cdot)}, \kappa)$ be Kähler. Then for a d-closed form ω of type (a,b), the following conditions are equivalent:*

i) The form ω is d-exact;
ii) the form ω is ∂-exact;
iii) the form ω is $\overline{\partial}$-exact;
iv) the form ω is $\partial\overline{\partial}$-exact.

Proof. We add another equivalent condition: v) The form ω is orthogonal to $\mathcal{H}^{(a,b)}$. The Kähler condition says that we don't have to specify with respect to which differential operator (d, ∂ or $\overline{\partial}$) harmonicity is considered.

Using Hodge decomposition, we see that v) is implied by any of the other conditions. Moreover, iv) implies i)-iii). Thus it suffices to show that v) implies iv).

If $\omega \in \Omega^{(a,b)}$ is d-closed (and thus ∂-closed) and orthogonal to the space of harmonic forms, then Hodge decomposition with respect to ∂ yields that $\omega = \partial(\eta)$. Now applying Hodge decomposition with respect to $\overline{\partial}$ to the form η yields that

$$\eta = \overline{\partial}(\nu) + \overline{\partial}^*(\nu') + \nu''$$

for some harmonic ν''. Thus $\omega = \partial\overline{\partial}(\nu) + \partial\overline{\partial}^*(\nu')$. Using $\partial\overline{\partial}^* = -\overline{\partial}^*\partial$ and $\overline{\partial}(\omega) = 0$ we conclude $\overline{\partial}\overline{\partial}^*\partial(\nu') = 0$. Since $\langle \overline{\partial}\overline{\partial}^*\partial(\nu'), \partial(\nu') \rangle = \|\overline{\partial}^*\partial(n')\|^2$, it follows that $\partial\overline{\partial}^*(\nu') = -\overline{\partial}^*\partial(\nu') = 0$. Therefore, $\omega = \partial\overline{\partial}(\nu)$. □

Corollary 4.4.6. *Let $(\Omega^{(\cdot,\cdot)}, \kappa)$ be Kähler. Then there exists a decomposition*

$$H_d^k = \oplus_{a+b=k} H_\partial^{(a,b)} = \oplus_{a+b=k} H_{\overline{\partial}}^{(a,b)}. \tag{4.4.14}$$

The decomposition does not depend on the chosen Kähler structure.

For foliated manifolds there are different ways of proving the decomposition; see for example [**CW91**, **EKA90**, **PR96**]. To use averaging as in [**B́17**], it turns out that the correct generalization of compact lie groups are proper étale groupoids. For proper étale groupoids, there are Haar systems and cut off functions, by which one can average sections to make them invariant; see for example [**Har15**]. Motivated by this, we make the following definition.

DEFINITION 4.4.7. *We say that H acts on $(\Omega,\, d)$ properly (or $(\Omega,\, d)$ is a proper H-module) if there is a graded $\mathbb{C}$-linear morphism $\pi : \Omega \to \Omega$ which is a self-adjoint idempotent with range Ω_0.*

So we are actually capturing orbit spaces for proper étale groupoids or orbifolds. Note that if the Hopf algebroid is assumed to be semisimple, i.e., there is an integral (see [**Böh09**]), then it acts properly on any module. See Proposition 4.4.19 for a sufficient condition (or rather the actual projection, the algebraisation of which is the above definition) for such a projection to exist.

COROLLARY 4.4.8. *For a d-regular hermitian structure on $(\Omega,\, d)$ which is also a proper H-module, any $\omega \in \Omega_0^k$ can be written as*

$$\omega = \Delta_d(\eta) + \nu\,, \tag{4.4.15}$$

where $\eta \in \Omega_0^k$ and $\nu \in \mathcal{H}_d^k \cap \Omega_0^k$. Hence Hodge decomposition hold for $(\Omega_0,\, d)$ under d-regularity.

Corollary 4.4.8 implies that the same proof as in Corollary 4.4.5 goes through and implies an analogue of Corollary 4.4.5 for $(\Omega_0,\, d)$ under the properness assumption.

PROOF OF COROLLARY 4.4.8. The result follows from Hodge decomposition once we show that Δ_d commutes with π. Now let $\omega \in \Omega$. Then

$$\begin{aligned}
\langle \eta, \Delta_d \pi(\omega)\rangle &= \langle \Delta_d(\eta), \pi(\omega)\rangle \\
&= \langle \pi\Delta_d(\eta), \omega\rangle \\
&= \langle \Delta_d(\eta), \omega\rangle \\
&= \langle \eta, \Delta_d(\omega)\rangle = \langle \pi(\eta), \Delta_d(\omega)\rangle \\
&= \langle \eta, \pi\Delta_d(\omega)\rangle
\end{aligned}$$

for all $\eta \in \Omega_0$. Hence $\Delta_d\pi(\omega) = \pi\Delta_d(\omega)$. Here we use that Δ_d is self-adjoint and it preserves Ω_0. □

4.4.2. Formality of noncommutative Kähler structures. In this section we prove an analogue of the classical result that says compact Kähler manifolds are formal. For foliated manifolds this was shown in [**EKA90**, **CW91**] and for orbifolds in [**BBF$^+$17**]. We start by recalling the definition of a formal differential graded algebra. We closely follow [**Huy05**] for the whole section.

DEFINITION 4.4.9. *Two differential graded algebras (X, d_X) and (Y, d_Y) are equivalent if there exists a sequence of differential graded algebra quasi-isomorphisms*

$$\begin{tikzcd}[column sep=small]
& (C_1, d_{C_1}) \arrow[dl] \arrow[dr] & & \cdots \arrow[dl] \arrow[dr] & & (C_n, d_{C_n}) \arrow[dl] \arrow[dr] & \\
(X, d_X) & & (C_2, d_{C_2}) & & \cdots & & (Y, d_Y).
\end{tikzcd}$$

DEFINITION 4.4.10. *A differential grade algebra (X, d_X) is called formal if (X, d_X) is equivalent to a differential graded algebra (Y, d_Y) with $d_Y = 0$.*

We note that (X, d_X) is formal if and only if (X, d_X) is equivalent to its cohomology differential graded algebra $(H^{\cdot}(X, d_X), d = 0)$.

Now in our setup, suppose that $(\Omega,\, d)$ admits an H-covariant complex structure. Introduce the operator $d^c : \Omega^k \to \Omega^{k+1}$ defined as $d^c = -\sqrt{-1}(\partial - \overline{\partial})$. Lemma 4.3.22 then implies that $dd^c = -d^c d = 2\sqrt{-1}\partial\overline{\partial}$ and $(d^c)^2 = 0$. By Lemma 4.3.24, we see that d^c descends to Ω_0. We prove below an analogue of the dd^c-lemma in the classical situation.

LEMMA 4.4.11. *Suppose (Ω, d) admits a d-regular Kähler (d-regular hermitian which is Kähler) structure. Let $\omega \in \Omega^k$ be a d^c-exact and d-closed form. Then there exists a form $\eta \in \Omega^{k-2}$ with $\omega = dd^c(\eta)$. The same holds for (Ω_0, d) if (Ω, d) is a proper H-module.*

PROOF. We write $\omega = d^c(\phi)$ and consider the Hodge decomposition $\phi = d(\eta) + \nu + d^*(\psi)$. The property of being Kähler implies that the harmonic part ν is also ∂-closed and $\overline{\partial}$-closed. Hence $d^c(\phi) = d^c d(\eta) + d^c d^*(\psi)$.

It suffices to show $d^c d^*(\psi) = 0$. We now use $0 = d(\omega) = dd^c d^*(\psi)$ and $d^c d^* = -d^* d^c$ as in the proof of Corollary 4.4.5. Hence,

$$0 = \langle dd^* d^c(\psi), d^c(\psi)\rangle = \|d^* d^c(\psi)\|^2$$

and thus $d^c d^*(\psi) = -d^* d^c(\psi) = 0$.

Now for the last statement, we observe that because of Corollary 4.4.8 and Lemma 4.3.55, the same proof as above gives the dd^c-lemma for (Ω_0, d). □

Lemma 4.4.11 implies the following corollary. Part of the proof was suggested by the referee.

COROLLARY 4.4.12. *If $\omega \in \Omega^k$ is a d^c-closed and d-exact form for a d-regular Kähler structure on (Ω, d), then $\omega = d^c d(\eta)$ for some $\eta \in \Omega^{k-2}$.*

PROOF. We introduce the operator $\mathrm{I} : \Omega \to \Omega$ defined by $\mathrm{I}(\omega) = \sum_{a,b} (-1)^{\frac{a-b}{2}} \operatorname{proj}_{\Omega^{(a,b)}}(\omega)$ and observe that

$$d^c = \mathrm{I}^{-1} d\,\mathrm{I}, \quad d\,\mathrm{I}^2 = -\,\mathrm{I}^2 d,$$

which follow from the bidegree decomposition. Using $d^c = \mathrm{I}^{-1} d\,\mathrm{I}$, we get that ω is d^c-closed if and only if $\mathrm{I}(\omega)$ is d-closed. The identity $d\,\mathrm{I}^2 = -\,\mathrm{I}^2 d$ implies ω is d-exact if and only if $\mathrm{I}(\omega)$ is d^c-exact. Then

$$\mathrm{I}^{-1} dd^c = \mathrm{I}^{-1} d\,\mathrm{I}^{-1} d\,\mathrm{I} = \mathrm{I}^{-1} d\,\mathrm{I}\,\mathrm{I}^{-2} d\,\mathrm{I} = -d^c d\,\mathrm{I}^{-1},$$

together with Lemma 4.4.11 yield the corollary. □

We next consider the sub differential graded algebra $(\Omega^c, d) \subset (\Omega, d)$ consisting of all d^c-closed forms. Since $dd^c = -d^c d$, we get that $d(\Omega^c) \subset \Omega^c$.

LEMMA 4.4.13. *For a d-regular Kähler structure on (Ω, d), the inclusion $j : (\Omega^c, d) \to (\Omega, d)$ is a differential graded algebra quasi-isomorphism. If the H-action is proper, then the same conclusion holds for (Ω_0, d).*

PROOF. Let $\omega \in (\Omega^k)^c$ be a d-exact form. Then by Lemma 4.4.11, we get that $\omega = dd^c(\eta)$ for some $\eta \in \Omega^{k-2}$. Injectivity of j^* is now clear because $d^c(\eta)$ is already d^c-closed.

By Corollary 4.4.4, any cohomology class in H_d^k can be represented by a d-harmonic form $\omega \in \Omega^k$. By Proposition 4.3.53, any d-harmonic form is also ∂-harmonic and $\overline{\partial}$-harmonic. Thus ω is d^c-closed and hence ω is in the range of j^*. This gives the surjectivity of j^*.

The last statement is obtained by the same proof and corresponding results for (Ω_0, d). □

Since $dd^c = -d^c d$, it follows that d induces a natural differential

$$d : H_{d^c}^k \to H_{d^c}^{k+1},$$

where $H_{d^c}^k$ is the k-th cohomology of (Ω^c, d^c).

LEMMA 4.4.14. *For a d-regular Kähler structure on (Ω, d), the natural projection $p : (\Omega^c, d) \to (H_{d^c}, d)$ is a differential graded algebra quasi-isomorphism. If the H-action is proper, then the same holds for (Ω_0, d).*

PROOF. Let $\omega \in \Omega^k$ be d-closed and d^c-exact. Then Lemma 4.4.11 implies that $\omega = dd^c(\eta)$. In particular, ω is in the image of $d : (\Omega^{k-1})^c \to (\Omega^k)^c$. Hence p^* is surjective.

Let an element in the cohomology of (H_{d^c}, d) be represented by the d^c-closed form ω. Then $d(\omega)$ is d-exact and d^c-closed. Thus $d(\omega) = dd^c(\eta)$ by Lemma 4.4.11. Hence $\omega - d^c(\eta)$ is both d^c-closed and d-closed and represents the same class as ω in H_{d^c}. This proves the surjectivity of p^*. □

COROLLARY 4.4.15. *For a d-regular Kähler structure on $(\Omega,\, d)$, the differential d is trivial on H_{d^c}.*

PROOF. If ω is d^c-closed, then $d(\omega)$ is d-exact and d^c-closed, and thus it is of the form $d(\omega) = d^c d(\eta)$ for some η. So $0 = [d(\omega)] \in K^{k+1}_{d^c}$. □

If the H-action is proper then the above corollary holds for $(\Omega_0,\, d)$.

THEOREM 4.4.16. *Any given $(\Omega,\, d)$ is a formal differential graded algebra if it admits a d-regular Kähler structure. The same conclusion holds for $(\Omega_0,\, d)$ if the H-action is assumed to be proper.*

PROOF. By Lemma 4.4.13 and Lemma 4.4.14 respectively, $j : (\Omega^c,\, d) \to (\Omega,\, d)$ and $p : (\Omega^c,\, d) \to (H_{d^c},\, d)$ are differential graded algebra quasi-isomorphisms. Thus, the diagram

$$\begin{tikzcd} & (\Omega^c,\, d) \arrow[dl, "p"'] \arrow[dr, "j"] & \\ (\Omega,\, d) & & (H_{d^c}, 0) \end{tikzcd}$$

it follows that $(\Omega,\, d)$ is equivalent to a differential graded algebra with a trivial differential. □

4.4.3. A sufficient condition for d-regularity. In this subsection we establish a sufficient condition for a hermitian structure to be d-regular in the sense of Definition 4.4.2. We also prove that the projection ϕ as in Definition 4.4.7 commutes with Δ_d.

Recall from (4.3.12) that for a positive definite hermitian structure, an inner product is given by

$$\langle \omega, \eta \rangle = \tau g(\omega \otimes \overline{\eta}) = \int_\tau \omega \wedge \star(\eta^*). \tag{4.4.16}$$

DEFINITION 4.4.17. *Define the Hilbert space of forms $L^2(\Omega)$ to be the completion of Ω with respect to the inner product given by* (4.4.16).

Then Δ_d becomes a non-negative (see the proof of Proposition 4.3.50) densely defined symmetric (see Corollary 4.3.48) operator on $L^2(\Omega)$. Thus Δ_d has a canonical self-adjoint extension called the Friedrichs extension which we again denote by Δ_d.

THEOREM 4.4.18. *Assume that $\cap_k \mathrm{dom}(\Delta_d^k) = \Omega$ and that Δ_d has purely discrete spectrum, in the sense that there is an orthonormal basis $\{\omega_j\}$ for the Hilbert space $L^2(\Omega)$ consisting of forms $\omega_j \in \Omega$ which are eigenforms for Δ_d:*

$$\Delta_d(\omega_j) = \lambda_j \omega_j, \text{ for some scalar } \lambda_j$$

such that $0 = \lambda_0 < \lambda_1 < \ldots \to \infty$ as $j \to \infty$. Then the hermitian structure is d-regular.

PROOF. We have to show that conditions i) and ii) in Definition 4.4.2 are satisfied. For i), suppose we are given $\eta \in \Omega$ and a weak solution l of $\Delta_d(\omega) = \eta$. Write $\eta = \sum_j c_j \omega_j$. Observe that (4.4.1) implies that $c_0 = 0$ and $\frac{c_j}{\lambda_j} = l(\omega_j)$. Hence $\omega = \sum_1^\infty \frac{c_j}{\lambda_j} \omega_j$ is the form we are looking for. All we have to show is that $\omega \in \Omega$, i.e., ω is "smooth". As in the classical situation this follows from the basic estimate: introduce the norms $\|v\|_k^2 = \|\Delta_d^k(v)\|^2 + \|v\|^2$ on $H^k = \mathrm{dom}(\Delta_d^k)$. Then these spaces become Hilbert spaces with respect to these norms [**Hig06**]. Now $\omega_j \in \Omega = \cap_k H^k$, hence any finite linear combination of ω_j's is in H^k, for all k. Observe that for $m > n$ large enough so that $\lambda_n > 1$,

$$\|\sum_n^m \frac{c_j}{\lambda_j}\omega_j\|_k^2 = \sum_n^m \frac{|c_j|^2 \lambda_j^{2k}}{\lambda_j^2} + \sum_n^m \frac{|c_j|^2}{\lambda_j^2} < \sum_n^m |c_j|^2 \lambda_j^{2k} + \sum_n^m |c_j|^2 = \|\sum_n^m c_j \omega_j\|_k^2.$$

Since $\eta \in \Omega = \cap_k H^k$, we get that $\omega \in H^k$, for all k, hence smooth. Thus we proved that condition i) holds.

For ii), fix $\lambda \in \rho(\Delta_d)$-the resolvent set, and observe that the resolvent $(\lambda - \Delta_d)^{-1}$ is a compact self-adjoint operator. By hypothesis, $\|(\lambda - \Delta_d)(\eta_n)\| \le c(|\lambda| + 1)$ for all n. So, by compactness, $\{\eta_n = (\lambda - \Delta_d)^{-1}(\lambda - \Delta_d)(\eta_n)\}$ has a norm-convergent subsequence, hence the subsequence is Cauchy.

Thus the hermitian structure is d-regular. □

Now let $L^2(\Omega_0)$ be the closure of Ω_0 in $L^2(\Omega)$, and let P be the orthogonal projection onto $L^2(\Omega_0)$. The following is extracted from Proposition 1.17 of [**Sch12**].

PROPOSITION 4.4.19. *Suppose $\Delta_d|_{\Omega_0}$ is essentially self-adjoint on $L^2(\Omega_0)$. Then P takes* $\mathrm{dom}(\Delta_d)$ *into* $\mathrm{dom}(\Delta_d)$*, and $\Delta_d P(\omega) = P\Delta_d(\omega)$ for all $\omega \in \mathrm{dom}(\Delta_d)$. Moreover, P takes Ω into Ω. Hence $P|_\Omega$ gives a projection in the sense of Definition 4.4.7.*

PROOF. Let ω be in $\mathrm{dom}(\Delta_d)$. Then

$$\begin{aligned}\langle \Delta_d|_{\Omega_0}(\eta), P(\omega)\rangle &= \langle P\Delta_d|_{\Omega_0}(\eta), \omega\rangle \\ &= \langle \Delta_d|_{\Omega_0}(\eta), \omega\rangle \\ &= \langle \eta, \Delta_d(\omega)\rangle = \langle P(\eta), \Delta_d(\omega)\rangle \\ &= \langle \eta, P\Delta_d(\omega)\rangle\end{aligned}$$

for all $\eta \in \Omega_0$. We use that Δ_d is symmetric and Δ_d preserves Ω_0. So $P(\omega) \in \mathrm{dom}((\Delta_d|_{\Omega_0})^*)$ and $(\Delta_d|_{\Omega_0})^*(P(\omega)) = P\Delta_d(\omega)$. By hypothesis, $\Delta_d|_{\Omega_0}$ is essentially self-adjoint, so we have $(\Delta_d|_{\Omega_0})^* = \overline{\Delta_d|_{\Omega_0}}$. But Δ_d is closed, hence $\overline{\Delta_d|_{\Omega_0}} \subset \Delta_d$. Therefore we have the first statement. The last statement follows from the assumption that $\cap_k \mathrm{dom}(\Delta_d^k) = \Omega$. □

A further weakening condition can be given for Proposition 4.4.19 to hold. Namely, we determine when $\Delta_d|_{\Omega_0}$ is essentially self-adjoint. For this we consider the strongly continuous one-parameter unitary group $U(t) = e^{it\Delta_d}$.

LEMMA 4.4.20. *Assume that $\mathcal{D} := \{\omega \in \Omega_0 \mid \frac{(i\Delta_d)^n(\omega)}{n!} \to 0$ as $n \to \infty\}$ is dense in $L^2(\Omega_0)$. Then $U(t)$ takes $L^2(\Omega_0)$ into $L^2(\Omega_0)$.*

PROOF. Pick ω from the dense set $\mathcal{D}$ above. Observe that $U(t)\omega - \omega = \int_0^t \frac{d}{ds}(U(s)\omega)ds = \int_0^t iU(s)\Delta_d(\omega)ds$. So for $\eta \in L^2(\Omega_0)^\perp$,

$$\begin{aligned}\langle U(t)\omega, \eta\rangle &= \langle U(t)\omega - \omega, \eta\rangle \\ &= \langle \int_0^t iU(s)\Delta_d(\omega)ds, \eta\rangle \\ &= \int_0^t \langle iU(s)\Delta_d(\omega), \eta\rangle \\ &= \int_0^t \langle iU(s)\Delta_d(\omega) - i\Delta_d(\omega), \eta\rangle \text{ (since } \Delta_d \text{ takes } \Omega_0 \text{ into } \Omega_0) \\ &= \int_0^t \int_0^s \langle U(r)(i\Delta_d)^2(\omega), \eta\rangle drds \text{ (by repeating the steps above)} \\ &= \vdots \text{ (inductively)} \\ &= \int_0^{t_1} \cdots \int_0^{t_n} \langle U(t)(i\Delta_d)^n(\omega), \eta\rangle dt dt_1 \ldots dt_n \\ &= \int_\sigma \langle U(t)\frac{(i\Delta_d)^n(\omega)}{n!}, \eta\rangle,\end{aligned}$$

where σ is the standard simplex. Now the result follows from the density assumption on $\mathcal{D}$. □

Before we go onto the next proposition, we observe that $U(t)$ takes Ω_0 and hence $\mathcal{D}$ into Ω_0 because of the assumption that Ω consists of “smooth vectors” and Lemma 4.4.20. We follow Proposition 6.3 of [**Sch12**].

PROPOSITION 4.4.21. *Under the hypothesis of Lemma 4.4.20, the operator $\Delta_d|_{\Omega_0}$ is essentially self-adjoint.*

Proof. Suppose that $\tau \in \{1, -1\}$ and $\eta \in \ker((\Delta_d|_{\Omega_0})^* - \tau i I)$. Let $\omega \in D$. Lemma 4.4.20 and remarks made above imply that $U(t)\omega \in \Omega_0$. Now,

$$\frac{d}{dt}\langle U(t)\omega, \eta\rangle = \langle i\Delta_d U(t)\omega, \eta\rangle = \langle iU(t)\omega, \tau i\eta\rangle = \tau\langle U(t)\omega, \eta\rangle.$$

Thus the function $g(t) = \langle U(t)\omega, \eta\rangle$ is real analytic (because ω is smooth) and satisfies $g' = \tau g$. Hence $g(t) = g(0)e^{\tau t}$ and so $\langle \omega, U(-t)\eta\rangle = \langle \omega, e^{\tau t}\eta\rangle$. Since D is dense in $L^2(\Omega_0)$, we get that $U(-t)\eta = e^{\tau t}\eta$. So $t \to U(-t)\eta$ is differentiable at $t = 0$ and $\frac{d}{dt}|_{t=0}U(-t)\eta = \tau\eta = -i\Delta_d(\eta)$. Because Δ_d is self-adjoint, it follows that $\eta = 0$. $\square$

4.5. More examples of Hopf algebroids

So far we have focused on a single example, that of coming from étale groupoids. We have also mentioned Hopf algebras and weak Hopf algebras and built our framework using these as guiding examples. In this section we describe another example, namely, the Connes-Moscovici Hopf algebroid, which is over a noncommutative base, thus providing wider scope of our framework. Before we plunge into the Connes-Moscovici Hopf algebroid, we describe a special case, namely the following.

4.5.1. The enveloping Hopf algebroid of an algebra. Given an arbitrary $\mathbb{C}$-algebra A, let $H = A \otimes_{\mathbb{C}} A^{op}$. The left bialgebroid structure over A is given as

$$s_l(a) = a \otimes_{\mathbb{C}} 1, \qquad t_l(b) = 1 \otimes_{\mathbb{C}} b; \tag{4.5.1a}$$

$$\Delta_l(a \otimes b) = (a \otimes_{\mathbb{C}} 1) \otimes_A (1 \otimes_{\mathbb{C}} b), \qquad \varepsilon_l(a \otimes_{\mathbb{C}} b) = ab; \tag{4.5.1b}$$

and the right bialgebroid structure over A^{op} is given as

$$s_r(b) = 1 \otimes_{\mathbb{C}} b, \qquad t_r(a) = a \otimes_{\mathbb{C}} 1; \tag{4.5.2a}$$

$$\Delta_r(a \otimes_{\mathbb{C}} b) = (a \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} b), \qquad \varepsilon_r(a \otimes_{\mathbb{C}} b) = ba; \tag{4.5.2b}$$

for $a, b \in A$. Finally, the antipode

$$S(a \otimes_{\mathbb{C}} b) = b \otimes_{\mathbb{C}} a \tag{4.5.3}$$

makes H into a Hopf algebroid. If A is a $*$-algebra then H is Hopf $*$-algebroid. Then an H-covariant differential calculus on A is just a differential calculus on A, Definition 4.3.6 is satisfied with H_0 being $\mathbb{C}$! Covariant complex and further structures are then described as in Remark 4.3.25. So we get back the usual (non-covariant) structures. We now come to

4.5.2. The Connes-Moscovici Hopf algebroid. Let Q be a Hopf algebra over $\mathbb{C}$ with antipode T satisfying $T^2 = \mathrm{id}$ and A a Q-module algebra. Consider $H = A \otimes_{\mathbb{C}} Q \otimes_{\mathbb{C}} A$ with multiplication given by

$$(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b)(a' \otimes_{\mathbb{C}} q' \otimes_{\mathbb{C}} b') = a(q_{(1)}a') \otimes_{\mathbb{C}} q_{(2)}q' \otimes_{\mathbb{C}} (q_{(3)}b')b. \tag{4.5.4}$$

for $a, b, a', b' \in A$ and $q, q' \in Q$. A left bialgebroid structure over A, known as the Connes-Moscovici bialgebroid, is given as

$$s_l(a) = a \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} 1, \qquad t_l(b) = 1 \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} b; \tag{4.5.5a}$$

$$\Delta_l(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) = (a \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_A (1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} b); \tag{4.5.5b}$$

$$\varepsilon_l(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) = a\varepsilon(q)b; \tag{4.5.5c}$$

for $a, b \in A$ and $q \in Q$. ε is the counit of Q and we have used Sweedler notation for the coproduct of Q. This much is in the literature, see for example [**Böh09**]. We now put a right bialgebroid structure on H over A^{op} as

$$s_r(b) = 1 \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} b, \qquad t_r(a) = a \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} 1; \tag{4.5.6a}$$

$$\Delta_r(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) = (a \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} b); \tag{4.5.6b}$$

$$\varepsilon_r(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) = T(q)(ba); \tag{4.5.6c}$$

for $a, b \in A$ and $q \in Q$. Observe that, for the above structure maps,

$$a_1 \cdot (b \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b') \cdot a_2 = b(q_{(1)} \cdot a_1) \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} (q_{(3)} \cdot a_2)b',$$

for $a_1, a_2, b, b' \in A$ and $q \in Q$. From this, it at once follows that Δ_r and ε_r are bimodule morphisms. Coassociativity of Δ_r and counitarity of ε_r are easy to verify. We now check the Takeuchi condition. Given $a, b, c \in A$ and $q \in Q$, we have

$$\begin{aligned}
& s_r(a)(b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} c) \\
=& ((1 \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} a)(b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1)) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} c) \\
=& (b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} a) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} c) \\
=& ((b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1)(1 \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} T(q_{(2)})a)) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} q_{(3)} \otimes_{\mathbb{C}} c) \\
=& (b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} (1 \otimes_{\mathbb{C}} q_{(3)} \otimes_{\mathbb{C}} c)(T(q_{(2)})a \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} 1) \\
=& (b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} (q_{(3)}T(q_{(2)})a \otimes_{\mathbb{C}} q_{(4)} \otimes_{\mathbb{C}} c) \\
=& (b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} (a \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} c) \quad \text{(we use that } T^2 = \mathrm{id}) \\
=& (b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} ((a \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} 1)(1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} c)) \\
=& (b \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1) \otimes_{A^{op}} t_r(a)(1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} c),
\end{aligned}$$

thus proving the Takeuchi condition. The verification of the character property of ε_r is left to the reader. So this proves that we indeed have a right bialgebroid. Now we define the antipode S as

$$S(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) = T(q_{(3)})b \otimes_{\mathbb{C}} T(q_{(2)}) \otimes_{\mathbb{C}} T(q_{(1)})a. \tag{4.5.7}$$

Again, the antipode axioms are straightforward to check. As an example we show that $\mu(S \otimes \mathrm{id}_H)\Delta_l = s_r\varepsilon_r$ holds:

$$\begin{aligned}
& \mu(S \otimes \mathrm{id}_H)\Delta_l(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) \\
=& S(a \otimes_{\mathbb{C}} q_{(1)} \otimes_{\mathbb{C}} 1)(1 \otimes_{\mathbb{C}} q_{(2)} \otimes_{\mathbb{C}} b) \\
=& (T(q_{(3)})1 \otimes_{\mathbb{C}} T(q_{(2)}) \otimes_{\mathbb{C}} T(q_{(1)})a)(1 \otimes_{\mathbb{C}} q_{(4)} \otimes_{\mathbb{C}} b) \\
=& (1 \otimes_{\mathbb{C}} T(q_{(2)}) \otimes_{\mathbb{C}} T(q_{(1)})a)(1 \otimes_{\mathbb{C}} q_{(3)} \otimes_{\mathbb{C}} b) \\
=& T(q_{(4)})1 \otimes_{\mathbb{C}} T(q_{(3)})q_{(5)} \otimes_{\mathbb{C}} T(q_{(2)})bT(q_{(1)})a \\
=& 1 \otimes_{\mathbb{C}} T(q_{(2)})q_{(3)} \otimes_{\mathbb{C}} T(q_{(1)})(ba) \\
=& 1 \otimes_{\mathbb{C}} 1 \otimes_{\mathbb{C}} T(q)(ba) \\
=& s_r\varepsilon_r(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b).
\end{aligned}$$

Thus we have:

THEOREM 4.5.1. *With the structures described above, H becomes a Hopf algebroid, which we call the Connes-Moscovici Hopf algebroid. Furthermore, if Q is a Hopf $*$-algebra and A is a Q-module $*$-algebra then H becomes a Hopf $*$-algebroid in our sense.*

REMARK 4.5.2. *Observe that taking $Q = \mathbb{C}$ gives the enveloping Hopf algebroid back and $A = \mathbb{C}$ reduces H to a Hopf algebra. Thus it is a simultaneous generalization of the cases discussed above.*

REMARK 4.5.3. *We have used $T^2 = \mathrm{id}$ to make H into a Hopf algebroid. We think that it is possible to remove this condition by introducing a "modular pair in involution", that in turn produces a "twisted antipode" for Q, hence for H.*

We end this section by a proposition.

PROPOSITION 4.5.4. *Let (Ω, d) be a Q-covariant differential calculus on A. Then (Ω, d) can be made into an H-covariant differential calculus on A in the sense of Definition 4.3.6. Furthermore, if Q is a Hopf $*$-algebra, A is a Q-module $*$-algebra and (Ω, d) is a Q-covariant $*$-differential calculus then it can be made into an H-covariant $*$-differential calculus in the sense of Definition 4.3.7.*

PROOF. We define the H-action on Ω as follows:

$$(a \otimes_{\mathbb{C}} q \otimes_{\mathbb{C}} b) \cdot \omega = a(q \cdot \omega)b. \tag{4.5.8}$$

The only non-trivial part to check is that (4.3.1) holds. This is easy because H_0 contains $1 \otimes_{\mathbb{C}} Q \otimes_{\mathbb{C}} 1$. $\square$

4.6. Further directions and comments

We end this paper by discussing some directions that we have not touched upon.

4.6.1. Comparison with Connes' approach. In [**Con82**, **Con85**, **Con86**], the approach taken to study singular spaces, in particular, the leaf space of a foliation is as follows. One models the singular space by a groupoid G and then considers the convolution algebra $C_c^\infty(G)$ as the function algebra of the space in question. We have considered the groupoid here also, but as symmetries. To consider noncommutative complex geometry on the singular space, we need a differential calculus on the algebra $C_c^\infty(G)$. Here there are many choices and it is a priori not clear what is the correct choice to make. In fact, if one takes a discrete group and view it as a groupoid then the convolution algebra is the group algebra and we don't know what a choice of differential calculus would be (neither the universal one nor a bicovariant one), let alone the study of noncommutative complex structure and the meaning of it. So before moving onto arbitrary groupoids, one needs to answer the following question.

QUESTION 4.6.1. *Construct (or even classify) differential calculi on the group algebra $\mathbb{C}\Gamma$ of a discrete group Γ. Are there any complex structures on it? If so, what does it mean to have a complex structure on $\mathbb{C}\Gamma$?*

4.6.2. Comparison with Fröhlich et al.'s approach. In [**FGR97**], they study spectral data associated to hermitian, Kähler structure. [**B́17**] already mentions this and it is being taken up by him and collaborators [**B́DS19**]. We sketch this in our set up. Note that H is represented on $L^2(\Omega)$ by unbounded operators with common domain Ω. We first show that these operators are closable, by exhibiting densely defined adjoint operators. Taking ideas from [**KP11**], we exploit the (A_r, A_r)-bimodule structure on $\Omega \otimes_B \overline{\Omega}$ which is given by (4.2.27) via $\theta^{-1} : A_r \to A_l^{op}$; explicitly,

$$a_1 \cdot (\omega \otimes \overline{\eta}) \cdot a_2 = S(s_r(a_2)) \cdot \omega \otimes s_r(a_1) \cdot \overline{\eta}, \tag{4.6.1}$$

for $a_1, a_2 \in A_r$ and $\omega, \eta \in \Omega$. We assume that the faithful state τ used to define the inner product (4.3.12) is right invariant, i.e.,

$$\tau(h \cdot b) = \tau(\varepsilon_r(h) \cdot b), \tag{4.6.2}$$

for $h \in H$ and $b \in B$. We have the following lemma.

LEMMA 4.6.2. *For $\omega, \eta \in \Omega$ and $h \in H$,*

$$\tau g(\omega \otimes S(h) \cdot \overline{\eta}) = \tau g(h \cdot \omega \otimes \overline{\eta}) \tag{4.6.3}$$

holds, where g is as in Definition 4.3.36. Thus $\langle h \cdot \omega, \eta \rangle = \langle \omega, (S^2(h))^* \cdot \eta \rangle$.

PROOF. The proof is essentially contained in [**KP11**]. We compute

$$\begin{aligned}
\tau g(\omega \otimes S(h) \cdot \overline{\eta}) &= \tau g(\omega \otimes s_r\varepsilon_r(h^{(1)}) S(h^{(2)}) \overline{\eta}) \quad (4.2.13)\\
&= \tau(\varepsilon_r(h^{(1)}) \cdot g(\omega \otimes S(h^{(2)}) \cdot \overline{\eta})) \quad (4.3.11)\\
&= \tau(h^{(1)} \cdot g(\omega \otimes S(h^{(2)}) \cdot \overline{\eta})) \quad (4.6.2)\\
&= \tau g(h_{(1)} \cdot \omega \otimes h^{(1)}_{(2)} S(h^{(2)}_{(2)}) \cdot \overline{\eta}) \quad (4.2.7)\\
&= \tau g(h_{(1)} \cdot \omega \otimes \varepsilon_l(h_{(2)}) \cdot \overline{\eta})\\
&= \tau g(t_l \varepsilon_l(h_{(2)}) h_{(1)} \cdot \omega \otimes \overline{\eta})\\
&= \tau g(h \cdot \omega \otimes \overline{\eta}).
\end{aligned}$$

The last statement follows from the definition of H-action on $\overline{\Omega}$. □

Thus H is represented by closable operators having a common dense domain. We denote the adjoint of $h \in H$ by $h^\dagger$ so that $h^\dagger = (S^2(h))^*$ on Ω. From now on, let us allow a notational abuse of denoting by h both the operator on Ω and its closure in $L^2(\Omega)$. At this point, we make an additional regularity assumption (similar to assumption in Lemma 4.4.20):

Assumption. Given $h \in H$, $\mathcal{D}_h = \{\omega \in \Omega \mid \sum_0^\infty \frac{\|h^n\omega\|}{n!} < \infty\}$ is dense in $L^2(\Omega)$.

Lemma 4.6.3. *For $h \in H$ with $h = h^\dagger$ and $\omega \in \mathcal{D}_h$, define U_h by*

$$U_h(\omega) = \sum_n \frac{i^n}{n!} h^n \omega,$$

which is well-defined by the above **Assumption**. *Then U_h extends to a unitary operator on $L^2(\Omega)$ denoted by e^{ih}.*

Proof. The result follows from the observations that for such an h, $\mathcal{D}_h = \mathcal{D}_{-h}$ and that $U_h U_{-h} = U_{-h} U_h = \mathrm{id}$. □

Lemma 4.6.4. *If the commutator $[h, d + d^*]$ extends to a bounded operator on $L^2(\Omega)$, then so does $[e^{ih}, d + d^*]$.*

Proof. Observe that

$$\begin{aligned} e^{ih}(d + d^*) - (d + d^*)e^{ih} &= \int_0^1 \frac{d}{ds}(e^{ish}(d + d^*)e^{i(1-s)h})ds \\ &= \int_0^1 (ihe^{ish}(d + d^*)e^{i(1-s)h} - ie^{ish}(d + d^*)e^{i(1-s)h}h)ds \\ &= \int_0^1 i(e^{ish}h(d + d^*)e^{i(1-s)h} - e^{ish}(d + d^*)he^{i(1-s)h})ds \\ &= \int_0^1 i(e^{ish}[h, d + d^*]e^{i(1-s)h})ds. \end{aligned}$$

As e^{ith} is unitary, the integrand is bounded and the result follows. □

Combining Lemma 4.6.3 and Lemma 4.6.4, we get the following proposition.

Proposition 4.6.5. *Let $\mathcal{A}$ be the $*$-algebra generated by operators of the form $ae^{i(h+h^\dagger)}b$ with $a, b \in A_l$ and $h \in H_0$ in $B(L^2(\Omega))$. Then $(\mathcal{A}, L^2(\Omega), d + d^*)$ forms a spectral triple.*

Proof. We first observe that the representation of A_l on $L^2(\Omega)$ is induced by restricting through $A_l \to B$, $a \mapsto s_l(a) \cdot 1_B$. For $b \in B$ and $\omega \in \Omega$,

$$\langle b\omega, b\omega\rangle = \int_\tau b\omega \wedge \star(\omega^* b^*) = \tau(bg(\omega, \omega)b^*) \le \|\omega\|^2 \tau(bb^*),$$

implying that left multiplication by b extends to a bounded operator. In the above estimate, the inequality comes from the fact that $g(\omega, \omega)$ is positive in B.

Next, observing that $[s_l(a), d]$ is left multiplication by $d(s_l(a) \cdot 1_B)$, we prove that left multiplication by $d(s_l(a) \cdot 1_B)$ is bounded on $L^2(\Omega)$. Again we do this for $b \in B$. Note that

$$\begin{aligned} \langle db \wedge \omega, db \wedge \omega\rangle &= \langle db \wedge \omega, d(b\omega) - bd\omega\rangle \\ &= \langle db \wedge \omega, d(b\omega)\rangle - \langle db \wedge \omega, bd\omega\rangle. \end{aligned}$$

The first term in the above expression is estimated as follows:

$$|\langle db \wedge \omega, d(b\omega)\rangle| = |\langle d^*(db \wedge \omega), b\omega\rangle| \le \|d^*(db \wedge \omega)\|\|bw\| \le \text{ const. } \|\omega\|, \tag{4.6.4}$$

where we used Cauchy-Schwarz inequality and that left multiplication by b is bounded. The second term is estimated as follows:

$$|\langle db \wedge \omega, bd\omega\rangle| \le \|db \wedge \omega\|\|bdw\| \le \text{ const. } \|d\omega\|, \tag{4.6.5}$$

where we again used the boundedness of left multiplication by $b \in B$. Combining the two, we conclude that left multiplication by db is bounded on $L^2(\Omega)$. Now observe that $s_l(a) \cdot 1_B$ is adjointable (see the discussion after Lemma 4.6.2) with adjoint again an element of B (use Eq. (4.2.12) and compute: $(S^2(s_l(a)))^* = (Ss_r\varepsilon_r s_l(a))^* = (s_l\varepsilon_l s_r\varepsilon_r s_l(a))^* = s_l((\varepsilon_l s_r \varepsilon_r s_l(a))^*))$ and for such an element the adjoint of $[d,b]$ is precisely $-[d^*,b^\dagger]$, $b^\dagger$ is the adjoint of b. Since adjoint of $[d,b]$ is bounded, we can conclude that $[b,d+d^*]$ extends to a bounded operator on $L^2(\Omega)$.

Finally, $[s_l(a),d+d^*]$ extends to a bounded operator yields that $[h,d+d^*]$ too extends to a bounded operator for $h\in H_0$. The result follows from 4.6.4. □

If we assume that Δ_d has purely discrete spectrum then we get a spectral triple of compact type. We also note that [**B́17**] computes the spectrum for the concrete examples. In our abstract setup, we propose a way of doing it generally. We have already assumed an analogue (or rather a corollary) of classical Sobolev embedding (see the remarks before Theorem 4.4.18). It would be interesting to know the answer of the following:

QUESTION 4.6.6. *If we assume an analogue of Relich's lemma ($H^k\hookrightarrow H^{k+2}$ is compact in the notation of the Proof of Theorem 4.4.18) then does it follow that Δ_d has purely discrete spectrum? See* [**Hig06**] *for the setup and more on abstract pseudo-differential calculi which has motivated this question.*

This would give a uniform way of proving that the Laplacian Δ_d has purely discrete spectrum in the setting of noncommutative differential calculi.

4.6.3. Further examples. As examples for our framework, we have mentioned étale groupoids, Hopf algebras, weak Hopf algebras and the Connes-Moscovici Hopf algebroid. There is another class of examples coming from Lie-Rinehart algebras and associated jet spaces; see [**KP11**]. It would be interesting to know the answer of the following

QUESTION 4.6.7. *Investigate if these examples fit into our framework. If so, what is the meaning of having a complex structure on a Lie-Rinehart algebra?*

On this note, we mention a result from an ongoing work that produces a left bialgebroid that is not of the form dealt with in this paper, see Subsection 5.3.2. Let X be the finite set $\{1,\dots,n\}$.

PROPOSITION 4.6.8. *There is a left bialgebroid H over $C(X)$ such that the action on $C(X)$ lifts to an action on the space of universal one forms in the sense of Definition 4.3.3. Moreover, it is not of the form $C(X)\#Q$ for any Hopf algebra Q.*

Finally, we ask a question which is not directly related to this work but interesting in its own right. In [**GJ18**], it is shown that a coaction of a compact quantum group on an algebra can be lifted to a differential calculus (at least in the classical situation) under some suitable (unitarity of the coaction, technically, see also (4.3.10)) conditions, like one expects from a group action. So we ask

QUESTION 4.6.9. *Is the above true for unitary action (i.e.,* (4.3.10) *is satisfied) of Hopf algebroids?*

We have shown above that if we have the action on the full differential calculus, then, under some more conditions, the action becomes unitary. So we are seeking a converse of this.

CHAPTER 5

Geometry on finite spaces

5.1. Introduction

Finite spaces or even discrete ones are uninteresting from the perspective of classical geometry. They are thoroughly understood. Surprisingly in the noncommutative realm, even these simple spaces hold such mysteries. This can be seen from the work of Connes himself, see Chapter 6 of his book [**Con94**]. After Woronowicz introduced compact quantum groups, on Connes' suggestion, Wang studied [**Wan98**] "quantum symmetries", the dominant theme of this thesis, of finite spaces. Wang's work introduced a new array of examples of compact quantum groups. It turned out that if the space has more than three points, it has infinite dimensional quantum symmetry group. In other words, the quantum symmetry groups holds significantly more information than the classical permutation groups.

This discovery set the stage for further investigation of finite spaces, graphs, finite dimensional algebras, etc. There is this whole new enterprise of matrix geometry, which approximates higher dimensional manifolds and studies these approximating spaces by the tools of noncommutative methods, discovering information hidden hitherto.

This chapter consists of materials from one short article [**BG19b**] and one preprint [**BG**]. The article was the author's warm-up project and first foray into the noncommutative realm. It was motivated by the classical fact that on a Kähler manifold, the Chern connection and the Levi-Civita connection on the underlying Riemannian manifold are intimately related. Vaguely, the Chern connection is the complexification of the Levi-Civita connection. More precisely, recall [**Huy05**] that for a hermitian manifold X with complex structure I, one can identify the complex bundles $T^{1,0}X$ and (TX, I). Under this identification, any hermitian connection on $T^{1,0}X$ induces a metric connection on the Riemannian manifold X. Then the Chern connection and the Levi-Civita connection on the underlying Riemannian manifold are related, as described in the following

Proposition 5.1.1. *Let ∇ be a torsion free hermitian connection on the hermitian bundle $(T^{1,0}, g_{\mathbb{C}})$.*

i) Then ∇ is the Chern connection on the holomorphic bundle $\mathcal{T}_X$ endowed with the hermitian structure $g_{\mathbb{C}}$.
ii) The induced connection D on the underlying Riemannian manifold is the Levi-Civita connection.
iii) The hermitian manifold (X, g) is Kähler.

The formulation and proving the existence-uniqueness of Levi-Civita connection in noncommutative geometry is a challenging problem and there have been some progress in this direction [**BGM18**]. The main motivation for my first project was to define and study the Levi-Civita connection for a noncommutative Kähler manifold as the Chern connection which always exists [**BM17**]. The testing ground was to see if the scalar curvature matched with intuition in simple examples.

We classified, using brute-force computation, noncommutative complex structures. Unaware of the developments in [**B́17**], the author introduced a version of a noncommutative Kähler structure, which surprisingly coincided with that of [**B́**]. But it turned out that none of them are compatible with a Kähler metric in that sense. The first section of this chapter describes this work.

The other preprint is a continuation of the philosophy of the previous chapter, 4. Various aspects of the internal structure of Hopf algebroids were analyzed in a subsequent series of works, [**BB06**, **KP11**, **KR04**], to name a few. The "symmetry" aspect, as observed in [**BNS99**], was yet to be investigated. This was started in the beautiful paper [**Har15**]. As expounded in the last chapter, we made a tentative start at using Hopf algebroids as generalized symmetry objects in our preprint [**BBG**], a viewpoint that was hinted at [**KP11**].

We continue viewing Hopf algebroids as symmetry objects. We obtain a universal left bialgebroid that acts on a finite set, the analogue of Wang's quantum permutation group. We define faithfulness of such action and show that the left bialgebroid coming from a certain étale groupoid over the finite set is actually the universal one in our sense. Wang's quantum permutation group produces a natural example of a faithful left bialgebroid acting on the finite set and we show that this bialgebroid is strictly "smaller" than our universal one.

We end the chapter by lifting the action to the space of universal one forms on the finite set, thus realizing the promised Proposition 4.6.8 of the last chapter.

Although, Hopf algebroids are the ones that should be viewed as symmetry objects, and since we are only able to produce left bialgebroids, a few words are in order. In the Hopf algebra case, there is no distinction between a left bialgebra and a right bialgebra. Thus the bialgebra structure of the quantum permutation group is completely determined by the given action data. Moreover, the antipode is determined too. This is something that does not happen in the algebroid setting. As observed in [**BS04**], the antipode connects the two bialgebroid structures in a Hopf algebroid. Given two of the three structures of the Hopf algebroid- the left and right bialgebroid and the antipode, one can recover the third. But without knowing a priori that they constitute a Hopf algebroid, one cannot pass from the two to the third structure. This, we hope, justifies our content with a left bialgebroid for the moment and we sincerely hope that, inspired by our work, further investigations along these lines will be undertaken. The final section describes this work.

5.2. Classification of noncommutative complex structures on a three point space

Our humble aim in this short section is to classify all the almost complex structures on this non-commutative manifold consisting of just three points. Surprisingly they also turn out to be complex structures, but none of them are Kähler in our sense.

5.2.1. Complex structures. We consider a space made of three points $Y = \{1, 2, 3\}$. The algebra $\mathcal{C}(Y)$ of continuous functions is the direct sum $\mathcal{C}(Y) = \mathbb{C} \oplus \mathbb{C} \oplus \mathbb{C}$ and any element $f \in \mathcal{C}(Y)$ is a triplet of complex numbers (f_1, f_2, f_3), with $f_i = f(i)$ the value of f at the point i. The functions χ_i defined by $\chi_i(j) = \delta_{ij}, \quad i, j = 1, 2, 3$ form a $\mathbb{C}$-basis for $\mathcal{C}(Y)$.

Recall the Definition 3.1.2 of a noncommutative differential calculus from Subsection 3.1.3. We will now produce a differential $*$-calculus on the $*$-algebra $\mathcal{C}(Y)$ of continuous functions on the three-point space.

Recall that 1.4.1, given a spectral triple (A, H, D) one constructs a compatible differential calculus on A, called the space of Connes' forms by means of a suitable representation of the universal algebra $\Omega_u A$ in the algebra of bounded operators on H.

Let us now describe a spectral triple on the three-point space $Y = \{1, 2, 3\}$. This is a special case of a class of spectral triples considered in [**CI07**, **Rie99**] on compact metric spaces.

PROPOSITION 5.2.1. *Put $A = \mathcal{C}(Y)$ and $H = \mathbb{C}^2_{12} \oplus \mathbb{C}^2_{23} \oplus \mathbb{C}^2_{13}$ (the subscript ij says that the Hilbert space is along the "edge" connecting the point i with j). Define $\pi : A \to B(H)$ by*

$$\pi(f) = \begin{bmatrix} f(1) & 0 \\ 0 & f(2) \end{bmatrix}_{12} \oplus \begin{bmatrix} f(2) & 0 \\ 0 & f(3) \end{bmatrix}_{23} \oplus \begin{bmatrix} f(1) & 0 \\ 0 & f(3) \end{bmatrix}_{13}, \tag{5.2.1}$$

for $f \in A$. And finally, define the operator D as

$$D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{12} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{23} \oplus \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}_{13}. \tag{5.2.2}$$

Then (A, H, D), as constructed above, is a spectral triple on the three-point space.

PROOF. The conditions of 1.4.1 are satisfied since H is a finite dimensional Hilbert space (D is manifestly self-adjoint). □

REMARK 5.2.2. *Let us denote the differential graded algebra of Connes' forms over the algebra $A = \mathcal{C}(Y)$ simply by Ω.*

THEOREM 5.2.3. *Let (A, H, D) be the spectral triple on the three-point space described in 5.2.1. Then $e_i = [D, \chi_i], i = 1, 2$ is a free right basis for Ω^1. The bimodule structures are given by*

i) $\chi_i e_i = e_i(1 - \chi_i)$
ii) $\chi_i e_j = -e_i \chi_j, \qquad i \neq j$

and

i) $e_i \chi_i = (1 - \chi_i)e_i$
ii) $e_i \chi_j = -\chi_i e_j \qquad i \neq j$

PROOF. We note that, by definition, the space of 1-forms consists of bounded operators on H of the form $\sum_j a_0^j [D, a_1^j]$, where $a_i^j \in A$. We recall that $\chi_i, \quad i = 1, 2, 3$ form a $\mathbb{C}$ basis of A and satisfies $\chi_1 + \chi_2 + \chi_3 = 1$. Since $[D, 1] = 0$, we get the first conclusion. The bimodule structure follows from Leibniz rule and the observations $\chi_i^2 = \chi_i$, $\chi_i \chi_j = 0$. □

REMARK 5.2.4. *The basis described above is also a left basis for Ω^1.*

REMARK 5.2.5. *It can be shown by computation that there are no junk forms. Also the higher spaces of forms are finite dimensional vector spaces.*

We present an alternate but equivalent definition of a complex structure which appeared in [**BPS13**].

DEFINITION 5.2.6. *Let $(\Omega A, d, *)$ be a $*$-differential calculus on A. An almost complex structure on $(\Omega A, d, *)$ is a degree zero derivation $J : \Omega A \to \Omega A$ such that*

i) J *is identically* 0 *on* A *and hence an* (A, A)*-bimodule endomorphism of* ΩA;
ii) $J^2 = -1$ *on* $\Omega^1 A$; *and*
iii) $J(\xi^*) = J(\xi)^*$ *for* $\xi \in \Omega^1 A$ *(*J *preserves* $*$, *i.e.,* $\overline{J}* = *J$*).*

Because $J^2 = -1$ on $\Omega^1 A$, there is an (A, A)-bimodule decomposition

$$\Omega^1 A = \Omega^{1,0} A \oplus \Omega^{0,1} A \tag{5.2.3}$$

where

$$\Omega^{1,0} A = \{ \ \omega \in \Omega^1 A \mid J\omega = \iota\omega\} \tag{5.2.4}$$

and

$$\Omega^{0,1} A = \{ \ \omega \in \Omega^1 A \mid J\omega = -\iota\omega\}. \tag{5.2.5}$$

Condition iii) implies $(\Omega^{0,1} A)^* = \Omega^{1,0} A$. Recall that the map $* : \Omega^{1,0} A \to \overline{\Omega^{0,1} A}$ is an isomorphism of (A, A)-bimodules.

For all $p, q \geq 0$ we define

$$\Omega^{p,q} A := \{ \ \xi \in \Omega^{p+q} A \mid J\xi = (p - q)\iota\xi\}. \tag{5.2.6}$$

Elements in $\Omega^{p,q} A$ are called (p, q) forms. It is a theorem that

$$\Omega^n A = \bigoplus_{p+q=n} \Omega^{p,q} A. \tag{5.2.7}$$

REMARK 5.2.7. *We call an endomorphism $J : \Omega^1 A \to \Omega^1 A$ satisfying the above conditions a first order almost complex structure. In most of the examples studied so far, one defines the endomorphism on $\Omega^1 A$ and then extends it to whole of ΩA using the derivation property and some "basis" of higher forms. In our case we use the free basis for Ω^1 and vector space basis for higher forms.*

We have actually proved one side of the equivalence with Definition 3.2.1. Recall the operator ∂ and $\overline{\partial}$ from Equation (3.2.1)

The following is the analogue of Newlander-Nirenberg theorem [**Huy05**], equivalent to Lemma 3.2.2.

DEFINITION 5.2.8. *An almost complex structure J on $(\Omega A, d, *)$ is integrable if $d\Omega^{1,0}A \subset \Omega^{2,0}A \oplus \Omega^{1,1}A$.*

DEFINITION 5.2.9. *A complex structure on $(\Omega A, d, *)$ is an almost complex structure J which is integrable.*

Now we explicitly determine all the complex structures on the three-point space.

THEOREM 5.2.10. *Let (A, H, D) be the spectral triple on the three-point space as described in Proposition 5.2.1. Then there are 8 complex structures for the calculus as in Theorem 5.2.3, as enumerated below:*

i) $i\begin{bmatrix} 1-2\chi_3 & 0 \\ 2\chi_1 & 2\chi_2-1 \end{bmatrix}, \qquad -i\begin{bmatrix} 1-2\chi_3 & 0 \\ 2\chi_1 & 2\chi_2-1 \end{bmatrix};$

ii) $i\begin{bmatrix} 1-2\chi_3 & -2\chi_2 \\ 2\chi_1 & 2\chi_3-1 \end{bmatrix}, \qquad -i\begin{bmatrix} 1-2\chi_3 & -2\chi_2 \\ 2\chi_1 & 2\chi_3-1 \end{bmatrix};$

iii) $i\begin{bmatrix} 2\chi_1-1 & 2\chi_2 \\ 0 & 1-2\chi_3 \end{bmatrix}, \qquad -i\begin{bmatrix} 2\chi_1-1 & 2\chi_2 \\ 0 & 1-2\chi_3 \end{bmatrix};$

iv) $i\begin{bmatrix} 2\chi_1 & 0 \\ 0 & 1-2\chi_2 \end{bmatrix}, \qquad -i\begin{bmatrix} 2\chi_1 & 0 \\ 0 & 1-2\chi_2 \end{bmatrix}.$

PROOF. By Remark 5.2.7, we let J be a first order almost complex structure. Let $Je_i = e_1J_{1i} + e_2J_{2i}, \quad i = 1, 2$ and extend right linearly. Then J is a left module morphism reads as $J(\chi_ie_j) = \chi_iJ(e_j)$ (we use the bimodule rules to take the χ_i to the other side). In coordinates, $J^2 = -1$ is $\sum_{j,k} e_kJ_{kj}J_{ji} = -e_i$. Finally, J preserves $*$ reads as $\sum_j e_jJ_{ji} = \sum_j \overline{J_{ji}}e_j$.

Now comparing coefficients and solving for J_{ij} from the above equations, we get the first order almost complex structures. Surprisingly, there are no more first order almost complex structures than the listed ones. Next we extend these according to Remark 5.2.7 and note that the restriction of any almost complex structure to the space of one forms has to be one of these and since there is only one way to (derivation property) extend the first order ones, we get all the almost complex structures.

For the integrability condition, we check it individually case by case. For example, for $J = i\begin{bmatrix} 2\chi_1 & 0 \\ 0 & 1-2\chi_2 \end{bmatrix}$, $\Omega^{1,0}$ consists of elements of the form $\alpha(e_1\chi_1) + \beta(e_2\chi_1) + \gamma(e_2\chi_3), \quad \alpha, \beta, \gamma \in \mathbb{C}$.

We apply D to an element of that form, followed by J, and find that the result is 0, i.e., the element lies in $\Omega^{1,1}$. So this J is integrable. And similarly for the other J's, which concludes the proof. □

5.2.2. Kähler structures. Let us now go over to a possible non-commutative version of Kähler geometry. We begin with a basic ingredient in the classical theory, namely, that of a metric.

DEFINITION 5.2.11. *Let $(\Omega A, d, *)$ be a $*$-differential calculus on A. A metric g on A is a non-degenerate*[1] *bimodule morphism*

$$g : \Omega^1A \otimes_A \Omega A^1 \to A \tag{5.2.8}$$

and a hermitian metric is a non-degenerate bimodule morphism

$$h : \Omega^1A \otimes_A \overline{\Omega^1A} \to A. \tag{5.2.9}$$

REMARK 5.2.12. *Given a hermitian metric $h : \Omega^1A \otimes_A \overline{\Omega^1A} \to A$, we get a metric $g : \Omega^1A \otimes_A \Omega A^1 \to A$ by defining $g = h(\mathrm{id} \otimes *)$. We say h is induced by g.*

Assume that Ω^1A is finitely generated and projective as a left A-module. Let (e_i, e^i) be a dual basis, where $e_i \in \Omega^1A$ and $e^i \in (\Omega^1A)'$. Then the coevaluation

$$\mathrm{coev} : A \to \Omega^1A \otimes_A (\Omega^1A)', \quad 1 \mapsto e_i \otimes e^i \tag{5.2.10}$$

[1]Non-degeneracy means $g(-, \xi)$ induces a bijection $\Omega^1A \to (\Omega^1A)'$, the $()'$ denoting the left A-dual. The isomorphism automatically becomes an (A, A)-bimodule morphism as g is assumed to be so.

is an (A, A)-bimodule morphism. Then using the metric, we obtain $f_i \in \Omega^1 A$ such that $e^i = g(-, f_i)$, and hence $\omega = \sum_i g(\omega, f_i)e_i$ for all $\omega \in \Omega^1 A$. Following [**BM17**], we call the element $\sum_i e_i \otimes f_i \in \Omega^1 A \otimes_A \Omega^1 A$ the inverse of g, to be denoted by $\mathfrak{g}$. Since the coevaluation does not depend on a particular choice of a dual basis and the isomorphism $\Omega^1 A \cong (\Omega^1 A)'$ depends only on g, $\mathfrak{g}$ depends only on g.

Now along the classical lines we have,

DEFINITION 5.2.13. *Let $(\Omega A, d, *)$ be a $*$-differential calculus with almost complex structure J. We say that a metric $g : \Omega^1 A \otimes_A \Omega A^1 \to A$ (respectively, a hermitian metric $h : \Omega^1 A \otimes_A \overline{\Omega^1 A} \to A$) is compatible with J if $g(J \otimes J) = g$ (respectively, $h(J \otimes \overline{J}) = h$).*

REMARK 5.2.14. *If g is induced by h and h is compatible with J, then g is automatically compatible with J.*

REMARK 5.2.15. *If a metric g is compatible with J then g is identically 0 on $\Omega^{1,0} A \otimes_A \Omega^{1,0}$ and $\Omega^{0,1} A \otimes_A \Omega^{0,1} A$.*

The very algebraic definition given in [**Huy05**] has the following counterpart.

DEFINITION 5.2.16. *Let $(\Omega A, d, *)$ be a $*$-differential calculus with a complex structure J and g be a compatible metric on A. Assume $\Omega^1 A$ is projective as a left module. Let $\mathfrak{g} \in \Omega^1 A \otimes_A \Omega A^1$ be the corresponding inverse of g. Then the fundamental form ω is defined to be the form $\wedge(J \otimes \mathrm{id})(\mathrm{proj}\,|_{\Omega^{(1,0)}} \otimes \mathrm{proj}\,|_{\Omega^{(0,1)}})(\mathfrak{g}) \in \Omega^{1,1} A$. The metric is said to be Kähler if $d\omega = 0$.*

Unfortunately, for the three-point space we have the following

THEOREM 5.2.17. *For the spectral triple (A, H, D) on the three-point space as above in Proposition 5.2.1 and the calculus obtained in Theorem 5.2.3, there are no compatible Kähler metric for the complex structures enumerated in Theorem 5.2.10.*

PROOF. This is also proved by case by case analysis. We outline the overall scheme.

We first find J compatible metric g. Let $g(e_i, e_j) = g_{ij}$. Since $e_i, i = 1, 2$ is both a right and left basis for Ω^1 and g is a bimodule morphism, g is determined by g_{ij}'s. The non-degeneracy condition turns into the invertibility of the matrix $\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$ (since A is commutative, this is equivalent to the determinant $g_{11}g_{22} - g_{12}g_{21}$ being an unit).

Then the compatibility condition reads as $g_{ij} = g(\sum_k e_k J_{ki}, \sum_l e_l J_{lj})$. We take the J_{ki}'s to the right entry of g and then use the bimodule rule to get those out of g. Solving, we get J compatible metrics.

The inverse $\mathfrak{g}$ of g takes the form $\sum_i e_i \otimes (\sum_j e_j g^{ji})$, where $\begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix}$ is the inverse of $\begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}$.

According to Definition 5.2.16, we apply the projections and then J on the first tensorand (which is just multiplication by i) and multiply them, thus getting the fundamental form. The condition $d\omega = 0$ then contradicts the non-degeneracy condition of g as can be seen by direct computation. $\square$

REMARK 5.2.18. *In the paper [**B́**] a similar definition of the fundamental form was proposed. But in [**B́17**], it is defined more conceptually and many of the standard classical theorems are proved in that paper. It is of interest to know whether the three point space admits a Kähler structure in the set-up of the paper [**B́17**].*

We end this section by noting

REMARK 5.2.19. *None of the complex structures is S_3-covariant, S_3 being the "quantum symmetry" group of the three-point space.*

5.3. The search for universal generalized symmetry of a finite space

In this section we construct a universal left bialgebroid acting on a finite set. We view this as continuation of Wang's work in the realm of generalized symmetries.

5.3.1. Universal bialgebroid acting on a finite set. Let X be the finite set $\{1, \ldots, n\}$. Consider the pair groupoid $X \times X$ over X, which is clearly an étale groupoid. Thus the convolution algebra gives a Hopf algebroid over $C(X)$ (see 4.2.2 for more details). We shall show that this Hopf algebroid is the universal left bialgebroid acting on $C(X)$ in a suitable sense. More precisely we have the following theorem.

THEOREM 5.3.1. *Let H be a left bialgebroid over $C(X)$ and acting on $C(X)$. Then there is a unique left bialgebroid morphism $(\Phi : H \to C(X \times X), \phi : C(X) \to C(X))$ such that $\phi = \mathrm{id}$.*

PROOF. We denote the structure maps of H by $s, t, \ldots$, etc., and that of $C(X \times X)$ by $s_X, t_X, \ldots$, etc. Let χ_i be the function defined as $\chi_i(j) = \delta_{ij}$ for $j = 1, \ldots, n$. These form a basis of $C(X)$. Thus for $h \in H$, we have

$$h \cdot \chi_j = \sum_i h_{ij} \chi_i, \tag{5.3.1}$$

for some $h_{ij} \in \mathbb{C}$. We write $\underline{h}$ for the matrix (h_{ij}). Now define $\Phi : H \to C(X \times X)$ by

$$\Phi(h) = \underline{h}^t. \tag{5.3.2}$$

Recall that, for $\alpha, \beta \in C(X \times X)$ the product is defined as

$$\alpha * \beta(m, n) = \sum_p \alpha(p, n) \beta(m, p), \tag{5.3.3}$$

where $m, n, p \in X$. Thus

$$\begin{aligned}
\Phi(h_1 h_2)(m, n) &= (\underline{h_1 h_2})^t(m, n) \\
&= \underline{h_2}^t \underline{h_1}^t(m, n) \\
&= \sum_p \underline{h_2}^t(m, p) \underline{h_1}^t(p, n) \\
&= \sum_p \underline{h_1}^t(p, n) \underline{h_2}^t(m, p) \\
&= \sum_p \Phi(h_1)(p, n) \Phi(h_2)(m, p) \\
&= \Phi(h_1) * \Phi(h_2)(m, n),
\end{aligned} \tag{5.3.4}$$

which shows that Φ is a ring homomorphism. Next observe that for $f \in C(X)$ the action of $s(f)$ on χ_i is given by

$$s(f) \cdot \chi_i = f \chi_i = f(i) \chi_i. \tag{5.3.5}$$

Similarly,

$$t(f) \cdot \chi_i = \chi_i f = f(i) \chi_i. \tag{5.3.6}$$

This clearly yields

$$\Phi s = s_X, \qquad \Phi t = t_X. \tag{5.3.7}$$

Recall that the counit ε_X of $C(X \times X)$ is defined as

$$\varepsilon_X(\alpha)(n) = \sum_m \alpha(m, n). \tag{5.3.8}$$

Now observe that

$$\begin{aligned}
\sum_i \varepsilon(h)(i) \chi_i = s\varepsilon(h) \cdot \sum_i \chi_i = s\varepsilon(h) \cdot 1 \\
= h \cdot 1 = h \cdot \sum_j \chi_j = \sum_{i,j} h_{ij} \chi_i = \sum_i (\sum_j h_{ij}) \chi_i,
\end{aligned} \tag{5.3.9}$$

which implies that

$$\varepsilon(h)(i) = \sum_j h_{ij}. \tag{5.3.10}$$

Thus

$$\begin{aligned}\varepsilon_X\Phi(h)(n) = \varepsilon_X(\underline{h}^t)(n) &= \sum_m \underline{h}^t(m,n) \\ &= \sum_m h_{nm} = \varepsilon(h)(n),\end{aligned} \tag{5.3.11}$$

i.e., $\varepsilon_X\Phi = \varepsilon$. Again recall that the coproduct on $C(X \times X)$ is given by

$$\Delta_X(\alpha)(m_1, m_2, n) = \begin{cases}\alpha(m,n), & \text{if } m_1 = m_2 = m \\ 0, & \text{else.}\end{cases} \tag{5.3.12}$$

On one hand, we have

$$\begin{aligned}h \cdot (\chi_i\chi_j) &= (h_1 \cdot \chi_i)(h_2 \cdot \chi_j) \\ &= (\sum_m (h_1)_{mi}\chi_m)(\sum_n (h_2)_{nj}\chi_n) \\ &= \sum_{m,n} (h_1)_{mi}(h_2)_{nj}\chi_m\chi_n \\ &= \sum_m (h_1)_{mi}(h_2)_{mj}\chi_m.\end{aligned} \tag{5.3.13}$$

On the other,

$$h \cdot (\chi_i\chi_j) = h \cdot (\delta_{ij}\chi_i) = \delta_{ij}\sum_m h_{mi}\chi_m. \tag{5.3.14}$$

Hence, we have

$$(h_1)_{mi}(h_2)_{mj} = \delta_{ij}h_{mi}. \tag{5.3.15}$$

Putting these together,

$$\begin{aligned}(\Phi(h_1) \otimes \Phi(h_2))(m_1, m_2, n) &= \Phi(h_1)(m_1, n)\Phi(h_2)(m_2, n) \\ &= \underline{h_1}^t(m_1, n)\underline{h_2}^t(m_2, n) \\ &= (h_1)_{n,m_1}(h_2)_{n,m_2} \\ &= \delta_{m_1m_2}h_{nm_1} \\ &= \delta_{m_1m_2}\underline{h}^t(m_1, n) \\ &= \Delta_X(\Phi(h))(m_1, m_2, n).\end{aligned} \tag{5.3.16}$$

which implies $\Delta_X\Phi = (\Phi \otimes \Phi)\Delta$. Finally, for the uniqueness, let (Φ', id) be another left bialgebroid morphism. Then

$$\begin{aligned}\sum_i h_{ij}\chi_i = h \cdot \chi_j &= \varepsilon(hs(\chi_j)) \\ &= \varepsilon_X\Phi'(hs(\chi_j)) \\ &= \varepsilon_X(\Phi'(h) * \Phi's(\chi_j)) \\ &= \varepsilon_X(\Phi'(h) * s_X(\chi_j)) \\ &= \sum_i \Phi'(h)(j,i)\chi_i,\end{aligned} \tag{5.3.17}$$

the last equality follows from

$$\begin{aligned} \Phi'(h) \cdot \chi_j(n) &= \varepsilon_X(\Phi'(h) * s_X(\chi_j))(n) \\ &= \sum_m \Phi'(h) * s_X(\chi_j)(m,n) \\ &= \sum_{m,p} \Phi'(h)(p,n) s_X(\chi_j)(m,p) \\ &= \sum_{m,p} \Phi'(h)(p,n) \delta_{mp}\delta_{jm} \\ &= \Phi'(h)(j,n). \end{aligned} \tag{5.3.18}$$

This yields that $h_{ij} = \Phi'(h)(j,i)$ implying $\Phi'(h) = \Phi(h)$ for all h. Hence we get $\Phi' = \Phi$ concluding the proof. □

Let H be a Hopf algebra such that $C(X)$ is an H-module algebra. Then the smashed product $C(X)\#H$ is a left bialgebroid in a canonical way, see Theorem 4.5.1. We conjecture that $C(X \times X)$ is not isomorphic to any of the left bialgebroids coming from this smashed product construction.

Conjecture 5.3.2. *The morphism $\Phi : C(X)\#H \to C(X \times X)$ is not an isomorphism for any Hopf algebra H making $C(X)$ a module-algebra.*

Remark 5.3.3. *We are grateful to the referee for pointing out an error in the "proof" of the conjecture above, which was previously a proposition. However, we still believe that it can be proven.*

5.3.2. Left bialgebroid covariance of universal 1-forms. In this subsection, we find a left bialgebroid over $C(X)$ whose action on $C(X)$ lifts to the space of universal one forms on $C(X)$, in the sense of Definition 4.3.3. For that, let us consider the left bialgebroid $C(X \times X \times X \times X)$ over $C(X \times X)$, constructed in the same way as above. We identify $C(X)$ as the first copy of the product $C(X) \otimes C(X) = C(X \times X)$. Let $f \in C(X)$ and $h \in C(X \times X \times X \times X)$. Then

$$\begin{aligned} h \cdot (f \otimes 1)(x,y) &= \sum_{z,w} h * s(f \otimes 1)(z,w,x,y) \\ &= \sum_{z,w,\alpha,\beta} h(\alpha,\beta,x,y) s(f \otimes 1)(z,w,\alpha,\beta) \\ &= \sum_{z,w,\alpha,\beta} h(\alpha,\beta,x,y) \delta_{z,\alpha}\delta_{w,\beta}(f \otimes 1)(z,w) \\ &= \sum_{z,w} h(z,w,x,y) f(z). \end{aligned} \tag{5.3.19}$$

So a sufficient condition that h takes $f \otimes 1$ to an element of the same form is that $\sum_w h(z,w,x,y)$ does not depend on y. Therefore we can write

$$\sum_w h(z,w,x,y) = \sum_w h(z,w,x,x). \tag{5.3.20}$$

Now the space of universal one forms can be identified with functions on $X \times X$ vanishing on the diagonal. If $\sum_i f_i \otimes g_i$ is such an element then

$$h \cdot (\sum_i f_i \otimes g_i)(x,y) = \sum_{i,z,w} h(z,w,x,y) f_i(z) g_i(w). \tag{5.3.21}$$

Thus a sufficient condition that h preserves this space is that

$$h(z,w,x,x) = 0 \tag{5.3.22}$$

for $z \neq w$. Observe that, (5.3.22) together with (5.3.20) imply

$$\sum_w h(z,w,x,y) = h(z,z,x,x). \tag{5.3.23}$$

Next, we find sufficient conditions on h such that $[h - s\varepsilon(h), d] = 0$ holds.

$$
\begin{aligned}
[h - s\varepsilon(h), d](f)(x,y) &= (h - s\varepsilon(h)) \cdot (f \otimes 1 - 1 \otimes f)(x,y) - (h - s\varepsilon(h)) \cdot f(x) \\
&+ (h - s\varepsilon(h)) \cdot f(y) \\
&= \sum_{z,w} (h - s\varepsilon(h))(z,w,x,y)(f(z) - f(w)) \\
&- \sum_{z,w} (h - s\varepsilon(h))(z,w,x,x) f(z) \\
&+ \sum_{z,w} (h - s\varepsilon(h))(z,w,y,y) f(z) \\
&= \sum_{z,w} \{h(z,w,x,y) - \delta_{z,x}\delta_{w,y} \sum_{\alpha,\beta} h(\alpha,\beta,x,y)\}(f(z) - f(w)) \\
&- \sum_{z,w} \{h(z,w,x,x) - \delta_{z,x}\delta_{w,x} \sum_{\alpha,\beta} h(\alpha,\beta,x,x)\} f(z) \\
&+ \sum_{z,w} \{h(z,w,y,y) - \delta_{z,y}\delta_{w,y} \sum_{\alpha,\beta} h(\alpha,\beta,y,y)\} f(z) \\
&= \sum_{z,w} h(z,w,x,y)(f(z) - f(w)) - \sum_{\alpha,\beta} h(\alpha,\beta,x,y)(f(x) - f(y)) \\
&- \sum_{z,w} h(z,w,x,x) f(z) + \sum_{\alpha,\beta} h(\alpha,\beta,x,x) f(x) \\
&+ \sum_{z,w} h(z,w,y,y) f(z) - \sum_{\alpha,\beta} h(\alpha,\beta,y,y) f(y) \\
&= \sum_{z,w} h(z,w,x,y)(f(z) - f(w)) - \sum_{\alpha,\beta} h(\alpha,\beta,x,y)(f(x) - f(y)) \\
&- \sum_{z} (\sum_{w} h(z,w,x,x)) f(z) + \sum_{\alpha} (\sum_{\beta} h(\alpha,\beta,x,x)) f(x) \\
&+ \sum_{z} (\sum_{w} h(z,w,y,y)) f(z) - \sum_{\alpha} (\sum_{\beta} h(\alpha,\beta,y,y)) f(y) \\
&= \sum_{z,w} h(z,w,x,y)(f(z) - f(w)) - \sum_{\alpha,\beta} h(\alpha,\beta,x,y)(f(x) - f(y)) \\
&- \sum_{z} h(z,z,x,x) f(z) + \sum_{\alpha} h(\alpha,\alpha,x,x) f(x) \\
&+ \sum_{z} h(z,z,y,y) f(z) - \sum_{\alpha} h(\alpha,\alpha,y,y) f(y).
\end{aligned}
$$

Now we put $f = \chi_a$ in the above expression and obtain

$$
\begin{aligned}
&\sum_{z,w} h(z,w,x,y)(\delta_{z,a} - \delta_{w,a}) - \sum_{\alpha,\beta} h(\alpha,\beta,x,y)(\delta_{x,a} - \delta_{y,a}) \\
&- \sum_{z} h(z,z,x,x)\delta_{z,a} + \sum_{\alpha} h(\alpha,\alpha,x,x)\delta_{x,a} \\
&+ \sum_{z} h(z,z,y,y)\delta_{z,a} - \sum_{\alpha} h(\alpha,\alpha,y,y)\delta_{y,a},
\end{aligned}
$$

which equals

$$\begin{aligned}
&\sum_w h(a,w,x,y) - \sum_z h(z,a,x,y) - \sum_{\alpha,\beta} h(\alpha,\beta,a,a)(\delta_{x,a}-\delta_{y,a})\\
&- h(a,a,x,x) + \sum_\alpha h(\alpha,\alpha,a,a)\delta_{x,a}\\
&+ h(a,a,y,y) - \sum_\alpha h(\alpha,\alpha,a,a)\delta_{y,a}\\
&= h(a,a,x,x) - \sum_z h(z,a,x,y) - \sum_\alpha h(\alpha,\alpha,a,a)(\delta_{x,a}-\delta_{y,a})\\
&- h(a,a,x,x) + h(a,a,y,y) + \sum_\alpha h(\alpha,\alpha,a,a)(\delta_{x,a}-\delta_{y,a})\\
&= -\sum_z h(z,a,x,y) + h(a,a,y,y).
\end{aligned}$$

So the condition

$$\sum_z h(z,w,x,y) = h(w,w,y,y) \tag{5.3.24}$$

gives that $[h - s\varepsilon(h), d] = 0$. Let H_0 be the set of all $h \in C(X\times X\times X\times X)$ such that

i) $\sum_w h(z,w,x,y) = h(z,z,x,x)$;

ii) $h(z,w,x,x) = 0$ for $z \neq w$;

iii) $\sum_z h(z,w,x,y) = h(w,w,y,y)$;

And let H be the smallest subalgebra of $C(X\times X\times X\times X)$ containing $C(X)$ and H_0.

Proposition 5.3.4. *H is a left bialgebroid over $C(X)$ such that the action on $C(X)$ lifts to an action on the space of universal one forms in the sense of Definition 4.3.3.*

Proof. We define

$$s_H : C(X) \to H, \quad f \mapsto s(f\otimes 1); \tag{5.3.25}$$

with t_H same as s_H. The counit of $C(X\times X\times X\times X)$ restricted to H actually takes value into $C(X)\otimes 1$. For this first observe that, for $h \in H_0$ and $f \in C(X)$

$$\begin{aligned}
\varepsilon(h * s_H(f))(x,y) &= \sum_{z,w} h(z,w,x,y)f(z)\\
&= \sum_z h(z,z,x,x)f(z),
\end{aligned} \tag{5.3.26}$$

the last expression being a function of x only. Now the character property shows that ε takes values inside $C(X)\otimes 1$.

We define the coproduct viewing elements of $\Delta(H)$ as functions of "seven variables":

$$\Delta h(z,w,x,y,z',w',x,y') = \delta_{z,z'}\delta_{w,w'}\delta_{y,y'}h(z,w,x,y). \tag{5.3.27}$$

Coassociativity is tedious but straightforward. Takeuchi condition holds which is easy to see this way: if $y \neq y'$ then both sides of the condition vanish. If $y = y'$ then the coproduct is nothing but the coproduct of $C(X\times X\times X\times X)$ which already satisfies the condition. The proof of counitarity is same as in the case of $C(X\times X\times X\times X)$. Thus H is left bialgebroid over $C(X)$.

Now the first two conditions in the definition of H_0 shows that H acts on $\Omega^1(C(X))$. That this action satisfies the conditions in Definition 4.3.3 follows from the third condition and the definition of H. $\square$

Conjecture 5.3.5. *H is not of the form $C(X)\#Q$ for any Hopf algebra Q acting on $C(X)$.*

Remark 5.3.6. *We remark that the space H_0 is at least n^2-dimensional and hence H is at least n^3-dimensional, sufficiently large for our purposes.*

We end with a

Question 5.3.7. *Give a nice characterization/description of H.*

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