Quantum Markov Maps: Structure and Asymptotics

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Indian Statistical Institute
2020

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Thesis submitted to the Indian Statistical Institute

in partial fulfilment of the requirements

for the award of the degree of

Doctor of Philosophy

in **Mathematics**

by

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Indian Statistical Institute

2020

Dedicated

to

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Acknowledgements

Firstly, I would like to thank Professor B. V. Rajarama Bhat, my supervisor, under whose guidance I learnt immensely. He never shied away from explaining to me even elementary concepts that I had issues with all these years. Altogether, he was very approachable and has provided me tremendous support in both technical and administrative issues.

I would also like to thank Professor Sushama Agrawal, Ramanujan Institute, Chennai, who inspired me to take up Ph.D. She supported me constantly, from the time I joined my masters degree. I wish to thank Professor Robin Hillier, Lancaster University and Dr. Nirupama Mallick, Chennai Mathematical Institute for all the discussions and fruitful collaboration, parts of which have gone into this thesis. Also, I have benefited from the discussions I had with Professor V. S. Sunder, Institute of Mathematical Sciences, Chennai and Professor G. Ramesh, IIT Hyderabad prior to joining ISI. I thank them for their support.

I would also like to thank all my friends at ISI with whom I had discussions both mathematical and otherwise. My family has been one of the most important sources of support. They have helped me cope with stress in the course of my PhD. I thank them for their role in the completion of my thesis.

Last but not least, I would like to thank the National Board for Higher Mathematics and the Bangalore Centre of Indian Statistical Institute for their financial support during my PhD.

Bangalore
23 January, 2020.

Vijaya Kumar U

Notations

$\mathbb{N}$	$\{1, 2, 3, \ldots\}$
$\mathbb{Z}_+$	$\{0, 1, 2, \ldots\}$
$\mathbb{R}_+$	Set of all non-negative real numbers
$\mathbb{T}$	$\mathbb{R}_+$ or $\mathbb{Z}_+$
M_n	Set of all $n \times n$ complex matrices
$\mathcal{B}(\mathcal{H})$	Set of all bounded linear operators on the Hilbert space $\mathcal{H}$
E_{ij}	Matrix units in M_n
$\mathbb{E}_{ij}$	$\mathbf{1} \otimes E_{ij}$ in $\mathcal{A} \otimes \mathrm{M}_n$
$\mathcal{B}^r(E, F)$	Set of all right linear maps from E to F
$\mathcal{B}^r(E)$	$\mathcal{B}^r(E, E)$
$\mathcal{B}^a(E, F)$	Set of all bounded adjointable maps from E to F
$\mathcal{B}^a(E)$	$\mathcal{B}^a(E, E)$
$\mathcal{B}^{a,bil}(E, F)$	Set of all bounded adjointable bilinear maps from E to F
$\mathcal{B}^{a,bil}(E)$	$\mathcal{B}^{a,bil}(E, E)$
$\overline{\mathcal{S}}^s$	SOT closure of $\mathcal{S}$
$E \odot F$	Interior tensor products of the Hilbert C^*-modules E and F
$E \bar{\odot}^s F$	The strong closure of $E \odot F$
$\mathcal{F}(E)$	The full Fock module over E
$\mathbb{I}\Gamma(E)$	The time ordered Fock module over E

Abbreviations

CP	Completely Positive
CB	Completely Bounded
CCP	Conditionally Completely Positive
UCP	Unital Completely Positive
UNCP	Unital Normal Completely Positive
QDS	Quantum Dynamical Semigroup(s)
QMS	Quantum Markov Semigroup(s)

Contents

CHAPTER 1

Introduction

In classical probability Markov processes are random processes where the future is dependent on the present but not on the past. The stochastic dependence of the future on the present is described through transition probabilities. Random walks, Brownian motion and so on are examples of such processes. Countable state Markov processes are known as Markov chains, and their transition probabilities are described through stochastic matrices. One may have the processes in discrete time (where the time is usually parametrized by $\mathbb{Z}_+$) or continuous time (with parametrization using $\mathbb{R}_+$), and accordingly one has discrete or continuous semigroups of stochastic matrices. These semigroups have non-commutative or quantum analogues known as *quantum dynamical semigroups* (QDS) (See Definition 2.3.5). In the non-commutative or quantum analogues, the role of transition probabilities (or stochastic matrices) is played by *completely positive* (CP) maps (See Definition 2.1.16) on C^*-algebras (cf. [EK98, Stø13]). So CP maps appear naturally in quantum probability (unital CP maps are known as *quantum Markov maps* in quantum probability). Trace preserving, unital CP maps are known as *quantum channels* in quantum information theory. In this thesis we study the following two problems about Quantum Markov Maps.

`Problem 1:` Quantum channels in quantum information theory, describe how quantum states get changed or transformed in open systems. In this context, it is important to know whether a given completely positive map admits square roots or higher order roots within the category of CP maps. Since completely positive maps are closed under composition, it makes sense to study the question of roots in this setting, namely: given a C*-algebra or von Neumann algebra $\mathcal{A}$, a number $n \in \mathbb{N}$, and a completely positive map $\phi : \mathcal{A} \to \mathcal{A}$, is there another completely positive map $\psi : \mathcal{A} \to \mathcal{A}$ such that $\phi = \psi^n$? One may

[0] 1991 *Mathematics Subject Classification.* primary: 46L57; secondary: 60J10, 81P45, 46L08, 81S22. *Key words and phrases:* completely positive maps, Markov chains, matrix algebras, operator algebras, product systems, Hilbert C^*-modules, quantum dynamical semigroups, dilation theory.

go further and ask whether the given CP map embeds in a one parameter semigroup of completely positive maps, that is, whether we can find a continuous time quantum dynamical semigroup $\tau = \{\tau_t : t \in \mathbb{R}_+\}$ such that $\phi = \tau_{t_0}$ for some t_0. We are also interested in knowing as to when does a CP map appear as a limit of a quantum dynamical semigroup and if so in how many different ways. This requires studying asymptotics of these semigroups. Surprisingly some quantum dynamical semigroups may reach an equilibrium state in finite time. Such phenomenon seems to be rare in classical Markov processes. However, this has been observed in [Bha12] by Bhat in the quantum case for a whole class of semigroups and it would be good to understand this phenomenon in a more general setup.

We introduce the concept of completely positive roots of completely positive maps on operator algebras. We do this in different forms: as asymptotic roots, proper discrete roots and as continuous one-parameter semigroups of roots. We present several general existence and non-existence results, some special examples in settings where we understand the situation better, and several open problems. Our study is closely related to Elfving's embedding problem in classical probability, which is about characterizing stochastic matrices which can be embedded in one parameter semigroups of stochastic matrices (See [Dav10, VB18, Kin62, G37]) and the divisibility problem of quantum channels, which is essentially about factorizing CP maps (See [Wol11, BC16, WC08]).

`Problem 2:` Semigroups of unital CP maps are known as *quantum Markov semigroups* (QMS) and semigroups of unital endomorphisms are known as E_0-*semigroups* in quantum probability. While studying units of E_0-semigroups of $\mathcal{B}(\mathcal{H})$ Powers was led into considering block CP semigroups (CP semigroups of block-wise acting maps) (See [Pow03] and [BLS08], [Ske10]). In [BM10], Bhat and Mukherjee proved a structure theorem for block QMS on $\mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. The main point is that when we have a block QMS, there is a contractive morphism between inclusion systems (synonymous with subproduct system) of diagonal CP semigroups. Moreover, this morphism lifts to associated product systems. Our main goal is to explore the structure of block quantum dynamical semigroups on general von Neumann algebras, using the technology of Hilbert C^*-modules.

W. Paschke's version (See [Pas73]) of Stinespring's theorem (See [Sti55]) associates a Hilbert C^*-module along with a generating vector to every completely positive map. Building on this, to every QDS on a C^*-algebra $\mathcal{B}$ one may associate an inclusion system $E = (E_t)$ of Hilbert $\mathcal{B}$-$\mathcal{B}$-modules with a generating unit $\xi = (\xi_t)$. The extension of the theory of block CP maps in [BM10] to the general case, is not straight forward for the following reason. In the case of $\mathcal{B}(\mathcal{H})$, we need only to consider product systems of Hilbert spaces, whereas now we need to deal with both product systems of Hilbert $\mathcal{B}$-modules and also product systems of Hilbert-$M_2(\mathcal{B})$ modules (See Theorem 4.2.1) and their interdependences. But a careful analysis of these modules does lead us to a morphism between inclusion systems as in the $\mathcal{B}(\mathcal{H})$ case and this morphism can also be lifted to a morphism at the level of associated product systems. At various steps we consider adjoints of maps between our modules and so it is convenient to have von Neumann modules. The picture is unclear for Hilbert C^*-modules.

This thesis contains four chapters including this chapter. The second chapter contains the preliminaries required for the next two core chapters. The third and fourth chapters are based on the two preprints mentioned in the Publications/Preprints. In the following we give brief details about the chapters:

`Chapter 2`: First we present the definitions and results about quantum Markov maps in Section 2.1. We give an introduction to the theory of Hilbert C^*-modules in Section 2.2, where we also explain the GNS-construction by Paschke [Pas73]. The GNS-construction is in a sense an extension of Stinespring dilation theorem (cf. Observation 2.2.4). The GNS-construction is more useful as the GNS-construction of the composition of two CP maps can be written as a submodule of the tensor product of their individual GNS-constructions (cf. Observation 2.2.5). Finally we give a brief introduction to the quantum dynamical semigroups in Section 2.3, where we also show the connection between QDS and product systems of Hilbert C^*-modules or von Neumann modules, and we briefly recall the construction of E_0-dilation through Hilbert C^*-modules, by Bhat and Skeide in [BS00], of a conservative QDS on a unital C^*-algebra or von Neumann algebra. At the end of this section we give a brief introduction to the time ordered Fock module.

`Chapter 3`: We give a complete characterization for the asymptotic roots in Theorem 3.2.1. As a byproduct, this theorem answers Problem 3 in [Arv03, p.387] affirmatively. We provide several existence and non-existence results under different additional assumptions, e.g. regarding the dimension or structure of the algebra or the range of the CP map. In particular, for the case of states on M_d or $\mathcal{B}(\mathcal{H})$ or $\mathbb{C}^d$ we have a complete characterization of existence of n-th roots (See Theorems 3.3.1, 3.3.2 and 3.3.3). We give few examples to indicate that a "complete and elegant" characterization of existence or non-existence of roots is expected to be complicated (See Examples 3.3.1, 3.3.2, 3.3.3 and 3.3.4). Using [Den88, Cor.4] we prove Proposition 3.4.1, which gives a connection between proper discrete roots and proper continuous roots in the finite dimensional case. Using the ideas used in [Bha12] we prove Theorem 3.4.1. This contains results on existence and non-existence of proper continuous roots of states on $\mathcal{B}(\mathcal{H})$.

`Chapter 4:` Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ be a von Neumann algebra. Suppose $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix} : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ is a block-wise acting CP map. In Theorem 4.2.1 we prove that ψ is determined by the diagonals ϕ_1 and ϕ_2 up to a adjointable bilinear contraction $T : E_2 \to E_1$, where E_i is a GNS-representation for $\phi_i, i = 1, 2$. Using this we prove a structure theorem for a block QDS on $M_2(\mathcal{B})$ in Theorem 4.3.1. This says that given a block QDS on $M_2(\mathcal{B})$ there is a contractive morphism between the inclusion systems associated to diagonal CP semigroups, determining the off-diagonal maps. Example 4.2.1, indicates that we can not replace the von Neumann algebra $\mathcal{B}$ in these theorems by an arbitrary C^*-algebra. In Theorem 4.4.1, we prove that if $\mathcal{B}$ is a von Neumann algebra, then any morphism between inclusion systems of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules can be lifted to a morphism between the product systems generated by these inclusion systems. In Theorem 4.3.2, we notice that the E_0-dilation of a block quantum Markov semigroup constructed in [BS00] by Bhat and Skeide is again a semigroup of block maps.

`Conventions:` Throughout this thesis, all Hilbert spaces are taken as complex and separable, with scalar products linear in the second variable. All C^*-algebras are complex vector spaces.

* * * * *

CHAPTER 2

Preliminaries

2.1 Introduction to quantum Markov maps

Quantum Markov maps are non-commutative analogues of transition probability matrices (or stochastic matrices) of Markov chains in classical probability. In the following subsections, we present the basic notions of this theory. We begin with recalling the concept of stochastic matrices.

2.1.1 Stochastic Matrices

Definition 2.1.1. Let $\{X_n : n \in \mathbb{N}\}$ be a set of random variables taking values in a countable set S, defined on a common probability space. The set of random variables is said to be a *Markov chain* if the following holds:

$$P(X_{n+1} = x_{n+1}|X_0 = x_0, X_1 = x_1, \cdots, X_n = x_n) = P(X_{n+1} = x_{n+1}|X_n = x_n) \tag{2.1.1}$$

for $x_j \in S, 1 \leq j \leq n+1$. The set S is known as the state space of the Markov chain.

A Markov chain can be interpreted as a set of random processes observed in discrete time intervals such that the outcome of the future depends only on the present.

Example 2.1.1. Suppose an urn initially consists of 3 red and 2 blue identical balls. At each time epoch a ball is picked at random and replaced with a ball of the other color. Let s_i denote the state that the urn contains i red balls and $(5-i)$ blue balls for $0 \leq i \leq 5$. Then the state space is given by

$$S = \{s_i : \ 0 \leq i \leq 5\}.$$

It is easy to see that (2.1.1) holds as the change in state is dependent on chances of picking either a blue ball or a red ball. This in turn purely depends on the current configuration of the urn. Also, for two states s and t, $P(X_{n+1} = s|X_n = t)$ is independent of n. This time invariance property is referred to as time homogeneity of the Markov chain.

In this example, we can thus represent the Markov chain with a finite matrix P whose $(i,j)^{th}$ element is given by $P(X_{n+1} = s_j|X_n = s_i)$ for each $0 \leq i,j \leq 5$. Some elementary computations yield us the transition probability matrix P of the Markov chain given by:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{1}{5} & 0 & \frac{4}{5} & 0 & 0 & 0 \\ 0 & \frac{2}{5} & 0 & \frac{3}{5} & 0 & 0 \\ 0 & 0 & \frac{3}{5} & 0 & \frac{2}{5} & 0 \\ 0 & 0 & 0 & \frac{4}{5} & 0 & \frac{1}{5} \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Observe that for any $n \in \mathbb{N}$, P^n is a matrix whose $(i,j)^{th}$ entry denotes the probability that the Markov chain transitions from state s_i to state s_j in exactly n steps. It is easy to see that the classical semigroup property

$$P^{n+m} = P^n P^m, \qquad \text{for all } n, m \in \mathbb{Z}_+, \tag{2.1.2}$$

holds for Markov chains with finite state spaces (cf. [KS76, Chu79]).

Observe that, the transition probability matrix P of a Markov chain with d states is a $d \times d$ stochastic matrix, that is

$$p_{ij} \geq 0, \text{ for } 1 \leq i,j \leq d, \qquad \text{and} \qquad \sum_{j=1}^{d} p_{ij} = 1 \text{ for all } i. \tag{2.1.3}$$

We can treat this P as a linear map on the commutative C^*-algebra $\mathbb{C}^d$ (See the definitions in the next subsections). In this setup, Eq. (2.1.3) is nothing but the statement that the map P is positive (indeed, completely positive cf. Theorem 2.1.4) and unital. Our interest is to study the non-commutative or quantum analogue of these maps. They are known as *quantum Markov maps* in quantum probability. We shall precisely define them in the following subsections.

In general, the semigroup property (2.1.2) holds even when the state space S of Markov chains is infinite. Further, when the indexing set of the random variables is uncountable i.e., for Markov processes, a generalization of Markov chains, the semigroup property still holds. The semigroup property is crucial for our purposes and we shall retain it in our non-commutative generalizations.

2.1.2 C^*-algebras

C^*-algebras are the non-commutative analogues of the function spaces $C(X)$, the space of all continuous functions on a locally compact Hausdorff space X. Quantum Markov maps would be maps acting on C^*-algebras. Here we recall the basic definition and we set up our notation. We refer to the following standard books for the proofs and details of this subsection [Con00, Sun97, KR97, Mur90, Tak02].

Definition 2.1.2. Let $\mathcal{A}$ be an algebra, an *involution* $* : \mathcal{A} \to \mathcal{A}$ is a map which maps $a \mapsto a^*$ such that for all $a, b \in \mathcal{A}, \alpha \in \mathbb{C}$ the following conditions hold:

(i) $(a^*)^* = a$,
(ii) $(ab)^* = b^* a^*$,
(iii) $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$.

Definition 2.1.3. An algebra $\mathcal{A}$ is said to a *normed algebra* if there is a norm on $\mathcal{A}$ satisfying:

$$\|ab\| \leq \|a\|\|b\|, \quad \text{for all } a, b \in \mathcal{A}. \tag{2.1.4}$$

A normed algebra $\mathcal{A}$ is said to be a *Banach algebra* if it is complete with respect to the norm.

Definition 2.1.4. A normed algebra with an involution is said to be a *pre-C^*-algebra* if

$$\|a^* a\| = \|a\|^2, \quad \text{for all } a \in \mathcal{A}. \tag{2.1.5}$$

Definition 2.1.5. A pre-C^*-algebra $\mathcal{A}$ is said to be a *C^*-algebra* if it is complete with respect to the norm. If $\mathcal{A}$ has a unit/identity $\mathbf{1}$ (i.e., $\mathbf{1}x = x\mathbf{1} = x \ \forall \ x \in \mathcal{A}$), then $\mathcal{A}$ is said to be a *unital* C^*-algebra.

Remark 2.1.1. Note that a C^*-algebra is a Banach algebra with an involution fulfilling (2.1.5). If $\mathcal{A}$ is a C^*-algebra and $a \in \mathcal{A}$, then $\|a^*\| = \|a\|$ and $\|aa^*\| = \|a\|^2$. If the multiplication in $\mathcal{A}$ is commutative, then $\mathcal{A}$ is said to be *commutative* or *abelian*. An algebraic homomorphism between two C^*-algebras, which respects the involutions is said to be a *$*$-homomorphism* . An *isomorphism* between two C^*-algebras is a bijective $*$-homomorphism.

Remark 2.1.2 (unitization)**.** Let $\mathcal{A}$ be a C^*-algebra. Consider $\tilde{\mathcal{A}} = \{(a, \lambda) : a \in \mathcal{A}, \lambda \in \mathbb{C}\}$ with addition $(a, \lambda) + (b, \mu) = (a + b, \lambda + \mu)$, multiplication $(a, \lambda)(b, \mu) = (ab + \mu a + \lambda b, \lambda\mu)$, involution $(a, \lambda)^* = (a^*, \bar{\lambda})$, and norm $\|(a, \lambda)\| = \sup_{b \in \mathcal{A}, \|b\| \leq 1} \|ab + \lambda b\|$. Then $\tilde{\mathcal{A}}$ is a unital C^*-algebra containing $\mathcal{A}$ as an ideal. If $\mathcal{A}$ has no unit, then $\tilde{\mathcal{A}}/\mathcal{A}$ is one dimensional. If $\mathcal{B}$ is a C^*-algebra with identity and if $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then $\varphi_1(a+\lambda) = \varphi(a)+\lambda$ defines a $*$-homomorphism $\varphi_1 : \tilde{\mathcal{A}} \to \mathcal{B}$.

Example 2.1.2. (i) $\mathbb{C}^n$ is a finite dimensional commutative, unital C^*-algebra with sup-norm.

(ii) $\mathrm{M}_n(\mathbb{C}), (n > 1)$ is a finite dimensional non-commutative, unital C^*-algebra.

(iii) If X is a compact Hausdorff space, $C(X)$, the collection all continuous functions on X, is a commutative, unital C^*-algebra. (It is infinite dimensional, if X is infinite).

(iv) $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on $\mathcal{H}$, is a non-commutative, unital C^*-algebra. (It is infinite dimensional, if $\dim \mathcal{H} = \infty$).

(v) If X is a locally compact Hausdorff space, $C_0(X)$, the algebra of continuous functions on X that vanish at infinity, is a commutative, C^*-algebra. $C_0(X)$ is unital if and only if X is compact. (It is infinite dimensional, if X is an infinite set).

(vi) $\mathcal{K}(\mathcal{H})$, the algebra of compact operators on $\mathcal{H}$, is a non-commutative, non-unital C^*-algebra. (It is infinite dimensional, if $\dim \mathcal{H} = \infty$).

Definition 2.1.6. If $\mathcal{A}$ is a pre-C^*-algebra and $a \in \mathcal{A}$, then we say that:

(i) a is *self adjoint* or *hermitian* if $a = a^*$,

(ii) a is *normal* if $a^*a = aa^*$,

(iii) when $\mathcal{A}$ has an identity $\mathbf{1}$, a is *unitary* if $a^*a = aa^* = \mathbf{1}$,

(iv) a is *positive* if $a = b^*b$ for some $b \in \mathcal{A}$. We write $a \geq 0$ to denote a is positive.

If $\mathcal{A}$ is a unital Banach algebra and $a \in \mathcal{A}$, the spectrum of a is denoted by $\sigma_{\mathcal{A}}(a)$ or simply by $\sigma(a)$ and the spectral radius of a is denoted by $r(a)$ and they are defined as

$$\sigma(a) = \{\lambda \in \mathbb{C} : a - \lambda\mathbf{1} \text{ is not invertible in } \mathcal{A}\}, \tag{2.1.6}$$

$$r(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}. \tag{2.1.7}$$

If $\mathcal{A}$ is a unital C^*-algebra and $a \in \mathcal{A}$ is self adjoint, then $\|a\| = r(a)$. If $\mathcal{A}$ and $\mathcal{B}$ are two unital C^* -algebras, $a \in \mathcal{A}$ and $\varphi : \mathcal{A} \to \mathcal{B}$ is a $*$-homomorphism, then it is easy to see that $\sigma(\varphi(a)) \subseteq \sigma(a)$. From these two facts it follows that, if φ is an isomorphism, then φ is an isometry. This in particular says that the norm in a C^*-algebra is unique.

The spectrum of elements in a C^*-algebra has the following nice property: If $\mathcal{B} \subseteq \mathcal{A}$ is a C^*-subalgebra of the C^*-algebra $\mathcal{A}$ with a common unit, then $\sigma_{\mathcal{B}}(a) = \sigma_{\mathcal{A}}(a)$ for any $a \in \mathcal{B}$.

Let $\mathcal{A}$ be a commutative Banach algebra with an identity. Let Σ be the maximal ideal space of $\mathcal{A}$. i.e.,

$$\Sigma = \{I \subseteq \mathcal{A} : I \text{ is a maximal ideal in } \mathcal{A}\}. \tag{2.1.8}$$

Recall that $\{\varphi : \mathcal{A} \to \mathbb{C} : \varphi \text{ is linear, multiplicative and } \varphi(\mathbf{1}) = 1\}$, the set of non-zero complex homomorphisms is identified with Σ (via. $\phi \mapsto \ker \phi \in \Sigma$). Each non-zero complex homomorphism has norm 1, so $\Sigma \subseteq \mathcal{A}_1^*$, the unit ball of $\mathcal{A}^*$ (the Banach space dual of $\mathcal{A}$). If we equip Σ with the relative weak* topology of $\mathcal{A}^*$, then Σ becomes a compact Hausdorff space. For $a \in \mathcal{A}$ define $\hat{a} : \Sigma \to \mathbb{C}$ by $\hat{a}(f) = f(a)$ for $f \in \Sigma$. Then $\hat{a}$ is continuous i.e., $\hat{a} \in C(\Sigma)$. This function $\hat{a}$ is called the *Gelfand transform* of a. Define $\gamma : \mathcal{A} \to C(\Sigma)$ by $\gamma(a) = \hat{a}$. Then γ is an algebraic homomorphism, with $\|\gamma\| = 1$. The map γ is called the *Gelfand transform* for the algebra $\mathcal{A}$.

Theorem 2.1.1. *Let $\mathcal{A}$ be a commutative unital C^*-algebra. Then the Gelfand transform $\gamma : \mathcal{A} \to C(\Sigma)$ defined above is an isomorphism.*

Now suppose $\mathcal{A}$ is a commutative C^*-algebra without an identity. Let $\tilde{\mathcal{A}}$ be the unitization of $\mathcal{A}$ as explained in Remark 2.1.2. Let Σ and $\tilde{\Sigma}$ denote the maximal ideal spaces of $\mathcal{A}$ and $\tilde{\mathcal{A}}$ respectively. As $\mathcal{A}$ has no identity, by Remark 2.1.2, $\mathcal{A}$ is a maximal ideal in $\tilde{\mathcal{A}}$. Let $h : \tilde{\mathcal{A}} \to \mathbb{C}$ be the unique homomorphism with $\ker h = \mathcal{A}$. Then we have $\Sigma = \tilde{\Sigma} \setminus \{h\}$. This observation with Theorem 2.1.1 gives us the following corollary.

Corollary 2.1.1. *Let $\mathcal{A}$ be a commutative C^*-algebra (without an identity). Then the Gelfand transform $\gamma : \mathcal{A} \to C_0(\Sigma)$ is an isomorphism, where Σ is the maximal ideal space of $\mathcal{A}$.*

We now move to maps on C^*-algebras.

Definition 2.1.7. If $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras, a linear map $\phi : \mathcal{A} \to \mathcal{B}$ is said to be *positive* if $\phi(a) \geq 0$ for all $a \geq 0$. We write $\phi \geq 0$ to mean ϕ is positive.

Notation. If $\tau, \psi : \mathcal{A} \to \mathcal{B}$ are linear maps, we denote $\psi \leq \tau$ if $\tau - \psi \geq 0$.

Definition 2.1.8. Let $\mathcal{A}$ be a C^*-algebra and $\mathcal{H}$ be a Hilbert space. A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be a *representation* of $\mathcal{A}$ in $\mathcal{H}$. If $\mathcal{A}$ is unital, then it is assumed that $\pi(1) = 1$.

Definition 2.1.9. A representation π of a C^*-algebra $\mathcal{A}$ in $\mathcal{H}$ is said to be *faithful* if it is injective, *non-degenerate* if $\overline{\operatorname{span}}\ \pi(\mathcal{A})\mathcal{H} = \mathcal{H}$ and *cyclic* if there is a vector $e \in \mathcal{H}$ such that $\{\pi(a)e : a \in \mathcal{A}\}$ is dense in $\mathcal{H}$. A vector $e \in \mathcal{H}$ that satisfies this condition is called a *cyclic vector*.

Definition 2.1.10. A (unital) map ϕ from $\mathcal{A}$ onto $\mathcal{B} \subseteq \mathcal{A}$ is called a *conditional expectation*, if $\phi^2 = \phi$ and $\|\phi\| = 1$.

Definition 2.1.11. A positive linear functional on a C^*-algebra $\mathcal{A}$ is said to be a *state* if it has norm 1.

Suppose $\mathcal{A}$ is a C^*-algebra and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation. If e is a unit vector in $\mathcal{H}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ is defined by $\varphi(a) = \langle e, \pi(a)e\rangle$, then φ is a state on $\mathcal{A}$. Conversely, we have the following GNS theorem:

Theorem 2.1.2 (Gelfand-Naimark-Segal (GNS) construction)**.** *Let φ be a state on a C^*-algebra $\mathcal{A}$. Then there is a cyclic representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ with a unit cyclic vector $e \in \mathcal{H}$ such that*

$$\varphi(a) = \langle e, \pi(a)e\rangle \quad \text{for all } a \in \mathcal{A}. \tag{2.1.9}$$

The triple $(\pi, \mathcal{H}, e)$ is called a GNS-triple *for φ.*

The GNS construction with some work gives us the following theorem, which says that every abstract C^*-algebra is isomorphic to a C^*-algebra of operators on a Hilbert space.

Theorem 2.1.3. *Every C^*-algebra $\mathcal{A}$ has a faithful representation. Moreover every separable C^*-algebra has a faithful representation on a separable Hilbert space.*

The following is a Radon-Nikodym type theorem for positive linear functionals.

Proposition 2.1.1 ([Con00, Proposition 32.1])**.** *Let φ be a state on a C^*-algebra $\mathcal{A}$, with a GNS triple $(\pi_\varphi, \mathcal{H}_\varphi, e_\varphi)$ and let ψ be a positive linear functional on $\mathcal{A}$, then $\psi \leq \varphi$ if and only if there is a unique operator T with $T\pi(a) = \pi(a)T$ for all $a \in \mathcal{A}$ and $0 \leq T \leq I$ such that $\psi(a) = \langle e_\varphi, \pi_\varphi(a)Te_\varphi\rangle$ for all $a \in \mathcal{A}$.*

The states where the collection of dominated positive linear functionals is trivial are known as pure states. They give rise to irreducible GNS representations. Here is the formal definition.

Definition 2.1.12. A state φ on $\mathcal{A}$ is called *pure* if for any positive linear functional ψ on $\mathcal{A}$ such that $\psi \leq \varphi$, there is a scalar λ such that $\psi = \lambda\varphi$.

The space of all states on a C^*-algebra $\mathcal{A}$, is a weak* compact convex subset of $\mathcal{A}^*$, the Banach space dual of $\mathcal{A}$.

Theorem 2.1.4 ([Con00, Theorem 32.7])**.** *Let φ be a state on $\mathcal{A}$. Then φ is pure if and only if φ is an extreme point of the space of all states on $\mathcal{A}$.*

2.1.3 von Neumann algebras

von Neumann algebras are the non-commutative analogues of the measurable function spaces $L^\infty(X, \mu)$, for measure spaces (X, μ). They are C^*-algebras with additional algebraic and topological properties. Often it is convenient to have the set up of von Neumann algebras.

Definition 2.1.13. If $\mathcal{S}$ is a subset of $\mathcal{B}(\mathcal{H})$, the *commutant* of $\mathcal{S}$, denoted by $\mathcal{S}'$ is defined by

$$\mathcal{S}' = \{T \in \mathcal{B}(\mathcal{H}) : TS = ST \text{ for all } S \in \mathcal{S}\}. \tag{2.1.10}$$

The double commutant of $\mathcal{S}$ is defined by $\mathcal{S}'' = \{\mathcal{S}'\}'$ and similarly $\mathcal{S}''' = \mathcal{S}^{(3)}, \cdots$.

Remark 2.1.3. Clearly $I \in \mathcal{S}'$ and $\mathcal{S} \subseteq \mathcal{S}''$ for any subset $\mathcal{S}$. If $\mathcal{N} \subseteq \mathcal{M}$ we have from the definition of commutants that $\mathcal{M}' \subseteq \mathcal{N}'$ and hence $\mathcal{N}'' \subseteq \mathcal{M}''$. From these observations we can see that, for any subset $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ we have $\mathcal{S}' = \mathcal{S}'''$. Moreover we see that

$$\mathcal{S}' = \mathcal{S}''' = \mathcal{S}^{(5)} = \cdots, \quad \text{and} \quad \mathcal{S} \subseteq \mathcal{S}'' = \mathcal{S}^{(4)} = \cdots.$$

Notation. For any $\mathcal{S} \subseteq \mathcal{B}(\mathcal{H})$ we denote the strong operator topology (SOT) closure of $\mathcal{S}$ as $\overline{\mathcal{S}}^s$.

Theorem 2.1.5 (Double Commutant Theorem)**.** *If $\mathcal{A}$ is a unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, then $\overline{\mathcal{A}}^s = \mathcal{A}''$.*

Definition 2.1.14. A strongly closed, unital C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ is called a *von Neumann algebra*

Example 2.1.3. (i) Any finite dimensional C^*-algebra is a von Neumann algebra.

(ii) Let (X, μ) be a measure space, then $L^\infty(X, \mu)$, the collection all essentially bounded, measurable functions on X is a commutative von Neumann algebra.

(iii) $\mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on $\mathcal{H}$, is a non-commutative, von Neumann algebra. (It is infinite dimensional, if $\dim \mathcal{H} = \infty$).

Theorem 2.1.6 (Vigier, see [Mur90]). *Let $(u_\lambda)_{\lambda\in\Lambda}$ be an increasing net of hermitian operators on a Hilbert space $\mathcal{H}$. Assume that it is bounded above. Then $(u_\lambda)_{\lambda\in\Lambda}$ is strongly convergent.*

Definition 2.1.15. Let $\mathcal{A}$ and $\mathcal{B}$ be von Neumann algebras. We say that a positive map $\phi : \mathcal{A} \to \mathcal{B}$ is *normal* if $\phi(u_\lambda) \uparrow \phi(u)$ for all bounded increasing net $(u_\lambda)_{\lambda\in\Lambda}$ of self-adjoint operators such that $u_\lambda \uparrow u$.

Example 2.1.4. Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{K}(\mathcal{H})$ denote the algebra of all compact operators on $\mathcal{H}$ let $\mathcal{C}$ denote the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. Let $q : \mathcal{B}(\mathcal{H}) \to \mathcal{C}$ be the quotient map. Let φ be any state on $\mathcal{C}$. Then $\rho : \mathcal{B}(\mathcal{H}) \to \mathbb{C}$ defined by $\rho = \varphi \circ q$ is not normal.

Notation. We introduce the bra-ket notations here. For a more general description of this notation look at Subsection 2.2.1.4. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. For $a \in \mathcal{H}, b \in \mathcal{K}$, we define the operator $|b\rangle\langle a| : \mathcal{H} \to \mathcal{K}$ as follows:

$$|b\rangle\langle a|\,(a') = \langle a, a'\rangle b, \quad \text{for } a' \in \mathcal{H}.$$

Theorem 2.1.7. *Let $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a normal representation. Then there exist a Hilbert space $\mathcal{P}$ and an isometry $W : \mathcal{H} \otimes \mathcal{P} \to \mathcal{K}$ such that*

$$\pi(X) = W(X \otimes I_{\mathcal{P}})W^*. \tag{2.1.11}$$

(If π is unital, we can choose W to be unitary).

Furthermore, there exists a collection of operators $\{V_n : \mathcal{H} \to \mathcal{K}\}_{n\geq 1}$ (finite or countably infinite) such that $V_m^* V_n = \delta_{mn}, \sum_m V_m V_m^* = \pi(\mathbf{1})$ and

$$\pi(X) = \sum_n V_n X V_n^*, \tag{2.1.12}$$

for all $X \in \mathcal{B}(\mathcal{H})$, where the sum in (2.1.12) is in SOT.

Proof. We shall give the sketch of the proof. Fix $a \in \mathcal{H}$ with $\|a\| = 1$ and consider the rank-one projection $P_a = |a\rangle\langle a|$. Take $\mathcal{P} = \operatorname{ran}(\pi(|a\rangle\langle a|))$. Now define $W : \mathcal{H} \otimes \mathcal{P} \to \mathcal{K}$ by

$$W(x \otimes \pi(|a\rangle\langle a|)y) = \pi(|x\rangle\langle a|)y \tag{2.1.13}$$

for $x, y \in \mathcal{H}$. Now we can verify the following:

(i) W is an isometry. (unitary, if π is unital)
(ii) $W^*z = \sum_k e_k \otimes \pi(|a\rangle\langle e_k|)z$, where $\{e_k\}_k$ is an orthonormal basis for $\mathcal{H}$.
(iii) $\pi(X) = W(X \otimes I_{\mathcal{P}})W^*$ for all $X \in \mathcal{B}(\mathcal{H})$.

For the second part, Fix an orthonormal basis $\{e_n\}$ for $\mathcal{P}$. Define $V_n : \mathcal{H} \to \mathcal{K}$ by

$$V_n x = W(x \otimes e_n), \quad \text{for } x \in \mathcal{H}. \tag{2.1.14}$$

□

2.1.4 Quantum Markov maps

Quantum Markov maps are special classes of completely positive (CP) maps. In this subsection we define completely positive (CP) maps and present a few important structure theorems for CP maps.

Recall that, if $\mathcal{H}$ is any Hilbert space, the natural identification $M_n(\mathcal{B}(\mathcal{H})) \simeq \mathcal{B}(\mathcal{H}^n)$ gives us a norm that makes $M_n(\mathcal{B}(\mathcal{H}))$ as a C^*-algebra. Now given any C^*-algebra $\mathcal{A}$, by Theorem 2.1.3 $\mathcal{A}$ can be identified as a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Therefore, as $\mathcal{A}$ acts on $\mathcal{H}, M_n(\mathcal{A})$ acts on $\mathcal{H}^n$ in the usual way. Using this we identify $M_n(\mathcal{A})$ as a C^*-subalgebra of $M_n(\mathcal{B}(\mathcal{H}))$.

Definition 2.1.16. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. If $\phi : \mathcal{A} \to \mathcal{B}$ is a linear map, then define $\phi_n : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ by

$$\phi_n((a_{ij})) = (\phi(a_{ij})), \quad \text{for } (a_{ij}) \in M_n(\mathcal{A}). \tag{2.1.15}$$

Then,

(i) ϕ is said to be *n-positive* if ϕ_n is positive.
(ii) ϕ is said to be *completely positive* (CP) if ϕ is n-positive for all $n \in \mathbb{N}$.
(iii) ϕ is said to be *completely bounded*(CB) if $\sup\limits_{n\in\mathbb{N}}\|\phi_n\| < \infty$, and in this case, we set

$$\|\phi\|_{\mathrm{cb}} = \sup_{n\in\mathbb{N}}\|\phi_n\|. \tag{2.1.16}$$

Remark 2.1.4. (i) Unital CP maps are known as *quantum Markov maps* in quantum probability. (ii) Trace preserving, unital CP maps are known as *quantum channels* in quantum information theory.

We can see that $\|\cdot\|_{\mathrm{cb}}$ is a norm on the space of completely bounded maps. It is easy to observe that if ϕ is n-positive, then ϕ is k-positive for $k \le n$. Also $\|\phi_k\| \le \|\phi_n\|$ for $k \le n$.

Remark 2.1.5. Note for any C^*-algebra $\mathcal{A}$ that we have the isomorphism $M_n(\mathcal{A}) \simeq \mathcal{A} \otimes \mathrm{M}_n$. In this identification $\phi_n = \phi \otimes \mathrm{id}_n : \mathcal{A} \otimes \mathrm{M}_n \to \mathcal{B} \otimes \mathrm{M}_n$, where id_n is the identity map on M_n. Therefore ϕ is n-positive if and only if $\phi \otimes \mathrm{id}_n$ is positive.

Proposition 2.1.2. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Then ϕ is n-positive if and only if $\sum_{i,j} b_i^* \phi(a_i^* a_j) b_j \geq 0$ for all $a_1, a_2, \ldots a_n \in \mathcal{A}, b_1, b_2, \ldots, b_n \in \mathcal{B}$.*

Proposition 2.1.3 ([Pau02, Proposition 3.6])**.** *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a CP map between C^*-algebras. Then ϕ is CB and $\|\phi\|_{cb} = \|\phi\| = \|\phi(1)\|$.*

Example 2.1.5.
(i) Any $*$-homomorphism between two C^*-algebras is a CP map.
(ii) Let $A = (a_{ij}) \in \mathrm{M}_n$ be a positive matrix. Define $\phi : \mathrm{M}_n \to \mathrm{M}_n$ by $\phi(X) = (a_{ij} x_{ij})$ the Schur product (entry-wise product) of A and $X = (x_{ij})$. Then ϕ is CP.
(iii) Let $P = (p_{ij}) \in \mathrm{M}_n$. Consider $\phi : \mathbb{C}^n \to \mathbb{C}^n$ defined by $\phi(x) = Px$. Then ϕ is positive (and hence CP by Proposition 2.1.4) if and only if the entries p_{ij} are non-negative. ϕ is unital if and only if P is a stochastic matrix.
(iv) Let $\mathcal{A}$ be a C^*-algebra. Fix $x, y \in \mathcal{A}$ and define $\phi : \mathcal{A} \to \mathcal{A}$ by $\phi(a) = xay$. Then ϕ is a CB map with $\|\phi\|_{\mathrm{cb}} \leq \|x\| \|y\|$. If $x = y^*$, then ϕ is CP.
(v) $\phi : \mathrm{M}_2 \to \mathrm{M}_2$ defined by $\phi(A) = A'$ is a positive map which is not 2-positive, where A' is the matrix *transpose* of A. Hence not CP. But this is a CB map.
(vi) If ϕ is CP. Note that $-\phi$ is CB but not CP.

Proposition 2.1.4 (Stinespring[Sti55])**.** *Every positive map on a commutative C^*-algebra is CP.*

Proposition 2.1.5 (Arveson [Arv69])**.** *Let $\mathcal{B}$ be a commutative C^*-algebra. Suppose $\phi : \mathcal{A} \to \mathcal{B}$ is a positive linear map, then ϕ is CP.*

Let $\mathcal{A}$ be a C^*-algebra, $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ be a $*$-homomorphism, and let $V : \mathcal{H} \to \mathcal{K}$ be a bounded linear map. Then $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by $\phi(a) = V^* \pi(a) V$ is a CP map. Conversely, we have the following characterization theorem (generalization of GNS construction) by Stinespring for CP maps from any C^*-algebra into $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Theorem 2.1.8 (Stinespring's dilation theorem [Sti55])**.** *Let $\mathcal{A}$ be a unital C^*-algebra and $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CP map. Then there exists a triple $(\mathcal{K}, \pi, V)$ of a Hilbert space $\mathcal{K}$, a unital $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and a bounded operator $V : \mathcal{H} \to \mathcal{K}$ with $\|\phi(1)\| = \|V\|^2$ such that*

$$\phi(a) = V^* \pi(a) V, \quad \textit{for all } a \in \mathcal{A}. \tag{2.1.17}$$

The triple $(\mathcal{K}, \pi, V)$ is called a Stinespring representation *or* Stinespring triple *for ϕ.*

Remark 2.1.6. (i) Note that if ϕ is unital, then V is an isometry. So in this case we may identify $\mathcal{H}$ as a subspace of $\mathcal{K}$ with $V(\mathcal{H})$. Hence V^* is the projection $P_{\mathcal{H}}$ of $\mathcal{K}$ onto $\mathcal{H}$ and we have ϕ as a compression of the $*$-homomorphism as follows:

$$\phi(a) = P_{\mathcal{H}}\pi(a)|_{\mathcal{H}}, \quad \text{for all } a \in \mathcal{A}. \tag{2.1.18}$$

(ii) Let $\hat{\mathcal{K}} = \overline{\text{span}}\, \pi(\mathcal{A})V\mathcal{H}$. Then $\hat{\mathcal{K}}$ reduces $\pi(\mathcal{A})$ and hence π restricted to $\hat{\mathcal{K}}$ defines a $*$-homomorphism $\pi|_{\hat{\mathcal{K}}} : \mathcal{A} \to \mathcal{B}(\hat{\mathcal{K}})$. Note that $V\mathcal{H} \subseteq \hat{\mathcal{K}}$. Therefore we have $\phi(a) = V^*\pi|_{\hat{\mathcal{K}}}(a)V$. That is, $(\hat{\mathcal{K}}, \pi|_{\hat{\mathcal{K}}}, V)$ is also a Stinespring triple for ϕ.

Definition 2.1.17. A Stinespring representation $(\mathcal{K}, \pi, V)$ of $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is called a *minimal Stinespring representation* if

$$\mathcal{K} = \overline{\text{span}}\, \pi(\mathcal{A})V\mathcal{H}. \tag{2.1.19}$$

From Remark 2.1.6 (ii), it follows that every CP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ has a minimal Stinespring representation. The following proposition shows that any two minimal representations are isomorphic.

Proposition 2.1.6. *If $(\mathcal{K}_i, \pi_i, V_i), i = 1, 2$ are two minimal Stinespring representations for a CP map $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ then the map $U : \mathcal{K}_1 \to \mathcal{K}_2$ defined by*

$$U\left(\sum_i \pi_1(a_i)V_1h_i\right) = \sum_i \pi_2(a_i)V_2h_i$$

is a unitary satisfying $UV_1 = V_2$ and $U\pi_1U^* = \pi_2$.

We have the following useful inequality as a corollary to the Stinespring's dilation theorem.

Proposition 2.1.7 (Kadison-Schwarz inequality)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a CP map. Then for every $a \in \mathcal{A}$ we have*

$$\phi(a^*)\phi(a) \le \phi(a^*a)\|\phi(\mathbf{1})\|. \tag{2.1.20}$$

Theorem 2.1.9. *Let $\phi : \mathcal{A} \to \mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ be a normal CP map where $\mathcal{A}$ and $\mathcal{B}$ are von Neumann algebras and let $(\mathcal{K}, \pi, V)$ be the minimal Stinespring representation of ϕ. Then the map $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ is normal.*

For normal CP maps between the algebras of operators on Hilbert spaces, we have the following result as a consequence of Theorems 2.1.7, 2.1.8 and 2.1.9.

Theorem 2.1.10. *Let $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ be a normal CP map. Then there exist $L_n \in \mathcal{B}(\mathcal{H}, \mathcal{K}), n \in \mathbb{N}$ such that*

$$\phi(X) = \sum_n L_n X L_n^* \quad \textit{for all } X \in \mathcal{B}(\mathcal{H}). \quad \textit{(SOT sum)} \tag{2.1.21}$$

For the CP maps on the space of $n \times n$ matrices M_n, we have the following characterization theorem by Choi.

Theorem 2.1.11 (Choi [Cho75])**.** *Let $\mathcal{B}$ be a C^*-algebra, let $\phi : \mathrm{M}_n \to \mathcal{B}$ and let $E_{ij}, 1 \leq i, j \leq n$ be the standard matrix units for M_n. Then the following are equivalent:*

(i) ϕ *is CP.*
(ii) ϕ *is n-positive.*
(iii) $(\phi(E_{ij}))_{i,j=1}^n$ *is positive in* $\mathrm{M}_n(\mathcal{B})$.

Notation. Let $\mathcal{A}, \mathcal{B}$ be C^*-algebras, and $\phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Define $\phi^* : \mathcal{A} \to \mathcal{B}$ by

$$\phi^*(a) = \phi(a^*)^*, \quad \text{for all } a \in \mathcal{A} \tag{2.1.22}$$

and

$$\operatorname{Re}\phi = \frac{\phi + \phi^*}{2}, \qquad \operatorname{Im}\phi = \frac{\phi - \phi^*}{2i}.$$

Then $\operatorname{Re}\phi$ and $\operatorname{Im}\phi$ are self-adjoint, linear maps such that $\phi = \operatorname{Re}\phi + i \operatorname{Im}\phi$.

Theorem 2.1.12 ([Pau02, Theorem 8.3])**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CB map. Then there exists CP maps ϕ_i with $\|\phi_i\|_{cb} = \|\phi\|_{cb}, i = 1, 2$ such that $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$ defined by*

$$\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi_1(a) & \psi(b) \\ \psi^*(c) & \phi_2(d) \end{pmatrix}$$

is CP.

The following theorem for CB maps, follows from the previous theorem and it is analogues to the Stinespring representation for CP maps.

Theorem 2.1.13 ([Pau02, Theorem 8.4])**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a CB map. Then there exists a tuple $(\mathcal{K}, \pi, V_1, V_2)$ of a Hilbert space $\mathcal{K}$, a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{K})$ and bounded operators $V_i : \mathcal{H} \to \mathcal{K}, i = 1, 2$ with $\|\psi\|_{cb} = \|V_1\|\|V_2\|$ such that*

$$\psi(a) = V_1^* \pi(a) V_2, \quad \textit{for all } a \in \mathcal{A}. \tag{2.1.23}$$

Definition 2.1.18. Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras and let $\alpha, \beta : \mathcal{A} \to \mathcal{B}$ be CP maps. Then α is said to *dominate* β if $\alpha - \beta$ is CP and we write $\alpha \geq \beta$ to mean α dominates β.

The following theorem of Arveson for CP maps is analogues to the Proposition 2.1.1.

Theorem 2.1.14 ([Arv69, Lemma 1.4.1])**.** *Suppose $\alpha, \beta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ are CP maps such that $\alpha \geq \beta$. Let $(\mathcal{K}, \pi, V)$ be a (minimal) Stinespring representation of α. Then there exists a unique $T \in \pi(\mathcal{A})'$ such that $0 \leq T \leq I$ and $\beta(\cdot) = V^*\pi(\cdot)TV$.*

Theorem 2.1.15 ([PS85, Corollary 2.7])**.** *Let $\mathcal{A}$ be a unital C^*-algebra and let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be CP and $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be CB. Let $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$ be defined by $\Phi = \begin{pmatrix} \phi & \psi \\ \psi^* & \phi \end{pmatrix}$ That is, $\Phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \phi(a) & \psi(b) \\ \psi^*(c) & \phi(d) \end{pmatrix}$. Suppose $(\mathcal{K}, \pi, V)$ is the minimal Stinespring representation of ϕ. Then Φ is CP if and only if there exists a contraction $T \in \pi(\mathcal{A})'$ such that $\psi(\cdot) = V^*T\pi(\cdot)V$.*

2.2 Hilbert C^*-modules

A *Hilbert C^*-module* is a right module over a C^*-algebra $\mathcal{B}$ with a $\mathcal{B}$-valued inner product fulfilling axioms (Definition 2.2.1) similar to the axioms of an inner product of a Hilbert space. Hilbert C^*-modules were first introduced by Kaplansky in [Kap53], where his idea was to generalize Hilbert space by allowing the inner product to take values in a (commutative unital) C^*-algebra (he called them as "C^*-modules"). Paschke [Pas73] and Rieffel [Rie74] introduced the Hilbert C^*-modules over non-commutative C^*-algebras.

The theory of Hilbert C^*-modules is already rich, well studied and is considered as a valuable tool in operator algebra theory. Many authors call them as C^*-correspondences. Here though we define (semi-) pre-Hilbert C^*-modules, we present the theory, restricted to the Hilbert C^*-modules only. We refer to [Ske01, Lan95] and the references from there for further details on the theory of Hilbert C^*-modules.

We use the theory of Hilbert C^*-modules, to study CP maps and the semigroups of CP maps. In [Pas73] Paschke obtained the *GNS-construction* for any CP map between two C^*-algebras, which says that: given a CP map $\phi : \mathcal{A} \to \mathcal{B}$ there exist a $\mathcal{A}$-$\mathcal{B}$-module E (called as "GNS-module") and a cyclic vector $\xi \in E$ such that $\phi(a) = \langle \xi, a\xi \rangle$ for all $a \in \mathcal{A}$. The advantage of the GNS-construction is that we can write the GNS-module of the composition of two CP maps, as a submodule of the tensor product of their individual GNS-modules (See Observation 2.2.5). This is not the case with the Stinespring's dilation. This helps us to connect the semigroups of CP maps on $\mathcal{B}$ with the product systems of

Hilbert $\mathcal{B}$-$\mathcal{B}$-modules. In the same paper Paschke proved also that von Neumann modules are self-dual.

Arveson in [Arv89] established the connection between product systems of Hilbert spaces and E_0-semigroups on $\mathcal{B}(\mathcal{H})$. In [BS00], Bhat and Skeide observe the connection between inclusion systems of Hilbert C^*-modules and CP semigroups and using that they constructed the E_0-dilation for unital CP semigroups on a unital C^*-algebra $\mathcal{B}$. Muhly and Solel [MS07] took a dual approach to achieve this, where they have called these Hilbert C^*-modules as C^*-correspondences.

Definition 2.2.1. Let $\mathcal{B}$ be a pre- C*-algebra. A complex vector space E is said to be an *inner product $\mathcal{B}$-module* or *pre-Hilbert $\mathcal{B}$-module* if E is a right $\mathcal{B}$-module, with a $\mathcal{B}$-valued inner product $\langle \cdot, \cdot \rangle : E \times E \to \mathcal{B}$ such that

(i) $\lambda(xb) = (\lambda x)b = x(\lambda b)$ for $x \in E, b \in \mathcal{B}, \lambda \in \mathbb{C}$, (compatibility)

(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$ for $x, y, z \in E, \alpha, \beta \in \mathbb{C}$,

(iii) $\langle x, yb \rangle = \langle x, y \rangle b$ for $x, y \in E, b \in \mathcal{B}$,

(iv) $\langle x, y \rangle^* = \langle y, x \rangle$ for $x, y \in E$,

(v) $\langle x, x \rangle \geq 0$ for all $x \in E$,

(vi) $\langle x, x \rangle = 0$ if and only if $x = 0$.

Observe that (ii) shows that the inner product is linear in second variable, and it follows from (iv) that the inner product is conjugate linear in the first variable. Also (iii) and (iv) together implies that $\langle xb, y \rangle = b^* \langle x, y \rangle$ for $x, y \in E, b \in \mathcal{B}$.

If E satisfies all the conditions of an inner product $\mathcal{B}$-module except (vi) then E is said to be a *semi-inner product $\mathcal{B}$-module* or *semi-Hilbert $\mathcal{B}$-module.*

Proposition 2.2.1. *Let E be a pre-Hilbert $\mathcal{B}$-module and let $x, x' \in E$. If*

$$\langle y, x \rangle = \langle y, x' \rangle, \quad \textit{for all } y \in E,$$

then $x = x'$. Consequently, if $\mathbf{1} \in \mathcal{B}$ is the unit of $\mathcal{B}$, then $w\mathbf{1} = w$ for all $w \in E$.

We have the following version of Cauchy-Schwarz inequality for the semi-inner product modules. Note that this inequality is not an inequality of numbers but of elements of the C^*-algebra $\mathcal{B}$.

Proposition 2.2.2 ([Lan95, Proposition 1.1])**.** *Let E be a semi-inner product $\mathcal{B}$-module. If $x, y \in E$, then*

$$\langle x, y \rangle \langle y, x \rangle \leq \|\langle y, y \rangle\| \langle x, x \rangle. \tag{2.2.1}$$

For $x \in E$ we define $\|x\| := \|\langle x, x\rangle\|^{\frac{1}{2}}$. The following proposition is immediate from this definition and Proposition 2.2.2.

Proposition 2.2.3. *Let E be a semi-Hilbert $\mathcal{B}$-module. For any $x, y \in E$ and $b \in \mathcal{B}$ we have the following inequalities:*

(i) $\|xb\| \leq \|x\|\|b\|$.
(ii) $\|\langle x, y\rangle\| \leq \|x\|\|y\|$.

Using the inequality (ii) of Proposition 2.2.3 we can easily prove that if E is a semi-inner product $\mathcal{B}$-module then $\|\cdot\|$ is a semi-norm on E and if E is an inner product $\mathcal{B}$-module then $\|\cdot\|$ is a norm on E.

Let E be a semi-inner product $\mathcal{B}$-module. Consider

$$N = \{x \in E : \langle x, x\rangle = 0\}.$$

Then N is a sub-$\mathcal{B}$-module ($\mathcal{B}$-submodule) of E. Define a $\mathcal{B}$-valued inner product on E/N by

$$\langle x + N, y + N\rangle = \langle x, y\rangle \text{ for } x, y \in E.$$

This inner product makes E/N into an inner product $\mathcal{B}$-module.

Definition 2.2.2. Let $\mathcal{B}$ be a C^*-algebra. An inner product $\mathcal{B}$-module is said to be a *Hilbert $\mathcal{B}$-module* or *Hilbert C^*-module over the C^*-algebra $\mathcal{B}$* if it is complete with respect to the norm: $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$.

Example 2.2.1. (i) Hilbert spaces are Hilbert $\mathbb{C}$-modules.

(ii) Any C^*-algebra $\mathcal{B}$ is a Hilbert $\mathcal{B}$-module with the inner product given by $\langle b, c\rangle = b^*c$.

(iii) Let $\mathcal{H}, \mathcal{G}$ be Hilbert spaces. Let $\mathcal{B}$ be a C^*-subalgebra of $\mathscr{B}(\mathcal{G})$. Let E any subspace of $\mathscr{B}(\mathcal{G}, \mathcal{H})$ such that $E\mathcal{B} \subset E$ and $E^*E \subset \mathcal{B}$. Then E becomes a Hilbert $\mathcal{B}$-module with the right action $SX = S \circ X$ for $S \in E, X \in \mathcal{B}$ and inner product $\langle S, T\rangle = S^* \circ T$ for $S, T \in E$. In particular, $\mathscr{B}(\mathcal{G}, \mathcal{H})$ is a Hilbert $\mathscr{B}(\mathcal{G})$-module (Note that the operator norm and the Hilbert module norm coincide).

(iv) If $\{E_\alpha\}_{\alpha \in \Lambda}$ is a family of Hilbert $\mathcal{B}$-modules. The direct sum of E_α's is defined as

$$\bigoplus_{\alpha \in \Lambda} E_\alpha = \left\{ x = (x_\alpha) : \sum_{\alpha \in \Lambda} \langle x_\alpha, x_\alpha\rangle \text{ converges in } \mathcal{B} \right\},$$

with the module action $(x_\alpha)b = (x_\alpha b)$ and the inner product $\langle (x_\alpha), (y_\alpha)\rangle = \sum_\alpha \langle x_\alpha, y_\alpha\rangle$. For $n \in \mathbb{N}$, we define $E^n := \oplus_{i=1}^n E$.

(v) Let $\mathcal{H}$ be a Hilbert space and $\mathcal{B}$ be a C^*-algebra. Then the closure of the algebraic tensor product $\mathcal{H} \otimes \mathcal{B}$ is a Hilbert $\mathcal{B}$-module with the module action $(h \otimes a)b = h \otimes ab$ for $h \in \mathcal{H}, a, b \in \mathcal{B}$ and the inner product $\langle h \otimes a, g \otimes b \rangle = \langle h, g \rangle a^* b$ for $h, g \in \mathcal{H}, a, b \in \mathcal{B}$.

2.2.1 Operators on Hilbert C^*-modules

Let E be a Hilbert $\mathcal{B}$-module. Then for any $x \in E$ we have, as in the Hilbert space case that

$$\|x\| = \sup\{\|\langle x, y \rangle\| : y \in E, \|y\| \leq 1\}.$$

Hence for any linear map $T : E \to F$ between Hilbert $\mathcal{B}$-modules we have

$$\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \sup_{\|x\|, \|y\| \leq 1} \|\langle y, Tx \rangle\|. \tag{2.2.2}$$

Though Hilbert modules have structures similar to Hilbert spaces they have the following significant differences.

(i) Hilbert spaces are self-dual, that is, each bounded linear functional on a Hilbert space arises by taking inner product with a unique fixed vector $x \in \mathcal{H}$ (namely, $\langle x| : \mathcal{H} \to \mathbb{C}$) but not all Hilbert C^* modules are self-dual (cf. [Pas73]).

(ii) The theory of Hilbert spaces is mainly based on the orthogonal complements of closed subspaces. (Example 2.2.2 shows that) In the module setup, unlike in the Hilbert space case, not all closed submodules are complemented.

(iii) Every bounded linear operator between Hilbert spaces has an (unique) adjoint but for operators between Hilbert modules it is not always the case (See Example 2.2.3).

We shall discuss these more precisely after defining the following natural definitions and notations which are motivated from the theory of Hilbert spaces, to build a theory of Hilbert C^*-modules (analogues to the theory of Hilbert spaces).

Definition 2.2.3. Let $T : E \to F$ be a map between the pre-Hilbert $\mathcal{B}$-modules E and F. We say that T is *right $\mathcal{B}$-linear* or *module map* if T is complex linear and $T(xb) = T(x)b$ for $x \in E, b \in \mathcal{B}$ and we say that T is *bounded* if $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| < \infty$.

2.2.1.1 Self-duality

Notation. For a Hilbert $\mathcal{B}$-module E we define

$$E^r = \{\varphi : E \to \mathcal{B} : \varphi \text{ is bounded and right } \mathcal{B}\text{-linear}\}.$$

$$E^* = \{x^* : E \to \mathcal{B} : x \in E; x^*(y) = \langle x, y\rangle, \text{ for all } y \in E\}.$$

The space E^r is the *space of bounded right $\mathcal{B}$-functionals* or just *$\mathcal{B}$-functionals* and E^* is the *dual* module of E. The map $x \mapsto x^*$ is an anti-linear Banach space isometry from E onto E^*. As every x^* is a $\mathcal{B}$-functional, we have $E^* \subseteq E^r$ and this can be a proper inclusion (cf. [Pas73]).

Definition 2.2.4. A Hilbert $\mathcal{B}$-module E is said to be *self-dual* if $E^r = E^*$.

Self-dual modules have properties in common with both Hilbert spaces and von Neumann algebras. We shall state those results after giving the required definitions.

2.2.1.2 Complementability

Let F be a closed submodule of a Hilbert $\mathcal{B}$-module E. We define the orthogonal complement $F^\perp$ of F, by

$$F^\perp = \{y \in E : \langle x, y\rangle = 0, \forall x \in F\}. \tag{2.2.3}$$

Then $F^\perp$ is also a closed submodule of E.

Definition 2.2.5. Let E be a Hilbert C^*-module over $\mathcal{B}$. A closed submodule F of E is *orthogonally complemented* or *complemented* if $E = F \oplus F^\perp$. F is *topologically complemented* if there is a closed submodule G such that $E = F + G$ and $F \cap G = \{0\}$.

Example 2.2.2 ([Lan95, page 7])**.** Let $\mathcal{B} = C(X)$ for some compact Hausdorff space X. Let Y be a closed nonempty subset of X such that $\overline{Y^c} = X$. Let $E = \mathcal{B}$ and $F = \{f \in \mathcal{B} : f(Y) = \{0\}\} \subseteq E$. Then $F^\perp = \{0\}$. Hence $E \neq F \oplus F^\perp$.

As we have mentioned already, Example 2.2.2 shows that not all closed submodules are complemented. Clearly every complemented submodule is topologically complemented. We shall discuss more about complementability in the subsection about projections.

2.2.1.3 Adjointability

Definition 2.2.6. Let E and F be Hilbert $\mathcal{B}$-modules. A map $T : E \to F$ is said to be *adjointable* if there exists a map $T^* : F \to E$ such that

$$\langle Tx, y\rangle = \langle x, T^*y\rangle \quad \text{for } x \in E, y \in F. \tag{2.2.4}$$

The map T^* is called the *adjoint* of T.

It is easy to see that if T has an adjoint then it is unique. Here we record some additional properties of the adjoint.

Observation 2.2.1. Let E and F be Hilbert $\mathcal{B}$-modules and let $T : E \to F$ be an adjointable map. Then

(i) T is right $\mathcal{B}$-linear.
(ii) T is bounded (follows from Banach-Steinhaus theorem).
(iii) T^* is also adjointable with $(T^*)^* = T$.
(iv) Using (2.2.2) we can prove the same way as it is for the operators on Hilbert spaces, that $\|T\|^2 = \|T^*T\| = \|T^*\|^2$.

Example 2.2.3 ([Lan95])**.** Let E, F be as in Example 2.2.2. Let $i : F \to E$ be the inclusion map. Suppose i were adjointable, then $(i^*(\mathbf{1}))^* f = \langle i^*(\mathbf{1}), f\rangle = \langle \mathbf{1}, f\rangle = f, \forall f \in F$ hence $i^*(\mathbf{1}) = \mathbf{1}$. But $\mathbf{1} \notin F$. Thus there is no adjoint for i.

Notation. Let E and F be Hilbert $\mathcal{B}$ modules.

$$\mathcal{B}^r(E, F) = \{T : E \to F : T \text{ is bounded right } \mathcal{B}\text{-linear}\}$$
$$\mathcal{B}^a(E, F) = \{T : E \to F : T \text{ is adjointable}\}$$

Also we write $\mathcal{B}^r(E)$ and $\mathcal{B}^a(E)$ for $\mathcal{B}^r(E, E)$ and $\mathcal{B}^a(E, E)$ respectively.

Let E and F be Hilbert $\mathcal{B}$-modules. Note that $\mathcal{B}^r(E, \mathcal{B}) = E^r$. Also note that every $x^* \in E^*$ is adjointable with adjoint $(x^*)^* : \mathcal{B} \to E$ given by $b \mapsto xb$. Thus $E^* \subseteq \mathcal{B}^a(E, B)$.

We have $\mathcal{B}^a(E, F) \subseteq \mathcal{B}^r(E, F)$ from the observation 2.2.1(ii). Example 2.2.3 shows that not all bounded maps between Hilbert modules are adjointable. $\mathcal{B}^a(E)$ is a closed subalgebra of the Banach algebra of all bounded maps on E. Indeed using the observation 2.2.1(iv) we can see that $\mathcal{B}^a(E)$ is a C^*-algebra.

Now we shall state some of the well-known results without proof.

Proposition 2.2.4 ([Pas73])**.** *Let E be a self-dual Hilbert $\mathcal{B}$-module and let F be any Hilbert $\mathcal{B}$-module. Then any bounded $\mathcal{B}$-linear map $T : E \to F$ is adjointable.*

Theorem 2.2.1 ([Pas73, Theorem 2.8])**.** *Let $\mathcal{B}$ be a unital C^*-algebra. For a linear map $T : E \to F$ between Hilbert $\mathcal{B}$-modules the following are equivalent:*

(i) $T \in \mathcal{B}^r(E, F)$.
(ii) There exists a real number $k \geq 0$ such that $\langle Tx, Tx\rangle \leq k\langle x, x\rangle$ for all $x \in E$.

Corollary 2.2.1 ([Pas73, Remark 2.9])**.** *Let E and F be Hilbert C^*-modules over a unital C^*-algebra $\mathcal{B}$. If $T \in \mathcal{B}^r(E, F)$ then $\|T\| = \inf\{k^{\frac{1}{2}} : \langle Tx, Tx\rangle \leq k\langle x, x\rangle, \forall x \in E\}$.*

2.2.1.4 Finite rank and compact operators

Now we shall define a class of operators which are analogues to finite rank operators on Hilbert spaces.

Notation. Let E be a Hilbert $\mathcal{B}$-module and let $x, y \in E$. Let

$$|x\rangle : \mathcal{B} \to E \text{ be defined by } b \mapsto xb,$$

$$\langle y| : E \to \mathcal{B} \text{ be defined by } z \mapsto \langle y, z\rangle.$$

Then we have $|x\rangle \in \mathcal{B}^a(\mathcal{B}, E)$ with $|x\rangle^* = \langle x| \in \mathcal{B}^a(E, \mathcal{B})$.

Now let E and F be Hilbert $\mathcal{B}$-modules and let $x \in E, y \in F$. We denote the composition of $|y\rangle$ with $\langle x|$ as $|y\rangle\langle x|$ that is,

$$|y\rangle\langle x| = |y\rangle \circ \langle x| : E \to F \text{ is the operator given by } x' \mapsto y\langle x, x'\rangle.$$

We have $|y\rangle\langle x| \in \mathcal{B}^a(E, F)$ and $|y\rangle\langle x|^* = |x\rangle\langle y| \in \mathcal{B}^a(F, E)$.

We identify $E \subset \mathcal{B}^a(\mathcal{B}, E)$ via $E \ni x \mapsto |x\rangle \in \mathcal{B}^a(\mathcal{B}, E)$ and as $\langle y| = y^* \in F^*$, clearly $F^* \subset \mathcal{B}^a(F, \mathcal{B})$. Therefore sometimes we may write $|x\rangle\langle y|$ as xy^*.

Definition 2.2.7. Let E and F be Hilbert $\mathcal{B}$-modules. The operators $|y\rangle\langle x| \in \mathcal{B}^a(E, F)$ defined above are called the *rank-one operators*. The linear span $\mathcal{F}(E, F)$ of all rank-one operators is called *finite rank operators* and its completion $\mathcal{K}(E, F)$ is called the space of *compact operators*. We write $\mathcal{F}(E)$ and $\mathcal{K}(E)$ instead of $\mathcal{F}(E, E)$ and $\mathcal{K}(E, E)$ respectively.

Remark 2.2.1. Notice that, neither the rank-one operators are of rank 1, nor the finite rank operators are of finite rank in the usual sense. Also in general, the compact operators are not compact in the sense of operators between Banach spaces. However, this space of 'compact operators' is also a Banach space.

Proposition 2.2.5. *Let E and F be Hilbert $\mathcal{B}$-modules. Then we have*

(i) $|y\rangle\langle x|\,|x'\rangle\langle y'| = |y\langle x, x'\rangle\rangle\langle y'| = |y\rangle\langle y'\langle x', x\rangle|$ *for* $x, x' \in E$ *and* $y, y' \in Y$.
(ii) $T\,|x\rangle\langle y| = |Tx\rangle\langle y|$ *for* $x \in E, y \in F$ *and* $T \in \mathcal{B}^a(E)$.
(iii) $|x\rangle\langle y|\,S = |x\rangle\langle S^*y|$ *for* $x \in E, y \in F$ *and* $S \in \mathcal{B}^a(F)$.

The Proposition 2.2.5 shows that $\mathcal{K}(E)$ is an ideal in $\mathcal{B}^a(E)$. Indeed it is a C^*-subalgebra of $\mathcal{B}^a(E)$.

2.2.1.5 Positive operators

Definition 2.2.8. Let E be a Hilbert C^*-module over a C^*-algebra $\mathcal{B}$. A $\mathcal{B}$-linear map $T : E \to E$ is said to be *positive* if $\langle x, Tx \rangle \geq 0$ for all $x \in E$.

If T is positive, T is adjointable and $T^* = T$. If T is positive we denote it by $T \geq 0$. Also if $T, S \in \mathcal{B}^a(E)$ such that $T - S \geq 0$ then we also write $T \geq S$ or $S \leq T$.

Proposition 2.2.6. *For $T \in \mathcal{B}^r(E)$, the following are equivalent:*

(i) T is positive.
(ii) T is positive in the C^*-algebra $\mathcal{B}^a(E)$.

2.2.1.6 Projections

Definition 2.2.9. Let E be a Hilbert $\mathcal{B}$-module. An adjointable linear map $P : E \to E$ is said to be a *projection* if $P = P^* = P^2$.

As P is adjointable it is $\mathcal{B}$-linear. Note that the identity $\|P\| = \|P^*P\| = \|P\|^2$ implies that $\|P\| = 1$ or $\|P\| = 0$.

Proposition 2.2.7. *Let F be a closed submodule of a Hilbert C^*-module E. Then F is complemented in E if and only if there exists a projection $P \in \mathcal{B}^a(E)$ onto F.*

Proposition 2.2.8. *Let E and F be Hilbert $\mathcal{B}$-modules. Suppose $T \in \mathcal{B}^a(E, F)$ has closed range. Then*

(i) $\ker(T)$ *and* $\operatorname{ran}(T)$ *are complemented submodules of E and F respectively.*
(ii) $T^* \in \mathcal{B}^a(F, E)$ has closed range.

Remark 2.2.2. Let E, F be Hilbert C^*-modules and let $T \in \mathcal{B}^a(E, F)$. Then

(i) It is easy to prove $\ker(T^*) = \operatorname{ran}(T)^\perp$. But unlike in the Hilbert spaces, it need not be the case that $\ker(T^*)^\perp = \overline{\operatorname{ran}}(T)$.
(ii) We always have $\overline{\operatorname{ran}}(T) = \overline{\operatorname{ran}}(TT^*)$ and hence $\overline{\operatorname{ran}}(T^*) = \overline{\operatorname{ran}}(T^*T)$.
(iii) If $\operatorname{ran}(T)$ is closed, then
  a) using Proposition 2.2.8 we can prove that $E = \ker(T) \oplus \operatorname{ran}(T^*)$ and $F = \operatorname{ran}(T) \oplus \ker(T^*)$
  b) $\operatorname{ran}(T) = \operatorname{ran}(TT^*)$ hence, since $\operatorname{ran}(T^*)$ is also closed, $\operatorname{ran}(T^*) = \operatorname{ran}(T^*T)$.
(iv) If $\operatorname{ran}(T)$ is not closed, Then neither $\ker(T)$ nor $\overline{\operatorname{ran}}(T)$ need to be complemented.

Definition 2.2.10. A Hilbert $\mathcal{B}$-module is said to be *complementary* if it is complemented in all Hilbert $\mathcal{B}$-modules where it appears as a submodule.

Proposition 2.2.9 ([Ske01, Proposition 1.5.9])**.** *Self-dual Hilbert C^*-modules are complementary.*

2.2.1.7 Isometries, unitaries and partial isometries

Definition 2.2.11. Let E and F be Hilbert $\mathcal{B}$-modules. A map $V : E \to F$ is said to be an *isometry* if

$$\langle Vx, Vy \rangle = \langle x, y \rangle, \quad \text{for all } x, y \in E. \tag{2.2.5}$$

Proposition 2.2.10. *For a map $V : E \to F$ the following are equivalent:*

(i) V is an isometry.
(ii) V is $\mathcal{B}$-linear and $\|Vx\| = \|x\|$ for all $x \in E$.

Isometries need not be adjointable always. So the identity $V^*V = I_E$ need not hold for isometries between Hilbert modules.

Proposition 2.2.11. *An isometry $V : E \to F$ is adjointable if and only if* $\operatorname{ran}(V)$ *is complemented in F.*

Corollary 2.2.2. *For $V : E \to F$ the following are equivalent:*

(i) V is an isometry with complemented range.
(ii) V is adjointable and $V^*V = I_E$.

Definition 2.2.12. A surjective isometry between Hilbert C^*-modules is said to be a *unitary.*

Proposition 2.2.12. *For $U : E \to F$ the following are equivalent.*

(i) U is a unitary.
(ii) U is adjointable and $U^*U = I_E, UU^* = I_F$.

Two Hilbert $\mathcal{B}$-modules E and F are said to be *isomorphic* if there is a unitary $U : E \to F$ and we write $E \simeq F$.

Definition 2.2.13. Let E and F be Hilbert $\mathcal{B}$-modules. A map $V \in \mathcal{B}^a(E, F)$ is said to be a *partial isometry* if $F_0 = \operatorname{ran}(V)$ is complemented in F and there exists a complemented submodule E_0 of E such that $V : E_0 \to F_0$ is unitary and V is zero on $E_0^\perp$.

The following proposition shows some equivalent conditions for a partial isometry (similar to the partial isometries in the Hilbert space operators).

Proposition 2.2.13 ([Lan95])**.** *For $V \in \mathcal{B}^a(E, F)$ the following are equivalent:*

(i) V *is a partial isometry.*
(ii) V^*V *is a projection in* $\mathcal{B}^a(E)$.
(iii) VV^* *is a projection in* $\mathcal{B}^a(F)$.
(iv) $V = VV^*V$.
(v) $V^*VV^* = V^*$.

We have polar decomposition for operators T on Hilbert C^*-modules with the property that the closures of the ranges of T and T^* are both complemented.

Theorem 2.2.2 ([Lan95])**.** *Let E and F be Hilbert $\mathcal{B}$-modules. Let $T \in \mathcal{B}^a(E, F)$ be such that both $\overline{\operatorname{ran}}(T)$ and $\overline{\operatorname{ran}}(T^*)$ are complemented. Then there exists a partial isometry $V : E \to F$ such that $T = V|T|$, where $|T|$ is the positive square root of the operator T^*T in the C^*-algebra $\mathcal{B}^a(E)$.*

2.2.2 Representations on Hilbert C^*-modules

Definition 2.2.14. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras and let E be a Hilbert $\mathcal{B}$-module. A *representation* of $\mathcal{A}$ on E is a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}^a(E)$, and it is *non-degenerate* if $\overline{\operatorname{span}}\, \pi(\mathcal{A})E = E$.

If $\mathcal{A}$ is unital then a representation π of $\mathcal{A}$ is non-degenerate if and only if π is unital. If π is any representation of $\mathcal{A}$ on E, then the submodule $\tilde{E} = \overline{\operatorname{span}}\, \pi(\mathcal{A})E$ is invariant under the action of $\mathcal{A}$ and hence $\pi : \mathcal{A} \to \mathcal{B}^a(\tilde{E})$ is a non-degenerate representation.

Definition 2.2.15. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras. A Hilbert $\mathcal{B}$-module E with a non-degenerate representation $\pi : \mathcal{A} \to \mathcal{B}^a(E)$ is said to be a *Hilbert $\mathcal{A}$-$\mathcal{B}$-module* or *$\mathcal{A}$-$\mathcal{B}$-correspondence.*

If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, π induces a left action on E by $\mathcal{A}$ and we denote the action of $a \in \mathcal{A}$ on x as ax instead of $\pi(a)x$ for all $a \in \mathcal{A}, x \in E$. Note that as π is a representation we have

$$\|ax\| \leq \|\pi(a)\|\|x\| \leq \|a\|\|x\|.$$

Definition 2.2.16. Let E and F be Hilbert $\mathcal{A}$-$\mathcal{B}$-modules. A linear map $T : E \to F$ is said to be $\mathcal{A}$-$\mathcal{B}$-*linear* or *bilinear* or *two sided* if $T(axb) = aT(x)b$ for all $x \in E, a \in \mathcal{A}, b \in \mathcal{B}$.

Notation. $\mathfrak{B}^{a,bil}(E,F)$ = the space of all bounded, adjointable, bilinear maps from E to F. When $F = E$ we write $\mathfrak{B}^{a,bil}(E)$ instead of $\mathfrak{B}^{a,bil}(E,E)$.

The complement of an $\mathcal{A}$-$\mathcal{B}$-submodule in E is an $\mathcal{A}$-$\mathcal{B}$-submodule. The range of a projection P is an $\mathcal{A}$-$\mathcal{B}$-submodule if and only if P is bilinear.

Example 2.2.4. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras.

(i) Any Hilbert $\mathcal{B}$-module E is a Hilbert $\mathfrak{B}^a(E)$-$\mathcal{B}$-module with the left action given by $\pi(\phi)x = \phi(x)$ for $\phi \in \mathfrak{B}^a(E), x \in E$. Moreover $E^n = \oplus_{i=1}^n E$ is an $M_n(\mathfrak{B}^a(E))$-$\mathcal{B}$-module with the similar natural left action.

(ii) Let $\{E_\alpha\}_{\alpha\in\Lambda}$ be a family of Hilbert $\mathcal{A}$-$\mathcal{B}$-modules. Then the direct sum $\bigoplus_{\alpha\in\Lambda} E_\alpha$ of E_α's as defined in Example 2.2.1(iv) is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module with the left action $a(x_\alpha) = (ax_\alpha)$.

(iii) If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then E^n is a Hilbert $M_n(\mathcal{A})$-$\mathcal{B}$-module.

(iv) If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then $E_{(n)}$ ($= E^n$ considered as row vectors) is a Hilbert $\mathcal{A}$-$M_n(B)$-module.

(v) If E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then $M_n(E)$ is a Hilbert $M_n(\mathcal{A})$-$M_n(\mathcal{B})$-module with usual matrix multiplication as module actions and $\langle (x_{ij}), (y_{ij})\rangle = (\sum_k \langle x_{ki}, y_{kj}\rangle)$.

2.2.3 Tensor products of Hilbert C^*-modules

For our purpose we need interior tensor products of Hilbert C^*-modules, hence in this section we present the definitions and some properties of them. For further details about interior tensor products look at [Lan95, Ske01].

2.2.3.1 Interior tensor products

Let E be a Hilbert $\mathcal{B}$-module and F be a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Consider the algebraic tensor product $E \otimes F$. Define right $\mathcal{C}$-module action on $E \otimes F$ by

$$(x \otimes y)c = x \otimes yc \quad \text{for } x \in E, y \in F \text{ and } c \in \mathcal{C},$$

and define a $\mathcal{C}$-valued sesquilinear form on $E \otimes F$ by

$$\langle x \otimes y, x' \otimes y'\rangle = \langle y, \langle x, x'\rangle y'\rangle \quad \text{for } x, x' \in E \text{ and } y, y' \in F. \tag{2.2.6}$$

Then with this sesquilinear form $E \otimes F$ becomes a semi-inner product $\mathcal{C}$-module. Let

$$N_{E\otimes F} = \{z \in E \otimes F : \langle z, z\rangle = 0\}. \tag{2.2.7}$$

The completion of inner product $\mathcal{C}$-module $E\otimes F/N_{E\otimes F}$ is called the *interior tensor product* of the Hilbert modules E and F, and it is denoted by $E \odot_{\mathcal{B}} F$ or just by $E \odot F$. We denote the coset of $x \otimes y$ in $E \odot F$ by $x \odot y$.

If $T : E \to E'$ and $S : F \to F'$ are bounded (bilinear) maps, then, $T \odot S : E \odot F \to E' \odot F'$ is a bounded bilinear map defined by $(T \odot S)(x \odot y) = Tx \odot Sy$ for $x \in E, y \in F$.

Proposition 2.2.14. *Let E_1, E_2 be Hilbert $\mathcal{B}$-modules and let $a \in \mathcal{B}^a(E_1, E_2)$. Let F be a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Then*

$$a \odot \mathrm{id} : (x \odot y) \mapsto ax \odot y \tag{2.2.8}$$

extends to a well-defined adjointable operator $a\odot \mathrm{id} : E_1 \odot F \to E_2 \odot F$ with adjoint $a^* \odot \mathrm{id}$ and

$$\|a \odot \mathrm{id}\| \leq \|a\|. \tag{2.2.9}$$

Moreover, if $E_1 = E_2$ the map $a \mapsto a \odot \mathrm{id}$ is a unital $*$-homomorphism from $\mathcal{B}^a(E) \to \mathcal{B}^a(E \odot F)$.

It follows from Proposition 2.2.14 that, in the above construction of tensor product, if E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, then we have $E \odot F$ as a Hilbert $\mathcal{A}$-$\mathcal{C}$-module with the left action of $\mathcal{A}$ given by

$$a(x \odot y) = ax \odot y \text{ for } x \in E, y \in F \text{ and } a \in \mathcal{A}. \tag{2.2.10}$$

Remark 2.2.3. Let $N_{\mathcal{B}}$ be the subspace of $E \otimes F$ generated by elements of the form

$$xb \otimes y - x \otimes by, \quad x \in E, y \in F, b \in \mathcal{B}.$$

Then we can show that (cf. [Lan95]) $N_{E\otimes F} = N_{\mathcal{B}}$. Therefore (2.2.6) defines an inner product on $E \otimes F/N_{\mathcal{B}}$. Thus we can consider the interior tensor product $E \odot F$ as the completion of the inner product module $E \otimes F/N_{\mathcal{B}}$.

Proposition 2.2.15. *The interior tensor product has the following properties:*

(i) associative, i.e.,

$$(E_1 \odot E_2) \odot E_3 \simeq E_1 \odot (E_2 \odot E_3) \text{ via } (x_1 \odot x_2) \odot x_3 \mapsto x_1 \odot (x_2 \odot x_3).$$

(ii) distributive over addition (direct sum), i.e.,

$$(E_1 \oplus E_2) \odot E \simeq (E_1 \odot E) \oplus (E_2 \odot E) \text{ via } (x_1 \oplus x_2) \odot x \mapsto (x_1 \odot x) \oplus (x_2 \odot x).$$

Observation 2.2.2. Let $\mathcal{B}$ be a C^*-algebra. We identify $E \odot \mathcal{B}$ and E (via $x \odot b \mapsto xb$), also we identify $\mathcal{B} \odot F$ and F (via $b \odot y \mapsto by$).

Observation 2.2.3. Let E be a Hilbert $\mathcal{B}$-module and let F be a Hilbert $\mathcal{B}$-$\mathcal{C}$-module. Let $x \in E$. Then

$$L_x : y \mapsto x \odot y \tag{2.2.11}$$

defines a mapping $F \to E \odot F$ with $\|L_x\| \leq \|x\|$ and $L_x \in \mathcal{B}^a(F, E \odot F)$ with the adjoint given by

$$L_x^* : x' \odot y \mapsto \langle x, x' \rangle y. \tag{2.2.12}$$

Observation 2.2.4. Let E, F, F', G be bilinear Hilbert modules and $\beta : F \to F'$ be a bilinear isometry. Then the map $\mathrm{id} \odot \beta \odot \mathrm{id} : E \odot F \odot G \to E \odot F' \odot G$ is also a bilinear isometry.

2.2.4 GNS-construction

The following construction is due to Paschke in [Pas73], which intimately connects interior tensor products of Hilbert C^*-modules and compositions of CP maps. For next two subsections we mostly follow the notation and set up of [BBLS04, BS00]. For any CP map into $\mathcal{B}(\mathcal{H})$ we have the Stinespring's dilation Theorem 2.1.8. The GNS-construction is a generalization of the Stinespring's dilation to CP maps between arbitrary C^*-algebras.

2.2.4.1 Construction

Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\phi : \mathcal{A} \to \mathcal{B}$ be a CP map. Consider the algebraic tensor product $\mathcal{A} \otimes \mathcal{B}$. For $a, a' \in \mathcal{A}, b, b' \in \mathcal{B}$, define

$$\langle a \otimes b, a' \otimes b' \rangle = b^* \phi(a^* a') b.$$

Then $\langle \cdot, \cdot \rangle$ makes $\mathcal{A} \otimes \mathcal{B}$ into a semi-Hilbert $\mathcal{A}$-$\mathcal{B}$-module in a natural way. Let E be the completion of $\mathcal{A} \otimes \mathcal{B} / \mathcal{N}_{\mathcal{A} \otimes \mathcal{B}}$, where $N_{\mathcal{A} \otimes \mathcal{B}} = \{z \in \mathcal{A} \otimes \mathcal{B} : \langle z, z \rangle = 0\}$ and let $\xi = 1 \otimes 1 + N_{\mathcal{A} \otimes \mathcal{B}}$, then we have

$$\phi(a) = \langle \xi, a\xi \rangle, \quad \text{for all } a \in \mathcal{A}. \tag{2.2.13}$$

Moreover, ξ is cyclic (i.e., $E = \overline{\text{span}}\ \mathcal{A}\xi\mathcal{B}$). The pair (E, ξ) is called the *GNS-construction* of ϕ and E is called the *GNS-module*. It is obvious that ϕ is unital if and only if $\langle \xi, \xi \rangle = 1$.

Definition 2.2.17. Let $\phi : \mathcal{A} \to \mathcal{B}$ be a CP map. Let E be a Hilbert $\mathcal{A}$-$\mathcal{B}$-module and $\xi \in E$, we call (E, ξ) as a *GNS-representation* for ϕ if $\phi(a) = \langle \xi, a\xi \rangle$ for all $a \in \mathcal{A}$. It is said to be *minimal* if

$$E = \overline{\text{span}}\ \mathcal{A}\xi\mathcal{B}.$$

If (E, ξ) and (F, ζ) are two minimal GNS-representations for ϕ then the map $\xi \mapsto \zeta$ extends as a bilinear unitary from E to F.

When we know the Stinespring's representations of two CP maps, the Stinespring's representation of the composition of these two CP maps, is not clear in terms of their individual Stinespring representations. But GNS-construction is very powerful in this regard, as we mentioned before. Namely, the GNS-module of the composition of two CP maps can be written as a submodule of the (interior) tensor product of the GNS-modules of those CP maps, as follows:

Observation 2.2.5. Let $\phi : \mathcal{A} \to \mathcal{B}$ and $\psi : \mathcal{B} \to \mathcal{C}$ be CP maps with GNS-representations (E, ξ) and (F, ζ) respectively. Let (K, κ) be the minimal GNS-representation of $\psi \circ \phi$. Note that

$$\langle \xi \odot \zeta, a\xi \odot \zeta \rangle = \langle \zeta, \langle \xi, a\xi \rangle \zeta \rangle = \langle \zeta, \phi(a)\zeta \rangle = \psi \circ \phi(a), \text{ for all } a \in \mathcal{A}. \tag{2.2.14}$$

This says that $(E \odot F, \xi \odot \zeta)$ is a GNS-representation (not necessarily minimal) for $\psi \circ \phi$. Thus the the mapping

$$\kappa \mapsto \xi \odot \zeta \tag{2.2.15}$$

extends as a unique bilinear isometry from K to $E \odot F$. Hence we may identify K as the submodule $\overline{\text{span}}(\mathcal{A}\xi \odot \zeta\mathcal{C})$ of $E \odot F$.

Note that $E \odot F = \overline{\text{span}}(\mathcal{A}\xi\mathcal{B} \odot \mathcal{B}\zeta\mathcal{C}) = \overline{\text{span}}(\mathcal{A}\xi \odot \mathcal{B}\zeta\mathcal{C}) = \overline{\text{span}}(\mathcal{A}\xi\mathcal{B} \odot \zeta\mathcal{C})$.

2.2.5 von Neumann modules

Let $\mathcal{B}$ be a C^*-algebra and E be a Hilbert $\mathcal{B}$-module. Suppose π is a non-degenerate representation of $\mathcal{B}$ on a Hilbert space $\mathcal{G}$ ($\mathcal{G}$ can be viewed as a Hilbert $\mathcal{B}$-$\mathbb{C}$-module).

Consider the (interior) tensor product $\mathcal{H} = E \odot \mathcal{G}$. Note that $\mathcal{H}$ is a Hilbert space as it is a Hilbert $\mathbb{C}$-module.

For $x \in E$ let $L_x : \mathcal{G} \to \mathcal{H}$ be defined by $L_x(g) = x \odot g$, then $L_x \in \mathcal{B}(\mathcal{G}, \mathcal{H})$ (See Observation 2.2.3) with $L_x^* : x' \odot g \mapsto \langle x, x' \rangle g$ $(= \pi(\langle x, x' \rangle)g)$. Define $\eta : E \to \mathcal{B}(\mathcal{G}, \mathcal{H})$ by $\eta(x) = L_x$. Then we have $L_x^* L_y = \pi(\langle x, y \rangle) \in \mathcal{B}(\mathcal{G})$, hence, if the representation of $\mathcal{B}$ on $\mathcal{G}$ is faithful then so is η. Also we have $L_{xb} = L_x b$ so that we may identify E as a concrete subset of $\mathcal{B}(\mathcal{G}, \mathcal{H})$.

Definition 2.2.18. The map η is referred as the *Stinespring representation of E* (associated with $\mathcal{G}$).

Moreover, when E is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, $\mathcal{H}$ is a Hilbert $\mathcal{A}$-$\mathbb{C}$-module. i.e., $\mathcal{H}$ is a Hilbert space with a (nondegenerate) representation $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ defined by

$$\rho(a)(x \odot g) = ax \odot g. \tag{2.2.16}$$

(Note that $\rho(a) = a \odot \mathrm{id} \in \mathcal{B}(\mathcal{H})$ in the notation of Proposition 2.2.14) In this case we have $L_{axb} = \rho(a) L_x b$ for $x \in E, a \in \mathcal{A}, b \in \mathcal{B}$. Therefore we may identify E as a concrete subset of $\mathcal{B}(\mathcal{G}, \mathcal{H})$ (cf. Example 2.2.1(iii)).

Definition 2.2.19. The map ρ defined above is called the *Stinespring representation of $\mathcal{A}$* (associated with E and $\mathcal{G}$). (See Remark 2.2.4 below)

Remark 2.2.4. Let $\phi : \mathcal{A} \to \mathcal{B}(\mathcal{G})$ be a CP map. Suppose (E, ξ) is the GNS-construction for ϕ. Let $\eta : E \to \mathcal{B}(\mathcal{G}, \mathcal{H})$ be the Stinespring representation of E as defined above, then

$$\phi(a) = \langle \xi, a\xi \rangle = L_\xi^* L_{a\xi} = L_\xi^* \rho(a) L_\xi.$$

Note that L_ξ is an isometry in $\mathcal{B}(\mathcal{G}, \mathcal{H})$ if and only if ϕ is unital. So we obtain the usual minimal Stinespring representation $(\mathcal{H}, \rho, L_\xi)$ of ϕ.

Conversely, if $(\mathcal{H}, \pi, V)$ is the minimal Stinespring representation for ϕ. Consider $\mathcal{B}(\mathcal{G}, \mathcal{H})$ as a Hilbert $\mathcal{A}$-$\mathcal{B}(\mathcal{G})$-module, where the left action of $\mathcal{A}$ is given by the representation π. Let $E = \overline{\operatorname{span}}\, \mathcal{A} V \mathcal{B}(\mathcal{G}) \subseteq \mathcal{B}(\mathcal{G}, \mathcal{H})$. Then (E, V) is a minimal GNS-representation for ϕ.

In Particular, if $\mathcal{B}$ is a von Neumann algebra on a Hilbert space $\mathcal{G}$, we always consider E as a concrete subset of $\mathcal{B}(\mathcal{G}, E \odot \mathcal{G})$.

Definition 2.2.20. Let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. A Hilbert $\mathcal{B}$-module E is said to be a *von Neumann $\mathcal{B}$-module* if E is strongly closed in $\mathcal{B}(\mathcal{G}, E \odot \mathcal{G})$.

Proposition 2.2.16 ([Pas73])**.** *If E is a von Neumann $\mathcal{B}$-module, then $\mathcal{B}^a(E)$ is a von Neumann algebra.*

Definition 2.2.21. Let $\mathcal{A}$ be a von Neumann algebra. A von Neumann $\mathcal{B}$-module E is said to be *von Neumann $\mathcal{A}$-$\mathcal{B}$-module* if it is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module such that the Stinespring representation ρ of $\mathcal{A}$ on $E \odot \mathcal{G}$ is normal.

Theorem 2.2.3 ([Pas73])**.** *von Neumann modules are self-dual. That is, for any $\varphi \in \mathcal{B}^a(E, \mathcal{B})$ there exists $x \in E$ such that $\varphi(y) = x^*(y) = \langle x, y\rangle$ for all $y \in E$. In particular, we have $\mathcal{B}^a(E) = \mathcal{B}^r(E)$.*

Proposition 2.2.17 ([Pas73])**.** *Every von Neumann module has a pre-dual.*

The following propositions are from [BS00].

Proposition 2.2.18. *Let E be a Hilbert $\mathcal{A}$-$\mathcal{B}$-module, where $\mathcal{A}$ is a C^*-algebra and $\mathcal{B}$ is a von Neumann algebra on a Hilbert space $\mathcal{G}$. Then the maps $x \mapsto xb : E \to E$ for $b \in \mathcal{B}$, $x \mapsto ax : E \to E$ for $a \in \mathcal{A}$ and the $\mathcal{B}$-functional $x \mapsto \langle y, x\rangle : E \to \mathcal{B}$ are strongly continuous. Hence $\overline{E}^s \subset \mathcal{B}(\mathcal{G}, \mathcal{H}) \subset \mathcal{B}(\mathcal{G} \oplus \mathcal{H})$ is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module and von Neumann $\mathcal{B}$-module, where $\mathcal{H} = E \odot \mathcal{G}$.*

Proposition 2.2.19. *Let $\phi : \mathcal{A} \to \mathcal{B}$ be a normal CP map between von Neumann algebras. Let E be the GNS-module for ϕ. Then $\overline{E}^s$ is a von Neumann $\mathcal{A}$-$\mathcal{B}$-module.*

Proposition 2.2.20. *Let E be a von Neumann $\mathcal{A}$-$\mathcal{B}$-module. Suppose π is a normal representation of $\mathcal{B}$ on a Hilbert space $\mathcal{G}$. Then the Stinespring representation ρ of $\mathcal{A}$ on $E \odot \mathcal{G}$ is normal.*

Proposition 2.2.21. *Let E be a von Neumann $\mathcal{A}$-$\mathcal{B}$-module and F be a von Neumann $\mathcal{B}$-$\mathcal{C}$-module where $\mathcal{C}$ acts on the Hilbert space $\mathcal{G}$. Then $\overline{E \odot F}^s$, the strong closure of $E \odot F$ in $\mathcal{B}(G, E \odot F \odot \mathcal{G})$ is a von Neumann $\mathcal{A}$-$\mathcal{C}$-module.*

Definition 2.2.22. Due to Propositions 2.2.19, 2.2.20 and 2.2.21 we make the following conventions:

(i) Whenever $\mathcal{B}$ is a von Neumann algebra and $\phi : \mathcal{A} \to \mathcal{B}$ is a CP map, by *GNS-module* we always mean $\overline{E}^s$, where E is the GNS-module, constructed above.

(ii) If E and F are von Neumann modules, by *tensor product* of E and F we mean the strong closure $E \bar{\odot}^s F = \overline{E \odot F}^s$ of $E \odot F$ and we still write $E \odot F$.

2.2.6 Inductive limits

Definition 2.2.23. Let $\mathbb{L}$ be a partially ordered set, which is directed increasingly. A family $(E_t)_{t\in\mathbb{L}}$ of vector spaces with a family $(\beta_{ts})_{t\geq s}$ of linear maps $\beta_{t,s} : E_s \to E_t$, is said to be an *inductive system*[1] over $\mathbb{L}$ if $\beta_{tr}\beta_{rs} = \beta_{ts}$ for all $t \geq r \geq s$ and $\beta_{tt} = \mathrm{id}_{E_t}$.

Let $\mathcal{N}$ denotes the subspace of $E^{\oplus} = \bigoplus\limits_{t\in\mathbb{L}} E_t$, consisting of all those $x = (x_t)$ for which there exists $s \in \mathbb{L}$ (with $s \geq t$ for all t with $x_t \neq 0$) such that $\sum_{t\in\mathbb{L}} \beta_{st}x_t = 0 \in E_s$. The *inductive limit* $\mathcal{E} = \liminf\limits_{t\in\mathbb{L}} E_t$ of the family $(E_t)_{t\in\mathbb{L}}$ is defined as the space $\mathcal{E} = E^{\oplus}/\mathcal{N}$.

Proposition 2.2.22 ([BS00])**.** *The canonical mappings $i_t : E_t \to \mathcal{E}$ have the property: $i_t\beta_{ts} = i_s$ for all $t \geq s$. Also $\mathcal{E} = \bigcup\limits_{t\in\mathbb{L}} i_tE_t$.*

Notation. Let $i : E^{\oplus} \to E$ denote the canonical mapping.

Definition 2.2.23 in a sense is algebraic. So it is referred as *algebraic inductive limit*. However when we have a inductive system of topological spaces, it is necessary to enlarge the inductive limits in order to preserve the structure. e.g. the inductive limits of Hilbert modules need not be complete always. So we make the following conventions.

Definition 2.2.24. (i) The *inductive limit* of an inductive system of Hilbert modules is defined as the norm completion of the algebraic inductive limit.

(ii) The *inductive limit* of an inductive system of von Neumann modules is defined as the strong completion of the algebraic inductive limit.

Observation 2.2.6 ([BS00, Proposition A.6])**.** Let (E_t, β_{ts}) be a inductive system, where E_t's are Hilbert C^*-modules and β_{ts} are isometries. Let $\mathcal{E}$ be the inductive limit of this inductive system. Define

$$\langle x, x'\rangle = \sum_{t,t'} \langle \beta_{st}x_t, \beta_{st'}x'_{t'}\rangle,$$

where $x = i((x_t)), x' = i((x'_t)) \in \mathcal{E}$ and s such that $bx_t = x'_t = 0$ whenever $t > s$. Then $\langle \cdot, \cdot\rangle$ defines an inner product on $\mathcal{E}$ and clearly the canonical maps i_t's are isometries.

Further, if E_t's are two-sided Hilbert C^* modules and β_{ts} are bilinear maps. Then $\mathcal{E}$ is also two-sided, and the canonical mappings i_t respect the left multiplications.

[1] We refer to [BS00, appendix A] for the detailed proofs of the results of this section.

2.3 Quantum dynamical semigroups

We begin with some general theory of one-parameter semigroups of linear maps on Banach spaces. Then in the next subsection we shall move on to our main interest of strongly continuous one-parameter semigroups of CP maps on C^*-algebras.

2.3.1 Some general theory

In this subsection we introduce the standard definitions and some results about the theory of one-parameter semigroups of (continuous) linear maps on Banach spaces. More generally, we have the theory of semigroups of linear maps on locally convex topological vector spaces, but we restrict to Banach spaces. For more details and the proofs of this subsection we refer the reader to [Yos95, Gol17, Dav80].

Definition 2.3.1. Let X be a Banach space. A family $\tau = (\tau_t)_{t\geq 0} \subset \mathcal{B}(X)$ of bounded linear maps in X is called a C_0*-semigroup* or *strongly continuous one-parameter semigroup* on X if:

(i) $\tau_{s+t} = \tau_s \tau_t$ for each $s, t \in \mathbb{R}_+$;
(ii) $\tau(0) = I$, identity operator on X;
(iii) The map $t \mapsto \tau_t(x)$ is continuous for each $x \in X$. (strong continuity)

Definition 2.3.2. A C_0-semigroup τ is said to be *uniformly continuous* if

$$\|\tau_t - I\| \to 0 \text{ as } t \to 0. \tag{2.3.1}$$

Definition 2.3.3. A C_0-semigroup τ on X is called a C_0*-contraction semigroup* on X if $\|\tau_t\| \leq 1$ for all $t \in \mathbb{R}_+$.

Definition 2.3.4. Let τ be a C_0-semigroup on X. The (infinitesimal) *generator* L of τ is defined by

$$L(x) = \lim_{t\to 0} \frac{\tau_t(x) - x}{t}, \quad \text{for } x \in \mathfrak{D}(L) \tag{2.3.2}$$

where, $\mathfrak{D}(L)$ is the domain of L, consisting of $x \in X$, for which the above limit (2.3.2) exists.

Here $\mathfrak{D}(L)$ is a subspace of X and L is a linear map from $\mathfrak{D}(L)$ to X. In general $\mathfrak{D}(L)$ is not equal to X but it is always a dense subspace of X. Also L is always a closed linear operator. A semigroup is uniquely determined by its generator.

Theorem 2.3.1. *Let $L \in \mathcal{B}(X)$. Then the family $\tau = (\tau_t)$ defined by $\tau_t = e^{tL}$ is a uniformly continuous C_0-semigroup with generator L.*

Conversely, if τ is a uniformly continuous C_0-semigroup, then the generator L of τ is a bounded linear operator on X and $\tau_t = e^{tL}$.

Theorem 2.3.1 shows that the C_0-semigroups (satisfying strong continuity, see Definition 2.3.1(iii)) which do not satisfy the uniform continuity condition (2.3.1) are precisely the ones having unbounded generators.

Theorem 2.3.2 (Hille-Yosida [Gol17])**.** *L is the generator of a C_0-contraction semigroup if and only if L is densely defined, closed operator fulfilling, for each $\lambda \in (0, \infty)$, $\lambda \in \sigma(L)^c$ and*[2]

$$\left\|\lambda(\lambda - L)^{-1}\right\| \leq 1.$$

Theorem 2.3.3 ([Gol17])**.** *Let τ be a C_0-semigroup. Then there exist constants $M \geq 1, \omega \geq 0$ such that*

$$\|\tau_t\| \leq Me^{\omega t} \text{ for all } t \in \mathbb{R}_+.$$

Theorem 2.3.4 (Stone's theorem [Gol17])**.** *L is the generator of a C_0-unitary group on a complex Hilbert space $\mathcal{H}$ if and only if L is skew-adjoint (i.e., iL is self-adjoint).*

2.3.2 Quantum dynamical semigroups

In this section we restrict to the case that the Banach space X is a C^*-algebra or a von Neumann algebra $\mathcal{A}$ and we study the semigroups of CP maps. In this section $\mathbb{T}$ denotes $\mathbb{R}_+$ or $\mathbb{Z}_+$. For the details and proofs look at [SG07] and its references.

Definition 2.3.5. Let $\mathcal{A}$ be a unital C^*-algebra. A one-parameter semigroup of contractive CP maps $\phi = (\phi_t)_{t \in \mathbb{T}}$ on $\mathcal{A}$ is said to be a *quantum dynamical semigroup* (QDS). That is, a family $(\phi_t)_{t \in \mathbb{T}}$ of linear maps on $\mathcal{A}$ is said to be a QDS if

(i) ϕ_t is CP for all $t \in \mathbb{T}$;
(ii) $\phi_{s+t} = \phi_s \circ \phi_t$ for all $t \in \mathbb{T}$;
(iii) $\phi_0(a) = a$ for all $a \in \mathcal{A}$;
(iv) $\phi_t(\mathbf{1}) \leq \mathbf{1}$ for all $t \in \mathbb{T}$; (contractivity)

It is said to be *conservative QDS* or *Quantum Markov semigroup* (QMS) if ϕ_t is unital for all $t \in \mathbb{T}$. In practice, in addition to (i)-(iv) we may assume continuity of $t \to \phi_t(a)$ in different topologies, depending upon the context.

[2] See (2.1.6) from Subsection 2.1.2 for the definition of $\sigma(L)$.

Definition 2.3.6. A QDS ϕ is said to be *uniformly continuous QDS* if

$$\|\phi_t - \mathrm{id}\| \to 0 \text{ as } t \to 0. \tag{2.3.3}$$

A QDS ϕ is uniformly continuous if and only if the generator is bounded (cf. Theorem 2.3.1)

Definition 2.3.7. Let $\mathcal{A}$ be a unital C^*-algebra. A one-parameter semigroup $\vartheta = (\vartheta_t)_{t\in\mathbb{T}}$ of unital endomorphisms of $\mathcal{A}$ is said to be an E_0-semigroup.

Clearly every E_0-semigroup is a Quantum Markov Semigroup. If ϕ is a semigroup of CP maps or endomorphisms on a von Neumann algebra $\mathcal{A}$, we typically assume that every ϕ_t is *normal*.

Definition 2.3.8. Let $\phi = (\phi_t)$ be a QDS on a C^*-algebra $\mathcal{A}$. An E_0-semigroup $\vartheta = (\vartheta_t)$ on a C^*-algebra $\mathcal{B} \supseteq \mathcal{A}$ is said to be an E_0*-dilation* of ϕ if there is a projection $P \in \mathcal{B}$ such that $\mathcal{A} = P\mathcal{B}P$ and $\phi_t(a) = P\vartheta_t(a)P$ for all $a \in \mathcal{A}, t \in \mathbb{T}$.

Definition 2.3.9. Let $\mathcal{A}$ be a C^*-subalgebra of a C^*-algebra $\mathcal{B}$. Let $L : \mathcal{A} \to \mathcal{B}$ be a bounded linear adjoint preserving map. Then L is said to be *conditionally completely positive* (CCP) if

$$\sum_{i,j=1}^{n} b_i^* L(a_i^* a_j) b_j \geq 0$$

for all $a_1, \ldots a_n$ in $\mathcal{A}$ and $b_1, \ldots, b_n$ in $\mathcal{B}$ satisfying $\sum_{i=1}^{n} a_i b_i = 0$ and for all $n \in \mathbb{N}$.

Theorem 2.3.5 ([SG07])**.** *A bounded linear adjoint preserving map L on a unital C^*-algebra is CCP if and only if e^{tL} is CP for all $t \geq 0$.*

The structure of generators of uniformly continuous quantum dynamical semigroups on matrix algebras was obtained by Gorini, Kossakowski and Sudershan. This was extended to type I von Neumann algebras by Lindblad. A detailed account of this can be seen in K R Parthasarathy [Par92]. The extension to general C^*-algebras was carried out by Christensen and Evans [CE79]. Here we just state the $\mathcal{B}(\mathcal{H})$ version.

Theorem 2.3.6. *The generator L of a uniformly continuous QDS on $\mathcal{B}(\mathcal{H})$ can be written as*

$$L(T) = \sum_{n=1}^{\infty} R_n^* T R_n + GT + TG^*$$

for $T \in \mathcal{B}(\mathcal{H})$, where $R_n, G \in \mathcal{B}(\mathcal{H})$ such that $-\mathrm{Re}(G)$ is a positive operator and the sum on the right-hand side converges strongly.

2.3.3 Product systems and morphisms

W. Arveson introduced tensor product system of Hilbert spaces in order to classify E_0-semigroups of $\mathcal{B}(\mathcal{H})$. We need to look at product systems of Hilbert C^*-modules to study E_0-semigroups of general C^*-algebras. Analyzing similar structures for quantum dynamical semigroups of C^*-algebras Bhat and Skeide ([BS00]) looked at inclusion systems of Hilbert C^*-modules and their inductive limits to product systems. The name 'inclusion systems' is from ([BM10], and similar objects were called subproduct systems in Solel, Shalit [SS09]). Let $\mathbb{T}$ denote the set of all non-negative real numbers $\mathbb{R}_+$ or the set of all non-negative integers $\mathbb{Z}_+$.

Definition 2.3.10. Let $\mathcal{B}$ be a C^*-algebra (von Neumann algebra). An *inclusion system* (E,β) is a family $E=(E_t)_{t\in\mathbb{T}}$ of Hilbert $\mathcal{B}$-$\mathcal{B}$-modules (von Neumann $\mathcal{B}$-$\mathcal{B}$-modules) with $E_0=\mathcal{B}$ and a family $\beta=(\beta_{s,t})_{s,t\in\mathbb{T}}$ of (adjointable) bilinear isometries $\beta_{s,t}:E_{s+t}\to E_s\odot E_t$ such that, for all $r,s,t\in\mathbb{T}$,

$$(\beta_{r,s}\odot \mathrm{id}_{E_t})\beta_{r+s,t}=(\mathrm{id}_{E_r}\odot\beta_{s,t})\beta_{r,s+t}.$$

It is said to be a *product system* if every β_{st} is unitary.

Recall that: if $\mathcal{B}$ is a von Neumann algebra, and if each E_t is a von Neumann $\mathcal{B}$-$\mathcal{B}$-modules, as we already mentioned, by $E_s\odot E_t$ we mean $E_s\bar{\odot}^s E_t$, the strong closure of $E_s\odot E_t$.

Definition 2.3.11. Let (E,β) be an inclusion system. A family $\xi^\odot=(\xi_t)_{t\in\mathbb{T}}$ of vectors $\xi_t\in E_t$ is called a *unit* for the inclusion system, if $\beta_{s,t}(\xi_{s+t})=\xi_s\odot\xi_t$. It is said to be *unital* if $\langle\xi_t,\xi_t\rangle=1$ for all $t\in\mathbb{T}$, and *generating* if ξ_t is cyclic in E_t for all $t\in\mathbb{T}$. Suppose (E,β) is a product system, a unit $\xi^\odot=(\xi_t)_{t\in\mathbb{T}}$ is said to be a *generating unit for the product system* (E,β) if E_t is spanned by images of elements $b_n\xi_{t_n}\odot\cdots\odot b_1\xi_{t_1}b_0$ $(t_i\in\mathbb{T},\sum t_i=t,b_i\in\mathcal{B})$ under successive applications of appropriate mappings $\mathrm{id}\odot\beta^*_{s,s'}\odot\mathrm{id}$.

Observation 2.3.1. Suppose $\xi^\odot$ is a unit for an inclusion system (E,β). Consider $\phi_t:\mathcal{B}\to\mathcal{B}$ defined by

$$\phi_t(b)=\langle\xi_t,b\xi_t\rangle \text{ for } b\in\mathcal{B}.$$

Then as $\beta_{s,t}$'s are bilinear isometries and $\xi^\odot$ is a unit, for $b\in\mathcal{B}$ we have

$$\phi_t\circ\phi_s(b)=\phi_t(\langle\xi_s,b\xi_s\rangle)=\langle\xi_t,\langle\xi_s,b\xi_s\rangle\xi_t\rangle=\langle\xi_s\odot\xi_t,b(\xi_s\odot\xi_t)\rangle=\langle\xi_{t+s},b\xi_{t+s}\rangle=\phi_{t+s}(b).$$

That is, $(\phi_t)_{t\in\mathbb{T}}$ is a CP semigroup. Further, (ϕ_t) is unital if and only if $\xi^\odot$ is a unital unit.

Example 2.3.1. Let $\phi = (\phi_t)_{t\in\mathbb{T}}$ be a CP semigroup on a C^*-algebra $\mathcal{B}$ and let (E_t, ξ_t) be the GNS-construction for ϕ_t. (If $\mathcal{B}$ is a von Neumann algebra, we assume that ϕ_t's are normal, $E_t = \overline{E}_t^s$ for all t and $E_t \odot E_s = E_t \bar{\odot}^s E_s$.)

Recall that ξ_t is a unit cyclic vector in E_t such that $\phi_t(b) = \langle \xi_t, b\xi_t\rangle$ for all $b \in \mathcal{B}$, and $E_0 = \mathcal{B}$ and $\xi_0 = 1$. Define $\beta_{s,t} : E_{s+t} \to E_s \odot E_t$ by

$$\xi_{t+s} \mapsto \xi_s \odot \xi_t. \tag{2.3.4}$$

Then by Observation 2.2.5, $\beta_{s,t}$'s are bilinear isometries. Now

$$\begin{aligned}(\beta_{r,s} \odot I_{E_t})\beta_{r+s,t}(\xi_{r+s+t}) &= (\beta_{r,s} \odot I_{E_t})(\xi_{r+s} \odot \xi_t) = (\xi_r \odot \xi_s) \odot \xi_t\\ &= \xi_r \odot (\xi_s \odot \xi_t) = (I_{E_r} \odot \beta_{s,t})(\xi_r \odot \xi_{s+t})\\ &= (I_{E_r} \odot \beta_{s,t})\beta_{r,s+t}(\xi_{r+s+t}),\end{aligned}$$

shows that (E, β) is an inclusion system. It is obvious to see that $\xi^\odot = (\xi_t)$ is generating unit for (E, β).

Definition 2.3.12. For a CP semigroup ϕ, the triple $(E, \beta, \xi^\odot)$ given in Example 2.3.1 is called the *inclusion system associated to the CP semigroup* ϕ. When β is clear from the context, we just write $(E, \xi^\odot)$ as the inclusion system associated to the CP semigroup ϕ.

Definition 2.3.13. Let (E, β) and (F, γ) be two inclusion systems. Let $T = (T_t)_{t\in\mathbb{T}}$ be a family of adjointable two-sided (bilinear) maps $T_t : E_t \to F_t$, satisfying $\|T_t\| \le e^{tk}$ for some $k \in \mathbb{R}$. Then T is said to be a *morphism* or a *weak morphism* from (E, β) to (F, γ) if every $\gamma_{s,t}$ is adjointable and

$$T_{s+t} = \gamma^*_{s,t}(T_s \odot T_t)\beta_{s,t} \text{ for all } s, t \in \mathbb{T}. \tag{2.3.5}$$

It is said to be a *strong morphism* if

$$\gamma_{s,t}T_{s+t} = (T_s \odot T_t)\beta_{s,t} \text{ for all } s, t \in \mathbb{T}. \tag{2.3.6}$$

Clearly a strong morphism is also a weak morphism.

2.3.4 E_0-dilation of quantum Markov semigroups

In this subsection we recall the E_0-dilation of a QMS as presented by Bhat and Skeide in [BS00]. Let $\phi = (\phi_t)_{t\in\mathbb{T}}$ be a conservative CP semigroup (QMS). Let (E, β) be the inclusion system associated to ϕ as explained in Definition 2.3.12. First goal is to show that this inclusion system leads to a product system by performing some inductive limits.

Notation. Let $0 < t \in \mathbb{T}$. For any $\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{T}^n$ we denote $|\mathfrak{t}| = \sum_{i=1}^n t_i$. Let

$$\mathbb{I}_t = \{\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{T}^n : n \in \mathbb{N}, t = t_n > \dots > t_1 > 0\}. \tag{2.3.7}$$

$\mathbb{I}_t$ has the natural notion of *inclusion, union* and *intersection* of tuples. Inclusion defines a *partial order* on $\mathbb{I}_t$. Let

$$\mathbb{J}_t = \{\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{T}^n : t_i > 0, |\mathfrak{t}| = t, n \in \mathbb{N}\}. \tag{2.3.8}$$

For $\mathfrak{s} = (s_m, \dots, s_1) \in \mathbb{J}_s$ and $\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{J}_t$ we define the *joint tuple* $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{s+t}$ by

$$\mathfrak{s} \smile \mathfrak{t} = ((s_m, \dots, s_1), (t_n, \dots, t_1)) = (s_m, \dots, s_1, t_n, \dots, t_1).$$

We define a partial order " $\geq$ " on $\mathbb{J}_t$ as follows: $\mathfrak{t} \geq \mathfrak{s} = (s_m, \dots, s_1)$, if for each j $(1 \leq j \leq m)$ there are (unique) $\mathfrak{s}_j \in \mathbb{J}_{s_j}$ such that $\mathfrak{t} = \mathfrak{s}_m \smile \dots \smile \mathfrak{s}_1$.

For $t = 0$ we extend the definitions of $\mathbb{I}_t$ and $\mathbb{J}_t$ as $\mathbb{I}_0 = \mathbb{J}_0 = \{()\}$, where $()$ is the empty tuple. Also for $\mathfrak{t} \in \mathbb{J}_t$ we put $\mathfrak{t} \smile () = \mathfrak{t} = () \smile \mathfrak{t}$.

Proposition 2.3.1 ([BS00, Proposition 4.1])**.** *The mapping* $(t_n, \dots, t_1) \mapsto (\sum_{i=1}^n t_i, \dots, \sum_{i=1}^1 t_i)$ *is an order isomorphism* $\mathbb{J}_t \to \mathbb{I}_t$.

Fix $t > 0$ in $\mathbb{T}$. For $\mathfrak{t} = (t_n, \dots, t_1) \in \mathbb{J}_t$ we define $E_\mathfrak{t} = E_{t_n} \odot \dots \odot E_{t_1}$ and $E_{()} = E_0$. In particular we have $E_{(t)} = E_t$. By Observations 2.2.4 and 2.2.5

$$\xi_t \mapsto \xi_\mathfrak{t} = \xi_{t_n} \odot \dots \odot \xi_{t_1} \tag{2.3.9}$$

defines a bilinear isometry $\beta_{\mathfrak{t}(t)} : E_t \to E_\mathfrak{t}$. Note that $\beta_{\mathfrak{t}(t)}$ is also given by

$$\beta_{\mathfrak{t}(t)} = (\beta_{t_n, t_{n-1}} \odot \mathrm{id})(\beta_{t_n + t_{n-1}, t_{n-2}} \odot \mathrm{id}) \dots (\beta_{t_n + \dots + t_3, t_2} \odot \mathrm{id})\beta_{t_n + \dots + t_2, t_1}.$$

Now suppose $\mathfrak{t} = (t_n, \dots, t_1) = \mathfrak{s}_m \smile \dots \smile \mathfrak{s}_1 \geq \mathfrak{s} = (s_m, \dots, s_1)$ with $|\mathfrak{s}_j| = s_j$, then we define

$$\beta_{\mathfrak{t}\mathfrak{s}} = \beta_{\mathfrak{s}_m(s_m)} \odot \dots \odot \beta_{\mathfrak{s}_1(s_1)}.$$

Then $\beta_{\mathfrak{t}\mathfrak{s}} : E_\mathfrak{s} \to E_\mathfrak{t}$ is a bilinear isometry. Clearly $\beta_{\mathfrak{t}\mathfrak{r}}\beta_{\mathfrak{r}\mathfrak{s}} = \beta_{\mathfrak{t}\mathfrak{s}}$ for all $\mathfrak{t} \geq \mathfrak{r} \geq \mathfrak{s}$. Thus $((E_\mathfrak{t})_{\mathfrak{t} \in \mathbb{J}_t}, (\beta_{\mathfrak{t}\mathfrak{s}})_{\mathfrak{t} \geq \mathfrak{s}})$ forms an inductive system of Hilbert $\mathcal{B}$-$\mathcal{B}$-modules.

Hence, the inductive limit $\mathcal{E}_t = \liminf_{\mathfrak{t} \in \mathbb{J}_t} E_\mathfrak{t}$ is a Hilbert $\mathcal{B}$-$\mathcal{B}$-module and the canonical mappings $i_\mathfrak{t} : E_\mathfrak{t} \to \mathcal{E}_t$ are bilinear isometries (cf. [BS00, Proposition 4.3]). Note in particular that $i_{(t)} : E_t \to \mathcal{E}_t$ is a bilinear isometry. The following observation shows that $\mathcal{E}_t$ has a special vector.

Observation 2.3.2. Let $\xi^t = i_{(t)}\xi_t$. Then it is easy to prove using Proposition 2.2.22 that $i_{\mathfrak{t}}\xi_{\mathfrak{t}} = \xi^t$ for all $\mathfrak{t} \in \mathbb{J}_t$. Moreover,

$$\langle \xi^t, b\xi^t \rangle = \langle i_{(t)}\xi_t, bi_{(t)}\xi_t \rangle = \langle i_{(t)}\xi_t, i_{(t)}b\xi_t \rangle = \langle \xi_t, b\xi_t \rangle = \phi_t(b). \tag{2.3.10}$$

Also as ϕ is unital, $\langle \xi^t, \xi^t \rangle = 1$.

For each $s \in \mathbb{T}$, recall from Proposition 2.2.22, that $\mathcal{E}_s = \overline{\operatorname{span}}\{i_{\mathfrak{s}}E_{\mathfrak{s}} : \mathfrak{s} \in \mathbb{J}_s\}$. For any given $\mathfrak{r} \in \mathbb{J}_{s+t}$, there exist $\mathfrak{s} \in \mathbb{J}_s$ and $\mathfrak{t} \in \mathbb{J}_t$ such that $\mathfrak{s} \smile \mathfrak{t} \geq \mathfrak{r}$. Therefore we have $\mathcal{E}_{s+t} = \overline{\operatorname{span}}\{i_{\mathfrak{s}\smile\mathfrak{t}}(E_{\mathfrak{s}\smile\mathfrak{t}}) : \mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t\}$.

For $\mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t$ clearly $E_{\mathfrak{s}} \odot E_{\mathfrak{t}} = E_{\mathfrak{s}\smile\mathfrak{t}}$. Using this trivial observation we define $u_{st} : \mathcal{E}_s \odot \mathcal{E}_t \to \mathcal{E}_{s+t}$ by

$$u_{st}(i_{\mathfrak{s}}x_{\mathfrak{s}} \odot i_{\mathfrak{t}}y_{\mathfrak{t}}) = i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \quad \text{for } x_{\mathfrak{s}} \in E_{\mathfrak{s}}, y_{\mathfrak{t}} \in E_{\mathfrak{t}}, \mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t. \tag{2.3.11}$$

Indeed, by repeated applications of Proposition 2.2.22, we can prove the following theorem.

Theorem 2.3.7 ([BS00, Theorem 4.8])**.** *With the above notations $(\mathcal{E}^{\odot} = (\mathcal{E}_t)_{t\in\mathbb{T}}, u = (u_{st})_{s,t\in\mathbb{T}})$ forms a product system. The family $\xi^{\odot} = (\xi^t)_{t\in\mathbb{T}}$ forms a generating, unital unit for this product system.*

Remark 2.3.1. Since u_{st} is a unitary for all $s, t \in \mathbb{T}$, we always make the identification:

$$\mathcal{E}_s \odot \mathcal{E}_t = \mathcal{E}_{s+t}. \tag{2.3.12}$$

Now we perform a second inductive limit to patch the $\mathcal{E}_t$'s together and using this new module we get, an E_0-dilation for the QMS ϕ.

Let $s, t \in \mathbb{T}$ with $t \geq s$. Let $\gamma_{ts} = L_{\xi^{t-s}} : \mathcal{E}_s \to \mathcal{E}_{t-s} \odot \mathcal{E}_s = \mathcal{E}_t$, namely $x \mapsto \xi^{t-s} \odot x$ (See Observation 2.2.3 for the notations). Now let $t \geq r \geq s$. As (ξ^t) is a unit we have $\gamma_{ts} = \xi^{t-s} \odot \mathrm{id} = \xi^{t-r} \odot \xi^{r-s} \odot \mathrm{id} = \gamma_{tr}\gamma_{rs}$. Therefore $((\mathcal{E}_t)_{t\in\mathbb{T}}, (\gamma_{ts})_{t\geq s})$ forms an inductive system. The inductive limit $\mathcal{E} = \liminf_{t\to\infty} \mathcal{E}_t$ is a Hilbert $\mathcal{B}$-module and the canonical mappings $k_t : \mathcal{E}_t \to \mathcal{E}$ are isometries (cf. [BS00, Proposition 5.1]). As in Observation 2.3.2, $\mathcal{E}$ also has a special vector: namely $\xi = k_0\xi^0$ satisfying $\xi = k_t\xi^t$, for all $t \in \mathbb{T}$, and $\langle \xi, \xi \rangle = 1$. In the following proposition we use notations from Subsection 2.2.1.4.

Proposition 2.3.2 ([BS00, Corollary 5.3])**.** *The map $j_0 : \mathcal{B} \to \mathcal{B}^a(\mathcal{E})$ defined by $j_0(b) = |\xi\rangle\, b \,\langle\xi|$ is a faithful representation, where ξ is as above. Moreover, the map $a \mapsto j_0(1)aj_0(1)$ defines a conditional expectation $\mathcal{B}^a(\mathcal{E}) \to j_0(\mathcal{B})$.*

Theorem 2.3.8 ([BS00, Theorem 5.4])**.** *For all $t \in \mathbb{T}$ we have*

$$\mathcal{E} \odot \mathcal{E}_t = \mathcal{E}. \tag{2.3.13}$$

extending (2.3.12) *in a natural way. Moreover, $\xi \odot \xi^t = \xi$ for all $t \in \mathbb{T}$.*

Using the identification (2.3.13) we define the following endomorphisms to get an E_0-dilation of the given CP semigroup.

Define $\vartheta_t : \mathcal{B}^a(\mathcal{E}) \to \mathcal{B}^a(\mathcal{E} \odot \mathcal{E}_t) = \mathcal{B}^a(\mathcal{E})$ by

$$\vartheta_t(a) = a \odot \mathrm{id}_{\mathcal{E}_t}, \quad \text{for all } a \in \mathcal{B}^a(\mathcal{E}).$$

Theorem 2.3.9 ([BS00])**.** *$\vartheta = (\vartheta_t)_{t \in \mathbb{T}}$ is an E_0-dilation of the CP semigroup $\phi = (\phi_t)_{t \in \mathbb{T}}$.*

Remark 2.3.2. Let $\mathcal{B}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{G}$. Let $(\phi_t)_{t \in \mathbb{T}}$ be a normal CP semigroup on $\mathcal{B}$. With the conventions of Definitions 2.2.22 and 2.2.24 the whole construction holds.

So in the above dilation theory, replacing C^*-albebra by von Neumann algebra $\mathcal{B}$ and unital CP semigroup by unital normal CP semigroup ϕ, we get the following theorem.

Theorem 2.3.10 ([BS00])**.** *Let (ϕ_t) be a conservative normal CP semigroup on a von Neumann algebra $\mathcal{B}$. Then with the above notations,*

(i) The family $E^{\odot} = (E_t)_{t \in \mathbb{T}}$ forms an inclusion system of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules and $\xi = (\xi_t)_{t \in \mathbb{T}}$ forms a unital, generating unit for this inclusion system. Also $\langle \xi_t, b\xi_t \rangle = \phi_t(b)$.

(ii) The family $\mathcal{E}^{\odot} = (\mathcal{E}_t)_{t \in \mathbb{T}}$ forms a product system of von Neumann modules and $\xi^{\odot} = (\xi^t)_{t \in \mathbb{T}}$ forms a unital, generating unit for this product system. Also $\langle \xi^t, b\xi^t \rangle = \phi_t(b)$.

(iii) We have $\mathcal{E} = \mathcal{E} \odot \mathcal{E}_t$ for all $t \in \mathbb{T}$ and in this identification $\xi = \xi \odot \xi^t$.

(iv) $j_0 : \mathcal{B} \to \mathcal{B}^a(\mathcal{E})$ defined by $j_0(b) = |\xi\rangle\, b\, \langle\xi|$ is a normal representation.

(v) $(\vartheta_t)_{t \in \mathbb{T}}$ (given by $\vartheta_t : \mathcal{B}^a(\mathcal{E}) \to \mathcal{B}^a(\mathcal{E}) \simeq \mathcal{B}^a(\mathcal{E} \odot \mathcal{E}_t)$ via., $(a \mapsto a \odot \mathrm{id}_{\mathcal{E}_t})$) is an E_0-dilation of (ϕ_t). i.e.,

$$j_0(1)\vartheta_t(j_0(b))j_0(1) = j_0(\phi_t(b)), \quad \textit{for all } b \in \mathcal{B}.$$

2.3.5 The time ordered Fock module

The time ordered Fock modules are one of the fundamental examples of product systems of Hilbert C^*-modules.

Definition 2.3.14. Let E be a Hilbert $\mathcal{B}$-$\mathcal{B}$-module. The *full Fock module* over E is defined as

$$\mathcal{F}(E) = \bigoplus_{n=0}^{\infty} E^{\odot n} \tag{2.3.14}$$

where $E^{\odot 0} = \mathcal{B}$ and $\omega = 1 \in E^{\odot 0}$ denotes the *vacuum*. If $\mathcal{B}$ is a von Neumann algebra, then $\mathcal{F}^s(E)$ denotes the von Neumann $\mathcal{B}$-$\mathcal{B}$-module $\overline{\mathcal{F}(E)}^s$.

Definition 2.3.15. Let $T \in \mathcal{B}^{a,bil}(E)$ be a contraction. The *second quantization* of T is defined as

$$\mathcal{F}(T) = \oplus_{n=0}^{\infty} T^{\odot n} \in \mathcal{B}^a(\mathcal{F}(E)),$$

where $T^{\odot 0} = \mathrm{id}$.

Definition 2.3.16. Let E be a Hilbert $\mathcal{B}$-$\mathcal{B}$-module. For $t \in \mathbb{R}$, the *time shift* $\mathcal{S}_t$ in $\mathcal{B}^{a,bil}(L^2(\mathbb{R}, E))$ is defined as $[\mathcal{S}_t f](s) = f(s-t)$.

Definition 2.3.17. Let E be a Hilbert $\mathcal{B}$-$\mathcal{B}$-module and (M, μ) be a measure space. Consider the algebraic tensor product $L^2(M) \otimes E$ as a pre-Hilbert $\mathcal{B}$-$\mathcal{B}$-module with right action $(f \otimes x)b = f \otimes xb$, left action $b(f \otimes x) = f \otimes bx$ and inner product $\langle f \otimes x, g \otimes y \rangle = \langle x, y \rangle \int_M \bar{f} g d\mu$. Let $L^2(M, E)$ denote the completion of this pre-Hilbert module. Note that $L^2(M, E)$ can be treated as the completion of the set of functions on M taking values in E. If E is von Neumann $\mathcal{B}$-$\mathcal{B}$-module, then by $L^{2,s}(M, E)$ we mean the strong closure of $L^2(M) \otimes E$, which is a von Neumann $\mathcal{B}$-$\mathcal{B}$-module.

Let E be a Hilbert $\mathcal{B}$-$\mathcal{B}$-module. Let $I \subseteq \mathbb{R}$ be a measurable subset of $\mathbb{R}$. Inner product in $L^2(I, E)$ is given by $\langle f \otimes x, g \otimes y \rangle = \langle x, y \rangle \int_I \overline{f(t)} g(t) dt$ and $L^2(I, E)^{\odot n} = L^2(I^n, E^{\odot n})$. For $n \in \mathbb{N}$, let Δ_n denote the indicator function of the subset $\{(t_n, \ldots, t_1) : t_n \geq \cdots \geq t_1\}$ of $\mathbb{R}^n$ and let $\Delta_0 = 1$. Then Δ_n can be treated as a multiplication operator on $L^2(I^n)$. Hence Δ_n acts as a projection on $L^2(I, E)^{\odot n}$ so that $\Delta = \oplus_{n=0}^{\infty} \Delta_n$ is the projection of the time ordered part of $\mathcal{F}(L^2(I, E))$.

Definition 2.3.18. The *time ordered Fock module* over E is the two-sided submodule

$$\mathbb{\Gamma}(E) = \Delta(\mathcal{F}(L^2(\mathbb{R}_+, E))) = \bigoplus_{n=0}^{\infty} \Delta_n L^2(\mathbb{R}_+, E)^{\odot n}$$

of $\mathcal{F}(L^2(\mathbb{R}_+, E))$ and the *extended time ordered Fock module* over E is defined as $\Delta\mathcal{F}(L^2(\mathbb{R}, E))$.

Notation. For $t \geq 0$, define $\mathbb{\Gamma}_t(E) := \Delta\mathcal{F}(L^2([0,t), E)) \subset \mathbb{\Gamma}(E)$.

If $\mathcal{B}$ is a von Neumann algebra on a Hilbert space $\mathcal{G}$ and if E is a von Neumann $\mathcal{B}$-$\mathcal{B}$-module, then we consider the strong closures, $\mathbb{\Gamma}^s(E) = \overline{\mathbb{\Gamma}(E)}^s$ and $\mathbb{\Gamma}^s_t(E) = \overline{\mathbb{\Gamma}(E)_t}^s$.

Define $u_{st} : \mathbb{I}\Gamma_s(E) \odot \mathbb{I}\Gamma_t(E) \to \mathbb{I}\Gamma_{s+t}(E)$ by

$$\begin{aligned}[u_{st}(X_s \odot Y_t)](s_m, \dots s_1, t_n \dots, t_n) &= [\mathcal{F}(\mathcal{S}_t)X_s](s_m, \dots, s_1) \odot Y_t(t_n \dots, t_1) \\ &= X_s(s_m - t, \dots, s_1 - t) \odot Y_t(t_n \dots, t_1),\end{aligned}$$

for $s + t \geq s_m \geq \cdots \geq s_1 \geq t \geq t_n \geq \cdots \geq t_1 \geq 0, X_s \in \Delta_m(L^2([0, s], E)^{\odot m}, Y_t \in \Delta_n(L^2([0, t], E)^{\odot n}$. Then u_{st} is unitary for all s, t.

The following theorem by Bhat and Skeide is analogues to the well-known factorization $\Gamma(L^2([0, s+t])) = \Gamma(L^2([t, s+t])) \otimes \Gamma(L^2([0, t]))$ of the symmetric Fock spaces.

Theorem 2.3.11 ([BS00]). *$(\mathbb{I}\Gamma^{\odot}(E) = (\mathbb{I}\Gamma_t(E))_{t \in \mathbb{R}_+}, u = (u_{st})_{s,t \in \mathbb{R}_+})$ is a product system.*

Corollary 2.3.1. *Let E be a von Neumann $\mathcal{B}$-$\mathcal{B}$-module. Let $\mathbb{I}\Gamma^{s\odot}(E) = (\mathbb{I}\Gamma^s_t(E))_{t \in \mathbb{R}_+}$ and let u_{st} be defined as above and extended to the strong closures. Then $(\mathbb{I}\Gamma^{s\odot}(E), u = (u_{st})_{s,t \in \mathbb{R}_+})$ is a product system of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules.*

Theorem 2.3.12 ([LS01]). *Let E be a Hilbert $\mathcal{B}$-$\mathcal{B}$-module. Let $\beta \in \mathcal{B}, \zeta \in E$ and let $\xi^0 = (\xi^0_t)_{t \in \mathbb{R}_+}$ with $\xi^0_t = e^{t\beta}$ be the uniformly continuous semigroup in $\mathcal{B}$ with generator β. Then $\xi^{(\beta,\zeta)\odot} = (\xi^{(\beta,\zeta)}_t)_{t \in \mathbb{R}_+}$ where the component ξ^n_t of $\xi_t = \xi^{(\beta,\zeta)}_t \in \mathbb{I}\Gamma_t(E)$ in the n-particle sector is defined by*

$$\xi^n_t(t_n, \dots, t_1) = \xi^0_{t-t_n}\zeta \odot \xi^0_{t_n - t_{n-1}}\zeta \odot \cdots \odot \xi^0_{t_2 - t_1}\zeta\xi^0_{t_1}, \tag{2.3.15}$$

is a unit for $\mathbb{I}\Gamma^{\odot}(E)$. Moreover, the function $t \mapsto \xi_t \in \mathbb{I}\Gamma_t(E) \subseteq \mathbb{I}\Gamma(E)$ and the CP semigroup $\phi^{(\beta,\zeta)}$ with $\phi^{(\beta,\zeta)}_t(b) = \langle \xi^{(\beta,\zeta)}_t, b\xi^{(\beta,\zeta)}_t \rangle$ are uniformly continuous and the generator $L^{(\beta,\zeta)}$ of $\phi^{(\beta,\zeta)}$ is

$$L^{(\beta,\zeta)}(b) = \langle \zeta, b\zeta \rangle + b\beta + \beta^* b. \tag{2.3.16}$$

Conversely, let $\xi^{\odot}$ be a unit such that $t \mapsto \xi_t \in \mathbb{I}\Gamma(E)$ is a continuous function. Then there exist unique $\beta \in \mathcal{B}$ and $\zeta \in E$ such that $\xi_t = \xi^{(\beta,\zeta)}_t$ as defined by (2.3.15).

Observation 2.3.3. Theorem 2.3.12 holds also for von Neumann $\mathcal{B}$-$\mathcal{B}$-modules E. (cf. [BBLS04, Observation 2.3.14]).

Theorem 2.3.12 says that the continuous units of $\mathbb{I}\Gamma^{\odot}(E)$ is parameterized by $\mathcal{B} \times E$. Let $\mathcal{U}_c(E)$ denote the set of continuous units of $\mathbb{I}\Gamma^{\odot}(E)$. Consider the algebraic subsystem $\mathbb{I}\Gamma^{\mathcal{U}_c\odot}(E) = (\mathbb{I}\Gamma^{\mathcal{U}_c\odot}_t(E))_{t \in \mathbb{R}_+}$ of the time ordered system $\mathbb{I}\Gamma^{\odot}(E)$ which is generated by $\mathcal{U}_c(E)$. Note that any morphism maps a unit to a unit. Using this observation the following theorem is proved in [BBLS04].

Theorem 2.3.13 ([BBLS04])**.** *Let E and E' be Hilbert $\mathcal{B}$-$\mathcal{B}$-modules. Then setting*

$$w_t(\xi_t^{(\beta,\zeta)}) = \xi_t^{(\gamma+\beta+\langle\eta,\zeta\rangle,\eta'+a\zeta)} \tag{2.3.17}$$

we establish a one-to-one correspondence between possibly unbounded continuous morphisms $w^{\odot} = (w_t)_{t\in\mathbb{R}_+}$ from $\mathbb{\Gamma}^{\mathfrak{U}_c\odot}(E)$ to $\mathbb{\Gamma}^{\mathfrak{U}_c\odot}(E')$ and matrices

$$\gamma = \begin{pmatrix} \gamma & \eta^* \\ \eta' & a \end{pmatrix} \in \mathcal{B}^{a,bil}(\mathcal{B}\oplus E, \mathcal{B}\oplus E').$$

Moreover, the adjoint of $w^{\odot}$ is given by the adjoint matrix $\Gamma^* = \begin{pmatrix} \gamma^* & \eta'^* \\ \eta & a^* \end{pmatrix}$.

Chapter

3

Roots of Completely Positive Maps

3.1 Introduction

In many mathematical settings, the concept of a square-root or higher order roots is familiar, e.g. in the context of real numbers, in the context of matrices, in the context of real-valued functions, or measures. What all of these settings have in common is the underlying structure of a semigroup. To be slightly more formal, given a semigroup $(A, \star)$ and given $a \in A$ and $n \in \mathbb{N}$, we can ask whether there exists some $x \in A$ such that $a = x \star x \star \cdots \star x$ (x appearing n times). Then we may call x an n-th root of a. If such a root exists for all n, we would call a infinitely divisible. We may also ask whether there is a one-parameter semigroup $(x_t)_{t\in\mathbb{R}_+}$ in A (namely $x_{s+t} = x_s \star x_t$, for all $s, t \in \mathbb{R}_+$), such that $x_{t_0} = a$ for some $t_0 > 0$. If this is the case then a may be called embeddable (into a continuous semigroup). Finally, if there is a topology on A, we may also look for asymptotic roots or asymptotic embeddability, that is, whether there is a one-parameter semigroup $(x_t)_{t\in\mathbb{R}_+}$, with $\lim_{t\to\infty} x_t = a$.

Yuan [Yua76] deals with some of these questions in the full generality of topological semigroups, but without further structure it seems that one cannot say too much. We would like to add such structure and look at unital normal completely positive (UNCP) maps on von Neumann algebras. They arise in many ways in operator algebras and quantum physics and are natural objects to study (cf. [EK98, Stø13, Wol11]). Since 'composition' is an associative operation on UNCP maps, it makes sense to study the question of roots also in this setting, namely: given a von Neumann algebra $\mathcal{A}$, a number $n \in \mathbb{N}$, and a UNCP map $\phi : \mathcal{A} \to \mathcal{A}$, is there another UNCP map $\psi : \mathcal{A} \to \mathcal{A}$ such that $\phi = \psi^n$? It turns out that currently surprisingly little is known in general.

However, there are a number of connection points with results in related areas. For example, we could specialize to commutative algebras. Then UNCP maps become stochastic maps of Markov chains in classical probability theory. A notable special case is that of (discrete or continuous) convolution semigroups of probability measures. Existence and uniqueness criteria for roots of stochastic maps have been studied earlier (See e.g. [HP01, HL11]). Suppose a given Markov chain on a countable state space converges to an invariant distribution (an absorbing state). Then typically such a convergence happens exponentially (i.e., asymptotically) over time. In discrete time there are some instances when the convergence takes place in finite time. An analysis of transition probability matrices of such Markov chains can be seen in [BG78] and [Sub76]. This does not work in continuous time if the semigroup generator is bounded, [GI88]. But the condition of boundedness of the generator might be too strong, and it is widely believed that under some very minimal continuity assumptions on the transition semigroup, convergence in finite time should be impossible. However, we are not aware of any proof. Surprisingly the non-commutative counterpart is more involved and convergence in finite continuous time to a given pure state is indeed possible as has been shown in [Bha12]; more precisely, for a given normal pure state φ on $\mathcal{B}(\mathcal{H})$, identified with the completely positive map $\phi = \varphi(\cdot)\mathbf{1}$, it is possible to construct a quantum Markov semigroup $(\tau_t)_{t\in\mathbb{R}_+}$ (a strongly continuous one-parameter semigroup of UNCP maps) on $\mathcal{B}(\mathcal{H})$ which coincides with ϕ at all times $t \geq 1$; in other words, for all states ψ, we get $\psi \circ \tau_t = \varphi$, for $t \geq 1$. Hence convergence in the continuous setting is possible in finite time, for all pure states on $\mathcal{B}(\mathcal{H})$. A trivial consequence is that ϕ has an n-th root, for every $n \in \mathbb{N}$. It is natural to ask what happens in the case of $\phi = \varphi(\cdot)\mathbf{1}$ where φ is a mixed state. What can be said about n-th roots or semigroups of roots of ϕ? And in light of our first observation, what can be said about n-th roots and semigroups of roots of more general completely positive maps ϕ, not only those arising from states?

If we drop the assumption of convergence to an invariant distribution, many things can happen. E.g. the question of continuous roots of a given stochastic map makes sense here and is known under the name of Elfving's embedding problem, dating back to 1937 [G37]: given a stochastic map S, when can we find a map L such that $e^L = S$ and all e^{tL}, for $t \geq 0$, are stochastic maps? A number of necessary and sufficient criteria have been found over the years, see e.g. [Dan10, Dav10, Kin62, VB18] for a non-exhaustive list. One particularly interesting condition is: if S is infinitely divisible, i.e., it has n-th roots for all n, then it is embeddable into a semigroup [Kin62]. The non-commutative analogue of this question is not so easy but we may restrict ourselves to finite dimensions to start with. Indeed it should be noted that completely positive (trace-preserving) maps in finite dimensions form the basis of quantum information theory (there termed "quantum channels" [Wol11]).

The question of asymptotic behavior of sequences of compositions of quantum channels appears relevant in quantum information problems, e.g. in the context of entanglement breaking maps [RJP18], and the question of "divisibility" of quantum channels, which is essentially the meaning of a root, has also been studied in a few places. A number of divisibility criteria can be found in [Den88, WC08, WECC08, BC16] and some of the questions we pose here have also been discussed in [WC08,BC16] but with slightly different terminology and complementary answers. Most notably though, it has been shown that the complexity of the problems of deciding whether a given quantum channel (or stochastic map) has a square-root or whether it is embeddable into a continuous semigroup are NP-hard [CEW12a, CEW12b, BC16]. This means that the set of such quantum channels has no simple expression other than explicit enumeration of its elements, and it is impossible to find a simply verifiable criterion for the existence of such roots as in the case of positive numbers or matrices, for example. However, this should not discourage from looking for interesting new relations or characterizations, at least for some special classes of UNCP maps, and that is what we would like to do here.

Our outline for this chapter is as follows. In Section 3.2, we start by describing the quantum analogue and generalization of the "exponential" convergence to a given invariant distribution: given a UNCP map, is there a continuous one-parameter semigroup that converges to ϕ as $t \to \infty$? We completely clarify this question. Such semigroups we will call asymptotic continuous roots. As a byproduct we obtain an affirmative answer to a question of Arveson (Problem 3 in [Arv03, p.387]) through very elementary methods.

We then move on to the question of proper roots in the finite-time setting, where Section 3.3 deals with the n-th root case while Section 3.4 deals with the continuous semigroup case. We are able to provide several existence and non-existence results under different additional assumptions, e.g. regarding the dimension or structure of the algebra or the range of the CP map. In particular, for the case of states on M_d or $\mathcal{B}(\mathcal{H})$ or $\mathbb{C}^d$ we have a complete characterization of existence of n-th roots. However, we are still far from a full understanding and have to leave some questions in the form of conjectures and open problems.

Our diversity of results together with the few related results in literature, e.g. [HL11, CEW12a, CEW12b, BC16], indicate that a "complete and elegant" characterization is unlikely to be found though. We are mainly concerned with the existence or non-existence of roots of UNCP maps. Whenever such roots exist, they are typically far from unique, and a subsequent natural question would be to find a useful characterization of all such roots for a given UNCP map. We deal with UNCP maps (in finite and infinite dimension) and to some minor extent with the commutative special case of stochastic maps though we do not look at the related question of nonnegative roots of (entry-wise) nonnegative matrices as can be found in other places, e.g. [Min88].

3.2 Asymptotic roots

In the present section we work in the C^*-algebraic setting because it appeared more natural to us; however, everything can be adjusted and translated in a straight-forward way to the von Neumann algebraic setting, cf. also Remark 3.2.1 below, which would also bring it more in line with the subsequent sections.

Definition 3.2.1. Given a unital C*-algebra $\mathcal{A}$ and a bounded unital completely positive (UCP) map $\phi : \mathcal{A} \to \mathcal{A}$,

(ad) an *asymptotic discrete root* of ϕ is a UCP map $\tau : \mathcal{A} \to \mathcal{A}$ such that $\tau^n \to \phi$ (pointwise in norm), as $n \to \infty$;

(ac) an *asymptotic continuous root* of ϕ is a uniformly continuous one-parameter semigroup $(\tau_t)_{t\geq 0}$ of UCP maps on $\mathcal{A}$ such that $\tau_t \to \phi$ (pointwise in norm), as $t \to \infty$.

We then have:

Theorem 3.2.1. *Let $\mathcal{A}$ be a unital C^*-algebra and ϕ a UCP map of $\mathcal{A}$. Then the following three statements are equivalent:*

(i) ϕ is idempotent, i.e., $\phi^2 = \phi$;

(ii) ϕ has an asymptotic continuous root;

(iii) ϕ has an asymptotic discrete root.

Proof. (i) $\Rightarrow$ (ii). Suppose $\phi^2 = \phi$. Then define the map

$$\mathcal{L} = \phi - \mathrm{id} = -(\mathrm{id} - \phi) : \mathcal{A} \to \mathcal{A},$$

which is bounded and conditionally completely positive [EK98, Section 4.5] and therefore generates a uniformly continuous UCP semigroup τ (cf. Theorem 2.3.5). We find

$$\mathcal{L}^n(x) = (-1)^n(\mathrm{id} - \phi)^n(x) = (-1)^n(\mathrm{id} - \phi)(x), \quad x \in \mathcal{A},$$

and therefore

$$\tau_t(x) = \sum_{n=0}^{\infty} \frac{(-t)^n(\mathrm{id} - \phi)^n}{n!} x = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!}(\mathrm{id} - \phi)(x) + \phi(x) = e^{-t}(\mathrm{id} - \phi)(x) + \phi(x).$$

We see

$$\|\tau_t(x) - \phi(x)\| = e^{-t}\|x - \phi(x)\| \leq e^{-t}(1 + \|\phi\|)\|x\|, \quad x \in \mathcal{A}, \tag{3.2.1}$$

so $\tau_t \to \phi$ uniformly, as $t \to \infty$, so τ is an asymptotic continuous root for ϕ.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i). Suppose now that there is an asymptotic discrete root τ of ϕ. Then the fact that $\tau^m \to \phi$ as $m \to \infty$ allows us to make the following manipulations:

$$\phi \circ \tau^n(x) = \lim_{m\to\infty} \tau^m \circ \tau^n(x) = \lim_{n+m\to\infty} \tau^{n+m}(x) = \phi(x), \quad x \in \mathcal{A},$$

and therefore

$$\phi^2(x) = \lim_{m\to\infty} \phi \circ \tau^m(x) = \lim_{m\to\infty} \phi(x) = \phi(x), \quad x \in \mathcal{A},$$

so $\phi^2 = \phi$. □

Remark 3.2.1. Let $\mathcal{A}$ be a unital C^*-algebra and ϕ a UCP map of $\mathcal{A}$.

(i) An asymptotic root of ϕ is in general not unique.

(ii) We did not specify the dimension of $\mathcal{A}$ and the Hilbert space $\mathcal{H}$ on which it acts. In fact, the statements are interesting in both finite and infinite dimensions.

(iii) If ϕ has an asymptotic (discrete/continuous) root with respect to the strong operator topology then the above proof shows that ϕ also has an asymptotic (discrete/continuous) root with respect to the uniform topology.

(iv) The definition, theorem and proof continue to hold true upon replacing C^*-algebras by von Neumann algebras, replacing UCP by UNCP, and replacing the uniform by the strong operator topology.

Remark 3.2.2. As a byproduct, the theorem answers Problem 3 in [Arv03, p.387] affirmatively, namely given an eigenvalue list $(\lambda_1, \lambda_2, \ldots)$ with $0 \leq \lambda_i \leq 1$ and $\sum_i \lambda_i = 1$ as in [Arv03, Section 12.4], consider the density matrix

$$D = \mathrm{Diag}(\lambda_1, \lambda_2, \ldots) \in \mathcal{B}(\mathcal{H}),$$

the normal state $\varphi = \mathrm{Tr}(D\cdot)$ on $\mathcal{B}(\mathcal{H})$, and the UNCP map

$$\phi = \varphi(\cdot)\mathbf{1} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}).$$

Then the asymptotic root in the above proof is a UNCP semigroup with bounded generator that has φ as absorbing state: for every normal state ψ on $\mathcal{B}(\mathcal{H})$ and every $x \in \mathcal{B}(\mathcal{H})$, we get

$$|\psi \circ \tau_t(x) - \varphi(x)| = |\psi(\tau_t(x) - \varphi(x)\mathbf{1})| \leq \|\tau_t(x) - \varphi(x)\mathbf{1}\| \leq 2e^{-t}\|x\|$$

where the last inequality follows from (3.2.1); thus,

$$\|\psi \circ \tau_t - \varphi\| \to 0, \quad t \to \infty,$$

meaning that φ is an absorbing state for $(\tau_t)_{t\geq 0}$, which answers Problem 3 in [Arv03, p.387].

3.3 Proper discrete roots

In this and the following section, we work exclusively with von Neumann algebras.

3.3.1 General statements

Our fundamental definition is the following:

Definition 3.3.1. Given a von Neumann algebra $\mathcal{A}$, a UNCP map $\phi : \mathcal{A} \to \mathcal{A}$ and an integer $n \in \mathbb{N} \setminus \{1\}$, a *proper n-th discrete root* of ϕ is a UNCP map $\tau : \mathcal{A} \to \mathcal{A}$ such that $\tau^n = \phi$ and $\tau^k \neq \phi$ for all $k < n$. We call n the *order* of τ.

We need a notational convenience which turns out very important in many proofs and characterizations:

Definition 3.3.2. For every UNCP map ϕ on a von Neumann algebra $\mathcal{A}$, we define the *support projection* as the smallest projection $p_\phi \in \mathcal{A}$ such that $\phi(p_\phi) = \mathbf{1}$. We write $p'_\phi := \mathbf{1} - p_\phi \in \mathcal{A}$.

The existence and the uniqueness of p_ϕ follow from [Dix81, Proposition I.4.3], roughly as follows: one first realizes that the set of $x \in \mathcal{A}$ such that $\phi(x^*x) = 0$ forms a σ-weakly closed left ideal in $\mathcal{A}$. For such ideals there exists a maximal projection p such that the ideal consists of all $x \in \mathcal{A}$ with $x = xp$. This is exactly the projection $p = p'_\phi = \mathbf{1} - p_\phi$ from the preceding definition.

We use the following block matrix decomposition of $x \in \mathcal{A}$:

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} p_\phi x p_\phi & p_\phi x p'_\phi \\ p'_\phi x p_\phi & p'_\phi x p'_\phi \end{pmatrix}. \tag{3.3.1}$$

A first useful fact is the following variation of [BM14, Theorem 4.2] about the relation with nilpotent NCP maps:

Lemma 3.3.1. *Let $\mathcal{A}$ be a von Neumann algebra, ϕ a UNCP map of $\mathcal{A}$ and $n \in \mathbb{N}$. Suppose there exists a proper n-th discrete root τ of ϕ. Then*

(i) $\tau(p_\phi) \geq p_\phi$;

(ii) there also exists a nilpotent NCP map $\alpha : p'_\phi \mathcal{A} p'_\phi \to p'_\phi \mathcal{A} p'_\phi$ of order at most n such that

$$\tau \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha(x) \end{pmatrix}, \quad x \in p'_\phi \mathcal{A} p'_\phi;$$

(iii) for every $\begin{pmatrix} 0 & x \\ y & z \end{pmatrix} \in \mathcal{A}$ w.r.t. to the above block decomposition, there is $\begin{pmatrix} 0 & x' \\ y' & z' \end{pmatrix} \in \mathcal{A}$ such that

$$\tau \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & x' \\ y' & z' \end{pmatrix};$$

(iv) $p_\phi \tau(\cdot) p_\phi$ restricts to a proper discrete root of $p_\phi \phi(\cdot) p_\phi$ on $p_\phi \mathcal{A} p_\phi$ of order at most n.

Proof. (i). We first notice that $\tau(p_\phi) \leq \mathbf{1}$ since τ was assumed to be UNCP. Therefore $0 \leq p_\phi \tau(p_\phi) p_\phi \leq p_\phi$. Let us write $b = p_\phi - p_\phi \tau(p_\phi) p_\phi \geq 0$. We would like to show that $b = 0$.

To start with,

$$\tau \circ \phi = \tau \circ \tau^n = \tau^{n+1} = \tau^n \circ \tau = \phi \circ \tau.$$

This implies

$$\phi(p_\phi) = \mathbf{1} = \tau(\mathbf{1}) = \tau(\phi(p_\phi)) = \phi(\tau(p_\phi)) = \phi(p_\phi \tau(p_\phi) p_\phi),$$

thus

$$\phi(b) = \phi(p_\phi - p_\phi \tau(p_\phi) p_\phi) = 0.$$

Let $e_b \in \mathcal{A}$ be the support projection of b, which can be defined through Borel functional calculus. Notice that $e_b \leq p_\phi$ because $b \leq p_\phi$, so $p_\phi - e_b$ is a subprojection of p_ϕ. Then it follows from the construction of the spectral theorem (in projection-valued measures form [RS72, Section 7.3]) that $\phi(b) = 0$ if and only if $\phi(e_b) = 0$. Since we have already proved $\phi(b) = 0$, we find

$$\phi(p_\phi - e_b) = \phi(p_\phi) - \phi(e_b) = \mathbf{1} - 0 = \mathbf{1}.$$

Thus $p_\phi - e_b$ fulfills the properties of a support projection of ϕ and therefore must be equal to p_ϕ due to its uniqueness, so $e_b = 0$, hence $b = 0$.

(ii). Unitality of τ together with part (i) implies $\tau(p'_\phi) \leq p'_\phi$. Thus, $\tau(p'_\phi \cdot p'_\phi)$ is a NCP map with image in $p'_\phi \mathcal{A} p'_\phi$, hence giving rise to an NCP map

$$\alpha = \tau|_{p'_\phi \mathcal{A} p'_\phi} : p'_\phi \mathcal{A} p'_\phi \to p'_\phi \mathcal{A} p'_\phi.$$

Since τ is an n-th root of ϕ, we have

$$0 = \phi \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \tau^n \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & \alpha^n(x) \end{pmatrix}, \quad x \in p'_\phi \mathcal{A} p'_\phi,$$

implying that α is nilpotent of order at most n.

(iii). Part (i) shows that $\tau(p'_\phi) \leq p'_\phi$. Using the block decomposition in (3.3.1), and Proposition 2.1.7 we can write

$$\begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \geq \tau \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix} \geq \tau\left[\begin{pmatrix} 0 & 0 \\ x^* & 0 \end{pmatrix} \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \right] \geq \tau \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}^* \tau \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix},$$

for every $x \in p_\phi \mathcal{A} p'_\phi$ with $x^* x \leq p'_\phi$. This means that

$$\tau \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x' \\ 0 & z' \end{pmatrix}$$

with certain $x' \in p_\phi \mathcal{A} p'_\phi$, $z' \in p'_\phi \mathcal{A} p'_\phi$. Together with part (ii) and the self-adjointness of τ, we have, for any $x \in p_\phi \mathcal{A} p'_\phi$, $y \in p'_\phi \mathcal{A} p_\phi$, $z \in p'_\phi \mathcal{A} p'_\phi$:

$$\tau \begin{pmatrix} 0 & x \\ y & z \end{pmatrix} = \begin{pmatrix} 0 & x' \\ y' & z' \end{pmatrix}$$

with certain $x' \in p_\phi \mathcal{A} p'_\phi$, $y' \in p'_\phi \mathcal{A} p_\phi$, $z' \in p'_\phi \mathcal{A} p'_\phi$.

(iv). Since $p_\phi \in \mathcal{A}$ and $p_\phi \phi(p_\phi) p_\phi = p_\phi = p_\phi \tau(p_\phi) p_\phi$ by part (i), it is clear that both $p_\phi \phi(\cdot) p_\phi$ and $p_\phi \tau(\cdot) p_\phi$ restrict to UNCP maps on $p_\phi \mathcal{A} p_\phi$. Moreover, it follows from part (iii) that

$$p_\phi \tau(p_\phi \tau(\cdot) p_\phi) p_\phi = p_\phi \tau^2(\cdot) p_\phi,$$

and by induction, since $\tau^n = \phi$, we get that $p_\phi \tau(\cdot) p_\phi$ is a proper discrete root of $p_\phi \phi(\cdot) p_\phi$ of order at most n. □

If ϕ is idempotent then there is generally more hope to say something about roots. A particularly nice case of idempotency is that where ϕ has rank one, namely $\phi = \varphi(\cdot)\mathbf{1}$ for some normal state φ on $\mathcal{A}$. In that case, we get the following easy correspondence:

Lemma 3.3.2. *Given a von Neumann algebra $\mathcal{A}$, a normal state φ on $\mathcal{A}$ and $n \in \mathbb{N}$, let $\phi = \varphi(\cdot)\mathbf{1}$, which is UNCP. Then a map τ on $\mathcal{A}$ is a proper n-th discrete root of ϕ if and only if $\tau = \phi + \alpha$ with α some normal nilpotent map of order n such that $\alpha \circ \phi = 0 = \phi \circ \alpha$ and $\phi + \alpha$ is CP.*

Proof. ($\Rightarrow$) Consider $\alpha = \tau - \phi$. Clearly α is normal. Since $\phi \circ \tau = \tau \circ \phi = \tau^{n+1} = \phi$ we have, for all $k \geq 1$ $\alpha^k = \tau^k - \phi$. Now since $\tau^k \neq \phi$ for $k < n$ and $\tau^n = \phi$ we have $\alpha^k \neq 0$ for $k < n$ and $\alpha^n = 0$. i.e., α is nilpotent of order n. Also $\phi \circ \alpha = \phi \circ (\tau - \phi) = 0 = (\tau - \phi) \circ \phi = \alpha \circ \phi$. The converse part ($\Leftarrow$) is trivial. $\square$

When $d = \dim \mathcal{A} < \infty$, Lemma 3.3.2 shows that any UNCP map arising from a state on $\mathcal{A}$ cannot have proper discrete roots of order higher than d. Indeed the following lemma shows that the order of such a root must be strictly less than d:

Proposition 3.3.1. *Let $\mathcal{A}$ be a finite dimensional von Neumann algebra of dimension d. Let τ be a UNCP map on $\mathcal{A}$. Then the following are equivalent:*

(i) $\tau^n = \phi = \varphi(\cdot)\mathbf{1}$ *for some state* φ *on* $\mathcal{A}$ *and for some* $n \in \mathbb{N}$.
(ii) $\tau = \phi + \alpha$ *for some nilpotent map* α *and* $\phi = \varphi(\cdot)\mathbf{1}$ *for some state* φ *with* $\alpha \circ \phi = 0 = \phi \circ \alpha$.
(iii) 0 *is an eigenvalue of* τ *with algebraic multiplicity* $d - 1$.
(iv) $\operatorname{Tr} \tau^k = 1$ *for all* $k \geq 1$. (Tr *denotes the trace of a matrix*)

In any of these equivalent cases, τ is a root of order at most $d - 1$.

Proof. The idea of the proof is to treat ϕ and τ as linear maps on $\mathbb{C}^d$.

(i) $\Leftrightarrow$ (ii) follows from Lemma 3.3.2.

(i) $\Rightarrow$ (iii). As $\tau^n = \phi$ has rank 1, 0 is an eigenvalue of τ^n of multiplicity $d - 1$, hence 0 is an eigenvalue of τ of multiplicity $d - 1$.

(iii) $\Rightarrow$ (i). Looking at the Jordan normal form of τ it is clear that τ^n has rank 1 for some $n \in \mathbb{N}$. Since τ^n is unital, there is a state φ on $\mathcal{A}$ such that $\tau^n(x) = \varphi(x)\mathbf{1}$ for all $x \in \mathcal{A}$.

(iii) $\Rightarrow$ (iv) is obvious as $\tau(\mathbf{1}) = \mathbf{1}$.

(iv) $\Rightarrow$ (iii). Let $\lambda_1 = 1, \lambda_2, \lambda_3, ..., \lambda_m$ be the distinct eigenvalues of τ with algebraic multiplicity $a_1, a_2, ..., a_m$, respectively. From (iv) we have $(a_1 - 1)\lambda_1^k + a_2\lambda_2^k + \cdots + a_m\lambda_m^k = 0$ for all $k \geq 1$. Consider the Vandermonde matrix $V = (\lambda_i^{j-1})_{1 \leq i,j \leq n} \in \mathrm{M}_m$. Then as the λ_i's are mutually distinct, we have $\det V \neq 0$. Also note that $V((a_1 - 1)\lambda_1, a_2\lambda_2, ..., a_m\lambda_m)' = 0$. This implies that $((a_1 - 1)\lambda_1, a_2\lambda_2, ..., a_m\lambda_m) = 0$. Hence $a_1 = 1, m = 2$ and $\lambda_2 = 0$. That means 0 is an eigenvalue of τ with algebraic multiplicity $d - 1$.

Now regarding our final statement, let τ be a proper n-th discrete root of $\phi = \varphi(\cdot)\mathbf{1}$ on $\mathcal{A}$. It is clear from (iii) $\Leftrightarrow$ (i) that n is the maximal possible size of all Jordan blocks of τ. Hence $n \leq d - 1$. $\square$

Remark 3.3.1. It is worth pointing out that a proper n-th discrete root τ for a state φ is "absorbing", namely $\psi \circ \tau^k = \varphi$, for all $k \geq n$ and all other states ψ. So in this case τ is also an asymptotic discrete root. The same is true for proper versus asymptotic continuous roots, as shall become clear from the following section, cf. Proposition 3.4.2. In general though, there is no clear relationship between proper and asymptotic roots.

Here are some examples regarding existence and non-existence of roots of UNCP maps in finite dimensions. We start with a map which has no nontrivial proper discrete roots at all.

Example 3.3.1. Let $\phi : \mathrm{M}_2 \to \mathrm{M}_2$ be the UNCP map defined by $\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$. We claim that ϕ has no proper discrete root. Suppose for contradiction there exists a proper n-th discrete root τ for ϕ, then $\tau^n = \phi$ and $\tau \circ \phi = \tau \circ \tau^n = \tau^{n+1} = \tau^n \circ \tau = \phi \circ \tau$. Let

$$\tau \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \tau \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix},$$
$$\tau \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}, \quad \tau \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix}.$$

Since $\tau \circ \phi = \phi \circ \tau$ and $\tau(\mathbf{1}) = \mathbf{1}$, we have $a_{12} = a_{21} = d_{12} = d_{21} = 0$ and $a_{11} = d_{22}$ and $d_{11} = a_{22} \neq 0$. It follows that

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \tau^n \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a_{11}^n + * & 0 \\ 0 & * \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \tau^n \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} * & 0 \\ 0 & d_{22}^n + * \end{pmatrix},$$

where all $*$'s are nonnegative terms depending on a_{11} and a_{22} only. In particular we see from these equalities that $a_{11} = d_{22} = 0$ and the only possible solution is

$$\tau \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tau \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

i.e., $\tau = \phi$. Thus ϕ has no proper n-th discrete root.

The following map has only a proper square root.

Example 3.3.2. Let $\phi : \mathrm{M}_2 \to \mathrm{M}_2$ be the idempotent UNCP map defined by $\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$. Then ϕ has a proper square root $\tau \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}$ but ϕ has no other proper discrete roots, which can be proved in the same style as Example 3.3.1.

Finally, a map with proper discrete roots of all orders:

Example 3.3.3. Let $\phi : \mathrm{M}_2 \to \mathrm{M}_2$ be the UNCP map defined by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{c}{2} & d \end{pmatrix}.$$

For every $n \in \mathbb{N}$, define

$$\tau_{1/n} : \mathrm{M}_2 \to \mathrm{M}_2, \quad \tau_{1/n} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \frac{b}{2^{\frac{1}{n}}} \\ \frac{c}{2^{\frac{1}{n}}} & d \end{pmatrix}.$$

Then $\tau_{1/n}$ is a UNCP map and $\tau_{1/n}^n = \phi$, so $\tau_{1/n}$ is a proper n-th discrete root of ϕ.

Example 3.3.4. Let $\phi : \mathrm{M}_2 \to \mathrm{M}_2$ be the UNCP map defined by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & \frac{c}{2} \\ \frac{b}{2} & a \end{pmatrix}.$$

Then ϕ has an n-th root for every odd $n \in \mathbb{N} \setminus \{1\}$ but not for even n. This is again proved in the same way as Example 3.3.1.

So we are led to the following problem:

Problem. Suppose $\mathcal{A} = \mathrm{M}_d$ or $\mathcal{B}(\mathcal{H})$ and ϕ is a UNCP map on $\mathcal{A}$. Then for which $n \in \mathbb{N}$ is there a proper n-th discrete root of ϕ?

Though we have got some illustrative examples here, a general characterization of existence and non-existence of proper discrete roots is expected to be complicated and does involve more details about the map ϕ, as the following subsection indicates. Similar facts have been pointed out in [BC16] and it matches the findings in [HL11, Section 4].

3.3.2 Proper discrete roots for states on M_d and $\mathcal{B}(\mathcal{H})$

We can say much more by specializing the results of the preceding subsection to the setting of normal states on $\mathcal{B}(\mathcal{H})$ or M_d, which we are going to do now.

Theorem 3.3.1. *Suppose $d < \infty$ and φ is a state on M_d of support dimension $r = \dim(p_\phi \mathbb{C}^d)$. Then $\phi = \varphi(\cdot)\mathbf{1}$ has a proper n-th discrete root on M_d if and only if $1 < n \leq d + r^2 - r - 1$.*

Proof. We split the proof into two steps, depending on r. First of all, we may choose and fix a basis $(e_k)_{k=1,\dots,d}$ such that φ is in diagonal form, so $\varphi = \sum_{k=1}^{d} \lambda_k \langle e_k, \cdot e_k\rangle$ and $\lambda_1 \geq \ldots \geq \lambda_r > \lambda_{r+1} = 0 = \ldots = \lambda_d$.

(Step 1) Suppose $r = d$, so φ is faithful. We have to prove that ϕ has a proper n-th discrete root if and only if $1 < n \leq d^2 - 1$. First we see from Lemma 3.3.1 that if τ is a proper n-th discrete root of ϕ then $n \leq d^2 - 1$. We write

$$\alpha = \tau - \phi,$$

which is nilpotent of order n with $\alpha(\mathbf{1}) = 0 = \phi \circ \alpha$ owing to Lemma 3.3.2.

Let us introduce the scalar product

$$\langle \cdot, \cdot \rangle_\varphi : (x, y) \in \mathrm{M}_d \times \mathrm{M}_d \mapsto \varphi(x^* y).$$

Then α restricts to a linear nilpotent map from $\mathrm{M}_d \ominus \mathbb{C}\mathbf{1}$ into itself, and this subspace has dimension $d^2 - 1$. The maximal order of nilpotency is therefore $d^2 - 1$, so $n \leq d^2 - 1$.

Next we would like show that we can actually attain this maximal order. To this end, consider an orthonormal basis $(\mathbf{1}, Y_1, \ldots Y_{d^2-1})$ of M_d with respect to $\langle \cdot, \cdot \rangle_\varphi$ such that $Y_i^* = Y_i$ and $\phi(Y_i) = 0$, for all i, which can always be achieved. Then define

$$\alpha(\mathbf{1}) = \alpha(Y_{d^2-1}) = 0, \quad \alpha(Y_i) = \varepsilon Y_{i+1}, \quad i = 1, \ldots, d^2 - 2,$$

with suitable $\varepsilon > 0$ still to be determined, and

$$\tau = \phi + \alpha.$$

Then it is clear that α is nilpotent of order d^2-1 and so $\tau^{d^2-1}=\phi$ because $\phi\circ\alpha=\alpha\circ\phi$ but $\tau^k\neq\phi$ for $k<d^2-1$. Moreover, α is self-adjoint, namely $\alpha(x^*)=\alpha(x)^*$ for all $x\in\mathrm{M}_d$, thus is τ. In order to show that τ is a proper discrete root, it remains to show that τ is CP. To this end, we compute the Choi matrix $C_\tau\in\mathrm{M}_d(\mathrm{M}_d)$ of τ, cf. [Stø13], and find

$$C_\tau=\begin{pmatrix}\tau(e_{11}) & \dots & \tau(e_{1d})\\ \vdots & & \vdots\\ \tau(e_{d1}) & \dots & \tau(e_{dd})\end{pmatrix}=\begin{pmatrix}\tau(e_{11}^*) & \dots & \tau(e_{d1}^*)\\ \vdots & & \vdots\\ \tau(e_{1d}^*) & \dots & \tau(e_{dd}^*)\end{pmatrix}=\begin{pmatrix}\tau(e_{11})^* & \dots & \tau(e_{d1})^*\\ \vdots & & \vdots\\ \tau(e_{1d})^* & \dots & \tau(e_{dd})^*\end{pmatrix}=C_\tau^*$$

so C_τ is self-adjoint for all ε. We notice that C_τ depends continuously on ε and that for $\varepsilon=0$, we get

$$\begin{pmatrix}\lambda_1\mathbf{1} & 0 & \dots & 0\\ \vdots & & & \vdots\\ 0 & \dots & 0 & \lambda_d\mathbf{1}\end{pmatrix}.$$

This matrix lies in the interior of the convex cone of positive matrices because all $\lambda_i>0$. Choosing $\varepsilon>0$ small enough, we therefore find that C_τ must still be inside this cone. By Choi's theorem, cf. [Stø13], this implies that τ is CP, hence it is a proper discrete root of order d^2-1.

In order to get a proper discrete root of order $n<d^2-1$, all we have to do is change the map α accordingly, e.g

$$\alpha(\mathbf{1})=\alpha(Y_n)=\ldots=\alpha(Y_{d^2-1})=0,\quad \alpha(Y_i)=\varepsilon Y_{i+1},\quad i=1,\ldots,n-1,$$

and proceed in the same way as above.

(Step 2) Next we examine the case $r<d$ and write M_r for $p_\phi\,\mathrm{M}_d\,p_\phi$. Suppose τ is a root of ϕ. Then by Lemma 3.3.1(iv),

$$\tau'=p_\phi\tau(\cdot)p_\phi:\mathrm{M}_r\to\mathrm{M}_r$$

defines a proper discrete root of the faithful state $\varphi|_{\mathrm{M}_r}$ on M_r, hence its maximal order is r^2-1 according to (Step 1) above. As shown in Lemma 3.3.1(iii), we have the following action in block decomposition:

$$\tau\begin{pmatrix}w & x\\ y & z\end{pmatrix}=\begin{pmatrix}\tau'(w) & *\\ * & *\end{pmatrix},$$

in particular

$$\tau^{r^2-1}\begin{pmatrix}w & x\\ y & z\end{pmatrix}=\begin{pmatrix}\varphi|_{\mathrm{M}_r}(w)\mathbf{1} & *\\ * & *\end{pmatrix}.$$

We therefore have to find the minimal number n' such that

$$\tau^{n'}\begin{pmatrix}\mathbf{1} & x\\ y & z\end{pmatrix} = \begin{pmatrix}\mathbf{1} & 0\\ 0 & \mathbf{1}\end{pmatrix}, \tag{3.3.2}$$

for all x, y, z, and we claim that it is at most $d - r$.

To this end, let us write

$$\tau = \sum_{i=1}^{N} L_i^*(\cdot)L_i, \quad L_i = \begin{pmatrix}A_i & B_i\\ C_i & D_i\end{pmatrix}.$$

Since

$$0 = \varphi\circ\tau\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}\end{pmatrix} = \sum_{i=1}^{N}\varphi\begin{pmatrix}C_i^*C_i & C_i^*D_i\\ D_i^*C_i & D_i^*D_i\end{pmatrix} = \sum_{i=1}^{N}\varphi|_{\mathrm{M}_r}(C_i^*C_i)$$

and $\varphi|_{\mathrm{M}_r}$ is faithful, we obtain $C_i = 0$ for all i. Moreover, it follows from Lemma 3.3.1 that $\tau|_{p'_\phi \mathrm{M}_d p'_\phi}$ is nilpotent and CP, and it follows from [BM14, Corollary 2.5] that the order of nilpotency is at most $\dim(p'_\phi\mathcal{H}) = d - r =: r'$. Therefore

$$0 = \tau^{r'}\begin{pmatrix}0 & 0\\ 0 & \mathbf{1}\end{pmatrix} = \sum_{i_1,\dots,i_{r'}=1}^{N}\begin{pmatrix}0 & 0\\ 0 & D_{i_{r'}}^*\cdots D_{i_1}^*D_{i_1}\cdots D_{i_{r'}}\end{pmatrix},$$

so

$$D_{i_1}\cdots D_{i_{r'}} = 0, \tag{3.3.3}$$

for all $i_1,\dots,i_{r'}\in\{1,\dots,N\}$. Moreover, unitality of τ' implies that

$$\sum_{i=1}^{N} A_i^*A_i = \mathbf{1}. \tag{3.3.4}$$

We have $L_{i_1}\cdots L_{i_{k-1}}L_{i_k} = \begin{pmatrix}A_{i_1}\cdots A_{i_k} & M_{i_1,i_2,\dots,i_k}\\ 0 & D_{i_1}D_{i_2}\cdots D_{i_k}\end{pmatrix}$ for every $k\in\mathbb{N}$, where

$$\begin{aligned} M_{i_1,i_2,\dots,i_k} =& A_{i_1}\cdots A_{i_{k-1}}B_{i_k} + A_{i_1}\cdots A_{i_{k-2}}B_{i_{k-1}}D_{i_k} + \dots\\ &+ A_{i_1}B_{i_2}D_{i_3}\cdots D_{i_k} + B_{i_1}D_{i_2}\cdots D_{i_k}.\end{aligned}$$

Now it follows from (3.3.3) and (3.3.4) that

$$\tau^{r'}\begin{pmatrix}\mathbf{1} & x\\ y & z\end{pmatrix} = \sum_{i_1,\dots,i_{r'}=1}^{N}\begin{pmatrix}A_{i_{r'}}^*\cdots A_{i_1}^*A_{i_1}\cdots A_{i_{r'}} & A_{i_{r'}}^*\cdots A_{i_1}^*M_{i_1,i_2,\dots,i_{r'}}\\ M_{i_1,i_2,\dots,i_{r'}}^*A_{i_1}\cdots A_{i_{r'}} & M_{i_1,i_2,\dots,i_{r'}}^*M_{i_1,i_2,\dots,i_{r'}}\end{pmatrix}.$$

Furthermore,

$$\begin{aligned}\sum_{i_0,i_1,\dots,i_{r'}=1}^{N} M^*_{i_0,i_1,i_2,\dots,i_{r'}} M_{i_0,i_1,i_2,\dots,i_{r'}} &= \sum_{i_0,i_1,\dots,i_{r'}=1}^{N} M^*_{i_1,i_2,\dots,i_{r'}} A^*_{i_0} A_{i_0} M_{i_1,i_2,\dots,i_{r'}} \\ &= \sum_{i_1,\dots,i_{r'}=1}^{N} M^*_{i_1,i_2,\dots,i_{r'}} M_{i_1,i_2,\dots,i_{r'}}.\end{aligned}$$

Similarly

$$\begin{aligned}\sum_{i_0,i_1,\dots,i_{r'}=1}^{N} M^*_{i_0,i_1,i_2,\dots,i_{r'}} A_{i_0} A_{i_1} \cdots A_{i_{r'}} &= \sum_{i_0,i_1,\dots,i_{r'}=1}^{N} M^*_{i_1,i_2,\dots,i_{r'}} A^*_{i_0} A_{i_0} A_{i_1} \cdots A_{i_{r'}} \\ &= \sum_{i_1,\dots,i_{r'}=1}^{N} M^*_{i_1,i_2,\dots,i_{r'}} A_{i_1} \cdots A_{i_{r'}}.\end{aligned}$$

By induction we find that

$$\tau^{r'+k}\begin{pmatrix} \mathbf{1} & x \\ y & z \end{pmatrix} = \tau^{r'}\begin{pmatrix} \mathbf{1} & x \\ y & z \end{pmatrix} = \begin{pmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix}, \quad \forall k \in \mathbb{N}.$$

and together with (3.3.2) we see that n' can be at most r', so the order of τ on M_d can be at most $r^2 - 1 + r' = r^2 - 1 + d - r$.

It remains to show that all orders $n = 2, \dots, r^2 - 1 + d - r$ can be attained. First of all, following the ideas in (Step 1) and given a root of order $n = 1, \dots, r^2 - 1$ on M_r, there is $l = 1, \dots, r$ and $w \in \mathrm{M}_r$ such that

$$\Big(\sum_{i_1,\dots,i_{n-2}=1}^{N} A^*_{i_{n-2}} \cdots A^*_{i_1} w A_{i_1} \cdots A_{i_{n-2}}\Big)_{ll} \neq \Big(\sum_{i_1,\dots,i_{n-1}=1}^{N} A^*_{i_{n-1}} \cdots A^*_{i_1} w A_{i_1} \cdots A_{i_{n-1}}\Big)_{ll}.$$

Then setting all $D_i = 0$ and $B_i = e_{l,i}$ for $i = 1, \dots r'$, we can obtain roots of orders $n+1 = 2, \dots, r^2$ on M_d. In order to get order $n = r^2 + n'$, we keep $B_i = e_{l,i}$ for $i = 1, \dots r'$ and choose for D_1 any contractive nilpotent matrix of order $n' + 1$ and all other $D_i = 0$. This way we achieve

$$\begin{aligned}\sum_{i_1,\dots,i_{n'+1}=1}^{N} M^*_{i_1,i_2,\dots,i_{n'+1}} M_{i_1,i_2,\dots,i_{n'+1}} &= \sum_{i_1,\dots,i_{n'}=1}^{N} M^*_{i_1,i_2,\dots,i_{n'}} M_{i_1,i_2,\dots,i_{n'}} \\ &\neq \sum_{i_1,\dots,i_{n'-1}=1}^{N} M^*_{i_1,i_2,\dots,i_{n'-1}} M_{i_1,i_2,\dots,i_{n'-1}},\end{aligned}$$

so in total we have a root τ of order $r^2 + n'$, completing the proof of the theorem. $\square$

We can adapt the construction in the preceding proof to obtain the corresponding statement in $\mathcal{B}(\mathcal{H})$ as follows:

Theorem 3.3.2. *Suppose $\mathcal{H}$ is infinite-dimensional separable and φ is a normal state on $\mathcal{B}(\mathcal{H})$. Then $\phi = \varphi(\cdot)\mathbf{1}$ has a proper n-th discrete root on $\mathcal{B}(\mathcal{H})$, for every $n \in \mathbb{N}$.*

Proof. Let $r = \dim(p_\phi \mathcal{H})$ and $r' = \dim(p'_\phi \mathcal{H})$. We distinguish two cases.

Case $r' = \infty$. Here we choose α as a contractive nilpotent CP map of order n on $B(p'_\phi \mathcal{H})$. We define

$$\tau \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} \varphi|_{B(p_\phi \mathcal{H})}(w)\mathbf{1} & 0 \\ 0 & \alpha(z) + \varphi|_{B(p_\phi \mathcal{H})}(w)(\mathbf{1} - \alpha(\mathbf{1})) \end{pmatrix}$$

Then τ is a proper n-th discrete root.

Case $r' < \infty$. Then $r = \infty$ and we may assume as in the proof of Theorem 3.3.1 that the density matrix is in diagonal form with respect to a fixed orthonormal basis (e_i) of $\mathcal{H}$ and with entries $\lambda_1 \geq \lambda_2 \geq \ldots$. Consider the projection p_n onto $\operatorname{span}\{e_1, \ldots, e_n\}$. Then

$$\varphi_n = \frac{1}{\varphi(p_n)} \varphi|_{B(p_n \mathcal{H})}$$

defines a faithful state on $B(p_n \mathcal{H})$. We may then proceed as in (Step 1) of the proof of Theorem 3.3.1 to find a nilpotent map $\alpha_n : B(p_n \mathcal{H}) \to B(p_n \mathcal{H})$ of order n such that $\alpha_n(p_n) = 0 = \varphi_n \circ \alpha_n$. We rescale α_n by $\varphi(p_n)$ and extend it trivially to $\mathcal{B}(\mathcal{H}) \ominus B(p_n \mathcal{H})$ and denote the resulting normal nilpotent map by α. Then

$$\tau = \phi + \alpha$$

is a proper n-th discrete root of ϕ. $\square$

3.3.3 Classical probability theory – proper discrete roots of states on finite-dimensional commutative von Neumann algebras

We would like to briefly specialize our general findings to the case of finite classical probability spaces because also here we get some interesting results. Note that a map $\tau : \mathbb{C}^d \to \mathbb{C}^d$ is UCP if and only if τ is a stochastic matrix[1] and a map $\varphi : \mathbb{C}^d \to \mathbb{C}$ is a state if and

[1] Look at Subsection 2.1.1 for the details about stochastic matrices.

only if there is a probability vector $p = (p_1, p_2, ..., p_d)' \in \mathbb{C}^d$ such that $\varphi(x) = \langle p, x\rangle$, for all $x \in \mathbb{C}^d$.

In this subsection, we will use the following special notation. For $x \in \mathbb{C}^n, y \in \mathbb{C}^m$ we define $|x\rangle\langle y| = xy^* \in \mathrm{M}_{n,m}$ (See the notations defined in Subsection 2.1.3). For any $x = (x_1, ..., x_d)' \in \mathbb{C}^d$ and $m < d$, we write $x^{(m)} = (x_1, ..., x_m)' \in \mathbb{C}^m$. We write $\mathbf{1}$ for the unit matrix but also for the unit vector $(1, \ldots, 1)' \in \mathbb{C}^d$. Sometimes we will add subscripts or superscripts to 0 and $\mathbf{1}$ in order to indicate the space on which it is acting but we try to avoid this when it is obvious from the context.

As according to Lemma 3.3.1, a state on $\mathbb{C}^d$ can have proper discrete roots only up to order $d-1$, the states on $\mathbb{C}$ and $\mathbb{C}^2$ will not have any proper discrete roots. The following example is a construction of proper n-th discrete roots of states on $\mathbb{C}^d$, for all $2 \le n \le d-1$ and $d > 2$.

Example 3.3.5. Let $d > 2$ and φ be a state on $\mathbb{C}^d$ given by a probability vector $p = (p_1, p_2, ..., p_d)'$. Let $\phi = \varphi(\cdot)\mathbf{1}$. Then ϕ is the stochastic matrix $\phi = |\mathbf{1}\rangle\langle p|$.

First let us consider the case when φ is faithful, i.e., $p = (p_1, p_2, ..., p_d)'$ with $p_i > 0$, for all i. Let $2 \le n \le d-1$. Note that $\phi = |\mathbf{1}\rangle\langle p|$ is diagonalizable and of rank one, so we can write $\phi = S \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} S^{-1}$, with a suitable invertible matrix S. Consider a nilpotent matrix $\alpha_0 \in \mathrm{M}_{d-1}$ of order n and let $\alpha = \varepsilon S \begin{pmatrix} 0 & 0 \\ 0 & \alpha_0 \end{pmatrix} S^{-1}$. If $\varepsilon > 0$ is small enough then all entries of $\phi + \alpha$ are non-negative because φ was assumed to be faithful. By construction we have got $\phi \circ \alpha = 0 = \alpha \circ \phi$ and hence by Lemma 3.3.2, $\tau = \phi + \alpha$ is a proper n-th discrete root of ϕ.

Now let us assume that φ is not faithful. Without loss of generality we can assume that $p = (p_1, p_2, ..., p_r, 0, ..., 0)', p_i > 0$ for all $i = 1, 2, ..., r < d$. Let us consider two separate cases, namely $r \le 2$ and $2 < r < d$, because our construction of n-th roots work differently in these two cases.

Case $r \le 2$. Given $r \le n \le d-1$, let

$$\tau = \begin{pmatrix} \left|\mathbf{1}^{(r)}\right\rangle\left\langle p^{(r)}\right| & 0 \\ \left|y(d-n)\right\rangle\left\langle e_1^{(r)}\right| & S_n \end{pmatrix}$$

where $y(d-n) = \begin{pmatrix} \mathbf{1}^{(d-n)} \\ 0^{(n-r)} \end{pmatrix} \in \mathbb{C}^{d-r}$, $S_n \in \mathrm{M}_{d-r}$ is the operator defined by $S_n(e_i^{(d-r)}) = 0$

for $i = d-r, 1, 2, , ..., d-n-1$ and $S_n(e_i^{(d-r)}) = e_{i+1}^{(d-r)}$ for $i = d-n, d-n+1, ..., d-r-1$ and $e_i^{(d-r)}$ is the i-th canonical basis vector in $\mathbb{C}^{d-r}$. Then τ is a proper n-th discrete root of ϕ. (Note that when $n = r$, we have $y(d-n) = \mathbf{1}^{(d-r)}$ and $S_n = 0$.)

Case $r > 2$. Given $2 \leq n \leq d-1$, decompose $n = n_1 + n_2$, with suitable $1 \leq n_1 \leq r-1$ and $1 \leq n_2 \leq d-r$. Let $\tau_{[r,n_1]}$ be an n_1-th root of $\left|\mathbf{1}^{(r)}\middle\rangle\middle\langle p^{(r)}\right|$ as in the case of faithful φ above. Then we define

$$\tau = \begin{pmatrix} \tau_{[r,n_1]} & 0 \\ \left|y(d-n_2)\middle\rangle\middle\langle e_j^{(r)}\right| & S_{n_2} \end{pmatrix}$$

where $y(d-n_2)$ and S_{n_2} are as in the previous case and j is chosen as follows: if $n_1 \geq 2$ then choose j such that the j-th row of $\tau_{[r,n_1]}^{n_1-1}$ is different from $p^{(r)\prime}$, while for $n_1 = 1$ we choose $j = 1$. Then τ is a proper n-th discrete root of ϕ.

We summarize the result of the preceding example as follows:

Theorem 3.3.3. *A state φ on $\mathbb{C}^d$ has a proper n-th discrete root if and only if $2 \leq n \leq d-1$. Or in more probabilistic terms: given a probability distribution p on a probability space with d elements, there is a stochastic map S that leaves p invariant and such that $S^n = |\mathbf{1}\rangle\langle p|$ and $S^k \neq |\mathbf{1}\rangle\langle p|$ for $k < n$ if and only if $2 \leq n \leq d-1$.*

Example 3.3.6. Let $\tau = \frac{1}{24}\begin{pmatrix} 5 & 5 & 14 \\ 11 & 11 & 2 \\ 8 & 8 & 8 \end{pmatrix}$. Then τ is a stochastic matrix such that $\tau^k = \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}$ for all $k \geq 2$.

In Theorem 3.3.3, φ may be regarded as a stochastic matrix of rank 1. For stochastic matrices of rank > 1, we have no complete and simple characterization though some partial characterizations with necessary or sufficient conditions are known, e.g. in [HL11]. The case of rank d is closely related to Elfving's embedding problem [Dan10, G37].

3.4 Proper continuous roots

We continue to use the notation from Section 3.3.

Definition 3.4.1. Given a von Neumann algebra $\mathcal{A}$ and a UNCP map $\phi : \mathcal{A} \to \mathcal{A}$, a *proper continuous root* of ϕ is a strongly-continuous one-parameter semigroup $(\tau_t)_{t\geq 0}$ of UNCP maps on $\mathcal{A}$ such that $\tau_1 = \phi$ and $\tau_t \neq \phi$, for all $0 < t < 1$.

In this definition one might also consider seemingly more general semigroups with $\tau_{t_0} = \phi$ for some $t_0 > 0$. However, since we can always reduce the situation to the case $t_0 = 1$ by rescaling, we decided to keep things simple and consider only the case $t_0 = 1$. For more information on strongly continuous one-parameter semigroups in general, we refer the reader to [Arv03, Dav80].

Proposition 3.4.1. *Let $\mathcal{A}$ be a finite-dimensional von Neumann algebra and $\phi : \mathcal{A} \to \mathcal{A}$ a UNCP map. Then the following are equivalent:*

(i) ϕ has a proper continuous root;
(ii) ϕ is bijective and has a proper n-th discrete root, for every $n \in \mathbb{N} \setminus \{1\}$.

Proof. (i) $\Rightarrow$ (ii). If $(\tau_t)_{t\geq 0}$ is a proper continuous root, then it must be a uniformly continuous UNCP semigroup, hence of the form $\tau_t = e^{t\mathcal{L}}$ with some (bounded) conditionally completely positive generator $\mathcal{L}$, cf. [EK98, Section 4.5], so $e^{-\mathcal{L}}$ is an inverse of ϕ (in the sense of linear maps on $\mathcal{A}$) and $\tau_{1/n}$ is a proper n-th discrete root of ϕ, for every $n \in \mathbb{N}$.

(ii) $\Rightarrow$ (i). If ϕ has a proper n-th discrete root for every $n \in \mathbb{N}$ (this is called infinitely divisible in [Den88]) then according to [Den88, Corollary 4] there are a conditional expectation $E : \mathcal{A} \to \mathcal{A}$ and a conditionally completely positive generator $\mathcal{L}$ such that $\phi = e^{\mathcal{L}}E$. Since ϕ is invertible, so is E and hence E must be the identity map because $\mathcal{A}$ is finite-dimensional. Thus we may choose $\tau_t = e^{t\mathcal{L}}$, for all $t \geq 0$, to obtain a proper continuous root of ϕ. $\square$

Remark 3.4.1. In the classical case, namely if $\mathcal{A}$ is commutative, ϕ is automatically bijective if it has a proper n-th discrete root for every $n \in \mathbb{N}$. This is one of the characterizations of Markovianity in the context of Elfving's embedding problem due to Kingman [Kin62, Proposition 7]. On the other hand, in the non-commutative case, bijectivity is not automatic. E.g. consider

$$\phi : \mathrm{M}_3 \to \mathrm{M}_3, \quad \phi(x) = \begin{pmatrix} x_{11} & 0 & 0 \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}.$$

This has proper n-th roots for all n but is clearly not bijective. Similarly, we see that the map

$$\phi : \mathrm{M}_2 \to \mathrm{M}_2, \quad \phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & c/2 \\ b/2 & a \end{pmatrix}$$

from Example 3.3.4 is bijective but has proper n-th roots only for odd $n \in \mathbb{N} \setminus \{1\}$, hence it has no proper continuous root.

The following example provides a bijective UNCP map in finite dimensions where the conditions in the proposition are verified. In fact, it is a simple "interpolation" of the construction in Example 3.3.3:

Example 3.4.1. Let $\phi : \mathrm{M}_2 \to \mathrm{M}_2$ be the UNCP map defined by

$$\phi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \frac{b}{2} \\ \frac{c}{2} & d \end{pmatrix}.$$

For every $t \in [0, \infty)$, define

$$\tau_t : \mathrm{M}_2 \to \mathrm{M}_2, \quad \tau_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & \frac{b}{2^t} \\ \frac{c}{2^t} & d \end{pmatrix}.$$

Then $(\tau_t)_{t\geq 0}$ is a proper continuous root of ϕ, namely $\tau_1 = \phi$ and the semigroup property and continuity are a straight-forward verification.

Embedding this example into a higher (possibly infinite) dimensional space, we can get continuous roots for certain UNCP maps in higher dimensions as well. A more complete criterion as to when such continuous roots exist seems out of reach. Notice that this might be even more difficult than Problem 3.3.1.

Yet if ϕ arises from a state, we can say a little bit more:

Proposition 3.4.2. *Let $\mathcal{A}$ be a von Neumann algebra, φ a state on $\mathcal{A}$ and $\phi = \varphi(\cdot)\mathbf{1}$. If $(\tau_t)_{t\geq 0}$ is a proper continuous root of ϕ then*

(i) $\varphi \circ \tau_t = \varphi$, for every $t \geq 0$, i.e., φ is τ-invariant;

(ii) $\psi \circ \tau_t = \phi$, for every $t \geq 1$ and every UNCP map ψ, i.e., all UNCP maps converge to ϕ in finite time: $\tau_t = \phi$, for all $t \geq 1$.

Proof. (i) Since $\phi = \tau_1$, for all $t \geq 1$, we get from the linearity and the semigroup properties of τ:

$$\mathbf{1}\varphi \circ \tau_t(x) = \tau_1 \circ \tau_t(x) = \tau_{t+1}(x) = \tau_t \circ \tau_1(x) = \tau_t(\varphi(x)\mathbf{1}) = \varphi(x)\tau_t(\mathbf{1}) = \varphi(x)\mathbf{1}, \quad x \in \mathcal{A}.$$

(ii) For all $t \geq 1$ and $x \in \mathcal{A}$, we have, using the unitality and the semigroup property of τ:

$$\psi \circ \tau_t(x) = \psi \circ \tau_{t-1}(\tau_1(x)) = \psi \circ \tau_{t-1}(\varphi(x)\mathbf{1}) = \varphi(x)\psi \circ \tau_{t-1}(\mathbf{1}) = \varphi(x)\psi(\mathbf{1}) = \phi(x).$$

□

The property that $(\tau_t)_{t\geq 0}$ stabilizes after time $t = 1$ is very particular to states, cf. Example 3.4.1 for a counter-example. In the special case where ϕ arises from a state and moreover $\mathcal{A} = \mathcal{B}(\mathcal{H})$, we can provide a partial classification of proper continuous roots:

Theorem 3.4.1. *Let $\mathcal{A} = \mathcal{B}(\mathcal{H})$ with $\mathcal{H}$ infinite-dimensional, φ a normal state on $\mathcal{A}$ and $\phi = \varphi(\cdot)\mathbf{1}$.*

(i) If $\dim(p_\phi\mathcal{H}) = 1$, i.e., φ is a pure state, then ϕ has a proper continuous root.
(ii) If $1 < \dim(p_\phi\mathcal{H}) < \infty$, i.e., φ is a finite convex combination of (at least two) pure states, then ϕ has no proper continuous root.
(iii) If $\dim(p_\phi\mathcal{H}) = \infty$, i.e., φ is an infinite convex combination of pure states, and moreover $0 < \dim(p'_\phi\mathcal{H}) < \infty$ then ϕ has no proper continuous root.

Proof. (i). This is taken from [Bha12, Ex.1.3]. Since φ is pure, we can write $\varphi = \langle \xi, \cdot \xi\rangle$, where ξ is a suitable vector in $\mathcal{H}$. We decompose $\mathcal{H} = \mathbb{C}\xi \oplus L^2[0,1]$, so p_ϕ is the projection onto the first, p'_ϕ the projection onto the second component. Let $(S_t)_{t\geq 0}$ be the standard nilpotent right-shift semigroup on $L^2[0,1]$ defined as follows: for $f \in L^2[0,1]$, $t \in [0,\infty)$ and $s \in [0,1]$, set

$$S_t(f)(s) = \begin{cases} f(s-t) : s-t \in [0,1] \\ 0 \qquad\quad\; : \text{otherwise.} \end{cases} \tag{3.4.1}$$

Then with respect to the decomposition $\mathcal{H} = \mathbb{C}\xi \oplus L^2[0,1]$, define

$$\tau_t : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H}), \qquad \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \mapsto \begin{pmatrix} x_{11} & x_{12}S_t^* \\ S_t x_{21} & S_t x_{22} S_t^* + x_{11}(\mathbf{1} - S_t S_t^*) \end{pmatrix}.$$

This can be written as

$$\tau_t \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \tau_t(x) = (1 \oplus S_t)x(1\oplus S_t)^* + \varphi(x)\big(\mathbf{1} - (1\oplus S_t)(1\oplus S_t)^*\big),$$

and it is straight-forward to verify that $(\tau_t)_{t\geq 0}$ is a strongly continuous semigroup, every τ_t is UNCP and $\tau_1(x) = \varphi(x)\mathbf{1} = \phi(x)$. Thus $(\tau_t)_{t\geq 0}$ forms a proper continuous root of ϕ.

(ii) Suppose a proper continuous root $(\tau_t)_{t\geq 0}$ of ϕ exists. As in Lemma 3.3.1(4) we see that $(p_\phi \tau_t(\cdot) p_\phi)_{t\geq 0}$ restricts to a continuous root of $p_\phi \phi(\cdot) p_\phi$ on $p_\phi \mathcal{A} p_\phi$. However, we know from Proposition 3.4.1 that such a continuous root cannot exist because $p_\phi\phi(\cdot)p_\phi$ is not bijective on $p_\phi\mathcal{A}p_\phi$, so we reach a contradiction. Thus ϕ cannot have a proper continuous root.

(iii) Suppose for contradiction a proper continuous root $(\tau_t)_{t\geq 0}$ of ϕ exists. Since $\tau_t(p'_\phi) \leq p'_\phi$ according to Lemma 3.3.1(i), we see that $(\tau_t|_{p'_\phi \mathcal{A} p'_\phi})_{t\geq 0}$ forms an NCP semigroup, and according to Lemma 3.3.1(ii), it is nilpotent with $\tau_1|_{p'_\phi \mathcal{A} p'_\phi} = 0$. If $0 < \dim(p'_\phi) < \infty$, a CP semigroup must be of the form $(e^{t\mathcal{L}})_{t\geq 0}$ with bounded conditionally CP map $\mathcal{L}$. Then $e^{-\mathcal{L}}$ is the inverse of $e^{\mathcal{L}}$ (as a linear map), so we get $0 = \tau_1|_{p'_\phi \mathcal{A} p'_\phi} = e^{\mathcal{L}} \neq 0$, which is a contradiction, so ϕ cannot have a proper continuous root.

□

Problem. In the setting of Theorem 3.4.1, does ϕ have a proper continuous root in the following two missing cases

(iv) $\dim p_\phi = \infty$ with $\dim p'_\phi = 0$;

(v) $\dim p_\phi = \infty$ with $\dim p'_\phi = \infty$?

We wish to point out that the two cases are equivalent, so it suffices to study (iv).

Remark 3.4.2. In [Bha12], the roots in case (i) of Theorem 3.4.1 have been completely classified in terms of E_0-semigroups in standard form, cf. [Arv03] and [Pow99, Definition 2.12].

Remark 3.4.3. A similar construction can be used in order to get a proper continuous root $(T_t)_{t\geq 0}$ of a pure state on an uncountable classical probability space $C([0,1])$, namely consider

$$T_t : C([0,1]) \to C([0,1]), \quad T_t f(s) = \begin{cases} f(s-t) : s-t \geq 0 \\ f(0) \quad\quad : \text{otherwise.} \end{cases}$$

A pure state on $C([0,1])$ corresponds to an evaluation functional ev_x, with some $x \in [0,1]$. Then $\mathrm{ev}_x \circ T_t$ equals a pure state at all times $t \in [0,1]$, in particular $\mathrm{ev}_x \circ T_1 = \mathrm{ev}_0$. In contrast, in the non-commutative case of $\mathcal{A} = \mathcal{B}(\mathcal{H})$ as in Theorem 3.4.1(i) suppose $\psi \neq \varphi$ is another pure state. Then $\psi \circ \tau_t$ equals the pure states ψ at time $t = 0$ and φ at $t = 1$ but in between it is a convex combination of two pure states depending on t. Moreover, for countable classical states space, we expect that no proper continuous root exists at all. This indicates a stark difference between the commutative and the non-commutative setting.

CHAPTER 4

Structure of Block Quantum Dynamical Semigroups and their Product Systems

4.1 Introduction

It is well-known that a block matrix $\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}$ of operators on a direct sum of Hilbert spaces $(\mathcal{H} \oplus \mathcal{K})$ is positive if and only if A, D are positive and there exists a contraction $K : \mathcal{K} \to \mathcal{H}$ such that $B = A^{\frac{1}{2}} K D^{\frac{1}{2}}$. This says that the positivity of a block matrix is determined up to a contraction by the positive diagonals. We want to look at the structure of block completely positive (CP) maps, that is, completely positive maps which send 2×2 block operators as above to 2×2 block operators. Such maps have already appeared in many different contexts. For example, Paulsen uses the block CP maps in [Pau84] to prove that every completely polynomially bounded operator is similar to a contraction. The structure of completely bounded (CB) maps are understood using the 2×2 block CP maps (See [Pau84, PS85, Sue85],[Pau02, Chapter 8]). The usual way to study the structure of CP maps into $\mathcal{B}(\mathcal{H})$ is via Stinespring dilation theorem [Sti55] (See Theorem 2.1.8). If $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix} : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$ is a block CP map, then the diagonals $\phi_i, i = 1, 2$ are also CP maps on $\mathcal{A}$. Also the Stinespring representation of Φ gives us natural Stinespring representations for ϕ_i by the appropriate compressions. In [PS85, Corollary 2.7], Paulsen and Suen proved that: if $\Phi = \begin{pmatrix} \phi & \psi \\ \psi^* & \phi \end{pmatrix} : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$ is CP and if ϕ has the minimal Stinespring representation $(\mathcal{K}, \pi, V)$ then there exists a contraction $T \in \pi(\mathcal{A})'$ such that $\psi(\cdot) = V^*\pi(\cdot)TV$. While studying units of E_0-semigroups of $\mathcal{B}(\mathcal{H})$ Powers was led into considering block CP semigroups (See [Pow03] and [BLS08], [Ske10]). In [BM10], Bhat and Mukherjee proved a structure theorem for block quantum Markov semigroups on

$\mathcal{B}(\mathcal{H}\oplus\mathcal{K})$. These results show the importance of studying block CP maps. In this chapter, our interest is to study the structure of one-parameter semigroups of block CP maps on general von Neumann algebras.

In [Bha96] Bhat proved that any QMS on $\mathcal{B}(\mathcal{H})$ admits a unique E_0-dilation, and in [Bha99], extended the result to QMS to unital C^*-subalgebras of $\mathcal{B}(\mathcal{H})$. Later in [BS00] Bhat and Skeide constructed the E_0-dilation for arbitrary quantum Markov semigroups (QMS) on abstract unital C^*-algebras, using the technology of Hilbert C^*-modules. Here one sees for the first time subproduct systems and product systems of Hilbert C^*-modules. Muhly and Solel [MS07] took a dual approach to achieve this, where they have called these Hilbert C^*-modules as C^*-correspondences. Subproduct systems and inclusion systems are synonyms. The word 'subproduct systems' seems to be better established now. Since we are mostly following the ideas and notations of [BM10], we will continue to call these objects as inclusion systems.

Here is an outline for this chapter. We prove in Theorem 4.2.1 that if $\mathcal{A}$ is a unital C^*-algebra, $\mathcal{B}$ is a von Neumann algebra and $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ is a block CP map then the CB map in the off diagonal corner can be determined by the GNS-representations of the diagonal CP maps up to an adjointable bilinear contraction. Also we give an example to indicate that the von Neumann algebra $\mathcal{B}$ in Theorem 4.2.1 can not be replaced by arbitrary C^*-algebras. We show in Theorem 4.3.1, that if $\mathcal{B}$ is a von Neumann algebra and if we have a block quantum dynamical semigroup on $M_2(\mathcal{B})$ then the CB semigroup sitting in the off-diagonal corner can be described by a unique morphism between the inclusion systems associated to the CP semigroups in the diagonals. We prove in Theorem 4.4.1 that if $\mathcal{B}$ is a von Neumann algebra, Then any morphism between inclusion systems of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules can be lifted as a morphism between the product systems generated by these inclusion systems. We notice in Theorem 4.3.2 that the E_0-dilation of a block QMS constructed by Bhat and Skeide in [BS00] is again a semigroup of block maps.

4.2 Block CP maps

Let $\mathcal{A}$ be a unital C^*-algebra. Let $p \in \mathcal{A}$ be a projection. Set $p' = \mathbf{1} - p$. Then for every $x \in \mathcal{A}$ we have the following block decomposition:

$$x = \begin{pmatrix} pxp & pxp' \\ p'xp & p'xp' \end{pmatrix} \in \begin{pmatrix} p\mathcal{A}p & p\mathcal{A}p' \\ p'\mathcal{A}p & p'\mathcal{A}p' \end{pmatrix}. \tag{4.2.1}$$

Definition 4.2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. Let $p \in \mathcal{A}$ and $q \in \mathcal{B}$ be projections. We say that a map $\Phi : \mathcal{A} \to \mathcal{B}$ is a *block map* (with respect to p and q) if Φ respects the above block decomposition. i.e., for all $x \in \mathcal{A}$ we have

$$\Phi(x) = \begin{pmatrix} \Phi(pxp) & \Phi(pxp') \\ \Phi(p'xp) & \Phi(p'xp') \end{pmatrix} \in \begin{pmatrix} q\mathcal{B}q & q\mathcal{B}q' \\ q'\mathcal{B}q & q'\mathcal{B}q' \end{pmatrix}. \tag{4.2.2}$$

If $\Phi : \mathcal{A} \to \mathcal{B}$ is a block map, we get the following four maps: $\phi_{11} : p\mathcal{A}p \to q\mathcal{B}q$, $\phi_{12} : p\mathcal{A}p' \to q\mathcal{B}q'$, $\phi_{21} : p'\mathcal{A}p \to q'\mathcal{B}q$, and $\phi_{22} : p'\mathcal{A}p' \to q'\mathcal{B}q'$. So we write Φ as

$$\Phi = \begin{pmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{pmatrix}.$$

Lemma 4.2.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras. For $i = 1, 2$, let $\phi_i : \mathcal{A} \to \mathcal{B}$ be a CP map with a GNS-representation (E_i, x_i). Suppose $T : E_2 \to E_1$ is an adjointable bilinear contraction and $\psi : \mathcal{A} \to \mathcal{B}$ is given by $\psi(a) = \langle x_1, Tax_2\rangle$. Then the block map $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix} : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ is CP.*

Proof. Set $y = Tx_2 \in E_1$. Then

$$\Phi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \langle x_1, ax_1\rangle & \langle x_1, by\rangle \\ \langle y, cx_1\rangle & \langle y, dy\rangle \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \langle x_2, d(\mathrm{id}_{E_2} - T^*T)x_2\rangle \end{pmatrix}.$$

Clearly $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \langle x_1, ax_1\rangle & \langle x_1, by\rangle \\ \langle y, cx_1\rangle & \langle y, dy\rangle \end{pmatrix}$ is CP. Since T is an adjointable bilinear contraction, $(\mathrm{id}_{E_2} - T^*T)$ is bilinear and positive. Hence $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & \langle x_2, d(\mathrm{id}_{E_2} - T^*T)x_2\rangle \end{pmatrix}$ is CP. Therefore Φ is CP. ☐

Let F be a Hilbert $M_2(\mathcal{B})$-module. Define a right $\mathcal{B}$-module action and a $\mathcal{B}$-valued semi-inner product $\langle\cdot,\cdot\rangle_\Sigma$ on F by

$$xb := x\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix} \text{ and } \langle x, y\rangle_\Sigma := \sum_{i,j=1}^{2} \langle x, y\rangle_{i,j} \quad \text{for } x, y \in F, b \in \mathcal{B}.$$

where $\langle x, y\rangle_{i,j}$ denotes the $(i, j)^{\text{th}}$ entry of $\langle x, y\rangle \in M_2(\mathcal{B})$.

Let $F^{(\mathcal{B})}$ denote the quotient space F/N where $N = \{x : \langle x, x\rangle_\Sigma = 0\}$. (We denote the coset $x + N$ of $x \in F$ by $[x]_F$ or just by $[x]$). Then $F^{(\mathcal{B})}$ is a pre-Hilbert $\mathcal{B}$-module with right $\mathcal{B}$-action and inner product given by

$$[x]b = [x\begin{pmatrix} b & 0 \\ 0 & b \end{pmatrix}] \text{ and } \langle [x], [y]\rangle = \langle x, y\rangle_\Sigma = \sum_{i,j=1}^{2} \langle x, y\rangle_{i,j} \quad \text{for } x, y \in F, b \in \mathcal{B}. \tag{4.2.3}$$

Proposition 4.2.1. *If F is a Hilbert (von Neumann) $M_2(\mathcal{B})$-module, then $F^{(\mathcal{B})}$ is a Hilbert (von Neumann) $\mathcal{B}$-module.*

Proof. Let F be a Hilbert $M_2(\mathcal{B})$-module. For each $x \in F$, we have $[x] = [x \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}]$ and

$$\|[x]\| = \left\| \sum_{i,j=1}^{2} \langle x, x \rangle_{i,j} \right\|^{\frac{1}{2}} = \left\| x \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\|. \tag{4.2.4}$$

Consider a Cauchy sequence $([x_n])_{n \geq 1}$ in $F^{(\mathcal{B})}$. Set $y_n = x_n \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in F$. Then by (4.2.4), $(y_n)_{n \geq 1}$ is a Cauchy sequence in F. Let $y = \lim_{n \to \infty} y_n$ in F. Then $y = y \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Take $x = \frac{y}{2}$. Then, again by using (4.2.4), we see that $([x_n])_{n \geq 1}$ converges to $[x]$ in $F^{(\mathcal{B})}$. Thus $F^{(\mathcal{B})}$ is complete.

Now assume that F is von Neumann $M_2(\mathcal{B})$-module. Let $\mathcal{B} \subseteq \mathbb{B}(\mathcal{G})$. So $F^{(\mathcal{B})} \subseteq \mathbb{B}(\mathcal{G}, F^{(\mathcal{B})} \odot \mathcal{G})$ and $F \subseteq \mathbb{B}(\mathcal{G}^2, F \odot \mathcal{G}^2)$ where $\mathcal{G}^2 = \mathcal{G} \oplus \mathcal{G}$. We have for $x \in F, g_1, g_2 \in \mathcal{G}$,

$$\|[x] \odot (g_1 + g_2)\| = \left\langle g_1 + g_2, \sum_{i,j=1}^{2} \langle x, x \rangle_{i,j} (g_1 + g_2) \right\rangle^{\frac{1}{2}} = \left\| x \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \odot \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\|. \tag{4.2.5}$$

Using (4.2.5), we can prove as in the above case, that $F^{(\mathcal{B})}$ is SOT closed in $\mathbb{B}(\mathcal{G}, F^{(\mathcal{B})} \odot \mathcal{G})$ and hence $F^{(\mathcal{B})}$ is a von Neumann $\mathcal{B}$-module. □

Let F be a Hilbert $M_2(\mathcal{B})$-module. Suppose F has a nondegenerate left action of $\mathcal{A}$, then (4.2.4) implies that the natural left action of $\mathcal{A}$ on $F^{(\mathcal{B})}$ given by

$$a[x] := [ax] \quad \text{for } a \in \mathcal{A}, x \in F \tag{4.2.6}$$

is a well defined nondegenerate action.

Proposition 4.2.2. *If F is a Hilbert (von Neumann) $\mathcal{A}$-$M_2(\mathcal{B})$-module, then $F^{(\mathcal{B})}$ is a Hilbert (von Neumann) $\mathcal{A}$-$\mathcal{B}$-module with the left action defined in (4.2.6).*

Proof. If F is a Hilbert $\mathcal{A}$-$M_2(\mathcal{B})$-module, then clearly $F^{(\mathcal{B})}$ is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module. We shall prove that if F is a von Neumann $\mathcal{A}$-$M_2(\mathcal{B})$-module, then $F^{(\mathcal{B})}$ is a von Neumann $\mathcal{A}$-$\mathcal{B}$-module. Let $\mathcal{B} \subseteq \mathbb{B}(\mathcal{G})$. So $F^{(\mathcal{B})} \subseteq \mathbb{B}(\mathcal{G}, F^{(\mathcal{B})} \odot \mathcal{G})$ and $F \subseteq \mathbb{B}(\mathcal{G}^2, F \odot \mathcal{G}^2)$ where

$\mathcal{G}^2 = \mathcal{G} \oplus \mathcal{G}$. We must show that the Stinespring representation $\rho : \mathcal{A} \to \mathcal{B}(F^{(\mathcal{B})} \odot \mathcal{G})$ of $\mathcal{A}$ given by $\rho(a)([x] \odot g) = a[x] \odot g$ is normal. For any $x \in F, g \in \mathcal{G}$, a computation similar to (4.2.5) implies that

$$\|[x] \odot g\| = \left\| x \odot \begin{pmatrix} g \\ g \end{pmatrix} \right\|. \tag{4.2.7}$$

As the Stinespring representation $\hat{\rho} : \mathcal{A} \to \mathcal{B}(F \odot \mathcal{G}^2)$ given by $\tilde{\rho}(a)(x \odot \underline{g}) = ax \odot \underline{g}$ for $a \in \mathcal{A}, \underline{g} \in \mathcal{G}^2$ is normal, using (4.2.7), we can see that ρ is normal. □

Remark 4.2.1. Suppose F is a Hilbert (von Neumann) $M_2(\mathcal{A})$-$M_2(\mathcal{B})$-module, then we can consider F as a Hilbert (von Neumann) $\mathcal{A}$-$M_2(\mathcal{B})$-module by considering the left action of $\mathcal{A}$ given by

$$ax := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} x \quad \text{for } x \in F, a \in \mathcal{A}. \tag{4.2.8}$$

Therefore, Proposition 4.2.2 shows that, if F is a Hilbert (von Neumann) $M_2(\mathcal{A})$-$M_2(\mathcal{B})$-module, then $F^{(\mathcal{B})}$ is a Hilbert (von Neumann) $\mathcal{A}$-$\mathcal{B}$-module.

Remark 4.2.2. Let $E \subseteq F$ be a $M_2(\mathcal{B})$-submodule of a $M_2(\mathcal{B})$-module F. Then

$$E^{(\mathcal{B})} \simeq \{[x]_F : x \in E\} \subseteq F^{(\mathcal{B})}.$$

Theorem 4.2.1. *Let $\mathcal{A}$ be a unital C^*-algebra and $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. For $i = 1, 2$, let $\phi_i : \mathcal{A} \to \mathcal{B}$ be a CP map with a GNS-representation (F_i, y_i). Suppose $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix} : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ is a block CP map[1] for some CB map $\psi : \mathcal{A} \to \mathcal{B}$ then, there is an adjointable bilinear contraction $T : F_2 \to F_1$ such that $\psi(a) = \langle y_1, Tay_2 \rangle$ for all $a \in \mathcal{A}$.*

Proof. Let (E, x) be the (minimal) GNS-construction for Φ. So, E is a von Neumann $M_2(\mathcal{B})$-module and Hilbert $M_2(\mathcal{A})$-$M_2(\mathcal{B})$-module. Let $\mathbb{E}_{ij} := \mathbf{1} \otimes E_{ij}$ in $\mathcal{A} \otimes M_2$, or $\mathcal{B} \otimes M_2$, depending upon the context, where $\{E_{ij}\}$'s are the matrix units in M_2. Set $\hat{E}_i := \mathbb{E}_{ii} E \subseteq E, i = 1, 2$. Then $\hat{E}_i$'s are SOT closed (as $\mathbb{E}_{ii}$'s are projections) $M_2(\mathcal{B})$-submodules of E such that $E = \hat{E}_1 \oplus \hat{E}_2$.

Let $x_i := \mathbb{E}_{ii} x \mathbb{E}_{ii} \in \hat{E}_i, i = 1, 2$. Clearly $\langle x_1, x_2 \rangle = 0$. Also for $i, j = 1, 2$ and $i \neq j$,

$$\|x_i - \mathbb{E}_{ii} x\|^2 = \|\mathbb{E}_{ii} x \mathbb{E}_{jj}\|^2 = \|\langle \mathbb{E}_{ii} x \mathbb{E}_{jj}, \mathbb{E}_{ii} x \mathbb{E}_{jj} \rangle\| = \|\mathbb{E}_{jj} \Phi(\mathbb{E}_{ii}) \mathbb{E}_{jj}\| = 0,$$

[1]By block CP map, we mean a CP map which acts block-wise.

and

$$\|x_i - x\mathbb{E}_{ii}\|^2 = \|\mathbb{E}_{jj}x\mathbb{E}_{ii}\|^2 = \|\langle \mathbb{E}_{jj}x\mathbb{E}_{ii}, \mathbb{E}_{jj}x\mathbb{E}_{ii}\rangle\| = \|\mathbb{E}_{ii}\Phi(\mathbb{E}_{jj})\mathbb{E}_{ii}\| = 0.$$

Thus

$$x_i = \mathbb{E}_{ii}x = x\mathbb{E}_{ii}, i = 1,2 \text{ and hence } x = (\mathbb{E}_{11} + \mathbb{E}_{22})x = x_1 + x_2. \tag{4.2.9}$$

As Φ is a block map, for $A \in M_2(\mathcal{A})$, using (4.2.9) we have

$$\Phi(A) = \langle x, Ax\rangle = \sum_{i,j=1}^{2} \langle x_i, Ax_j\rangle = \begin{pmatrix} \langle x_1, Ax_1\rangle_{11} & \langle x_1, Ax_2\rangle_{12} \\ \langle x_2, Ax_1\rangle_{21} & \langle x_2, Ax_2\rangle_{22} \end{pmatrix},$$

where $\langle a, b\rangle_{ij}$ denotes the $(i,j)^{\text{th}}$ entry of $\langle a, b\rangle \in M_2(\mathcal{B})$.

Consider the Hilbert $\mathcal{A}$-$\mathcal{B}$-module and von Neumann $\mathcal{B}$-module $E^{(\mathcal{B})}$ (as described in Remark[2] 4.2.1), and consider the von Neumann $\mathcal{B}$-modules $\hat{E}_i^{(\mathcal{B})}, i = 1,2$. Observe that $\hat{E}_i$ has a non-degenerate left action of $\mathcal{A}$ given by

$$ax := \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} x \quad \text{for } a \in \mathcal{A}, x \in \hat{E}_i. \tag{4.2.10}$$

Therefore, Proposition 4.2.2 shows that $\hat{E}_i^{(\mathcal{B})}$ is also a Hilbert $\mathcal{A}$-$\mathcal{B}$-module for $i = 1,2$.

We have $E^{(\mathcal{B})} \simeq \hat{E}_1^{(\mathcal{B})} \oplus \hat{E}_2^{(\mathcal{B})}$ (via $[y]_E \mapsto [\mathbb{E}_{11}y]_{\hat{E}_1} + [\mathbb{E}_{22}y]_{\hat{E}_2}$ for $y \in E$). For $a \in \mathcal{B}$ and $i = 1,2$ see that,

$$\langle [x_i], a[x_i]\rangle = \sum_{r,s=1}^{2} \left\langle \mathbb{E}_{ii}x, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbb{E}_{ii}x \right\rangle_{r,s} = \sum_{r,s=1}^{2} \Phi\left(\mathbb{E}_{ii} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbb{E}_{ii}\right)_{r,s} = \phi_i(a).$$

This shows that $(\hat{E}_i^{(\mathcal{B})}, [x_i])$ is a GNS-representation (not necessarily minimal) for $\phi_i, i = 1,2$. Define $U : \hat{E}_2^{(\mathcal{B})} \to \hat{E}_1^{(\mathcal{B})}$ by $U[w] = [\mathbb{E}_{12}w]$ for all $w \in \hat{E}_2$. Then, for all $z, w \in \hat{E}_2$,

$$\langle U[z], U[w]\rangle = \sum_{i,j=1}^{2} \langle \mathbb{E}_{12}z, \mathbb{E}_{12}w\rangle_{i,j} = \sum_{i,j=1}^{2} \langle z, \mathbb{E}_{21}\mathbb{E}_{12}w\rangle_{i,j} = \sum_{i,j=1}^{2} \langle z, w\rangle_{i,j} = \langle [z], [w]\rangle,$$

also for $y \in \hat{E}_1$ we have $\mathbb{E}_{21}y \in \hat{E}_2$ such that

$$U[\mathbb{E}_{21}y] = [\mathbb{E}_{12}\mathbb{E}_{21}y] = [\mathbb{E}_{11}y] = [y].$$

Therefore U is a unitary from the von Neumann $\mathcal{B}$-module $\hat{E}_2^{(\mathcal{B})}$ to the von Neumann $\mathcal{B}$-module $\hat{E}_1^{(\mathcal{B})}$. Now for $a \in \mathcal{A}, w \in \hat{E}_2$,

$$Ua[w] = U[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} w] = [\mathbb{E}_{12} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} w] = [\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbb{E}_{12}w] = a[\mathbb{E}_{12}w] = aU[w].$$

[2] See also Propositions 4.2.1,4.2.2.

Thus $U : \hat{E}_2^{(\mathcal{B})} \to \hat{E}_1^{(\mathcal{B})}$ is a bilinear (adjointable) unitary between the Hilbert $\mathcal{A}$-$\mathcal{B}$ modules.

Let $\tilde{F}_i := \overline{\text{span}}^s \mathcal{A} y_i \mathcal{B} \subseteq F_i$ and $\tilde{E}_i = \overline{\text{span}}^s \mathcal{A}[x_i]\mathcal{B} \subseteq \hat{E}_1^{(\mathcal{B})}$, so that $(\tilde{F}_i, y_i)$ and $(\tilde{E}_i, [x_i])$ are minimal GNS-representations for $\phi_i, i = 1, 2$. Therefore, $\tilde{V}_i : \tilde{F}_i \to \tilde{E}_i$ given by

$$\tilde{V}_i(a y_i b) = a[x_i]b, \quad a \in \mathcal{A}, b \in \mathcal{B},$$

extends to a bilinear (adjointable) unitary. Let $V_i : F_i \to \hat{E}_i^{(\mathcal{B})}$ be the extension of $\tilde{V}_i$, by defining it to be zero on the complement $\tilde{F}_i^{\perp}$ of $\tilde{F}_i$. Note that V_i is a bilinear partial isometry with initial space $\tilde{F}_i$ and final space $\tilde{E}_i$ for $i = 1, 2$. Take $T := V_1^* U V_2$.

Now consider, for $a \in \mathcal{A}$,

$$\begin{aligned}
\langle y_1, T a y_2 \rangle &= \langle y_1, V_1^* U V_2 a y_2 \rangle = \langle V_1 y_1, U V_2 a y_2 \rangle = \langle \tilde{V}_1 y_1, U \tilde{V}_2 a y_2 \rangle \\
&= \langle [x_1], U a [x_2] \rangle = \sum_{i,j=1}^{2} \left\langle \mathbb{E}_{11} x, \mathbb{E}_{12} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbb{E}_{22} x \right\rangle_{i,j} \\
&= \sum_{i,j=1}^{2} \left\langle x, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} x \right\rangle_{i,j} = \sum_{i,j=1}^{2} \Phi \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}_{i,j} = \sum_{i,j=1}^{2} \begin{pmatrix} 0 & \psi(a) \\ 0 & 0 \end{pmatrix}_{i,j} = \psi(a).
\end{aligned}$$

This completes the proof. □

Remark 4.2.3 (Uniqueness)**.** With the same hypothesis and notations of Theorem 4.2.1 let $T, T' : F_2 \to F_1$ be any two adjointable bilinear contractions such that $\psi(a) = \langle y_1, T a y_2 \rangle = \langle y_1, T' a y_2 \rangle$ for all $a \in \mathcal{A}$, then

$$\begin{aligned}
\langle a_1 y_1 b_1, T(a_2 y_2 b_2) \rangle &= b_1^* \langle y_1, T((a_1^* a_2) y_2) \rangle b_2 \\
&= b_1^* \langle y_1, T'((a_1^* a_2) y_2) \rangle b_2 \\
&= \langle a_1 y_1 b_1, T'(a_2 y_2 b_2) \rangle
\end{aligned}$$

for $a_1, a_2 \in \mathcal{A}, b_1, b_2 \in \mathcal{B}$ and hence $P_{\tilde{F}_1} T P_{\tilde{F}_2} = P_{\tilde{F}_1} T' P_{\tilde{F}_2}$ where $P_{\tilde{F}_i} : F_i \to F_i$ is the projection onto $\tilde{F}_i$. This in particular shows that the contraction T in Theorem 4.2.1 is *unique* if F_i's are minimal GNS-modules.

Corollary 4.2.1. *Let $\mathcal{A}$ be a unital C^*-algebra. For $i = 1, 2$, let $\varphi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ be a completely positive map with the minimal Stinespring representation $(\mathcal{K}_i, \pi_i, V_i)$. Suppose $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B}(\mathcal{H}))$, defined by $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$ is block CP for some CB map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, then there is a unique contraction $T : \mathcal{K}_2 \to \mathcal{K}_1$ with $\pi_1(a) T = T \pi_2(a)$ for all $a \in \mathcal{A}$ such that $\psi(a) = V_1^* T \pi_2(a) V_2$ for all $a \in \mathcal{A}$.*

Proof. Given that $(\mathcal{K}_i, \pi_i, V_i)$ is a minimal Stinespring representation for $\phi_i, i = 1, 2$. Let (E_i, V_i) be the minimal GNS-representation for $\phi_i, i = 1, 2$ as explained in Remark 2.2.4, where $E_i = \overline{\operatorname{span}}^s \pi_i(\mathcal{A}) V_i \mathcal{B}(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}, \mathcal{K}_i)$. By Theorem 4.2.1, there exists an adjointable bilinear contraction $\hat{T} : E_2 \to E_1$ such that

$$\psi(a) = \langle V_1, \hat{T}\pi_2(a)V_2 \rangle = V_1^* \hat{T}\pi_2(a)V_2 \quad \text{for all } a \in \mathcal{A}. \tag{4.2.11}$$

As $(\mathcal{K}_i, \pi_i, V_i)$ is the minimal Stinespring representation for ϕ, we have $\mathcal{K}_i = \overline{\pi_i(\mathcal{A})V_i\mathcal{H}}$. Define $T : \mathcal{K}_2 \to \mathcal{K}_1$ by

$$T(\pi_2(a)V_2 h) = (\hat{T}(\pi_2(a)V_2))h \quad \text{for all } a \in \mathcal{A}, h \in \mathcal{H}. \tag{4.2.12}$$

Let h be a non-zero vector in $\mathcal{H}$. As $\hat{T}$ is right $\mathcal{B}(\mathcal{H})$-linear and contraction, we have for $a \in \mathcal{A}, h \in \mathcal{H}$, (In the following, we use the bra-ket notations, defined in Subsection 2.1.3)

$$\begin{aligned}\left\| \left| (\hat{T}(\pi_2(a)V_2))h \middle\rangle \middle\langle h \right| \right\| = \left\| \hat{T}(\pi_2(a)V_2) \left| h \rangle\langle h \right| \right\| &= \left\| \hat{T}(\pi_2(a)V_2 \left| h \rangle\langle h \right|) \right\| \\ &\leq \left\| \pi_2(a)V_2 \left| h \rangle\langle h \right| \right\| = \left\| \left| (\pi_2(a)V_2)h \rangle\langle h \right| \right\|.\end{aligned}$$

This implies

$$\left\| (\hat{T}(\pi_2(a)V_2))h \right\| \leq \|\pi_2(a)V_2 h\| \quad \text{for all } a \in \mathcal{A}, h \in \mathcal{H}. \tag{4.2.13}$$

Therefore T is a well-defined contraction. Now as $\hat{T}$ is left $\mathcal{A}$-linear, for all $a, b \in \mathcal{A}$ and $h \in \mathcal{H}$, we have

$$\begin{aligned}T\pi_2(a)(\pi_2(b)V_2 h) = T(\pi_2(ab)V_2 h) = \hat{T}(\pi_2(ab)V_2)h &= \hat{T}(\pi_2(a)\pi_2(b)V_2)h \\ &= \pi_1(a)\hat{T}(\pi_2(b)V_2)h = \pi_1(a)T(\pi_2(b)V_2 h).\end{aligned}$$

Thus $T\pi_2(a) = \pi_1(a)T$, for all $a \in \mathcal{A}$. Now (4.2.11) shows that $\psi(a)h = V_1^* T\pi_2(a)V_2 h$ for all $h \in \mathcal{H}$. For the uniqueness of T, let T' be another contraction such that $T'\pi_2(a) = \pi_1(a)T'$ and $\psi(a) = V_1^* T'\pi_2(a)V_2$ for all $a \in \mathcal{A}$. Consider for $a, b \in \mathcal{A}$ and $h, g \in \mathcal{H}$,

$$\begin{aligned}\langle T\pi_2(b)V_2 g, \pi_1(a)V_1 h \rangle &= \langle V_1^* T\pi_2(a^*b)V_2 g, h \rangle \\ &= \langle \psi(a^*b)g, h \rangle = \langle V_1^* T'\pi_2(a^*b)V_2 g, h \rangle \\ &= \langle T'\pi_2(b)V_2 g, \pi_1(a)V_1 h \rangle.\end{aligned}$$

This proves the uniqueness of T. $\square$

Remark 4.2.4. (i). Corollary 4.2.1 can be proved directly (without deducing from Theorem 4.2.1).

(ii). Given two CP maps $\phi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}), i = 1, 2$. Let $(\mathcal{K}_i, \pi_i, V_i)$ be the minimal Stinespring representation for $\phi_i, i = 1, 2$. Suppose the block map $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$ is CP for some CB map $\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. Then Furuta in [Fur94, Proposition 6.1] proved that: ψ is non-trivial (non-zero) if and only if there exists a non-zero operator $T : \mathcal{K}_2 \to \mathcal{K}_1$ such that $T\pi_1(a) = \pi_2(a)T$ for all $a \in \mathcal{A}$. On the other hand, Corollary 4.2.1 explicitly tells us the structure of ψ from the minimal Stinespring representations of ϕ_i's.

(iii). Corollary 4.2.1 is a generalization of [PS85, Corollary 2.7] (namely, when $\phi_1 = \phi_2$ in Corollary 4.2.1, we get the result of Paulsen and Suen [PS85, Corollary 2.7]).

The following example shows that we cannot replace the von Neumann algebra $\mathcal{B}$ in Theorem 4.2.1 by an arbitrary C^*-algebra.

Example 4.2.1. Let $\mathcal{A} = \mathcal{B} = C([0, 1])$, the commutative unital C^*-algebra of continuous functions on $[0, 1]$. Let $E = C([0, 1])$. It is a Hilbert $\mathcal{A}$-$\mathcal{B}$-module with the natural actions and standard inner product: $\langle f, g\rangle = f^*g$. Let

$$h_1(t) = t, \quad h_2(t) = 1 \quad \text{ for } t \in [0, 1].$$

Consider the CP map $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ defined by

$$\Phi \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} = \begin{pmatrix} h_1^* & 0 \\ 0 & h_2^* \end{pmatrix} \begin{pmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{pmatrix} \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix} = \begin{pmatrix} h_1^* f_{11} h_1 & h_1^* f_{12} h_2 \\ h_2^* f_{21} h_1 & h_2^* f_{22} h_2 \end{pmatrix}.$$

Note that Φ is the block CP map $\begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$, where $\phi_i, \psi : \mathcal{A} \to \mathcal{B}$ are given by

$$\psi(f) = \langle h_1, fh_2\rangle \text{ and } \phi_i(f) = \langle h_i, fh_i\rangle \quad \text{for } f \in \mathcal{A}, i = 1, 2. \tag{4.2.14}$$

Therefore, (E, h_i) is a GNS-representation for $\phi_i, i = 1, 2$. Let $E_i = \overline{\text{span}}\, \mathcal{A}h_i\mathcal{B} \subseteq E$. Then (E_i, h_i) is the minimal GNS-representation for $\phi_i, i = 1, 2$. Note that

$$E_1 = \{f \in C([0, 1]) : f(0) = 0\} \quad \text{ and } \quad E_2 = \mathcal{A}.$$

Now suppose that there exists a bilinear contraction $T : E_2 \to E_1$ such that $\psi(f) = \langle h_1, Tfh_2\rangle$ for all $f \in \mathcal{A}$. Then $\psi(h_2) = h_1 = h_1T(h_2)$. That is, $t = tT(h_2)(t)$ for all $t \in [0, 1]$. This implies that $T(h_2)(t) = 1$ for all $t \neq 0$. This is a contradiction to $T(h_2) \in E_1$.

Remark 4.2.5. Let $\mathcal{A}$ and $\mathcal{B}$ be unital C^*-algebras and let $\Phi : M_2(\mathcal{A}) \to M_2(\mathcal{B})$ be a block CP map $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$. Suppose $\mathcal{B}$ is a unital subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Let $\mathcal{C}$ be the von Neumann algebra $\overline{\mathcal{B}}^s$. Now enlarge the codomain of Φ to $M_2(\mathcal{C})$. That is, consider the block CP map $\tilde{\Phi} : M_2(\mathcal{A}) \to M_2(\mathcal{C})$, such that $\tilde{\Phi}(A) = \Phi(A)$.

Let $\tilde{\Phi} = \begin{pmatrix} \tilde{\phi}_1 & \tilde{\psi} \\ \tilde{\psi}^* & \tilde{\phi}_2 \end{pmatrix}$. Then, by Theorem 4.2.1 we get a bilinear contraction $\tilde{T} : \tilde{E}_2 \to \tilde{E}_1$ such that $\psi(a) = \tilde{\psi}(a) = \langle x_1, \tilde{T} a x_2 \rangle$ for all $a \in \mathcal{A}$, where $(\tilde{E}_i, x_i)$ is the GNS-construction for $\tilde{\phi}_i, i = 1, 2$. Note that $(\tilde{E}_i, x_i)$ is not a GNS-representation for $\phi_i, i = 1, 2$ as $\tilde{E}_i$ is an Hilbert $\mathcal{A}$-$\mathcal{C}$-module which need not be an Hilbert $\mathcal{A}$-$\mathcal{B}$ module.

In particular, in Example 4.2.1 if we enlarge the codomain $\mathcal{B}$ to $\mathcal{C} = \overline{\mathcal{B}}^s = L^\infty([0,1])$, then with the above notations, we have $\tilde{E}_i = \overline{\text{span}}\ \mathcal{A} h_i \mathcal{C} = L^\infty([0,1]), i = 1, 2$. Note also that there exists a bilinear contraction $\tilde{T} : \tilde{E}_2 \to \tilde{E}_1$ given by $\tilde{T} f = f, f \in E_2$ such that $\psi(f) = \tilde{\psi}(f) = \langle h_1, T f h_2 \rangle$ for all $f \in \mathcal{A}$.

The following example is a modification of Example 4.2.1 to get an example of a unital block CP map $\Phi : M_2(\mathcal{B}) \to \mathcal{M}_2(\mathcal{B})$.

Example 4.2.2. Let $\mathcal{A}$ be the unital C^*-algebra $C([0,1])$. Let $\mathcal{B} = \mathcal{A} \oplus \mathcal{A}$ and let $F = \mathcal{A} \oplus \mathcal{A}$ be the Hilbert $\mathcal{B}$-$\mathcal{A}$-module with the module actions and inner product given by

$$\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} k = \begin{pmatrix} f_1 k \\ f_2 k \end{pmatrix}, \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} k_1 f_1 \\ k_2 f_2 \end{pmatrix} \text{ and } \left\langle \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \right\rangle = f_1^* g_1 + f_2^* g_2$$

for $k \in \mathcal{A}$, $\begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathcal{B}$, $\begin{pmatrix} f_1 \\ f_2 \end{pmatrix}, \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \in F$. Consider $E = F \oplus F$ as a Hilbert $\mathcal{B}$-$\mathcal{B}$-module with right action

$$\begin{pmatrix} x \\ y \end{pmatrix} f = \begin{pmatrix} x f_1 \\ y f_2 \end{pmatrix} \quad \text{where } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in \mathcal{B}, \begin{pmatrix} x \\ y \end{pmatrix} \in E,$$

inner product

$$\left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\rangle = \begin{pmatrix} \langle x_1, y_1 \rangle \\ \langle x_2, y_2 \rangle \end{pmatrix},$$

and the left action

$$f \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} f x \\ f y \end{pmatrix} \quad \text{for } f \in \mathcal{B}, \begin{pmatrix} x \\ y \end{pmatrix} \in E.$$

Let $h_{11}(t) = t, h_{12}(t) = \sqrt{1-t^2}, h_{21}(t) = 1, h_{22}(t) = 0$ for $t \in [0,1]$. Let

$$h_1 = \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} \oplus \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}, h_2 = \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} \oplus \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} \in E = F \oplus F$$

Let $\Phi : M_2(\mathcal{B}) \to M_2(\mathcal{B})$ be the block CP map $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$, where $\phi_i, \psi : \mathcal{B} \to \mathcal{B}$ are defined by

$$\phi_i(f) = \langle h_i, f h_i \rangle \text{ and } \psi(f) = \langle h_1, f h_2 \rangle \quad \text{for } f \in \mathcal{B}, i = 1, 2.$$

Let $E_i = \overline{\operatorname{span}} \mathcal{B} h_i \mathcal{B} \subseteq E, i = 1, 2$. Then $E_1 = F_1 \oplus F_1$ with $F_1 = C_0([0,1]) \oplus C_1([0,1])$ where $C_j([0,1]) = \{f \in C([0,1]) : f(j) = 0\}$ for $j = 0, 1$, and $E_2 = F_2 \oplus F_2$ with $F_2 = C([0,1]) \oplus 0$. Now suppose there exists a bilinear contraction $T : E_2 \to E_1$ such that $\psi(f) = \langle h_1, T f h_2 \rangle$ for all $f \in \mathcal{B}$. Then for $f = \begin{pmatrix} h_{21} \\ h_{22} \end{pmatrix} \in \mathcal{B}$,

$$\begin{pmatrix} h_{11} \\ h_{22} \end{pmatrix} = \langle h_1, f h_2 \rangle = \psi(f) = \langle h_1, f T h_2 \rangle = \left\langle \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix} \oplus \begin{pmatrix} h_{11} \\ h_{12} \end{pmatrix}, \begin{pmatrix} l_{11} \\ h_{22} \end{pmatrix} \oplus \begin{pmatrix} l_{21} \\ h_{22} \end{pmatrix} \right\rangle$$

where $T h_2 = \begin{pmatrix} l_{11} \\ l_{12} \end{pmatrix} \oplus \begin{pmatrix} l_{21} \\ l_{22} \end{pmatrix} \in E_1$. Therefore $h_{11} = h_{11} l_{11} + h_{12} h_{22}$. Hence $t = t l_{11}(t)$ for all $t \in [0,1]$. Hence $l_{11}(t) = 1$ for $t \neq 0$. This is a contradiction to the assumption that $T h_2 \in E_1$. So no such T exists.

We could not get any reasonable answer to the following question.

Problem. Let $\mathcal{A}, \mathcal{B}$ be unital C^*-algebras and let $p \in \mathcal{A}, q \in \mathcal{B}$ be projections. Let $\Phi = \begin{pmatrix} \phi_1 & \psi \\ \psi^* & \phi_2 \end{pmatrix}$ be a block CP map from $\mathcal{A}$ to $\mathcal{B}$ with respect to p and q. Let (E_i, ξ_i) be GNS-representation of $\phi_i, i = 1, 2$. Can we prove a theorem similar to Theorem 4.2.1? In other words what is the structure of ψ in terms of (E_i, ξ_i)?

4.3 Semigroups of block CP maps

4.3.1 Structure of block quantum dynamical semigroups

In this subsection, we shall prove a structure theorem similar to (or using) Theorem 4.2.1 for semigroup of block CP maps. We shall start with a few basic examples of semigroups of block CP maps, which are of interest.

Example 4.3.1. Let $\mathcal{H}$ be a Hilbert space. Let $(\theta_t)_{t\geq 0}$ be an E_0-semigroup on $\mathcal{B}(\mathcal{H})$. Let $(U_t)_{t\geq 0}$ be a family of unitaries in $\mathcal{B}(\mathcal{H})$ forming left cocycle for θ, that is, $U_0 = I, U_{s+t} = U_s\theta_s(U_t)$, $t \mapsto U_t$ continuous in SOT. Let $\psi_t(X) = U_t\theta_t(X)U_t^*$ for $X \in \mathcal{B}(\mathcal{H})$. Then $(\psi_t)_{t\geq 0}$ is an E_0-semigroup, cocycle conjugate to $(\theta_t)_{t\geq 0}$. Define $\tau_t : \mathcal{B}(\mathcal{H}\oplus\mathcal{H}) \to \mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ by

$$\tau_t\begin{pmatrix} X & Y \\ Z & W\end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & U_t\end{pmatrix}\begin{pmatrix} \theta_t(X) & \theta_t(Y) \\ \theta_t(Z) & \theta_t(W)\end{pmatrix}\begin{pmatrix} I & 0 \\ 0 & U_t^*\end{pmatrix} = \begin{pmatrix} \theta_t(X) & \theta_t(Y)U_t^* \\ U_t\theta_t(Z) & U_t\theta_t(W)U_t^*\end{pmatrix}.$$

Then clearly $(\theta_t)_{t\geq 0}$ is a block E_0-semigroup.

Example 4.3.2. Let $(a_t)_{t\geq 0}$ and $(b_t)_{t\geq 0}$ be semigroups on a C^*-algebra $\mathcal{B}$ and let $(\phi_t^i)_{t\geq 0}, i = 1, 2$, be two QDSs on $\mathcal{B}$ such that $\phi_t^1(\cdot) - a_t(\cdot)a_t^*$ and $\phi_t^2(\cdot) - b_t(\cdot)b_t^*$ are CP maps (cf. Definition 2.1.18). Define $\tau_t : M_2(\mathcal{B}) \to M_2(\mathcal{B})$ by

$$\tau_t\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} \phi_t^1(a) & a_tbb_t^* \\ b_tca_t^* & \phi_t^2(d)\end{pmatrix}.$$

Then τ_t is CP, for all $t \geq 0$, as

$$\tau_t\begin{pmatrix} a & b \\ c & d\end{pmatrix} = \begin{pmatrix} a_t & 0 \\ 0 & b_t\end{pmatrix}\begin{pmatrix} a & b \\ c & d\end{pmatrix}\begin{pmatrix} a_t^* & 0 \\ 0 & b_t^*\end{pmatrix} + \begin{pmatrix} \phi_t^1(a) - a_taa_t^* & 0 \\ 0 & \phi_t^2(d) - b_tdb_t^*\end{pmatrix}.$$

Clearly $(\tau_t)_{t\geq 0}$ is a block QDS.

Recall the following from Subsection 2.3.3 (Example 2.3.1 and Definition 2.3.12): Let $\phi = (\phi_t)_{t\geq 0}$ be a QDS on a C^*-algebra/von Neumann algebra $\mathcal{B}$ and let (E_t, ξ_t) be the GNS-construction for ϕ_t. For $t, s \geq 0$ define $\beta_{s,t} : E_{s+t} \to E_s \odot E_t$ by

$$\xi_{t+s} \mapsto \xi_s \odot \xi_t. \tag{4.3.1}$$

(Note that $E_t = \overline{E_t}^s = \overline{\operatorname{span}}^s\mathcal{B}\xi_t\mathcal{B}$ and $E_s \odot E_t = E_s\bar{\odot}^s E_s$ if $\mathcal{B}$ is a von Neumann algebra). Then $\beta_{s,t}$'s are two-sided isometries such that

$$(\beta_{r,s} \odot I_{E_t})\beta_{r+s,t} = (I_{E_r} \odot \beta_{s,t})\beta_{r,s+t}. \tag{4.3.2}$$

That is, $(E = (E_t)_{t\geq 0}, \beta = (\beta_{t,s})_{t,s\geq 0})$ is an *inclusion system* of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules with *generating unit* $\xi^\odot = (\xi_t)_{t\geq 0}$ for (E, β). The triple $(E, \beta, \xi^\odot)$ or just the pair $(E, \xi^\odot)$ (when β is clear from the context) is called the *inclusion system (or subproduct system) associated to the QDS* ϕ.

Lemma 4.3.1. *Let $\mathcal{B}$ be a unital C^*-algebra. Given two inclusion systems $(E^i, \beta^i, \xi^{\odot i})$ associated to a pair of CP semigroups $\phi^i = (\phi^i_t)_{t\geq 0}, i = 1, 2$ on $\mathcal{B}$ and a contractive morphism[3] $T = (T_t) : E^2 \to E^1$, there is a block CP semigroup[4] $\Phi = (\Phi_t)_{t\geq 0}$ on $M_2(\mathcal{B})$ such that $\Phi_t = \begin{pmatrix} \phi^1_t & \psi_t \\ \psi^*_t & \phi^2_t \end{pmatrix}$ and $\psi_t(a) = \langle \xi^1_t, T_t(a\xi^2_t)\rangle$.*

Proof. Define $\Phi_t : M_2(\mathcal{B}) \to M_2(\mathcal{B})$ as the block maps $\Phi_t = \begin{pmatrix} \phi^1_t & \psi_t \\ \psi^*_t & \phi^2_t \end{pmatrix}$, where $\psi_t(b) = \langle \xi^1_t, T_t(b\xi^2_t)\rangle$. Then, as $T_t : E^2_t \to E^1_t$ is an adjointable bilinear contraction, Φ_t is CP for all $t \geq 0$ (see the proof of Lemma 4.2.1).

Now we shall show that $\Phi = (\Phi_t)_{t\geq 0}$ is a semigroup on $M_2(\mathcal{B})$. We have

$$\Phi_s \circ \Phi_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \Phi_s \begin{pmatrix} \phi^1_t(a) & \psi_t(b) \\ \psi^*_t(c) & \phi^2_t(d) \end{pmatrix} = \begin{pmatrix} \phi^1_{s+t}(a) & \psi_s(\psi_t(b)) \\ \psi^*_s(\psi^*_t(c)) & \phi^2_{s+t}(d) \end{pmatrix}.$$

It is clear from this, that to show Φ is a semigroup, it is enough to show that $(\psi_t)_{t\geq 0}$ is a semigroup. Now as T is a morphism it is easy to see that $(\psi_t)_{t\geq 0}$ is a semigroup. □

When $\mathcal{B}$ is a von Neumann algebra, we have the converse of Lemma 4.3.1. Example 4.2.1 says that we cannot take $\mathcal{B}$ as an arbitrary C^*-algebra.

Theorem 4.3.1. *Let $\mathcal{B}$ be a von Neumann algebra. Let $\Phi = (\Phi_t)_{t\geq 0}$ be a semigroup of block normal CP maps on $M_2(\mathcal{B})$ with $\Phi_t = \begin{pmatrix} \phi^1_t & \psi_t \\ \psi^*_t & \phi^2_t \end{pmatrix}$. Then, there are inclusion systems $(E^i, \beta^i, \xi^{\odot i}), i = 1, 2$ associated to ϕ^i (canonically arising from the inclusion system associated to Φ) and a* unique *contractive (weak) morphism $T = (T_t) : E^2 \to E^1$ such that $\psi_t(a) = \langle \xi^1_t, T_t a\xi^2_t\rangle$ for all $a \in \mathcal{B}, t \geq 0$.*

Proof. We shall prove this extending the same ideas of the proof of Theorem 4.2.1 to the semigroup level. Let $(E = (E_t), \beta = (\beta_{t,s}), \eta^{\odot} = (\eta_t))$ be the inclusion system associated to Φ. Note that E_t's are von Neumann $M_2(\mathcal{B})$-$M_2(\mathcal{B})$-modules. Let $\mathbb{E}_{ij} := \mathbf{1} \otimes E_{ij} \in \mathcal{B} \otimes M_2$, where E_{ij}'s are the matrix units in M_2. Let $\hat{E}^i_t := \mathbb{E}_{ii}E_t \subseteq E_t, i = 1, 2$. Then $\hat{E}^i_t$'s are SOT closed $M_2(\mathcal{B})$-submodules of E_t such that $E_t = \hat{E}^1_t \oplus \hat{E}^2_t$ for all $t \geq 0$. Let $\eta^i_t := \mathbb{E}_{ii}\eta_t\mathbb{E}_{ii} \in \hat{E}^i_t$, $i = 1, 2$. Then we have (as in the proof of Theorem 4.2.1)

$$\eta_t = \eta^1_t + \eta^2_t \text{ with } \langle \eta^1_t, \eta^2_t\rangle = 0 \text{ and } \eta^i_t = \mathbb{E}_{ii}\eta_t = \eta_t\mathbb{E}_{ii} \quad \text{for all } t \geq 0, i = 1, 2. \tag{4.3.3}$$

[3]Note that, T_t's are bilinear (cf. Definition 2.3.13).

[4]By a block CP semigroup, we mean, a semigroup of block CP maps.

As $\beta_{t,s} : E_{t+s} \to E_t \odot E_s$ are the canonical maps: $\eta_{t+s} \mapsto \eta_t \odot \eta_s$, using (4.3.3) we have,

$$\beta_{t,s}(\eta^i_{t+s}) = \beta_{t,s}(\mathbb{E}_{ii}\eta_{t+s}\mathbb{E}_{ii}) = \mathbb{E}_{ii}\eta_t \odot \eta_s\mathbb{E}_{ii} = \eta^i_t \odot \eta^i_s \quad \text{for } t, s \geq 0, i = 1, 2. \tag{4.3.4}$$

Consider the von Neumann $\mathcal{B}$-$\mathcal{B}$-modules $E^{(\mathcal{B})}_t$ (as described in Remark 4.2.1) and the von Neumann $\mathcal{B}$-modules $\hat{E}^{i(\mathcal{B})}_t$ (see Proposition 4.2.1). Notice that $\hat{E}^{i(\mathcal{B})}_t$ is also a von Neumann $\mathcal{B}$-$\mathcal{B}$-module for $i = 1, 2$ with the left action of $\mathcal{B}$ given by

$$a[x] := [\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} x] \quad \text{for } a \in \mathcal{B}, x \in \hat{E}^i_t. \tag{4.3.5}$$

Then, we have $E^{(\mathcal{B})}_t \simeq \hat{E}^{1(\mathcal{B})}_t \oplus \hat{E}^{2(\mathcal{B})}_t$ (as two-sided von Neumann modules) for all $t \geq 0$ (as in the proof of Theorem 4.2.1). Let $\xi^i_t = [\eta^i_t] \in \hat{E}^{i(\mathcal{B})}_t, i = 1, 2$. Then for $a \in \mathcal{B}, i = 1, 2$, we have

$$\langle \xi^i_t, a\xi^i_t\rangle = \sum_{r,s=1}^{2} \left\langle \mathbb{E}_{ii}\eta_t, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbb{E}_{ii}\eta_t \right\rangle_{r,s} = \sum_{r,s=1}^{2} \Phi_t \left(\mathbb{E}_{ii} \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mathbb{E}_{ii} \right)_{r,s} = \phi^i_t(a).$$

Therefore, $(\hat{E}^{i(\mathcal{B})}_t, \xi^i_t)$ is a GNS-representation (not necessarily minimal) for $\phi^i_t, i = 1, 2$. Let $E^i_t = \overline{\operatorname{span}}^s \mathcal{B}\xi^i_t\mathcal{B} \subseteq \hat{E}^{i(\mathcal{B})}_t$ be the minimal GNS-module for ϕ^i_t for $i = 1, 2$. Let $\beta^i_{t,s} : E^i_{t+s} \to E^i_t \odot E^i_s$ be the canonical maps (as in Remark 2.3.1) given by

$$\xi^i_{t+s} \mapsto \xi^i_t \odot \xi^i_s \quad \text{for } t, s \geq 0, i = 1, 2,$$

so that $(E^i = (E^i_t), \beta^i = (\beta^i_{t,s}), \xi^{\odot i} = (\xi^i_t))$ is the inclusion system associated to $\phi^i, i = 1, 2$. (Equation (4.3.4) shows that, we get the inclusion systems associated to ϕ^i's in a canonical way from the inclusion system associated to Φ.)

Let $V^i_t : E^i_t \to \hat{E}^{i(\mathcal{B})}_t$ be the inclusion maps and let $U_t : \hat{E}^{2(\mathcal{B})}_t \to \hat{E}^{1(\mathcal{B})}_t$ be defined by

$$U_t[w] = [\mathbb{E}_{12}w] \quad \text{for } w \in \hat{E}^2_t.$$

Then V^i_t's are adjointable, bilinear isometries and U_t's are bilinear unitaries (as in the proof of Theorem 4.2.1). Take $T_t := V^{1*}_t U_t V^2_t$. Then $T_t : E^2_t \to E^1_t$ is an adjointable, bilinear contraction such that for $a \in \mathcal{B}$,

$$\begin{aligned} \langle \xi^1_t, T_t a\xi^2_t\rangle &= \langle \xi^1_t, V^{1*}_t U_t V^2_t a\xi^2_t\rangle = \langle V^1_t[\eta^1_t], U_t V^2_t a[\eta^2_t]\rangle = \left\langle [\eta^1_t], U_t[\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \eta^2_t] \right\rangle \\ &= \left\langle [\eta^1_t], [\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \eta^2_t] \right\rangle = \sum_{i,j=1}^{2} \left\langle \eta^1_t, \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \eta^2_t \right\rangle_{i,j} \end{aligned}$$

$$= \sum_{i,j=1}^{2} \left(\mathbb{E}_{11} \Phi_t \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mathbb{E}_{22} \right)_{i,j} = \sum_{i,j=1}^{2} \begin{pmatrix} 0 & \psi_t(a) \\ 0 & 0 \end{pmatrix}_{i,j} = \psi_t(a).$$

For $a, b, c, d \in \mathcal{B}$,

$$\begin{aligned}
\langle a\xi^1_{t+s} b, T_{t+s}(c\xi^2_{t+s} d)\rangle &= b^* \psi_{t+s}(a^*c)d \\
&= b^* \psi_s(\psi_t(a^*c))d \\
&= b^* \psi_s(\langle \xi^1_t, a^*cT_t\xi^2_t\rangle)d \\
&= b^* \langle \xi^1_s, T_s(\langle \xi^1_t, a^*cT_t\xi^2_t\rangle \xi^2_s)\rangle d \\
&= b^* \langle \xi^1_s, \langle \xi^1_t, a^*cT_t\xi^2_t\rangle T_s\xi^2_s\rangle d \\
&= b^* \langle \xi^1_t \odot \xi^1_s, a^*c(T_t \odot T_s)(\xi^2_t \odot \xi^2_s)\rangle d \\
&= \langle \beta^1_{t,s}(a\xi^1_{t+s}b), (T_t \odot T_s)\beta^2_{t,s}(c\xi^2_{t+s}d)\rangle \\
&= \langle a\xi^1_{t+s}b, \beta^{1*}_{t,s}(T_t \odot T_s)\beta^2_{t,s}(c\xi^2_{t+s}d)\rangle,
\end{aligned}$$

shows that $T := (T_t)_{t\geq 0}$ is a morphism of inclusion systems from (E^2, β^2) to (E^1, β^1).

To prove the uniqueness of T, let $T' = (T'_t)_{t\geq 0}$ be another morphism of inclusion systems from $(E^2, \beta^2, \xi^{\odot 2})$ to $(E^1, \beta^1, \xi^{\odot 1})$ such that $\psi_t(a) = \langle \xi^1_t, T'_t(a\xi^2_t)\rangle$ for all $a \in \mathcal{B}, t \geq 0$, then

$$\langle a_1\xi^1_t b_1, T(a_2\xi^2_t b_2)\rangle = b_1^*\psi_t(a_1^*a_2)b_2 = \langle a_1\xi^1_t b_1, T'(a_2\xi^2_t b_2)\rangle$$

for $a_1, a_2, b_1, b_2 \in \mathcal{B}$ and hence $T_t = T'_t$ for all $t \geq 0$. □

Example 4.3.3. Let $\mathcal{B}$ be a von Neumann algebra. Let E be a von Neumann $M_2(\mathcal{B})$-$M_2(\mathcal{B})$-module. Take $\beta = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_2 \end{pmatrix}$ in $M_2(\mathcal{B})$ and $\zeta \in E$ such that $\zeta = \mathbb{E}_{11}\zeta\mathbb{E}_{11} + \mathbb{E}_{22}\zeta\mathbb{E}_{22}$, where $\mathbb{E}_{ij} = \mathbf{1} \otimes E_{ij} \in \mathcal{B} \otimes M_2$ and $\{E_{ij}\}_{i,j=1}^2$ are the matrix units in M_2.

Let $\xi^{\odot}(\beta, \zeta) = (\xi_t(\beta, \zeta))_{t\in\mathbb{R}_+} \in \mathbb{\Gamma}^{\odot}(E)$, the product system of time ordered Fock module[5] over E, where the component ξ^n_t of $\xi_t(\beta, \zeta) \in \mathbb{\Gamma}_t(E)$ in the n-particle ($n > 0$) sector is defined as

$$\xi^n_t(t_n, \ldots, t_1) = e^{(t-t_n)\beta}\zeta \odot e^{(t_n - t_{n-1})\beta}\zeta \odot \cdots \odot e^{(t_2-t_1)\beta}\zeta e^{t_1\beta}. \tag{4.3.6}$$

and $\xi^0_t = e^{t\beta}$. Then it follows from [LS01, Theorem 3] that, $\xi^{\odot}(\beta, \zeta)$ is a unit for the product system $\mathbb{\Gamma}^{\odot}(E)$. Further if $\Phi^{(\beta,\zeta)}_t : M_2(\mathcal{B}) \to M_2(\mathcal{B})$ is defined by

$$\Phi^{(\beta,\zeta)}_t(A) = \langle \xi_t(\beta, \zeta), A\xi_t(\beta, \zeta)\rangle \quad \text{for } A \in M_2(\mathcal{B}), \tag{4.3.7}$$

[5] See Subsection 2.3.5 for the details about the time ordered Fock module.

then $\Phi := (\Phi_t)_{t\geq 0}$ is a uniformly continuous CP-semigroup on $M_2(\mathcal{B})$, with bounded generator

$$L(A) = L^{(\beta,\zeta)}(A) = A\beta + \beta^* A + \langle \zeta, A\zeta\rangle \quad \text{for } A \in M_2(\mathcal{B}). \tag{4.3.8}$$

Let $\zeta_i = \mathbb{E}_{ii}\zeta\mathbb{E}_{ii}, i = 1, 2$, then $\zeta = \zeta_1 + \zeta_2, \langle \zeta_1, \zeta_2\rangle = 0$. Let $\tau : M_2(\mathcal{B}) \to M_2(\mathcal{B})$ be defined by $\tau(A) = \langle \zeta, A\zeta\rangle, A \in M_2(\mathcal{B})$, then τ is a block CP map, say $\tau = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12}^* & \tau_{22} \end{pmatrix}$.

Note that $(E^{(\mathcal{B})}, [\zeta_i])$ is a GNS-representation for $\tau_{ii}, i = 1, 2$, where $E^{(\mathcal{B})}$ is the von Neumann $\mathcal{B}$-$\mathcal{B}$-module as described[6] in Remark 4.2.1.

Let $E_i = \overline{\operatorname{span}}^s \mathcal{B}[\zeta_i]\mathcal{B} \subseteq E^{(\mathcal{B})}$ be the minimal GNS-representation for $\tau_{ii}, i = 1, 2$ and let $T : E_2 \to E_1$ be the unique bilinear, adjointable contraction such that $\tau_{12}(a) = \langle [\zeta_1], Ta[\zeta_2]\rangle$ as given in Theorem 4.2.1. Therefore, we have

$$\tau(A) = \langle \zeta, A\zeta\rangle = \begin{pmatrix} \langle [\zeta_1], a_{11}[\zeta_1]\rangle & \langle [\zeta_1], Ta_{12}[\zeta_2]\rangle \\ \langle [\zeta_2], T^* a_{21}[\zeta_1]\rangle & \langle [\zeta_2], a_{22}[\zeta_2]\rangle \end{pmatrix}, \text{ for } A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathcal{B})$$

and hence

$$\begin{aligned} L(A) &= A\beta + \beta^* A + \langle \zeta, A\zeta\rangle \\ &= \begin{pmatrix} a_{11}\beta_1 + \beta_1^* a_{11} + \langle [\zeta_1], a_{11}[\zeta_1]\rangle & a_{12}\beta_2 + \beta_1^* a_{12} + \langle [\zeta_1], Ta_{12}[\zeta_2]\rangle \\ a_{21}\beta_1 + \beta_2^* a_{21} + \langle [\zeta^2], T^* a_{21}[\zeta_1]\rangle & a_{22}\beta_2 + \beta_2^* a_{22} + \langle [\zeta_2], a_{22}[\zeta_2]\rangle \end{pmatrix} \\ &= \begin{pmatrix} L_{11}^{(\beta_1,[\zeta_1])}(a_{11}) & L_{12}^{(\beta_1,\beta_2,[\zeta_1],[\zeta_2],T)}(a_{12}) \\ L_{21}^{(\beta_1,\beta_2,[\zeta_1],[\zeta_2],T)}(a_{21}) & L_{22}^{(\beta_2,[\zeta_2])}(a_{22}) \end{pmatrix}, \end{aligned}$$

where

$$L_{ii}(a) = L_{ii}^{(\beta_i,[\zeta_i])}(a) = a\beta_i + \beta_i^* a + \langle [\zeta_i], a[\zeta_i]\rangle, \quad i = 1, 2,$$

and

$$L_{12}(a) = L_{12}^{(\beta_1,\beta_2,[\zeta_1],[\zeta_2],T)}(a) = a\beta_2 + \beta_1^* a + \langle [\zeta_1], Ta[\zeta_2]\rangle_{\mathcal{B}}, \tag{4.3.9}$$

$$L_{21}(a) = L_{12}(a^*)^*,$$

for $a \in \mathcal{B}$. Therefore, for $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(\mathcal{B})$,

$$\Phi_t(A) = e^{tL(A)} = \begin{pmatrix} e^{tL_{11}^{(\beta_1,[\zeta_1])}(a_{11})} & e^{tL_{12}^{(\beta_1,\beta_2,[\zeta_1],[\zeta_2],T)}(a_{12})} \\ e^{tL_{21}^{(\beta_1,\beta_2,[\zeta_1],[\zeta_2],T)}(a_{21})} & e^{tL_{22}^{(\beta_2,[\zeta_2])}(a_{22})} \end{pmatrix}.$$

[6]See also Propositions 4.2.1,4.2.2.

Now note that the inclusion system $(E^i = (E^i_t), \xi^i = (\xi^i_t(\beta_i, [\zeta_i])))$ associated to $\phi^i = (e^{tL_{ii}^{(\beta_i,[\zeta_i])}})_{t\geq 0}$ is a subsystem of the product system of time-ordered Fock module $\mathbb{\Gamma}^{\odot}(E_i)$ over $E_i, i = 1, 2$.

Let $w = (w_t)_{t\geq 0}$ be the contractive morphism from (E^2, ξ^2) to (E^1, ξ^1) such that

$$e^{tL_{12}}(a) = \langle \xi^1_t(\beta_1, [\zeta_1]), a w_t(\xi^2_t(\beta_2, [\zeta_2]))\rangle, \quad \text{for all } a \in \mathcal{B}. \tag{4.3.10}$$

As any morphism maps a unit to a unit we have

$$w_t(\xi^2_t(\beta_2, [\zeta_2])) = \xi^1_t(\gamma_w(\beta_2, [\zeta_2]), \eta_w(\beta_2, [\zeta_2])) \tag{4.3.11}$$

for some $\gamma_w, \eta_w : \mathcal{B} \times E_2 \to \mathcal{B} \times E_1$. Hence from (4.3.10) and (4.3.11) we have

$$e^{tL_{12}}(a) = \langle \xi^1_t(\beta_1, [\zeta_1]), \xi^1_t(\gamma_w(\beta_2, [\zeta_2]), \eta_w(\beta_2, [\zeta_2]))\rangle.$$

Now by differentiating (4.3.10), we get

$$L_{12}(a) = \langle \zeta_1, a\eta_w(\beta_2, [\zeta_2])\rangle + a\gamma_w(\beta_2, [\zeta_2]) + \beta_1^* a. \tag{4.3.12}$$

Therefore as (4.3.9)=(4.3.12) we have $\gamma_w(\beta_2, [\zeta_2]) = \beta_2$ and $\eta_w(\beta_2, [\zeta_2]) = T[\zeta_2]$. Thus, the unique morphism (w_t) is given by

$$w_t \xi^2_t(\beta_2, [\zeta_2]) = \xi^1_t(\beta_2, T[\zeta_2]).$$

4.3.2 E_0-dilation of block quantum Markov semigroups

In this subsection we shall prove that if we have a block QMS on a unital C^*-algebra then the E_0-dilation constructed in [BS00] is also a semigroup of block maps.

Let $\mathcal{B}$ be a unital C^*-algebra. Let $p \in \mathcal{B}$ be a projection. Denote $p' = \mathbf{1} - p$. Let $\Phi = (\Phi_t)_{t\geq 0}$ be a block QMS[7] on $\mathcal{B}$ with respect to p.

Let $(E = (E_t), \xi^{\odot} = (\xi_t))$ be the inclusion system associated to Φ. Recall from Subsection 2.3.4 (cf. [BS00, Sections 4, 5]) that

$$\begin{array}{c} (E_t, \xi_t) \\ \Big\downarrow \text{first inductive limit} \\ (\mathcal{E}_t, \xi^t) \\ \Big\downarrow \text{second inductive limit} \\ (\mathcal{E}, \xi) \end{array}$$

[7] a block QMS, is a QMS of block maps.

That is, we have a $\mathcal{B}$-module $\mathcal{E}$ with $\mathcal{E} \simeq \mathcal{E} \odot \mathcal{E}_t$, a representation $j_0 : \mathcal{B} \to \mathcal{B}^a(\mathcal{E})$ ($b \mapsto |\xi\rangle\, b\, \langle\xi|$) and endomorphisms $\vartheta_t : \mathcal{B}^a(\mathcal{E}) \to \mathcal{B}^a(\mathcal{E})$ defined by $\vartheta_t(a) = a \odot \mathrm{id}_{\mathcal{E}_t}$ such that $(\vartheta_t)_{t\geq 0}$ is an E_0-dilation of $(\Phi_t)_{t\geq 0}$. Moreover, we have the Markov property

$$j_0(\mathbf{1})\vartheta_t(j_0(x))j_0(\mathbf{1}) = j_0(\Phi_t(x)), \quad x \in \mathcal{B}. \tag{4.3.13}$$

This implies that $j_0(\mathbf{1})\vartheta_t(j_0(\mathbf{1}))j_0(\mathbf{1}) = j_0(\Phi_t(\mathbf{1})) = j_0(\mathbf{1})$. Since $j_0(\mathbf{1})$ is a projection, we have $j_0(\mathbf{1}) \leq \vartheta_t(j_0(\mathbf{1}))$ and hence $(\vartheta_t(j_0(\mathbf{1})))_{t\geq 0}$ is an increasing family of projections. Hence it converges in SOT. Now if $k_s : \mathcal{E}_s \to \mathcal{E}$ are the canonical maps ($x_s \mapsto \xi \odot x_s$) then

$$\overline{\operatorname{span}}\, k_s(\mathcal{E}_s) = \mathcal{E}. \tag{4.3.14}$$

Hence $\vartheta_t(j_0(\mathbf{1}))(\mathcal{E}_t) = \vartheta_t(j_0(\mathbf{1}))(\xi \odot \mathcal{E}_t) = (|\xi\rangle\langle\xi| \odot \mathrm{id}_{\mathcal{E}_t})(\xi \odot \mathcal{E}_t) = \xi \odot \mathcal{E}_t$ shows that $\vartheta_t(j_0(\mathbf{1}))_{t\geq 0}$ is converging in SOT to $\mathrm{id}_{\mathcal{E}}$, the identity on $\mathcal{E}$.

Now for $q = p$ or p', consider $\vartheta_t(j_0(q)) = \vartheta_t(|\xi\rangle q\langle\xi|)$. Note that since Φ is a unital block semigroup $\Phi_t(q) = q$ for $q = p, p'$. Hence by the Markov property (4.3.13) we have

$$j_0(\mathbf{1})\vartheta_t(j_0(q))j_0(\mathbf{1}) = j_0(\Phi_t(q)) = j_0(q), \quad \text{for } q = p, p'. \tag{4.3.15}$$

Note that $j_0(\mathbf{1}) = j_0(p) + j_0(p')$ and $j_0(p)j_0(p') = j_0(p')j_0(p) = 0$. Hence multiplying by $j_0(q)$ on both sides of Equation (4.3.15) we get

$$j_0(q)\vartheta_t(j_0(q))j_0(q) = j_0(q), \quad \text{for } q = p, p'.$$

Since $j_0(q)$ is a projection, we have $j_0(q) \leq \vartheta_t(j_0(q))$ for all t, hence $\vartheta_s(j_0(q)) \leq \vartheta_t(j_0(q))$ for $s \leq t$. Therefore $(\vartheta_t(j_0(q)))_{t\geq 0}$ is an increasing family of projections in $\mathcal{B}^a(\mathcal{E})$. Say $(\vartheta_t(j_0(p)))_{t\geq 0}$ converges to P. Then as $(\vartheta_t(j_0(\mathbf{1}))_{t\geq 0}$ converges to $\mathrm{id}_{\mathcal{E}}$, $(\vartheta_t(j_0(p')))_{t\geq 0}$ will converge to $P' = \mathrm{id}_{\mathcal{E}} - P$. Note that we have $PP' = 0$ and

$$\overline{\operatorname{span}}^s \vartheta_t(j_0(p))(\mathcal{E}) = P(\mathcal{E}) \quad \text{and} \quad \overline{\operatorname{span}}^s \vartheta_t(j_0(p'))(\mathcal{E}) = P'(\mathcal{E}). \tag{4.3.16}$$

Thus, we have $\mathcal{E} = \mathcal{E}^{(1)} \oplus \mathcal{E}^{(2)}$ where $\mathcal{E}^{(1)} = P(\mathcal{E})$ and $\mathcal{E}^{(2)} = P'(\mathcal{E})$.

Lemma 4.3.2. *$P(\mathcal{E}_t) = \vartheta_t(j_0(p))(\mathcal{E}_t)$ and $P'(\mathcal{E}_t) = \vartheta_t(j_0(p'))(\mathcal{E}_t)$ for all $t \geq 0$.*

Proof. Fix $t \geq 0$. It is enough to prove for $q = p, p'$ that $\vartheta_s(j_0(q))(x) = \vartheta_t(j_0(q))(x)$ if $x \in \mathcal{E}_t$, $s \geq t$. Note that (since $\langle\xi^t, q\xi^t\rangle = \Phi_t(q) = q$ for $q = p$ or p') we have

$$\left\|p\xi^t - p\xi^t p\right\|^2 = \left\|p\xi^t p'\right\|^2 = \|p'\Phi_t(p)p'\| = 0 = \|p\Phi_t(p')p\| = \left\|p'\xi^t p\right\|^2 = \left\|\xi^t p - p\xi^t p\right\|^2.$$

This implies that $p\xi^t = \xi^t p = p\xi^t p$. Similarly we have $p'\xi^t = \xi^t p' = p'\xi^t p'$.

Let $q = p$ or p' and let $s \geq t$. If $x \in \mathcal{E}_t$, then $\xi^{s-t} \odot x \in \mathcal{E}_s$ and

$$\begin{aligned}\vartheta_s(j_0(q))(x) &= (|\xi\rangle\, q\, \langle\xi| \odot \mathrm{id}_{\mathcal{E}_s})(\xi \odot \xi^{s-t} \odot x) = \xi \odot q\xi^{s-t} \odot x = \xi \odot \xi^{s-t} q \odot x \\ &= \xi \odot \xi^{s-t} \odot qx = \xi \odot qx = \vartheta_t(j_0(q))(x).\end{aligned}$$

□

We have from [BS00, Theorem 5.4] that

$$\mathcal{E} \simeq \mathcal{E} \odot \mathcal{E}_t, \quad \text{for all } t \geq 0. \tag{4.3.17}$$

Now we shall prove a similar result for $\mathcal{E}^{(i)}$'s by recalling the proof of this result. It is important to note that we are not getting something like $\mathcal{E}^{(i)} = \mathcal{E}^{(i)} \odot \mathcal{E}_t^{(i)}$, and we have not even bothered to define $\mathcal{E}_t^{(i)}$.

Lemma 4.3.3. *$\mathcal{E}^{(i)} \simeq \mathcal{E}^{(i)} \odot \mathcal{E}_t$, for $i = 1, 2, t \geq 0$.*

Proof. Let $k_t : \mathcal{E}_t \to \mathcal{E}$ be the canonical maps (isometries). Then $u_t : \mathcal{E} \odot \mathcal{E}_t \to \mathcal{E}$ defined by

$$u_t(k_s(x_s) \odot y_t) = k_{s+t}(x_s \odot y_t) \tag{4.3.18}$$

for $x_s \in \mathcal{E}_s, y_t \in \mathcal{E}_t$, is a unitary ([BS00, Theorem 5.4]). Hence, we have $\mathcal{E} \simeq \mathcal{E} \odot \mathcal{E}_t$. Since $\mathcal{E} = \mathcal{E}^{(1)} \oplus \mathcal{E}^{(2)}$, we have, $\mathcal{E} \simeq \mathcal{E}^{(1)} \odot \mathcal{E}_t \oplus \mathcal{E}^{(2)} \odot \mathcal{E}_t$.

We shall prove that, the restriction of this unitary u_t to $\mathcal{E}^{(i)} \odot \mathcal{E}_t$ is a unitary from $\mathcal{E}^{(i)} \odot \mathcal{E}_t$ onto $\mathcal{E}^{(i)}$. It is enough to prove that $u_t(\mathcal{E}^{(i)} \odot \mathcal{E}_t) \subseteq \mathcal{E}^{(i)}$. To prove this, (from (4.3.14), (4.3.16) and Lemma 4.3.2) it is sufficient to prove that $u_t(\vartheta_s(j_0(p))k_s(\mathcal{E}_s) \odot \mathcal{E}_t) \subseteq \mathcal{E}^{(1)}$ and $u_t(\vartheta_s(j_0(p'))k_s(\mathcal{E}_s) \odot \mathcal{E}_t) \subseteq \mathcal{E}^{(2)}$. To prove this consider for $q = p$ or p' and $x_s \in \mathcal{E}_s$

$$\begin{aligned}u_t(\vartheta_s(j_0(q))k_s(x_s) \odot y_t) &= u_t((\xi q \odot x_s) \odot y_t) = u_t((\xi \odot qx_s) \odot y_t) \\ &= u_t(k_s(qx_s) \odot y_t) \\ &= k_{s+t}(qx_s \odot y_t) \\ &= \xi \odot qx_s \odot y_t = \xi q \odot x_s \odot y_t \\ &= \vartheta_{s+t}(j_0(q))k_{s+t}(x_s \odot y_t),\end{aligned}$$

which is in $\mathcal{E}^{(1)}$ if $q = p$ and is in $\mathcal{E}^{(2)}$ if $q = p'$. □

Theorem 4.3.2. *The E_0-dilation $\vartheta = (\vartheta_t)_{t \geq 0}$ of Φ is a semigroup of block maps with respect to the projection P defined above.*

Proof. As $\mathcal{E} = \mathcal{E}^{(1)} \oplus \mathcal{E}^{(2)}$, we have

$$\mathcal{B}^a(\mathcal{E}) = \begin{pmatrix} \mathcal{B}^a(\mathcal{E}^{(1)}) & \mathcal{B}^a(\mathcal{E}^{(2)}, \mathcal{E}^{(1)}) \\ \mathcal{B}^a(\mathcal{E}^{(1)}, \mathcal{E}^{(2)}) & \mathcal{B}^a(\mathcal{E}^{(2)}) \end{pmatrix}.$$

For any $i, j \in \{1, 2\}$, let $a \in \mathcal{B}^a(\mathcal{E}^{(i)}, \mathcal{E}^{(j)})$, then

$$\vartheta_t(a) = a \odot \mathrm{id}_{\mathcal{E}_t} \in \mathcal{B}^a(\mathcal{E}^{(i)} \odot \mathcal{E}_t, \mathcal{E}^{(j)} \odot \mathcal{E}_t) = \mathcal{B}^a(\mathcal{E}^{(i)}, \mathcal{E}^{(j)}).$$

Therefore ϑ_t acts block-wise. □

4.4 Lifting of morphisms

In this section we will show that any (weak) morphism between two inclusion systems of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules can be always lifted as a morphism between the product systems generated by them.

We shall recall some notations and results from Subsection 2.3.4 (cf. [BS00] and [BM10]). For all $t > 0$ we define

$$\mathbb{J}_t = \{\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{T}^n : t_i > 0, |\mathfrak{t}| = t, n \in \mathbb{N}\} \tag{4.4.1}$$

and for $\mathfrak{s} = (s_m, \ldots, s_1) \in \mathbb{J}_s$ and $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$ we define the *joint tuple* $\mathfrak{s} \smile \mathfrak{t} \in \mathbb{J}_{s+t}$ by

$$\mathfrak{s} \smile \mathfrak{t} = ((s_m, \ldots, s_1), (t_n, \ldots, t_1)) = (s_m, \ldots, s_1, t_n, \ldots, t_1).$$

We have a partial order " $\geq$ " on $\mathbb{J}_t$ as follows: $\mathfrak{t} \geq \mathfrak{s} = (s_m, \ldots, s_1)$, if for each j $(1 \leq j \leq m)$ there are (unique) $\mathfrak{s}_j \in \mathbb{J}_{s_j}$ such that $\mathfrak{t} = \mathfrak{s}_m \smile \cdots \smile \mathfrak{s}_1$ (In this case we also write $\mathfrak{s} \leq \mathfrak{t}$ to mean $\mathfrak{t} \geq \mathfrak{s}$).

For $t = 0$ we extend the definition of $\mathbb{J}_t$ as $\mathbb{J}_0 = \{()\}$, where $()$ is the empty tuple. Also for $\mathfrak{t} \in \mathbb{J}_t$ we put $\mathfrak{t} \smile () = \mathfrak{t} = () \smile \mathfrak{t}$.

Now we will describe the construction of product system generated by an inclusion system of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules using the inductive limits. (This construction holds also for Hilbert $\mathcal{B}$-$\mathcal{B}$-modules along the same lines, but as we are going to prove the lifting theorem only for von Neumann $\mathcal{B}$-$\mathcal{B}$-modules, we confine ourselves to von Neumann modules.)

Let $(E = (E_t)_{t\geq 0}, \beta = (\beta_{s,t})_{s,t\geq 0})$ be an inclusion system of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules. Fix $t \in \mathbb{T}$. Let $E_{\mathfrak{t}} = E_{t_n} \odot \cdots \odot E_{t_1}$ for $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$. For all $\mathfrak{t} = (t_n, \ldots, t_1) \in \mathbb{J}_t$ we define $\beta_{\mathfrak{t}(t)} : E_t \to E_{\mathfrak{t}}$ by

$$\beta_{\mathfrak{t}(t)} = (\beta_{t_n,t_{n-1}} \odot \mathrm{id})(\beta_{t_n+t_{n-1},t_{n-2}} \odot \mathrm{id}) \ldots (\beta_{t_n+\cdots+t_3,t_2} \odot \mathrm{id})\beta_{t_n+\cdots+t_2,t_1},$$

and for $\mathfrak{t} = (t_n, \ldots, t_1) = \mathfrak{s}_m \smile \cdots \smile \mathfrak{s}_1 \geq \mathfrak{s} = (s_m, \ldots, s_1)$ with $|\mathfrak{s}_j| = s_j$, we define $\beta_{\mathfrak{t}\mathfrak{s}} : E_{\mathfrak{s}} \to E_{\mathfrak{t}}$ by

$$\beta_{\mathfrak{t}\mathfrak{s}} = \beta_{\mathfrak{s}_m(s_m)} \odot \cdots \odot \beta_{\mathfrak{s}_1(s_1)}.$$

Then it is clear from the definitions that $\beta_{\mathfrak{t}\mathfrak{s}}, \mathfrak{t} \geq \mathfrak{s}$ are bilinear isometries and $\beta_{\mathfrak{t}\mathfrak{s}}\beta_{\mathfrak{s}\mathfrak{r}} = \beta_{\mathfrak{t}\mathfrak{r}}$ for $\mathfrak{t} \geq \mathfrak{s} \geq \mathfrak{r}$. This says that, the family $(E_{\mathfrak{t}})_{\mathfrak{t}\in\mathbb{J}_t}$ with $(\beta_{\mathfrak{t}\mathfrak{s}})_{\mathfrak{s}\leq\mathfrak{t}}$ is an inductive system of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules. Hence the inductive limit $\mathcal{E}_t = \limind_{\mathfrak{t}\in\mathbb{J}_t} E_{\mathfrak{t}}$ is also a von Neumann $\mathcal{B}$-$\mathcal{B}$-module and the canonical mappings $i_{\mathfrak{t}} : E_{\mathfrak{t}} \to \mathcal{E}_t$ are bilinear isometries.

For $\mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t$ it is clear that $E_{\mathfrak{s}} \odot E_{\mathfrak{t}} = E_{\mathfrak{s}\smile\mathfrak{t}}$. Using this observation we define $B_{st} : \mathcal{E}_s \odot \mathcal{E}_t \to \mathcal{E}_{s+t}$ by

$$B_{st}(i_{\mathfrak{s}}x_{\mathfrak{s}} \odot i_{\mathfrak{t}}y_{\mathfrak{t}}) = i_{\mathfrak{s}\smile\mathfrak{t}}(x_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \text{ for } x_{\mathfrak{s}} \in E_{\mathfrak{s}}, y_{\mathfrak{t}} \in E_{\mathfrak{t}}, \mathfrak{s} \in \mathbb{J}_s, \mathfrak{t} \in \mathbb{J}_t.$$

Then $(\mathcal{E} = (\mathcal{E}_t)_{t\in\mathbb{T}}, B = (B_{st})_{s,t\in\mathbb{T}})$ forms a product system (Bhat and Skeide [BS00]).

Definition 4.4.1. Given an inclusion system (E, β), the product system $(\mathcal{E}, B)$ described above is called the product system generated by the inclusion system (E, β).

We also recall the following from Subsection 2.2.5: Let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. Let E be a von Neumann $\mathcal{B}$-module. Then $\mathcal{H} = E \odot \mathcal{G}$ is a Hilbert space such that $E \subseteq \mathcal{B}(\mathcal{G}, \mathcal{H})$ via $E \ni x \mapsto L_x \in \mathcal{B}(\mathcal{G}, \mathcal{H})$, where $L_x : \mathcal{G} \to \mathcal{H}$ is defined by $L_x(g) = x \odot g$ for $g \in \mathcal{G}$. Note that E is strongly closed in $\mathcal{B}(\mathcal{G}, \mathcal{H})$. Sometimes we write xg instead of $x \odot g$ with the above identification in mind.

Lemma 4.4.1. *Let $(\mathcal{E}, B)$ be the product system generated by the inclusion system (E, β) on a von Neumann algebra $\mathcal{B} \subseteq \mathcal{B}(\mathcal{G})$. Let $i_{\mathfrak{t}} : E_{\mathfrak{t}} \to \mathcal{E}_t, \mathfrak{t} \in \mathbb{J}_t$ be the canonical two-sided isometries. Then $i_{\mathfrak{t}}i_{\mathfrak{t}}^*$ increases to identity in strong operator topology, that is, for all $x \in \mathcal{E}_t$ and $g \in \mathcal{G}$ we have*

$$\lim_{\mathfrak{t}\in\mathbb{J}_t} \|xg - i_{\mathfrak{t}}i_{\mathfrak{t}}^*(x)g\| = 0. \tag{4.4.2}$$

Proof. Note that we have $E_{\mathfrak{t}} \subseteq \mathcal{B}(\mathcal{G}, H_{\mathfrak{t}})$ and $\mathcal{E}_t \subseteq \mathcal{B}(\mathcal{G}, \mathcal{H}_t)$ where $H_{\mathfrak{t}} = E_{\mathfrak{t}} \odot \mathcal{G}$ and $\mathcal{H}_t = \mathcal{E}_t \odot \mathcal{G}$. Let $v_{\mathfrak{t}} = i_{\mathfrak{t}} \odot \mathrm{id} : H_{\mathfrak{t}} \to \mathcal{H}_t$. Then $v_{\mathfrak{t}}$'s are isometries. Note that for $\mathfrak{s} \leq \mathfrak{t} \in \mathbb{J}_t$, as $i_{\mathfrak{t}}i_{\mathfrak{t}}^*i_{\mathfrak{s}}i_{\mathfrak{s}}^* = i_{\mathfrak{s}}i_{\mathfrak{s}}^* = i_{\mathfrak{s}}i_{\mathfrak{s}}^*i_{\mathfrak{t}}i_{\mathfrak{t}}^*$ we have

$$v_{\mathfrak{t}}v_{\mathfrak{t}}^*v_{\mathfrak{s}}v_{\mathfrak{s}}^* = v_{\mathfrak{s}}v_{\mathfrak{s}}^* = v_{\mathfrak{s}}v_{\mathfrak{s}}^*v_{\mathfrak{t}}v_{\mathfrak{t}}^* \quad \text{ for } \mathfrak{s} \leq \mathfrak{t} \in \mathbb{J}_t. \tag{4.4.3}$$

Also note that as $\overline{\text{span}}^s\{i_{\mathfrak{t}}(E_{\mathfrak{t}}) : \mathfrak{t} \in \mathbb{J}_t\} = \mathcal{E}_t$ we have

$$\overline{\text{span}}\,\{v_{\mathfrak{t}}(H_{\mathfrak{t}}) : \mathfrak{t} \in \mathbb{J}_t\} = \mathcal{H}_t. \tag{4.4.4}$$

Now (4.4.3) and (4.4.4) implies that $\lim_{\mathfrak{t}\in\mathbb{J}_t} v_{\mathfrak{t}}v_{\mathfrak{t}}^*h = h$ for all $h \in \mathcal{H}_t$. Observe that for $x \in \mathcal{E}_t, \mathfrak{t} \in \mathbb{J}_t$ and $g \in \mathcal{G}$ we have

$$v_{\mathfrak{t}}v_{\mathfrak{t}}^*(xg) = (i_{\mathfrak{t}}i_{\mathfrak{t}}^* \odot \text{id})(x \odot g) = i_{\mathfrak{t}}i_{\mathfrak{t}}^*(x) \odot g = i_{\mathfrak{t}}i_{\mathfrak{t}}^*(x)g.$$

Thus

$$\lim_{\mathfrak{t}\in\mathbb{J}_t} \|xg - i_{\mathfrak{t}}i_{\mathfrak{t}}^*(x)g\| = \lim_{\mathfrak{t}\in\mathbb{J}_t} \|xg - v_{\mathfrak{t}}v_{\mathfrak{t}}^*(xg)\| = 0.$$

□

Now we shall prove the lifting theorem along the same lines of the proof of [BM10, Thorem 11]

Theorem 4.4.1. *Let $\mathcal{B}$ be a von Neumann algebra on a Hilbert space $\mathcal{G}$. Let (E, β) and (F, γ) be two inclusion systems of von Neumann $\mathcal{B}$-$\mathcal{B}$-modules generating two product systems $(\mathcal{E}, B), (\mathcal{F}, C)$ respectively. Let i, j be their respective inclusion maps. Suppose $T : (E, \beta) \to (F, \gamma)$ is a (weak) morphism then there exists a unique morphism $\hat{T} : (\mathcal{E}, B) \to (\mathcal{F}, C)$ such that $T_s = j_s^*\hat{T}_s i_s$ for all $s \in \mathbb{T}$.*

Proof. Given that $T : (E, \beta) \to (F, \gamma)$ is a morphism. Let k be such that $\|T_s\| \leq e^{ks}$ for all $s \in \mathbb{T}$. For $\mathfrak{s} = (s_n, ..., s_1) \in \mathbb{J}_s$, define $T_{\mathfrak{s}} : E_{\mathfrak{s}} \to F_{\mathfrak{s}}$ by $T_{\mathfrak{s}} = T_{s_n} \odot \cdots \odot T_{s_1}$. Let $i_{\mathfrak{s}} : E_{\mathfrak{s}} \to \mathcal{E}_s$ and $j_{\mathfrak{s}} : F_{\mathfrak{s}} \to \mathcal{F}_s$ be the canonical two-sided isometries. Then for $\mathfrak{s} \leq \mathfrak{t}$ in $\mathbb{J}_s$ we have

$$\gamma_{\mathfrak{t}\mathfrak{s}}^* T_{\mathfrak{t}} \beta_{\mathfrak{t}\mathfrak{s}} = T_{\mathfrak{s}}. \tag{4.4.5}$$

Consider for $\mathfrak{s} \in \mathbb{J}_s, \Phi_{\mathfrak{s}} = j_{\mathfrak{s}}T_{\mathfrak{s}}i_{\mathfrak{s}}^*$. Set $P_{\mathfrak{s}} = j_{\mathfrak{s}}j_{\mathfrak{s}}^*$ and $Q_{\mathfrak{s}} = i_{\mathfrak{s}}i_{\mathfrak{s}}^*$. Then by Lemma 4.4.1 $(P_{\mathfrak{s}})_{\mathfrak{s}\in\mathbb{J}_s}$ and $(Q_{\mathfrak{s}})_{\mathfrak{s}\in\mathbb{J}_s}$ are families of increasing projections. Now for $\mathfrak{r} \leq \mathfrak{s}$, $i_{\mathfrak{r}} = i_{\mathfrak{s}}\beta_{\mathfrak{s}\mathfrak{r}}$, $j_{\mathfrak{r}} = j_{\mathfrak{s}}\gamma_{\mathfrak{s}\mathfrak{r}}$ implies that $\beta_{\mathfrak{s}\mathfrak{r}} = i_{\mathfrak{s}}^*i_{\mathfrak{r}}, \gamma_{\mathfrak{s}\mathfrak{r}} = j_{\mathfrak{s}}^*j_{\mathfrak{r}}$, hence it follows from (4.4.5) that $P_{\mathfrak{r}}\Phi_{\mathfrak{s}}Q_{\mathfrak{r}} = \Phi_{\mathfrak{r}}$.

For all $s \geq 0$, $\mathcal{E}_s \subseteq \mathcal{B}(\mathcal{G}, \mathcal{E}_s \odot \mathcal{G})$ and $\mathcal{F}_s \subseteq \mathcal{B}(\mathcal{G}, \mathcal{F}_s \odot \mathcal{G})$. Fix $s \in \mathbb{T}$. Let $x \in \mathcal{E}_s, g \in \mathcal{G}$ and let $\epsilon > 0$. Using (4.4.2) choose $\mathfrak{r}_0 \in \mathbb{J}_s$ such that

$$e^{ks}\|Q_{\mathfrak{r}_0}(x)g - xg\| < \frac{\epsilon}{3}. \tag{4.4.6}$$

Then, for any $\mathfrak{s} \in \mathbb{J}_s$, we have

$$\|\Phi_{\mathfrak{s}}(x)g - \Phi_{\mathfrak{s}}Q_{\mathfrak{r}_0}(x)g\| = \|\Phi_{\mathfrak{s}}(x) \odot g - \Phi_{\mathfrak{s}}Q_{\mathfrak{r}_0}(x) \odot g\|$$

$$\begin{aligned}&= \|(\Phi_{\mathfrak{s}} \odot \mathrm{id}_{\mathcal{G}})(x \odot g - Q_{\mathfrak{r}_0}(x) \odot g)\| \\ &\leq \|\Phi_{\mathfrak{s}} \odot \mathrm{id}_{\mathcal{G}}\| \|xg - Q_{\mathfrak{r}_0}(x)g\| \\ &\leq e^{ks} \, \|xg - Q_{\mathfrak{r}_0}(x)g\| < \frac{\epsilon}{3}. \quad \text{(by (4.4.6))} \end{aligned} \tag{4.4.7}$$

Let $\mathfrak{t} \geq \mathfrak{s} \geq \mathfrak{r}_0 \in \mathbb{J}_s$. As $(P_{\mathfrak{s}})_{\mathfrak{s}\in\mathbb{J}_s}$ and $(Q_{\mathfrak{s}})_{\mathfrak{s}\in\mathbb{J}_s}$ are increasing families of projections, we have

$$\begin{aligned}\|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 &= \|P_{\mathfrak{t}}\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 \\ &= \|P_{\mathfrak{s}}\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g + (P_{\mathfrak{t}} - P_{\mathfrak{s}})\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 \\ &= \|P_{\mathfrak{s}}\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 + \|(P_{\mathfrak{t}} - P_{\mathfrak{s}})\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 \\ &= \|P_{\mathfrak{s}}\Phi_{\mathfrak{t}} Q_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2 + \|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g - P_{\mathfrak{s}}\Phi_{\mathfrak{t}} Q_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2 \\ &= \|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2 + \|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g - \Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2. \end{aligned} \tag{4.4.8}$$

Hence for $\mathfrak{t} \geq \mathfrak{s} \geq \mathfrak{r}_0 \in \mathbb{J}_s$, we have $\|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 \geq \|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2$. Also

$$\|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2 \leq \|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}\|^2 \|x\|^2 \|g\|^2 \leq e^{2ks} \|x\|^2 \|g\|^2$$

for all $\mathfrak{s} \in \mathbb{J}_s$. Thus $(\|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2)_{\mathfrak{s}\in\mathbb{J}_s}$ is a Cauchy net, hence choose $\mathfrak{r}_1 \in \mathbb{J}_s, \mathfrak{r}_1 \geq \mathfrak{r}_0$ such that

$$|\|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 - \|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2| < \left(\frac{\epsilon}{3}\right)^2 \quad \text{for } \mathfrak{t} \geq \mathfrak{s} \geq \mathfrak{r}_1 \geq \mathfrak{r}_0 \in \mathbb{J}_s. \tag{4.4.9}$$

Therefore for $\mathfrak{t} \geq \mathfrak{s} \geq \mathfrak{r}_1$ in $\mathbb{J}_s$, from (4.4.8) and (4.4.9) we have

$$\|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g - \Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\| = |\|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\|^2 - \|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\|^2|^{\frac{1}{2}} < \frac{\epsilon}{3}. \tag{4.4.10}$$

Now for $\mathfrak{t} \geq \mathfrak{s} \geq \mathfrak{r}_1$ in $\mathbb{J}_s$, from (4.4.7) and (4.4.10) we have

$$\begin{aligned}&\|(\Phi_{\mathfrak{t}} - \Phi_{\mathfrak{s}})(x)g\| \\ &\leq \|\Phi_{\mathfrak{t}}(x)g - \Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g\| + \|\Phi_{\mathfrak{t}} Q_{\mathfrak{r}_0}(x)g - \Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g\| + \|\Phi_{\mathfrak{s}} Q_{\mathfrak{r}_0}(x)g - \Phi_{\mathfrak{s}}(x)g\| < \epsilon.\end{aligned}$$

Thus $\lim_{\mathfrak{s}\in\mathbb{J}_s} \Phi_{\mathfrak{s}}(x)g$ exists. Define $\hat{T}_s(x)g = \lim_{\mathfrak{s}\in\mathbb{J}_s} \Phi_{\mathfrak{s}}(x)g$ for $s > 0$. This defines a bounded bilinear map $\hat{T}_s : \mathcal{E}_s \to \mathcal{F}_s$ for all $s \in \mathbb{T}$.

Now for $\mathfrak{s} \in \mathbb{J}_s$ and for all $x_{\mathfrak{s}} \in E_{\mathfrak{s}}, g \in \mathcal{G}$, we have

$$j_{\mathfrak{s}}^* \hat{T}_s i_{\mathfrak{s}}(x_{\mathfrak{s}})g = \lim_{\mathfrak{r}\in\mathbb{J}_s} j_{\mathfrak{s}}^* \Phi_{\mathfrak{r}} i_{\mathfrak{s}}(x_{\mathfrak{s}})g = \lim_{\mathfrak{r}\in\mathbb{J}_s} j_{\mathfrak{s}}^* j_{\mathfrak{r}} T_{\mathfrak{r}} i_{\mathfrak{r}}^* i_{\mathfrak{s}}(x_{\mathfrak{s}})g = \lim_{\mathfrak{r}\in\mathbb{J}_s} \gamma_{\mathfrak{r}\mathfrak{s}}^* T_{\mathfrak{r}} \beta_{\mathfrak{r}\mathfrak{s}}(x_{\mathfrak{s}})g = T_{\mathfrak{s}}(x_{\mathfrak{s}})g.$$

Thus $T_{\mathfrak{s}} = j_{\mathfrak{s}}^* \hat{T}_s i_{\mathfrak{s}}$ for all $\mathfrak{s} \in \mathbb{J}_s$ and $s \in \mathbb{T}$. In particular $T_s = j_s^* \hat{T}_s i_s$ for all $s \in \mathbb{T}$.

Now we shall prove that $(\hat{T}_t)_{t \geq 0}$ is a morphism of product systems. For $\mathfrak{t} \in \mathbb{J}_t, \mathfrak{s} \in \mathbb{J}_s$ and $x_{\mathfrak{t}} \in E_{\mathfrak{t}}, x_{\mathfrak{s}} \in E_{\mathfrak{s}}, y_{\mathfrak{t}} \in F_{\mathfrak{t}}, y_{\mathfrak{s}} \in F_{\mathfrak{s}}$ consider,

$$\begin{aligned}
\langle C_{s,t}^* (\hat{T}_s \odot \hat{T}_t) B_{s,t} i_{\mathfrak{s} \smile \mathfrak{t}} (x_{\mathfrak{s}} \odot x_{\mathfrak{t}}), j_{\mathfrak{s} \smile \mathfrak{t}} (y_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \rangle &= \langle (\hat{T}_s \odot \hat{T}_t)(i_{\mathfrak{s}} \odot i_{\mathfrak{t}})(x_{\mathfrak{s}} \odot x_{\mathfrak{t}}), j_{\mathfrak{s}} \odot j_{\mathfrak{t}} (y_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \rangle \\
&= \langle \hat{T}_s i_{\mathfrak{s}} x_{\mathfrak{s}} \odot \hat{T}_t i_{\mathfrak{t}} x_{\mathfrak{t}}, j_{\mathfrak{s}} y_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \rangle \\
&= \langle j_{\mathfrak{t}}^* \hat{T}_t i_{\mathfrak{t}} x_{\mathfrak{t}}, \langle j_{\mathfrak{s}}^* \hat{T}_s i_{\mathfrak{s}} x_{\mathfrak{s}}, y_{\mathfrak{s}} \rangle y_{\mathfrak{t}} \rangle \\
&= \langle T_{\mathfrak{s}} x_{\mathfrak{s}} \odot T_{\mathfrak{t}} x_{\mathfrak{t}}, (y_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \rangle \\
&= \langle (T_{\mathfrak{s}} \odot T_{\mathfrak{t}})(x_{\mathfrak{s}} \odot x_{\mathfrak{t}}), y_{\mathfrak{s}} \odot y_{\mathfrak{t}} \rangle \\
&= \langle T_{\mathfrak{s} \smile \mathfrak{t}} (x_{\mathfrak{s}} \odot x_{\mathfrak{t}}), (y_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \rangle \\
&= \langle \hat{T}_{s+t} i_{\mathfrak{s} \smile \mathfrak{t}} (x_{\mathfrak{s}} \odot x_{\mathfrak{t}}), j_{\mathfrak{s} \smile \mathfrak{t}} (y_{\mathfrak{s}} \odot y_{\mathfrak{t}}) \rangle
\end{aligned}$$

Thus $\hat{T}_{s+t} = C_{s,t}^* (\hat{T}_s \odot \hat{T}_t) B_{s,t}$ for all $s, t > 0$. $\square$

Publications/Preprints

This thesis is based on the following articles.

(1) B. V. Rajarama Bhat, Robin Hillier, Nirupama Mallick and Vijaya Kumar U, *Roots of Completely Positive Maps*, Linear Algebra Appl., 587 (2020), 143–165. MR 4030295

(2) B. V. Rajarama Bhat and Vijaya Kumar U, *Structure of Block Quantum Dynamical Semigroups and their Product Systems*, arXiv:1908.04098, Preprint.

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