Essays in Behavioral Social Choice Theory

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To my teachers,
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Contents

Chapter 1

Introduction

This thesis comprises four essays on social choice theory. The first three essays/chapters consider models where voters follow "non-standard" rules for decision making. The last chapter considers the binary social choice model and analyzes the consequences of a new axiom.

The first chapter introduces a new axiom for manipulability when voters incur a cost if they misreport their true preference ordering. The second chapter considers the random voting model with strategic voters where standard stochastic dominance strategy-proofness is replaced by strategy-proofness under two lexicographic criteria. The third chapter also considers the random voting model but from a non-strategic perspective. It introduces a new "robustness to small mistakes" by voters. The last chapter provides a characterization of the status quo rule. We provide a brief description of each chapter below.

1.1 Strategy-Proof Voting with Lying Costs

Chapter 2 considers the usual model of strategic voting where voters have private information about their preference orderings. A social choice function (SCF) is manipulable if a voter can obtain a better outcome by lying about her preference ordering than telling the truth. The standard assumption is that if a voter has any opportunity to gain (howsoever small) she will manipulate. It implicitly assumes that there is no cost of lying. There is a large experimental literature that suggests that agents are averse to lying - see for instance, Abeler et al. (2019), Gneezy et al. (2013), Kajackaite and Gneezy (2015) Lundquist et al. (2007) and Lundquist et al. (2009). Motivated by the experimental evidence, we say that a SCF is K-manipulable ($K \geq 1$) if the voter can improve by *at least* K ranks in her true preference ordering by lying. If $K > 1$, we have a weaker notion of strategy-proofness that captures the idea that a voter manipulates only when the gains from lying are "substantial". If K

equals the number of alternatives, K-strategy-proofness will be trivially satisfied. We can therefore reasonably expect a wider class of SCFs to be strategy-proof as the "reluctance to lie" increases.

Our main result is that this optimism is somewhat misplaced. Our main result is that under various notions of efficiency and for a sufficiently large number of alternatives, a SCF is K-strategy-proof only if it is K-dictatorial, i.e. there exists a voter such that the outcome at every profile is one of her top K-ranked alternatives. As K increases, the strength of strategy-proofness decreases but so does the power of K-dictatorship. Rather surprisingly, they exactly counterbalance each other. We also show that the well-known equivalence of ontoness, unanimity and efficiency for strategy-proof SCFs breaks down when strategy-proofness is replaced by K-strategy-proofness. We also have various results on the lower bound of number of alternatives required to prove the K-dictatorial SCF.

1.2 Random Strategy-Proof Voting with Lexicographic Extension

An important issue in the random environments, especially in strategic models is that preferences in the voting model are ordinal rankings while the outcome of voting is a probability distribution over alternatives. In order to compare the outcomes for different voting profiles, it is necessary to specify an appropriate extension from an (ordinal) preference ordering to lotteries - in other words, to extend preferences over degenerate lotteries to preferences over all lotteries. The choice of an extension has profound implications for the analysis.

The standard notion of strategy-proofness is proposed by Gibbard (1977). According to Gibbard, a RSCF is *sd*-strategy-proof if the truth telling lottery *stochastically dominates* all lotteries obtained via misrepresentation. Gibbard (1977) showed that all *sd* -strategy-proof RSCFs which satisfy the additional (mild) property of *unanimity*, must be a *random dictatorship.* We replace the *sd*-extension by two simple lottery extensions based on lexicographic comparisons. The first is the *downward lexicographic* or *dl*-extension and the second is the *upward lexicographic* or *ul*-extension. While comparing two lotteries in the former case, the voter will prefer the lottery which has the higher probability on the first-ranked alternative. If they are the same, the voter will consider probabilities assigned to the second-ranked alternative, preferring the lottery which has higher probability. If they are the same, she will consider the third-ranked alternative and so on till the last ranked alternative. The voter in this case cares "much more" about a higher ranked alternative than a lower-ranked alternative.

There are two broad sets of results in this chapter. The first concerns *ul*-strategy-proofness. We show that the Gibbard (1977) random dictatorship result continues to hold, i.e. every RSCF satisfying *ul*-strategy-proofness and unanimity must be a random dictatorship. This is rather surprising in view of the fact that *ul*-strategy-proofness is significantly weaker than *sd*-strategy-proofness. The second set of results concern RSCFs that satisfy *dl*-strategy-proofness. We show that a wider class of RSCFs beyond random dictatorship satisfy unanimity and *dl*-strategy-proofness. However, if unanimity is strengthened to *efficiency*, *dl*-strategy-proof RSCFs must be *top-support* rules, i.e. they can give strictly positive probability in a profile only to alternatives that are ranked first by some voter. We show that a class of RSCFs that we call *top-weight rules*, are characterized by *dl*-strategy-proofness, efficiency and an additional but familiar property of *tops-onlyness*. In the case of two voters, we show that the tops-onlyness property is implied by the other two requirements.

1.3 Stochastic same-sidedness in random voting models

In this chapter, we study the standard random voting model. In this framework, we propose an axiom called *stochastic same-sidedness* (SSS) and explore its consequences. Consider a preference profile and suppose a voter changes her preference ordering to an adjacent one by swapping two consecutively ranked alternatives. Then SSS imposes two restrictions on the RSCF. First, the sum of probabilities of the alternatives which are ranked strictly higher than the swapped pair, should remain the same. Second, the sum probabilities assigned to the swapped pair, should also remain the same. In the deterministic framework Muto and Sato (2017) showed that this innocuous-looking property has strong negative implications. The key question addressed in this paper is the following: does randomization significantly expand the class of RSCFs satisfying SSS relative to the deterministic case?

We show that in the two voters case, every RSCF that satisfies *efficiency* and SSS, is a *random dictatorship*. The result does not hold if we replace *efficiency* by *unanimity*. If there are more than two voters, *efficiency* and SSS do not imply *random dictatorship*. However, if RSCFs are required to satisfy *tops-onlyness* in addition to *efficiency* and SSS, we have *random dictatorship* again. The weakest form of strategy-proofness is the *weak sd-strategy-proofness*. We also show that SSS and weak *sd*-strategy-proofness are independent i.e. neither implies the other. In particular, SSS allows instances where the truth-telling lottery gives (strictly) lower expected utility than a lottery obtained via a misreport for every utility representation of voter's true ordinal preference. In other words, truth-telling lottery is stochastically dominated by a lottery obtained by lying. As a consequence, the SSS axiom cannot be interpreted as an incentive-compatible property.

1.4 A characterization of the status quo rule in the binary social choice model

This chapter considers the model of binary social choice. Each voter can have one of the three preferences - one alternative can be strictly preferred to the other or they could be indifferent to each other. The status quo rule identifies one of the two alternatives as the status quo alternative. The rule picks this alternative at all profiles except the one where all voters rank the non-status quo alternative strictly better than the status quo alternative. It is a conservative rule which is “almost” constant. However, it is an appealing rule in certain circumstances where change from the status quo can impose losses on a large number of voters. Examples of such policies in India in recent years have been the Citizen Amendment Act, the Goods and Services tax, the demonetization policy (2016) and the four-year undergraduate program at Delhi University.

We use three axioms for our characterization. Two of these properties, ontoness and strategy-proofness are well-known in the axiomatic literature. The third one is a new axiom introduced by us, which we call *Positive Welfare Association* (PWA). To understand the axiom, consider a profile where a particular voter, say i is indifferent. Suppose i changes her preference from indifference to a strict preference. The new outcome differs from the earlier one and is i's strictly preferred outcome in the new preference ordering. Then, PWA requires all other voters not to be made worse-off at the new profile. The PWA axiom is key to our result and cannot be replaced by welfare dominance in the characterization. We also show that our characterization is tight by providing examples of non-status quo rules that satisfy all but one of the axioms.

Chapter 2

Strategy-Proof Voting with Lying Costs

2.1 Introduction

In this chapter we study the standard strategic voting model in deterministic framework. It analyzes situations where a group of agents/voters want to decide what outcome should be selected for the whole group from the set of available alternatives based on the their (ordinal) preferences over the alternatives. Our goal in this chapter is to examine the consequences of departing from the following hypothesis that underlies virtually all of mechanism design theory: a voter will lie/manipulate whenever she can gain by doing so. The hypothesis implicitly assumes that lying is costless. We will assume instead that lying has moral or psychological costs so that an agent will lie only if the gains from lying are "sufficiently large".

There is a large experimental literature that suggests that agents are averse to lying - see for instance, Abeler et al. (2019), Gneezy et al. (2013), Kajackaite and Gneezy (2015) Lundquist et al. (2007) and Lundquist et al. (2009). To quote Kajackaite and Gneezy (2015): "we find that people lie more, and in particular lie more when the incentives to do so increase". We will make a further assumption about the nature of lying - an agent may not lie if the gains from lying are "small" but when she lies, she does so in a manner that maximizes her gain from lying. The assumption is consistent with constant costs of lying. There is experimental evidence to support this hypothesis as well. Quoting Kajackaite and Gneezy (2015) again: "Our results reject the common assumption of "small lies" due to convex cost of lying. By contrast, our data are consistent with a fixed intrinsic cost of lying: when our participants lie, they do so to the full extent, whereas partial lying is rare. Combined, our results show that for many participants, the decision to lie follows a simple cost-benefit analysis: they compare the intrinsic cost of lying with the incentives to lie; once the incentives are higher than the cost, they switch from telling the truth to lying to the full extent".

In the standard model of strategic voting, voters have a ranking or a preference ordering over a finite number of alternatives. A *social choice function* or SCF associates an alternative with every profile of preference orderings. A voter's preferences are private information. A SCF is *strategy-proof* if no voter can gain (according to her true preferences) by lying about her preference ordering at every profile of preference orderings for the other voters. The fundamental result of (Gibbard (1973) and Satterthwaite (1975)) states that every SCF that is strategy-proof and satisfies the mild requirement of unanimity, is *dictatorial*. In other words, there exists a voter whose first-ranked alternative is always picked as the outcome of the SCF at every preference profile.

We say that a SCF is K-manipulable ($K \geq 1$) if the voter can improve by *at least* K ranks in her true preference ordering by lying. A SCF is K-strategy-proof if it is not K-manipulable. If $K = 1$, K-strategy-proofness is standard strategy-proofness. However, if $K > 1$, we have a weaker notion of strategy-proofness that captures the idea that a voter manipulates only when the gains from lying are "substantial". If K equals the number of alternatives, K-strategy-proofness will be trivially satisfied. We can therefore reasonably expect a wider class of SCFs to be strategy-proof as the "reluctance to lie" increases.

Our main result is that this optimism is somewhat misplaced. We show that under various notions of efficiency and for a sufficiently large number of alternatives, a SCF is K-strategy-proof only if it is K-dictatorial, i.e. there exists a voter such that the outcome at every profile is one of her top K-ranked alternatives. As K increases, the strength of strategy-proofness decreases but so does the power of K-dictatorship. Rather surprisingly, they exactly counterbalance each other. For instance, if K is equal to the number of alternatives, K- strategy-proofness is vacuously satisfied but so is K-dictatorship. If $K = 2$, the SCF must always pick either the dictator's first or second-ranked alternatives. This choice can depend on the preferences of other voters. Our result establishes a robustness property of the Gibbard-Satterthwaite result.

We also show that the well-known equivalence of ontoness, unanimity and efficiency for strategy-proof SCFs breaks down when strategy-proof is replaced by K-strategy-proofness. Along with this, we have various results on the lower bound of number of alternatives required to prove the K-dictatorial SCF.

We would like to contrast our approach with that taken in the literature on local-global equivalence (Carroll (2012), Sato (2013), Kumar et al. (2019)). In these models, agents are restricted to "local lies", i.e the *inputs* provided by the voters are constrained to be "close" to the true private information they possess. Their main question is to determine whether immunity to local lies guarantees that no lie will be beneficial. Our proposal is to focus instead on the *output* of the SCF. A voter is not constrained in the manner in which she lies but will lie only when the rewards from doing so are suitably high. As we have mentioned earlier, there is experimental evidence to support our approach.

There is a large theoretical literature on strategic communication, cheap talk and implementation which explicitly takes into account, the cost of lying (see Sobel (2020), Kartik (2009), Khalmetski and Sliwka (2019) and Dutta and Sen (2012), Matsushima (2008a) and Matsushima (2008b)). The papers on implementation, especially Dutta and Sen (2012) are close in spirit to our work since they are also concerned with ordinal environments. However, the actual assumptions on the costs of lying are quite different. For instance, Dutta and Sen (2012) assume a lexicographic cost structure - agents always lie if they can improve their material payoff; however they strictly prefer to tell the truth if truth-telling and lying yield the same material payoff.

Two papers in the social choice literature that bear some resemblance to ours are Campbell and Kelly (2009) and Campbell and Kelly (2010). Both papers consider the standard notion of manipulability. The former investigates the relation between the maximal gain that any voter can obtain by manipulation and a measure of the "degree of dictatorship" of particular class of social choice functions. The second paper conducts a similar analyze for the maximum loss that a manipulating voter can impose on other voters. Our approach differs fundamentally from these papers in the sense that we depart from the standard notion of manipulability and consider a behavioural approach to truth telling. The results in our chapter are also unrelated to those in the papers mentioned above.

The rest of the chapter is organized as follows. Section 2.2 introduces the basic notation and definitions. Section 2.3 contains the main results. Section 2.4 discusses various aspects of our results while Section 2.5 concludes the chapter. Section 2.6 is the Appendix and contains the proofs.

2.2 Preliminaries

Let $N = \{1, \dots, n\}$ be a finite set of voters and A be a finite set of m alternatives i.e. $|A| = m$. We will write i, j and a, x, y, x_k, etc. for generic elements in N and A respectively. Each voter $i \in N$ has a (linear) preference ordering P_i over the elements of the set A[1]. For distinct $a, b \in A$ by aP_ib we mean : a is strictly preferred to b by voter i according to her preference ordering P_i. Let $\mathbb{P}$ denote the set of all linear orderings over the elements of A.

For any preference ordering P_i and integer $k = 1, \dots, m$, $r_k(P_i)$ denotes the k^{th} ranked alternative in P_i, i.e. $|\{a \in A : aP_ir_k(P_i)\}| = k - 1$ and $r(P_i, a) \in \{1, 2, \dots, m\}$ denotes the rank of alternative a at P_i. Note that for any $P_i \in \mathbb{P}$, $k \in \{1, 2, \dots, m\}$ and $a \in A$,

[1] Linear order is a binary relation which satisfies completeness, transitivity and anti-symmetry.

$r_k(P_i) = a$ if and only if $r(P_i, a) = k$. It will be helpful to write P_i as an ordered tuple, $P_i \equiv \Big(r_1(P_i),\ r_2(P_i),\ \ldots,\ r_k(P_i),\ r_{k+1}(P_i),\ \ldots r_{m-1}(P_i),\ r_m(P_i)\Big)$.

A profile is a list $P = (P_1, \ldots, P_n) \in \mathbb{P}^n$ of voters' preference orderings. For any coalition $S \subset N$, let $P_S \equiv (P_i)_{i \in S}$ and $P_{-S} \equiv (P_i)_{i \in N \setminus S}$. For simplicity, we write P_{-i} for $P_{-\{i\}}$ and P_{-ij} for $P_{-\{i,j\}}$ and so on. A profile P is also denoted by (P_i, P_{-i}), more generally (P_S, P_{-S}) for any $S \subset N$. Some standard definitions are as follows:

Definition **2.1** *A social choice function (SCF) f is a mapping $f : \mathbb{P}^n \to A$.*

A SCF picks an alternative at every preference profile. Note that all preference profiles are admissible i.e. the preference domain is "unrestricted" throughout the chapter.

In the standard model of strategic voting, a voter's preference ordering is private information. A desirable property for a SCF is *strategy-proofness.* A strategy-proof SCF has the property that no voter can gain by misreporting her true ordering, irrespective of the reports of other voters. In other words, truth telling is a weakly dominant strategy for each voter in the direct revelation game. This is formally stated below.

Definition **2.2** *The SCF f is manipulable at profile P via P_i' if $f(P_i, P_{-i})\ P_i\ f(p)$. The SCF is strategy-proof if it is not manipulable by any voter at any profile.*

In this chapter, we explore the consequences of introducing a fixed cost of lying for every voter. Our model (which is the standard voting model) is ordinal. It is however natural to interpret ranks as utilities. With this understanding we assume that the fixed cost of lying for every voter is $K-1$ ranks in her preference ordering. A voter will therefore will lie only if the lie leads to an improvement of at least K ranks according to her true preference ordering. This leads to the definition of K-strategy-proofness.

Definition **2.3** *Pick an integer K in the set $\{1, 2, \ldots, m\}$. The SCF f is K-manipulable at profile P via P_i' if $r(P_i, f(P)) - r(P_i, f(P_i', P_{-i})) \geq K$. The SCF is K-strategy-proof if it is not K-manipulable by any voter at any profile.*

Figure 2.1 diagrammatically illustrates K-manipulability of SCF f at a profile P via P_i'.

We make two important remarks regarding K-strategy-proofness.

Remark **2.1** The notion of K-strategy-proofness reduces to strategy-proofness in the case $K = 1$. This is the case where cost of lying is zero.

$$\begin{array}{rcl} & & P_i \\ & & \vdots \\ r[P_i, f(P_i', P_{-i})] = L - K & \longrightarrow & f(P_i', P_{-i}) \\ & & \left.\begin{array}{c}\cdot\\ \cdot\\ \cdot\\ \cdot\end{array}\right\} = K-1 \\ r[P_i, f(P_i, P_{-i})] = L & \longrightarrow & f(P_i, P_{-i}) \\ & & \vdots \end{array}$$

Figure 2.1: K-manipulability

REMARK **2.2** As K increases, the cost of lying for every voter increases. The incentive for truth telling therefore increases with K. If $K = m$ the cost of lying is so high that no voter has incentive to lie i.e. all SCFs are m-strategy-proof. Thus K-strategy-proofness reflects the behaviour of an agent who has an intrinsic cost of lying and misreports her true preference only when the gain from lying exceeds her intrinsic cost of lying.

As noted in the Remark 2.2 and also mentioned in the introduction that K-strategy-proofness can be given a behavioral justification. A voter may not choose to misreport if she can only gain by a "small" or "imperceptible" amount.

Our goal is to characterize the class of K-strategy-proof SCFs in conjunction with some other axioms. The benchmark result in strategic social choice theory is the celebrated Gibbard-Satterthwaite theorem for strategy-proof SCFs. In order to state the result some additional definitions are required.

DEFINITION **2.4** *A SCF f satisfies unanimity if $f(P) = x$ for all profiles P where $r_1(P_i) = x$ for all $i \in N$.*

A SCF satisfies unanimity if it always picks the commonly first ranked alternative, whenever it exists.

DEFINITION **2.5** *A SCF f is dictatorial if there exists a voter i (called a dictator) if $f(P) = r_1(P_i)$ for all profiles P.*

A dictatorial SCF always picks the first ranked alternative of a pre-specified voter called the dictator. It is easy to verify that a dictatorial SCF is strategy-proof and satisfies unanimity.

The Gibbard-Satterthwaite theorem states that dictatorial rules are the only SCFs that satisfy these properties, provided there are at least three alternatives.

THEOREM [Gibbard (1973) and Satterthwaite (1975)] *Assume $m \geq 3$. A SCF f is strategy-proof and satisfies unanimity if and only if it is dictatorial.*

It is well-known that the Gibbard-Satterthwaite theorem continues to hold when unanimity is replaced by a weaker requirement on the range of the SCF.[2]

DEFINITION **2.6** *A SCF f is onto if, for all $a \in A$ there exists a profile $P \in \mathbb{P}^n$ such that $f(P) = a$.*

THEOREM [Gibbard (1973) and Satterthwaite (1975)] *Assume $m \geq 3$. A SCF f is strategy-proof and satisfies ontoness if and only if it is dictatorial.*

A counterpart of the dictatorial SCF in our context is the K-dictatorial SCF which is defined below.

DEFINITION **2.7** *A SCF f is K-dictatorial if there exists a voter i (called a K-dictator) if $f(P) \in \{r_1(P_i), r_2(P_i), \ldots, r_K(P_i)\}$ for all profiles P.*

The K-dictatorship SCF picks one of the top K ranked alternatives of a pre-specified voter called the K-dictator. Again, it is obvious that the K-dictator can never K-manipulate at any profile. An important difference between a dictatorial and a K-dictatorial SCF is that the outcome of the latter at any profile may depend on the preferences of voters other than that of the K-dictator. This is illustrated in the Example 2.1.

EXAMPLE **2.1** Pick K such that $2 < K \leq m$. Consider the following SCF: at every profile, pick the highest ranked alternative in voter 2's preference ordering among the top K ranked alternatives in voter 1's preference ordering. As we have remarked earlier, voter 1 cannot K-manipulate. In addition, voter 2 also cannot K-manipulate. The SCF also satisfies unanimity. However, it is not strategy-proof. For example, consider the profile P where $P_1 = (x_1 x_2 x_3 \ldots x_k x_{K+1} \ldots)$[3] and $P_2 = (x_2 x_1 x_3 \ldots x_k x_{K+1} \ldots)$. The outcome at this profile is x_2. Consider the misreport $P_1' = (x_{K+1} x_1 x_3 \ldots x_K x_2 \ldots)$ by voter 1. Now x_2 is no longer among the K best alternatives reported by the voter 1. The outcome therefore changes to x_1. However, this is a manipulation for voter 1.

[2] An even weaker notion of the Gibbard-Satterthwaite theorem can be stated where the range of the SCF is required to be three. However, in this case the definition of a dictator has to be modified to be a voter who always gets the maximal alternative in the range of the SCF.

[3] Recall that P_1 is the ordering where x_1 is ranked first, x_2 second, x_3 third and so on.

Remark 2.3 An important feature of the K-dictatorial SCF is that the dependence of its outcome at any profile on the K-dictator's preferences decreases, as K increases. In the case when $K = m$, the outcome does not depend at all on the K-dictatorship's preferences. On the other hand, when $K = 1$ the outcome is completely determined by the K-dictator's preferences.

In the next section, we will show that there is a tight relationship between K-strategy-proof SCFs and K-dictatorship.

2.3 Main Results

Our first result shows that unanimity and K-strategy-proofness imply K-dictatorship for sufficiently large number of alternatives.

Theorem 2.1 *Let K be an integer in the set $\{1, 2, \ldots, m\}$ and assume $m \geq 5K$. If a SCF is K-strategy-proof and unanimous then it is a K-dictatorship.*

The proof of the theorem is relegated to the Appendix. Our result is surprising in one respect which we state below.

Remark 2.4 As we have observed earlier in Remark 2.2, the requirement of K-strategy-proofness weakens as K increases. We may therefore expect many more rules to be K-strategy-proof for large values of K. Interestingly, all these extra possibilities are exactly captured within the class of K-dictatorships SCFs. As we have also observed earlier in Remark 2.3, the class of K-dictatorship expands as K increases. This expansion exactly accommodates the extra possibilities afforded by the K-strategy-proofness.

Remark 2.5 Our earlier Remake 2.4 suggests that Theorem 2.1 is a generalization of the Gibbard-Satterthwaite Theorem. However, this is not exactly true since the lower bound on the number of alternatives for the case $K = 1$ in Theorem 2.1 is 5. However, in the Gibbard-Satterthwaite Theorem this lower bound is 3. We have not been able to reconcile this difference.

The bound on the number of alternatives can be improved to $3K$ if unanimity is strengthened to efficiency.

Definition 2.8 *Let P be a profile. Alternative a dominates alternative b at P if aP_ib for all $i \in N$. The SCF f is efficient if at all profiles P and alternatives b, $f(P) \neq b$ if there exists an alternative a which dominates b.*

Now we can state our next result.

Theorem **2.2** *Let K be an integer in the set $\{1, 2, \dots, m\}$ and assume $m \geq 3K$. If a SCF is K-strategy-proof and efficient then it is a K-dictatorship.*

Remark **2.6** Note that, K-dictatorship does not imply K-strategy-proofness. In a K-dictatorial SCF, the outcome may depend on the preference orderings of other voters as well. If the outcome is not selected carefully, then the SCF may become K-manipulable. Consider a K-dictatorial SCF, where voter 1 is the K-dictator. Suppose the SCF selects the least preferred alternative for voter 2 between the first and second ranked alternatives in the preference ordering of voter 1. This SCF is K-manipulable by voter 2. On the other hand, consider a different SCF which selects the most preferred alternative for voter 2 between the first and second ranked alternatives in the preference ordering of voter 1. Now, the new SCF is K-strategy-proof. This complication makes it harder to provide a complete characterization of K-strategy-proof SCFs satisfying unanimity or efficiency.

The proof of Theorem 2.2 is also contained in the Appendix. It broadly follows the arguments in the proof of Theorem 2.1 except that many of the steps are considerably simplified. The next section discuses several aspects of our results.

2.4 Discussion

This section contains two subsections. The first discusses the relation between various axioms and K-strategy-proofness. The second subsection discusses the lower bounds on the number of alternatives in Theorems 2.1 and 2.2.

2.4.1 The Relationship between Ontoness, Unanimity and Efficiency

It is well-known that the notions of ontoness, unanimity and efficiency coincide for strategy-proof SCFs. We show that this equivalence breaks down for values of K strictly greater than 1. Of course efficiency implies unanimity which in turn implies ontoness. Below we provide an example of a SCF which satisfies K-strategy-proofness and ontoness but violates unanimity.

Example **2.2** Let $K \geq 2$. Define a preference ordering $\bar{P}_1$ of voter 1 as follows : $a_l \, \bar{P}_1 \, a_{l+1}$ for $l = 1, 2, \dots, m-1$ i.e. $\bar{P}_1 = (a_1, a_2, \dots, a_m)$. The SCF f is defined as follows :

$$f(P) = \begin{cases} r_{K-1}(P_1) & \text{if } r_K(P_1) = a_1 \\ \underset{P_2}{\text{Max}} \{a_1, a_{K+1}\} & \text{if } P_1 = \bar{P}_1 \\ r_K(P_1) & \text{otherwise} \end{cases}$$

We first argue that f is not a K-dictatorial SCF. Note that no voter other than 1 can be a K-dictator. To see this consider a profile P where $r_K(P_1) = a_m$ and $r_m(P_i) = a_m$ for all $i \neq 1$. We have $f(P) = a_m$ which shows that voters other than 1 do not get an alternative in the set of their top K alternatives (since $m > K$). Voter 1 is not a K-dictator either because in the profile P where $P_1 = \bar{P}_1$ and $a_{K+1}\, P_2\, a_1$ (the second case mentioned in the SCF), she gets her K+1 ranked alternative a_{K+1}.

It is easy to see that f is onto. To show that it violates unanimity, consider the profile P' where $r_1(P'_i) = a_2$ for all $i \in N$ and $r_K(P'_1) = a_3$. According to the SCF we have $f(P') = a_3$. However, unanimity requires that outcome should be a_2.

Finally, we will show that no voter can K-manipulate at any profile. At all profiles no voter other than 1 or 2 can influence the outcome. Voter 2 chooses the best alternative from the set which is specified independently of her ordering. Moreover, at every profile voter 1 always gets an alternative which is either ranked $K-1$ or K in her ordering except when $P_1 = \bar{P}_1$ and $a_{K+1}\, P_2\, a_1$. In this case, she gets her K+1 ranked alternative. The only way she can K-manipulate is if she can get her top-ranked alternative in $\bar{P}_1$ i.e. a_1. This is ruled out since $f(P) \neq a_1$ when $a_{K+1}\, P_2\, a_1$. We have covered all profiles. Hence, f is K-strategy-proof.

Remark **2.7** The Example 2.2 also highlights the fact that K-strategy-proofness and ontoness do not imply K-dictatorship. We note this fact in the next proposition.

Proposition **2.1** *Suppose $m > K \geq 2$. There exist non K-dictatorial SCFs satisfying K-strategy-proofness and ontoness.*

The next example demonstrates the existence of a SCF which satisfies K-strategy-proofness and unanimity but violates efficiency.

Example **2.3** Let $K \geq 2$. Consider a K-dictatorial SCF f where voter 1 is the K-dictator.

$$f(P) = \begin{cases} b & \text{if } r_1(P_1) = r_{m-1}(P_2) = a \text{ and } r_2(P_1) = r_m(P_2) = b \\ \underset{P_2}{\text{Max}}\{r_1(P_1), r_2(P_1)\} & \text{otherwise} \end{cases}$$

It is easy to verify that f satisfies K-strategy-proofness and unanimity. However, it violates efficiency. To see this, consider a profile where every voter prefers a over b. The specific ranking of a and b in preference orderings P_1 and P_2 coincides with the first case in the specification of the SCF. At that profile the efficient outcome is a but it selects b. Thus, violating efficiency.

Remark **2.8** Both Examples 2.2 and 2.3 show that the breakdown of the equivalence between ontoness, unanimity and efficiency under K-strategy-proofness does not depend on the value of either m or K, as long as $K \geq 2$.

2.4.2 Tightening the lower bound on number of alternatives

In this subsection, we discuss why K-strategy-proofness requires a large number of alternatives. In the case of efficiency (Theorem 2.2), we have shown that K-strategy-proofness implies K-dictatorship, if there are at least $3K$ alternatives. The next example shows that there exists a non-K-dictatorial, efficient and K-strategy-proof SCF, if the number of alternatives is at most $2K$.

We introduce some new notation for this purpose. For any coalition $S \subset N$ and sub-profile P_S we let $\mathcal{T}^K(P_S) = \bigcap_{i \in S} \{r_1(P_i), \ldots, r_K(P_i)\}$ i.e. $\mathcal{T}^K(P_S)$ contains the alternatives that are ranked among the first K alternatives by all voters in S in the profile P_S. All the examples in this section have the feature that the outcome at any profile depends only on the preference orderings of voters 1 and 2. Therefore, in order to show that K-strategy-proofness is satisfied, we only need to argue for voters 1 and 2.

Example **2.4** Let $K < m < 2K$. Pick P_i' and P_i'' such that $a_l \; P_i' \; a_{l+1}$ and $a_{l+1} \; P_i'' \; a_l$ respectively, for all $l = 1, \ldots, m-1$ i.e.$P_i' = (a_1, a_2, \ldots, a_m)$ and $P_i'' = (a_m, a_{m-1}, \ldots, a_1)$. The SCF f is defined as follows :

$$f(P) = \begin{cases} a_{k+1} & \text{if } \; P_{\{1,2\}} = (P_1', P_2'') \text{ or } (P_1'', P_2') \\ \underset{P_2}{\text{Max}} \, \mathcal{T}^K(P_1, P_2) & \text{otherwise} \end{cases}$$

Pick an arbitrary profile. If the profile is such that the outcome of f is specified by the first case in the description of f then the outcome a_{K+1} is efficient because at this profile every alternative is efficient. For any other profile, voter 2 chooses her most preferred outcome from the set of alternatives which are top K-ranked by both voters. Since $m < 2K$, the set $\mathcal{T}^K(P_1, P_2)$ is non-empty. Clearly f is efficient.

We consider the K-strategy-proofness of f. Pick a profile where the outcome of f is specified by the second case in the description of f. Here, the outcome belongs to the top K-ranked alternatives of both voters 1 and 2. Therefore, they cannot K-manipulate. If a profile P is such that $P_{\{1,2\}} = (P_1', P_2'')$ then outcome is a_{K+1}, which belongs to top K-ranked alternatives of voter 2 and she cannot K-manipulate here (to be precise a_{K+1} is $m - K + 2$ ranked alternative for voter 2 and we have assumed $m < 2K$ so $m-K+2 < K$). For voter 1 it is her $K+1$ ranked alternative. In order for her to K-manipulate, voter 1 must obtain the first ranked alternative which is a_1. According to the SCF if $P_2 = P_2''$ then for any preference ordering of voter 1 the outcome can never be a_1. Hence she cannot K-manipulate either. At profile P where $P_{\{1,2\}} = (P_1', P_2'')$ a similar argument applies after reversing the roles of voter 1 and voter 2.

Finally, we show that f is not K-dictatorial. It is obvious that voters other than 1 and 2 cannot be K-dictators. At profiles where the sub-profile of voter 1 and 2 is either (P_1', P_2'') or

(P_1'', P_2'), the outcome is a_{K+1}, which is ranked $K+1$ by voters 1 and voter 2 at preference ordering P_1' and P_2' respectively. Therefore, neither 1 nor 2 can be a K-dictator.

EXAMPLE **2.5** Let $m = 2K$. Fix a set of alternatives $A_K \subset A$ such that $|A_K| = K$ and denote $A_K = \{a_1, a_2, \ldots, a_K\}$. The SCF f is defined as follows :

$$f(P) = \begin{cases} \underset{P_2}{\text{Max}}\, \mathcal{T}^K(P_1, P_2) & \text{if } \mathcal{T}^K(P_1, P_2) \neq \emptyset \\ \underset{P_2}{\text{Max}}\, \mathcal{T}^K(P_1) & \text{if } \mathcal{T}^K(P_1, P_2) = \emptyset \text{ and } \mathcal{T}^K(P_1) \cap A_K \neq \emptyset \\ \underset{P_1}{\text{Max}}\, \mathcal{T}^K(P_2) & \text{otherwise} \end{cases}$$

The SCF f is efficient because there is always a voter (either 1 or 2) who selects the best alternative according to her preference ordering from the top K-ranked alternatives in the other voter's preference ordering. We will show that it satisfies K-strategy-proofness. It is obvious that if a voter gets an outcome from her top K-ranked alternatives, then she cannot K-manipulate. Therefore, we focus only on voters 1 and 2 when they get an alternative which is not top K-ranked. In the second case, voter 2 is getting her K+1-ranked alternative, but by misreporting she can only become worse-off. Thus, she cannot K-manipulate. Similarly, in the third case, when voter 1 gets her K+1 ranked alternative, she cannot manipulate. Finally, we argue that it is not a K-dictatorial rule. In the second and third cases of the rule, voters 2 and 1 get an alternative which is K+1-ranked. So, neither of them can be a K-dictator.

REMARK **2.9** It is clear from results in the examples that we have complete answers for the cases when $m \leq 2K$ or $m \geq 3K$. In the case of two voters we can show efficiency and K-strategy-proofness imply K-dictatorship whenever $m \geq 2K + 1$.[4] Unfortunately, we are unable to settle the issue for more than two voters and $2K < m < 3K$.

Example 2.6 shows that a non-dictatorial, unanimous and K-strategy-proof SCF exists when $2K < m < \frac{5K}{2}$. In conjunction with Examples 2.4 and 2.5 we can conclude that there exists a non-dictatorial, unanimous and K-strategy-proof when $K < m < \frac{5K}{2}$. Note that inequality $2K < m < \frac{5K}{2}$ can be satisfied only when $K \geq 3$.

EXAMPLE **2.6** Let $K \geq 3$. Fix an integer $\bar{K} < \frac{K}{2}$. Let m be such that $2K < m \leq 2K + \bar{K}$. Fix a set $X \subset A$, such that $|X| = \bar{K}$. For any preference ordering P_1, define $l^\star(P_1) = \text{Min}\{r(P_1, a) : a \in A \backslash X\}$. i.e. $l^\star(P_1)$ is rank of the most preferred alternative, according to P_1 which does not belong to X. Define $\mathcal{T}^K_{l^\star(P_1)}(P_1) = \{r_{l^\star}(P_1), r_{l^\star+1}(P_1), \ldots, r_{l^\star+K-1}(P_1)\} \backslash X$.

[4] We omit the proof of this claim. It is available with the author on request.

The SCF f is defined as follows:

$$f(P) = \begin{cases} \underset{P_2}{\text{Max}}\, \mathcal{T}^K(P_1, P_2) & \text{if } \mathcal{T}^K(P_1, P_2) \neq \emptyset \\ \underset{P_2}{\text{Max}}\, \mathcal{T}^K_{l^\star(P_1)}(P_1) & \text{otherwise.} \end{cases}$$

It says that if there are any common alternatives among the top K ranked alternatives of voters 1 and 2, then f picks the most preferred alternative from them, according to the preference ordering of voter 2. When there is no such alternative, it picks the most preferred alternative from the set $\mathcal{T}^K_{l^\star(P_1)}(P_1)$, according to the preference ordering of voter 2. In other words, start from the top in preference ordering of voter 1 and pick the first alternative which is not in the set X. Now including this alternative, consider the next $K-1$ alternatives in preference ordering of voter 1 (which may include alternatives from X). Out of this collection of K alternatives, select the most preferred alternative in the complement of X, according to the preference ordering of voter 2.

According to the SCF, if there is any alternative which is ranked first by both the voters 1 and 2, then it should be selected at that profile. Thus, it satisfies unanimity. We claim it also satisfies K-strategy-proofness. It is obvious that a voter cannot K-manipulate, if she is getting an alternative from her top K-ranked alternatives, so we only focus on profiles which come under the second case (in the description of f).

Pick any profile P, where $\mathcal{T}^K(P_1, P_2) = \emptyset$. According to the SCF, an alternative from the set X can be obtained only if it is commonly ranked in the top K-ranked alternatives by both the voters 1 and 2. Therefore, fixing P_{-1}, voter 1 can never obtain an alternative $x_i \in X$, which belongs to her top K-ranked alternatives, since $\mathcal{T}^K(P_1, P_2) = \emptyset$ (in particular, alternative x_i which is ranked higher than $r_{l^\star}(P_1)$). Moreover, the outcome must belong to the set $\mathcal{T}^K_{l^\star(P_1)}(P_1)$. It follows that at every profile voter 1 always gets an outcome which belongs to the top K alternatives within her opportunity set. So, she cannot K-manipulate.

A similar argument applies to voter 2. At profile P, she cannot obtain an alternative which belongs to top K-ranked alternatives in her preference ordering because we have assumed $\mathcal{T}^K(P_1, P_2) = \emptyset$. Moreover at every profile she selects the most preferred outcome from her opportunity set. It implies that by misreporting, *at best*, she can obtain an alternative which is ranked at $K+1$. Fix the sub-profile P_{-2}, because of our assumption of $m < \frac{5K}{2}$, voter 2 *at most* can gain $K-2$ places in her preference ordering. We highlight this specific case in Figure 2.2. Therefore, voter 2 also can never K-manipulate. Hence, f is K-strategy-proof SCF.

It is easy to see that no voter can be a K-dictator because according to the SCF, for every voter there exist profiles where she obtains an alternative below her top K-ranked alternatives.

$$
\begin{array}{ccl}
P_1 & P_2 & \\
a_1 & a_{20} & \\
\cdot & \cdot & \\
\cdot & \cdot & \\
\cdot & \cdot & \\
a_{10-\bar{K}} & a_{11+\bar{K}} & \\
x_1 & a_{10+\bar{K}} & \\
\cdot & \cdot & \\
\cdot & \cdot & \\
x_{\bar{K}} & a_{11} & \\
\hline
a_{11-\bar{K}} & x_1 & \\
a_{12-\bar{K}} & x_2 & \\
\cdot & \cdot & \\
\cdot & \cdot & \left.\right\} = \bar{K}-1 \\
a_{10} & x_{\bar{K}} & \\
\hline
a_{11} & a_{10} & \\
a_{12} & a_9 & \\
\cdot & \cdot & \left.\right\} = \bar{K} \\
\cdot & \cdot & \\
a_{10+\bar{K}} & a_{11-\bar{K}} & \\
\hline
a_{11+\bar{K}} & a_{10-\bar{K}} & \\
\cdot & \cdot & \\
\cdot & \cdot & \\
a_{20} & a_1 &
\end{array}
$$

Figure 2.2: The profile P where voter 2 has maximum possible gain via a misreporting.

We provide a profile P, where voter 2 has the *maximum* scope of manipulation. We argue that this manipulation can lead to *at most* a gain of $2\bar{K}-1$ places, which is at most $K-2$. Therefore, she cannot K-manipulate. For simplicity, assume $K = 10$.[5] This makes $m = 20+\bar{K}$, where $\bar{K} \in \{1, 2, 3, 4\}$. Consider a profile P such that P_1 and P_2 are as follows :

(i) $P_1 = (a_1, \dots, a_{10-\bar{K}}, x_1, \dots, x_{\bar{K}}, a_{11-\bar{K}}, \dots, a_{20})$.

(ii) $P_2 = (a_{20}, \dots, a_{11}, x_1, \dots, x_{\bar{K}}, a_{10}, \dots\dots, a_1)$.

The profile P is shown in Figure 2.2. At this profile, we have $\mathcal{T}^K(P_1, P_2) = \emptyset$. According to the SCF, P comes under the second case. Therefore, we need to compute $l^\star(P_1)$, which is

[5]This particular instance can easily be generalized to any value of K.

equal to 1 because a_1 is the first alternative from the top, which does not belong to X. It implies that

$$\mathcal{T}^K_{l^\star(P_1)}(P_1) = \{r_{l^\star}(P_1), r_{l^\star+1}(P_1), \ldots, r_{l^\star+K-1}(P_1)\}\backslash X = \{a_1, a_2, \ldots, a_{10-\bar{K}}\}$$

Hence, $f(P) = \underset{P_2}{\text{Max}}\, \mathcal{T}^K_{l^\star(P_1)}(P_1) = a_{10-\bar{K}}$. The *most* profitable manipulation for voter 2 is when she obtains x_1 via a misreport. But, we have assumed that $\bar{K} < \frac{K}{2}$. Therefore, this improvement at *maximum* could be of $2\bar{K}-1$ places, which is strictly lower than $K-1$. So, at *best* she can $K-1$ manipulate but not K-manipulate. This completes our claim.

2.5 Conclusion

We have shown the robustness of the Gibbard-Satterthwaite Theorem when voters incur a cost of lying. We have introduced the new notion of K-strategy-proofness. We have shown that for a sufficiently large number of alternatives K-strategy-proofness in conjunction with either efficiency or unanimity leads to the K-dictatorship. This is surprising in view of the fact that K-strategy-proofness is a significant weakening of strategy-proofness.

2.6 Appendix

We provide a proof of Theorem 2.1. We begin with a lemma which we will use frequently in the proof. It is a "lifting" lemma of the sort which is common in the arguments involving strategy-proofness.

Let S be a non-empty subset of voters. Let (A_1, A_2, A_3) be a partition of the set A with $|A_2| \geq K$ and $A_3 \neq \emptyset$. Let $\hat{P}$ be a profile such that all voters in S rank all alternatives in A_1 above all alternatives in A_2 and all alternatives in A_2 above all alternatives in A_3.

For any voter $i \in S$ and let $\hat{\mathbb{P}}_{i(A_2)}$ be the set of preference orderings obtained from the $\hat{P}_i$, by only reshuffling the alternatives from A_2, while keeping the ranks of all the alternatives in A_1 and A_3 unchanged i.e. for $i \in S$ denote $\hat{\mathbb{P}}_{i(A_2)} = \{P_i \in \mathbb{P} \,:\, r(P_i, x) = r(\hat{P}_i, x)$ for all $x \in A_1 \cup A_3\}$.

Let $\hat{\mathbb{P}}^n_{S(A_2)}$ be the set of all profiles P such that $P_i \in \mathbb{P}_{i(A_2)}$ for all $i \in S$ and $P_j = \hat{P}_j$ for all $j \in N\backslash S$ i.e. $P_{N\backslash S} = \hat{P}_{N\backslash S}$. In other words, every P in $\hat{\mathbb{P}}^n_{S(A_2)}$ is obtained from the profile $\hat{P}$, by only reshuffling the alternatives from set A_2 and only by voters in S. Let $\mathbb{P}^n_{S(A)}$ be the set of all profiles P' such that $P'_{N\backslash S} = \hat{P}_{N\backslash S}$. In other words, every P' in $\hat{\mathbb{P}}^n_{S(A)}$ is obtained from the profile $\hat{P}$ by keeping the orderings of voters in $N\backslash S$ the same, while the orderings of voters in S can be chosen arbitrarily. It is obvious that $\hat{P}^n_{S(A_2)} \subsetneq \hat{P}^n_{S(A)}$.

Lemma **2.1** *If* $f(\hat{P}) \in A_3 \;\Rightarrow\; f(P) \in A_3$ *for all* $P \in \hat{\mathbb{P}}^n_{S(A_2)}$ *then* $f(P') \in A_3$ *for all* $P' \in \mathbb{P}^n_{S(A)}$.

Proof: Suppose the Lemma is false. Let $\mathcal{S}$ be a collection of subsets of set S as follows :

$$\mathcal{S} = \{T \subseteq S \ : \exists P \text{ such that } (i) P_i \in \hat{P}_{i(A_2)} \forall i \in S \backslash T \ (ii) P_{N \backslash S} = \hat{P}_{N \backslash S} \ (iii) f(P) \notin A_3\}.$$

The collection $\mathcal{S}$ contains all sets T of voters from set S such that there exists a profile where voters in $S \backslash T$ report a preference ordering obtained by only reshuffling A_2 and $N \backslash S$ report exactly same as $\hat{P}_{N \backslash S}$ and voters in T can obtain an outcome outside A_3 by reporting some sub-profile. As we have assumed that the Lemma is false then $\mathcal{S}$ cannot be non-empty, it must include at least the set S.

Pick a set of minimal size from $\mathcal{S}$ i.e. $S^\star \in \mathcal{S}$ such that $|S^\star| \leq |T|$ for all $T \in \mathcal{S}$. Let $|S^\star| = l^\star > 0$. Assume w.l.o.g. that $S^\star = \{1, 2, \ldots, l^\star\}$. The set $S^\star$ must satisfy the following property : there exists a profile $P^\star$ such that (i) $f(P^\star) \notin A_3$ when $P_i^\star \in \hat{\mathbb{P}}_{i(A_2)} \forall i \in S \backslash S^\star$ and $P^\star_{N \backslash S} = \hat{P}_{N \backslash S}$, (ii) $f(P_{l^\star}, P^\star_{-l^\star}) \in A_3$ for every $P_{l^\star} \in \hat{\mathbb{P}}_{l^\star(A_2)}$. Fix this profile $P^\star$. We obtain a contradiction by showing that $S^\star$ is not the minimum.

Our first claim is that $f(P^\star) \notin A_1$. Let $P^{l^\star}$ be the profile obtained from $P^\star$ by replacing $P^\star_{l^\star}$ with $\hat{P}_{l^\star}$ i.e. $P^{l^\star} = (\hat{P}_{l^\star}, P^\star_{-l^\star})$. By the fact that $S^\star$ is minimal we have $f(P^{l^\star}) \in A_3$. If $f(P^\star) \in A_1$ then voter $l^\star$ can K-manipulate at profile $P^{l^\star}$ via $P^\star_{l^\star}$ because of the assumption $|A_2| \geq K$. Therefore, $f(P^\star) \in A_2$.

Let $P''_{l^\star}$ be an ordering where (i) all alternatives in A_1 are ranked above those in A_2, which in turn are ranked above those in A_3 and (ii) $x^\star$ is the highest ranked alternative within A_2. Formally, $P''_{l^\star} \in \hat{\mathbb{P}}_{l^\star(A_2)}$ and $r_{|A_1|+1}(P''_{l^\star}) = x^\star$.

Our next claim is that $l^\star \neq 1$. Suppose $l^\star = 1$ i.e. $S^\star = \{1\}$ and $f(P^\star) = x^\star \in A_2$. Observe that $(P''_1, P^\star_{-1}) \in \hat{\mathbb{P}}^n_{S(A_2)}$. By our assumption $f(P''_1, P^\star_{-1}) \in A_3$. So, if $f(P^\star) = x^\star$ and $|A_2| \geq K$ then voter 1 can K-manipulate at profile $(P''_1, P^\star_{-1})$ via $P^\star_1$.

The remaining case to consider is $l^\star > 1$. As we have argued previously that $l^\star$ is the cardinality of the set which is minimum in the size in the collection $\mathcal{S}$. Thus, it must be the case that $f(P''_{l^\star}, P^\star_{-l^\star}) \in A_3$; otherwise $l^\star - 1$ becomes the cardinality of minimal set i.e. $l^\star - 1$ number of voters are enough in the set S to obtain an outcome outside A_3. As $f(P^\star) = x^\star$ and $|A_2| \geq K$, the voter $l^\star$ can K-manipulate at profile $(P''_{l^\star}, P^\star_{-l^\star})$ via $P^\star_{l^\star}$ because $x^\star$ is ranked at least $K-1$ places above every alternative in A_3 in the preference ordering $P''_{l^\star}$. Therefore, we arrive at a contradiction and it completes the proof. ■

Let G_1, G_2 and G_3 be a partition of N such that G_1 is non-empty. Let $\bar{P}_{G_3}$ be a sub-profile for voters in G_3. Let $X \subset A$ such that $|X| = K$. We denote P_i^X for a ranking of alternatives only in set X.[6] We say G_1 is decisive over X given $\bar{P}_{G_3}$, if there exists some ranking $\bar{P}_i^X$ such that we have $f(P) \in X$ for all profiles P such that

[6]In other words, P_i^X is linear order over the set X. In this terminology any preference ordering P_i can also be written as $P_i \equiv P_i^A$.

(i) all voters in G_1 rank alternatives in X above all alternatives in $A \backslash X$.

(ii) all voters in G_1 have the common ranking $\bar{P}_i^X$ over X.

(iii) the ordering of voters in G_3 is $\bar{P}_{G_3}$ i.e. $P_{G_3} = \bar{P}_{G_3}$.

According to the definition, coalition G_1 can "force" the outcome in X *irrespective* of the preferences of G_2 provided that preferences of G_1 and G_3 satisfy certain conditions.[7] When $G_3 = \emptyset$, the condition reduces to saying G_1 is decisive over X.

Next, we prove an important proposition which says that if N is partitioned into two coalitions, a single voter and the rest, then either the single voter is decisive over some set or the remainder coalition is decisive over another disjoint set.

PROPOSITION **2.2** *Let X and Y be arbitrary disjoint sets each with K alternatives. Let $(\{i\}, N \backslash \{i\})$ be an arbitrary partition of N. If a SCF satisfies unanimity and K-strategy-proofness then either voter i is decisive over Y or coalition $N \backslash \{i\}$ is decisive over X.*

Proof: Assume without loss of generality that $i = 1$. Let (X, Y, B, C, D) be a partition of set A such that (i) $|Z| = K$ for $Z = X, Y$. (ii) $|Z| \geq K$ for $Z = B, C, D$. This is feasible since $|A| \geq 5K$. Let P be the profile such that

(i) voter 1 ranks all alternatives in Y on top, followed by all alternatives in D, followed by all alternatives in X, followed by all alternatives in B and finally followed by all alternatives in C.

(ii) all voters $2, 3, \ldots, n$ rank all alternatives in X on top, followed by all alternatives in B, followed by all alternatives in Y, followed by all alternatives in C and finally followed by all alternatives in D.

(iii) all voters rank alternatives in X in the same way. We assume without loss of generality that this common ranking is $(x_1, x_2, \ldots, x_K)$. Similarly, all the voters rank alternatives in Y the same way. We assume without loss of generality that this common ranking is $(y_1, y_2, \ldots, y_K)$.

The profile P is shown in Figure 2.3. Note that X and Y are shown in bold to signify that all voters have a common ranking over X and Y as defined above. We emphasize that all voters can rank alternatives arbitrarily in B, C and D, different from each other.[8]

[7] To clarify the nature of the condition let's assume $G_1 = \{1\}$ and $G_2 = \{2, \ldots, n-1\}$ and $G_3 = \{n\}$. Let $\bar{P}_i^X = (x_1, \ldots, x_K)$ and $\bar{P}_n = (a_1, a_2, \ldots, a_m)$. Suppose voter 1 is decisive over X given $\bar{P}_1^X$ and $\bar{P}_n$. The definition does not restrict the outcome if voter 1 changes her preference ordering by swapping x_1 and x_2. In this case, the outcome may no longer belong to X. A similar implication applies to voter n. Therefore,

$$P = \begin{array}{c} \begin{array}{cccccc} P_1 & P_2 & P_3 & \cdots & P_n \end{array} \\ \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ D & B & B & \cdots & B \\ \boldsymbol{X} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\ B & C & C & \cdots & C \\ C & D & D & \cdots & D \end{pmatrix} \end{array}$$

Figure 2.3

Step 1 : We claim that $f(P) \in X \cup Y$.
In order to prove the claim, we argue that $f(P) \notin B \cup C \cup D$. Suppose $f(P) \in B \cup C$. Consider a misreport by voter 1 of the type $P_1' = (x_1, \ldots\ldots)$ i.e. $r_1(P_1') = x_1$. By unanimity $f(P_1', P_{-1}) = x_1$. It is clear that voter 1 can K-manipulate at profile P via P_1'.

If $f(P) \in D$ then we show a contradiction using Lemma 2.1. Suppose $f(P) \in D$. Consider changes in the preferences of voters other than 1 in a manner where only ranking of alternatives in C change i.e. only reshuffling the alternatives from C. Suppose a voter $i \neq 1$ can shift the alternative from D. If the resulting alternative is above C then we have a K-manipulation by voter i since $|C| \geq K$. If the outcome is in C then the argument in the previous paragraph applies and voter 1 can K-manipulate. Hence, the outcome must remain in D for all such profiles.

All the conditions of Lemma 1 are satisfied with $A_1 = X \cup B \cup Y,\ A_2 = C,\ A_3 = D$ and coalition $S = N \setminus \{1\}$. Consider P' where $P_1' = P_1$ and for $i \neq 1$, $P_i' = (y_1, \ldots\ldots)$ i.e all voters from 2 to n have y_1 on top. Applying Lemma 2.1 we have $f(P') \in D$. Moreover, at this profile y_1 is commonly ranked first by all voters. However, this contradicts unanimity which requires $f(P') = y_1$. This completes the proof of Step 1.

Step 2 : If $f(P) \in Y$ then voter 1 is decisive over Y.
Consider a profile P' such that

(i) $P_1 = P_1'$.

(ii) all voters $2, 3, \ldots, n$ rank all alternatives in X on top, followed by all alternatives in B, followed by alternative y_1, followed by all alternatives in C and finally followed by all alternatives in $D \cup Y \setminus \{y_1\}$.

---

even a "slight change" in either $\bar{P}_i^X$ via any voter in G_1 or $\bar{P}_{G_3}$ via any voter in G_3 can lead to a drastic change in the outcome.

[8] In all the figures, wherever we depict a set in bold in a preference ordering of a voter, it means alternatives of that set are ranked in a specific way by that voter. Whereas, a non-bold set signifies that alternatives in that set are ranked arbitrarily.

(iii) all voters $2, 3, \ldots, n$ rank alternatives in X according to $(x_1, x_2, \ldots, x_K)$ and Y according to $(y_1, y_2, \ldots, y_K)$.

The profile P' is shown in Figure 2.4. We claim $f(P') = y_1$.

$$f(P') = f\begin{array}{c} \begin{array}{ccccc} P'_1 & P'_2 & P'_3 & \cdots & P'_n \end{array} \\ \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \mathbf{X} \\ D & B & B & \cdots & B \\ \boldsymbol{X} & y_1 & y_1 & \cdots & y_1 \\ B & C & C & \cdots & C \\ C & \vdots & \vdots & \cdots & \vdots \end{pmatrix} \end{array} = y_1 \tag{2.1}$$

Figure 2.4

Consider a sequence which starts from P and terminates at P' such that voters from 2 to n progressively change their preference ordering from P_i to P'_i. Formally, consider the sequence $\{P^1, P^2, \ldots, P^n\}$ where $P^1 = P$ and $P^k = (P'_k, P^{k-1}_{-k}) = (P'_1 \ldots P'_k, P_{k+1}, \ldots, P_n)$ for $k = 2, \ldots, n$ i.e profile P^k is obtained from P^{k-1} by replacing P_k with P'_k. Note that $P^n = P'$. We claim that for any $k \geq 2$,

(i) if $f(P^{k-1}) \in Y \backslash \{y_1\}$ then $f(P^k) \in Y \cup D$.

(ii) if $f(P^{k-1}) = y_1$ then $f(P^k) = y_1$.

We first show (i). If $f(P^k) \in B \cup C$ then an argument identical to the one used in Step 1 applies. Voter 1 K-manipulates by misreporting via an ordering of type $P''_1 = (x_1 \ldots x_K \ldots\ldots)$. Suppose $f(P^k) \in X$. As per ordering P_k every alternative in X is ranked at least K places above every alternative in Y. Thus, voter k can K-manipulate at profile P^{k-1} via P'_k. Therefore, $f(P^k) \in Y \cup D$. This proves part (i).

Suppose $f(P^{k-1}) = y_1$ but $f(P^k) \neq y_1$. According to part (i) it has to belong to $Y \cup D$. In ordering P'_k the alternative y_1 is ranked at least K places above every alternative in $D \cup Y \backslash \{y_1\}$. Therefore, at profile $P^k = (P'_1 \ldots P'_k, P_{k+1}, \ldots, P_n)$ voter k can K-manipulate via misreporting P_k to obtain y_1 instead of an alternative from $D \cup Y \backslash \{y_1\}$. This is a contradiction to K-strategy-proof and completes the proof of part (ii).

It is obvious that (i) and (ii) imply $f(P') \in Y \cup D$. Suppose $f(P') \in D \cup Y \backslash \{y_1\}$. Consider any profile obtained by changing the preference ordering of voters other than 1, *only* over alternative in C i.e. keeping the ranking of alternatives in $A \backslash C$ unchanged. Arguments[9] in Step 1 can be replicated to show that the outcome in this profile remains in $D \cup Y \backslash \{y_1\}$.

[9]The arguments showed that $f(P) \notin D$ in Step 1.

Thus, all conditions of Lemma 2.1 are satisfied with $A_1 = X \cup B \cup \{y_1\},\ A_2 = C,\ A_3 = D \cup Y \setminus \{y_1\}$ and $S = N \setminus \{1\}$. Lemma 2.1 would imply that for any profile $\hat{P}$ where $\hat{P}_1 = P'_1$, we have $f(\hat{P}) \in D \cup Y \setminus \{y_1\}$. Pick $\hat{P}_{-1}$ such that $\hat{P}_i = (y_1 \ldots y_K \ldots\ldots)$ for each $i \neq 1$, unanimity implies $f(\hat{P}) = y_1$. However, this contradicts our conclusion $f(P') \in D \cup Y \setminus \{y_1\}$ in previous paragraph. Therefore, $f(P') = y_1$. We will use this conclusion later and record it as Equation (2.1) Figure 2.4.

Consider a profile $(\tilde{P}_1, P'_{-1})$ where $\tilde{P}_1 = (y_1 \ldots y_K, x_1 \ldots x_K, \ldots)$, it is shown in Figure 2.5. We claim $f(\tilde{P}) = y_1$. If $f(\tilde{P}_1, P'_{-1}) \notin Y \setminus \{y_1\}$, we can apply the same arguments as in the previous paragraph to show that all conditions of Lemma 1 are satisfied with $A_1 = X \cup B \cup \{y_1\},\ A_2 = C,\ A_3 = D \cup Y \setminus \{y_1\}$ and $S = N \setminus \{1\}$, again arriving at a contradiction. Therefore, $f(\tilde{P}_1, P'_{-1}) = y_1$.

$$
(\tilde{P}_1, P'_{-1}) = \begin{array}{c} \begin{array}{ccccc} \tilde{P}_1 & P'_2 & P'_3 & \cdots & P'_n \end{array} \\ \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \mathbf{X} \\ \boldsymbol{X} & B & B & \cdots & B \\ \vdots & y_1 & y_1 & \cdots & y_1 \\ \vdots & C & C & \cdots & C \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix} \end{array}
$$

Figure 2.5

Now consider a profile P'' such that

(i) $P''_1 = \tilde{P}_1$.

(ii) all voters $2, 3, \ldots, n$ rank all alternatives in X on top, followed by all alternatives in $B \cup C \cup D$, followed by all alternatives in Y.

(iii) all voters rank alternatives in X according to the common ranking $(x_1, x_2, \ldots, x_K)$. Similarly, alternatives in Y have the common ranking $(y_1, y_2, \ldots, y_K)$.

The profile P'' is shown in Figure 2.6.

We claim $f(P'') \in Y$. In order to see this, start from profile P' and progressively change the preference ordering of voters 2 through n from P'_i to P''_i. Note that $f(P') = y_1$ i.e. $\in Y$. Suppose that voter k changes from P'_k to P''_k and the outcome no longer belongs to Y.

If it belongs to X then voter k can K-manipulate at $(P''_1, \ldots, P''_{k-1}, P'_k, \ldots, P'_n)$ via P''_k. If outcome belongs to $B \cup C \cup D$ then voter 1 K-manipulates at $(P''_1, \ldots, P''_k, P'_{k+1}, \ldots, P'_n)$ via any ordering of type $\hat{P}_1 = (x_1 \ldots\ldots)$. Therefore, we have a contradiction, establishing that $f(P'') \in Y$.

$$
P'' = \begin{array}{c}
\begin{array}{ccccc} \tilde{P}_1 & P''_2 & P''_3 & \cdots & P''_n \end{array} \\
\begin{pmatrix}
\boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
\boldsymbol{X} & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y}
\end{pmatrix}
\end{array}
$$

Figure 2.6

In profile P'' note that the ranking of voters 2 through n over $B \cup C \cup D$ is arbitrary. Since $f(P'') \in Y$, all conditions of Lemma 2.1 are satisfied with $A_1 = X$ $A_2 = B \cup C \cup D$, $A_3 = Y$ and $S = N \setminus \{1\}$. Applying Lemma 2.1 we have $f(P''') \in Y$, where P''' is shown in Figure 2.7 along with its outcome.

Note that in profile P''' the preference orderings of voters 2 through n are arbitrary. While the preference ordering of voter 1 is such that she ranks $(y_1, y_2, \dots, y_K)$ on top, immediately followed by $(x_1, x_2, \dots, x_K)$ and the ranking over $B \cup C \cup D$ is arbitrary.

$$
f \begin{array}{c}
\begin{array}{ccccc} P'''_1 & P'''_2 & P'''_3 & \cdots & P'''_n \end{array} \\
\begin{pmatrix}
\boldsymbol{Y} & \vdots & \vdots & \cdots & \vdots \\
\boldsymbol{X} & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots
\end{pmatrix}
\end{array} \in Y
$$

Figure 2.7

Our final step is to show that voter 1 is decisive over set Y i.e. for any profile $\hat{P}$, where $\hat{P}_1 = (y_1 \dots y_K \dots\dots)$, we have $f(\hat{P}) \in Y$.

Consider any such ordering $\hat{P}_1$ without loss of generality assume $\hat{P}_1 = (y_1 \dots y_K, t_1, \dots, t_K, \dots\dots)$ i.e. $r_l(\hat{P}_1) = y_l$ for $l = 1, 2, \dots, K$ and $r_l = t_{l-K}$ for $l = 1+K, 2+K, \dots, 2K$.

Let (E, F, G) be a partition of $A \setminus [Y \cup T]$ such that each has at least K elements i.e. $|V| \geq K$ for $V = E, F, G$. Let P be the profile such that

(i) voter 1 ranks all alternatives in Y on top, followed by all alternatives in G, followed by all alternatives in T, followed by all alternatives in E and finally followed by all alternatives in F.

(ii) all voters $2, 3, \dots, n$ rank all alternatives in T on top, followed by all alternatives in E, followed by all alternatives in Y, followed by all alternatives in F and finally followed by all alternatives in G.

(iii) all voters rank alternatives in T and Y in the same way with common ranking $(t_1, t_2, \ldots, t_K)$ and $(y_1, y_2, \ldots, y_K)$ respectively.

Profile P is shown in Figure 2.8.

$$\begin{array}{c c} & \begin{array}{ccccc} P_1 & P_2 & P_3 & \cdots\cdot & P_n \end{array} \\ P = & \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{T} & \boldsymbol{T} & \cdots\cdot & \boldsymbol{T} \\ G & E & E & \cdots\cdot & E \\ \boldsymbol{T} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \\ E & F & F & \cdots\cdot & F \\ F & G & G & \cdots\cdot & G \end{pmatrix} \end{array}$$

Figure 2.8

It follows from Step 1 that $f(P) \in Y \cup T$. Suppose $f(P) \in T$. At this profile consider a misreport $P_1''' = (\boldsymbol{Y}, \boldsymbol{X}, \ldots)$ via voter 1. By our previous arguments, we have $f(P_1''', P_{-1}) \in Y$. This implies voter 1 can K-manipulate. Hence, $f(P) \in Y$.

Again replicating our earlier arguments, it follows that $f(\hat{P}) \in Y$, where $\hat{P}_1 = (\boldsymbol{Y}, \boldsymbol{T} \ldots)$. Since choice of ordering $\hat{P}_1$ is arbitrary, it completes the proof.

Step 3 : If $f(P) \in X$ then coalition $\{2, 3, \ldots, n\}$ is decisive over X.
Suppose $f(P) \in X$. Consider any ordering $\hat{P}_1$ of voter 1 where she reshuffles the ordering of alternatives in $Y \cup D$ while keeping $Y \cup D$ above $X \cup C \cup B$ and the ranking of all alternatives in $X \cup C \cup B$ the same. Profile $(\hat{P}_1, P_{-1})$ with its outcome is shown in Figure 2.9.

$$\begin{array}{c c c} & \begin{array}{ccccc} \hat{P}_1 & P_2 & P_3 & \cdots\cdot & P_n \end{array} & \\ f & \begin{pmatrix} \vdots & \boldsymbol{X} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ \vdots & B & B & \cdots\cdot & B \\ \boldsymbol{X} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \\ B & C & C & \cdots\cdot & C \\ C & D & D & \cdots\cdot & D \end{pmatrix} & \in X \end{array}$$

Figure 2.9

Suppose $f(\hat{P}_1, P_{-1}) \in Y$, then voter 1 can K-manipulate at P via $\hat{P}_1$. Suppose $f(\hat{P}_1, P_{-1}) = d_l$ for some d_l in D. We can argue without loss of generality that d_l is the first ranked alternative in D according to P_1 in profile P i.e $r_{K+1}(P_1) = d_l$. Now, voter 1 can

K-manipulate at P via $\hat{P}_1$. If $f(\hat{P}_1, P_{-1}) \in B \cup C$ then voter 1 can K-manipulate at $(\hat{P}_1, P_{-1})$ via misreporting a preference ordering which puts x_1 on top. Therefore, $f(\hat{P}_1, P_{-1}) \in X$.

Consider any arbitrary partition (T, W) of $Y \cup D$ where both have cardinality of at least K. Let $\tilde{P}_1$ be an ordering such that voter 1 ranks all alternatives in T on top, followed by an alternative x_1,[10] followed by all alternatives in W, followed by the remaining alternatives, which are $B \cup C \cup X \backslash \{x_1\}$. We claim $f(\tilde{P}_1, P_{-1}) = x_1$. See Figure 2.10.

$$\begin{array}{c} \\ f(\tilde{P}_1, P_{-1}) = f \end{array} \begin{array}{c} \begin{matrix} \tilde{P}_1 & P_2 & P_3 & \cdots & P_n \end{matrix} \\ \begin{pmatrix} T & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ x_1 & B & B & \cdots & B \\ W & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\ \vdots & C & C & \cdots & C \\ \vdots & D & D & \cdots & D \end{pmatrix} \end{array} = x_1$$

Figure 2.10: Note that $T \cup W = Y \cup D$

Suppose $f(\tilde{P}_1, P_{-1}) = z \in Y \cup D$. Using the previous argument we can assume without loss of generality that z is the top ranked alternative in $\hat{P}_1$ i.e. $r_1(\hat{P}_1) = z$. By construction alternative z is ranked at least $2K - 1$ places above every alternative in X. Moreover, we have shown that $f(\hat{P}_1, P_{-1}) \in X$. Therefore, voter 1 can K-manipulate at profile $(\hat{P}_1, P_{-1})$ via P_1'. Suppose $f(\tilde{P}_1, P_{-1}) \in X \cup B \cup C \backslash \{x_1\}$ i.e. the outcome is an alternative below W. Then voter 1 can K-manipulate via misreporting x_1 on top. Therefore, $f(P_1', P_{-1}) = x_1$.

Consider an arbitrary profile P' where, $P_1' = \tilde{P}_1$ and voters 2 through n rank X on top with the common ordering $(x_1, x_2, \ldots, x_n)$. The ordering over all other alternatives i.e $A \backslash X$ is arbitrary. We claim $f(P') = x_1$. See Figure 2.11.

$$\begin{array}{c} \\ f(P') = f \end{array} \begin{array}{c} \begin{matrix} \tilde{P}_1 & P_2' & P_3' & \cdots & P_n' \end{matrix} \\ \begin{pmatrix} T & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ x_1 & \vdots & \vdots & \cdots & \vdots \\ W & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix} \end{array} \in X$$

Figure 2.11

To see this, suppose voter 2 changes her preference from P_2 to P_2'. Suppose the outcome

[10] Recall that x_1 is top ranked alternative of all voters other than voter 1.

is no longer x_1. If the outcome is in $X\backslash\{x_1\}$ then voter 1 can K-manipulate here via misreporting x_1 on top (since $|W| \geq K$). On the other hand, if the outcome is not in X then voter 2 can K-manipulate at $(\tilde{P}_1, P_2', P_3, \ldots, P_n)$ via P_2 to obtain x_1 again. Hence, the outcome remains x_1 at this profile. Now, repeating the same argument for voters 3 through n we establish the claim.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} P_1'' & P_2'' & P_3'' & \cdots & P_n'' \end{array} \\
P'' = & \begin{pmatrix} \vdots & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix}
\end{array}
$$

Figure 2.12

Consider any profile P'' (see Figure 2.12), where P_1'' is arbitrary and $P_i'' = P_i'$ for all $i \neq 1$. We claim $f(P'') \in X \cup B \cup C$. Suppose the claim is false and $f(P'') = z \in Y \cup D$. As (T, W) is an arbitrary partition of $Y \cup D$, we can assume without loss of generality that $z \in T$ and z is the top-ranked alternative in P_1'. Since T has at least K alternatives, z is ranked $K-1$ places above x_1. The argument in the previous paragraph shows that $f(P') = x_1$. Therefore, voter 1 can K-manipulate at P' via P_1''. So, $f(P'') \in X \cup B \cup C$.

$$
\begin{array}{cc}
 & \begin{array}{ccccc} P_1^\star & P_2^\star & P_3^\star & \cdots & P_n^\star \end{array} \\
P^\star = & \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ B & D & D & \cdots & D \\ \boldsymbol{X} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\ D & C & C & \cdots & C \\ C & B & B & \cdots & B \end{pmatrix}
\end{array}
\qquad
\begin{array}{cc}
 & \begin{array}{ccccc} P_1 & P_2 & P_3 & \cdots & P_n \end{array} \\
P^{\star\star} = & \begin{pmatrix} \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ C & B & B & \cdots & B \\ \boldsymbol{X} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\ B & D & D & \cdots & D \\ D & C & C & \cdots & C \end{pmatrix}
\end{array}
$$

Figure 2.13

Consider the profiles $P^\star$ and $P^{\star\star}$ shown in Figure 2.13. As we have shown $f(P'') \in X \cup B \cup C$, it follows immediately that $f(P^\star) \in X \cup B \cup C$. However, in Step 1 we have already shown that $f(P^\star) \in X \cup Y$. Thus, $f(P^\star) \in X$.[11]

Starting from profile $P^\star$ instead of P and using the arguments in Step 3 we can deduce that $f(P'') \in X \cup D \cup C$ because B and D have been interchanged in P to obtain $P^\star$. By an identical argument for $P^{\star\star}$, we can show that $f(P'') \in X$.[12]

[11]This is so because $P^\star$ is the type of profile P''.

[12]The sets C and D have been interchanged in P to obtain $P^{\star\star}$.

Since B, C and D are mutually disjoint sets, it follows that $f(P'') \in X$. This shows that coalition $\{2, 3, \ldots, n\}$ is decisive over X. ■

Proposition **2.3** *Let f be a unanimous and K-strategy-proof SCF. If there exists $X \subset A$ such that a voter i is decisive over X then f is K-dictatorial and i is the K-dictator.*

Proof: We have to prove that for any profile $P = (P_i, P_{-i})$, we have $f(P) \in \{r_1(P_i), r_2(P_i) \ldots r_K(P_i)\}$. Pick an arbitrary preference ordering P_i. Suppose W contains the top K alternatives in this preference ordering and assume without loss of generality that they are ranked according to $(w_1, w_2, \ldots, w_n)$ i.e. $r_l(P_i) = w_l$ for all $l = 1, \ldots, K$. Let Z be the set of alternatives such that $Z \subset A \backslash W \cup Y$ and $|Z| = K$.

$$P^* = \begin{array}{c} \begin{array}{ccccccc} P_1^* & \cdots & P_{i-1}^* & P_i^* & P_{i+1}^* & \cdots & P_n^* \end{array} \\ \begin{pmatrix} \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{W} & \boldsymbol{Z} & \cdots & \boldsymbol{Z} \\ E & \cdots & E & G & E & \cdots & E \\ \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{Z} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\ F & \cdots & F & E & F & \cdots & F \\ G & \cdots & G & F & G & \cdots & G \end{pmatrix} \end{array}$$

Figure 2.14

Consider a partition (Z, W, E, F, G) of A such that $|V| = K$ for $V = Z, W$ and $|V| \geq K$ for $V = E, F, G$. Pick a profile P^* (shown in Figure 2.14) such that

(i) voter i ranks all alternatives in W on top, followed by all alternatives in G, followed by all alternatives in Z, followed by all alternatives in E and finally followed by all alternatives in F.

(ii) all voters in $N \backslash \{i\}$ rank all alternatives in Z on top, followed by all alternatives in E, followed by all alternatives in W, followed by all alternatives in F and finally followed by all alternatives in G.

(iii) all voters rank alternatives in W in the same way with the common ranking being $(w_1, w_2, \ldots, w_K)$. Similarly, alternatives in Z are ranked with the common ranking $(z_1, z_2, \ldots, z_K)$.[13]

Comparing profile P^* with profile P in the Figure 2.3, we see that they are "the same" except that voter 1 and partition (X, Y, B, C, D) are replaced by voter i and partition (Z, W, E, F, G) respectively.

[13]Note that voters can rank alternatives in E, F and G differently from each other.

$$
f\begin{array}{c} \begin{array}{ccccccc} P_1^* & \cdots & P_{i-1}^* & P_i^* & P_{i+1}^* & \cdots & P_n^* \end{array} \\ \begin{pmatrix} \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{W} & \boldsymbol{Z} & \cdots & \boldsymbol{Z} \\ E & \cdots & E & G & E & \cdots & E \\ \boldsymbol{W} & \cdots & \boldsymbol{W} & \boldsymbol{Z} & \boldsymbol{W} & \cdots & \boldsymbol{W} \\ F & \cdots & F & E & F & \cdots & F \\ G & \cdots & G & F & G & \cdots & G \end{pmatrix} \end{array} \in W \;\Rightarrow\; f\begin{array}{c} \begin{array}{ccccccc} P_1'' & \cdots & P_{i-1}'' & P_i'' & P_{i+1}'' & \cdots & P_n'' \end{array} \\ \begin{pmatrix} \vdots & \cdots & \vdots & \boldsymbol{W} & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix} \end{array} \in W
$$

Figure 2.15

Proposition 2.2 implies that either voter i is decisive over W or coalition $N \backslash \{i\}$ is decisive over Z. If coalition $N \backslash \{i\}$ is decisive over Z then it is a contradiction to the hypothesis that voter i is decisive over X. Therefore, voter i is decisive over W (see Figure 2.15). Note that the choice of the preference ordering P_i was arbitrary. It follows that if voter i has preference ordering P_i then i is decisive over set of alternatives $\{r_1(P_i), r_2(P_i) \ldots r_K(P_i)\}$ i.e. outcome must belong to her top K alternatives. Hence, voter i is K-dictator and f is a dictatorial SCF. ■

Proposition 2.4 *Suppose f be a unanimous and K-strategy-proof SCF. Let X and Y be disjoint sets each with K alternatives. Fix any $i \in N$, then either (i) voter i is decisive over both X and Y or (ii) coalition $N \backslash \{i\}$ is decisive over both X and Y.*

Proof: Assume w.l.o.g. that $i = 1$. Proposition 2.2 implies that either voter 1 is decisive over Y or coalition $N \backslash \{1\}$ is decisive over X. If voter 1 is decisive over Y then Proposition 2.3 implies that voter 1 is the K-dictator which means she is also decisive over Y. This covers the first possibility.

$$
P' = \begin{array}{c} \begin{array}{cccccc} P_1' & P_2' & P_3' & \cdots\cdot & P_n' \end{array} \\ \begin{pmatrix} \boldsymbol{X} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \\ D & B & B & \cdots\cdot & B \\ \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ B & C & C & \cdots\cdot & C \\ C & D & D & \cdots\cdot & D \end{pmatrix} \end{array}
$$

Figure 2.16

Suppose coalition $N \backslash \{1\}$ is decisive over X. In order to show that it is also decisive over Y, interchange X and Y in the profile P considered in Proposition 2.2 (Part 1)-Figure 2.3 to obtain profile P' shown in Figure 2.16.

$$
f\begin{pmatrix}
\boldsymbol{X} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\
D & B & B & \cdots & B \\
\boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
B & C & C & \cdots & C \\
C & D & D & \cdots & D
\end{pmatrix}_{P'_1\ P'_2\ P'_3\ \cdots\ P'_n} \in Y \;\Rightarrow\; f\begin{pmatrix}
\vdots & \boldsymbol{Y} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \vdots & \vdots & \cdots & \vdots
\end{pmatrix}_{P''_1\ P''_2\ P''_3\ \cdots\ P''_n} \in Y
$$

Figure 2.17

From Proposition 2.2-Step 1, we know that $f(P') \in X \cup Y$. We want to argue that $f(P) \in Y$. Suppose $f(P) \in X$. Then, using the arguments of Proposition 2.2-Step 2, we can conclude that voter 1 is decisive over X. If voter 1 becomes decisive over X then Proposition 2.3 implies that she is the K-dictator. This contradicts the hypothesis that coalition $N\backslash\{1\}$ is decisive over X. Therefore, $f(P) \in Y$. Repeat the arguments of Proposition 2.2-Step 3, with the only modification being X and Y are interchanged. This leads to the conclusion that coalition $N\backslash\{i\}$ is decisive over set Y (see Figure 2.17). This completes the proof. ■

We will define some notation for the next proposition. Let $\{X, Y, B, C, D\}$ be a partition of A such that $|Z| = K$ for $Z = X, Y, B$ and $|Z| \geq K$ for $Z = C, D$. For any positive integer k define $G_{k^-} = \{1, 2, \ldots, k-1\}$, $G_k = \{k, k+1, \ldots, n\}$ and $G_{k^+} = \{k+1, k+2, \ldots, n\}$. Let l be a positive integer less than equal to $n-1$. Let $\bar{P}_{G_{l^-}}$ be a sub-profile for voters in G_{l^-} such that all voters $1, 2, \ldots, l-1$ rank alternatives in B on top with the common ranking $(b_1, b_2, \ldots, b_K)$, followed by all alternatives in D, followed by alternatives in $\{x_1, y_1\}$, followed by all alternatives in C and finally followed by remaining alternatives i.e. in $X \cup Y \backslash \{x_1, y_1\}$[14]. The sub-profile $\bar{P}_{G_{l^-}}$ is shown Figure 2.18.

$$
\begin{array}{c} \\ \bar{P}_{G_{l^-}} = \end{array}
\begin{array}{c}
\begin{array}{ccccc} \bar{P}_1 & \bar{P}_2 & \cdots & \bar{P}_{l-1} \end{array} \\
\begin{pmatrix}
\boldsymbol{B} & \boldsymbol{B} & \cdots & \boldsymbol{B} \\
D & D & \cdots & D \\
\{x_1, y_1\} & \{x_1, y_1\} & \cdots & \{x_1, y_1\} \\
C & C & \cdots & C \\
\vdots & \vdots & \cdots & \vdots
\end{pmatrix}
\end{array}
$$

Figure 2.18

[14]Note that voters in G_{l^-} can rank alternatives in D, $\{x_1, y_1\}$, C, $X \cup Y \backslash \{x_1, y_1\}$ differently from each other.

Proposition **2.5** *Let f be a unanimous and K-strategy-proof SCF. Fix L such that $1 \leq L \leq n-1$. Suppose coalition G_l is decisive over both X and Y given $\bar{P}_{G_{l^-}}$ for every $l \leq L$. Then either (i) voter L is decisive over both X and Y or (i) coalition G_{L^+} is decisive over both X and Y given $\bar{P}_{G_{L^-}}$.*[15]

Proof: The proof will be similar to that of Proposition 2.2. It is divided into three steps and a one preliminary step called Step 0. The preliminary step is similar to unanimity but restricted to coalition G_L. The first step is concerned with the possible outcomes at a specific profile. Depending on the outcome at this profile, the second and third step show that either voter L is decisive over X and Y or coalition G_{L^+} is decisive over X and Y.

Consider an arbitrary preference profile $P^\star$ where coalition G_{L^-} has the sub-profile $\bar{P}_{G_{L^-}}$. All voters in G_L rank X on top and alternatives in X are ranked according to the common ranking $(x_1, x_2, \ldots, x_K)$. Similarly, let P^{**} be an arbitrary profile where coalition G_{L^-} has a profile $\bar{P}_{G_{L^-}}$. In addition, all voters in G_L rank Y on top and alternatives in Y are ranked according to the common ranking $(y_1, y_2, \ldots, y_n)$.[16] The profiles P^* and P^{**} are shown in Figure 2.19.

Step 0 : We claim $f(P^*) = x_1$ and $f(P^{**}) = y_1$.

Since G_L is decisive over X given $\bar{P}_{G_{L^-}}$, the outcome at P^* must belong to X. Suppose $f(P^*) \in X \backslash \{x_1\}$. Consider the profile obtained when voter $L-1$ replaces ordering $\bar{P}_{L-1}$ by an ordering $\hat{P}_{L-1}$, which places X on top such that alternatives in X are ranked according to the common ranking $(x_1, x_2, \ldots, x_K)$. By the hypothesis, G_{L-1} is also decisive over X given $\bar{P}_{G_{L-1^-}}$. Therefore, the outcome at this new profile $(\hat{P}_{L-1}, P^*_{-(L-1)})$, must belong to X. If the outcome is x_1 then there is a K-manipulation at the original profile P^* by voter $L-1$ via $\hat{P}_{L-1}$ (since $|C| \geq K$). Therefore, the outcome at the new profile also belongs to $X \backslash \{x_1\}$.

We can continue in the same way changing the preferences of voters $L-2$ through 1 by an ordering that places X on top such that alternatives in X are ranked according to the common ranking $(x_1, x_2, \ldots, x_K)$. At every step in this sequence the outcome must belong to $X \backslash \{x_1\}$. However, at the end of the sequence unanimity will imply that the outcome is x_1. We have a contradiction. Therefore, $f(P^*) = x_1$. By an identical argument where X is replaced by Y, we have $f(P^{**}) = y_1$. This completes the claim.

Consider the preference profile P such that (see Figure 2.20)

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[15] According to the Proposition, if voter L is decisive then there is no restriction. Whereas coalition G_{L^+} becomes decisive only with respect to the sub-profile $\bar{P}_{G_{L^-}}$. To clarify further, suppose $n = 6$ and $L = 3$. The Proposition says : Suppose it is true that $N, \{2,3,4,5,6\}, \{3,4,5,6\}$ is decisive over X and Y given $\emptyset, \bar{P}_{\{1\}}, \bar{P}_{\{1,2\}}$ respectively. Then either (i) 3 is decisive over X and Y or (ii) $\{4,5,6\}$ is decisive over X and Y given $\bar{P}_{\{1,2,3\}}$.

[16] We are assuming w.l.o.g. a coalition is decisive over X and Y w.r.t. $\bar{P}_i^X = (x_1, x_2, \ldots, x_K)$ and $\bar{P}_i^Y = (y_1, y_2, \ldots, y_n)$ respectively.

$$P^* = \begin{array}{c} \begin{array}{cccccccc} \bar{P}_1 & \dots & \bar{P}_{L-1} & P^*_L & P^*_{L+1} & \cdots\cdot & P^*_n \end{array} \\ \begin{pmatrix} \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{X} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ D & \cdots & D & \vdots & \vdots & \cdots\cdot & \vdots \\ \{x_1, y_1\} & \cdots & \{x_1, y_1\} & \vdots & \vdots & \cdots\cdot & \vdots \\ C & \cdots & C & \vdots & \vdots & \cdots\cdot & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots\cdot & \vdots \end{pmatrix} \end{array}$$

$$P^{**} = \begin{array}{c} \begin{array}{cccccccc} \bar{P}_1 & \dots & \bar{P}_{L-1} & P^{**}_L & P^{**}_{L+1} & \cdots\cdot & P^{**}_n \end{array} \\ \begin{pmatrix} \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \\ D & \cdots & D & \vdots & \vdots & \cdots\cdot & \vdots \\ \{x_1, y_1\} & \cdots & \{x_1, y_1\} & \vdots & \vdots & \cdots\cdot & \vdots \\ C & \cdots & C & \vdots & \vdots & \cdots\cdot & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots\cdot & \vdots \end{pmatrix} \end{array}$$

Figure 2.19

(i) $P_{G_{L^+}} = \bar{P}_{G_{L^+}}$

(ii) voter L ranks all alternatives in Y on top, followed by all alternatives in D, followed by all alternatives in X, followed by all alternatives in B and finally followed by all alternatives in C.

(iii) all voters in $G_{L^+} = \{L+1, L+2, \dots, n\}$ rank all alternatives in X on top, followed by all alternatives in B, followed by all alternatives in Y, followed by all alternatives in C and finally followed by all alternatives in D.

(iv) all voters in $G_L = \{L, L+1, \dots, n\}$ rank alternatives in X in the same way according to common ranking $(x_1, x_2, \dots, x_K)$. Similarly, alternatives in Y are ranked in the same way according to common ranking $(y_1, y_2, \dots, y_K)$.[17]

Step 1′ : We claim that $f(P) \in X \cup Y$.

We prove the claim by showing $f(P) \notin B \cup C \cup D$. Suppose $f(P) \in B \cup C$. Consider a misreport by voter L that puts X on top which are ranked according to $(x_1, x_2, \dots, x_K)$ i.e. an ordering of type $P'_L = (x_1 \dots x_K \dots\dots)$. We thereby obtain the profile P^*. Since $f(P^*) = x_1$, voter L is able to K-manipulate at P.

[17]Note that voters in G_L can rank alternatives in B, C and D differently from each other.

$$P = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccccc} \bar{P}_1 & \dots & \bar{P}_{L-1} & P_L & P_{L+1} & \cdot\cdot\cdot\cdot & P_n \end{array}$$

$$P = \begin{pmatrix} \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ D & \cdots & D & D & B & \cdots\cdot & B \\ \{x_1, y_1\} & \cdots & \{x_1, y_1\} & \boldsymbol{X} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \\ C & \cdots & C & B & C & \cdots\cdot & C \\ \vdots & \cdots & \vdots & C & D & \cdots\cdot & D \end{pmatrix}$$

Figure 2.20

Suppose $f(P) \in D$. Consider any profile where voters in G_{L^+} reshuffle *only* alternatives in C i.e. keeping the position of all alternatives in $A \setminus C$ unchanged.

Suppose a voter $i \in G_{L^+}$ can shift the outcome away from D. If the resulting outcome is in C then the argument in the previous paragraph applies and voter L will K-manipulate because she can obtain outcome x_1 via misreporting $(x_1, x_2, \dots, x_K)$ on top. If on the other hand, the outcome is above C then we have a K-manipulation by voter i since $|C| \geq K$. Hence, the outcome must remain in D for all such profiles.

Lemma 2.1 can therefore be applied with $A_1 = X \cup B \cup Y,\ A_2 = C,\ A_3 = D$ and $S = G_{L^+}$ to conclude that the outcome at a profile where voters $L+1$ through n have Y on top and alternatives in Y are ranked according to $(y_1, y_2, \dots, y_n)$, the outcome is in D. This implies that $P^{**} \in D$, which contradicts our conclusion in Step 0. This completes Step $1'$.

Step 2′ : If $f(P) \in Y$ then L is decisive over Y.
The arguments will be similar to that of Proposition 2.2-Step 2 with the modifications that voter L and coalition G_{L^+} are treated like voter 1 and coalition G_{1^+} respectively.[18]

Consider the profile P' such that

(i) $P'_{G_{L^-}} = \bar{P}_{G_{L^-}}$.

(ii) $P'_L = P_L$.

(iii) all voters in G_{L^+} rank all alternatives in X on top, followed by all alternatives in B, followed by alternative y_1, followed by all alternatives in C and finally followed by all remaining alternatives i.e. $D \cup Y \setminus \{y_1\}$.

(iv) all voters in G_{L^+} rank alternatives in X according to $(x_1, x_2, \dots, x_K)$ and Y according to $(y_1, y_2, \dots, y_K)$.

The profile P' is shown in Figure 2.21.

[18]We will keep the profile of G_{L^-} as $\bar{P}_{G_{L^-}}$.

$$P' = \begin{array}{c} \begin{array}{cccccccc} \bar{P}_1 & \dots & \bar{P}_{L-1} & P_L & P'_{L+1} & \cdots & P'_n \end{array} \\ \left(\begin{array}{cccccccc} \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\ D & \cdots & D & D & B & \cdots & B \\ \{x_1, y_1\} & \cdots & \{x_1, y_1\} & \boldsymbol{X} & y_1 & \cdots & y_1 \\ C & \cdots & C & B & C & \cdots & C \\ \vdots & \cdots & \vdots & C & \vdots & \cdots & \vdots \end{array}\right) \end{array}$$

Figure 2.21

We claim $f(P') = y_1$. Consider a sequence of profiles $\{P^L, P^{L+1}, \dots, P^n\}$, where $P^L = P$ and $P^k = (P'_k, P^{k-1}_{-k}) = (P'_1, \dots, P'_k, P_{k+1}, \dots, P_n)$ i.e profile P^k is obtained from P^{k-1} by replacing P_k with P'_k for $k = L+1, \dots, n$. Note that $P^n = P'$. The sequence starts with profile P and voters $L+1$ through n progressively change their preference ordering from P_i to P'_i. We claim that for any $k \geq 2$,

(i) if $f(P^{k-1}) \in Y \backslash \{y_1\}$ then $f(P^k) \in Y \cup D$.

(ii) if $f(P^{k-1}) = y_1$ then $f(P^k) = y_1$.

We first show Part (i). If $f(P^k) \in B \cup C$ then an argument identical to the one used in Step $1'$ applies and voter L is able to K-manipulate via preference ordering of type $P''_L = (x_1 \dots x_K \dots \dots)$. Suppose $f(P^k) \in X$. As per preference ordering P_k every alternative in X is ranked at least K places above every alternative in Y (since $|B| = K$). Thus, voter k is able to K-manipulate at profile P^{k-1} via P'_k. Therefore, $f(P^k) \in Y \cup D$. This proves part (i).

Suppose $f(P^{k-1}) = y_1$ but $f(P^k) \neq y_1$. According to part (i) it has to belong to $Y \cup D$. In preference ordering P'_k, the alternative y_1 is ranked at least K places above every alternative in $D \cup Y \backslash \{y_1\}$ (since $|C| \geq K$). Therefore, at profile $P^k = (P'_1 \dots P'_k, P_{k+1}, \dots, P_n)$, voter k is able to K-manipulate via misreporting P_k to obtain y_1 instead of an alternative from $D \cup Y \backslash \{y_1\}$. This is a contradiction to K-strategy-proof and it completes the proof of part (ii).

Suppose $f(P') \in D \cup Y \backslash \{y_1\}$. Consider any profile obtained by changing the preference orderings of voters in coalition G_{L^+}, *only* over alternatives in C i.e. keeping the position of alternatives in $A \setminus C$ unchanged. Arguments in Step $1'$ can be replicated to show that the outcome at this profile remains in $D \cup Y \backslash \{y_1\}$.[19] Thus, all conditions of Lemma 2.1 are satisfied with $A_1 = X \cup B \cup \{y_1\}$, $A_2 = C$, $A_3 = D \cup Y \setminus \{y_1\}$ and $S = G_{L^+}$. Lemma

[19] We refer to the arguments which showed that $f(P) \notin D$.

$$
(\tilde{P}_1, P'_{-L}) = \begin{array}{c} \\ \left(\begin{array}{ccccccc} \bar{P}_1 & \dots & \bar{P}_{L-1} & \tilde{P}_L & P'_{L+1} & \cdots\cdot & P'_n \\ \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ D & \cdots & D & \boldsymbol{X} & B & \cdots\cdot & B \\ \{x_1, y_1\} & \cdots & \{x_1, y_1\} & \vdots & y_1 & \cdots\cdot & y_1 \\ C & \cdots & C & \vdots & C & \cdots\cdot & C \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots\cdot & \vdots \end{array}\right) \end{array}
$$

Figure 2.22

2.1 implies that for any profile $\hat{P}$, where $\hat{P}_{N\backslash G_{L^+}} = P'_{N\backslash G_{L^+}}$, we have $f(\hat{P}) \in D \cup Y\backslash\{y_1\}$. Pick $\hat{P}_{G_{L^+}}$ such that $\hat{P}_i = (y_1 \dots y_K \dots\dots)$ for each $i \in G_{L^+}$. The Step 0 implies $f(\hat{P}) = y_1$. However, this contradicts our conclusion $f(P') \in D \cup Y\backslash\{y_1\}$ in the previous paragraph. Therefore, $f(P') = y_1$.

Consider the profile $(\tilde{P}_1, P'_{-L})$ shown in Figure 2.22. Note that $\tilde{P}_L = (y_1 \dots y_K, x_1 \dots x_K, \dots\dots)$. We claim $f(\tilde{P}_1, P'_{-L}) = y_1$. If $f(\tilde{P}_1, P'_{-1}) \notin Y$ then voter L is able to K-manipulate at profile $(\tilde{P}_1, P'_{-1})$ via P'_L to obtain y_1. If on the other hand, $f(\tilde{P}_1, P'_{-1}) \in Y\backslash\{y_1\}$, we can apply the same arguments as in the previous paragraph to show that all conditions of Lemma 2.1 are satisfied with $A_1 = X \cup B \cup \{y_1\}$, $A_2 = C$, $A_3 = D \cup Y\backslash\{y_1\}$ and $S = G_{L^+}$. Again, it implies a contradiction. Therefore, $f(\tilde{P}_1, P'_{-1}) = y_1$.

Now consider a profile P'' (see Figure 2.23) such that

(i) $P''_{G_{L^-}} = \bar{P}_{G_{L^-}}$.

(ii) $P''_L = \tilde{P}_L$.

(iii) all voters in G_{L^+} rank all alternatives in X on top, followed by all alternatives in $B \cup C \cup D$, followed by all alternatives in Y.

(iv) all voters rank alternatives in X according to the common ranking $(x_1, x_2, \dots, x_K)$. Similarly, alternatives in Y have the common ranking $(y_1, y_2, \dots, y_K)$.

We claim $f(P'') \in Y$. In order to see this, start from profile P' and progressively change the preference ordering of voters $L+1$ through n from P'_i to P''_i. Note that $f(P') = y_1 \in Y$. Suppose that k is the first voter whose change from P'_k to P''_k leads the outcome to no longer belong to Y. If it belongs to X then voter k is able to K-manipulate at $(P''_1, \dots, P''_{k-1}, P'_k, \dots, P'_n)$ via P''_k. On the other hand, if the outcome belongs to $B \cup C \cup D$ then voter L can K-manipulate at $(P''_1, \dots, P''_k, P'_{k+1}, \dots, P'_n)$ via any ordering of type $\hat{P}_1 = (x_1 \dots x_K \dots\dots)$. Therefore, we have a contradiction, establishing $f(P'') \in Y$.

$$
P'' = \begin{array}{c}
\begin{array}{cccccccc} \bar{P}_1 & \cdots & \bar{P}_{L-1} & \tilde{P}_L & P''_{L+1} & \cdots\cdot & P''_n \end{array} \\
\begin{pmatrix}
\boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\
D & \cdots & D & \boldsymbol{X} & \vdots & \cdots\cdot & \vdots \\
\{x_1, y_1\} & \cdots & \{x_1, y_1\} & \vdots & \vdots & \cdots\cdot & \vdots \\
C & \cdots & C & \vdots & \vdots & \cdots\cdot & \vdots \\
\vdots & \cdots & \vdots & \vdots & Y & \cdots\cdot & Y
\end{pmatrix}
\end{array}
$$

Figure 2.23

In profile P'', note that the ranking of voters in G_{L^+} over $B \cup C \cup D$ is arbitrary. Since $f(P'') \in Y$, all conditions of Lemma 2.1 are satisfied with $A_1 = X,\ A_2 = B \cup C \cup D,\ A_3 = Y$ and $S = G_{L^-}$. Moreover, the ranking of voter L over $B \cup C \cup D$ is also arbitrary. Applying Lemma 2.1, we have $f(P''') \in Y$, where P''' is shown in Figure 2.24.

$$
P''' = \begin{array}{c}
\begin{array}{cccccccc} \bar{P}_1 & \cdots & \bar{P}_{L-1} & \tilde{P}_L & P'''_{L+1} & \cdots\cdot & P'''_n \end{array} \\
\begin{pmatrix}
\boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \vdots & \cdots\cdot & \vdots \\
D & \cdots & D & \boldsymbol{X} & \vdots & \cdots\cdot & \vdots \\
\{x_1, y_1\} & \cdots & \{x_1, y_1\} & \vdots & \vdots & \cdots\cdot & \vdots \\
C & \cdots & C & \vdots & \vdots & \cdots\cdot & \vdots \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots\cdot & \vdots
\end{pmatrix}
\end{array}
$$

Figure 2.24

Note that in profile P''', the preference orderings of voters $L+1$ through n are arbitrary, while voter L ranks $(y_1, y_2, \ldots, y_K)$ on top, followed by $(x_1, x_2, \ldots, x_K)$, followed by an arbitrary ordering over $B \cup C \cup D$.

Next, consider a profile $P^\star$ such that

(i) $P^\star_{G_{L^-}} = \bar{P}_{G_{L^-}}$.

(ii) voter L ranks all alternatives in Y on top according to the ranking $(y_1, y_2, \ldots, y_K)$, followed by all alternatives in X according to the ranking $(y_1, y_2, \ldots, y_K)$, followed by all alternatives in B, followed by all alternatives in D and finally followed by all alternatives in C.

(iii) all voters in G_{L^+} rank all alternatives in B on top, followed by all alternatives in D, followed by alternative y_1, followed by all alternatives in C and finally, followed by all remaining alternatives i.e. $D \cup Y \backslash \{y_1\}$.

(iv) all voters rank alternatives in B, according to the common ranking $(b_1, b_2, \ldots, b_K)$.

The profile $P^\star$ is shown in Figure 2.25. We claim $f(P^\star) = y_1$.

$$
\begin{array}{cccccccc}
 & \bar{P}_1 & \ldots & \bar{P}_{L-1} & P^\star_L & P^\star_{L+1} & \cdots & P^\star_n
\end{array}
$$
$$
P^\star = \begin{pmatrix}
\boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{B} & \cdots & \boldsymbol{B} \\
D & \cdots & D & \boldsymbol{X} & D & \cdots & D \\
\{x_1, y_1\} & \cdots & \{x_1, y_1\} & \boldsymbol{B} & y_1 & \cdots & y_1 \\
C & \cdots & C & D & C & \cdots & C \\
\vdots & \cdots & \vdots & C & \vdots & \cdots & \vdots
\end{pmatrix}
$$

Figure 2.25

The arguments in the previous paragraph (see Figure 2.24) imply that $f(P^\star) \in Y$. Suppose $f(P^\star) \in Y \backslash \{y_1\}$. As in Step 0, consider the profile when voter $L+1$ replaces ordering $P^\star_{L+1}$ by an ordering which places Y on top according to $(y_1, y_2, \ldots, y_n)$. The outcome at this new profile belongs to Y because of the argument in the previous paragraph. If the outcome is y_1 then it is a K-manipulation (since $|C| \geq K$) at the original profile $P^\star$ by voter $L+1$. Therefore, the outcome at the new profile must belong to $Y \backslash \{y_1\}$.

We can continue in the same way by changing the preference orderings of voters $L+2$ through n by a preference ordering that places Y on top, such that alternatives in Y are ranked according to $(y_1, y_2, \ldots, y_n)$. At every step in this sequence the outcome must belong to $Y \backslash \{y_1\}$. However, at the end of sequence, the profile so obtained is P^{**}. Step 0 will imply that the outcome is y_1 at this profile. So, we have a contradiction. Therefore, $f(P^\star) = y_1$.

Next, consider a profile $P^{\star\star}$ obtained from $P^\star$, where voters 1 through $L-1$ place x_1 below C. Formally, profile $P^{\star\star}$ (see Figure 2.26) is such that

(i) all voters in G_{L^-} rank all alternatives in B on top, according to the common ranking $(b_1, b_2, \ldots, b_K)$, followed by all alternatives in D, followed by the alternative y_1, followed by all alternatives in C and finally, followed by all remaining alternatives i.e. $D \cup Y \backslash \{y_1\}$.

(ii) $P^{\star\star}_{G_L} = P^*_{G_L}$.

We claim that $f(P^{\star\star}) = y_1$. Suppose $f(P^{\star\star}) \neq y_1$. Start with profile $P^\star$ and consider the profile obtained, when voter 1 replaces ordering $P^\star_1$ by $P^{\star\star}_1$. If the outcome at new profile $(P^{\star\star}_1, P^\star_{-1})$ belongs to $C \cup D$ then voter L can K-manipulate by misreporting an ordering of type $(b_1 \ldots b_k \ldots \ldots)$ to obtain b_1. If the outcome belongs to X then voter 1 is able to K-manipulate at $P^\star$ via ordering $P^{\star\star}_1$, since $|D| \geq K$. Suppose the outcome is an alternative

$$P^{\star\star} = \begin{array}{c} \begin{array}{cccccccc} P_1^{\star\star} & \dots & P_{L-1}^{\star\star} & P_L^{\star} & P_{L+1}^{\star} & \cdots & P_n^{\star} \end{array} \\ \begin{pmatrix} \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{Y} & \boldsymbol{B} & \cdots & \boldsymbol{B} \\ D & \cdots & D & \boldsymbol{X} & D & \cdots & D \\ y_1 & \cdots & y_1 & \boldsymbol{B} & y_1 & \cdots & y_1 \\ C & \cdots & C & D & C & \cdots & C \\ \vdots & \cdots & \vdots & C & \vdots & \cdots & \vdots \end{pmatrix} \end{array}$$

Figure 2.26: Profile $P^{\star\star}$

below C. Then, voter 1 can K-manipulate at profile $(P_1^{\star\star}, P_{-1}^{\star})$ via $P_1^{\star}$ by obtaining y_1, since $(|C| \geq K)$. Therefore, the outcome at the new profile is y_1 i.e. $f(P_1^{\star\star}, P_{-1}^{\star}) = y_1$.

We can continue in the same way, by changing progressively the preference ordering $P_i^{\star}$ of voters 2 through $L-1$, by the ordering $P_i^{\star\star}$. An identical argument implies that at every step in this sequence the outcome is y_1. Note that sequence ends at the profile $P^{\star\star}$. Therefore, we have $f(P^{\star\star}) = y_1$.

Our final step is to show that voter L is decisive over Y. Compare profile $P^{\star\star}$ with profile P' in the Proposition 2.2-Step 2 (Figure 2.4). We see that they are "the same" except that voter 1 and sets (D, X and B) are replaced by voter L and sets (X, B and D) respectively. Following a similar argument from there on, we can conclude that voter L is decisive over Y. Since voter L is decisive over Y, the Proposition 2.4 implies that L is also decisive over X. This completes the proof of this step.

Step 3′ : If $f(P) \in X$ then coalition G_{L^+} is decisive over X and Y given $\bar{P}_{G_{L^-}}$.
This step can be proved using the same arguments used in Proposition 2.2-Step 3 with some suitable modifications. In particular, voter L is replaced by voter 1 and coalition G_{L^+} is replaced by G_{1^+}. In all these arguments fix the sub-profile of coalition G_{L^-} equal to $\bar{P}_{G_{L^-}}$. We omit the details of the arguments, since they are essentially the same as those in Step 3 of Proposition 2.2. By doing this, we get the conclusion that coalition G_{L^+} is decisive over X given $\bar{P}_{G_{L^-}}$.[20]

Now apply the arguments similar to Proposition 2.4 by replacing X and Y in preference orderings of voters in G_{L^+} in profile P to conclude that with this interchange outcome belongs to Y. This will eventually lead to the conclusion that G_{L^+} is decisive over Y given $\bar{P}_{G_{L^-}}$. This completes the proof. ■

Proof: [*Theorem 2.1*] Proposition 2.5 implies that either there exists a voter $L \in \{1, 2, \dots, n-1\}$, who is decisive over X and Y or voter n is decisive over X and Y, given

[20]The Step 0 is important to establish this result.

$\bar{P}_{G_{(n-1)^-}}$.

If voter L is decisive then Proposition 2.3 implies that L is also a K-dictator and it completes the proof. For the latter case, we show that voter n is decisive over Y. Consider the profile P' (see Figure 2.27) such that

(i) $P'_{G_{(n-1)^-}} = \bar{P}_{G_{(n-1)^-}}$

(ii) voter $n-1$ ranks all alternatives in B on top, according to the common ranking $(b_1, b_2, \dots, b_K)$, followed by all alternatives in D, followed by alternative y_1, followed by all alternatives in C and finally, followed by all remaining alternatives i.e. $D \cup Y \backslash \{y_1\}$.

(iii) voter n ranks all alternatives in Y on top, according to the ranking $(y_1, y_2, \dots, y_K)$, followed by all alternatives in X according to the ranking $(y_1, y_2, \dots, y_K)$, followed by all alternatives in B, followed by all alternatives in D and finally, followed by all alternatives in C.

$$P' = \begin{array}{c} \\ \\ \\ \\ \\ \\ \end{array} \begin{array}{cccccc} \bar{P}_1 & \bar{P}_2 & \cdots & \bar{P}_{n-2} & P'_{n-1} & P'_n \end{array}$$
$$P' = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{B} & \boldsymbol{Y} \\ D & D & \cdots & D & D & \boldsymbol{X} \\ \{x_1, y_1\} & \{x_1, y_1\} & \cdots & \{x_1, y_1\} & y_1 & \boldsymbol{B} \\ C & C & \cdots & C & C & D \\ \vdots & \vdots & \cdots & \vdots & \vdots & C \end{pmatrix}$$

Figure 2.27

We claim $f(P') = y_1$. Since, G_n is decisive over Y, given $\bar{P}_{G_{n-1^-}}$, we have $f(P') \in Y$. Suppose $f(P') \in Y \backslash \{y_1\}$. Consider an ordering of voter $n-1$, which puts Y on top and alternatives in Y are ranked according to $(y_1, y_2, \dots, y_n)$. By such a preference ordering, voter $n-1$ obtains P^{**} and Step 0 implies the outcome is y_1. This implies that voter $n-1$ can K-manipulate by a misreport at profile P'. Thus, we have $f(P') = x_1$

Compare the profile P' and its outcome with profile $P^{\star}$ and its outcome in Figure 2.25. We see that they are " the same" except that voter L is replaced by voter n and coalition G_{L^+} is replaced by voter $n-1$. Replicating the arguments of Proposition 2.5-Step $2'$, we can conclude that n is decisive over Y. If n is decisive over Y, applying Proposition 2.3 will imply that n is also the K-dictator. This completes the proof. ∎

Recall, that for any L such that $1 \leq L \leq n-1$, coalitions G_{L^-}, G_L and G_{L^+} are $\{1, 2, \dots, L-1\}$, $\{L, L+1, \dots, n\}$ and $\{L+1, L+2, \dots, n\}$ respectively. In what follows, we use the same notion of decisiveness as in the proof of Proposition 2.2.

Consider a partition (X, Y, Z) of A, where $|X| = |Y| = K$. Since $m \geq 3K$, we have $|Z| \geq K$.

Proposition 2.6 *Assume $|A| \geq 3K$. Let f be an efficient and K-strategy-proof SCF. Fix L such that $1 \leq L \leq n-1$. Suppose coalition G_l is decisive over X, for every $l \leq L$. Then, either voter L is decisive over Y or coalition G_{L^+} is decisive over X.*[21]

Proof: The proof will closely follow that of Proposition 2.5. It is however considerably simpler because of the additional power of the efficiency axiom. As in Proposition 2.5 the proof is divided into three steps. The first step is concerned with possible outcomes at a specific profile. Depending on the outcome at this profile, the second and third step show that either voter L is decisive over Y or coalition G_{L^+} is decisive over X.

Consider the preference profile P such that (see Figure 2.28)

(i) all voters in G_{L^-} rank all alternatives in Z on top, followed by all alternatives in Y and finally, followed by all alternatives in X.

(ii) voter L ranks all alternatives in Y on top, followed by all alternatives in X and finally, followed by all alternatives in Y.

(iii) all voters in $G_{L^+} = \{L{+}1, L{+}2, \ldots, n\}$ rank all alternatives in X on top, followed by all alternatives in Y and finally, followed by all alternatives in Z.

(iv) all voters rank alternatives in X, Y and Z in the same way according to the common rankings $(x_1, x_2, \ldots, x_K)$, $(y_1, y_2, \ldots, y_K)$ and $(z_1, z_2, \ldots, z_{|Z|})$ respectively.

$$
\begin{array}{cccccccc}
 & P_1 & \ldots & P_{L-1} & P_L & P_{L+1} & \cdots & P_n \\
P = & \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{Y} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
 & \boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\
 & \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{Z} & \boldsymbol{Z} & \cdots & \boldsymbol{Z}
\end{array}
$$

Figure 2.28

Step 1 : We claim that $f(P) \in \{x_1, y_1\}$.
Efficiency implies that $f(P) \in \{x_1, y_1, z_1\}$. Suppose $f(P) = z_1$. Consider a misreport P'_L by voter L, which puts X on top and alternatives in X are ranked according to common ranking $(x_1, x_2, \ldots, x_K)$. Since, G_L is decisive over X, the outcome after misreport at profile $(P'_L, P_{{}_L})$ must belong to X. At this profile, x_1 dominates every other alternative in X. Therefore,

[21] Note that when G_{L^+} is decisive there is no restriction regarding the preferences of other voters, which differs from the claim in Proposition 2.5

$f(P'_L, P_L) = x_1$. Since, x_1 is ranked $K-1$ places above z_1, voter L can K-manipulate at P via P'_L. This is a contradiction. It completes the proof.

Step 2 : If $f(P) = x_1$ then coalition G_{L^+} is decisive over X.
Consider an ordering P'_i, which puts all alternatives in Z on top, followed by all alternatives in X and finally, followed by all alternatives in Y. Alternatives in X, Y and Z are ranked according to the common rankings $(x_1, x_2, \ldots, x_K)$, $(y_1, y_2, \ldots, y_K)$ and $(z_1, z_2, \ldots, z_{|Z|})$ respectively. Now, construct a profile P' from P by replacing P_i with P'_i for voters 1 through L. In profile P', the sub-profile of voters other than $\{1, 2, \ldots, L\}$ remains the same as in P. The profile P' is described below and shown in Figure 2.29.

(i) all voters in $\{1, 2, \ldots, L\}$ rank all alternatives in Z on top, followed by all alternatives in X and finally, followed by all alternatives in Y. The alternatives in X, Y and Z are ranked according to the common rankings $(x_1, x_2, \ldots, x_K)$, $(y_1, y_2, \ldots, y_K)$ and $(z_1, z_2, \ldots, z_{|Z|})$ respectively.

(ii) $P'_{G_{L^+}} = P_{G_{L^+}}$

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} P'_1 & \ldots & P'_{L-1} & P'_L & P_{L+1} & \cdots\cdot & P_n \end{array} \\
P' = & \begin{pmatrix} \boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{Z} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \\ \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{X} & \boldsymbol{Z} & \cdots\cdot & \boldsymbol{Z} \end{pmatrix}
\end{array}
$$

Figure 2.29

We claim $f(P') = x_1$. Consider the profile (P'_1, P_{-1}). Efficiency implies that $f(P'_1, P_{-1}) \in \{x_1, y_1, z_1\}$. If $f(P'_1, P_{-1}) = y_1$ then we have a contradiction because voter 1 can K-manipulate at profile P via P'_1 as y_1 is ranked $K-1$ places above x_1. If on the other hand, $f(P'_1, P_{-1}) = z_1$ then as we have argued in Step 1, voter L is able to K-manipulate at (P'_1, P_{-1}) by any ordering that puts X on top, which are ranked according to $(x_1, x_2, \ldots, x_K)$. Therefore, we have $f(P'_1, P_{-1}) = x_1$.

Now, progressively change P_i with P'_i for voters 2 through L. An identical argument applies till voter $L-1$, and we have $f(P'_1, \ldots, P'_{L-1}, P_L, \ldots, P_n) = x_1$. When voter L switches from P_L to P'_L, the outcome cannot be z_1 because at profile P' all the alternatives in Z are dominated by any $y_l \in Y$. The outcome cannot be y_1 either because it leads to a K-manipulation by voter L. So, we have $f(P') = x_1$.

Consider an arbitrary profile P'' obtained from P' where voters 1 through L reshuffle *only* alternatives in Z i.e. keeping the position of all alternatives in $A \backslash Z$ unchanged. The profile P'' is shown in Figure 2.30.

$$
\begin{array}{cccccccc}
 & P_1'' & \dots & P_{L-1}'' & P_L'' & P_{L+1} & \cdots & P_n \\
P'' = & \multicolumn{7}{l}{\left(\begin{array}{ccccccc}
\boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{Y} & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
\vdots & \cdots & \vdots & \vdots & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\
\boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{X} & \boldsymbol{Z} & \cdots & \boldsymbol{Z}
\end{array}\right)}
\end{array}
$$

Figure 2.30

At profile P'', efficiency will imply that $f(P) \in \{x_1, y_1\}$. Note that $f(P') = x_1$. If voter k changes the outcome from x_1 to y_1 then she can K-manipulate at $(P_1'', \dots, P_{k-1}'', P_k', P_{k+1}', \dots, P_n')$ via P_k''. It implies $f(P'') = x_1$. Since, $x_1 \in X$, all the conditions of Lemma 2.1 are satisfied with $A_1 = Y$, $A_2 = Z$, $A_3 = X$ and $S = \{1, 2, \dots, L\}$. Lemma 2.1 implies that for any profile where profile of coalition G_{L^+} is $P_{G_{L^+}}$, the outcome remains in X. This is shown through profile P''' in Figure 2.31.

$$
\begin{array}{cccccccc}
 & P_1''' & \dots & P_{L-1}''' & P_L''' & P_{L+1} & \cdots & P_n \\
P''' = & \multicolumn{7}{l}{\left(\begin{array}{ccccccc}
\vdots & \cdots & \vdots & \vdots & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
\vdots & \cdots & \vdots & \vdots & \boldsymbol{Y} & \cdots & \boldsymbol{Y} \\
\vdots & \cdots & \vdots & \vdots & \boldsymbol{Z} & \cdots & \boldsymbol{Z}
\end{array}\right)}
\end{array}
$$

Figure 2.31: $f(P''') \in X$

Consider an arbitrary profile $\hat{P}$ (shown in Figure 2.32) such that

(i) all voters in $\{1, 2, \dots, L\}$ rank all alternatives in $A \backslash X$ above all alternatives in X.

(ii) all voters in G_{L^+} rank all alternatives in X above all alternatives in $A \backslash X$.

(iii) all voters rank alternatives in X, according to the common ranking $(x_1, x_2, \dots, x_K)$. The ranking over $A \backslash X$ can be different for each voter.

$$
\begin{array}{cccccccc}
 & \hat{P}_1 & \dots & \hat{P}_{L-1} & \hat{P}_L & \hat{P}_{L+1} & \cdots & \hat{P}_n \\
\hat{P} = & \multicolumn{7}{l}{\left(\begin{array}{ccccccc}
\vdots & \cdots & \vdots & \vdots & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{X} & \vdots & \cdots & \vdots
\end{array}\right)}
\end{array}
$$

Figure 2.32

We claim $f(\hat{P}) = x_1$. Start from profile P'' and consider the sequence of profiles such that voters 1 through n progressively replace P_i'' with $\hat{P}_i$. The sequence of profiles thus obtained is $\{P^0, P^1, \ldots, P^n\}$, where $P^0 = P''$ and profile P^k is obtained from P^{k-1} by replacing P''_k with $\hat{P}_k$. Note that $P^n = \hat{P}$.

The step L in the sequence is profile $P^L = (\hat{P}_1, \ldots, \hat{P}_L, P_{L+1}, \ldots, P_n)$. The argument in previous paragraph (see Figure 2.31) implies that $f(P^L) \in X$. Efficiency[22] will imply $f(P^L)) = x_1$. Consider profile $P^{L+1} = (\hat{P}_{L+1}, P^L_{-(L+1)})$. If $f(P^{L+1}) \notin X$ then voter $L+1$ can K-manipulate at profile P^{L+1} via P''_L, as x_1 is ranked at least $K-1$ places above every alternative in $A \backslash X$. At profile P^k for $k \geq L$, the alternative x_1 dominates all other alternatives in X. Therefore, it leads to $f(P^{L+1}) = x_1$. The same argument applies to the remaining profiles in the sequence. This implies $f(\hat{P}) = x_1$.

The final argument in this Step is to show that coalition G_{L^+} is decisive over X. Consider an arbitrary profile $P^*_{N \backslash G_{L^+}}$ for voters in $\{1, 2, \ldots, L\}$. Let $(P^*_1, \hat{P}_{-1})$ be the profile obtained from profile $\hat{P}$, when voter 1 replaces $\hat{P}_1$ with P^*_1. We claim $f(P^*_1, \hat{P}_{-1}) \in X$. Suppose for some $a_l \in A \backslash X$, we have $f(P^*_1, \hat{P}_{-1}) = a_l$. Note that in ordering $\hat{P}_1$, voter 1 can arbitrarily rank alternatives in $A \backslash X$. We can therefore, assume that $r_1(\hat{P}_1) = a_l$. This allows voter 1 to K-manipulate at profile $(P^*_1, \hat{P}_{-1})$ via P^*_1 because a_l is at least $2K - 1$ places above every alternative in X according to $\hat{P}_1$. Hence, $f(P^*_1, \hat{P}_{-1}) \in X$. An identical argument applies when voters 2 through L replace $\hat{P}_i$ with P^*_i. Thus, we have $f(P^*_{N \backslash G_{L^+}}, \hat{P}_{G_{L^+}}) \in X$.

The ranking of alternatives in $A \backslash X$ in sub-profile $\hat{P}_{G_{L^+}}$ is also arbitrary. Therefore, for an arbitrary profile P^{**}, where every voter in coalition G_{L^+} ranks X on top with common ranking $(x_1, x_2, \ldots, x_K)$, we have $f(P^{**}) \in X$. It is shown in the Figure 2.33. Therefore, coalition G_{L^+} is decisive over X and it completes the proof of Step 2.

$$
\begin{array}{cc}
 & \begin{array}{cccccccc} P^{**}_1 & \ldots & P^{**}_{L-1} & P^{**}_L & P^{**}_{L+1} & \cdots & P^{**}_n \end{array} \\
P^{**} = & \begin{pmatrix}
\vdots & \cdots & \vdots & \vdots & \boldsymbol{X} & \cdots & \boldsymbol{X} \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\
\vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots
\end{pmatrix}
\end{array}
$$

Figure 2.33: $f(P^{**}) \in X$

Step 3 : If $f(P) = y_1$ (see Figure 2.28) then voter L is decisive over Y.
Consider the profile P' such that (shown in Figure 2.34),

(i) $P'_i = P_i$, for all voters $i \in \{1, 2, \ldots, L\}$.

(ii) all voters in G_{L^+} rank all alternatives in X on top, followed by all alternatives in Z and finally, followed by all alternatives in Y. The alternatives in X and Y are ranked according to the common ranking $(x_1, x_2, \ldots, x_K)$ and $(y_1, y_2, \ldots, y_K)$ respectively.

[22] At profile P^L, the alternative x_1 dominates all other alternatives in X.

$$P' = \begin{array}{c} \begin{array}{ccccccc} P_1 & \dots & P_{L-1} & P_L & P'_{L+1} & \cdots\cdot & P'_n \end{array} \\ \begin{pmatrix} \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{Y} & \boldsymbol{X} & \cdots\cdot & \boldsymbol{X} \\ \boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{X} & \vdots & \cdots\cdot & \vdots \\ \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{Z} & \boldsymbol{Y} & \cdots\cdot & \boldsymbol{Y} \end{pmatrix} \end{array}$$

Figure 2.34

We claim that $f(P') = y_1$. Consider the profile $(P'_{L+1}, P_{-(L+1)})$ obtained from profile P by replacing P_{L+1} with P'_{L+1}. At profile $(P'_{L+1}, P_{-(L+1)})$, efficiency implies that outcome must belong to $\{x_1, y_1\} \cup Z$. If $f(P'_{L+1}, P_{-(L+1)}) = x_1$ then voter $L+1$ can K-manipulate at P via P'_{L+1}. If on the other hand, $f(P'_{L+1}, P_{-(L+1)}) \in Z$ then consider a misreport by voter L of a preference, where X is on top and alternatives in X are ranked according to $(x_1, x_2, \dots, x_K)$. Since, coalition G_L is decisive over X, the outcome in the new profile after the misreport belongs to X. The alternative x_1 dominates all other alternatives in X. Thus, voter L is able to K-manipulate at $(P'_{L+1}, P_{-(L+1)})$ by obtaining x_1. Thus, we have $f(P'_{L+1}, P_{-(L+1)}) = y_1$. The same argument applies when voters $L+2$ through n progressively replace P_i with P'_i. At each step, the outcome remains y_1 and hence at the end, we have $f(P') = y_1$.

Since $f(P') = y_1 \in Y$, $|Z| \geq K$ and the ranking over Z is arbitrary for voters of coalition G_{L^+}, all conditions of Lemma 2.1 are satisfied with $A_1 = X$, $A_2 = Z$, $A_3 = Y$ and $S = G_{L^+}$. Lemma 2.1 implies that for all profiles where preferences of coalition $N \backslash G_{L^+}$, are $P_{N \backslash G_{L^+}}$ the outcome remains in Y. This is shown through profile $\tilde{P}$ in Figure 2.35.

$$\tilde{P} = \begin{array}{c} \begin{array}{ccccccc} P_1 & \dots & P_{L-1} & P_L & \tilde{P}_{L+1} & \cdots\cdot & \tilde{P}_n \end{array} \\ \begin{pmatrix} \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{Y} & \vdots & \cdots\cdot & \vdots \\ \boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{X} & \vdots & \cdots\cdot & \vdots \\ \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{Z} & \vdots & \cdots\cdot & \vdots \end{pmatrix} \end{array}$$

Figure 2.35: $f(\tilde{P}) \in Y$

Consider a profile P'' (see Figure 2.36) such that

(i) $P''_i = P_i$, for all voters $i \in \{1, 2, \dots, L\}$.

(ii) all voters in G_{L^+} rank all alternatives in Z on top, followed by all alternatives in X and finally, followed by all alternatives in Y. The alternatives in X, Y and Z are ranked according to the common ranking $(x_1, x_2, \dots, x_K)$ and $(y_1, y_2, \dots, y_K)$ and $(z_1, z_2, \dots, z_{|Z|})$ respectively.

$$
\begin{array}{c}
\begin{array}{ccccccc} P_1 & \dots & P_{L-1} & P_L & P''_{L+1} & \dots\dots & P''_n \end{array} \\
P'' = \begin{pmatrix} \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{Y} & \boldsymbol{Z} & \cdots\cdots & \boldsymbol{Z} \\ \boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{X} & \cdots\cdots & \boldsymbol{X} \\ \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{Z} & \boldsymbol{Y} & \cdots\cdots & \boldsymbol{Y} \end{pmatrix}
\end{array}
$$

Figure 2.36

As argued in previous paragraph (Figure 2.35), we have $f(P'') \in Y$. Since, y_1 dominates every other alternative in Y, it must be true that $f(P'') = y_1$.

Now, consider the profile $\hat{P}$ (see Figure 2.37) such that

(i) all voters in G_{L^-} rank all alternatives in Z on top, followed by all alternatives in X and finally, by all alternatives in Y.

(ii) voter L in rank all alternatives in Y on top, followed by all alternatives in Z and finally followed by all alternatives in X.

(iii) The alternatives in X, Y and Z are ranked, according to the common ranking $(x_1, x_2, \dots, x_K)$, $(y_1, y_2, \dots, y_K)$ and $z_1, z_2, \dots, z_{|Z|}$ respectively.

(iii) $\hat{P}_{G_{L^+}} = P_{G_L}$.

$$
\begin{array}{c}
\begin{array}{ccccccc} \hat{P}_1 & \dots & \hat{P}_{L-1} & \hat{P}_L & P''_{L+1} & \dots\dots & P''_n \end{array} \\
\hat{P} = \begin{pmatrix} \boldsymbol{Z} & \cdots & \boldsymbol{Z} & \boldsymbol{Y} & \boldsymbol{Z} & \cdots\cdots & \boldsymbol{Z} \\ \boldsymbol{X} & \cdots & \boldsymbol{X} & \boldsymbol{Z} & \boldsymbol{X} & \cdots\cdots & \boldsymbol{X} \\ \boldsymbol{Y} & \cdots & \boldsymbol{Y} & \boldsymbol{X} & \boldsymbol{Y} & \cdots\cdots & \boldsymbol{Y} \end{pmatrix}
\end{array}
$$

Figure 2.37

We claim that $f(\hat{P}) = y_1$. Consider the profile $(\hat{P}_L, P''_{-L})$ obtained from profile P'' by replacing $\hat{P}_L$ with P''_L by voter L. At this profile, efficiency implies that the outcome belongs to $\{y_1, z_1\}$. Note that, $f(P'') = y_1$ and y_1 is ranked $K-1$ places above z_1, according to $\hat{P}_L$. If $f(\hat{P}_L, P''_{-L}) = z_1$ then voter L can K-manipulate at profile $(\hat{P}_L, P''_{-L})$ via P''_L.

Now, start from profile $(\hat{P}_L, P''_{-L})$ and consider the sequence of profiles, where voters 1 through $L-1$ progressively replace P''_i with $\hat{P}_i$ i.e. the sequence of profiles is $\{P^0, P^1, \dots, P^{L-1}\}$, where $P^0 = (\hat{P}_L, P''_{-L})$ and P^k is obtained from P^{k-1} by replacing P''_k with $\hat{P}_k$. Note that $P^{L-1} = \hat{P}$.

Observe that, for the all profiles considered in the sequence, efficiency will imply that the outcome belongs to $\{y_1, z_1\}$. If P^k is the first profile in the sequence where $f(P^k) = z_1$ then voter k can K-manipulate at profile P^{k-1} via $\hat{P}_k$ because z_1 is ranked at least $K-1$ places above y_1 in the ordering P''_k. Since $f(P^0) = y_1$, we have $f(P^{L-1}) = f(\hat{P}) = y_1$.

Now, we can show that voter L is decisive over Y by the same arguments used in Step 2 with some modifications. We briefly outline the argument. Comparing profile $\hat{P}$ with profile P' in Figure 2.29, we see that they are " the same" except that coalitions $\{L\}$ and $N \backslash L$ are replaced by coalitions G_{L^+} and coalition $N \backslash G_{L^+}$ respectively. In addition, sets X, Y and Z in profile $\hat{P}$ are interchanged with Y, Z and X in profile P', respectively. Replicating the arguments of Proposition 2.6-Step 2, we can conclude that L is decisive over Y. ■

Proposition **2.7** *Assume $|A| \geq 3K$. Let f be an efficient and K-strategy-proof SCF. If there exists $X \subset A$ such that voter i is decisive over X then f is K-dictatorial and i is the K-dictator.*

Proof: The proof will be similar to that of Proposition 2.3. We have to show that for any profile $P = (P_i, P_{-i})$, $f(P) \in \{r_1(P_i), r_2(P_i) \ldots r_K(P_i)\}$. Pick any arbitrary ordering P_i. Suppose B consists of the top K alternatives in P_i and assume without loss of generality they are ranked $(b_1, b_2, \ldots, b_K)$. Let C be the set of alternatives such that $C \subset A \backslash B \cup Y$ and $|C| = K$. This is feasible since $|A| \geq 3K$ and $|Y| = |B| = K$

Consider a partition (B, C, D) of A such that $D = A \backslash B \cup C$. Pick a profile P^* (shown in Figure 2.38) such that

(i) voter i ranks all alternatives in B on top, followed by all alternatives in C and followed by all alternatives in D.

(ii) all voters in $N \backslash \{i\}$ rank all alternatives in C on top, followed by all alternatives in B, and followed by all alternatives in D.

(iii) all voters rank alternatives in B, C and D in the same way with the common ranking being $(b_1, b_2, \ldots, b_K)$, $(c_1, c_2, \ldots, c_K)$ and $(d_1, d_2, \ldots, d_K)$.

$$\begin{array}{cc} & \begin{array}{ccccccc} P_1^* & \cdots & P_{i-1}^* & P_i^* & P_{i+1}^* & \cdots & P_n^* \end{array} \\ P^* = & \begin{pmatrix} \boldsymbol{C} & \cdots & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{C} & \cdots & \boldsymbol{C} \\ \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{B} & \cdots & \boldsymbol{B} \\ \boldsymbol{D} & \cdots & \boldsymbol{D} & \boldsymbol{D} & \boldsymbol{D} & \cdots & \boldsymbol{D} \end{pmatrix} \end{array}$$

Figure 2.38

Comparing profile P^* with profile P in the Figure 2.28, we see that they are "the same" except that voter i and coalition $N\backslash\{i\}$ are replaced with voter L and coalition G_{L^+}, respectively, for the value $L = 1$.[23] In addition, the partition (B, C, D) is also replaced with the partition (X, Y, Z).

$$
f\begin{array}{c} \begin{array}{ccccccc} P_1^* & \cdots & P_{i-1}^* & P_i^* & P_{i+1}^* & \cdots & P_n^* \end{array} \\ \begin{pmatrix} \boldsymbol{C} & \cdots & \boldsymbol{C} & \boldsymbol{B} & \boldsymbol{C} & \cdots & \boldsymbol{C} \\ \boldsymbol{B} & \cdots & \boldsymbol{B} & \boldsymbol{C} & \boldsymbol{B} & \cdots & \boldsymbol{B} \\ \boldsymbol{D} & \cdots & \boldsymbol{D} & \boldsymbol{D} & \boldsymbol{D} & \cdots & \boldsymbol{D} \end{pmatrix} \end{array} = b_1 \Rightarrow f\begin{array}{c} \begin{array}{ccccccc} P_1'' & \cdots & P_{i-1}'' & P_i'' & P_{i+1}'' & \cdots & P_n'' \end{array} \\ \begin{pmatrix} \vdots & \cdots & \vdots & \boldsymbol{B} & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \end{pmatrix} \end{array} \in B
$$

Figure 2.39

Proposition 2.6 implies that either voter i is decisive over B or coalition $N\backslash\{i\}$ is decisive over C. If coalition $N\backslash\{i\}$ is decisive over C then we have a contradiction to the hypothesis that voter i is decisive over Y. Therefore, voter i is decisive over B (see Figure 2.39). As the choice of preference ordering P_i was arbitrary, it follows that the voter i is decisive over the set $\{r_1(P_i), r_2(P_i) \ldots r_K(P_i)\}$. Hence, voter i is the K-dictator. ■

Proof: [*Theorem 2.2*] Proposition 2.6 implies that there exists either a voter $L \in \{1, 2, \ldots, n-1\}$, who is decisive over Y or the voter n who is decisive over X. In addition, Proposition 2.7 implies that this voter is also the K-dictator. This completes the proof. ■

[23] The coalition G_{L^-} is empty for $L = 1$.

Chapter 3

Random Strategy-Proof Voting with Lexicographic Extension

3.1 Introduction

In social choice theory, the problem of collective decision making has been analyzed more extensively in deterministic than in random environments. An important issue in random environments, especially in strategic models is that preferences in the voting model are ordinal rankings while the outcome of voting is a probability distribution over alternatives (an exception is Benoit (2002)). In order to compare the outcomes for different voting profiles, it is necessary to specify an appropriate extension from an (ordinal) preference ordering to lotteries - in other words, to extend preferences over degenerate lotteries to preferences over all lotteries. The choice of an extension has profound implications for the analysis; however there are several extensions that can be justifiably chosen. The literature has almost exclusively used the *stochastic dominance* or *sd* criterion introduced by Gibbard (1977). Our goal in this chapter is to explore the consequences in mechanism design of replacing the *sd*-extension by alternative but natural extensions.

The *sd*-extension designates a lottery L as preferred to another lottery L' at some ordinal preference if the expected utility from L is greater than that from L' with respect to *every* utility representation of the ordinal preference. An equivalent formulation is that the probability weight assigned to all alternatives in the upper contour set of any alternative (according to the given preference) is at least as high in L as in L'. An important feature of the *sd*-extension is that it is *incomplete*. It is possible to find lotteries L and L' that are not comparable.

A *Random Social Choice Function* or *RSCF* assigns a lottery over the set of alternatives with every profile of ordinal preferences of voters. Gibbard (1977) provided a complete answer to the following question: which RSCFs have the property that the lottery obtained by truth-telling at every profile is *sd* preferred to any lottery that a voter can obtain by misreporting her preference. In other words, what is the class of *strategy-proof* RSCFs under the *sd*-extension? Gibbard (1977) showed that all *sd*-strategy-proof RSCFs which satisfy the additional (mild) property of *unanimity*, must be a *random dictatorship*. In such a RSCF, every first-ranked alternative is given a fixed probability weight with the sum of weights being one. The requirement for truth-telling in this model is strong - non-comparability of the truth-telling lottery and the lottery obtained by misrepresentation is not permitted. [1]

We replace the *sd*-extension by two simple lottery extensions based on lexicographic comparisons. The first is the *downward lexicographic* or *dl*-extension and the second is the *upward lexicographic* or *ul*-extension. While comparing two lotteries in the former case, the voter will prefer the lottery which has higher probability on the first-ranked alternative. If they are the same, the voter will consider probabilities assigned to the second-ranked alternative, preferring the lottery which has higher probability. If they are the same, she will consider the third-ranked alternative and so on till the last ranked alternative. The voter in this case cares "much more" about a higher ranked alternative than a lower-ranked alternative. For example, a voter will prefer lottery L over L' if the former puts "slightly more" weight on her first-ranked alternative, even though it may put "much more" weight on her worst-ranked alternative.

In contrast, the *ul*-extension captures the behaviour of a voter who wishes to "avoid the possibility" of getting lower-ranked alternatives. While comparing two lotteries, she will prefer the one which has lower probability on the last-ranked alternative. If they are the same, she will consider the alternative that is ranked second-last and so on. Once the *dl* and *ul* extensions are defined, the concept of strategy-proofness for the respective extensions naturally extend from *sd*-strategy-proofness. Thus, for any $e \in \{sd, dl, ul\}$, the e-strategy-proofness means that lottery obtained by truth-telling at every profile is e-preferred to any lottery that a voter can obtain by misreporting her true preference.

The *ul* and *dl* extensions are simple and natural criteria for decision-making which have experimental validity - see Campbell et al. (2006), Tversky and Kahneman (1974) and Starmer (2000). They have been used in various contexts as mentioned in the literature review. Mennle and Seuken (2014) highlight the importance of *dl*-strategy-proofness by showing that it is the lower bound of a generalization of *sd*-strategy-proofness, which they call *partial strategy-proofness*. The *ul* and *dl* extensions generate *complete* orderings over the set of lotteries in contrast to the *sd*-extension. Suppose a voter prefers lottery L to L' according to the *dl*-extension. As Cho (2016) shows, there exists a utility representation of the

[1] A weaker notion of strategy-proofness would require that no lottery obtained by misrepresentation be sd-preferred to the truth-telling lottery.

voter's ordinal preference according to which L has a higher expected utility than L'. Therefore, L' can never be preferred to L according to the *sd*-extension. A similar observation holds for the *ul*-extension. A successful misrepresentation according to either the *ul* or the *dl*-extension will also be a successful misrepresentation according to the *sd*-extension, but the reserve implication may not hold. It follows immediately that the *ul* and *dl* extensions allow for (in principle) a larger class of strategy-proof RSCFs than does the *sd*-extension.

There are two broad sets of results in this chapter. The first concerns *ul*-strategy-proofness. We show that the Gibbard (1977) random dictatorship result continues to hold, i.e. every RSCF satisfying *ul*-strategy-proofness and unanimity must be a random dictatorship. This is rather surprising in view of the fact that *ul*-strategy-proofness is significantly weaker than *sd*-strategy-proofness. The second set of results concern RSCFs that satisfy *dl*-strategy-proofness. We show that a wider class of RSCFs beyond random dictatorship satisfy unanimity and *dl*-strategy-proofness. However, if unanimity is strengthened to *efficiency*, *dl*-strategy-proof RSCFs must be *top-support* rules, i.e. they can give strictly positive probability in a profile only to alternatives that are ranked first by some voter. The weights given to these alternatives can vary across preference profiles. We show that a class of RSCFs that we call *top-weight rules*, are characterized by *dl*-strategy-proofness, efficiency and an additional but familiar property of *tops-onlyness*. In the case of two voters, we show that the tops-onlyness property is implied by the other two requirements. In this case, top-weight rules are characterized by efficiency and *dl*-strategy-proofness. Overall, our results show that the relationship between the random dictatorship and the lottery extension, is rather subtle.

3.1.1 Literature

The seminal work in strategic social choice theory in random environments is Gibbard (1977). The paper shows that *sd*-strategy-proofness and unanimity imply random dictatorship. This result has been extended to restricted domains of preferences in Chatterji et al. (2014) and Chatterji and Zeng (2018)) using the *sd*-extension.

There are several papers that use the *dl*-extension in private good object allocation models such as Bogomolnaia (2015), Alcalde et al. (2013), Saban and Sethuraman (2014), Schulman and Vazirani (2012), Aziz and Stursberg (2014) and Aziz et al. (2015). Cho (2016) and Aziz et al. (2014) consider lexicographic extensions in voting models. Cho (2016) provides conditions on the domain that make e-strategy proofness equivalent to the weaker notions of e-adjacent strategy-proofness and e-mistake monotonicity where $e \in \{sd, dl, ul\}$.

Two papers related to ours are Aziz et al. (2014) and Aziz and Stursberg (2014). Both consider the full domain with indifference. The first contains several results with different extensions and shows the incompatibility of *ul*-strategy-proofness and *ul*-efficiency. This result can be easily derived as a corollary to our Theorem 3.1. Aziz and Stursberg (2014) propose a rule called *the egalitarian simultaneous reservation rule* or (ESR) that satisfies dl-efficiency and two fairness properties. If ESR is restricted to the voting domain of strict preferences, it is weak-sd-strategy-proof. The ESR rule belongs to the class of top-weight rules.

3.2 Preliminaries

Let $N = \{1, \ldots, n\}, n \geq 2$ be a finite set of voters and A be a finite set of m alternatives i.e. $|A| = m \geq 3$. We will write $i, j...$ and $a, b, x, y...$ etc. for generic elements in N and A respectively. Let ΔA denote the set of all probability distributions or lotteries over A. It is the unit simplex of dimension $m-1$. For any lottery $L \in \Delta A$, L_a denotes the probability that alternative a gets under lottery L . Of course $L_a \geq 0$ for all $a \in A$ and $\sum_{a \in A} L_a = 1$.

Each voter $i \in N$ has a preference ordering P_i, which is a linear order over the elements of the set A[2]. For distinct $a, b \in A$, by aP_ib we mean: a is strictly preferred to b by voter i, according to her preference ordering P_i. Let $\mathbb{P}$ denote the set of all linear orderings over the elements of A.

For any ordering P_i and integer $k = 1, \ldots, m$, $r_k(P_i)$ denotes the k^{th} ranked alternative in P_i, i.e. $|\{a \in A : aP_i r_k(P_i)\}| = k - 1$. Also, $r(P_i, a) \in \{1, 2, \ldots, m\}$ will be referred to as the rank of a at P_i. Note that for any $P_i \in \mathbb{P}$, $k \in \{1, 2, \ldots, m\}$ and $a \in A$, $r_k(P_i) = a$ if and only if $r(P_i, a) = k$. We shall occasionally write P_i^a for an ordering where a is ranked first. Similarly, P_i^{ab} will denote a preference ordering where a is ranked first and b second. Let $B(a, P_i)$ denote the set $\{b \in A \mid b = a \text{ } or \text{ } bP_ia\}$ i.e. $B(a, P_i)$ is the set of (weakly) better alternatives than a in the preference ordering P_i.

A profile is a list $P = (P_1, \ldots, P_n) \in \mathbb{P}^n$ of voters' preference orderings. For any coalition $S \subset N$, let $P_S \equiv (P_i)_{i \in S}$ and $P_{-S} \equiv (P_i)_{i \in N \setminus S}$. For simplicity, we write P_{-i} for $P_{-\{i\}}$ and P_{-ij} for $P_{-\{i,j\}}$ and so on. A profile P is also denoted by (P_i, P_{-i}), more generally (P_S, P_{-S}) for any $S \subset N$.

Sometimes it will be useful to construct a preference ordering P_i' from P_i, where the relative ranking of some alternatives (from set X) remains the same. For any two preference orderings P_i, P_i' and set of alternatives $X \subseteq A$, if $xP_iy \Leftrightarrow xP_i'y$, for all $x, y \in X$, then we write $P_i(X) = P_i'(X)$.

Definition **3.1** *A random social choice function (RSCF) (or simply a rule) φ is a mapping $\varphi : \mathbb{P}^n \to \Delta A$.*

A RSCF picks a lottery at every preference profile. It is a standard concept in random mechanism design (for example Gibbard (1977)). The inputs to a RSCF are ordinal preference

[2] A linear order is a binary relation which satisfies completeness, transitivity and anti-symmetry

profile, while the output is a lottery. Note that all preference profiles are admissible i.e. the preference domain is "unrestricted" throughout the chapter. We will write $\varphi(P)$ for the lottery assigned by RSCF φ for profile P and $\varphi_a(P)$ for the probability of alternative a being chosen at profile P.

The objective of the mechanism designer is to incentivize voters to report their true preference orderings, irrespective of their believes about the reports of the other voters. This objective is referred to as strategy-proofness. In this framework, it is essential to introduce assumptions regarding the comparison of different lotteries for a voter with an ordinal ranking over alternatives. There are several ways to extend an ordinal preference over alternatives to preferences over lotteries over these alternatives. We define some of these ways below using the terminology of Cho (2016).

Let $\mathbb{R}(\Delta A)$ be the set of all preference orderings over ΔA. An *extension* is a mapping $e : \mathbb{P} \to \mathbb{R}(\Delta A)$ such that for each $P_i \in \mathbb{P}$, the restriction of $e(P_i)$ to A coincides with P_i. For each $P_i \in \mathbb{P}$, let $R_i^e \equiv e(P_i)$. The strict preference relation associated with R_i^e is denoted by P_i^e. For any two lotteries L and L', $L \; R_i^e \; L'$ means lottery L is (weakly) preferred to lottery L' under extension e when voter i has (ordinal) preference ordering P_i, in short L e-dominates L'. Similarly, for any two distinct lotteries L and L', $L \; P_i^e \; L'$ means L is strictly preferred to L', in short L (strictly) e-dominates L'. We can define strategy-proofness with respect to an extension.

Definition **3.2** *Let e be an extension. A RSCF $\varphi : \mathbb{P}^n \to \triangle A$ is e-strategy-proof if for all $i \in N$, $P \in \mathbb{P}^n$, $P_i' \in \mathbb{P}$ we have $\varphi(P_i, P_{-i}) \; R_i^e \; \varphi(P_i', P_{-i})$.*

The most widely used notion of an extension is the *stochastic dominance* extension (or sd-extension) introduced by Gibbard (1977) (also see Postlewaite and Schmeidler (1986) and Levy (1992)). We denote this extension by R_i^{sd} and is defined as follows: for any pair of lotteries L and L' and a voter i with ordering P_i, $L \; R_i^{sd} \; L'$ iff for all $a \in A$ we have $\sum\limits_{b \in B(a,P_i)} L_b \geq \sum\limits_{b \in B(a,P_i)} L'_b$.

Pick any two lotteries L and L'. Then, $L \; R_i^{sd} \; L'$ iff the expected utility of L is at least as high as that of L' with respect to any cardinal representation of the underlying ordering P_i. It is clear that R_i^{sd} is incomplete in the sense that there exist lotteries L and L' such that neither $L \; R_i^{sd} \; L'$ nor $L' \; R_i^{sd} \; L$ hold. Thus, sd-strategy-proofness requires that the lottery obtained by truth telling be comparable to the lottery obtained by any manipulation. Moreover, the expected utility of the former is greater than or equal to the latter for all utility representations of the true ordering.

We analyze strategy-proof RSCFs under two alternative extensions related with lexicographic preferences (Hausner (1954), Chipman (1960)). They have recently been analyzed in Cho (2016) and Aziz et al. (2014).[3] The first is the *downward lexicographic extension* (or *dl*-extension). For every $P_i \in \mathbb{P}$ and every pair L and $L' \in \triangle A$, $L\, R_i^{dl}\, L'$ if either (i) there exists $k \in \{1, 2, ..., m\}$ such that for each $h \leq k-1$, $L_{r_h(P_i)} = L'_{r_h(P_i)}$ and $L_{r_k(P_i)} > L'_{r_k(P_i)}$ or (ii) $L = L'$.

The *dl*-extension "strongly favors" lotteries that put higher weights on higher ranked alternatives. It can also be rationalized as an expected utility maximization with respect to a narrow but important class of utility representations. As Cho (2016) notes, the *dl*-extension represents "agents whose von Neumann-Morgenstern (vNM) utility functions assign 1 to the most preferred alternative, α to the second most preferred alternative, α^2 to the third most preferred alternative, and so on, where $\alpha \to 0^+$".[4]

The dual to the *dl*-extension is the *upward lexicographic extension* (or *ul*-extension) where voters lexicographically minimize probabilities for less preferred alternatives. For every $P_i \in \mathbb{P}$ and every pair L and $L' \in \triangle A$, $L\, R_i^{ul}\, L'$ if either (i) there exists $k \in \{1, 2, ..., m\}$ such that for each $h \geq k+1$, $L_{r_h(P_i)} = L'_{r_h(P_i)}$ *and* $L_{r_k(P_i)} < L'_{r_k(P_i)}$ or (ii) $L = L'$.

Once again, the *ul*-extension can be rationalized as utility maximizer over a narrow class of utility functions. These are the vNM utility functions which assign -1 to the least preferred alternative, $-\alpha$ to the second least preferred alternative, $-\alpha^2$ to the third least preferred alternative, and so on, where $\alpha \to 0^+$ (see Cho (2016)).

Definitions make it clear that *dl* and *ul* extensions generate complete ordering over lotteries unlike the *sd*-extension. It is also clear that if $L\, R_i^{sd}\, L'$ then both $L\, R_i^{dl}\, L'$ and $L\, R_i^{ul}\, L'$ hold. However, *dl* and *ul* extensions can disagree as the following example shows.

Example **3.1** Suppose $a\, P_i\, b\, P_i\, c$. Let $L = (\frac{1}{2}, 0, \frac{1}{2})$ and $L' = (0, 1, 0)$ be two lotteries. Here the first, second and third components refer to probabilities of a, b and c respectively. Thus, $LP_i^{dl}L'$ and $L'P_i^{ul}L$, while they are not comparable by *sd*-extension.

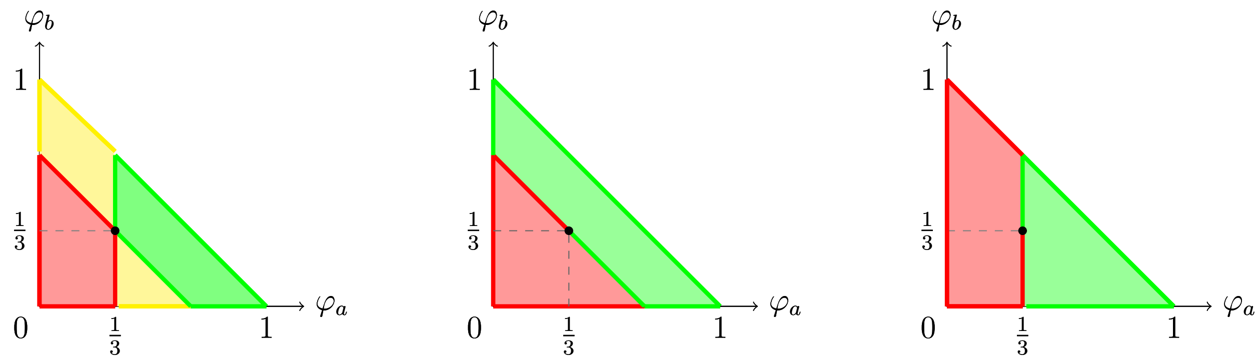
The *ul* and *dl* extensions allow for extra possibilities for the construction of strategy-proof rules. This is illustrated in Figure 3.1.

Let voter i's preference ordering be $P_i : a\, P_i\, b\, P_i\, c$. Suppose that the outcome of a RSCF at profile (P_i, P_{-i}) is $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ for some P_{-i}.[5]

[3]This extension has also been analyzed by Bogomolnaia (2015), Alcalde et al. (2013), Saban and Sethuraman (2014), Schulman and Vazirani (2012), Aziz and Stursberg (2014) and Aziz et al. (2015) in the context of object allocation models.

[4]"For any pair of lotteries L and L' we have $LP^{dl}L'$ iff there exists $\hat{\alpha} \in (0,1)$ such that for all $\alpha \in (0, \hat{\alpha})$ the expected utility from L is higher than L' where von Neumann-Morgenstern (vNM) utility functions assign 1 to the most preferred alternative, α to the second most preferred alternative, α^2 to the third most preferred alternative, and so on".

[5]The first, second and third components of λ denote the probabilities of a, b and c respectively



1(A): sd-extension 1(B): ul-extension 1(C): dl-extension

Figure 3.1: Lottery comparison under different lottery extensions

Figure 1(A) shows lottery comparisons with λ under the sd-extension: (i) the red region shows the set of lotteries that are sd-dominated by λ (ii) yellow region shows the set of lotteries that are not comparable using sd-extension and (iii) the green region shows lotteries that sd-dominate λ.

Similarly, Figure 1(B) and Figure 1(C) show lottery comparisons with λ under the ul-extension and dl-extension respectively: (i) the red region shows the set of lotteries that are dominated by λ and (ii) the green region shows lotteries that dominate λ. Note that there is no yellow region in either 1(B) or 1(C), since ul and dl extensions generate compete orderings over ΔA.

In order for a RSCF to be sd, ul and dl strategy-proof, any misreport P_i' by voter i must lead to a lottery in the red regions of Figures 1(A), 1(B) and 1(C) respectively[6]. It is apparent that the red region in Figure 1(A) is a strict subset of the red regions of Figure 1(B) and 1(C) respectively. This suggests, it ought to be the case that there exists a larger class of ul and dl strategy-proof RSCFs as compared to sd-strategy-proof RSCFs.

3.3 Strategy-Proofness with ul-extension

In this section, we provide a characterization result under strategy-proof with respect to ul extension in conjunction with a mild efficiency axiom. To proceed further we provide following definitions.

Definition **3.3** *Let P be a profile. Alternative a dominates alternative b at P if aP_ib for all $i \in N$. The RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is (ex-post) efficient if $\varphi_b(P) = 0$ whenever there exists*

[6]The axiom of e-strategy-proofness requires that truth telling lottery always e-dominates a lottery received under any lie where $e \in \{sd, ul, dl\}$. Thus, for a RSCF to be e-strategy-proof, it is needed that if the lottery λ is selected at profile P then at any profile (P_i', P_{-i}), it must select a lottery which is e-dominated by λ i.e. a lottery in red region.

an alternative a that dominates b in profile P.

An alternative that is (Pareto) dominated by another alternative in a profile is never chosen by an efficient RSCF at that profile. A weaker notion of efficiency is the standard axiom of unanimity which only requires that an alternative that is ranked-first by all voters is chosen for sure.

DEFINITION **3.4** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is unanimous if $\varphi_a(P) = 1$ at all profiles P where $r_1(P_i) = a$ for all $i \in N$.*

It is possible to define stronger notions of efficiency such as ordinal efficiency (see Bogomolnaia and Moulin (2001a) and Abdulkadiroğlu and Sönmez (2003) for details) and ex-ante efficiency (see Gibbard (1977)). In addition, we can define *e-efficiency* with respect to any *e-extension*. For further discussion refer to Aziz et al. (2014) and Brandt et al. (2016).

DEFINITION **3.5** *A RSCF $\varphi : \mathbb{P}^n \to \triangle A$ is a random dictatorship if there exist weights $\beta_1, ..., \beta_n \in [0, 1]$ with $\sum_{i=1}^{n} \beta_i = 1$ such that for all $P \in \mathbb{P}^n$ and $a \in A$ we have,*

$$\varphi_a(P) = \sum_{\{i \in N \,:\, r_1(P_i) = a\}} \beta_i$$

If any $\beta_i = 1$ for some i, the RSCF is the usual dictatorship which selects the top ranked alternative of voter i. Random dictatorship is a probability distribution over (deterministic) dictatorships, where β_i is the probability of voter i being the dictator.

The classical result for *sd*-strategy-proof RSCF is Gibbard (1977). [7]

Theorem [Gibbard (1977)] *Assume $|A| \geq 3$. A RSCF is sd-strategy-proof and satisfies unanimity if and only if it is a random dictatorship.*

Our first result shows that the Gibbard's result continues to hold, if *sd*-strategy-proofness is weakened to *ul*-strategy-proofness.

THEOREM **3.1** *Suppose $|A| \geq 3$. A RSCF is unanimous and ul-strategy-proof if and only if it is a random dictatorship.*

This result is surprising because *ul*-strategy-proofness is weaker than *sd*-strategy-proofness. As we have argued in the previous section, the following example shows that there are RSCFs which are *ul*-strategy-proof but not *sd*-strategy-proof. We know from our result that they must violate unanimity.

[7] The result in Gibbard (1977) is more general than the result stated in the Theorem 1. In particular, it does not assume unanimity and characterize the entire class of *sd*-strategy-proof rules.

Example **3.2** Let $A = \{a, b, c\}$. The RSCF depends only on the preference ordering of voter 1. If a is ranked third (or last) then the RSCF gives probability weights $(0.6, 0.25, 0.15)$ to the first, second and third ranked alternatives respectively. If b is ranked last then probability weights are $(0.55, 0.35, 0.1)$. If c is ranked last then the lottery is $(0.5, 0.45, 0.05)$. Suppose a is ranked last in 1's preference ordering it receives probability 0.15. If voter 1 misrepresents via an ordering that puts b or c at the bottom, the probability of a strictly increases. A similar argument can be made when b and c are ranked last in voter 1's preference. Suppose a is third ranked while b is second ranked. One cannot reduce the probability of b while keeping the probability of a the same. Therefore, it is ul-strategy-proof. However, it is not sd-strategy-proof because preference ordering $P'_1 : a\ P'_1\ c\ P'_1\ b$ is a manipulation at the profile where $P_1 :\ a\ P_1\ b\ P_1\ c$. Note that these lotteries are not sd comparable which is a violation of sd-strategy-proofness.

We conclude this section with the following remark.

Remark **3.1** Theorem 3 of Aziz et al. (2014) states the following : "there is no anonymous, ul-efficient and ul-strategy-proof RSCF for $n \geq 2$ and $m \geq 3$". This result can be obtained as a corollary of our Theorem 3.1. To see this, observe that our theorem uses a weaker notion of efficiency (ex-post efficiency), which is implied by ul-efficiency. Moreover, random dictatorship violates ul-efficiency except in the case when $\beta_i = 1$ for some voter i. However this violates anonymity.

3.4 Strategy-Proofness with dl-Extension

This section deals with various results concerning with dl-strategy-proofness. The first subsection shows that dl-strategy-proofness is more permissive than ul-strategy-proofness or sd-strategy-proofness .

3.4.1 Top-Support Rules

Unlike the Theorem 3.1, under dl-strategy-proofness unanimity does not imply random dictatorship. The class of dl-strategy-proof and efficient RSCFs is very rich. Unfortunately we cannot provide its characterization. However, all efficient and dl-strategy-proof RSCFs must belong to the class of *Top-Support (TS) rules.*

Definition **3.6** *A RSCF φ is a Top-Support rule if for any alternative a and profile P, $\varphi_a(P) > 0$ implies $a \in \{r_1(P_1), \ldots, r_1(P_n)\} \equiv \mathcal{T}(P)$.*

Only alternatives that are ranked first by some voter can get positive probability in a TS rule. A random dictatorship is a TS rule. However, the probability that a voter assigns to her top-ranked alternative can vary across profiles in a TS rule.

We provide an example of a *dl*-strategy-proof and efficient RSCF that is not a random dictatorship. Aziz and Stursberg (2014) introduced a rule called *egalitarian simultaneous reservation* (ESR)[8] in the full domain which includes indifferences. If ESR is restricted only to our domain i.e. domain of strict preferences then it reduces to a simple rule, which selects all distinct alternatives that are ranked first by some voter with equal probability. Formally,

EXAMPLE **3.3** The ESR rule φ^E is as follows [9] :

$$\varphi^E_x(P) = \begin{cases} \frac{1}{k} & \text{if } x \in \mathcal{T}(P) \text{ and } k = |\mathcal{T}(P)| \\ 0 & \text{otherwise} \end{cases}$$

ESR is a top-support rule. It is therefore efficient. Consider an arbitrary profile where the set of top ranked alternatives is of size k, where $k \in \{1, 2, \ldots, n\}$. A voter's first-ranked alternative gets probability $\frac{1}{k}$ in this profile. Any misreport will either reduce this probability or leave it unchanged. If a misreport does not the change the probability of a voter's first-ranked alternative, then the probability of all other alternatives also remains the same. Therefore, the RSCF is *dl*-strategy-proof .

Our next result shows that the top-support feature of the Example 3.1 is true in general.

PROPOSITION **3.1** *An efficient and dl-strategy-proof RSCF is a top-support rule.*

The proof of Proposition 3.1 is provided in Appendix. Although, the proposition significantly narrows down the class of admissible RSCFs, it is far from a characterization. In the rest of the chapter, we shall attempt to provide sharper results by making further assumptions on the number of voters and the properties of admissible RSCFs.

3.4.2 Top-Weight Rules

In this subsection, we introduce and characterize a new class of rules called top-weight rules. For the purpose of characterization we require the axiom of "tops-onlyness". It requires the

[8]They have defined a more general class of rules called simultaneous reservation (SR). Their objective is to characterized all *dl*-efficient RSCFs in full domain through SR.

[9]Recall that $\mathcal{T}(P)$ is the set of all top ranked alternatives at profile P i.e. $\mathcal{T}(P) = \{r_1(P_1), r_1(P_2), \ldots, r_1(P_n)\}$.

outcome to be the same at any two profiles, where the first-ranked alternatives for voters coincide. This axiom has been widely studied in the context of both deterministic and random environments- see Chatterji and Sen (2011) and Chatterji and Zeng (2018). In many contexts tops-onlyness is a consequence of strategy-proofness together with axioms like unanimity. In our setting however, tops-onlyness is not a consequence of dl-strategy-proofness and efficiency. We impose it nevertheless on the grounds of informational parsimony. In other words the outcome of the rule at any profile can be computed using a limited amount of revealed information. The class of efficient dl-strategy-proof rules is large; tops-onlyness allows us to make a natural and convenient selection from this class.

Definition **3.7** *A RSCF φ is a Tops-Only rule, if for any two profiles P and P' where $r_1(P_i) = r_1(P'_i)$ for all $i \in N$, we have $\varphi(P) = \varphi(P')$.*

To define the class of top-weight rules we further need some notations. Let $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in A^n$ be an ordered n-tuple of alternatives. Denote $[\boldsymbol{x}]$ to be the set of alternatives that appear in $\boldsymbol{x}$. Note that $\boldsymbol{x}$ can contain repetitions which are removed in $[\boldsymbol{x}]$. For example, let $n = 6$ and if $\boldsymbol{x} = (a, a, b, c, a, b) \in A^6$ then $[\boldsymbol{x}] = \{a, b, c\}$. On the other hand, if $\boldsymbol{x} = (a, b, c, d, e, f)$ then $[\boldsymbol{x}] = \{a, b, c, d, e, f\}$. For any $i \in \{1, 2, \ldots, n\} = N$ we will write $\boldsymbol{x} = (x_i, \boldsymbol{x}_{-i})$. Also, for any subset $S \subseteq N$, we write $\boldsymbol{x}_S = (x_i)_{i \in S}$.

Definition **3.8** *A probability assignment map ξ is a function $\xi : A^n \to \Delta(A)$ such that $\sum_{a \in [\boldsymbol{x}]} \xi_a(\boldsymbol{x}) = 1$.*

For any tuple $\boldsymbol{x}$, the output of ξ is a lottery $\xi(\boldsymbol{x})$ and $\xi_a(\boldsymbol{x})$ is the probability assigned to alternative a under this lottery. A probability assignment map assigns a lottery which gives positive probability only to those alternatives which belong to the tuple.

Definition **3.9** *A probability assignment map ξ is monotone if for all $\boldsymbol{x}_{-i} \in A^{n-1}$, any $i \in N$ and any distinct $x_i, x'_i \in A$.*

1. $\xi_{x_i}(x_i, \boldsymbol{x}_{-i}) \geq \xi_{x_i}(x'_i, \boldsymbol{x}_{-i})$.
2. *(i) If* $\xi_{x_i}(x_i, \boldsymbol{x}_{-i}) = \xi_{x_i}(x'_i, \boldsymbol{x}_{-i})$ *then* $\xi(x_i, \boldsymbol{x}_{-i}) = \xi(x'_i, \boldsymbol{x}_{-i})$.

   (ii) If $\xi_{x_i}(x_i, \boldsymbol{x}_{-i}) > \xi_{x_i}(x'_i, \boldsymbol{x}_{-i})$ *then* $\xi_{x'_i}(x_i, \boldsymbol{x}_{-i}) < \xi_{x'_i}(x'_i, \boldsymbol{x}_{-i})$.

Consider the tuple $(x'_i, \boldsymbol{x}_{-i})$. Suppose x'_i is replaced by x_i. According to the definition, the probability of x_i after replacement does not decrease. If it remains the same, then the probability of all alternatives must also remain the same i.e. the lottery should remain unchanged. If the probability of x_i increases then probability of x'_i must decrease. In this case, there are no restrictions on the probability of other alternatives.

Next, we define a class of rules which are generated by monotone probability assignment maps.

Definition **3.10** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is a* top-weight rule *if there exists a monotone probability assignment map ξ such that $\varphi(P) = \xi(r_1(P_1), r_1(P_2), \ldots, r_1(P_n))$ for all $P \in \mathbb{P}^n$.*

The next result shows that top-weight rules are the only *dl*-strategy-proof rules which satisfy the efficiency and tops-onlyness. In the case of two voters the tops-only assumption can be dropped.

Theorem **3.2** *Let $\varphi : \mathbb{P}^n \to \Delta A$ be a RSCF.*

(a) If $n \geq 3$, φ is tops-only, efficient and dl-strategy-proof if and only if it is a top-weight rule.

(b) If $n = 2$, φ is efficient and dl-strategy-proof if and only if it is a top-weight rule.

The proof of Theorem 3.2 is provided in the Appendix. The class of top-weight rules is very rich. It includes random dictatorship as a special case. To see this, pick a vector $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$ where $\beta_i \geq 0$ for all $i \in N$ and $\sum_{i \in N} \beta_i = 1$. Let $\xi^{\boldsymbol{\beta}}$ be the probability assignment map defined as $\xi^{\boldsymbol{\beta}}_a(\boldsymbol{x}) = \sum_{\{i \in N \,:\, x_i = a\}} \beta_i$ for any tuple $\boldsymbol{x} \in A^n$ and $a \in A$. It is easy to verify that $\xi^{\boldsymbol{\beta}}$ satisfies the monotonicity requirements of Definition 3.9. A RSCF generated by ξ is a random dictatorship where each voter i has weight β_i.

Top-weight rules can accommodate the following kind of non-monotonicity: the probability received by an alternative may decline, as the number of voters who rank it first increases. We show this with an example.

Example **3.4** Suppose $N = \{1, 2, 3\}$ and $|A| \geq 3$. Let $\boldsymbol{\beta} = (0.2, 0.2, 0.6)$ and $\boldsymbol{\beta'} = (0.6, 0.2, 0.2)$. Fix an alternative a, for any tuple $\boldsymbol{x} \in A^3$, the probability assignment map ξ is as follows:

$$\xi(\boldsymbol{x}) = \begin{cases} \xi^{\boldsymbol{\beta}}(\boldsymbol{x}) & \text{if } x_1 = a \\ \xi^{\boldsymbol{\beta'}}(\boldsymbol{x}) & \text{if } x_1 \neq a \end{cases}$$

As described earlier, $\xi^{\boldsymbol{\beta}}$ and $\xi^{\boldsymbol{\beta'}}$ are the probability assignment maps associated with random dictatorship rules with weights $\boldsymbol{\beta}$ and $\boldsymbol{\beta'}$ respectively. The probability assignment map in this example is obtained by combining these two probability assignment maps. In particular, $\xi^{\boldsymbol{\beta}}$ is chosen when voter 1 first-ranked alternative is a and $\xi^{\boldsymbol{\beta'}}$ when it is not. It is easy to verify that $\xi^{\boldsymbol{\beta}}$ satisfies monotonicity. Consequently, it generates a top-weight rule.

Consider two profiles P and P' such that the profile of first-ranked alternatives are (a, b, c) and (b, c, c) respectively. According to the rule, associated with the probability assignment map ξ, the probability of alternative c at both profiles are $\varphi_c(P) = 0.6$ and $\varphi_c(P') = 0.4$. In profile P, only voter 3 has c on top, whereas in P' voters 2 and 3 have c on top. However, probability of c declines in P' as compared to P.

It is possible to construct efficient dl-strategy-proof rules that are not top-weight rules. In particular, they will violate the tops-only axiom. This is shown in the example below.

EXAMPLE **3.5** Suppose $N = \{1,2,3\}$ and $|A| \geq 3$. Let $\boldsymbol{\beta} = (0.4, 0.1, 0.5)$ and $\boldsymbol{\beta'} = (0.1, 0.4, 0.5)$. Consider the RSCF φ^{ξ} generated by the probability assignment map ξ defined as follows:

$$\xi(\boldsymbol{x}) = \begin{cases} \xi^{\boldsymbol{\beta}}(\boldsymbol{x}) & \text{if } r_1(P_1)\, P_3\, r_1(P_2) \\ \xi^{\boldsymbol{\beta'}}(\boldsymbol{x}) & \text{otherwise} \end{cases}$$

This RSCF as the one in the previous example is a combination of two random dictatorships. However, particular choice of random dictatorship weights can depend on the ranking of alternatives that are not ranked first by voter 3. This rule is clearly not tops-only. However, it is efficient and dl-strategy-proof .

We conclude this section with two remarks. The first one is regarding the ESR rule and the second one is about various notions of efficiency and their relationship with our Theorem 3.2.

REMARK **3.2** The ESR rule is a top-support rule. It is easy to verify that following probability assignment map ξ^E generates the ESR rule φ^E defined in Example 3.3 :

$$\xi^E_a(\boldsymbol{x}) = \begin{cases} \frac{1}{|[\boldsymbol{x}]|} & \text{if} \quad a \in [\boldsymbol{x}] \\ 0 & \text{otherwise} \end{cases}$$

REMARK **3.3** Several papers use different notions of efficiency such as sd, dl and ul efficiency. It is easy to verify that the dl and ul efficiency notions are independent of each other. They both imply sd-efficiency which in turn implies our notion of efficiency[10]. The top-weight rule satisfies dl-efficiency but not ul-efficiency. Thus, we can replace efficiency axiom in our Theorem 3.2 with either dl-efficiency or sd-efficiency.

3.5 Conclusion

In this chapter, we have analyzed the structure of random social choice functions using variants of the standard stochastic dominance lottery comparisons. We show that the ul-extension leads to the same characterization as that under the sd-extension. However, the dl-extension allows for a richer class of strategy-proof random social choice functions. We show that dl-strategy-proofness in conjunction with efficiency implies a top-support rule. We further characterize a sub-class of such random social functions with an addition axiom of tops-onlyness.

[10] Our notion of efficiency is often referred to as ex-post efficiency.

3.6 Appendix

3.6.1 Appendix : Proof with ul-extension

In this appendix we provide a proof of Theorem 3.1. We begin with a couple of lemmas. Let P_i and P_i' be two distinct preference orderings that agree on the bottom k alternatives. Then ul-strategy-proofness implies that $\varphi(P_i, P_{-i})$ and $\varphi(P_i', P_{-i})$ assign the same probabilities to the commonly ranked bottom k alternatives, at all P_{-i}.

Lemma **3.1 (Monotonicity)** Suppose φ is a ul-strategy-proof RSCF. Let (P_i, P_{-i}) be a profile and let P_i' be a preference ordering such that $r_l(P_i) = r_l(P_i')$ for all $l \geq k$ for some $k \in \{1, 2, \ldots, m\}$. Then $\varphi_{r_l(P_i)}(P) = \varphi_{r_l(P_i')}(P_i', P_{-i})$ for all $l \geq k$.

Proof: Suppose the Lemma is false. Let $r_l(P_i) = r_l(P_i')$ for all $l \geq k$ for some $k \in \{1, 2, \ldots, m\}$ and $\varphi_{r_{\hat{l}}(P_i)}(P) \neq \varphi_{r_{\hat{l}}(P_i')}(P_i', P_{-i})$ for some $\hat{l} \geq k$. Let l' be the greatest integer for which $\varphi_{r_{l'}(P_i)}(P) \neq \varphi_{r_{l'}(P_i')}(P_i', P_{-i})$. Either $\varphi_{r_{l'}(P_i)}(P) < \varphi_{r_{l'}(P_i')}(P_i', P_{-i})$ or $\varphi_{r_{l'}(P_i)}(P) > \varphi_{r_{l'}(P_i')}(P_i', P_{-i})$ must hold. In the former case, $\varphi(P)\ P_i'^{ul}\ \varphi(P_i', P_{-i})$ and voter i manipulates at (P_i', P_{-i}) via P_i. In the latter case, $\varphi(P_i', P_{-i})\ P_i^{ul}\ \varphi(P)$ and voter i manipulates at P via P_i'. This establishes the Lemma. ■

Lemma **3.2** If a RSCF is ul-strategy-proof and satisfies unanimity, it satisfies efficiency.

Proof: Suppose not i.e. there exists a profile P and alternatives a and b such that aP_ib for all $i \in N$ and $\varphi_b(P) > 0$. Let P' be the profile, where each P_i' is constructed from P_i by moving a to the top rank and leaving the ordering of all other alternatives unchanged i.e. $r_1(P_i') = a$ and for all $x, y \in A\backslash\{a\}$, we have $xP_i'y \iff xP_iy$. Note that P_i and P_i' agree on the ranking of alternatives b and below.

Consider the sequence of profiles $\{P^0, P^1, \ldots, P^n\}$, where $P^0 = P$ and $P^k = (P_1' \ldots P_k', P_{k+1}, \ldots, P_n)$ for $k = 1, 2, \ldots, n$. The sequence starts with profile P and voters from 1 through n progressively change their preference ordering from P_i to P_i'. It ends at profile P'. We claim that $\varphi_b(P') > 0$.

Since the ranking of b and all alternatives below it is exactly the same in P_k and P_k', Lemma 3.1 implies that $\varphi_b(P^{k-1}) = \varphi_b(P^k)$. Repeatedly applying the argument for all $k \geq 1$, we get $\varphi_b(P) = \varphi_b(P') > 0$. Since a is ranked-first by all voters at P' and $\varphi_b(P') > 0$, we have a contradiction to unanimity. ■

To prove our Theorem 3.1, we follow the technique applied in Sen (2011). The proof uses induction on the number of voters. The base case in the induction is $n = 2$. In subsequent arguments, we will use efficiency instead of unanimity because of Lemma 3.2.

Proposition **3.2** Suppose $N = \{1, 2\}$ and $|A| \geq 3$. A RSCF is unanimous and ul-strategy-proof if and only if it is a random dictatorship.

We proceed in a sequence of lemmas.

Lemma **3.3** Fix a pair of alternatives a and b. For all profiles P, P' such that $P = (P_1^a, P_2^{ba})$ and $\hat{P} = (\hat{P}_1^{ab}, \hat{P}_2^b)$ we have $\varphi(P) = \varphi(\hat{P})$.

Proof: Fix a profile $\bar{P}$ such that $\bar{P} = (\bar{P}_1^{ab}, \bar{P}_2^{ba})$. We claim that $\varphi(\bar{P}_1^{ab}, \bar{P}_2^{ba}) = \varphi(P_1^a, \bar{P}_2^{ba}) = \varphi(P_1^a, P_2^{ba})$. Efficiency implies that at all three profiles only a and b get positive probability.

Suppose the first equality does not hold. If $\varphi_b(\bar{P}_1^{ab}, \bar{P}_2^{ba}) > \varphi_b(P_1^a, \bar{P}_2^{ba})$ then voter 1 manipulates at $(\bar{P}_1^{ab}, \bar{P}_2^{ba})$ via P_1^a. If $\varphi_b(\bar{P}_1^{ab}, \bar{P}_2^{ba}) < \varphi_b(P_1^a, \bar{P}_2^{ba})$, voter 1 manipulates at $(P_1^a, \bar{P}_2^{ba})$ via $\bar{P}_1^{ab}$. Second equality follows using virtually identical arguments with voter 1 being replaced by voter 2.

The proof of lemma is completed by showing $\varphi(\bar{P}_1^{ab}, \bar{P}_2^{ba}) = \varphi(\hat{P}_1^{ab}, \bar{P}_2^b) = \varphi(\hat{P}_1^{ab}, \hat{P}_2^b)$. This be done by using the same arguments as those in the earlier paragraph. ■

For any $x, y \in A$ and a profile $P = (P_1^{xy}, P_2^{yx})$, denote $\varphi_x(P) = \lambda^{xy}$ and $\varphi_y(P) = 1 - \lambda^{xy}$ where $0 \leq \lambda^{xy} \leq 1$. Lemma 3.3 implies that λ^{xy} does not depend on the ranking of alternatives other than x and y. Note that order is important and potentially λ^{xy} and λ^{yx} could be different. But next lemma shows that λ^{xy} is the same irrespective of any x and y, in any order.

Lemma **3.4** For any $a, b, c, d \in A$ where $a \neq b$ and $c \neq d$ we have $\lambda^{ab} = \lambda^{cd}$.

Proof: Pick three distinct alternatives $a, b, c \in A$. Consider the profile $P = (P_1^{ac}, P_2^{ba})$ and preference ordering $\bar{P}_1 = \bar{P}_1^{cb}$.

Lemma 3.3 and ul-strategy-proofness imply that $1 - \lambda^{ab} = \varphi_b(P_1^{ac}, P_2^{ba}) \leq \varphi_b(\bar{P}_1^{cb}, P_2^{ba}) = 1 - \lambda^{cb}$, otherwise voter 1 manipulates at (P_1^{ac}, P_2^{ba}) via $\bar{P}_1^{cb}$. This is true since at the former profile voter 1 prefers any lottery which gives zero probability to alternatives below b and a lower probability to alternative b. This is exactly what happens if the inequality above does not hold.

A similar argument holds at profile $P = (P_1^{ca}, P_2^{bc})$ and preference ordering $\bar{P}_1 = \bar{P}_1^{ab}$. Lemma 3.3 and ul-strategy-proofness imply that $1 - \lambda^{cb} = \varphi_b(P_1^{ca}, P_2^{bc}) \leq \varphi_b(\bar{P}_1^{ab}, P_2^{bc}) = 1 - \lambda^{ab}$, else voter 1 manipulates at (P_1^{ca}, P_2^{bc}) via $\bar{P}_1^{ab}$. These two inequalities imply that $\lambda^{ab} = \lambda^{cb}$. This is summarized below.

$$1-\lambda^{ab} = \varphi_b \begin{pmatrix} a & b \\ c & a \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} \leq \varphi_b \begin{pmatrix} c & b \\ b & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = 1-\lambda^{cb} \text{ and } 1-\lambda^{cb} = \varphi_b \begin{pmatrix} c & b \\ a & c \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} \leq \varphi_b \begin{pmatrix} a & b \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = 1-\lambda^{ab}$$

Pick an alternative $d \neq b, c$. A similar argument for voter 2 shows that $\lambda^{cb} = \lambda^{cd}$. Thus we have established that $\lambda^{ab} = \lambda^{ab} = \lambda^{cd}$. This completes the proof except the case when $r_1(P_1) = b$. This can easily be shown using an earlier argument for voter 1 to show that $\lambda^{cd} = \lambda^{bd}$. ■

Lemma 3.4 implies that λ^{ab} is same for any a and b. Thus, we simply write λ instead of λ^{ab}.

Lemma **3.5** For all profiles P such that $P = (P_1^{ax}, P_2^{bx})$ for any distinct $a, b, x \in A$ we have $\varphi_a(P) = \lambda$ and $\varphi_b(P) = 1 - \lambda$.

Proof: Consider an arbitrary profile $\hat{P} = (\hat{P}_1^{xa}, \hat{P}_2^{bx})$. Lemma 3.4 will imply that $\varphi_x(\hat{P}) = \lambda$ and $\varphi_b(\hat{P}) = 1-\lambda$. Now swap x and a. Note because of Lemma 3.1, probability of alternative b should remain the same i.e. $\varphi_b(P) = 1 - \lambda$.

Similarly start with profile $\bar{P} = (\bar{P}_1^{ax}, \bar{P}_2^{xb})$ we know $\varphi_a(P) = \lambda$ and $\varphi_x(\bar{P}) = 1 - \lambda$. Now swap x and b. Note because of the previous Lemma, probability of alternative a should remain the same i.e $\varphi_a(P) = \lambda$. Thus we have established the Lemma. ■

Lemma **3.6** For any profile $P \in \mathbb{P}^2$, we have $\varphi_{r_1(P_1)}(P) = \lambda$ and $\varphi_{r_1(P_2)}(P) = 1 - \lambda$.

Proof: Take any profile P and assume w.l.o.g. that $P = (P_1^{ax}, P_2^{by})$. If $x = y$, the result immediately follows by an application of Lemma 3.5. Suppose $x \neq y$. The probabilities of alternatives which are ranked below b by voter 1 and below a by voter 2 are zero by virtue of efficiency.

Suppose $\varphi_b(P_1^{ax}, P_2^{by}) > 1 - \lambda$. Then voter 1 can manipulate here at profile P via $\hat{P}_1^{ay}$. This is because the $\varphi(\hat{P}_1^{ay}, P_2)$ provides probability of b equal to $1 - \lambda$ and everything below it zero.

Suppose $\varphi_b(P_1^{ax}, P_2^{by}) < 1 - \lambda$. Consider the preference ordering $\tilde{P}_1^{ay}$ such that $\tilde{P}_1^{ay}(A\backslash\{a, y\} = P_1^{ax}(A\backslash\{a, y\})$.[11] Since Lemma 3.5 implies that $\varphi_b(\tilde{P}_1^{ay}, P_2^{by}) = 1 - \lambda$, it makes voter 1 to manipulate at $(\tilde{P}_1^{ay}, P_2^{by})$ via P_1^{ax}. But it contradicts the hypothesis that RSCF φ is *ul*-strategy-proof. Thus we have $\varphi_b(P_1^{ax}, P_2^{by}) = 1 - \lambda$.[12]

[11] Remember that for any two preference orderings P_i, P_i' and set of alternatives $X \subseteq A$, if $xP_iy \Leftrightarrow xP_i'y$ for all $x, y \in X$, then we write $P_i(X) = P_i'(X)$.

[12] The preference ordering $\tilde{P}_1$ is selected such that $\tilde{P}_1^{ay}$ holds. We want to emphasize why $\tilde{P}_1$ has an extra condition of $\tilde{P}_1(A\backslash\{a, y\}) = P_1(A\backslash\{a, y\})$. Suppose we have selected an arbitrary preference ordering $P_1' = P_1^{'ay}$ without this extra condition. Let there exist an alternative z such that $z\ P_1\ b$ but $b\ P_1'\ z$. Suppose $\varphi_z(P) > \varphi_z(P_1', P_2) = 0$, this is (potentially) possible because we have not ruled it out yet. If this be the case, then P_1' is not a manipulation at P because $\varphi(P_1, P_2)\ P_1^{'ul}\ \varphi(P_1', P_2)$ does not hold any more. In that case we do not have a contradiction to *ul*-strategy-proofness. To avoid this case we have used that extra condition.

A similar argument follows for agent 2 and implies that $\varphi_a(P_1^{ax}, P_2^{by}) = \lambda$. It completes the proof of Proposition 3.2. ■

Thus, we have established that when there are two voters, unanimity and ul-strategy-proofness imply random dictatorship. Now we use the induction argument. Assume that for all integers $k < n$, the following statement is true:

Induction Hypothesis (IH): Assume $m \geq 3$. If $\varphi : \mathbb{P}^k \to \Delta A$ satisfies unanimity and ul-strategy-proofness then it is a random dictatorship.

Let $\hat{N} = \{0, 3, \ldots, n\}$ be a set of voters where $3, \ldots, n \in N$. Define a RSCF $g : \mathbb{P}^{n-1} \to \Delta A$ for the set of voters in $\hat{N}$ as follows:

For all $(P_0, P_3, \ldots, P_n) \in \mathbb{P}^{n-1}$, $g(P_0, P_3, ..., P_n) = \varphi(P_1, P_2, P_3, ..., P_n)$ such that $P_1 = P_2 = P_0$.

Alternatively, we will write $g(P_0, P_3, ..., P_n) = g(P_0, P_{-12}) = \varphi(P_0, P_0, P_{-12})$ (where it is obvious that $P_1 = P_2 = P_0$ in $\varphi(\cdot)$).

Voter 0 in the RSCF g is obtained by "cloning" voters 1 and 2 in N. Thus if voters 1 and 2 in N have a common ordering P_i, then voter 0 in $\hat{N}$ also has ordering $P_0 = P_i$.

Lemma **3.7** *The RSCF g is a random dictatorship.*

Proof: It is easy to see that g is unanimous. To show that it is ul-strategy-proof , it is sufficient to show that voter 0 cannot manipulate since other voters cannot manipulate g because φ is ul-strategy-proof . Take any profile $P = (P_0, P_{-12}) \in \mathbb{P}^{n-1}$ and a preference ordering $\hat{P}_0 \in \mathbb{P}$. Using the definition of g and φ being ul-strategy-proof we can establish that following ranking of lotteries:

$$g(P_0, P_{-12}) = \varphi(P_0, P_0, P_{-12})\ R_1^{ul}\ \varphi(\hat{P}_0, P_0, P_{-12})\ R_2^{ul}\ \varphi(\hat{P}_0, \hat{P}_0, P_{-12}) = g(\hat{P}_0, P_{-12})\ ^{13}$$

So we have shown $g(P_0, P_{-12})\ R_0^{ul}\ g(\hat{P}_0, P_{-12})$ for preference profiles. Thus voter 0 cannot manipulate. Induction hypothesis implies that g is random dictatorship. ■

Let $\beta, \beta_3, ..., \beta_n$ be the weights associated with the random dictatorship g i.e. β_i, is the weight associated with voter $i = 3, ..., n$ and β is the weight associated with voter 0. For any profile P, let $\beta_{-12}^x(P) = \sum_{\{i \neq 1,2\,:\,r_1(P_i) = a\}} \beta_i$. So for any distinct alternatives x, y and profile P we have $g_x(P_0^x, P_{-12}) = \beta_{-12}^x(P) + \beta$ and $g_y(P_0^x, P_{-12}) = \beta_{-12}^y(P)$. To establish that φ is random dictatorship we will divide the proof in various lemmas.

[13]Remember that for any (ordinal) preference ordering P_i the associated ordering over lotteries under dl-extension is denoted by R_i^{dl} and it's strict part by P_i^{dl}. Similar notation follows for any P_i' as $\hat{R}_i^{ul}$ and $\hat{P}_i^{ul}$.

LEMMA **3.8** For an profile $P \in \mathbb{P}^n$ such that $r_1(P_1) = r_1(P_2)$ we have $\varphi(P) = g(P_0, P_{-12})$ where $r_1(P_0) = r_1(P_1)$.

Proof: Let $r_1(P_1) = r_1(P_2) = a$. We apply the definition of g and the fact that φ is *ul*-strategy-proof to conclude that for any $P_0^a, \hat{P}_0^a$ and P_{-12} we have

$$g(P_0^a, P_{-12}) = \varphi(P_0^a, P_0^a, P_{-12})\ R_1^{ul}\ \varphi(\hat{P}_0^a, P_0^a, P_{-12})\ R_2^{ul}\ \varphi(\hat{P}_0^a, \hat{P}_0^a, P_{-12}) = g(\hat{P}_0^a, P_{-12})$$

In the above expression first and third lottery are identical and because *ul* extension is a linear order then above relation can hold if only if the middle lottery is also the same. This will imply that $\varphi(P_1^a, P_2^a, P_{-12}) = g(P_0^a, P_{-12})$ and it completes the proof. This lemma imply that for any distinct x and y we have $\varphi_y(P_1^x, P_2^x, P_{-12}) = \beta_{-12}^y(P)$. ■

LEMMA **3.9** Fix an arbitrary sub-profile $P_{-12} \in \mathbb{P}^{n-2}$ and two different alternatives a and b. Pick any two profiles P' and $\hat{P}$ such that $P' = (P_1'^a, P_2'^{ba}, P_{-12})$ and $\hat{P} = (\hat{P}_1^{ab}, \hat{P}_2^b, P_{-12})$ then we have $\varphi(P') = \varphi(\hat{P})$.

Proof: We first prove a claim that for any profile $P = (P_1^a, P_2^{ba}, P_{-12})$ we have $\varphi_x(P) = \beta_{-12}^x(P)$ for all $x \in A\backslash\{a, b\}$. Consider a preference ordering P_2'' for voter 2 obtained from P_2^{ba} simply by swapping a and b i.e. $P_2''(A\backslash\{a, b\}) = P_2(A\backslash\{a, b\})$ and $(r_1(P_2''), r_2(P_2'')) = (a, b)$. Applying Lemma 3.1 and previous Lemma 3.8 we get $\varphi_x(P) = \varphi_x(P_2'', P_{-2}) = \beta_{-12}^x(P_2'', P_{-2}) = \beta_{-12}^x(P)$ for all $x \in A\backslash\{a, b\}$.

Now fix a profile $\bar{P}$ such that $\bar{P} = (\bar{P}_1^{ab}, \bar{P}_2^{ba}, P_{-12})$. We have just proved that for such a profile, probability of any alternative other than a and b is determined w.r.t. β_i's i.e. $\varphi_x(\bar{P}) = \beta_{-12}^x(\bar{P})$ for all $x \neq a, b$ which in turn imply that $\varphi_a(\bar{P}) + \varphi_b(\bar{P}) = \beta + \beta_{-12}^a(\bar{P}) + \beta_{-12}^b(\bar{P})$. We will make the following claim regarding the individual probabilities of a and b.

We claim $\beta_{-12}^a(\bar{P}) \leq \varphi_a(\bar{P}) \leq \beta + \beta_{-12}^a(\bar{P})$. Suppose not, if we assume $\varphi_a(\bar{P}) < \beta_{-12}^a(\bar{P})$. Consider a preference ordering P_1'' such that $r_1(P_1'') = b$. Now replace this preference ordering in profile $\bar{P}$ to obtain the profile $(P_1^b, \bar{P}_{-1})$. Since both voters 1 and 2 share a common alternative at top at this profile, Lemma 3.8 implies that $\varphi_a(P_1^b, \bar{P}_{-1}) = \beta_{-12}^a(P_1^b, \bar{P}_{-1}) = \beta_{-12}^a(\bar{P})$. This makes voter 1 worse-off compared to profile $\bar{P}$ i.e. $\varphi(\bar{P})\ \bar{P}_1^{ul}\ \varphi(P_1^b, \bar{P}_{-1})$. Thus voter 1 can manipulate at $(P_1^b, \bar{P}_{-1})$ via $\bar{P}_1$. On the other hand, if $\varphi_a(\bar{P}) > \beta + \beta^a$ then, by applying similar argument, we can obtain a manipulation by voter 2 at $(P_2^a, \bar{P}_{-2})$ via $\bar{P}_2$. The similar inequality holds, $\beta^b \leq \varphi_b(\bar{P})\beta + \beta^b$.[14]

Now we claim $\varphi(P') = \varphi(\bar{P}) = \varphi(\hat{P})$. We only show first equality the other can be shown with virtually similar argument. We have already shown that for profile of type P' we have $\varphi_x(P') = \beta^x$ for all $x \neq a, b$. Consider the sequence which starts from $\bar{P}$ and successively voter

[14]For simplicity, we will drop the indexation of P and simply write β^x instead of $\beta^x(P)$.

1 and 2 change their respective preference orderings from $\bar{P}_i$ to P'_i. We claim the outcome remain same throughout the sequence i.e. $\varphi(\bar{P}) = \varphi(P'_1, \bar{P}_{-1}) = \varphi(P'_1, P'_2, \bar{P}_{-12}) \equiv \varphi(P')$.

To see first equality, note that $\varphi_x(\bar{P}) = \varphi_x(P'_1, \bar{P}_{-1}) = \beta^x$ for all $x \neq a, b$. If $\varphi_b(\bar{P}) \neq \varphi_b(P'_1, \bar{P}_{-1})$ then either $\varphi_b(\bar{P}) < \varphi_b(P'_1, \bar{P}_{-1})$ or $\varphi_b(\bar{P}) > \varphi_b(P'_1, \bar{P}_{-1})$ must hold. In the former case $\varphi(\bar{P})\ P_i'^{ul}\ \varphi(P'_1, \bar{P}_{-1})$ and voter i manipulates at profile $(P'_1, \bar{P}_{-i})$ via preference ordering $\bar{P}_i$. In the latter case $\varphi(P'_1, \bar{P}_{-1})\ \bar{P}_i^{ul}\ \varphi(\bar{P})$ and i manipulates at profile $\bar{P}$ via preference ordering P'_i. Thus $\varphi(\bar{P}) = \varphi(P'_1, \bar{P}_{-1})$.

The second equality can be argued similarly for voter 2. Hence $\varphi(P') = \varphi(\bar{P})$ and it completes the proof of lemma. ■

Lemma 3.9 implies that all profiles where top two alternatives of voter 1 and 2 coincides then it has a special structure. For distinct $x, y \in A$ and a profile P such that $P_1 = P_1^{xy}$ and $P_2 = P_2^{yx}$ let $\varphi_x(P) = \lambda^{xy}\beta + \beta^x$ and $\varphi_y(P) = (1-\lambda^{xy})\beta + \beta^y$. As we earlier argued $\beta^z < \varphi_z(P) < \beta + \beta^z$ for $z = x$ and y, it implies that $\lambda^{xy} \in [0, 1]$. Lemma 3.9 implies that λ^{xy} does not depend on the ranking of alternatives other than x and y. It signifies that β is divided into ratio of λ^{xy} and $1 - \lambda^{xy}$ among voter 1 and 2 for the profiles where x and y are ranked in a certain way in their preference orderings. Note that order is important and potentially λ^{xy} and λ^{yx} could be different. But next lemma shows λ^{xy} is same irrespective of any x and y in any order.

Lemma **3.10** For any $a, b, c, d \in A$ where $a \neq b$ and $c \neq d$ we have $\lambda^{ab} = \lambda^{cd}$

Proof: Pick three distinct alternatives $a, b, c \in A$. Consider two profiles $P = (P_1^{ac}, P_2^{ba}, P_{-12})$ and $\bar{P} = (\bar{P}_1, P_{-1})$ where preference ordering $\bar{P}_1 = \bar{P}_1^{cb}$.

Lemma 3.9 and ul-strategy-proofness imply that $(1 - \lambda^{ab})\beta + \beta^b = \varphi_b(P) \leq \varphi_b(\bar{P}) = (1 - \lambda^{cb})\beta + \beta^b$, otherwise voter 1 manipulates at profile P via $\bar{P}_1$. This is true since at the former profile voter 1 prefers any lottery which gives probability β^x to all x ranked below b and a lower probability to alternative b. This is exactly what happens if the inequality above does not hold.

A similar argument holds at profiles $P = (P_1^{ca}, P_2^{bc}, P_{-12})$ and $\bar{P} = (\bar{P}_1, P_{-1})$ where preference ordering $\bar{P}_1 = \bar{P}_1^{ab}$. Lemma 3.3 and ul-strategy-proofness imply that $(1 - \lambda^{cb})\beta + \beta^b = \varphi_b(P) \leq \varphi_b(\bar{P}) = (1 - \lambda^{ab})\beta + \beta^b$ else voter 1 manipulates at profile P via $\bar{P}_1$. These two inequalities imply that $(1 - \lambda^{cb})\beta + \beta^b = (1 - \lambda^{ab})\beta + \beta^b$ and if $\beta > 0$ then $\lambda^{ab} = \lambda^{cb}$. This is summarized below (we have highlighted only preference ordering of voters 1 and 2, the sub-profile P_{-12} is kept same throughout).

$$(1-\lambda^{ab})\beta + \beta^b = \varphi_b \begin{pmatrix} a & b \\ c & a \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} \leq \varphi_b \begin{pmatrix} c & b \\ b & a \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = (1-\lambda^{cb})\beta + \beta^b \text{ and}$$

$$(1-\lambda^{cb})\beta + \beta^b = \varphi_b \begin{pmatrix} c & b \\ a & c \\ b & \cdot \\ \vdots & \vdots \end{pmatrix} \leq \varphi_b \begin{pmatrix} a & b \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = (1-\lambda^{ab})\beta + \beta^b$$

$$\Rightarrow \lambda^{ab} = \lambda^{cb}$$

To complete the claim, pick an alternative $d \neq b, c$. A similar argument for voter 2 shows that $\lambda^{cb} = \lambda^{cd}$. Thus we have established $\lambda^{ab} = \lambda^{cb} = \lambda^{cd}$

$$\lambda^{cb}\beta + \beta^c = \varphi_c \begin{pmatrix} c & b \\ b & d \\ \cdot & c \\ \vdots & \vdots \end{pmatrix} \leq \varphi_c \begin{pmatrix} c & d \\ b & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} = \lambda^{cd}\beta + \beta^c$$

$$\lambda^{cb}\beta + \beta^c = \varphi_c \begin{pmatrix} c & b \\ d & c \\ \vdots & \vdots \\ \vdots & \vdots \end{pmatrix} \geq \varphi_c \begin{pmatrix} c & d \\ d & b \\ \cdot & c \\ \vdots & \vdots \end{pmatrix} = \lambda^{cd}\beta + \beta^c$$

The earlier arguments cover all cases except when $r_1(P_1) = b$. This can also easily be done using the earlier argument for voter 1 to show that $\lambda^{cd} = \lambda^{bd}$. This completes the proof. ■

LEMMA **3.11** For any profile P where $r_2(P_1) = r_2(P_2)$ we have $\varphi_y(P) = \beta^y$ for all $y \in A$.

Proof: The profiles where $r_1(P_1) = r_1(P_2)$ are covered in Lemma 3.8, so here we consider the case when they are not equal. Pick any three distinct alternatives $a, b, x \in A$. We need to show that for any profile P where $P_1 = P_1^{ax}$ and $P_2 = P_2^{bx}$ we have $\varphi_y(P) = \beta^y(P)$ for all $y \in A$.

Consider the preference ordering P_1' such that $r_1(P_1') = x$ and $P_1'(A\backslash\{x\}) = P_1(A\backslash\{x\})$.[15] Lemma 3.10 will imply that $\varphi_y(P_1', P_{-1}) = \beta^y(P_1', P_{-1})$ for all $y \in A$. Now swap x and

[15]This implies that $r_2(P_1') = a$ and $r_l(P_1') = r_l(P_1)$ for all $l \geq 3$.

a in preference ordering of voter 1, in other words replace P_1' with P_1 to obtain profile P. Because of Lemma 3.1 all the alternatives that are ranked below a and x, should get the same probability as before i.e. $\varphi_z(P) = \beta^z(P_1', P_{-1}) = \beta^z(P)$ for all $z \neq a, x$.

Similarly argument applies when voter 2 has a preference ordering P_2' where $r_1(P_2') = x$ and $P_2'(A\backslash\{x\}) = P_2(A\backslash\{x\})$. Start with profile (P_2', P_{-2}) and replace P_2' with P_2 to obtain $\varphi_z(P) = \beta^z(P_2', P_{-1}) = \beta^z(P)$ for all $z \neq b, x$. Both these equalities imply $\varphi_y(P) = \beta^y(P)$ for all $y \in A\backslash\{x\}$. Since they all add to 1, we have $\varphi_x(P) = \beta^x(P)$ also. This completes the proof. ■

LEMMA **3.12** For any profiles $P \in \mathbb{P}^n$ we have $\varphi_x(P) = \beta^x$ for all $x \in A$.

Proof: We assume w.l.o.g. that $P_1 = P_1^{ax}$ and $P_2 = P_2^{by}$. If $x = y$, the result immediately follows by an application of Lemma 3.11. To complete the result we assume $x \neq y$.

Our arguments will focus on voter 1. First we claim that probability of every alternative z which is ranked below b in P_1 equals to $\beta^z(P)$. Suppose it is not true. Consider the preference ordering P_1' such that $P_1' = P_1^{'ab}$ and $P_1'(A\backslash\{a,b\}) = P_1(A\backslash\{a,b\})$. If $\varphi_z(P) > \beta^z(P)$ then voter 1 manipulates at profile P via P_1' because $\varphi(P_1', P_{-1})$ selects a lottery which gives same probability as $\varphi(P)$ to all alternatives below z but strictly lesser to z. On the other hand if we have $\varphi_z(P) < \beta^z(P)$ then it makes voter 1 to manipulate at profile (P_1', P_{-1}) via P_1 because of $\varphi(P)\ P_i^{'ul}\ \varphi(P_1', P_{-1})$. In both the cases we have a contradiction to φ being ul-strategy-proof.[16]

Next we claim $\varphi_b(P) = \beta^b$. If it is not true then voter 1 will manipulate at profile P via P_1'' when $\varphi_b(P) > \beta^b$ where $P_1'' = P_1^{''ay}$ and $P_1''(A\backslash\{a,y\}) = P_1(A\backslash\{a,y\})$.[17] On the other hand if $\varphi_b(P) < \beta^b$ then voter 1 manipulates at profile (P_1'', P_{-1}) via P_1, contradicting the dl-strategy-proofness of φ. In this argument the ranking of alternative y in P_1 is not important. If $b\ P_1\ y$ then its probability remains the same as β^y before and after the manipulation. And if $y\ P_1\ b$ then its probability is not relevant for manipulation.

Finally we claim $\varphi_z(P) = \beta^z$ for all z such that $z\ P_1\ b$. Suppose $\varphi_w(P) \neq \beta^w$ such that $w\ P_1\ b$. Consider the preference ordering P_1''' such that $P_1''' = P_1^{'''ab}$ and $P_1'''(A\backslash\{a,b\}) = P_1(A\backslash\{a,b\})$. If $\varphi_w(P) > \beta^w(P)$ then voter 1 manipulates at profile P via P_1''' because $\varphi(P_1''', P_{-1})$ selects a lottery which gives the same probability as $\varphi(P)$ to all alternatives

[16] The preference ordering P_1' is selected such that $P_1^{'ab}$. We want to emphasis that why P_1' has an extra condition of $P_1'(A\backslash\{a,b\}) = P_1(A\backslash\{a,b\})$. Suppose we have selected an arbitrary preference ordering $\hat{P}_1^{ab}$ without this extra condition. Let there exist a w such that $w\ P_1\ b$ and $z\ \hat{P}_1\ w$ and $\varphi_w(P) > \beta^w$. This is potentially possible because we have not ruled it out yet. If this be the case then $\hat{P}_1$ is not a manipulation at P because $\varphi(\hat{P}_1, P_{-1})\ P_1^{ul}\ \varphi(P)$ does not hold any more. In that case we do not have a contradiction to ul-strategy-proofness.

[17] The relevance of $P_1''(A\backslash\{a,y\}) = P_1(A\backslash\{a,y\})$ is similar to that of the previous paragraph.

below w but strictly lesser to w. On the other hand if we have $\varphi_z(P) < \beta^z(P)$ then it makes voter 1 to manipulate at profile (P_1''', P_{-1}) via P_1 because of $\varphi(P)\ P_i^{'''ul}\ \varphi(P_1''', P_{-1})$. In both the cases we have a contradiction to φ being ul-strategy-proof. ■

3.6.2 Appendix : Proofs with dl-extension

Lemma **3.13 (dl-monotonicity)** Suppose φ is a dl-strategy-proof RSCF. Let (P_i, P_{-i}) be a profile and let P_i' be a preference ordering such that $r_l(P_i) = r_l(P_i')$ for all $l \leq \bar{k}$, for some $\bar{k} \in \{1, 2, \ldots, m\}$. Then $\varphi_{r_l(P_i)}(P) = \varphi_{r_l(P_i')}(P_i', P_{-i})$ for all $l \leq \bar{k}$.

Proof: We show it by contradiction. Suppose $r_l(P_i) = r_l(P_i')$ for all $l \leq k$ and $\varphi_{r_l(P_i)}(P) \neq \varphi_{r_l(P_i')}(P_i', P_{-i})$ for some $\hat{l} \leq k$ for some $k \in \{1, 2, \ldots, m\}$. Let l' be the smallest integer for which $\varphi_{r_{l'}(P_i)}(P) \neq \varphi_{r_{l'}(P_i')}(P_i', P_{-i})$. Either $\varphi_{r_{l'}(P_i)}(P) > \varphi_{r_{l'}(P_i')}(P_i', P_{-i})$ or $\varphi_{r_{l'}(P_i)}(P) < \varphi_{r_{l'}(P_i')}(P_i', P_{-i})$ must hold. In the former case $\varphi(P)\ P_i^{'dl}\ \varphi(P_i', P_{-i})$ and voter i manipulates at (P_i', P_{-}) via P_i. In the latter case $\varphi(P_i', P_{-i})\ P_i^{dl}\ \varphi(P)$ and i manipulates at P via P_i'. This establishes the lemma. ■

Lemma **3.14** Suppose φ is a dl-strategy-proof RSCF. Let P_i' is obtained from preference ordering P_i by only improving the ranking of an alternative x and keeping everything else same i.e. $k' = r(P_i', x) < r(P_i, x) = k$ and $P_i(A \backslash \{x\}) = P_i'(A \backslash \{x\})$. Then for any sub-profile P_{-i} we have : (i) $\varphi_{r_l(P_i)}(P) = \varphi_{r_l(P_i')}(P_i', P_{-i})$ for all $l < k'$ and (ii) either $\varphi_x(P_i', P_{-i}) > \varphi_x(P)$ or $\varphi(P) = \varphi(P_i, P_{-i})$.

Proof: Part (i) is immediately follow from Lemma 3.13. If $\varphi_x(P_i', P_{-i}) < \varphi_x(P)$ then voter i manipulates at profile (P_i', P_{-i}) via P_i. Thus we have $\varphi_x(P_i', P_{-i}) \geq \varphi_x(P)$. Suppose $\varphi_x(P_i', P_{-i}) = \varphi_x(P)$ and $\varphi(P_i', P_{-i}) \neq \varphi_x(P)$. Pick the first alternative for which they are not equal i.e. $\varphi_y(P_i', P_{-i}) \neq \varphi_y(P)$ and $\varphi_z(P_i', P_{-i}) = \varphi_z(P)$ for all $z\ P_i'\ y$. Either $\varphi_y(P_i', P_{-i}) < \varphi_y(P)$ or $\varphi_y(P_i', P_{-i}) > \varphi_y(P)$. In the former case $\varphi(P)\ P_i^{'dl}\ \varphi(P_i', P_{-i})$ and voter i manipulates at (P_i', P_{-i}) via P_i. In the latter case $\varphi(P_i', P_{-i})\ P_i^{dl}\ \varphi(P)$ and i manipulates at P via P_i'. This establishes the Lemma. It is immediate that $\varphi_x(P_i', P_{-i}) \geq \varphi_x(P)$. ■

Lemma **3.15** Suppose φ is a dl-strategy-proof RSCF. Let P and P' be two profiles such that, every P_i' is obtained from P_i by only (weakly) improving the ranking of an alternative x and keeping everything else the same i.e. $k' = r(P_i', x) \leq r(P_i, x) = k$ and $P_i(A \backslash \{x\}) = P_i'(A \backslash \{x\})$ for all $i \in N$. Then either (i) $\varphi_x(P') > \varphi_x(P)$ or (ii) $\varphi(P) = \varphi(P_i, P_{-i})$.

Proof: Consider a sequence where voters 1 through n progressively replacing P_i with P'_i. Applying Lemma 3.14 at each step and hence we obtain the following inequality :

$$\begin{aligned}
\varphi_x(P) &= \varphi_x(P_1, P_2, P_3, \ldots, P_{n-1}, P_n) \\
&\leq \varphi_x(P'_1, P_2, P_3, \ldots, P_{n-1}, P_n) \\
&\leq \varphi_x(P'_1, P'_2, P_3, \ldots, P_{n-1}, P_n) \\
&\quad \vdots \\
&\leq \varphi_x(P'_1, P'_2, P'_3, \ldots, P'_{n-1}, P_n) \\
&\leq \varphi_x(P'_1, P'_2, P'_3 \ldots, P'_{n-1}, P'_n) = \varphi_x(P')
\end{aligned}$$

It is obvious that $\varphi_x(P) \leq \varphi_x(P)$. If $\varphi_x(P) = \varphi_x(P')$ then all inequalities hold with equality and at each step again applying Lemma 3.14-Part (ii), we have $\varphi(P) = \varphi(P')$. This completes the lemma. ■

LEMMA **3.16** *For any profile P, if $r_2(P_i) = x$ for all $i \in N$ then we have $\varphi_x(P) = 0$.*

Proof: Pick an arbitrary profile $\tilde{P}$ such that $r_2(\tilde{P}_i) = x$. We define few notations used in this proof. Assume w.l.o.g. that $\mathcal{T}(\tilde{P}) = \{a_1, a_2, \ldots, a_k\} = A_k$[18] i.e. there are k distinct alternatives in total that are ranked-first at profile $\tilde{P}$. Denote $K = \{1, 2, \ldots, k\}$ for indexation of these alternatives and G_l be set of voters who have alternative a_l as their first-ranked alternative in $\tilde{P}$ i.e. $G_l = \{i \in N : r_1(\tilde{P}_i) = a_l\}$ for all $l \in K$. It is obvious that each group G_l is non-empty and groups G_1 through G_k forms a partition of N.

Pick any set $\bar{K} \subseteq K$ and an arbitrary profile $\bar{P}$ such that

(i) $r_1(\bar{P}_i) = a_l$ and $r_2(\bar{P}_i) = x$ for all $i \in G_l$, for all $l \in \bar{K}$[19]

(ii) $r_1(\bar{P}_i) = a_t$ and $\{r_1(\bar{P}_i), r_2(\bar{P}_i), \ldots, r_k(\bar{P}_i)\} = A_k$ for all $i \in G_t$, for all $t \in K \backslash \bar{K}$.

In other words, at profile $\bar{P}$ all voters in group G_l (for $l \in \bar{K}$) has a_l first and x second in their preference ordering. All voters in group G_t (for $t \in K \backslash \bar{K}$) has a_t on first and their top-k alternatives are from set A_k.

Claim 1: If $|\bar{K}| = 1$ then $\varphi_x(\bar{P}) = 0$. W.l.o.g. assume $\bar{K} = \{l\}$. At profile $\bar{P}$ we have $a_l \ P_i \ x$ for all $i \in N$. Since φ is efficient, it implies $\varphi_x(P) = 0$.

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[18]Remember that for any profile P we denote $\mathcal{T}(P) = \{r_1(P_1), r_1(P_2), \ldots, r_1(P_n)\}$

[19]We want to emphasis that condition $r_2(\bar{P}_i)$ can easily be replace with $r(\bar{P}_i, x) \geq 2$. We can start with preference ordering P_i where $r_2(P_i) = 2$ and keep lowering the rank of x. By application of Lemma 3.15, probability of x can only decrease by lowering the ranking of x. Since it is already 0 thus it remains at 0.

Claim 2 (Induction Step): Suppose $\varphi_x(\bar{P}) = 0$ for every $\bar{K}$ such that $|\bar{K}| \leq L-1$. We claim for any $\bar{K}$ such that $|\bar{K}| = L$ we also have $\varphi_x(\bar{P}) = 0$.

Assume w.l.o.g. that $\bar{K} = \{1, 2, \ldots, L\}$. Consider a profile P such that

(i) All voters in G_1 rank alternative a_1 first, followed by a_2, followed by a_3 and so on till a_k i.e. $r_l(P_i) = a_l$ for all $l \in K$, for all $i \in G_1$.

(ii) All voters in group G_l rank alternative a_l first, a_1 second and x third from group G_2 through G_L i.e. $\big(r_1(P_i), r_2(P_i), r_3(P_i)\big) = (a_l, a_1, x)$ for all $i \in G_l$, for all $l \in \bar{K}\backslash\{1\} = \{2, \ldots, L\}$.

(iii) All voters from group G_{L+1} through G_k has same preference ordering as that of in profile $\bar{P}$ i.e. $P_i = \bar{P}_i$ for all $i \in G_t$, for all $t \in K\backslash\bar{K} = \{L+1, L+2, \ldots, k\}$.

At profile P every voter prefers a_1 over x. Thus efficiency implies $\varphi_x(P) = 0$. Let $\varphi_{a_1}(P) = \varepsilon_{a_1} \in [0, 1]$. Let $P_i^\star$ be the preference ordering obtained from P_i by raising x to the second rank while keeping everything else the same i.e. $r_2(P_i^\star) = x$ and $P_i^\star(A\backslash\{x\}) = P_i(A\backslash\{x\})$. Now consider three different profiles P', P'' and P''' each obtained from P. In profile P' only voters from G_1 have replaced their P_i with $P_i^\star$. In P'' only voters from group G_2 through G_L have replaced P_i with $P_i^\star$. Finally in P''' all voters from group G_1 through G_L have replaced P_i with $P_i^\star$. Formally,

(i) In profile P' we have $P_i' = P_i^\star$ for all $i \in G_1$ and $P_i' = P_i$ for all $i \in N\backslash G_1$.

(ii) In profile P'' we have $P_i' = P_i^\star$ for all $i \in G_2 \cup G_3 \ldots \cup G_L$ and $P_i' = P_i$ for all $i \in N\backslash G_2 \cup G_3 \ldots \cup G_L$

(iii) In profile P''' we have $P_i' = P_i^\star$ for all $i \in G_1 \cup G_2 \ldots \cup G_L$ and $P_i' = P_i$ for all $i \in N\backslash G_1 \cup G_2 \ldots \cup G_L$.

It is important to note that each profile P', P'' and P''' is obtained from P by raising only the ranking of alternative x for voters belong to certain groups. We show that $\varphi_x(P') = \varphi_x(P'') = \varphi_x(P''') = 0$. And then finally we obtain profile $\bar{P}$ from P''' to establish that $\varphi_x(\bar{P}) = 0$.

Claim 2.1 : $\varphi(P') = \varphi(P)$. Note that at profile P' each voter ranks a_1 above x. Thus efficiency implies $\varphi_x(P') = 0$. Note profile P' is obtained from P by raising only the ranking of x for voters in G_1. Therefore, Lemma 3.15 implies that $\varphi(P') = \varphi(P)$ because we have already shown that $\varphi_x(P') = \varphi_x(P)$. We note that $\varphi_{a_1}(P') = \varepsilon_{a_1}$.

Claim 2.2 : $\varphi(P'') = \varphi(P)$. An inspection of profile P'' reveals that there are $L-1$ groups in total (from G_2 through G_L), such that each group G_l ranks a_l first then x second and all other G_t groups (G_1, G_{L+1} through G_k) rank a_t at first followed by next $k-1$ alternatives only from the set A_k. Therefore, the profile P'' satisfies the conditions of induction hypothesis. Thus by our assumption we have $\varphi_x(P'') = 0$. Again Lemma 3.15 implies that $\varphi(P'') = \varphi(P)$. We note that $\varphi_{a_1}(P'') = \varepsilon_{a_1}$.

Claim 2.3 : $\varphi_x(P''') = 0$. Note that the only difference between profile P' and P''' is preference orderings of voters in $G_2 \cup G_3 \ldots \cup G_L$. Consider a sequence from P' to P''' such that each voter from group G_2 through G_L successively replaces P_i with $P_i^\star$ (while keeping everything as the same). To be precise each of them are raising x from third to the second rank and lower a_1 from second to the third. Lemma 3.14 implies that after each such change the probability of x can only increases. Moreover, it can increase if and only if the probability of a_1 decreases.[20] Thus at the end we have either (i) $\varphi_x(P''') > \varphi_x(P') = 0$ and $\varphi_{a_1}(P''') < \varphi_{a_1}(P') = \varepsilon_{a_1}$ or (ii) $\varphi(P''') = \varphi(P')$.

Suppose $\varphi_x(P''') > 0$ and $\varphi_{a_1}(P''') < \varepsilon_{a_1}$. We show that it leads to a contradiction to Lemma 3.13. Consider a sequence from P'' to P''' such each voter in group G_1 replaces P_i with $P_i^\star$. Note that first-ranked alternative is the same in both P_i and $P_i^\star$. Therefore, we have a contradiction to Lemma 3.13 because it implies that $\varphi_{a_1}(P''') = \varphi_{a_1}(P'')$. Thus, we have $\varphi_{a_1}(P''') = \varepsilon_{a_1}$ and which in turn implies that $\varphi_x(P''') = 0$. This completes the proof of the Claim

The Claim 1 and Claim 2 in together implies when $|\bar{K}| = k$ i.e. $\bar{K} = K$ we have $\varphi_x(\bar{P}) = 0$. Note that in preference ordering $\bar{P}_i$, the ranking of alternatives which are below x could be arbitrary. Therefore, at any profile where each group G_l ranks a_l first and x second, the probability of x is zero i.e $\varphi_x(\tilde{P}) = 0$. This completes the proof of Lemma. ■

Proof of Proposition 3.1: Pick an arbitrary profile $\hat{P}$, we will show that for any $x \notin \mathcal{T}(\hat{P})$ we have $\varphi_x(\hat{P}) = 0$. Consider the profile $\bar{P}$ such $r_2(\bar{P}_i) = x$ and $\bar{P}_i(A \backslash \{x\}) = \hat{P}_i(A \backslash \{x\})$ for all $i \in N$. Lemma 3.16 implies that $\varphi_x(\bar{P}) = x$.

Consider a sequence from profile $\bar{P}$ to $\hat{P}$ such that each voter 1 through n successively replaces $\bar{P}_i$ with $\hat{P}_i$ i.e. each of them lowers the ranking of alternative x, while keeping the relative ranking of all other alternatives the same. By an application of Lemma 3.15 the probability of x can only decrease with such replacement. Thus, at the end of the sequence

[20] If it is not the case, then an argument similar to Lemma 3.14 applies and a manipulation to *dl*-strategy-proofness can be shown.

we have $\varphi_x(\hat{P}) \leq \varphi_x(\bar{P}) = 0$, which implies that $\varphi_x(\hat{P}) = 0$. This completes the proof. ∎

Proof of Theorem 3.2 [Sufficiency] The tops-onlyness and efficiency of a top-weight is obvious. We only show *dl*-strategy-proofness.

Let φ be a top-weight rule. Suppose it is generated by a monotone probability assignment map ξ. Consider an arbitrary voter i and two profiles P and P' such that $P_{-i} = P'_{-i}$. We claim $\varphi(P)\ R_i^{dl}\ \varphi(P')$.

Suppose $a = r_1(P_i)$ and $r_1(P'_i) = b$. If $a = b$ then tops-onlyness of φ implies that $\varphi(P) = \varphi(P')$ and the claim is trivially true.

Suppose $a \neq b$. If $\varphi_b(P) = \varphi_b(P')$ then part 2(i) in the definition of monotonicity of ξ (Definition 3.9) implies that $\varphi(P) = \varphi(P')$. Once again we have $\varphi(P)\ R_i^{dl}\ \varphi(P')$.

Suppose $a \neq b$. If $\varphi_b(P) \neq \varphi_b(P')$ then part 2(ii) in the definition of monotonicity of ξ (Definition 3.9) implies that $\varphi_b(P) < \varphi_b(P')$ and $\varphi_a(P) > \varphi_a(P')$. Since a is first-ranked in P_i, we have $\varphi(P)\ P_i^{dl}\ \varphi(P')$. This completes the proof.

[*Necessity.*] First consider the case when $n \geq 3$. Let φ be a RSCF which satisfies tops-only, *dl*-strategy-proof and efficiency. We construct a monotone probability assignment map ξ which generates φ. Define a mapping $\xi : A^n \to \Delta A$ as follows: for any tuple $\boldsymbol{x} \in A^n$ we have $\xi(\boldsymbol{x}) = \varphi(P)$ where $x_i = r_1(P_i)$ for all $i \in N$. Since φ is tops-only ξ is well-defined.

Applying Proposition 3.1, it follows that φ is a top support rule. This implies that $\sum_{a\in\mathcal{T}(P)} \varphi_a(P) = 1$ or $\sum_{a\in[\boldsymbol{x}]} \xi_a(\boldsymbol{x}) = 1$. Hence ξ is a probability assignment map. It only remains to show that ξ satisfies monotonicity.

Pick any $\boldsymbol{x}_{-i} \in A^{n-1}$ and distinct $x_i, x'_i \in A$ for any $i \in N$. Consider a profile P and a preference ordering P'_i such that $r_1(P_j) = x_j$ for all $j \in N$ and $x'_i = r_1(P'_i)$. Suppose $x_i = r_1(P_i) = a \neq x'_i = r_1(P'_i) = b$. We claim either $\varphi(P) = \varphi(P')$ holds or $\varphi_a(P) > \varphi_a(P')$ and $\varphi_b(P) < \varphi_b(P')$ is true. Since φ is *dl*-strategy-proof, $\varphi_a(P) \geq \varphi_a(P')$ follows immediately.

Case 1: Suppose $\varphi_a(P) = \varphi_a(P')$. We show $\varphi(P) = \varphi(P')$. Pick an arbitrary $c \in A\backslash\{a\}$. Since φ is tops-only we can assume w.l.o.g. that $r_2(P_i) = c$. Since φ is *dl*-strategy-proof , it follows that $\varphi_c(P) \geq \varphi_c(P')$; otherwise i can misreport P'_i at profile P to obtain a (strictly) better lottery. Since $\varphi_a(P) = \varphi_a(P') =$ and $\varphi_c(P) \geq \varphi_c(P')$ for all $c \in A\backslash\{a\}$, it must be the case that $\varphi(P) = \varphi(P')$.

Case 2: Suppose $\varphi_a(P) > \varphi_a(P')$. We claim $\varphi_b(P) < \varphi(P')$. Suppose not i.e. $\varphi_b(P) \geq \varphi_b(P')$. Assume w.l.og. that $r_2(P'_i) = a$. Thus the probability of b is (weakly) higher and that of a strictly higher in lottery $\varphi(P)$ as compared to $\varphi(P')$. Thus $\varphi(P)$ is strictly preferred to $\varphi(P')$ at P'_i according to *dl*-extension. Hence voter i manipulates at profile P' via P_i. Thus $\varphi_b(P) < \varphi(P')$ as claimed. This completes the proof for the case $n \geq 3$.

We now turn to the case of two voters ($n = 2$). We will show that tops-onlyness is redundant because *dl*-strategy-proofness and efficiency imply tops-onlyness.

Pick two distinct alternatives a and b and consider a profile $P = (P_1^{ab}, P_2^{ba})$.[21] Let $\varphi_a(P) = \lambda$. Efficiency implies that $\varphi_b(P) = 1 - \lambda$. Pick an arbitrary profile $\bar{P} = (\bar{P}_1^a, \bar{P}_2^b)$. If we can show that $\varphi(P) = \varphi(\bar{P})$ it completes the proof. The first step in that direction we claim that $\varphi(\bar{P}_1, P_2) = \varphi(P)$.

Suppose $\varphi_a(\bar{P}_1, P_2) \neq \varphi_a(P)$. If $\varphi_a(\bar{P}_1, P_2) > \varphi_a(P)$ then voter 1 manipulates at profile P via $\bar{P}_1$. On the other hand if $\varphi_a(\bar{P}_1, P_2) < \varphi_a(P)$ then voter 1 manipulates at profile $(\bar{P}_1, P_2)$ via P_1. Thus we have $\varphi_a(\bar{P}_1, P_2) = \varphi_a(P)$. Applying efficiency at $(\bar{P}_1, P_2)$ implies $\varphi_b(\bar{P}_1, P_2) = 1 - \varphi_a(\bar{P}_1, P_2) = 1 - \lambda$. By a similar argument we show $\varphi(P_1, \bar{P}_2) = \varphi(P)$.

Finally we claim that $\varphi(\bar{P}) = \varphi(P)$. Suppose not. If $\varphi_a(\bar{P}) < \varphi_a(P_1, \bar{P}_2) = \lambda$ then voter 1 manipulates at profile $\bar{P}$ via P_1. On the other hand, if $\varphi_b(\bar{P}) < \varphi_b(\bar{P}_1, P_2) = 1 - \lambda$ then voter 2 manipulates at profile $\bar{P}$ via P_2. This completes the proof. ■

[21] Remember that We shall write P_i^a for a preference ordering where a is ranked first. Similarly P_i^{ab} will denote a preference ordering where a is ranked first and b second.

Chapter 4

Stochastic same-sidedness in random voting models

4.1 Introduction

In this chapter, we study the standard random voting model. A group of agents (or voters) have to choose a lottery over the set of alternatives (or candidates) based on their ordinal preferences over the alternatives. A random social choice function (RSCF) maps each profile of agents' preferences to a lottery. In this framework, we propose an axiom called *stochastic same-sidedness* (SSS) and explore its consequences.

Consider a preference profile and suppose a voter changes her preference ordering to an adjacent one by swapping two consecutively ranked alternatives. Then, SSS imposes two restrictions on the RSCF. First, the sum of probabilities of the alternatives which are ranked strictly higher than the swapped pair, should remain the same. Second, the sum probabilities assigned to the swapped pair, should also remain the same.

The SSS axiom can be motivated as an axiom of robustness to "small mistakes". Consider a deterministic SCF where a voter makes a small mistake in reporting y better than x when x is better than y and consecutively ranked in her true preference and misreport. Suppose truth-telling outcome z was below y but is above x after the misreport. Then this could be interpreted as a "large" change as consequence of small mistake since the outcome "jumps" over a number of alternatives including x and y. A similar interpretation could be given if z was above x but new outcome is below y. In addition if z is either x or y then it must remain so after the misreport.

Muto and Sato (2017) introduced an axiom called same-sidedness (SS) which ensures that a SCF does not respond to small mistakes in this way. Our SSS axiom is the stochastic counterpart of the same-sidedness condition. In particular, probability weight should not be transferred from an outcome ranked strictly above x to either x or any outcome worse than x. Similarly, probability weight should not be transferred from an outcome ranked strictly below y to either y or any outcome better than y. The SSS axiom also requires that the sum of the probabilities of x and y is the same before and after a misreport. The same-sidedness condition of Muto and Sato (2017) is a weaker version of a condition they called *bounded response* (BR). Our SSS condition cannot be interpreted as a generalization of BR. However, our results show that even this form of weaker immunity to small mistakes leads to negative results.

Muto and Sato (2017) showed that the mild SS condition has strong negative implications. For more than two voters, unanimity and SS do not imply dictatorship. However, if unanimity is replaced by the requirement of efficiency, then it is not possible to escape dictatorship.

The key question addressed in this chapter is the following: does randomization significantly expand the class of RSCFs satisfying SSS relative to the deterministic case? In particular, can we escape the negative conclusions of Muto and Sato (2017)? The simple answer is that we cannot. We show that in the two voters case, every RSCF that satisfies *efficiency* and SSS, is a *random dictatorship*. The result does not hold if we replace *efficiency* by *unanimity*. If there are more than two voters, *efficiency* and SSS do not imply *random dictatorship*. However, if RSCFs are required to satisfy *tops-onlyness* in addition to *efficiency* and SSS, we have *random dictatorship* again. We note that results for deterministic SCFs do not always immediately translate into results for RSCFs. An illustration of this fact is that much stronger restrictions may be required for a *dictatorial domain* to also be a *random dictatorial domain* - see Chatterji et al. (2014).

It is important to clarify the precise relationship between SSS and various notions of incentive compatibility. The standard notion of incentive-compatibility is *sd*-strategy-proofness introduced in Gibbard (1977) (for detailed explanation refer Chapter 3 of the dissertation). The SSS axiom is much weaker than *sd*-strategy-proofness. Suppose for instance, a voter misrepresents her true preference ordering by swapping the 7^{th} and 8^{th} ranked alternatives, say x and y respectively. According to *sd*-strategy-proofness, the probability of all alternatives other than x and y should remain the same; furthermore the probability of y must not decline while the probability of x must not increase. On the other hand, SSS merely requires the *sum* of probabilities of alternatives above x and y (i.e. the top six alternatives) to remain unchanged. In addition, the *sum* of the probabilities of x and y must remain unchanged.

There are several weaker versions of *sd*-strategy-proofness that have been used in random voting model and random object assignment models (for instance, Bogomolnaia and Moulin (2001b), Aziz et al. (2014), Balbuzanov (2016), Sen (2011), Brandt (2017). See also

Chapter 3 for further discussion on this issue). The weakest among these notions is *weak sd-strategy-proofness*. It requires the existence of at least one utility representation of the voter's true ordinal preference for which the expected utility from truth-telling is higher than lying. In other words, the truth-telling lottery should not be stochastically dominated by any other lottery obtained by misrepresentation. We show that SSS and weak *sd*-strategy-proofness are independent i.e. neither implies the other. In particular, SSS allows instances where the truth-telling lottery gives (strictly) lower expected utility than a lottery obtained via a misreport for every utility representation of voter's true ordinal preference. In other words, truth-telling lottery is stochastically dominated by a lottery obtained by lying. As a consequence, the SSS axiom cannot be interpreted as an incentive-compatible property. We conclude this section by noting that recently, Chun and Yun (2020) introduced a weakening of *sd*-strategy-proofness, called *upper-contour strategy-proofness* to study random object assignment model. Our SSS axiom is weaker than their axiom, as former is immediately implied by the latter.

This chapter is organized as follows. Section 4.2 formally introduces the model. Section 4.3 discusses the relationship between SSS and incentive-compatibility. Section 4.4 contains the main results. Section 4.5 concludes the chapter. All proofs are contained in the Appendix (Section 4.6).

4.2 The Framework

Let $N = \{1, \ldots, n\}$ be a finite set of voters and A be a finite set of alternatives where $|A| = m \geq 3$. We denote ΔA the set of probability distributions or lotteries over the elements of A. Each voter $i \in N$ has a linear ordering P_i over the elements of the set A. Let $\mathbb{P}$ denote the set of all linear orderings over the elements of A. A preference profile is a list $P = (P_1, \ldots, P_n) \in \mathbb{P}^n$ of voters preferences. For any profile $P \in \mathbb{P}^n$ and voter $i \in N$, let P_{-i} denote the $n-1$ voters profile $(P_1, \ldots, P_{i-1}, P_{i+1}, \ldots, P_n)$. For any ordering P_i and integer $k = 1, \ldots, m$, we denote $r_k(P_i)$ the k^{th} ranked alternative in P_i, i.e. $|\{a \in A : aP_i r_k(P_i)\}| = k - 1$. For any ordering P_i and $a \in A$, we denote $r(P_i, a) \in \{1, 2, \ldots, m\}$ as the rank of a at P_i. Note that for any $P_i \in \mathbb{P}$, $k \in \{1, 2, \ldots, m\}$ and $a \in A$, $r_k(P_i) = a$ if and only if $r(P_i, a) = k$.

Definition **4.1** *A random social choice function (RSCF) φ is a mapping from $\mathbb{P}^n$ to ΔA i.e. $\varphi : \mathbb{P}^n \to \Delta A$.*

For any $P \in \mathbb{P}^n$, $\varphi(P)$ is a lottery over A. For any alternative $a \in A$, $\varphi_a(P)$ denotes the probability assigned to a at the lottery $\varphi(P)$. Clearly, $\varphi_a(P) \geq 0$ and $\sum\limits_{a \in A} \varphi_a(P) = 1$.

Throughout this paper, attention is restricted to RSCFs satisfying the efficiency property. This requires that if $a \in A$ is preferred to $b \in A$ by all voters in any profile, then the RSCF must assign zero probability to the alternative b in that profile.

Definition **4.2** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is efficient if for all $P \in \mathbb{P}^n$ and for all $a, b \in A$ such that aP_ib for all $i \in N$, we have $\varphi_b(P) = 0$.*

A much weaker notion of efficiency is unanimity. This requires that an alternative that is first-ranked by all voters in any profile be selected with probability one in that profile.

Definition **4.3** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is unanimous if for all $P \in \mathbb{P}^n$ and for all $a \in A$ such that $r_1(P_i) = a$ for all $i \in N$, we have $\varphi_a(P) = 1$.*

Now we introduce the key axiom of the paper, which we call stochastic same-sidedness. Before providing a formal definition, we need to introduce further notations. For any $P_i, P'_i \in \mathbb{P}$ and $B \subseteq A$, if $xP_iy \Leftrightarrow xP'_iy$ for all $x, y \in B$, then we write $P_i(B) = P'_i(B)$. Two orderings P and P' are adjacent if there exist two distinct alternatives $x, y \in A$ such that (1) $r_k(P_i) = x = r_{k+1}(P'_i)$ and $r_{k+1}(P_i) = y = r_k(P'_i)$, $k \in \{1, 2, \ldots, m-1\}$ and (2) $P_i(A \setminus \{x, y\}) = P'_i(A \setminus \{x, y\})$. If P_i and P'_i are adjacent and two distinct alternatives $x, y \in A$ satisfy xP_iy and yP'_ix, the set of two alternatives $\{x, y\}$ is denoted by $A(P_i, P'_i)$.

Two alternatives x and y are adjacent at P_i if they are consecutively ranked in P_i, i.e., $r(P_i, x) - r(P_i, y)$ is either 1 or -1. For any $P_i \in \mathbb{P}$ and for any two alternatives x and y at P_i, $U(P_i, \{x, y\})$ denotes the set of alternatives which are ranked above x and y at P_i i.e. $U(P_i, \{x, y\}) = \{a \in A : aP_ix \text{ and } aP_iy\}$. A sequence of distinct orderings $(P^1, \ldots, P^k)$ is a path from P^1 to P^k if for every $j \in \{1, \ldots, k-1\}$, P^j and P^{j+1} are adjacent.

Now we state stochastic same-sidedness formally below.

Definition **4.4** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ satisfies stochastic same-sidedness (SSS) if for all $P \in \mathbb{P}^n$, $i \in N$ and $P'_i \in \mathbb{P}$ such that P_i and P'_i are adjacent with $A(P_i, P'_i) = \{x, y\}$[1], we have*

a. $\sum_{a \in U(P_i, \{x,y\})} \varphi_a(P) = \sum_{a \in U(P_i, \{x,y\})} \varphi_a(P'_i, P_{-i})$.

b. $\sum_{a \in \{x,y\}} \varphi_a(P) = \sum_{a \in \{x,y\}} \varphi_a(P'_i, P_{-i})$.

Stochastic same-sidedness puts the following restrictions on RSCFs. At $P \in \mathbb{P}^n$ if voter $i \in N$ replaces her preference by P'_i which is adjacent to P_i, then the total probabilities to each of the two sets $U(P_i, A(P_i, P'_i)$ and $\{x, y\}$ at (P'_i, P_{-i}) should remain same as at P. The main objective of this paper is to study RSCFs that satisfy efficiency and SSS.

[1] Note that if P_i and P'_i are adjacent with $A(P_i, P'_i) = \{x, y\}$, then $U(P_i, x, y) = U(P'_i, x, y)$

4.3 Incentive-compatibility and stochastic same-sidedness

In this section we explore the connection between stochastic same-sidedness and incentive-compatibility. In this framework, there are several notions of incentive-compatibility. We first describe the approach of Gibbard (1977) which is, in fact the standard approach in probabilistic voting theory.

Definition **4.5** *A utility function $u : A \to \mathbb{R}$ represents the ordering P_i over A if for all $a, b \in A$,*

$$[aP_ib] \Leftrightarrow [u(a) > u(b)]$$

We let $U(P_i)$ denote the set of utility functions that represent P_i.

Definition **4.6** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is strategy-proof if, for all $i \in N$, for all $P \in \mathbb{P}^n$, for all $P_i' \in \mathbb{P}$ and all $u \in U(P_i)$, we have*

$$\sum_{a \in A} u(a)\varphi_a(P_i, P_{-i}) \geq \sum_{a \in A} u(a)\varphi_a(P_i', P_{-i})$$

.

A RSCF is strategy-proof if telling the truth maximizes a voter's expected utility for every utility representation of her ordinal preferences, irrespective of the announcements of the other voters. It is well-known that this is equivalent to requiring that the probability distribution from truth-telling stochastically dominates the probability distribution from misrepresentation in terms of a voter's true preferences. This is also known as stochastic dominance strategy-proofness (*sd*-strategy-proofness) and is stated formally below.

For any $i \in N$, $P_i \in \mathbb{P}$ and $k = 1, \ldots, m$, let $B(k, P_i) = \{a \in A : aP_ir_k(P_i)\} \cup \{r_k(P_i)\}$, i.e. $B(k, P_i)$ denotes the set of alternatives that are weakly preferred to the k^{th} ranked alternative in P_i.

Definition **4.7** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is sd-strategy-proof if for all $i \in N$, for all $P \in \mathbb{P}^n$, for all $P_i' \in \mathbb{P}$ and for all $k = 1, \ldots, m-1$, we have*

$$\sum_{a \in B(k,P_i)} \varphi_a(P_i, P_{-i}) \geq \sum_{a \in B(k,P_i)} \varphi_a(P_i', P_{-i})$$

.

In the following, we show that stochastic same-sidedness is weaker than *sd*-strategy-proofness.

Proposition **4.1** *Let $\varphi : \mathbb{P}^n \to \Delta A$ be a sd-strategy-proof RSCF. Then φ satisfies SSS.*

The proof of Proposition 4.1 is in the Appendix. However, SSS does not imply *sd*-strategy-proofness (see example 4.2).

The weakest form of incentive-compatibility is weak *sd*-strategy-proofness which we define formally below.

DEFINITION **4.8** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is weakly sd-strategy-proof if for all $i \in N$, for all $P \in \mathbb{P}^n$, for all $P_i' \in \mathbb{P}$*

$$\Big[\sum_{a \in B(k,P_i)} \varphi_a(P_i', P_{-i}) \geq \sum_{a \in B(k,P_i)} \varphi_a(P_i, P_{-i}) \text{ for all } k = 1, \ldots, m-1\Big] \Rightarrow \Big[\varphi(P) = \varphi(P_i', P_{-i})\Big]$$

.

Weak *sd*-strategy-proofness is equivalent to requiring the truth-telling lottery not to be first order stochastically dominated by all lotteries obtained by misrepresentation of preferences. In other words, it requires the existence of one utility representation of voter's true ordinal preference for which the expected utility from truth-telling is higher than lying. However, SSS is not a strategic property of RSCFs. It allows the fact that truth-telling lottery can be first order stochastically dominated by a lottery obtained by misrepresentation of preferences. In particular, in the below, we show that SSS and weak *sd*-strategy-proofness are independent.

EXAMPLE **4.1** *(Weak sd-strategy-proofness does not imply SSS)* Let $|A| \geq 3$ and $N = \{1, 2\}$. Fix $a \in A$. We define $\varphi : \mathbb{P}^2 \to \Delta A$ as follows: For any $P \in \mathbb{P}^2$ and for any $x \in A$,

$$\varphi_x(P) = \begin{cases} 1 & \text{if } x = r_1(P_1) = r_1(P_2) \\ \frac{1}{2} & \text{if } x = a = r_1(P_1) \text{ and } r_1(P_1) \neq r_1(P_2) \\ \frac{1}{2} & \text{if } a = r_1(P_1) \neq r_1(P_2) = x \\ \frac{3}{5} & \text{if } x = r_1(P_1) \neq a \text{ and } r_1(P_1) \neq r_1(P_2) \\ \frac{2}{5} & \text{if } r_1(P_1) \neq a \text{ and } r_1(P_1) \neq r_1(P_2) = x \\ 0 & \text{otherwise} \end{cases}$$

It is easy to verify that φ satisfies weak *sd*-strategy-proofness. Now consider the case where $A = \{a, b, c\}$. Let P_1, P_1' and P_2 be the following linear orders.

P_1	P_1'	P_2
a	b	c
b	a	b
c	c	a

Note that $\varphi_a(P_1, P_2) = \frac{1}{2}$ and $\varphi_b(P_1, P_2) = 0$. Therefore, $\varphi_a(P_1, P_2) + \varphi_b(P_1, P_2) = \frac{1}{2}$. Now note that P_1 and P_1' are adjacent and $A(P_i, P_i') = \{a, b\}$. But $\varphi_a(P_1', P_2) = 0$ and

$\varphi_b(P_1', P_2) = \frac{3}{5}$. So $\varphi_a(P_1', P_2) + \varphi_b(P_1', P_2) = \frac{3}{5}$. Therefore we have $\varphi_a(P_1, P_2) + \varphi_b(P_1, P_2) \neq \varphi_a(P_1', P_2) + \varphi_b(P_1', P_2)$. This is a violation of SSS. ■

EXAMPLE **4.2** *(SSS does not imply weak sd-strategy-proofness)* Let $|A| \geq 3$ and $N = \{1, 2\}$. Fix $a, b \in A$, $a \neq b$ and $\epsilon \in (0, 0.5)$. We define $\varphi^{a,b} : \mathbb{P}^2 \to \Delta A$ as follows: For any $P \in \mathbb{P}^2$ and for any $x \in A$,

$$\varphi_x^{a,b}(P) = \begin{cases} \frac{1}{2} - \epsilon & \text{if } x = a = r_1(P_1) \text{ and } r_m(P_1) = b \\ \epsilon & \text{if } a = r_1(P_1), x = r_2(P_1) \text{ and } r_m(P_1) = b \\ \frac{1}{2} & \text{if } a = r_1(P_1) \text{ and } r_m(P_1) = b = x \\ \frac{1}{2} & \text{if } x \in \{a, b\} \text{ and } a \neq r_1(P_1) \text{ or } b \neq r_m(P_1) \\ 0 & \text{otherwise} \end{cases}$$

Note that $\varphi^{a,b}$ satisfies SSS. Now consider the case where $A = \{a, b, c\}$. Let P_1, P_1' and P_2 be the following linear orders.

P_1	P_1'	P_2
a	a	c
c	b	b
b	c	a

Note that $\varphi_a^{a,b}(P_1, P_2) = \frac{1}{2} - \epsilon$, $\varphi_b^{a,b}(P_1, P_2) = \frac{1}{2}$ and $\varphi_c^{a,b}(P_1, P_2) = \epsilon$. Also $\varphi_a^{a,b}(P_1', P_2) = \frac{1}{2}$, $\varphi_b^{a,b}(P_1', P_2) = \frac{1}{2}$ and $\varphi_c^{a,b}(P_1, P_2) = 0$. Now assume that P_1 be the true preference of agent 1. Then we have

$$\varphi_a^{a,b}(P_1', P_2) > \varphi_a^{a,b}(P_1, P_2).$$

$$\varphi_a^{a,b}(P_1', P_2) + \varphi_c^{a,b}(P_1', P_2) = \varphi_a^{a,b}(P_1, P_2) + \varphi_c^{a,b}(P_1, P_2).$$

$$\varphi_a^{a,b}(P_1', P_2) + \varphi_c^{a,b}(P_1', P_2) + \varphi_b^{a,b}(P_1', P_2) = \varphi_a^{a,b}(P_1, P_2) + \varphi_c^{a,b}(P_1, P_2) + \varphi_b^{a,b}(P_1, P_2).$$

This shows that $\varphi^{a,b}(P_1', P_2)$ stochastically dominates $\varphi^{a,b}(P_1, P_2)$. This implies that $\varphi^{a,b}$ violates weak *sd*-strategy-proofness. ■

It is important to mention that a prominent class of unanimous and *sd*-strategy-proof RSCFs is the class of random dictatorships (Gibbard (1977)). Each voter first is assigned a non-negative weight such that the sum of all weights equals one. In a random dictatorship, at each preference profile, the probability received by an alternative is determined by the set of voters who prefer this alternative the most, and equals the sum of these voters' weights.

DEFINITION **4.9** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is a random dictatorship if there exist non-negative real numbers β_i, $i \in N$ with $\sum_{i \in N} \beta_i = 1$ such that for all $P \in \mathbb{P}^n$ and all $a \in A$,*

$$\varphi_a(P) = \sum_{\{i : r_1(P_i) = a\}} \beta_i$$

Note that random dictatorships are *sd*-strategy-proof. In the view of Proposition 4.1, we can conclude that random dictatorship satisfies SSS. We conclude this section by stating this observation formally below.

LEMMA **4.1** *Let the RSCF $\varphi : \mathbb{P}^n \to \Delta A$ be a random dictatorship. Then φ satisfies SSS.*

4.4 The Results

In this section we present the main results of the paper. For the two voters case, we show that efficiency and SSS imply random dictatorship.

THEOREM **4.1** *Let $n = 2$. A RSCF $\varphi : \mathbb{P}^2 \to \Delta A$ satisfies efficiency and SSS if and only if it is random dictatorship.*

The proof of Theorem 4.1 is in the Appendix. The following example shows that Theorem 4.1 does not hold if we replace efficiency by unanimity.

EXAMPLE **4.3** Let $A = \{a, b, c, d\}$ and $N = \{1, 2\}$. We define $\bar{\mathbb{P}}_1, \bar{\mathbb{P}}_2 \subseteq \mathbb{P}$ as follows:

$$\bar{\mathbb{P}}_1 = \{P_1 \in \mathbb{P} : r_1(P_1) = a, r_4(P_1) = b\}$$

$$\bar{\mathbb{P}}_2 = \{P_2 \in \mathbb{P} : r_1(P_2) = b, r_4(P_2) = a\}$$

Let $\bar{\mathbb{P}}^s = \bar{\mathbb{P}}_1 \times \bar{\mathbb{P}}_2$ and $0 < \epsilon < \frac{1}{2}$. We define $\varphi : \mathbb{P}^2 \to \Delta A$ as follows: For any $P \in \mathbb{P}^2$ and for any $x \in A$,

$$\varphi_x(P) = \begin{cases} \frac{1}{2} - \epsilon & \text{if } P \in \bar{\mathbb{P}}^s \text{ and } x \in \{a, b\} \\ \epsilon & \text{if } P \in \bar{\mathbb{P}}^s \text{ and } x \in \{c, d\} \\ \frac{k}{2} & \text{if } P \notin \bar{\mathbb{P}}^s \text{ and } k = |\{i \in \{1, 2\} : r_1(P_i) = x\}| \end{cases}$$

Note that φ does not satisfy random dictatorship. It can be verified that that φ satisfies unanimity and SSS. Now consider the profile $P = (P_1, P_2)$ such that

P_1	P_2
a	b
c	c
d	d
b	a

Note that $\varphi_d(P) = \epsilon > 0$. This violates efficiency as for all agents $i \in N$, cP_id. ■

For the more than two voters case, efficiency and SSS do not imply random dictatorship. However, if we impose the additional requirement of tops-onlyness on RSCFs, then efficiency and SSS imply random dictatorship. A RSCF satisfies the tops-only property if the lottery under each preference profile depends only on the top ranked alternatives of voters' preferences.

Definition **4.10** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ satisfies tops-onlyness if for all $P, P' \in \mathbb{P}^n$ where $r_1(P_i) = r_1(P'_i)$ for all $i \in N$, we have $\varphi(P) = \varphi(P')$.*

Tops-onlyness has the following implication on efficient RSCFs. If a RSCF satisfies efficiency and tops-onlyness, then the support of the lottery under each preference profile is a subset of the set of the top ranked alternatives of voters' preferences. We call this property as only-topness and provide a formal definition in the below.

Definition **4.11** *A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ satisfies only-topness if for all $P \in \mathbb{P}^n$ and $a \in A$, $\varphi_a(P) > 0$ implies $a = r_1(P_i)$ where $i \in N$.*

Proposition **4.2** *If a RSCF $\varphi : \mathbb{P}^n \to \Delta A$ is efficient and tops-only, then it is a only-top RSCF.*

Proof: Suppose not i.e. there exist $P \in \mathbb{P}^n$ and $a \in A$ such that $\varphi_a(P) > 0$ and $a \neq r_1(P_i)$ for all $i \in N$. Let $P' \in \mathbb{P}^n$ be such that for all $i \in N$, $r_1(P_i) = r_1(P_i)$ and $r_m(P'_i) = a$. By tops-onlyness, $\varphi_a(P) = \varphi_a(P') > 0$. This contradicts efficiency of φ at P'. Therefore, φ is only-top. ■

The state our main result for the case of more than two voters in below.

Theorem **4.2** *Let $n \geq 3$. A RSCF $\varphi : \mathbb{P}^n \to \Delta A$ satisfies efficiency, tops-onlyness and SSS if and only if it is random dictatorship.*

The proof of Theorem 4.2 is in the Appendix. The following example illustrates that for the more than two voters case, efficiency and SSS do not imply random dictatorship.

Example **4.4** Let $A = \{a, b, c, x, y, z\}$ and $N = \{1, 2, 3\}$. We define $\bar{\mathbb{P}}_1, \bar{\mathbb{P}}_2, \bar{\mathbb{P}}_3 \subseteq \mathbb{P}$ as follows:

$$\bar{\mathbb{P}}_1 = \{P_1 \in \mathbb{P} : r_1(P_1) = a, \{r_2(P_1), r_3(P_1)\} = \{x, z\}, \{r_4(P_1), r_5(P_1)\} = \{b, c\}, r_6(P_1) = y\}$$

$$\bar{\mathbb{P}}_2 = \{P_2 \in \mathbb{P} : r_1(P_2) = b, \{r_2(P_2), r_3(P_2)\} = \{x, y\}, \{r_4(P_2), r_5(P_2)\} = \{a, c\}, r_6(P_2) = z\}$$

$$\bar{\mathbb{P}}_3 = \{P_3 \in \mathbb{P} : r_1(P_3) = c, \{r_2(P_3), r_3(P_3)\} = \{y, z\}, \{r_4(P_3), r_5(P_3)\} = \{a, b\}, r_6(P_3) = x\}$$

Let $\bar{\mathbb{P}}^s = \bar{\mathbb{P}}_1 \times \bar{\mathbb{P}}_2 \times \bar{\mathbb{P}}_3$ and $0 < \epsilon < \frac{1}{3}$. We define $\varphi : \mathbb{P}^3 \to \Delta A$ as follows: For any $P \in \mathbb{P}^3$ and for any $w \in A$,

$$\varphi_w(P) = \begin{cases} \frac{1}{3} - \epsilon & \text{if } P \in \bar{\mathbb{P}}^s \text{ and } w \in \{a, b, c\} \\ \epsilon & \text{if } P \in \bar{\mathbb{P}}^s \text{ and } w \in \{x, y, z\} \\ \frac{k}{3} & \text{if } P \notin \bar{\mathbb{P}}^s \text{ and } k = |\{i \in \{1, 2, 3\} : r_1(P_i) = w\}| \end{cases}$$

Note that φ does not satisfy random dictatorship. It can be verified that that φ satisfies efficiency and SSS.

Now consider the profiles $P = (P_1, P_2, P_3)$ and $P' = (P'_1, P'_2, P'_3)$.

P_1	P_2	P_3	P'_1	P'_2	P'_3
a	b	c	a	b	c
x	x	y	b	a	b
z	y	z	c	c	a
b	a	a	x	x	x
c	c	b	y	y	y
y	z	x	z	z	z

Note that $\varphi_a(P) = \varphi_b(P) = \varphi_c(P) = \frac{1}{3} - \epsilon$ and $\varphi_x(P) = \varphi_y(P) = \varphi_z(P) = \epsilon$. On the other hand $\varphi_a(P') = \varphi_b(P') = \varphi_c(P') = \frac{1}{3}$ and $\varphi_x(P') = \varphi_y(P') = \varphi_z(P') = 0$. So $\varphi(P) \neq \varphi(P')$. This violates tops-onlyness as for all agents $i \in N$, $r_1(P_i) = r_1(P'_i)$. ■

The following example shows that Theorem 4.2 does not hold if we replace top-onlyness by only-topness.

Example **4.5** Let $A = \{a, b, c, d\}$ and $N = \{1, 2, 3, 4\}$. We define $\bar{\mathbb{P}}_1, \bar{\mathbb{P}}_2, \bar{\mathbb{P}}_3, \bar{\mathbb{P}}_4 \subseteq \mathbb{P}$ as follows:

$$\bar{\mathbb{P}}_1 = \{P_1 \in \mathbb{P} : r_1(P_1) = a, r_4(P_1) = b\}$$

$$\bar{\mathbb{P}}_2 = \{P_2 \in \mathbb{P} : r_1(P_2) = b, r_4(P_2) = a\}$$

$$\bar{\mathbb{P}}_3 = \{P_3 \in \mathbb{P} : r_1(P_3) = c, r_4(P_3) = d\}$$

$$\bar{\mathbb{P}}_4 = \{P_4 \in \mathbb{P} : r_1(P_4) = d, r_4(P_4) = c\}$$

Let $\bar{\mathbb{P}}^s = \bar{\mathbb{P}}_1 \times \bar{\mathbb{P}}_2 \times \bar{\mathbb{P}}_3 \times \bar{\mathbb{P}}_4$ and $0 < \epsilon < \frac{1}{2}$. We define $\varphi : \mathbb{P}^4 \to \Delta A$ as follows: For any $P \in \mathbb{P}^3$ and for any $x \in A$,

$$\varphi_x(P) = \begin{cases} \frac{1}{2} - \epsilon & \text{if } P \in \bar{\mathbb{P}}^s \text{ and } x \in \{a, b\} \\ \epsilon & \text{if } P \in \bar{\mathbb{P}}^s \text{ and } x \in \{c, d\} \\ \frac{k}{2} & \text{if } P \notin \bar{\mathbb{P}}^s \text{ and } k = |\{i \in \{1, 2\} : r_1(P_i) = x\}| \end{cases}$$

Note that φ does not satisfy random dictatorship. It can be verified that that φ satisfies efficiency, only-topness and SSS. ■

In the following, we show that Theorem 4.2 does not hold if we replace efficiency by unanimity.

Example **4.6** Let $A = \{a, b, c, d\}$ and $N = \{1, 2, 3\}$. We partition $\mathbb{P}^3$ into five sets as follows:

$$\mathbb{P}^3_1 = \{P \in \mathbb{P}^3 : r_1(P_1) = r_1(P_2) \neq r_1(P_3)\}$$

$$\mathbb{P}^3_2 = \{P \in \mathbb{P}^3 : r_1(P_1) = r_1(P_3) \neq r_1(P_2)\}$$

$$\mathbb{P}^3_3 = \{P \in \mathbb{P}^3 : r_1(P_1) \neq r_1(P_2) = r_1(P_3)\}$$

$$\mathbb{P}^3_4 = \{P \in \mathbb{P}^3 : r_1(P_1) \neq r_1(P_2), r_1(P_1) \neq r_1(P_3), r_2(P_1) \neq r_1(P_3)\}$$

$$\mathbb{P}^3_5 = \{P \in \mathbb{P}^3 : r_1(P_1) = r_1(P_2) = r_1(P_3)\}$$

We define $\varphi : \mathbb{P}^3 \to \Delta A$ as follows: For any $P \in \mathbb{P}^3$ and for any $x \in A$,

$$\varphi_x(P) = \begin{cases} \frac{1}{2} & \text{if } P \in \mathbb{P}_1^3 \text{ and } x = r_1(P_1) \\ \frac{1}{2} & \text{if } P \in \mathbb{P}_1^3 \text{ and } x = r_1(P_3) \\ \frac{3}{5} & \text{if } P \in \mathbb{P}_2^3 \text{ and } x = r_1(P_1) \\ \frac{2}{5} & \text{if } P \in \mathbb{P}_2^3 \text{ and } x = r_1(P_2) \\ \frac{3}{10} & \text{if } P \in \mathbb{P}_3^3 \text{ and } x = r_1(P_1) \\ \frac{7}{10} & \text{if } P \in \mathbb{P}_3^3 \text{ and } x = r_1(P_2) \\ \frac{2}{10} & \text{if } P \in \mathbb{P}_4^3 \text{ and } x = r_1(P_1) \\ \frac{3}{10} & \text{if } P \in \mathbb{P}_4^3 \text{ and } x = r_1(P_2) \\ \frac{4}{10} & \text{if } P \in \mathbb{P}_4^3 \text{ and } x = r_1(P_3) \\ \frac{1}{10} & \text{if } P \in \mathbb{P}_4^3 \text{ and } x \notin \{r_1(P_1), r_1(P_2), r_1(P_3)\} \\ 1 & \text{if } P \in \mathbb{P}_5^3 \text{ and } x = r_1(P_1) \\ 0 & \text{otherwise} \end{cases}$$

Note that φ does not satisfy random dictatorship. It can be verified that that φ satisfies unanimity, SSS and tops-onlyness. Now consider the profiles $P = (P_1, P_2, P_3)$. Note that

P_1	P_2	P_3
a	b	c
b	c	a
c	a	b
d	d	d

$\varphi_d(P) = \frac{1}{10}$. This violates efficiency as for all agents $i \in N$, aP_id. ■

4.5 Conclusion

This chapter explores the consequences of introducing a new axiom *stochastic same-sidedness* in the random voting model. The foundations for this axiom are non-strategic. We have shown that the axiom has strong implications for the structure of random social choice functions. Our results show that there is no escape from random dictatorship if a RSCF is required to satisfy a minimal robustness to small mistakes in conjunction with efficiency and tops-onlyness.

4.6 Appendix

The Proof of Proposition 4.1

Proof: Consider an $i \in N$, $P = (P_i, P_{-i}) \in \mathbb{P}^n$ and $P_i' \in \mathbb{P}$ such that P_i and P_i' are adjacent with $A(P_i, P_i') = \{x, y\}$. Without loss of generality, assume that $r(P_i, x) = r(P_i', y) = k$ and $r(P_i, y) = r(P_i', x) = k+1$. Note that for any $l \in \{1, 2, \ldots, k-1\}$, $B(l, P_i) = B(l, P_i')$. So applying sd-strategy-proofness for the deviation from P_i to P_i' and from P_i' to P_i, we have

$$\sum_{a \in B(l,P_i)} \varphi_a(P_i, P_{-i}) = \sum_{a \in B(l,P_i)} \varphi_a(P_i', P_{-i})$$

for all $l \in \{1, 2, \ldots, k-1\}$. Also note that $B(k+1, P_i) = B(k+1, P_i')$. So again applying sd-strategy-proofness for the deviation from P_i to P_i' and from P_i' to P_i we have

$$\sum_{a \in B(k+1,P_i)} \varphi_a(P_i, P_{-i}) \quad = \quad \sum_{a \in B(k+1,P_i)} \varphi_a(P_i', P_{-i})$$

Now, it follows that

$$\begin{aligned}
\sum_{a \in B(k+1,P_i)} \varphi_a(P_i, P_{-i}) &= \left\{ \sum_{a \in B(k-1,P_i)} \varphi_a(P_i, P_{-i}) \right\} + \varphi_x(P_i, P_{-i}) + \varphi_y(P_i, P_{-i}) \\
\sum_{a \in B(k+1,P_i)} \varphi_a(P_i', P_{-i}) &= \left\{ \sum_{a \in B(k-1,P_i)} \varphi_a(P_i', P_{-i}) \right\} + \varphi_x(P_i', P_{-i}) + \varphi_y(P_i', P_{-i})
\end{aligned}$$

Combining we have

$$\varphi_x(P_i, P_{-i}) + \varphi_y(P_i, P_{-i}) \quad = \quad \varphi_x(P_i', P_{-i}) + \varphi_y(P_i', P_{-i})$$

Note that $B(k-1, P_i) = B(k-1, P_i') = U(P_i, \{x, y\})$. Hence, φ satisfies SSS. ■

The Proof of Theorem 4.1

Proof: It is straightforward that random dictatorship satisfies efficiency and SSS. Therefore we show the only if part. Let $\varphi : \mathbb{P}^2 \to \Delta A$ satisfies efficiency and SSS. We will show that φ is a random dictatorship.

We complete the proof by showing following two lemmas.

Lemma **4.2** *For any $P, P' \in \mathbb{P}^2$ and $a, b \in A$ such that $r_1(P_1) = r_1(P_1') = a \neq b = r_1(P_2) = r_1(P_2')$, we have*

1. $\varphi_a(P) + \varphi_b(P) = \varphi_a(P') + \varphi_b(P') = 1$.

2. $\varphi_a(P) = \varphi_a(P')$ *and* $\varphi_b(P) = \varphi_b(P')$.

Proof: Consider any two alternatives $a, b \in A$ and a preference profile $\bar{P} \in \mathbb{P}^2$ such that $r_1(\bar{P}_1) = r_2(\bar{P}_2) = a \neq b = r_1(\bar{P}_2) = r_2(\bar{P}_1)$. By efficiency, $\varphi_a(\bar{P}) + \varphi_b(\bar{P}) = 1$. We assume that $\varphi_a(\bar{P}) = \beta_1$ and $\varphi_b(\bar{P}) = \beta_2$ where $0 \leq \beta_1, \beta_2 \leq 1$ and $\beta_1 + \beta_2 = 1$. We complete the proof by showing following claims.

Claim **4.1** *For any* $P \in \mathbb{P}^2$ *such that* $r_1(P_1) = a \neq b = r_1(P_2) = r_2(P_1)$ *or* $r_1(P_1) = r_2(P_2) = a \neq b = r_1(P_2)$*, we have* $\varphi_a(P) = \beta_1$ *and* $\varphi_b(P) = \beta_2$.

Proof: **Case** 1: $P \in \mathbb{P}^2$ be such that $r_1(P_1) = a \neq b = r_1(P_2) = r_2(P_1)$.

Let P_1' be an ordering where $\bar{P}_1(A \setminus b) = P_1'(A \setminus b)$ and $r_m(P_1') = b$. Let $\bar{P}_1 = P^1, P^2, \ldots, P^k = P_1'$ be a path from $\bar{P}_1$ to P_1' where $P^j(A \setminus b) = P^{j+1}(A \setminus b)$ for all $j \in \{1, 2, \ldots, k-1\}$. By efficiency and SSS, we have $\varphi(\bar{P}) = \varphi(P^2, \bar{P}_2) = \varphi(P^3, \bar{P}_2) = \ldots = \varphi(P_1', \bar{P}_2)$.

Let P_1'' be an ordering where $P_1(A \setminus b) = P_1''(A \setminus b)$ and $r_m(P_1'') = b$. Let $P_1' = P^1, P^2, \ldots, P^l = P_1''$ be a path from P_1' to P_1'' where $r_1(P^j) = a$ and $r_m(P^j) = b$ for all $j \in \{1, 2, \ldots, l\}$. Again, by efficiency and SSS, we have $\varphi(P_1', \bar{P}_1) = \varphi(P^2, \bar{P}_2) = \varphi(P^3, \bar{P}_2) = \ldots = \varphi(P_1'', \bar{P}_2)$.

Let $P_1'' = P^1, P^2, \ldots, P^r = P_1$ be a path from P_1'' to P_1 where $P^j(A \setminus b) = P^{j+1}(A \setminus b)$ for all $j \in \{1, 2, \ldots, r-1\}$. Applying efficiency and SSS, we get $\varphi(P_1'', \bar{P}_1) = \varphi(P^2, \bar{P}_2) = \varphi(P^3, \bar{P}_2) = \ldots = \varphi(P_1, \bar{P}_2)$.

Let P_2' be an ordering where $\bar{P}_2(A \setminus a) = P_2'(A \setminus a)$ and $r_m(P_2') = a$. Let $\bar{P}_2 = P^1, P^2, \ldots, P^{k'} = P_2'$ be a path from $\bar{P}_2$ to P_2' where $P^j(A \setminus a) = P^{j+1}(A \setminus a)$ for all $j \in \{1, 2, \ldots, k'-1\}$. By efficiency and SSS, we have $\varphi(P_1, \bar{P}_1) = \varphi(P_1, P^2) = \varphi(P_1, P^3) = \ldots = \varphi(P_1, P_2')$.

Let P_2'' be an ordering where $P_2(A \setminus a) = P_2''(A \setminus a)$ and $r_m(P_2'') = a$. Let $P_2' = P^1, P^2, \ldots, P^{l'} = P_2''$ be a path from P_2' to P_2'' where $r_1(P^j) = b$ and $r_m(P^j) = a$ for all $j \in \{1, 2, \ldots, l'\}$. Again, by efficiency and SSS, we have $\varphi(P_1, P_2') = \varphi(P_1, P^2) = \varphi(P_1, P^3) = \ldots = \varphi(P_1, P_2'')$.

Let $P_2'' = P^1, P^2, \ldots, P^{r'} = P_2$ be a path from P_2'' to P_2 where $P^j(A \setminus a) = P^{j+1}(A \setminus a)$ for all $j \in \{1, 2, \ldots, r'-1\}$. Applying efficiency and SSS, we get $\varphi(P_1, P_2'') = \varphi(P_1, P^2) = \varphi(P_1, P^3) = \ldots = \varphi(P_1, P_2)$.

Case 2: $P \in \mathbb{P}^2$ be such that $r_1(P_1) = r_2(P_2) = a \neq b = r_1(P_2)$. Using same arguments as in Case 1, we can get $= \varphi(\bar{P}_1, \bar{P}_2) = \varphi(P_1, P_2)$. ■

Claim **4.2** *For any $P \in \mathbb{P}^2$ and $x \in A$ such that $r_1(P_1) = a \neq b = r_1(P_2)$ and $r_2(P_1) = r_2(P_2) = x$, we have $\varphi_a(P) = \beta_1$ and $\varphi_b(P) = \beta_2$.*

Proof: Consider any $P \in \mathbb{P}^2$ and $x \in A$ such that $r_1(P_1) = a \neq b = r_1(P_2)$ and $r_2(P_1) = r_2(P_2) = x$. We assume for contradiction that either $\varphi_a(P) \neq \beta_1$ or $\varphi_b(P) \neq \beta_2$ or both. We assume that $\varphi_a(P) \neq \beta_1$ (a similar argument will lead to a contradiction if we assume $\varphi_b(P) \neq \beta_2$).

Let P_2' be an ordering where $P_2(A \setminus a) = P_2'(A \setminus a)$ and $r_3(P_2') = a$. Let $P_2 = P^1, P^2, \ldots, P^k = P_2'$ be a path from P_2 to P_2' where $P^j(A \setminus a) = P^{j+1}(A \setminus a)$ for all $j \in \{1, 2, \ldots, k-1\}$. By efficiency and SSS, $\varphi_a(P_1, P^j) = \varphi_a(P_1, P^{j+1})$ for all $j \in \{1, 2, \ldots, k-1\}$. Therefore we have $\varphi_a(P_1, P_2) = \varphi_a(P_1, P_2')$.

Let P_1' be an ordering where $P_1(A \setminus x) = P_1'(A \setminus x)$ and $r(P_1', x) = r(P_1', b) + 1$. Let $P_1 = P^1, P^2, \ldots, P^l = P_1'$ be a path from P_1 to P_1' where $P^j(A \setminus x) = P^{j+1}(A \setminus x)$ for all $j \in \{1, 2, \ldots, l-1\}$. By efficiency and SSS, $\varphi_a(P^j, P_2') = \varphi_a(P^{j+1}, P_2')$ for all $j \in \{1, 2, \ldots, l-1\}$. Therefore we have $\varphi_a(P_1, P_2) = \varphi_a(P_1, P_2') = \varphi_a(P_1', P_2')$. Also, by efficiency, we have $\varphi_b(P_1', P_2') = 1 - \varphi_a(P_1', P_2')$.

Let P_2'' be an adjacent ordering to P_2' where $A(P_2', P_2'') = \{x, a\}$. By efficiency and SSS, $\varphi_a(P_1', P_2') = \varphi_a(P_1', P_2'')$. Since, $\varphi_a(P_1', P_2'') \neq \beta_1$, we have a contradiction to Claim 4.1. ■

Claim **4.3** *For any $P \in \mathbb{P}^2$ such that $r_1(P_1) = a \neq b = r_1(P_2)$, we have $\varphi_a(P) = \beta_1$ and $\varphi_b(P) = \beta_2$.*

Proof: Consider any profile $P \in \mathbb{P}^2$ such that $r_1(P_1) = a \neq b = r_1(P_2)$. Let $B_1 = \{x \in A : aP_1x \textit{ and } xP_1b\}$ i.e B_1 is the set of alternatives which are preferred to b and worse than a at P_1. Similarly, let $B_2 = \{x \in A : bP_2x \textit{ and } xP_2a\}$. We consider following two cases to complete the proof.

Case 1: $B_1 \cap B_2 = \emptyset$. In this case, by efficiency, we have $\varphi_a(P) + \varphi_b(P) = 1$. If $\varphi_a(P) = \beta_1$, then $\varphi_b(P) = \beta_2$ and we are done. We assume for contradiction that $\varphi_a(P) \neq \beta_1$.

Let P_2' be an ordering where $P_2(A \setminus a) = P_2'(A \setminus a)$ and $r_2(P_2') = a$. Let $P_2 = P^1, P^2, \ldots, P^l = P_2'$ be a path from P_2 to P_2' where $P^j(A \setminus a) = P^{j+1}(A \setminus a)$ for all $j \in \{1, 2, \ldots, l-1\}$. By efficiency and SSS, $\varphi_a(P_1, P^j) = \varphi_a(P_1, P^{j+1})$ for all $j \in \{1, 2, \ldots, l-1\}$. Therefore we have $\varphi_a(P_1, P_2) = \varphi_a(P_1, P_2') \neq \beta_1$. This contradicts Claim 4.1.

Case 2: $B_1 \cap B_2 \neq \emptyset$. Let $B = B_1 \cap B_2$. By efficiency, if $\varphi_x(P) > 0$ for some $x \in A$, then $x \in B \cup \{a, b\}$. We consider following two sub-cases.

Sub-case 2.1:$\{x \in B : \varphi_x(P) > 0\} = \emptyset$. In this sub-case, by efficiency, we have $\varphi_a(P) + \varphi_b(P) = 1$. If $\varphi_a(P) = \beta_1$ and $\varphi_b(P) = \beta_2$, then we are done. We assume for contradiction that either $\varphi_a(P) > \beta_1$ or $\varphi_b(P) > \beta_2$. Let $\varphi_a(P) > \beta_1$ (a similar argument will lead to a contradiction if $\varphi_b(P) > \beta_2$)

Let P_2' be an ordering where $P_2(A \setminus a) = P_2'(A \setminus a)$ and $r_2(P_2') = a$. Let $P_2 = P^1, P^2, \dots, P^l = P_2'$ be a path from P_2 to P_2' where $P^j(A \setminus a) = P^{j+1}(A \setminus a)$ for all $j \in \{1, 2, \dots, l-1\}$. By efficiency and SSS, $\varphi_a(P_1, P^j) \leq \varphi_a(P_1, P^{j+1})$ for all $j \in \{1, 2, \dots, l-1\}$. Therefore we have $\beta_1 < \varphi_a(P_1, P_2) \leq \varphi_a(P_1, P_2')$. This contradicts Claim 4.1.

Sub-case 2.2: $\{x \in B : \varphi_x(P) > 0\} \neq \emptyset$. Let $S = \{x \in B : \varphi_x(P) > 0\}$ and $y \in S$ such that for all $x \in S \setminus y$, xP_1y.

Let P_1' be an ordering where $P_1(A \setminus y) = P_1'(A \setminus y)$ and $r_2(P_1') = y$. Let $P_1 = P^1, P^2, \dots, P^l = P_1'$ be a path from P_1 to P_1' where $P^j(A \setminus y) = P^{j+1}(A \setminus y)$ for all $j \in \{1, 2, \dots, l-1\}$. By efficiency and SSS, $\varphi_y(P^j, P_2) \leq \varphi_y(P^{j+1}, P_2)$ for all $j \in \{1, 2, \dots, l-1\}$. Therefore, we have $0 < \varphi_y(P_1, P_2) \leq \varphi_y(P_1', P_2)$.

Let P_2' be an ordering where $P_2(A \setminus y) = P_2'(A \setminus y)$ and $r_2(P_2') = y$. Let $P_2 = P^1, P^2, \dots, P^k = P_2'$ be a path from P_2 to P_2' where $P^j(A \setminus y) = P^{j+1}(A \setminus y)$ for all $j \in \{1, 2, \dots, k-1\}$. By efficiency and SSS, $\varphi_y(P_1', P^j) \leq \varphi_y(P_1', P^{j+1})$ for all $j \in \{1, 2, \dots, k-1\}$. Therefore we have $0 < \varphi_y(P_1, P_2) \leq \varphi_y(P_1', P_2) \leq \varphi_y(P_1', P_2')$. This contradicts Claim 4.2. ■

Claims 4.1, 4.2 and 4.3 establish Lemma 4.2. ■

Lemma **4.3** *For any $P, \bar{P} \in \mathbb{P}^2$ such that $r_1(P_1) = a \neq b = r_1(P_2)$ and $r_1(\bar{P}_1) = c \neq d = r_1(\bar{P}_2)$, we have $\varphi_a(P_1, P_2) = \varphi_c(\bar{P}_1, \bar{P}_2)$ and $\varphi_b(P_1, P_2) = \varphi_d(\bar{P}_1, \bar{P}_2)$.*

Proof: We consider $P, \bar{P} \in \mathbb{P}^2$ such that $r_1(P_1) = a \neq b = r_1(P_2)$ and $r_1(\bar{P}_1) = c \neq d = r_1(\bar{P}_2)$. We consider following two cases.

Case 1: $c \neq b$. Now we consider following four sub-cases.

Sub-case 1.1: $a = c$ and $b = d$. By Lemma 4.2, we are done with this sub-case.

Sub-case 1.2: $a = c$ and $b \neq d$. W.o.l.o.g we consider $P, \bar{P} \in \mathbb{P}^2$ such that $P_1 = \bar{P}_1$ and P_2 and $\bar{P}_2$ are adjacent with $A(P_2, \bar{P}_2) = \{b, d\}$. Applying Lemma 4.2 and SSS, we have $\varphi_a(P_1, P_2) = \varphi_c(\bar{P}_1, \bar{P}_2)$ and $\varphi_b(P_1, P_2) = \varphi_d(\bar{P}_1, \bar{P}_2)$.

Sub-case 1.3: $a \neq c$ and $b = d$. W.o.l.o.g we consider $P, \bar{P} \in \mathbb{P}^2$ such that $P_2 = \bar{P}_2$ and P_1 and $\bar{P}_1$ are adjacent with $A(P_1, \bar{P}_1) = \{a, c\}$. Again applying Lemma 4.2 and SSS, we have $\varphi_a(P_1, P_2) = \varphi_c(\bar{P}_1, \bar{P}_2)$ and $\varphi_b(P_1, P_2) = \varphi_d(\bar{P}_1, \bar{P}_2)$.

Sub-case 1.4: $a \neq c$ and $b \neq d$. By sub-case 1.3, for $P', P'' \in \mathbb{P}^2$ such that $r_1(P_1') = a \neq b = r_1(P_2')$ and $r_1(P_1'') = c \neq b = r_1(P_2'')$, we have $\varphi_a(P') = \varphi_c(P'')$ and $\varphi_b(P') = \varphi_b(P'')$. W.o.l.o.g. we assume $r_2(P_2'') = d$ and consider $P, \bar{P} \in \mathbb{P}^2$ such that $P = P'$, $P_1'' = \bar{P}_1$ and P_2'' and $\bar{P}_2$ are adjacent with $A(P_2'', \bar{P}_2) = \{b, d\}$. Applying Lemma 4.2 and SSS, we have $\varphi_a(P) = \varphi_c(\bar{P})$ and $\varphi_b(P) = \varphi_d(\bar{P})$.

Case 2: $c = b$. Now we consider following two sub-cases.

Sub-case 2.1: $d = a$. Let $x \neq a, b$. Consider $P, P' \in \mathbb{P}^2$ such that $r_1(P_1) = a \neq b = r_2(P_2)$, $P_2 = P'_2$ and P_1 and P'_1 are adjacent with $A(P_1, P'_1) = \{a, x\}$. Again applying Lemma 4.2 and SSS, we have $\varphi_a(P) = \varphi_x(P')$ and $\varphi_b(P) = \varphi_b(P')$. W.o.l.o.g we assume that $r_2(P'_2) = a$.

Now we consider $P'' \in \mathbb{P}^2$ such that $P'_1 = P''_1$ and P'_2 and P''_2 are adjacent with $A(P'_2, P''_2) = \{a, b\}$. Applying Lemma 4.2 and SSS, we have $\varphi_x(P') = \varphi_x(P'')$ and $\varphi_b(P') = \varphi_a(P'')$. Let P'''_1 be an ordering such that $r_1(P'''_1) = x$ and $r_2(P'''_1) = b$. By Lemma 4.2, $\varphi_x(P'') = \varphi_x(P'''_1, P''_2)$ and $\varphi_a(P'') = \varphi_a(P'''_1, P''_2)$.

Now we consider $\bar{P} \in \mathbb{P}^2$ such that $\bar{P}_2 = P''_2$ and $\bar{P}_1$ and P'''_1 are adjacent with $A(\bar{P}_1, P'''_1) = \{x, b\}$. Applying Lemma 4.2 and SSS, we have $\varphi_x(P'''_1, P''_2) = \varphi_b(\bar{P})$ and $\varphi_a(P'''_1, P''_2) = \varphi_a(\bar{P})$.

Since $\varphi_a(P) = \varphi_b(\bar{P})$ and $\varphi_b(P) = \varphi_a(\bar{P})$, we are done by Lemma 4.2.

Sub-case 2.2: $d \neq a$. By sub-case 2.1, for any $P, P' \in \mathbb{P}^2$ such that $r_1(P_1) = r_1(P'_2) = a \neq b = r_1(P_2) = r_1(P'_1)$ we have $\varphi_a(P) = \varphi_b(P')$ and $\varphi_b(P) = \varphi_a(P')$. For any $y \neq a, b$, w.o.l.o.g we can assume that $r_2(P'_2) = y$.

Now we consider $\bar{P} \in \mathbb{P}^2$ such that $\bar{P}_1 = P'_1$ and $\bar{P}_2$ and P'_2 are adjacent with $A(\bar{P}_2, P'_2) = \{y, a\}$. Applying Lemma 4.2 and SSS, we have $\varphi_b(P') = \varphi_b(\bar{P})$ and $\varphi_a(P') = \varphi_y(\bar{P})$.

Since $\varphi_a(P) = \varphi_b(\bar{P})$ and $\varphi_b(P) = \varphi_y(\bar{P})$ where $y \neq a, b$, we are done by Lemma 4.2. ■

Lemma 4.2 and 4.3 establish that φ is a random dictatorship. ■

The Proof of Theorem 4.2

Proof: It is straightforward that random dictatorship satisfies efficiency and SSS. Therefore we show the only if part. Let $\varphi : \mathbb{P}^n \to \Delta A$ satisfies efficiency, tops-onlyness and SSS. It will be shown that φ is a random dictatorship.

The arguments in the proof closely follow counterparts in Sen (2011). In what follows, we will use induction arguments. Assume that for all integers $k < n$, the following statement is true:

Induction Hypothesis (IH): Assume $m \geq 3$. If $\varphi : \mathbb{P}^k \to \Delta A$ satisfies efficiency, tops-onlyness and SSS, then it is a random dictatorship.

Let $\hat{N} = \{\hat{1}, 3, \ldots, n\}$ be a set of voters where $3, \ldots, n \in N$. A RSCF $g : \mathbb{P}^{n-1} \to \triangle(A)$ for the set of voters $\hat{N}$ is defined as follows: For all $(P_{\hat{1}}, P_3, \ldots, P_n) \in \mathbb{P}^{n-1}$,

$$g(P_{\hat{1}}, P_3, \ldots, P_n) = \varphi(P_1, P_1, P_3, \ldots, P_n)$$

Voter $\hat{1}$ in the RSCF g is obtained by "cloning" voters 1 and 2 in N . Thus if voters 1 and 2 in N have a common ordering P_1, then voter $\hat{1}$ in $\hat{N}$ has ordering $P_{\hat{1}}$.

Lemma **4.4** *The RSCF g is a random dictatorship.*

Proof: It is easy to verify that g satisfies efficiency and tops-only. We will show that g satisfies SSS. Consider any $P \in \mathbb{P}^{n-1}$, $i \in \hat{N}$ and $P'_i \in \mathbb{P}$ such that P_i and P'_i are adjacent with $A(P_i, P'_i) = \{x, y\}$. Let voter i be the voter $\hat{1}$. Since φ satisfies SSS, we have

$$\begin{aligned}\sum_{\{a \in U(P_i,x,y)\}} g_a(P_{\hat{1}}, P_3, \ldots, P_n) &= \sum_{\{a \in U(P_i,x,y)\}} \varphi_a(P_1, P_1, P_3, \ldots, P_n) \\ &= \sum_{\{a \in U(P_i,x,y)\}} \varphi_a(P'_1, P_1, P_3, \ldots, P_n) \\ &= \sum_{\{a \in U(P'_i,x,y)\}} \varphi_a(P'_1, P'_1, P_3, \ldots, P_n) \\ &= \sum_{\{a \in U(P_i,x,y)\}} g_a(P'_{\hat{1}}, P_3, \ldots, P_n)\end{aligned}$$

and

$$\begin{aligned}\sum_{a \in \{x,y\}} g_a(P_{\hat{1}}, P_3, \ldots, P_n) &= \sum_{a \in \{x,y\}} \varphi_a(P_1, P_1, P_3, \ldots, P_n) \\ &= \sum_{a \in \{x,y\}} \varphi_a(P'_1, P_1, P_3, \ldots, P_n) \\ &= \sum_{a \in \{x,y\}} \varphi_a(P'_1, P'_1, P_3, \ldots, P_n) \\ &= \sum_{a \in \{x,y\}} g_a(P'_{\hat{1}}, P_3, \ldots, P_n)\end{aligned}$$

For $i \in \{3, \ldots, n\}$ it is straightforward that above qualities holds because φ satisfies SSS. Hence, g satisfies SSS. Therefore, the IH will imply that g is a random dictatorship. ■

Let $\beta, \beta_3, \ldots, \beta_n$ be the weights associated with the random dictatorship g; i.e. β_i is the weight associated with voter i, $i = 3, \ldots, n$ and β is the weight associated with voter $\hat{1}$.

Lemma **4.5** *Let $P \in \mathbb{P}^n$ be an arbitrary profile. Let $a = r_1(P_1)$ and $b = r_1(P_2)$ and let $\beta^x = \sum_{\{i \in \{3,\ldots,n\}: r_1(P_i)=x\}} \beta_i$ for all $x \in A$.*[2] *Then,*

(i) $\varphi_a(P) = \beta + \beta^a$ *if* $a = b$.

(ii) $\varphi_c(P) = \beta^c$ *for all* $c \neq a = b$.

[2]For simplicity, we will drop the indexation of P and just write β^x instead of $\beta^x(P)$.

(iii) $\varphi_a(P) + \varphi_b(P) = \beta + \beta^a + \beta^b$ *if* $a \neq b$.

Proof: *Proof of Part* (i): By tops-onlyness and the fact that $r_1(P_1) = r_1(P_2) = a$, we have

$$\begin{aligned}\varphi_a(P_1, P_2, P_3, \ldots, P_n) &= \varphi_a(P_1, P_1, P_3, \ldots, P_n)\\ &= g_a(P_{\hat{1}}, P_3, \ldots, P_n)\\ &= \beta + \beta^a\end{aligned}$$

This establishes (i).

Proof of Part (ii): By tops-onlyness and the fact that $r_1(P_1) = r_1(P_2) = a$, we have

$$\begin{aligned}\varphi_c(P_1, P_2, P_3, \ldots, P_n) &= \varphi_c(P_1, P_1, P_3, \ldots, P_n)\\ &= g_c(P_{\hat{1}}, P_3, \ldots, P_n)\\ &= \beta^c\end{aligned}$$

This establishes the proof of part (ii).

Proof of Part (iii): Note that a and b are distinct. Let P_2' and P_2'' be two adjacent orderings where $r_1(P_2') = a = r_2(P_2'')$ and $r_2(P_2') = b = r_1(P_2'')$. In the view of part (i) and (ii), it can be deduced that $\varphi_a(P_1, P_2', P_3, \ldots, P_n) + \varphi_b(P_1, P_2', P_3, \ldots, P_n) = \beta + \beta^a + \beta^b$. By tops-onlyness and SSS, we can conclude that

$$\begin{aligned}\varphi_a(P_1, P_2, P_3, \ldots, P_n) + \varphi_b(P_1, P_2, P_3, \ldots, P_n) &= \varphi_a(P_1, P_2'', P_3, \ldots, P_n) + \varphi_b(P_1, P_2'', P_3, \ldots, P_n)\\ &= \varphi_a(P_1, P_2', P_3, \ldots, P_n) + \varphi_b(P_1, P_2', P_3, \ldots, P_n)\\ &= \beta + \beta^a + \beta^b\end{aligned}$$

This completes the proof of part (iii). ∎

LEMMA **4.6** *Fix* $a, b \in A$, $a \neq b$. *Let* $P \in \mathbb{P}^n$ *be a preference profile where* $r_1(P_1) = a$ *and* $b = r_1(P_2)$ *and let* $\beta^x = \sum_{\{i \in \{3,\ldots,n\} : r_1(P_i) = x\}} \beta_i$ *for all* $x \in A$. *Then, there exist* $0 \leq \beta_1^a, \beta_2^b \leq \beta$, $\beta_1^a + \beta_2^b = \beta$ *such that* $\varphi_a(P) = \beta_1^a + \beta^a$, $\varphi_b(P) = \beta_2^b + \beta^b$ *and* $\varphi_c(P) = \beta^c$ *for all* $c \neq a, b$.

Proof: Fix an alternative $d \neq a, b$. Let $P^* \in \mathbb{P}^n$ be such that $r_1(P_1^*) = a$, $r_1(P_2^*) = b$ and for all $j \in \{3, \ldots, n\}$, $r_1(P_j^*) = d$. In the view of Part (iii) of Lemma 4.5, we have that $\varphi_a(P^*) + \varphi_b(P^*) = \beta$. W.l.o.g we assume that $\varphi_a(P^*) = \beta_1^a$ and $\varphi_b(P^*) = \beta_2^b$ where $0 \leq \beta_1^a, \beta_2^b \leq \beta$ and $\beta_1^a + \beta_2^b = \beta$. Now we show the following claims.

Claim **4.4** *Let $r_1(P_j) \in \{a, b\}$ for all $j \in \{3, \ldots, n\}$. Then $\varphi_a(P) = \beta_1^a + \beta^a$, $\varphi_b(P) = \beta_2^b + \beta^b$ and $\varphi_c(P) = 0$ for all $c \neq a, b$.*

Proof: By Proposition 4.2, $\varphi_c(P) = 0$ for all $c \neq a, b$. We complete the proof by showing that $\varphi_a(P) = \beta_1^a + \beta^a$ and $\varphi_b(P) = \beta_2^b + \beta^b$. W.o.l.g. we assume that $r_1(P_j) = a$ for all $j \in \{3, \ldots, k\}$ and $r_1(P_j) = b$ for all $j \in \{k+1, \ldots, n\}$, $k \leq n$. We assume for contradiction that $\varphi_a(P) \neq \beta_1^a + \beta^a$.

Let $\bar{P} \in \mathbb{P}^n$, such that $r_1(\bar{P}_i) = r_1(P_i)$ for all for all $i \in N$ and $r_2(\bar{P}_i) = d$ for all $i \in \{3, \ldots, n\}$. By tops-onlyness, $\varphi(\bar{P}) = \varphi(P)$. For all $i \in \{3, \ldots, k\}$, let $\bar{P}_i$ and $\hat{P}_i$ be two adjacent ordering where $r_1(\bar{P}_i) = r_2(\hat{P}_i) = a$ and $r_2(\bar{P}_i) = r_1(\hat{P}_i) = d$. Also, For all $i \in \{k+1, \ldots, n\}$, let $\bar{P}_i$ and $\hat{P}_i$ be two adjacent ordering where $r_1(\bar{P}_i) = r_2(\hat{P}_i) = b$ and $r_2(\bar{P}_i) = r_1(\hat{P}_i) = d$. By SSS and the fact that $\varphi_a(\bar{P}) \neq \beta_1^a + \beta^a$ and $\varphi_d(\bar{P}) = 0$, we have

$$
\begin{aligned}
\varphi_a(\bar{P}) + \varphi_d(\bar{P}) &= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) + \varphi_d(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) \\
&\vdots \\
&= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) + \varphi_d(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) \\
&\neq \beta_1^a + \beta^a \\
&= \beta_1^a + \sum_{i=3}^{k} \beta_i
\end{aligned}
$$

By Proposition 4.2 and part (iii) of Lemma 4.5,

$$\varphi_d(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) = \sum_{i=3}^{k} \beta_i$$

Therefore, we can conclude that

$$\varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) \neq \beta_1^a$$

By Proposition 4.2 and SSS,

$$
\begin{aligned}
\varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) &= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \hat{P}_{k+1}, \ldots, \bar{P}_n) \\
&\vdots \\
&= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \hat{P}_{k+1}, \ldots, \hat{P}_n) \\
&\neq \beta_1^a
\end{aligned}
$$

However, by tops-onlyness, $\varphi_a(P^*) = \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \hat{P}_{k+1}, \ldots, \hat{P}_n) = \beta_1^a$ - a contradiction. Hence, $\varphi_a(P) = \beta_1^a + \beta^a$. Since $\varphi_a(P) + \varphi_b(P) = 1$, we have $\varphi_b(P) = \beta_1^b + \beta^b$. ∎

Claim **4.5** *Let $|\{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\}| = 1$. Then, $\varphi_a(P) = \beta_1^a + \beta^a$, $\varphi_b(P) = \beta_2^b + \beta^b$ and $\varphi_c(P) = \beta^c$ for all $c \neq a, b$.*

Proof: Let $\{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\} = e$. By Proposition 4.2, $\varphi_c(P) = 0$ for all $c \neq a, b, e$. W.l.o.g we assume that at P, $r_1(P_i) = e$ for $i \in \{3, \ldots, k\}$, $r_1(P_i) = a$ for $i \in \{k+1, \ldots, l\}$ and $r_1(P_i) = b$ for $i \in \{l+1, \ldots, n\}$, $k \leq l \leq n$. By Lemma 4.5 and Proposition 4.2, $\varphi_e(P) = \beta^e = \sum_{i=3}^{k} \beta_i$.

Next we show that $\varphi_a(P) = \beta_1^a + \beta^a = \beta_1^a + \sum_{i=k+1}^{l} \beta_i$. We assume for contradiction that $\varphi_a(P) \neq \beta_1^a + \sum_{i=k+1}^{l} \beta_i$.

Let $\bar{P} \in \mathbb{P}^n$, such that $r_1(\bar{P}_i) = r_1(P_i)$ for all for all $i \in N$ and $r_2(\bar{P}_i) = a$ for all $i \in \{3, \ldots, k\}$. By tops-onlyness, $\varphi(\bar{P}) = \varphi(P)$. For all $i \in \{3, \ldots, k\}$, let $\bar{P}_i$ and $\hat{P}_i$ be two adjacent ordering where $r_1(\bar{P}_i) = r_2(\hat{P}_i) = e$ and $r_2(\bar{P}_i) = r_1(\hat{P}_i) = a$. By SSS and the fact that $\varphi_a(\bar{P}) \neq \beta_1^a + \sum_{i=k+1}^{l} \beta_i$ and $\varphi_e(\bar{P}) = \sum_{i=3}^{k} \beta_i$, we have

$$\begin{aligned}
\varphi_a(\bar{P}) + \varphi_e(\bar{P}) &= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) + \varphi_e(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) \\
&\vdots \\
&= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) + \varphi_e(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) \\
&\neq \beta_1^a + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

The above inequality contradicts Claim 4.4. Therefore, $\varphi_a(P) = \beta_1^a + \beta^a = \beta_1^a + \sum_{i=k+1}^{l} \beta_i$.

Since, by Lemma 4.5, $\varphi_a(P) + \varphi_b(P) = \beta + \sum_{i=k+1}^{n} \beta_i$, we have $\varphi_b(P) = \beta_1^b + \sum_{i=l+1}^{n} \beta_i$. This completes the proof of the claim. ∎

Finally, we complete the proof of the lemma by induction on $|\{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\}|$.

Induction Hypothesis (IH): Let $|\{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\}| = k \in \{1, \ldots, m-3\}$. Then, $\varphi_a(P) = \beta_1^a + \beta^a$, $\varphi_b(P) = \beta_2^b + \beta^b$ and $\varphi_c(P) = \beta^c$ for all $c \neq a, b$.

We will show that if $|\{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\}| = k + 1$, then $\varphi_a(P) = \beta_1^a + \beta^a$, $\varphi_b(P) = \beta_2^b + \beta^b$ and $\varphi_c(P) = \beta^c$ for all $c \neq a, b$.

Note that by Proposition 4.2, $\varphi_c(P) = 0$ if $c \notin \{r_1(P_3), \ldots, r_1(P_n)\} \cup \{a, b\}$. First we show that $\varphi_c(P) = \beta^c$ for all $c \in \{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\}$. We assume for contradiction that there exists an alternative $c' \in \{r_1(P_3), \ldots, r_1(P_n)\} \setminus \{a, b\}$ such that $\varphi_{c'}(P) \neq \beta^{c'}$. We consider the following two cases.

Case I: $\varphi_{c'}(P) > \beta^{c'}$. In the view of part (iii) of Lemma 4.5, either (i) $\varphi_a(P) \geq \beta_1^a + \beta^a$ and $\varphi_b(P) \leq \beta_2^b + \beta^b$ or (ii) $\varphi_a(P) \leq \beta_1^a + \beta^a$ and $\varphi_b(P) \geq \beta_2^b + \beta^b$. First we will consider the case where (i) $\varphi_a(P) \geq \beta_1^a + \beta^a$ and $\varphi_b(P) \leq \beta_2^b + \beta^b$.

W.o.l.g. we assume that $r_1(P_j) = c'$ for all $j \in \{3, \ldots, k\}$ and $r_1(P_j) = a$ for all $j \in \{k+1, \ldots, l\}$, $k \leq l \leq n$. Let $\bar{P} \in \mathbb{P}^n$, such that $r_1(\bar{P}_i) = r_1(P_i)$ for all for all $i \in N$ and $r_2(\bar{P}_i) = a$ for all $i \in \{3, \ldots, k\}$. By tops-onlyness and the fact that $\varphi_{c'}(P) > \beta^{c'}$ and $\varphi_a(P) \geq \beta_1^a + \beta^a$, we have

$$\begin{aligned}
\varphi_{c'}(\bar{P}) + \varphi_a(\bar{P}) &= \varphi_{c'}(P) + \varphi_a(P) \\
&> \beta_1^a + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

For all $i \in \{3, \ldots, k\}$, let $\bar{P}_i$ and $\hat{P}_i$ be two adjacent ordering where $r_1(\bar{P}_i) = r_2(\hat{P}_i) = c'$ and $r_2(\bar{P}_i) = r_1(\hat{P}_i) = a$. By SSS and the induction hypothesis, we have

$$\begin{aligned}
\varphi_{c'}(\bar{P}) + \varphi_a(\bar{P}) &= \varphi_{c'}(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) + \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) \\
&\vdots \\
&= \varphi_{c'}(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) + \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) \\
&= \beta_1^a + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

This leads to a contradiction. Similarly, in the case where $\varphi_a(P) \leq \beta_1^a + \beta^a$ and $\varphi_b(P) \geq \beta_2^b + \beta^b$, a contradiction can be established by replacing the role of a in (i) by b.

Case II: $\varphi_{c'}(P) < \beta^{c'}$. In the view of part (iii) of Lemma 4.5, either (i) $\varphi_a(P) \geq \beta_1^a + \beta^a$ and $\varphi_b(P) \leq \beta_2^b + \beta^b$ or (ii) $\varphi_a(P) \leq \beta_1^a + \beta^a$ and $\varphi_b(P) \geq \beta_2^b + \beta^b$. First we will consider the case where (i) $\varphi_a(P) \geq \beta_1^a + \beta^a$ and $\varphi_b(P) \leq \beta_2^b + \beta^b$.

W.o.l.g. we assume that $r_1(P_j) = c'$ for all $j \in \{3, \ldots, k\}$ and $r_1(P_j) = b$ for all $j \in \{k+1, \ldots, l\}$, $k \leq l \leq n$. Let $\bar{P} \in \mathbb{P}^n$, such that $r_1(\bar{P}_i) = r_1(P_i)$ for all for all $i \in N$ and $r_2(\bar{P}_i) = b$ for all $i \in \{3, \ldots, k\}$. By tops-onlyness and the fact that $\varphi_{c'}(P) < \beta^{c'}$ and $\varphi_b(P) \leq \beta_2^b + \beta^b$, we have

$$\begin{aligned}
\varphi_{c'}(\bar{P}) + \varphi_b(\bar{P}) &= \varphi_{c'}(P) + \varphi_b(P) \\
&< \beta_2^b + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

For all $i \in \{3, \ldots, k\}$, let $\bar{P}_i$ and $\hat{P}_i$ be two adjacent ordering where $r_1(\bar{P}_i) = r_2(\hat{P}_i) = c'$

and $r_2(\bar{P}_i) = r_1(\hat{P}_i) = b$. By SSS and the induction hypothesis, we have

$$\begin{aligned}
\varphi_{c'}(\bar{P}) + \varphi_b(\bar{P}) &= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) + \varphi_b(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) \\
&\vdots \\
&= \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) + \varphi_b(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) \\
&= \beta_2^b + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

This leads to a contradiction. In the case where $\varphi_a(P) \leq \beta_1^a + \beta^a$ and $\varphi_b(P) \geq \beta_2^b + \beta^b$, a contradiction can be established by replacing the role of b in (i) by a.

In the view of cases I and II, we have that $\varphi_c(P) = \beta^c$ for all $c \neq a, b$.

Next we show that $\varphi_a(P) = \beta_1^a + \beta^a$. We assume for contradiction that $\varphi_a(P) \neq \beta_1^a + \beta^a$. W.o.l.g. we assume that $r_1(P_j) = d' \neq a, b$ for all $j \in \{3, \ldots, k\}$ and $r_1(P_j) = a$ for all $j \in \{k+1, \ldots, l\}$, $k \leq l \leq n$. Let $\bar{P} \in \mathbb{P}^n$, such that $r_1(\bar{P}_i) = r_1(P_i)$ for all for all $i \in N$ and $r_2(\bar{P}_i) = a$ for all $i \in \{3, \ldots, k\}$. In the view of what we have shown earlier and tops-onlyness, we have

$$\begin{aligned}
\varphi_{d'}(\bar{P}) + \varphi_a(\bar{P}) &= \varphi_{d'}(P) + \varphi_a(P) \\
&\neq \beta_1^a + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

For all $i \in \{3, \ldots, k\}$, let $\bar{P}_i$ and $\hat{P}_i$ be two adjacent ordering where $r_1(\bar{P}_i) = r_2(\hat{P}_i) = d'$ and $r_2(\bar{P}_i) = r_1(\hat{P}_i) = a$. By SSS and the induction hypothesis, we have

$$\begin{aligned}
\varphi_{d'}(\bar{P}) + \varphi_a(\bar{P}) &= \varphi_{d'}(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) + \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \bar{P}_n) \\
&\vdots \\
&= \varphi_{d'}(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) + \varphi_a(\bar{P}_1, \bar{P}_2, \hat{P}_3, \ldots, \hat{P}_k, \bar{P}_{k+1}, \ldots, \bar{P}_n) \\
&= \beta_1^a + \sum_{i=3}^{l} \beta_i
\end{aligned}$$

This leads to a contradiction. therefore, $\varphi_a(P) = \beta_1^a + \beta^a$. Finally, by part (iii) of Lemma 4.5, we have $\varphi_b(P) = \beta_2^b + \beta^b$. This completes the proof of the lemma. ■

The proof is now completed by considering two mutually exhaustive cases.

Case I: $\beta > 0$

Fix $(P_3, \ldots, P_n) \in \mathbb{P}^{n-2}$ and let $\beta^x = \sum_{\{i \in \{3,\ldots,n\}: r_1(P_i) = x\}} \beta_i$ for all $x \in A$. We define the function $h : \mathbb{P}^2 \to \mathbb{R}^m$ below: for all $(P_1, P_2) \in \mathbb{P}^2$ and $a \in A$,

$$h_a(P_1, P_2) = \frac{1}{\beta}[\varphi_a(P_1, P_2, P_3 \ldots, P_n) - \beta^a]$$

Lemma **4.7** *The function h is a RSCF and satisfies efficiency and SSS.*

Proof: Pick an arbitrary profile $(P_1, P_2) \in \mathbb{P}^2$. Let $a \in A$. If $r_1(P_1) = r_1(P_2) = a$, then $\varphi_a(P_1, P_2, P_3, \ldots, P_n) = \beta + \beta^a$ according to Lemma 4.5 part (i). Hence $h_a(P_1, P_2) = 1$. Suppose $r_1(P_1) = a \neq b = r_1(P_2)$. From Lemma 4.6, $\varphi_a(P_1, P_2, P_3, \ldots, P_n) \geq \beta^a$ and $\varphi_b(P_1, P_2, P_3, \ldots, P_n) \geq \beta^b$. Hence $h_a(P_1, P_2) \geq 0$ and $h_b(P_1, P_2) \geq 0$. If $a \notin \{r_1(P_1) \cup r_1(P_2)\}$, then $\varphi_a(P_1, P_2, P_3, \ldots, P_n) = \beta^a$. Hence $h_a(P_1, P_2) = 0$, i.e $h_a(P_1, P_2) \geq 0$ for all $a \in A$. Note also that $\sum_{a\in A} \varphi_a(P_1, P_2, P_3, \ldots, P_n) = \beta + \sum_{a\in A} \beta^a = 1$, so that $\sum_{a\in A} h_a(P_1, P_2) = 1$. Therefore, h is a RSCF.

First we show that h satisfies efficiency. Pick an arbitrary profile $(P_1, P_2) \in \mathbb{P}^2$. Let $x, y \in A$ such that xP_iy for all $i \in \{1, 2\}$. From Lemma 4.6, $\varphi_b(P_1, P_2, P_3, \ldots, P_n) = \beta^y$. Hence $h_y(P_1, P_2) = 0$. This concludes that h is efficient.

Next we show that h satisfies SSS. Pick an arbitrary profile $(P_1, P_2) \in \mathbb{P}^2$. Let P_i and P_i' be two adjacent orderings where $i \in \{1, 2\}$. W.o.l.g we assume that $i = 1$. Let $A(P_1, P_1') = \{x, y\}$. We will show that

(i). $\displaystyle\sum_{\{a\in U(P_i,x,y)\}} h_a(P_1, P_2) = \sum_{\{a\in U(P_i',x,y)\}} h_a(P_i', P_2)$.

(ii). $\displaystyle\sum_{a\in\{x,y\}} h_a(P_1, P_2) = \sum_{a\in\{x,y\}} h_a(P_1', P_2)$.

Note that for any $a \in A$, we have

$$\varphi_a(P_1, P_2, P_3, \ldots, P_n) = \beta \cdot h_a(P_1, P_2) + \beta^a$$

and

$$\varphi_a(P_1', P_2, P_3, \ldots, P_n) = \beta \cdot h_a(P_1', P_2) + \beta^a$$

Since φ satisfies SSS and $\beta > 0$, it can be easily verified that equalities in (i) and (ii) hold. Hence, h satisfies SSS. ■

Since h satisfies efficiency and SSS, it follows from Theorem 4.1 that h is a random dictatorship. Assume that the weights associated with h are γ_1 and γ_2 for voters 1 and 2 respectively. It follows from the definition of h that for all $P_1, P_2 \in \mathbb{P}$ and $a \in A$,

$$\varphi_a(P_1, P_2, P_3, \ldots, P_n) = \sum_{\{i\in\{1,2\}:r_1(P_i)=a\}} \beta\gamma_i + \sum_{\{i\in\{3,\ldots,n\}:r_1(P_i)=a\}} \beta_i$$

Therefore φ is a random dictatorship with weights $\beta\gamma_1, \beta\gamma_2, \beta_3, \ldots, \beta_n$ if the weights for the random dictatorship do not depend on the initial choice of the profile $(P_3, \ldots, P_n)$ for voters $3, \ldots, n$. Lemma 4.6 establishes that this is indeed the case.

This completes the proof of random dictatorship in Case I.

Case II: $\beta = 0$.

Let $P_1, P_2 \in \mathbb{P}$. Applying Lemma 4.5 and 4.6, it follows that $\varphi_a(P_1, P_2, P_3, \ldots, P_n) = \beta^a$ for all $a \in A$. But this implies that φ is a random dictatorship with weights $\beta_1 = \beta_2 = 0$ and $\beta_i, i = 3, \ldots, n$. This concludes the proof. ■

Chapter 5

A characterization of the status quo rule in the binary social choice model

5.1 Introduction

This chapter considers the following a model in which there is a finite set of voters who have to choose an alternative from the set of two alternatives, based on their preferences over the alternatives. Each voter can have one of the three preferences - one alternative can be strictly preferred to the other or they could be indifferent to each other. A social choice function (or simply, a rule) assigns an alternative to each profile. Our goal in this chapter is to characterize a salient rule in this model, the *status quo rule*.

The status quo rule identifies one of the two alternatives as the status quo alternative. The rule picks this alternative at all profiles except the one where all voters rank the non-status quo alternative strictly better than the status quo alternative. It is a conservative rule which is "almost" constant. However, it is an appealing rule in certain circumstances where change from the status quo can impose losses on a large number of voters. Examples of such policies in India in recent years have been the Citizen Amendment Act, the Goods and Services tax, the demonetization policy (2016) and the four-year undergraduate program at Delhi University. Since these decisions have irreversible and long-lasting consequences, they typically require super-large majorities to be passed as a law. Other examples of such decisions are Constitutional Amendments and jury decisions to acquit or convict.

We use three axioms for our characterization. Two of these properties, ontoness and strategy-proofness are well-known in the axiomatic literature. The third one is a new axiom introduced by us, which we call *Positive Welfare Association* or PWA. To understand the axiom, consider a profile where a particular voter, say i is indifferent. Suppose i changes her preference from indifference to a strict preference. The new outcome differs from the earlier one and is i's strictly preferred outcome in the new preference. Then, PWA requires all other voters not to be made worse-off at the new profile. The justification for the axiom is the principle that a change that does not make a voter unhappy (this is well-defined in a two-alternative model), must not make anyone else unhappy.

The PWA axiom is closely related to the various solidarity axioms in the fairness literature. The most relevant solidarity axiom for us is the *Welfare Dominance Under Preference Replacement* or WD. The axiom has been extensively analyzed in the binary choice model in Harless (2015). Suppose the preference ordering of a single voter changes. The WD axiom requires the welfare of all other voters to move in the same direction, i.e they must all either (weakly) gain or lose. The PWA axiom differs from WD in the respect that it relates the welfare of the voter who initiates the change with the rest of the voters. The WD axiom also allows for the initiating voter to be made better-off while all the remaining voters are made worse-off. This may be natural in allocation models where the voter whose preferences change, obtains an object that is highly valued by all other voters. This argument is less compelling in a public good model such as ours. We believe therefore that our axiom is entirely consistent with the notion of solidarity. The PWA axiom is key to our result and cannot be replaced by WD in the characterization. We also show that our characterization is tight by providing examples of non-status quo rules that satisfy all but one of the axioms.

5.1.1 Related Literature

The binary model was introduced by May (1952). This paper characterizes simple majority voting rules using anonymity, monotonicity, and neutrality. Fishburn (2015) provides a characterization of anonymous, neutral and monotonic rules which are further analyzed in Llamazares (2013). As mentioned earlier, Harless (2015) provides a complete characterization of rules satisfying WD. Recently, Lahiri and Pramanik (2019) characterize the class of onto, anonymous and strategy-proof rules. On the other hand, Moulin (1987) studies the binary model with money and quasi-linear utilities. The status quo rule in the Arrowian framework is studied in Bossert and Sprumont (2014), Harless (2016) and Athanasoglou (2019). Gordon (2007) provides conditions which guarantee the existence of a status quo alternative.

The rest of the chapter is organized as follows. Section 5.2 introduces the basic notation and definitions. Section 5.3 contains the main result. Section 5.4 contains discussion of the result. Section 5.5 concludes the chapter. The proof of our main result is provided in the Appendix (Section 5.6).

5.2 The Framework

Let $A = \{a, b\}$ and $N = \{1, ..., n\}$, $n \geq 2$ denote the set of two alternatives and the set of voters respectively. There are three possible preference orderings over A: (i) a is better than b, denoted by R^a, (ii) b is better than a, denoted by R^b and (iii) a and b are indifferent to each other, denoted by R^{ab}. Every voter $i \in N$ has a preference relation R_i over A where $R_i \in \{R^a, R^b, R^{ab}\} = \mathbb{R}$. We denote aP_ib when $R_i = R^a$, aR_ib when $R_i \in \{R^a, R^{ab}\}$ and aI_ib when $R_i = R^{ab}$. Similarly notation for $R_i = R^b$ or $R_i \in \{R^b, R^{ab}\}$.

A preference profile is a list $R = (R_1, R_2, \ldots, R_n) \in \mathbb{R}^n$ of voter preferences. A profile $R = (R_1, R_2, \ldots, R_n)$ is also denoted by (R_i, R_{-i}) for $i \in N$, or more generally, (R_S, R_{-S}) for $S \subseteq N$.

For every profile $R \in \mathbb{R}^n$, let $N_a(R)$, $N_b(R)$ and $N_{ab}(R)$ be the set of voters who prefer a to b, b to a and indifferent between a and b at R. A profile R can therefore be represented by the triple $N_a(R)$, $N_b(R)$ and $N_{ab}(R)$. Some standard definitions are below:

Definition **5.1** *A social choice function (SCF) (or simply a rule) f is a mapping $f : \mathbb{R}^n \to A$.*

Definition **5.2** *A rule f is onto if there exist $R', R'' \in \mathbb{R}$ such that $f(R') = a$ and $f(R'') = b$.*

Note that if a rule is not onto, it must be constant i.e. it picks the same alternative at every profile.

Definition **5.3** *A rule f is strategy-proof if, for all $i \in N$, for all $R_i, R'_i \in \mathbb{R}$, we have $f(R_i, R_{-i})R_if(R'_i, R_{-i})$ for all $R_{-i} \in \mathbb{R}^{n-1}$.*

Strategy-proofness is a standard requirement for social choice functions. It considers an environment where voters' preferences are private information. If a rule is strategy-proof then no individual can obtain a strictly better alternative by misrepresenting her preferences for any possible announcement of the preferences by other individuals.

Definition **5.4** *A SCF f satisfies positive welfare association (PWA) if, for any $R \in \mathbb{R}^n$, for any $i \in N_{ab}(R)$ and any $R'_i \in \{R^a, R^b\}$, if $f(R'_i, R_{-i}) \neq f(R)$ and $f(R'_i, R_{-i})\ R'_i\ f(R)$, then $f(R'_i, R_{-i})\ R_j\ f(R)$ for each $j \in N \backslash \{i\}$.*

Suppose a voter changes her preference from indifference to a strict preference. This results in a change in the alternative chosen with the strictly preferred alternative chosen in the new profile. The voter moves from being indifferent to being "strictly" happy. Then PWA requires all other voters not to be made worse-off. The axiom is clearly related to the solidarity axioms in the fairness literature. We discuss this issue further in Section 5.4.

5.3 The Main Result

We formally introduce the status quo rule below.

Definition **5.5** *A rule is a status quo rule, if there exists an alternative $x \in A$ such that for all $R \in \mathbb{R}^n$,*

$$f(R) = \begin{cases} y & \text{if } N_y(R) = N \text{ where } \{y\} = A \setminus \{x\} \\ x & \text{otherwise} \end{cases}$$

We say that $x \in A$ is the status-quo alternative. We denote by f^x the status-quo rule with status-quo alternative x. It is a simple rule gives status quo alternative x at all profiles except when all voters agree on best alternative being y. Our main result shows that the status quo rule can be characterized by PWA in conjunction with onto and strategy-proofness .

Theorem **5.1** *A rule is onto, strategy-proof and satisfies positive welfare association if and only if it is the status quo rule.*

The proof of the theorem is provided in the Appendix. In the next section we discuss various aspects of result and that of our key property we have introduced in this chapter.

5.4 Discussion

In this section explore the relationship between a solidarity condition and PWA. In addition we show independence of our axioms in our characterization and also provide an alternative characterization of the status quo rule.

5.4.1 Relationship between positive welfare association and solidarity

A solidarity property used widely in the fairness literature (see Thomson (1999) for a survey) is *welfare dominance under preference replacement* (or simply welfare dominance WD). According to the axiom any change in the outcome resulting from change in the preference of a voter must impact all other voters in the same direction i.e. they must all be made (weakly) better-off or all they must be made (weakly) worse-off.

Definition **5.6** *A SCF f satisfies welfare dominance under preference replacement (or simply welfare dominance WD) if for any $R \in \mathbb{R}^n$, $i \in N$ and $R'_i \in \mathbb{R}$, either $f(R) R_j f(R'_i, R_{-i})$ for all $j \in N \setminus \{i\}$ or $f(R'_i, R_{-i}) R_j f(R)$ for all $j \in N \setminus \{i\}$.*

As mentioned in the Introduction, Harless (2015) employs this axiom in our model, that of binary social choice. We believe that our PWA axiom captures the idea of solidarity better than WD in our model for two reasons. The first is that WD excludes the voter making the preference change from consideration. The second is that it allows for the possibility that the welfare of the voter whose preference changes, improves but makes others worse-off.

We now discuss the relationship between the WD and PWA axioms. At first glance it seems that the two are independent. The PWA axiom is concerned only with profiles where an indifferent voter changes to some strict preference ordering while WD axiom applies to all profiles. On the other hand PWA requires all other voters to be weakly better-off while WD says all other voters are either weakly better-off or worse-off. Thus the antecedent of PWA is weaker and its consequent stronger than WD. Indeed we show through examples that the axioms are independent and neither implies the other. The first examples shows that WD does not imply PWA.

EXAMPLE **5.1** Consider the SCF f that selects b at profiles where at least some voters strictly prefers b to a and others are indifferent, otherwise it selects a. For any $x, y \in A$ and $R \in \mathbb{R}^n$:

$$f(R) = \begin{cases} a & \text{if } \emptyset \neq N_a(R) \subsetneq N \\ b & \text{if } \emptyset \neq N_a(R) \subsetneq N \\ x & \text{if } N_a(R) \neq \emptyset \text{ and } N_b(R) \neq \emptyset \\ y & \text{if } N_{ab}(R) = N \end{cases}$$

Consider $R \in \mathbb{R}^n$ where $R_1 = R^{ab}$ and $R_i = R^b$ for all $i \in N\backslash\{1\}$ and preference ordering $R_1' = R^a$. According to the rule we have $f(R_1', R_{-i})\ P_1'\ f(R)$ but $f(R)\ P_i\ f(R_1', R_{-1})$ for all $i \in N\backslash\{1\}$. Thus f violates PWA. It is easy to see that it satisfies the WD. In Harless (2015), this rule is called “Consensus rule with default x and y”. It belongs to a larger class of rules characterized solely with the axiom of WD.

The next example demonstrates that PWA does not imply WD.

EXAMPLE **5.2** Consider the following anti-dictatorial rule f (where voter 1 is the anti-dictator). For each $R \in \mathbb{R}^n$,

$$f(R) = \begin{cases} b & \text{if } R_1 = R^a \\ a & \text{if } R_1 = R^b \\ a & \text{if } R_1 = R^{ab} \end{cases}$$

Consider the profile R where $R_2 = R^a$, $R_3 = R^b$ and $R_i = R^{ab}$ for all $i \in N\backslash\{2, 3\}$ and a preference ordering $R_1' = R^a$. The rule selects $f(R) = a$ and $f(R_1', R_{-1}) = b$. Thus we have $f(R)\ P_2\ f(R_1', R_{-1})$ while $f(R_1', R_{-1})\ P_3\ f(R)$. Clearly WDis violated.

We only need to focus on voter 1 to check whether PWA is satisfied - other voters cannot change the outcome by changing their preference ordering. Take any profile R where $R_1 = R^{ab}$. For any preference ordering R'_1 either $f(R) = f(R'_1, R_{-1})$ or $f(R)\ P'_1\ f(R'_1, R_{-1})$. Thus the antecedent of PWA is false; thus PWA holds vacuously.

Next we demonstrate that PWA cannot be replaced by WD in the characterization of the status quo rule. We construct an example which satisfies WD along with strategy-proofness and ontoness but does not satisfy PWA.

EXAMPLE **5.3** Consider the SCF f which selects a for all profiles where at least one voter strictly prefers a to b and for the remaining profiles, selects alternative b. For all $R \in \mathbb{R}^n$,

$$f(R) = \begin{cases} a & \text{if } N_a(R) \neq \emptyset \\ b & \text{otherwise} \end{cases}$$

This SCF is not a status quo rule. The status quo cannot be a because at the profile where all voters are indifferent f picks b. The status quo cannot be b either because at the profile where a single voter prefers a to b while others b to a, it picks a. This rule is clearly onto. In order to see that it satisfies strategy-proofness consider any profile R, voter i and an ordering R'_i such that $f(R) \neq f(R'_i, R_{-i})$ else it is trivially satisfied. This is possible only when $N_a(R_{-i}) = \emptyset$. In this case $f(R)\ R_i\ f(R'_i, R_{-i})$. Thus strategy-proofness is satisfied.

Consider the profile $R = (R^{ab}, R^b, \ldots, R^b)$ and the preference ordering $R'_1 = R^a$. According to the rule, we have $a = f(R'_1, R_{-1})\ P'_1\ f(R) = b$ but $b = f(R)\ P_j\ f(R'_1, R_{-1}) = a$ for all $j \in N \backslash \{1\}$. Thus it violates PWA.

On the other hand it satisfies WD. Consider any profile R, voter i and an ordering R'_i such that $f(R) \neq f(R'_i, R_{-i})$. As just explained we have either $f(R)\ R_j\ f(R'_i, R_{-i})$ or $f(R'_i, R_{-i})\ R_j\ f(R)$ for all $j \in N \backslash \{i\}$.

An alternative PWA axiom is the following.

DEFINITION **5.7** *A SCF f satisfies minus positive welfare association (PWA$^-$) if, for any $R \in \mathbb{R}^n$, for any $i \in N_{ab}(R)$ and any $R'_i \in \{R^a, R^b\}$, if $f(R'_i, R_{-i}) \neq f(R)$ and $f(R)\ R'_i\ f(R'_i, R_{-i})$, then $f(R)\ R_j\ f(R'_i, R_{-i})$ for each $j \in N \backslash \{i\}$.*

Suppose an indifferent voter changes her preference ordering to a strict ordering. Suppose further that this change leads to a change in the outcome. The new outcome is voter i's worst alternative according to i's new preference ordering. Then PWA$^-$ requires all other voters to be weakly worse-off. The PWA$^-$ axiom is the symmetric counterpart of the PWA axiom.[1] However it is not useful for our analysis. This is so because it an immediate implication of strategy-proofness.

[1] We can show that PWA and PWA$^-$ are independent. To see this, the Example 5.2 satisfies the former but violates the latter. On the other hand the Example 5.6 satisfies latter but fails the former.

5.4.2 Independence of the Axioms

In this subsection we show that our characterization is tight. We provide examples of non-status quo rules that satisfy all but one of the axioms which is specified in parentheses.

EXAMPLE **5.4 (Onto)** The constant rule satisfies strategy-proofness and PWA but fails ontoness.

EXAMPLE **5.5 (Strategy-proofness)** Consider the SCF f that selects a for all profiles except when everyone is indifferent, rule selects alternative b. This rule satisfies ontoness and PWA but does not satisfy strategy-proofness. For all $R \in \mathbb{R}^n$,

$$f(R) = \begin{cases} b & \text{if } N_{ab}(R) = N \\ a & \text{otherwise} \end{cases}$$

EXAMPLE **5.6 (PWA)** Consider the following dictatorial rule f (where voter 1 is the dictator). For each $R \in \mathbb{R}^n$,

$$f(R) = \begin{cases} a & \text{if } R_1 = R^a \\ b & \text{if } R_1 = R^b \\ a & \text{if } R_1 = R^{ab} \end{cases}$$

It is immediate to see that it satisfies strategy-proofness and onto. If voter 1 changes her preference ordering from a profile where she is indifferent while some voters prefer a over b to a preference ordering where she prefers b. This shift makes some voters worse-off. Thus violating PWA.

5.4.3 An alternative characterization

In this section, we consider a modified framework where voters can report either one of the two strict preferences or abstain from voting. A voter's report $R_i \in \bar{\mathbb{R}} = \{R^a, R^b, R^\emptyset\}$ where $R^\emptyset$ signifies abstention. We characterize the status-quo rule in terms of ontoness, the participation property and PWA.

DEFINITION **5.8** *A SCF $f : \bar{\mathbb{R}}^n \to A$ satisfies Participation (PART) if for any $i \in N$, $R_{-i} \in \mathbb{R}^{n-1}$, $R_i \in \{R^a, R^b\}$ and $R'_i = R^\emptyset$ we have $f(R_i, R_{-i}) \neq f(R'_i, R_{-i}) \Rightarrow f(R) P_i f(R'_i, R_{-i})$.*

The Participation Property was introduced in Moulin (1991) to avoid the no-show paradox. It prevents a voter from manipulating by abstention i.e. no voter can gain by abstaining. In general it is weaker than strategy-proofness. However in this framework the two conditions are equivalent as is shown in Lahiri and Pramanik (2019).

PROPOSITION **5.1** *A SCF $f : \mathbb{R}^n \to A$ satisfies PART if and only if it satisfies strategy-proofness.*

In view of Proposition 5.1, we can replace strategy-proofness by PART in Theorem 5.1.

THEOREM **5.2** *Let $n \geq 2$. A rule f satisfies ontoness, PART and PWA if and only if $f \equiv f^x$ where $x \in A$.*

5.5 Conclusion

In this chapter, we have provided a characterization of the status quo rule in binary social choice model. The key to the characterization is a new axiom, positive welfare association. We also show that our characterization is tight.

5.6 Appendix

We provide a proof of Theorem 5.1.

Proof: **Only if.** Let f be a onto, strategy-proof rule which satisfies PWA. We show that $f \equiv f^x$ for some $x \in A$.

Fix a profile $R' \in \mathbb{R}^n$ such that $R'_1 = R^a$, $R'_2 = R^b$ and $R'_i = R^{ab}$ for all $iN \backslash \{1, 2\}$. There are only two cases to consider: $f(R') = a$ and $f(R') = b$. We only consider $f(R') = a$. The case where $f(R') = b$ is identical and is therefore omitted.

Let $f(R') = a$. Now we will show that f is a status quo rule with a being the status quo i.e. $f \equiv f^a$. To show this, there are four types of profiles to consider: (i) $N_{ab} = N$ i.e. everyone is indifferent (ii) $N_b = \emptyset$ i.e. no voter prefers b to a, (iii) $N_a = \emptyset$ and $N_b \neq N$ i.e. no voter prefers a to b; however not all voters prefer b to a and (iv) $N_a \neq \emptyset$ and $N_b \neq \emptyset$ i.e. some voters prefer a to b and some prefer b to a. In each case we show $f(R) = a$. The only remaining profile is the one where $N_b(R) = N$. Since f is onto it follows that $f(R) = b$. This establishes that $f \equiv f^a$.

CASE **5.1 :** Fix the profile $\bar{R}$ where $N_{ab}(\bar{R}) = N$ i.e. $\bar{R}_i = R^{ab}$ for all $i \in N$. As we assumed at profile R' we have $N_a(R') = \{1\}$ and $N_b(R') = \{2\}$. Suppose $f(\bar{R}_1, R'_{-1}) = b$. Then we have $f(R') = a \; P'_1 \; b = f(\bar{R}_1, R'_{-1})$ but also $f(\bar{R}_1, R'_{-1}) \; P'_2 \; f(R')$, which contradicts PWA. Therefore, $f(\bar{R}_1, R'_{-1}) = a$. By strategy-proofness, $f(\bar{R}_1, \bar{R}_2, R'_3, \ldots, R'_n) = a$, otherwise voter 2 would manipulate at $(\bar{R}_1, R'_{-1})$ via $\bar{R}_2$. Since, $(\bar{R}_1, \bar{R}_2, R'_3, \ldots, R'_n) = \bar{R}$, we have $f(\bar{R}) = a$.

CASE **5.2 :** Consider a profile R where $N_b(R) = \emptyset$. W.l.o.g., let $N_a(R) = \{1, \ldots, k\}$. Suppose $f(R) = b$. By strategy-proofness, $f(\bar{R}_1, R_{-1}) = b$, otherwise agent 1 would manipulate at R via $\bar{R}_1$. Applying this argument repeatedly, we get

$$\begin{aligned} f(\bar{R}_1, R_{-1}) &= f(\bar{R}_1, \bar{R}_2, R_{-\{1,2\}}) \\ &\vdots \\ &= f(\bar{R}_1, \ldots, \bar{R}_k, R_{k+1}, \ldots, R_n) \\ &= f(\bar{R}) \\ &= b \end{aligned}$$

This contradicts our conclusion in Case 5.1. Therefore, we have $f(R) = a$.

CASE **5.3 :** Consider a profile R where $N_a(R) = \emptyset$ and $N_b(R) \neq N$. We will show if $f(R) \neq a$ it leads to contradiction.

Suppose $f(R) = b$. Assume w.l.o.g. $N_b(R) = \{1, \ldots, k\}$ and $N_{ab}(R) = \{k{+}1, \ldots, n\}$. Note that $k < n$. Pick $R''_{k+1} = R^a$. The PWA will imply that $f(R''_{k+1}, R_{-\{k+1\}}) = b$. Otherwise, $f(R''_{k+1}, R_{-\{k+1\}})\ P''_{K+1}\ f(R)$ but also $f(R)\ P_j\ f(R''_{k+1}, R_{-\{k+1\}})$ where $j \in \{1, \ldots, k\}$ which contradicts PWA.

Now, we show that $f(\bar{R}_1, R''_{k+1}, R_{-\{1,k+1\}}) = b$. Suppose it is not true and we have $f(\bar{R}_1, R''_{k+1}, R_{-\{1,k+1\}}) = a$. This implies $f(R''_{k+1}, R_{-\{k+1\}}) = b\ P_1\ a = f(\bar{R}_1, R''_{k+1}, R_{-\{1,k+1\}})$ along with $f(\bar{R}_1, R''_{k+1}, R_{-\{1,k+1\}}) P''_{k+1} f(R''_{k+1}, R_{-\{k+1\}})$. We have a contradiction to PWA. Therefore $f(\bar{R}_1, R''_{k+1}, R_{-\{1,k+1\}}) = b$.

Applying PWA repeatedly, we get

$$\begin{aligned} f(\bar{R}_1, R''_{k+1}, R_{-\{1,k+1\}}) &= f(\bar{R}_1, \bar{R}_2, R''_{k+1}, R_{-\{1,2,k+1\}}) \\ &\vdots \\ &= f(\bar{R}_1, \ldots, \bar{R}_k, R''_{k+1}, R_{k+2} \ldots, R_n) \\ &= b \end{aligned}$$

However, this contradicts the conclusion in Case 5.2. Therefore $f(R) = a$.

CASE **5.4 :** Let R be a profile where $N_a(R) \neq \emptyset$ and $N_b(R) \neq \emptyset$ (it is obvious that $1 \leq k \leq n{-}1$). W.l.o.g., let $N_a(R) = \{1, \ldots, k\}$ and $N_b(R) = \{k+1, \ldots, p\}$. We assume for contradiction that $f(R) = b$. Suppose that $f(\bar{R}_1, R_{-1}) = a$. Since $P_1 = R^a$ we have $f(\bar{R}_1, R_{-1}) P_1 f(R)$ which contradicts strategy-proof. Thus $f(\bar{R}_1, R_{-1}) = b$. Applying strategy-proofness repeatedly, we get

$$\begin{aligned} f(\bar{R}_1, R_{-1}) &= f(\bar{R}_1, \bar{R}_2, R'_{-\{1,2\}}) \\ &\vdots \\ &= f(\bar{R}_1, \ldots, \bar{R}_k, R_{k+1}, \ldots, R_p, R_{p+1}, \ldots, R_n) \\ &= b \end{aligned}$$

However, we reach a contradiction because $N_a(\bar{R}_1, \ldots, \bar{R}_k, R_{k+1}, \ldots, R_p, R_{p+1}, \ldots, R_n) = \emptyset$. In Case 5.3 we have shown that at this profile outcome should be a. Therefore starting assumption of this case is false. Hence $f(R) = a$.

Therefore, Case 5.1, 5.2, 5.3 and 5.4 establish that for all $R \in \mathbb{R}^n$ except when $N_b(R) = N$ we have $f(R) = a$. Since f is onto, there must be a profile where $f(R) = b$ and the only profile left is where $N_b(R) = N$. Therefore, $f(R) = b$ where $N_b(R) = N$. Hence, $f \equiv f^a$.

If. Let $f \equiv f^x$ where $x \in A$. We will show that f^x is onto, strategy-proof and satisfies PWA. It is straightforward to see that f^x satisfies ontoness.

In order to show strategy-proofness, assume w.l.o.g. that $x = a$. Pick any $R \in \mathbb{R}^n$, $i \in N$ and $R'_i \in \mathbb{R}$. Note that if $f^a(R) \neq f^a(R'_i, R_{-i})$ then R_{-i} is such that $R_j = R^b$ for all $j \neq i$. In that case $f^a(R)\ R_i\ f^a(R'_i, R_{-i})$. Therefore, f^a is strategy-proof.

Pick any $R \in \mathbb{R}^n$ and $i \in N$ such that $R_i = R^{ab}$. Note that for any $R'_i \in \mathbb{R}$, if $f^a(R) \neq f^a(R'_i, R_{-i})$ then $f^a(R) = a$ and $f^a(R'_i, R_{-i}) = b$. However, in that case, $f^a(R'_i, R_{-i})\ P_j\ f^a(R)$ for all $j \in N \setminus i$. Therefore, f^a satisfies PWA. This completes the proof. ■

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