A study of operators on the discrete analogue of Hardy spaces on homogeneous trees and on other structures

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Dedicated to my teachers

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Contents

Notations & Abbreviations

$\mathbb{N}$	Set of all natural numbers
$\mathbb{N}_0$	Set of all whole numbers; $\mathbb{N} \cup \{0\}$
$\mathbb{C}$	Set of all complex numbers; Complex plane
$\mathbb{D}$	Unit disk $\{z \in \mathbb{C} : \ |z| < 1\}$
$\mathbb{T}$	Unit circle $\{z \in \mathbb{C} : \ |z| = 1\}$
$\mathbb{C}_\theta$	Half-plane $\{s \in \mathbb{C} : \ \operatorname{Re} s > \theta\}$
T	Homogeneous rooted tree with root o
v^-	Parent of a vertex v
$|v|$	Number of edges between v and o
ζ	Riemann zeta function
χ_A	Characteristic function on a set A
χ_v	Characteristic function on the set $\{v\}$; $\chi_{\{v\}}$
M_ψ	Multiplication operator induced by ψ
C_ϕ	Composition operator induced by ϕ
$W_{\psi,\phi}$	Weighted composition operator induced by ψ and ϕ
$\sigma_e(A)$	Point spectrum of an operator A
$\sigma_a(A)$	Approximate point spectrum of an operator A
$\sigma(A)$	Spectrum of an operator A
$a_N(A)$	N^{th} approximation number of an operator A
$\mathcal{B}(X)$	Set of all bounded linear operators on X
$\mathcal{K}(X)$	Set of all compact operators on X
$\mathcal{H}(\Omega)$	Space of all analytic functions defined on Ω.

Chapter 1

Introduction

1.1 Outline of the thesis

In analytic function theory, the study of multiplication and composition operators has a rich structure for various analytic function spaces of the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ such as the Hardy spaces H^p, the Bergman spaces A^p and the Bloch space $\mathcal{B}$. This theory connects the operator theoretic properties such as boundedness, compactness, spectrum, invertibility, isometry with that of the function theoretic properties of the inducing map (symbol) such as bijectivity, boundary behaviour and vise versa.

Literature on multiplication operators is exhaustive. See for example the survey articles [3, 18, 52, 70] on multiplication operators on various function spaces of the unit disk. The study of composition operators on various analytic function spaces defined on $\mathbb{D}$ becomes a popular branch of analytic function theory and operator theory. There are excellent text books and articles on composition operators, see [29, 68, 69] and the references therein.

Around 50 years ago, E. A. Nordgren begun a systematic study of composition operators. Composition operators have arisen in the study of commutants of multiplication operators and more general operators. Also, they play a role in the theory of dynamical systems. De Brange's original proof of the Bieberbach conjecture relied on substitution operators (another name for composition operators). Ergodic transformations are sometimes thought of as inducing composition operators on L^p spaces.

In 1955, Banach begun the study of weighted composition operators. Banach [14] proved the classical Banach-Stone theorem which asserts that the surjective isometries between the spaces of continuous real-valued functions on a closed and bounded interval are certain weighted composition operators. In [37], Forelli proved that the isometric isomorphism of the Hardy space H^p ($p \neq 2$) are also weighted composition operators. The same result for the case of Bergman space is proved by Kolaski in [47].

The study of weighted composition operators can be viewed as a natural generalization of the composition operators. Moreover, weighted composition operators appear in applied areas such as dynamical systems and evolution equations. For example, classification of dichotomies in certain dynamical systems is connected with weighted composition operators, see [21].

In the recent years, there has been a considerable interest in the study of function spaces on discrete sets such as tree (more generally on graphs). For example, Lipschitz space of a tree (discrete analogue of Bloch space) [24], weighted Lipschitz space of a tree [6], iterated logarithmic Lipschitz space of a tree [5], weighted Banach spaces of an infinite tree [9] and H^p spaces on trees [48] are some in this line of investigation. In [48], the H^p spaces on trees are defined by means of certain maximal or square function operators associated with a nearest neighbour transition operator which is very regular, and this study was further developed in [30].

Multiplication operators were considered on various discrete function spaces on infinite tree such as Lipschitz space, weighted Lipschitz space, iterated logarithmic Lipschitz spaces and weighted Banach spaces of an infinite tree. See [4, 5, 6, 8, 9, 24, 25] for more details. The study of composition operators on discrete function spaces was first initiated by Colonna et al. in [7] in which the Lipschitz space (discrete analogue of Bloch space) of a tree was investigated. Composition operators on weighted Banach spaces of an infinite tree were considered in [10]. Recently, some classes of operators including Toeplitz operators with symbol from the Lipschitz space of a tree were considered in [26]. The study of composition operators is not well developed unlike multiplication operators in the discrete settings.

In this thesis, we mainly study the composition operators in three different settings, namely, the following:

1. Discrete analogue of generalized Hardy spaces ($\mathbb{T}_p$). See Chapter 4.
2. The Hardy-Dirichlet space $\mathcal{H}^2$, the space of Dirichlet series with square summable coefficients. See Chapter 5.
3. The class $\mathcal{P}_\alpha$ of analytic functions subordinate to $\frac{1+\alpha z}{1-z}$ for $|\alpha| \leq 1, \alpha \neq -1$. See Chapter 6.

All these settings are very different in nature. In particular, for each p, $\mathbb{T}_p$ is a Banach space but is not a Hilbert space. Observe that $\mathcal{H}^2$ has a Hilbert space structure by its own, whereas the class $\mathcal{P}_\alpha$ does not have a linear space structure. We note that $\mathcal{P}_\alpha$ is a compact convex family in $\mathcal{H}(\mathbb{D})$.

In Chapter 2, we define discrete analogue of generalized Hardy spaces ($\mathbb{T}_p$) and their separable subspaces ($\mathbb{T}_{p,0}$) on a homogenous rooted tree and study some of their properties such as completeness, inclusion relations with other spaces, separability and growth estimate for functions in these spaces and their consequences.

In Chapter 3, we obtain equivalent conditions for multiplication operators M_ψ on $\mathbb{T}_p$ and $\mathbb{T}_{p,0}$ to be bounded and compact. Furthermore, we discuss point spectrum, approximate point spectrum and spectrum of multiplication operators and discuss when a multiplication operator is an isometry.

In Chapter 4, we give an equivalent conditions for the composition operator C_ϕ to be bounded on $\mathbb{T}_p$ and on $\mathbb{T}_{p,0}$ spaces and compute their operator norms. We have considered the composition operators induced by special symbols such as univalent and multivalent maps and automorphism of a homogenous tree. We also characterize invertible composition operators and isometric composition operators on $\mathbb{T}_p$ and on $\mathbb{T}_{p,0}$ spaces. Also, we discuss the compactness of C_ϕ on $\mathbb{T}_p$ spaces and finally we prove that there are no compact composition operators on $\mathbb{T}_{p,0}$ spaces.

In Chapter 5, we consider the composition operators on the Hardy-Dirichlet space $\mathcal{H}^2$, the space of Dirichlet series with square summable coefficients. By using the Schur test, we give some upper and lower estimates on the norm of a composition operator on $\mathcal{H}^2$, for the affine-like inducing symbol $\varphi(s) = c_1 + c_q q^{-s}$, where $q \geq 2$ is a fixed integer. We also give an estimate for approximation numbers of a composition operators in our $\mathcal{H}^2$ setting.

In Chapter 6, we study the weighted composition operators preserving the class $\mathcal{P}_\alpha$. Some of its consequences and examples of certain special cases are presented. Furthermore, we discuss about the fixed points of weighted composition operators.

1.2 Basic definitions and results

We refer to the books [27, 61, 67] for basic definitions and results on functional analysis and [60, 63] for basic notions of complex analysis. Let us now recall several definitions and some basic results that are needed in the sequel.

Definition 1.2.1. A *normed space* is a pair $(X, \|\cdot\|)$, where X is a vector space and $\|\cdot\|$ is a norm on X. A *Banach space* is a normed space that is complete with respect to metric defined by $d(x,y) = \|x-y\|$ for $x, y \in X$. If X has a countable dense subset A i.e., $\overline{A} = X$, then X is said to be *separable*.

Definition 1.2.2. A *Hilbert space* is a vector space H together with an inner product $\langle \cdot, \cdot \rangle$ such that $(H, \|\cdot\|)$ is Banach space, where the norm is induced by the inner product, i.e., $\|a\| := \sqrt{\langle a, a \rangle}$ for all $a \in H$.

Proposition 1.2.3. (Cauchy-Schwarz Inequality) *For any f and g in a Hilbert space H, we have*

$$|\langle f, g \rangle| \leq \|f\| \, \|g\| \, .$$

Proposition 1.2.4. (Parallelogram Law) *If H is a Hilbert space and $f, g \in H$, then*

$$\|f+g\|^2 + \|f-g\|^2 = 2(\|f\|^2 + \|g\|^2).$$

Definition 1.2.5. A linear map A on a normed space X is said to be *bounded linear operator* or *bounded operator* if the operator norm $\|A\| := \sup\{\|Ax\| : \|x\| \leq 1\}$ is finite. The set of all bounded linear operators on X is denoted by $\mathcal{B}(X)$.

Remark 1.2.6. $\mathcal{B}(X)$ becomes a Banach space under the operator norm defined above if X is a Banach space.

Definition 1.2.7. A linear map A on a normed space X is said to be *closed operator* if its graph $G(A) = \{(x, Ax) : x \in X\}$ is closed in $X \times X$, or equivalently, $x_n \to x$ and $Ax_n \to y$ imply $y = Ax$.

Theorem 1.2.8. (Closed graph theorem) *Every closed operator A on a Banach space X is a bounded linear operator on X.*

Definition 1.2.9. A bounded linear operator A on a normed space X is said to be a *compact operator* if the closure of the image of the closed unit ball $\{Ax : \|x\| \leq 1\}$ is compact. The set of all compact operators on X is denoted by $\mathcal{K}(X)$.

Proposition 1.2.10. *$\mathcal{K}(X)$ is a closed subspace of $\mathcal{B}(X)$. In particular, limit of a sequence of compact operators is a compact operator.*

Definition 1.2.11. A bounded linear operator A on a normed space X is said to be a *finite rank operator* if the range $\{Ax : x \in X\}$ has finite dimension.

Remark 1.2.12. It is easy to see that, every finite rank operator is compact operator. Consequently, the limit of finite rank operators is also a compact operator.

Definition 1.2.13. The *essential norm* $\|A\|_e$ of a bounded operator A on X is defined to be the distance between A and $\mathcal{K}(X)$:

$$\|A\|_e = \operatorname{dist}(A, \mathcal{K}(X)) = \inf\{\|A - K\| : \, K \in \mathcal{K}(X)\}.$$

Definition 1.2.14. A bounded operator A on a normed space X is said to be an *isometry* if $\|Ax\| = \|x\|$ for all $x \in X$.

Definition 1.2.15. A bounded linear operator A on a normed space X is said to be an *invertible* if there exists a bounded linear operator B on X such that $B(A(x)) = A(B(x)) = x$ for all $x \in X$. Such an operator B is called the inverse of A and is denoted by A^{-1}.

Definition 1.2.16. A bounded linear operator A on an inner product space X is said to have an *adjoint* if there exists a bounded linear operator B on X such that

$$\langle Ax, y\rangle = \langle x, By\rangle \text{ for all } x, y \in X.$$

Such an operator B is called an adjoint of A and is denoted by A^*.

Remark 1.2.17. Every bounded linear operator A on a Hilbert space has the adjoint. Moreover, $\|A\| = \|A^*\|$ for all $A \in \mathcal{B}(X)$.

Definition 1.2.18. A bounded linear operator A on a Hilbert space H is called a *normal operator* if it commutes with its adjoint, i.e., $AA^* = A^*A$.

Definition 1.2.19. Let A be a bounded linear operator on a normed space X. The *point spectrum* $\sigma_e(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not injective. Thus, $\lambda \in \sigma_e(A)$ if and only if there exists a nonzero $x \in X$ such that $Ax = \lambda x$.

Definition 1.2.20. Let A be a bounded linear operator on a normed space X. The *approximate point spectrum* $\sigma_a(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not bounded below. Thus, $\lambda \in \sigma_a(A)$ if and only if there is a sequence $\{x_n\}$ in X such that $\|x_n\| = 1$ for each n and $\|A(x_n) - \lambda x_n\| \to 0$ as $n \to \infty$.

Definition 1.2.21. Let A be a bounded linear operator on a normed space X. The *spectrum* $\sigma(A)$ of A consists of all $\lambda \in \mathbb{C}$ such that $A - \lambda I$ is not invertible.

Definition 1.2.22. For a bounded linear operator A, the *spectral radius*, denoted by $r(A)$ is defined by

$$r(A) := \sup\{|\lambda| : \lambda \in \sigma(A)\}.$$

If $r(A) = \|A\|$, then the operator A is called *normaloid* operator. For example, every normal operator is normaloid.

Proposition 1.2.23. *Let A be a bounded linear operator on a complex Banach space X. Then,*

(1) $\sigma_e(A) \subseteq \sigma_a(A) \subseteq \sigma(A)$.

(2) *The boundary of the spectrum of A is contained in the approximate point spectrum of A.*

(3) *Approximate point spectrum $\sigma_a(A)$ is a closed subset of $\mathbb{C}$.*

(4) $\sigma(A)$ *is a nonempty compact subset of* $\mathbb{C}$ (Gelfand-Mazur theorem).

(5) $r(A) \leq \|A\|$. *That is,* $|\lambda| \leq \|A\|$ *for all* $\lambda \in \sigma(A)$.

Proposition 1.2.24. *Let A be a compact linear operator on normed space X. Then,*

(1) $\sigma_e(A) \setminus \{0\} = \sigma(A) \setminus \{0\}$.

(2) 0 *is the only possible limit point of* $\sigma(A)$.

(3) $\sigma(A)$ *is countable set.*

Chapter 2

Discrete Hardy spaces ($\mathbb{T}_p$)

2.1 Introduction

The theory of function spaces defined on the unit disk $\mathbb{D}$ is particulary a well developed subject. The recent book by Pavlović [58] and the book of Zhu [71] provide us with a solid foundation in studying various function spaces on the unit disk. One can also refer [33] for Hardy spaces (H^p), [43] for Bergman spaces (A^p), [36] for Dirichlet spaces ($\mathcal{D}_p$) and [11] for Bloch space ($\mathcal{B}$).

In the recent years, there has been a considerable interest in the study of function spaces on discrete sets such as tree (more generally on graphs). For example, Lipschitz space of a tree (discrete analogue of Bloch space) [24], weighted Lipschitz space of a tree [6], iterated logarithmic Lipschitz space of a tree [5] and H^p spaces on trees [48] are some in this line of investigation. In [48], the H^p spaces on trees are investigated by means of certain maximal or square function operators associated with a nearest neighbour transition operator which is very regular, and this study was further developed in [30].

In this chapter, we define discrete analogue of generalized Hardy spaces ($\mathbb{T}_p$) and their separable subspaces ($\mathbb{T}_{p,0}$) on a homogenous rooted tree and study some of their properties such as completeness, inclusion relations with other spaces, separability and growth estimate for functions in these spaces and their consequences. This chapter is based on our article [54].

2.2 Preliminaries

A *graph* G is a pair $G = (V, E)$ of sets satisfying $E \subseteq V \times V$. The elements of V and E are called vertices and edges of the graph G, respectively. Two vertices $x, y \in V$ (with the abuse of language, one can write as $x, y \in G$) are said to be *neighbours* or *adjacent* (denoted by $x \sim y$) if there is an edge connecting them. A *regular* (homogeneous) graph is a graph in which every vertex has the same number of neighbours. If every vertex has k neighbours, then the graph is said to be $k-$*regular* ($k-$homogeneous) graph. A *path* is part of a graph with finite or infinite sequence of distinct vertices $[v_0, v_1, v_2, \ldots]$ such that $v_n \sim v_{n+1}$. If $P = [v_0 - v_1 - v_2 - \cdots - v_n]$ is a path then the graph $C = [v_0 - v_1 - v_2 - \cdots - v_n - v_0]$ (path P with an additional edge $v_n - v_0$) is called a *cycle*. A non-empty graph G is called *connected* if any two of its vertices are linked by a path in G. A connected and locally finite (every vertex has finite number of neighbours) graph without cycles is called a *tree*. A *rooted tree* is a tree in which a special vertex (called root) is singled out. The distance between any two vertex of a tree is the number of edges in the unique path connecting them. If G is a rooted tree with root o, then $|v|$ denotes the distance between o and v. Further the *parent* (denoted by v^-) of a vertex $v \neq o$ is the unique vertex $w \in G$ such that $w \sim v$ and $|w| = |v| - 1$. For basic issues concerning graph theory, one can refer to standard texts such as [31].

Let T be a rooted tree. By a function on a graph, we mean a function defined on its vertices. The Lipschitz space [24] and the weighted Lipschitz space [6] of T are denoted by $\mathcal{L}$ and $\mathcal{L}_w$, respectively. These are defined as follows:

$$\mathcal{L} = \left\{ f : T \to \mathbb{C} : \sup_{v \in T,\, v \neq o} |f(v) - f(v^-)| < \infty \right\}$$

and

$$\mathcal{L}_w = \left\{ f : T \to \mathbb{C} : \sup_{v \in T, v \neq o} |v|\, |f(v) - f(v^-)| < \infty \right\},$$

respectively. Throughout the discussion, T denotes a $(q+1)-$homogeneous rooted tree for some $q \in \mathbb{N}$. For $n \in \mathbb{N}_0$, let D_n denote the set of all vertices $v \in T$ with $|v| = n$ and denote the number of elements in D_n by c_n. Thus,

$$c_n = \begin{cases} (q+1)q^{n-1} & \text{if } \ n \in \mathbb{N}, \\ 1 & \text{if } \ n = 0. \end{cases} \tag{2.2.1}$$

For $p \in (0, \infty]$, the generalized Hardy space $H_g^p(\mathbb{D})$ consists of all those measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that $M_p(r, f)$ exists for all $r \in [0, 1)$ and $\|f\|_p < \infty$, where

$$\|f\|_p = \begin{cases} \sup\limits_{0 \le r < 1} M_p(r, f) & \text{if } p \in (0, \infty), \\ \sup\limits_{z \in \mathbb{D}} |f(z)| & \text{if } p = \infty, \end{cases}$$

and

$$M_p^p(r, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta.$$

The classical Hardy space H^p is a subspace of $H_g^p(\mathbb{D})$ consisting of only analytic functions. See [20] for recent investigation on $H_g^p(\mathbb{D})$ and some related function spaces.

For our investigation, this definition has an analog in the following form.

Definition 2.2.1. Let T be a $(q+1)-$homogeneous tree rooted at o. For every $n \in \mathbb{N}$, we introduce

$$M_p(n, f) := \begin{cases} \left(\dfrac{1}{c_n} \sum\limits_{|v|=n} |f(v)|^p \right)^{\frac{1}{p}} & \text{if } p \in (0, \infty), \\ \max\limits_{|v|=n} |f(v)| & \text{if } p = \infty, \end{cases}$$

$M_p(0, f) := |f(o)|$ and

$$\|f\|_p := \sup_{n \in \mathbb{N}_0} M_p(n, f). \tag{2.2.2}$$

The discrete analogue of the generalized Hardy space, denoted by $\mathbb{T}_{q,p}$, is then defined by

$$\mathbb{T}_{q,p} := \{f : T \to \mathbb{C} \ \text{ such that } \ \|f\|_p < \infty\}.$$

Similarly, the discrete analogue of the generalized little Hardy space, denoted by $\mathbb{T}_{q,p,0}$, is defined by

$$\mathbb{T}_{q,p,0} := \{f \in \mathbb{T}_{q,p} : \lim_{n \to \infty} M_p(n, f) = 0\}$$

for every $p \in (0, \infty]$. For the sake of simplicity, we shall write $\mathbb{T}_{q,p}$ and $\mathbb{T}_{q,p,0}$ as $\mathbb{T}_p$ and $\mathbb{T}_{p,0}$, respectively. Unless otherwise stated explicitly, throughout $\|\cdot\|_p$ is defined as above.

2.3 Completeness

Theorem 2.3.1. *For* $1 \le p \le \infty$, $\|\cdot\|_p$ *induces a Banach space structure on the space* $\mathbb{T}_p$.

Proof. First we begin with the case $p = \infty$. In this case, (2.2.2) reduces to $\|f\|_p = \sup_{v\in T} |f(v)|$ and thus, the space $\mathbb{T}_\infty$ coincides with the set of all bounded functions on T with sup-norm which is known to be a Banach space.

Next, we consider the case $1 \le p < \infty$. We have the following.

(i) If $f \equiv 0$, then $\|f\|_p = 0$. Conversely, if $\|f\|_p = 0$ then $M_p(n, f) = 0$ for all $n \in \mathbb{N}_0$ showing that $\sum_{|v|=n} |f(v)|^p = 0$ for all $n \in \mathbb{N}_0$ and thus, $f \equiv 0$.

(ii) For each $n \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$, it is easy to see by the definition that $M_p(n, \alpha f) = |\alpha| M_p(n, f)$ and thus, $\|\alpha f\|_p = |\alpha|\, \|f\|_p$.

(iii) For each $n \in \mathbb{N}$ and $f, g \in \mathbb{T}_p$, one has (since $p \ge 1$)

$$
\begin{aligned}
M_p(n, f+g) &\le \left\{ \frac{1}{c_n} \sum_{|v|=n} (|f(v)| + |g(v)|)^p \right\}^{\frac{1}{p}} \\
&\le \left(\frac{1}{c_n} \sum_{|v|=n} |f(v)|^p \right)^{\frac{1}{p}} + \left(\frac{1}{c_n} \sum_{|v|=n} |g(v)|^p \right)^{\frac{1}{p}} \\
&= M_p(n, f) + M_p(n, g).
\end{aligned}
$$

The last inequality trivially holds for $n = 0$ and thus, $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

Hence $(\mathbb{T}_p, \|\cdot\|_p)$ is a normed space. In order to prove that $\mathbb{T}_p$ is a Banach space, we begin with a Cauchy sequence $\{f_k\}$ in $\mathbb{T}_p$. Then $\{f_k(v)\}$ is a Cauchy sequence in $\mathbb{C}$ for every $v \in T$ and thus, $\{f_k\}$ converges pointwise to a function f. Now, for a given $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that $M_p(n, f_k - f_l) < \epsilon$ for all $k, l \ge N$ and $n \in \mathbb{N}_0$. Letting $l \to \infty$, we get $M_p(n, f_k - f) \le \epsilon$ for all $k \ge N$ and $n \in \mathbb{N}_0$. Hence $\|f_k - f\|_p \le \epsilon$ for all $k \ge N$, which gives that $f_k \to f$. The triangle inequality $\|f\|_p \le \|f - f_N\|_p + \|f_N\|_p$ gives that $f \in \mathbb{T}_p$. This completes the proof of the theorem. □

Remark 2.3.2. For $0 < p < 1$, $d(f, g) = \|f - g\|_p^p$ defines a complete metric on $\mathbb{T}_p$.

A function $f\colon T \to \mathbb{C}$ is said to be a *radial constant function* if $f(v) = f(w)$ whenever $|v| = |w|$.

Remark 2.3.3. Since the integral means $M_p(r, f)$ of an analytic function f defined on $\mathbb{D}$ is an increasing function of r, we have $\|f\|_p = \lim_{r\to 1^-} M_p(r, f)$. Thus, the little Hardy space H_0^p, defined by

$$H_0^p := \{f \in H^p : M_p(r, f) \to 0 \ \text{ as } \ r \to 1^-\},$$

consists of only a single element, namely, the zero function. But this is not the case in the generalized Hardy space $H_g^p(\mathbb{D})$ of measurable functions, This is because the maximum modulus principle is not valid for a general element in $H_g^p(\mathbb{D})$. Consequently, the generalized little Hardy space $H_{0,g}^p$ is non-trivial (i.e., not a zero subspace), where

$$H_{0,g}^p := \{f \in H_g^p(\mathbb{D}) : M_p(r, f) \to 0 \ \text{ as } \ r \to 1^-\}.$$

For example, if

$$f_\alpha(z) = \begin{cases} 1 & \text{for } |z| \le \alpha, \\ 0 & \text{for } \alpha < |z| < 1, \end{cases}$$

then $f \in H_{0,g}^p$ for each $0 \le \alpha < 1$.

In the discrete case, $\mathbb{T}_{p,0}$ is non-trivial. In fact, the set of all radial constant functions in $\mathbb{T}_{p,0}$ is isometrically isomorphic to the sequence space c_0 (set of all sequences that converge to zero).

Here are few questions that arise naturally.

It is natural to ask whether $\mathbb{T}_2$ is a Hilbert space or not. The answer is indeed no!. For example, choose two vertices v_1 and v_2 such that $|v_1| = 1$ and $|v_2| = 2$. Take $f = \sqrt{q+1}\chi_{v_1}$ and $g = \sqrt{q(q+1)}\chi_{v_2}$, where χ_v denotes characteristic function on the set $\{v\}$. Then it is easy to see that $f, g \in \mathbb{T}_2$ with

$$\|f\|_2 = \|g\|_2 = \|f+g\|_2 = \|f-g\|_2 = 1$$

and hence the parallelogram law

$$\|f+g\|_2^2 + \|f-g\|_2^2 = 2(\|f\|_2^2 + \|g\|_2^2)$$

is not satisfied. Therefore, $\mathbb{T}_2$ cannot be a Hilbert space under $\|\cdot\|_2$.

Remark 2.3.4. In the classical Hardy space H^2 of the unit disk, it is known that

$$\sup_{0\le r<1} M_2(r,f) = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta,$$

which is due to Littlewood's subordination theorem and mean convergence theorem (see [33, Section 2.3]). Therefore H^2 becomes a Hilbert space in a natural way. On the other hand a similar situation does not occur in the $\mathbb{T}_p$ spaces.

Question 1. As with the l^p and the H^p spaces, it is natural to ask whether $\mathbb{T}_s$ is not isomorphic to $\mathbb{T}_r$ when $s \neq r$? What can be said about the dual of $\mathbb{T}_s$?

These questions remain open.

Theorem 2.3.5. *For $1 \le p \le \infty$, $\|\cdot\|_p$ induces a Banach space structure on $\mathbb{T}_{p,0}$.*

Proof. For $n \in \mathbb{N}_0$ and $f, g \in \mathbb{T}_p$, we easily have

$$M_p(n, \alpha f) = |\alpha| M_p(n,f) \quad \text{and} \quad M_p(n, f+g) \le M_p(n,f) + M_p(n,g)$$

so that $\mathbb{T}_{p,0}$ is a subspace of $\mathbb{T}_p$. Suppose $\{f_k\}$ is a Cauchy sequence in $\mathbb{T}_{p,0}$. Since $\mathbb{T}_p$ is a Banach space, $\{f_k\}$ converges to some function $f \in \mathbb{T}_p$. Next, we need to prove that $f \in \mathbb{T}_{p,0}$, i.e., $M_p(n,f) \to 0$ as $n \to \infty$. To do this, let $\epsilon > 0$ be given. Then there exists a $k \in \mathbb{N}$ such that $\|f_k - f\|_p < \epsilon/2$. Since $f_k \in \mathbb{T}_{p,0}$, we can choose $N \in \mathbb{N}$ so that $M_p(n, f_k) < \epsilon/2$ for all $n \ge N$. From the inequality

$$M_p(n,f) \le M_p(n, f - f_k) + M_p(n, f_k),$$

it follows that $M_p(n,f) < \epsilon$ for all $n \ge N$. Thus, $f \in \mathbb{T}_{p,0}$ and this completes the proof. □

2.4 Inclusion relations

Lemma 2.4.1. *For $0 < r < s \le \infty$ and for every complex-valued function f on T, $M_r(n,f) \le M_s(n,f)$ holds for all $n \in \mathbb{N}_0$.*

Proof. The result for $s = \infty$ follows from the definition of $M_r(n, f)$ and thus, it suffices to prove the lemma for the case $0 < r < s < \infty$. Again by Definition 2.2.1, we see that

$$M_r(0, f) = |f(o)| = M_s(0, f).$$

For $n \in \mathbb{N}$, we have c_n vertices with $|v| = n$. Recall that on the Euclidean space $\mathbb{C}^N$, the following norm equivalence is well-known for $0 < r < s$:

$$\|x\|_s \leq \|x\|_r \leq N^{\frac{1}{r}-\frac{1}{s}} \|x\|_s , \tag{2.4.1}$$

where p-norm $\|\cdot\|_p$ on $\mathbb{C}^N$ is given by $\|x\|_p^p = \sum_{k=1}^{N} |x_k|^p$. The second inequality in (2.4.1), is an easy consequence of Hölder's inequality for finite sum. We may now use this with $N = c_n$. As a consequence of it, we have

$$\left(\sum_{|v|=n} |f(v)|^r \right)^{\frac{1}{r}} \leq (c_n)^{\frac{1}{r}-\frac{1}{s}} \left(\sum_{|v|=n} |f(v)|^s \right)^{\frac{1}{s}}$$

which may be rewritten as

$$\left(\frac{1}{c_n} \sum_{|v|=n} |f(v)|^r \right)^{\frac{1}{r}} \leq \left(\frac{1}{c_n} \sum_{|v|=n} |f(v)|^s \right)^{\frac{1}{s}} .$$

This shows that $M_r(n, f) \leq M_s(n, f)$ for all $n \in \mathbb{N}_0$. The proof is complete. □

As an immediate consequence of Lemma 2.4.1, one has the following.

Theorem 2.4.2. *For $0 < r < s \leq \infty$, we have $\mathbb{T}_s \subset \mathbb{T}_r$ and $\mathbb{T}_{s,0} \subset \mathbb{T}_{r,0}$.*

We now show by an example that the inclusions in Theorem 2.4.2 are proper. Let $0 < r < p < s \leq \infty$. Choose a sequence of vertices $\{v_n\}$ such that $|v_n| = n$ for all $n \in \mathbb{N}$. Consider the function f defined by

$$f(v) = \begin{cases} (c_n)^{\frac{1}{p}} & \text{if } v = v_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then, for $n \in \mathbb{N}$, one has

$$M_r(n, f) = (c_n)^{\frac{1}{p}-\frac{1}{r}}$$

so that $M_r(n,f) \to 0$ as $n \to \infty$, since $p > r$. Also, we have

$$M_s(n,f) = \begin{cases} (c_n)^{\frac{1}{p}-\frac{1}{s}} & \text{if } s < \infty, \\ (c_n)^{\frac{1}{p}} & \text{if } s = \infty, \end{cases}$$

and in either case, we find that $M_s(n,f) \to \infty$ as $n \to \infty$. This example shows that $\mathbb{T}_{s,0}$ is a proper subspace of $\mathbb{T}_{r,0}$ for $r < s$. From this example, it can be also seen that $\mathbb{T}_s$ is a proper subspace of $\mathbb{T}_r$ for $r < s$.

Lemma 2.4.3. *For $f \in \mathbb{T}_\infty$, we have* $\lim_{s\to\infty} \|f\|_s = \|f\|_\infty$.

Proof. For $n \in \mathbb{N}_0$ and $0 < s < t \le \infty$, we see that $M_s(n,f) \le M_t(n,f)$ and thus, $\|f\|_s \le \|f\|_t$ for $s < t$ which in turn gives that

$$\limsup_{s\to\infty} \|f\|_s \le \|f\|_\infty .$$

On the other hand, for each $n \in \mathbb{N}_0$, we find that

$$(c_n)^{-1/s} M_\infty(n,f) \le M_s(n,f) \le \|f\|_s .$$

Now, by letting $s \to \infty$ and taking supremum over $n \in \mathbb{N}_0$, we get

$$\|f\|_\infty \le \liminf_{s\to\infty} \|f\|_s .$$

Hence, $\lim_{s\to\infty} \|f\|_s = \|f\|_\infty$. □

Remarks 2.4.4. 1. The unbounded function $f(v) = |v|$ belongs to $\mathcal{L}$ but is not in $\mathbb{T}_p$ for any $0 < p < \infty$. On the other hand, let us now fix an infinite path $o - v_1 - v_2 \cdots$ with $|v_k| = k$. Define $g(v) = (c_k)^{\frac{1}{p}}$ if $v = v_k$ and 0 otherwise. It is then easy to check that g belongs to $\mathbb{T}_p$ for all $0 < p < \infty$ but is not in $\mathcal{L}$. Thus, $\mathcal{L}$ is not comparable with $\mathbb{T}_p$ for all $0 < p < \infty$.

2. Consider the radial constant function h defined by $h(o) = 0$ and

$$h(v) = \sum_{k=1}^{|v|} \frac{1}{k} \quad \text{if } |v| \ge 1.$$

By simple calculations, we find that h belongs to $\mathcal{L}_w$ but is not in $\mathbb{T}_p$ for any $0 < p < \infty$. For the other direction, we fix an infinite path $o - v_1 - v_2 \cdots$ with $|v_k| = k$. If $A = \{o, v_1, v_2, \ldots\}$ then the characteristic function χ_A, namely, $\chi_A(v) = 1$ for $v \in A$ and zero elsewhere, belongs to $\mathbb{T}_p$ for all $0 < p < \infty$ but is not in $\mathcal{L}_w$. This concludes the proof that $\mathcal{L}_w$ is not comparable with $\mathbb{T}_p$ for all $0 < p < \infty$.

3. Clearly, $\mathbb{T}_\infty \subseteq \left(\bigcap_{0<p<\infty} \mathbb{T}_p \right) \bigcap \mathcal{L}$, whereas $\mathcal{L}_w$ is not comparable with $\mathbb{T}_\infty$. For 2−homogeneous trees, this inclusion relation becomes an equality. This is because of the fact that there is no unbounded function in $\mathbb{T}_p$ for 2−homogeneous trees, which can be observed from the definition of $\mathbb{T}_p$.

2.5 Separability

In order to state results about the separability of $\mathbb{T}_{p,0}$ and $\mathbb{T}_p$, we need to introduce the following. Denote by $C_c(T)$ the set of all functions $f\colon T \to \mathbb{C}$ such that $M_p(n, f) = 0$ for all but finitely many n's.

Lemma 2.5.1. *For $0 < p \leq \infty$, closure of $C_c(T)$ under $\|\cdot\|_p$ is $\mathbb{T}_{p,0}$.*

Proof. Let $f \in \mathbb{T}_{p,0}$ and, for each n, define f_n by

$$f_n(v) = \begin{cases} f(v) & \text{if } |v| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $f_n \in C_c(T)$ for each $n \in \mathbb{N}$ and

$$M_p(k, f - f_n) = \begin{cases} M_p(k, f) & \text{if } k > n, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, we see that

$$\|f - f_n\|_p = \sup_{m \in \mathbb{N}_0} M_p(m, f - f_n) = \sup_{m > n} M_p(m, f)$$

and, because $M_p(m, f) \to 0$ as $m \to \infty$, it follows that $\|f - f_n\|_p \to 0$ as $n \to \infty$. This completes the proof. $\square$

Theorem 2.5.2. *For $0 < p \leq \infty$, $\mathbb{T}_{p,0}$ is a separable space.*

Proof. It is easy to verify that $B = \{\chi_v : v \in T\}$ is a basis for $C_c(T)$. Since B is countable, it follows that $C_c(T)$ is separable. Since $C_c(T)$ is dense in $\mathbb{T}_{p,0}$, we conclude that $\mathbb{T}_{p,0}$ is separable and the theorem follows. □

We remark that $C_c(T)$ cannot be a Banach space with respect to any norm, since it has a countably infinite basis.

Theorem 2.5.3. *For $0 < p \leq \infty$, $\mathbb{T}_p$ is not separable.*

Proof. Let $E \subset \mathbb{T}_p$ denote the set of all radial constant functions f whose range is a subset of $\{0, 1\}$. Let $f, g \in E$ and $f \neq g$. Then there exists a $v \in T$ such that $f(v) \neq g(v)$. Since $f, g \in E$, we have $M_p(n, f - g) \leq 1$ for all n. On the other hand, $M_p(|v|, f - g) = 1$ and hence,

$$\|f - g\|_p = \sup_{n \in \mathbb{N}_0} M_p(n, f - g) = 1.$$

It is easy to check that E is an uncountable subset of $\mathbb{T}_p$. Since any two distinct elements of E must be of distance 1 apart and E is uncountable, it follows that any dense subset of $\mathbb{T}_p$ cannot be countable. Consequently, $\mathbb{T}_p$ is not a separable space. □

2.6 Growth estimate and consequences

Lemma 2.6.1. *Let T be a $(q+1)-$homogeneous tree rooted at o and $0 < p < \infty$. Then, for $v \in T \setminus \{o\}$, we have the following:*

(a) *If $f \in \mathbb{T}_p$, then $|f(v)| \leq ((q+1)q^{|v|-1})^{\frac{1}{p}} \|f\|_p$.*

(b) *If $f \in \mathbb{T}_{p,0}$, then*
$$\lim_{|v| \to \infty} \frac{f(v)}{((q+1)q^{|v|-1})^{\frac{1}{p}}} = 0.$$

The results are sharp.

Proof. Fix $v \in T \setminus \{o\}$ and let $n = |v|$. Then,

$$|f(v)|^p \leq \sum_{|w|=n} |f(w)|^p = c_n M_p^p(n, f)$$

so that $|f(v)| \leq (c_n)^{\frac{1}{p}} M_p(n, f)$ and thus,

$$\frac{|f(v)|}{(c_n)^{\frac{1}{p}}} \leq M_p(n, f) \leq \|f\|_p .$$

The desired results follow.

In order to prove the sharpness, we fix $v \in T \setminus \{o\}$. Define $f(v) = ((q+1)q^{|v|-1})^{\frac{1}{p}}$ for the fixed v and 0 elsewhere. We now let $m = |v|$ so that $M_p(n, f) = 0$ for every $n \neq m$ and

$$M_p(m, f) = \left(\frac{1}{c_m} \sum_{|w|=m} |f(w)|^p \right)^{\frac{1}{p}} = \left(\frac{1}{c_m} |f(v)|^p \right)^{\frac{1}{p}} = 1.$$

We obtain that $\|f\|_p = \sup_{n \in \mathbb{N}_0} M_p(n, f) = 1$ and hence,

$$|f(v)| = f(v) = ((q+1)q^{|v|-1})^{\frac{1}{p}} \|f\|_p .$$

We conclude the proof. □

Remark 2.6.2. For $v = o$, we have $|f(o)| \leq \|f\|_p$. Sharpness of this inequality is easy to verify (for example, $f = \chi_o$).

Proposition 2.6.3. *Convergence in* $\|\cdot\|_p$ $(0 < p \leq \infty)$ *implies uniform convergence on compact subsets of* T.

Proof. The edge counting distance on T induces the discrete metric. So, finite subsets are the only compact sets in T. Let K be an arbitrary compact subset of T. Then there exists an $N \in \mathbb{N}$ such that $|v| \leq N$ for every $v \in K$. The proposition trivially holds for the case $p = \infty$, because given a function f and a sequence $\{f_n\}$ converging to f in norm,

$$\sup_{v \in K} |(f_n - f)(v)| \leq \|f_n - f\|_\infty .$$

Next, we consider the case $0 < p < \infty$. From Lemma 2.6.1, given a function f in $\mathbb{T}_p$, we have

$$|f(v)| \leq ((q+1)q^{|v|-1})^{\frac{1}{p}} \|f\|_p \leq (c_N)^{\frac{1}{p}} \|f\|_p \text{ for every } v \in K.$$

This gives

$$\sup_{v\in K} |f(v)| \leq (c_N)^{\frac{1}{p}} \|f\|_p$$

and thus, by replacing f by $f_n - f$, we conclude that the convergence in $\|\cdot\|_p$ implies the uniform convergence on compact subsets of T. □

Uniform convergence on compact subsets of T does not necessarily imply the convergence in $\|\cdot\|_p$ $(0 < p \leq \infty)$ as can be seen from the following example.

Example 2.6.4. Consider the function $f \equiv 1$. For each n, define f_n by

$$f_n(v) = \begin{cases} f(v)\ (=1) & \text{if } |v| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$M_p(k, f - f_n) = \begin{cases} M_p(k, f)\ (=1) & \text{if } k > n, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be a compact subset of T. Then there exists an $N \in \mathbb{N}$ such that $|v| \leq N$ for every $v \in K$ and $\sup_{v\in K} |(f_n - f)(v)| = 0$ for every $n > N$. It follows that $\{f_n\}$ converges uniformly on compact subsets of T to f. On the other hand,

$$\|f - f_n\|_p = \sup_{m\in\mathbb{N}_0} M_p(m, f - f_n) = 1 \text{ for every } n \in \mathbb{N}.$$

Hence, $\{f_n\}$ does not converge to f in $\|\cdot\|_p$.

From Proposition 2.6.3, we observe that the topology of uniform convergence on the compact subsets of T on $\mathbb{T}_p$ is similar to that of analytic cases such as H^p spaces. This observation raises a natural question. Is $\mathbb{T}_p$ complete in the topology of uniform convergence on compact sets? The following example shows that the answer is negative. For each n, define f_n by

$$f_n(v) = \begin{cases} |v| & \text{if } |v| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

Let K be a compact subset of T. Then there exists an $N \in \mathbb{N}$ such that $|v| \leq N$ for every $v \in K$. For $N < n < m$, $f_n(v) = f_m(v)$ for all $v \in K$. It is easy to see that $\{f_n\}$ is a Cauchy sequence in the topology of uniform convergence on compact sets and $\{f_n\}$ converges pointwise to the function $f(v) = |v|$. Note that f can be the only possible limit of $\{f_n\}$ in the topology of uniform convergence on compact sets. Since $\mathbb{T}_p$ contains the sequence $\{f_n\}$ but not f, $\mathbb{T}_p$ cannot be complete under the topology of uniform convergence on compact sets.

Chapter 3

Multiplication operators on $\mathbb{T}_p$

3.1 Introduction

Let Ω be a nonempty set and X a complex Banach space of complex-valued functions f defined on Ω. For a given complex-valued function ψ on Ω, the multiplication operator M_ψ induced by the symbol ψ is defined as

$$M_\psi(f) = \psi f \quad \text{for} \quad f \in X.$$

In the study of operators on analytic function spaces, multiplication and composition operators arise naturally and play an important role. Literature on these topics are exhaustive. See for example the survey articles [3, 18, 52, 70] on multiplication operators on various function spaces of the unit disk $\mathbb{D}$. The systematic study of operator theory on discrete structure specially on infinite trees has been the subject of several recent papers [4, 5, 6, 7, 8, 9, 10, 24, 25, 26, 54, 53]. Discrete function spaces are mostly defined to be analogs of analytic function spaces.

Multiplication and composition operators are mainly considered on discrete function spaces. The basic questions such as boundedness, compactness, estimates for operator norm and essential norm, isometry and spectrum were considered for multiplication operators between the various discrete function spaces on infinite trees such as Lipschitz space, weighted Lipschitz space, iterated logarithmic Lipschitz spaces and weighted Banach spaces of an infinite tree. See [4, 5, 6, 8, 9, 24, 25] for more details.

In this chapter, we obtain equivalent conditions for multiplication operators on $\mathbb{T}_p$ and $\mathbb{T}_{p,0}$ to be bounded and compact. Furthermore, we discuss about point spectrum, approximate point spectrum and spectrum of multiplication operators and discuss when a multiplication operator is an isometry. This chapter is based on our article [54].

3.2 Bounded multiplication operators

A Banach space X of functions on Ω is said to be a *functional Banach space* if for each $v \in \Omega$, the point evaluation map $e_v : X \to \mathbb{C}$ defined by $e_v(f) = f(v)$ is a bounded linear functional on X. The following result is well-known.

Lemma 3.2.1. *[35, Lemma 11] Let X be a functional Banach space on the set Ω and ψ be a complex-valued function on Ω such that M_ψ maps X into itself. Then M_ψ is bounded on X and $|\psi(v)| \leq \|M_\psi\|$ for all $v \in \Omega$. In particular, ψ is a bounded function.*

Proof. The boundedness of M_ψ follows from the closed graph theorem. For each $v \in \Omega$, there exists a $f \in X$ with $f(v) \neq 0$ so that $\|e_v\| > 0$ for all $v \in \Omega$. For $f \in X$ and $v \in \Omega$, we have

$$|\psi(v)f(v)| = |M_\psi f(v)| \leq \|M_\psi f\| \, \|e_v\| \leq \|M_\psi\| \, \|f\| \, \|e_v\| \, .$$

By taking supremum over $\|f\| = 1$, we get

$$|\psi(v)| \, \|e_v\| \leq \|M_\psi\| \, \|e_v\| \, ,$$

which completes the proof. □

It is natural to ask whether $\mathbb{T}_p$ and $\mathbb{T}_{p,0}$ are functional Banach spaces or not.

Proposition 3.2.2. *For $1 \leq p \leq \infty$, $\mathbb{T}_p$ and $\mathbb{T}_{p,0}$ are functional Banach spaces.*

Proof. First we consider the case when $p = \infty$. Since $|e_v(f)| = |f(v)| \leq \|f\|_\infty$ for every $v \in T$, it follows that $\mathbb{T}_\infty$ is a functional Banach space.

Let us now consider the case when $1 \leq p < \infty$. The point evaluation map at o, namely, e_o, is a bounded linear functional on $\mathbb{T}_p$ because of the fact that $|f(o)| \leq \|f\|_p$

for every f in $\mathbb{T}_p$. Now, we fix $v \in T$ and $v \neq o$. Then, from Lemma 2.6.1, we have

$$|e_v(f)| = |f(v)| \leq ((q+1)q^{|v|-1})^{\frac{1}{p}} \|f\|_p \quad \text{for every} \quad f \in \mathbb{T}_p.$$

So e_v is a bounded linear functional on $\mathbb{T}_p$ with $\|e_v\| \leq ((q+1)q^{|v|-1})^{\frac{1}{p}}$. Hence $\mathbb{T}_p$ is a functional Banach space. A similar proof works also for the space $\mathbb{T}_{p,0}$. The proof is complete. □

Throughout this chapter, X denotes either $\mathbb{T}_p$ or $\mathbb{T}_{p,0}$ with the norm $\|\cdot\|_p$ defined by (2.2.2), where $1 \leq p \leq \infty$.

Theorem 3.2.3. *Let T be a $(q+1)-$homogeneous tree rooted at o and ψ be a complex-valued function on T. Then the following are equivalent (compare with [70, Proposition 2]).*

(a) *M_ψ is a bounded linear operator on X,*

(b) *ψ is a bounded function on T, i.e., $\psi \in \mathbb{T}_\infty$.*

Moreover, $\|M_\psi\| = \|\psi\|_\infty$.

Proof. (a) $\Rightarrow$ (b) : We will prove this implication by a method of contradiction. Suppose that ψ is an unbounded function on T. Then there exists a sequence of vertices $\{v_k\}$ such that

$$|v_1| < |v_2| < |v_3| \cdots, \quad \text{and} \quad |\psi(v_k)| \geq k.$$

For each k, define $f_k : T \to \mathbb{C}$ by $f_k = C_{k,p}\chi_{v_k}$, where the constants $C_{k,p}$'s are chosen in such a way that $\|f_k\|_p = 1$. Note that $f_k \in X$ for each $k \in \mathbb{N}$. We obtain that

$$k = k\|f_k\|_p \leq |\psi(v_k)| \, \|f_k\|_p = M_p(|v_k|, \psi f_k) \leq \|\psi f_k\|_p = \|M_\psi f_k\|_p$$

for every $k \in \mathbb{N}$, which gives a contradiction to our assumption. Hence, ψ is a bounded function on T.

(b) $\Rightarrow$ (a) : Suppose that ψ is a bounded function on T and $1 \leq p < \infty$. Then for any $f \in X$,

$$\begin{aligned} M_p(n, M_\psi f) &= \left(\frac{1}{c_n} \sum_{|v|=n} |(M_\psi f)(v)|^p \right)^{\frac{1}{p}} \\ &= \left(\frac{1}{c_n} \sum_{|v|=n} |\psi(v)|^p |f(v)|^p \right)^{\frac{1}{p}}, \end{aligned}$$

which shows that

$$M_p(n, M_\psi f) \leq \|\psi\|_\infty M_p(n, f). \tag{3.2.1}$$

For $p = \infty$, the inequality (3.2.1) is trivially holds. From (3.2.1), one can also observe that $M_\psi f \in X$ whenever $f \in X$. Taking the supremum over $n \in \mathbb{N}_0$ on both sides of (3.2.1), we deduce that $\|M_\psi f\|_p \leq \|\psi\|_\infty \|f\|_p$ and thus, M_ψ is bounded linear operator from X to X with $\|M_\psi\| \leq \|\psi\|_\infty$.

Since X is a functional Banach space, by Lemma 3.2.1, one has $|\psi(v)| \leq \|M_\psi\|$ for all $v \in T$ which by taking the supremum gives $\|\psi\|_\infty \leq \|M_\psi\|$. Therefore, it follows that $\|M_\psi\| = \|\psi\|_\infty$. □

3.3 Spectrum

In this section, we compute point spectrum, approximate point spectrum and spectrum of multiplication operator M_ψ.

Theorem 3.3.1. *Let M_ψ be a bounded multiplication operator on X. Then*

(a) $\sigma_e(M_\psi) =$ *Range of* $\psi = \psi(T)$*;*

(b) $\sigma(M_\psi) = \sigma_a(M_\psi) = \overline{\psi(T)}$*.*

Proof. In order to prove (a), we begin by letting $\lambda \in \sigma_e(M_\psi)$. Then there exists a non-zero function $f \in X$ such that $\psi f = M_\psi f = \lambda f$. Since $f \neq 0$, there is a vertex v such that $f(v) \neq 0$ and $(\psi(v) - \lambda) f(v) = 0$. Thus, $\lambda = \psi(v) \in \psi(T)$ and therefore, $\sigma_e(M_\psi) \subseteq \psi(T)$.

Conversely, suppose that $\alpha \in \psi(T)$. Then there exists a vertex v such that $\psi(v) = \alpha$. Thus, $M_\psi(\chi_v) = \alpha\chi_v$ and $0 \neq \chi_v \in X$. This gives $\alpha \in \sigma_e(M_\psi)$. Hence, $\sigma_e(M_\psi) = \psi(T)$.

Before proving (b), we observe that for every $\lambda \in \mathbb{C}, M_\psi - \lambda I = M_{\psi-\lambda}$. Thus, $\lambda \in \sigma(M_\psi)$ if and only if $M_{\psi-\lambda}$ is not invertible if and only if $\frac{1}{\psi-\lambda}$ is not a bounded function on T (see Theorem 3.2.3). Now, we let $\lambda \notin \overline{\psi(T)}$. Since the complement of $\overline{\psi(T)}$ is open, there exists an $r > 0$ such that the disk of radius r centered at λ is a subset of $\mathbb{C} \setminus \overline{\psi(T)}$. So, $|\psi(v) - \lambda| \geq r$ for every $v \in T$. Thus $\frac{1}{\psi-\lambda}$ is a bounded function and therefore, by Theorem 3.2.3, $M_{\frac{1}{\psi-\lambda}}$ is a bounded operator on X. It is easy to verify that $M_{\frac{1}{\psi-\lambda}}$ is the inverse of $M_{\psi-\lambda}$ and hence, $M_{\psi-\lambda}$ is invertible. We conclude that λ cannot be in the spectrum, which in turn implies that $\sigma(M_\psi) \subseteq \overline{\psi(T)}$.

On the other hand, $\psi(T) = \sigma_e(M_\psi) \subseteq \sigma_a(M_\psi) \subseteq \sigma(M_\psi) \subseteq \overline{\psi(T)}$ and the fact that the approximate point spectrum and the spectrum are closed subsets of $\mathbb{C}$, give that $\sigma(M_\psi) = \sigma_a(M_\psi) = \overline{\psi(T)}$. □

Remark 3.3.2. The operator $M_\psi : X \to X$ is not injective if and only if $0 \in \sigma_e(M_\psi) = \psi(T)$. So, 0 is in the range of ψ is a necessary and sufficient condition for M_ψ not being injective on X.

3.4 Compact multiplication operators

Theorem 3.4.1. *Let M_ψ be a bounded multiplication operator on X. Then M_ψ is a compact operator on X if and only if $\psi(v) \to 0$ as $|v| \to \infty$.*

Proof. Let M_ψ be a compact operator on X. Then, from [67, Theorem 4.25], $\sigma_e(M_\psi) = \psi(T)$ (as well as $\sigma(M_\psi)$) is a countable set with 0 as the only possible limit point. Suppose $\psi(v) \not\to 0$ as $|v| \to \infty$. Then there exist an $\epsilon > 0$ and a sequence $\{v_k\}$ in T such that $|v_k| \to \infty$ and $|\psi(v_k)| \geq \epsilon$ for all k. By Bolzano-Weierstrass theorem, $\{\psi(v_k)\}$ has a limit point. Because $|\psi(v_k)| \geq \epsilon$ for all k, we obtain a contradiction to the fact that 0 is the only possible limit point. Hence, $\psi(v) \to 0$ as $|v| \to \infty$.

For the proof of the converse part, we use the fact that the set of compact operators is a closed subspace of the set of all bounded operators (see [67, Theorem 4.18 part (c)]).

Consider a function ψ from $C_c(T)$. Then there exists an $N \in \mathbb{N}$ such that $\psi(v) = 0$ for every $|v| > N$. So, $(M_\psi f)(v) = \psi(v)f(v) = 0$ for every $|v| > N$ and for every $f \in X$. Thus, the range of M_ψ is a finite dimensional subspace, which shows that M_ψ is a compact operator ([67, Theorem 4.18 part (a)]).

Let ψ be a arbitrary function such that $\psi(v) \to 0$ as $|v| \to \infty$. For each n, define $\{f_n\}$ by

$$f_n(v) = \begin{cases} \psi(v) & \text{if } |v| \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

By definition $f_n \in C_c(T)$ for every n. Thus, M_{f_n} is a compact operator for every n. Moreover,

$$\|M_{f_n} - M_\psi\| = \|M_{f_n-\psi}\| = \|f_n - \psi\|_\infty = \sup_{|v|>n} |\psi(v)|,$$

which approaches zero as $|v| \to \infty$, because $\psi(v) \to 0$ as $|v| \to \infty$. Thus, M_ψ is the limit (in the operator norm) of a sequence $\{M_{f_n}\}$ of compact operators, and hence, M_ψ is compact on X. The proof is now complete. □

Lemma 3.4.2. *If M_ψ is a compact operator on X, then, for every bounded sequence $\{f_n\}$ in X converging to 0 pointwise, the sequence $\|\psi f_n\| \to 0$ as $n \to \infty$.*

Proof. Suppose that $\{g_n\}$ in X is a bounded sequence converging to 0 pointwise. Since M_ψ is a compact operator, there is a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ such that $\{\psi g_{n_k}\} = \{M_\psi(g_{n_k})\}$ converges in $\|\cdot\|_p$ to some function, say, g. It follows that $\{\psi g_{n_k}\}$ converges to g pointwise. Since the convergence of $\{g_n\}$ to 0 implies that $g \equiv 0$, we deduce that $\{\psi g_{n_k}\}$ converges to 0 in $\|\cdot\|_p$.

Let $\{f_n\}$ be a bounded sequence in X converging to 0 pointwise. We claim that $\|M_\psi(f_n)\| = \|\psi f_n\| \to 0$ as $n \to \infty$. Suppose that $\|M_\psi(f_n)\| \not\to 0$ as $n \to \infty$. Then there exists a subsequence $\{f_{n_j}\}$ and an $\epsilon > 0$ such that $\left\|M_\psi(f_{n_j})\right\| \geq \epsilon$ for all j. By taking $g_n = f_{n_j}$ in the last paragraph, we find that $\{\psi g_{n_k}\}$ converges to 0 in $\|\cdot\|_p$, which is not possible because $\left\|M_\psi(f_{n_j})\right\| \geq \epsilon$ for all j. Hence, $\|M_\psi(f_n)\| = \|\psi f_n\| \to 0$ as $n \to \infty$, and the proof is complete. □

3.5 Upper bound for the essential norm

The following theorem is a natural generalization of Theorem 3.4.1.

Theorem 3.5.1. *Let M_ψ be a bounded multiplication operator on X. Then*

$$\|M_\psi\|_e \le \limsup_{n\to\infty} M_\infty(n,\psi) = \lim_{n\to\infty} \sup_{|v|\ge n} |\psi(v)|.$$

Proof. For each $n \in \mathbb{N}$, define $\{\psi_n\}$ by

$$\psi_n(v) = \begin{cases} \psi(v) & \text{if } |v| < n, \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, M_{ψ_n} is a compact operator for every $n \in \mathbb{N}$ and thus, for every $n \in \mathbb{N}$,

$$\begin{aligned} \|M_\psi\|_e &= \inf\{\|M_\psi - K\| : K \in \mathcal{K}(X)\} \\ &\le \|M_\psi - M_{\psi_n}\| = \|\psi - \psi_n\|_\infty \\ &= \sup_{|v|\ge n} |\psi(v)| = \sup_{m\ge n} M_\infty(m,\psi). \end{aligned}$$

Hence, $\|M_\psi\|_e \le \inf_{n\in\mathbb{N}} \sup_{|v|\ge n} |\psi(v)| = \limsup_{n\to\infty} M_\infty(n,\psi)$ and the proof is complete. □

We would like to point out that one way implication of Theorem 3.4.1 follows from Theorem 3.5.1. Indeed if $\psi(v) \to 0$ as $|v| \to \infty$, then Theorem 3.5.1 gives that $\|M_\psi\|_e = 0$ and thus, M_ψ is a compact operator.

3.6 Isometry

The following result tells us that isometric multiplication operators are induced by unimodular symbols.

Theorem 3.6.1. *Let $M_\psi : X \to X$ be a bounded multiplication operator. Then M_ψ is an isometry on X if and only if $|\psi(v)| = 1$ for all $v \in T$.*

Proof. Suppose that $|\psi(v)| = 1$ for all $v \in T$. Then $M_p(n, \psi f) = M_p(n, f)$ for all n, which shows that $\|f\|_p = \|\psi f\|_p = \|M_\psi(f)\|_p$ and thus, M_ψ is an isometry on X.

Conversely, suppose that M_ψ is an isometry on X. First we consider the case $p=\infty$. Let f be χ_v. Because M_ψ is an isometry on X, we have

$$|\psi(v)| = \|M_\psi(f)\|_\infty = \|\psi f\|_\infty = \|f\|_\infty = 1,$$

which holds for every $v \in T$. Hence $|\psi(v)| = 1$ for all $v \in T$.

Next, we consider the case $1 \leq p < \infty$. Let f be χ_o. Since M_ψ is an isometry, we obtain that

$$|\psi(o)| = \|M_\psi(f)\|_\infty = \|\psi f\|_\infty = \|f\|_\infty = 1,$$

Next, take an arbitrary element $v \in T$ with $|v| \geq 1$ and let f be χ_v. Moreover, since M_ψ is an isometry, we have

$$\left(\frac{|\psi(v)|^p}{c_n}\right)^{\frac{1}{p}} = \|M_\psi(f)\|_p = \|f\|_p = \left(\frac{1}{c_n}\right)^{\frac{1}{p}},$$

which shows that $|\psi(v)| = 1$ for all $v \in T$. It completes the proof. $\square$

Chapter 4

Composition operators on $\mathbb{T}_p$

4.1 Introduction

Let Ω be a nonempty set and X be a complex Banach space of complex-valued functions defined on Ω. For a self-map ϕ of Ω, the composition operator C_ϕ induced by the symbol ϕ is defined as

$$C_\phi(f) = g \quad \text{where} \quad g(x) = f(\phi(x)) \quad \text{for all } x \in \Omega \text{ and } f \in X.$$

The study of composition operators on analytic function spaces has a rich structure. In the classical case, Ω is the unit disk $\mathbb{D}$ and the choices for X are analytic functions spaces, eg. the Hardy spaces H^p, the Bergman spaces A^p, the Bloch space $\mathcal{B}$. The study of composition operators on various analytic function spaces defined on $\mathbb{D}$ is well known. There are excellent books on composition operators, see [29, 68, 69] and the references therein. The approach in the first two books [29, 68] are function theoretic whereas [69] deals in measure theoretic point of view.

Book of Cowen and MacCluer [29] deals with composition operators defined on various spaces of analytic functions on the unit disk, whereas the book of Shapiro [68] is devoted mainly to composition operators on the Hardy space H^2. The composition operators on various measure spaces are discussed in the book of Singh and Manhas [69]. These books bring together many well-developed aspects of the subject along with several open problems. Also, there is a number of articles dealing with composition operators on different transform spaces, see for example [1, 2, 22].

The study of composition operators on discrete function space was first initiated by Colonna et al. [7]. In that paper, Lipschitz space of a tree was investigated. Composition operators on weighted Banach spaces of an infinite tree were considered in [10].

In this chapter, we give equivalent conditions for the composition operator C_ϕ to be bounded on $\mathbb{T}_p$ and on $\mathbb{T}_{p,0}$ spaces and compute their operator norms. We also characterize invertible composition operators and isometric composition operators on $\mathbb{T}_p$ and on $\mathbb{T}_{p,0}$ spaces. Also, we discuss the compactness of C_ϕ on $\mathbb{T}_p$ spaces and finally we prove that there are no compact composition operators on $\mathbb{T}_{p,0}$ spaces.

This chapter is based on our articles [53, 56].

4.2 Bounded composition operators on $\mathbb{T}_p$

Before we proceed to discuss our results, it is appropriate to recall some basic results about bounded composition operators in the classical setting. For example (see [29, Corollary 3.7]), every analytic self-map ϕ of $\mathbb{D}$ induces bounded composition operator C_ϕ on H^p, $1 \leq p < \infty$. Moreover,

$$\|C_\phi\|^p \leq \frac{1 + |\phi(0)|}{1 - |\phi(0)|}. \tag{4.2.1}$$

It is also known that (see [29, Theorem 3.8]) equality holds in (4.2.1) for every inner function of $\mathbb{D}$ (for example, for every automorphism of $\mathbb{D}$). For the case $p = \infty$, it is easy to see that $\|C_\phi\| = 1$ for every analytic self-map ϕ of $\mathbb{D}$.

In this section, we discuss boundedness of composition operator C_ϕ on $\mathbb{T}_p$ spaces and compute their norm. Before we move on to further discussion, let us fix some notation.

We let ϕ be a self-map of $(q+1)-$homogeneous rooted tree T. For $n \in \mathbb{N}_0$ and $w \in T$, let $N_\phi(n, w)$ denote the number of pre-images of w for ϕ in $|v| = n$. That is, $N_\phi(n, w)$ is the number of elements in $\{\phi^{-1}(w)\} \bigcap \{|v| = n\}$. Finally, for each m and $n \in \mathbb{N}_0$, let $N_{m,n}$ denote the maximum of $N_\phi(n, w)$ over $|w| = m$. It is obvious that $\sum\limits_{m=0}^{\infty} N_{m,n} \leq c_n$

for each n. For $w \in T$, we define the weight function W as follows:

$$W(w) := \begin{cases} (q+1)q^{|w|-1} & \text{if } w \in T \setminus \{o\}, \\ 1 & \text{if } w = o. \end{cases} \tag{4.2.2}$$

Note that $W(w)$ is nothing but $c_{|w|}$. For the boundedness of C_ϕ, we will discuss case by case.

Theorem 4.2.1. *Every self-map ϕ of T induces bounded composition operator C_ϕ on $\mathbb{T}_\infty$ with $\|C_\phi\| = 1$.*

Proof. For each $f \in \mathbb{T}_\infty$ and every self-map ϕ of T, we have

$$\|C_\phi(f)\|_\infty = \|f \circ \phi\|_\infty = \sup_{w \in \phi(T)} |f(w)| \leq \|f\|_\infty .$$

Thus, C_ϕ is bounded on $\mathbb{T}_\infty$ with $\|C_\phi\| \leq 1$.

For the converse, note that

$$\|C_\phi(\chi_v)\|_\infty = \|\chi_v \circ \phi\|_\infty = 1 = \|\chi_v\|_\infty$$

for each $v \in \phi(T)$, where χ_v denotes the characteristic function on $\{v\}$. It gives that $\|C_\phi\| \geq 1$. Hence the result follows. ☐

In order to study the boundedness of the composition operators on $\mathbb{T}_p$ for $1 \leq p < \infty$, it is convenient to deal with the case $q = 1$ and $q \geq 2$ independently. First, we begin with the case $q = 1$.

Next, we consider composition operators on $\mathbb{T}_p$ for $1 \leq p < \infty$ over $2-$homogeneous trees.

Theorem 4.2.2. *Let T be a $2-$homogeneous tree with root o and let $D_n = \{a_n, b_n\}$ for each $n \in \mathbb{N}$. Furthermore, let ϕ be a self-map of T and C_ϕ be the composition operator on $\mathbb{T}_p$, $1 \leq p < \infty$. We have the following:*

(1) *If $\phi(o) \neq o$, then $\|C_\phi\|^p = 2$.*

(2) *If $\phi(o) = o$, then any one of the following distinct cases must occur:*

(a) *Either* $\phi \equiv o$ *or for every* $n \in \mathbb{N}$*, if* ϕ *maps* D_n *bijectively onto* D_m *for some* $m \in \mathbb{N}$ *then* $\|C_\phi\|^p = 1$.

(b) *If* ϕ *maps exactly one element of* D_n *to* o *for each* $n \in \mathbb{N}$ *then* $\|C_\phi\|^p = \frac{3}{2}$.

(c) *Either there exists an* $n \in \mathbb{N}$ *such that* $\phi(a_n) = \phi(b_n) \neq o$ *or if there exists an* $n \in \mathbb{N}$ *such that* $|\phi(a_n)|$ *and* $|\phi(b_n)|$ *are not equal and different from* 0 *then* $\|C_\phi\|^p = 2$.

Proof. From the growth estimate (see Lemma 2.6.1) for 2−homogeneous trees, it follows that for each $n \in \mathbb{N}_0$,

$$M_p^p(n, C_\phi f) = \frac{1}{c_n} \sum_{|v|=n} |f(\phi(v))|^p \leq 2\,\|f\|^p \quad \text{for every} \quad f \in \mathbb{T}_p.$$

This yields that $\|C_\phi\|^p \leq 2$. Thus every self-map ϕ of T induces a bounded C_ϕ on $\mathbb{T}_p$ with $\|C_\phi\|^p \leq 2$.

Suppose that $w = \phi(o) \neq o$. For $f = 2^{\frac{1}{p}}\chi_w$, we have $\|f\| = 1$ and $\|C_\phi(f)\|^p = 2$ and hence, $\|C_\phi\|^p = 2$.

Now suppose that $\phi(o) = o$. Then we need to consider all the five possible cases.

Suppose that $\phi \equiv o$. Then for each $n \in \mathbb{N}_0$,

$$M_p^p(n, C_\phi f) = |f(o)|^p \leq \|f\|^p \quad \text{for every} \quad f \in \mathbb{T}_p.$$

This yields that $\|C_\phi\|^p \leq 1$. For $f = \chi_o$, we obtain that $\|f\|^p = \|C_\phi(f)\|^p$ and thus, $\|C_\phi\|^p = 1$.

Suppose that for every $n \in \mathbb{N}$, ϕ maps D_n bijectively onto D_m for some $m \in \mathbb{N}$. Then, $M_p^p(n, C_\phi f) = M_p^p(m, f)$ for every $n \in \mathbb{N}$ and for some $m \in \mathbb{N}$. Thus

$$\|C_\phi f\|^p \leq \|f\|^p \quad \text{for every} \quad f \in \mathbb{T}_p,$$

which gives that $\|C_\phi\|^p \leq 1$. As in the previous case, by considering $f = \chi_o$, we get $\|C_\phi\|^p = 1$.

Suppose that ϕ maps exactly one element of D_n to o for each $n \in \mathbb{N}$. Then, in view of growth estimate for 2−homogeneous trees along with this assumption, we see that

$$\|C_\phi f\|^p \leq \frac{3}{2}\,\|f\|^p \quad \text{for every} \quad f \in \mathbb{T}_p$$

which gives $\|C_\phi\|^p \leq 3/2$. On the other hand, by assumption, either a_1 or b_1 maps to o. Without loss of generality, we assume that $\phi(a_1) = o$. Take $\phi(b_1) = w$ and $f = \chi_o + 2^{\frac{1}{p}}\chi_w$. Then, $\|f\| = 1$ and

$$M_p^p(1, C_\phi f) = \frac{3}{2} = \|C_\phi(f)\|^p\,.$$

Thus, $\|C_\phi\|^p = 3/2$.

Now assume that there exists an $n \in \mathbb{N}$ such that $w = \phi(a_n) = \phi(b_n) \neq o$. We have already observed that $\|C_\phi\|^p \leq 2$. For $f = 2^{\frac{1}{p}}\chi_w$, we have

$$\|f\| = 1 \quad \text{and} \quad \|C_\phi(f)\|^p = 2$$

and therefore, $\|C_\phi\|^p = 2$.

Finally, assume that there exists an $n \in \mathbb{N}$ such that $|\phi(a_n)|$ and $|\phi(b_n)|$ are not equal and are different from 0. Now, we take

$$f = 2^{\frac{1}{p}}(\chi_u + \chi_v),$$

where $\phi(a_n) = u$ and $\phi(b_n) = v$. It follows that $\|f\| = 1$ and $\|C_\phi(f)\|^p = 2$, which gives that $\|C_\phi\|^p = 2$. The proof is complete. □

Corollary 4.2.3. *For every self-map ϕ of 2−homogeneous tree T, C_ϕ is bounded on $\mathbb{T}_p$ with $\|C_\phi\|^p \leq 2$, $1 \leq p < \infty$.*

Theorem 4.2.4. *If T is a $(q+1)$−homogeneous tree with $q \geq 2$ such that*

$$\sup_{n\in\mathbb{N}} \left(\sum_{|v|=n} q^{|\phi(v)|-n} \right) < \infty, \tag{4.2.3}$$

then C_ϕ is bounded on $\mathbb{T}_p$, $1 \leq p < \infty$.

Proof. For $n \in \mathbb{N}$, $w \in T$ and $f \in \mathbb{T}_p$, by Lemma 2.6.1 on growth estimate, we have

$$M_p^p(n, C_\phi f) \leq \frac{1}{c_n} \sum_{|v|=n} (q+1)q^{|\phi(v)|-1} \|f\|^p = \sum_{|v|=n} q^{|\phi(v)|-n} \|f\|^p.$$

Moreover,

$$M_p^p(0, C_\phi f) = |f(\phi(o))|^p \leq (q+1)q^{|\phi(o)|-1} \|f\|^p$$

and thus,

$$\|C_\phi f\|^p \leq \max \left\{ (q+1)q^{|\phi(o)|-1}, \quad \sup_{n\in\mathbb{N}} (\sum_{|v|=n} q^{|\phi(v)|-n}) \right\} \|f\|^p$$

showing that C_ϕ is bounded on $\mathbb{T}_p$. $\square$

Theorem 4.2.5. *Let T be a $(q+1)-$homogeneous tree and $1 \leq p < \infty$. If C_ϕ is bounded on $\mathbb{T}_p$, then*

$$\sup_{w\in T} \sup_{n\in\mathbb{N}_0} \left\{ \frac{W(w)}{c_n} N_\phi(n,w) \right\} \leq \|C_\phi\|^p.$$

Proof. For each $w \in T$, define $f_w = \{W(w)\chi_w\}^{\frac{1}{p}}$, where W is defined in (4.2.2). It is easy to verify that for every $w \in T$, $M_p(n, f_w) = 1$ when $n = |w|$ and 0 otherwise. This observation gives that $\|f_w\| = 1$ for all $w \in T$. Now, for each fixed $w \in T$, we have for $n \in \mathbb{N}_0$,

$$\begin{aligned} M_p^p(n, C_\phi f_w) &= \frac{1}{c_n} \sum_{|v|=n} W(w)\chi_w(\phi(v)) \\ &= \frac{1}{c_n} \sum_{\substack{|v|=n \\ \phi(v)=w}} W(w) = \frac{W(w)}{c_n} N_\phi(n,w) \end{aligned}$$

which yields that

$$\|C_\phi f_w\|^p = \sup_{n\in\mathbb{N}_0} \left\{ \frac{W(w)}{c_n} N_\phi(n,w) \right\}.$$

Consequently,

$$\|C_\phi\|^p = \sup_{\|f\|=1} \|C_\phi(f)\|^p \geq \sup_{w\in T} \|C_\phi(f_w)\|^p = \sup_{w\in T} \sup_{n\in\mathbb{N}_0} \left\{ \frac{W(w)}{c_n} N_\phi(n,w) \right\}$$

and the desired conclusion follows. $\square$

Corollary 4.2.6. *If C_ϕ is bounded on $\mathbb{T}_p$, then*

$$\sup\left\{q^{|w|-n}N_\phi(n,w) : w \in T\setminus\{o\}, n\in\mathbb{N}\right\}$$

is finite.

Proof. For $w \in T\setminus\{o\}$ and $n\in\mathbb{N}$, we note that

$$\frac{W(w)}{c_n} = q^{|w|-n}.$$

The desired result follows by Theorem 4.2.5. ☐

Corollary 4.2.7. *If ϕ fixes the root, namely, $\phi(o) = o$, then $\|C_\phi\| \geq 1$.*

Proof. Let f be the characteristic function on the root o. Clearly, $\|f\| = 1$ and $M_p(0, C_\phi f) = |f(\phi(o))| = |f(o)| = 1$. We see that

$$\|C_\phi\| = \sup_{\|g\|=1} \|C_\phi(g)\| \geq \|C_\phi(f)\| \geq M_p(0, C_\phi f) = 1$$

and the result follows. ☐

Corollary 4.2.8. *If ϕ does not fix the root, i.e. $\phi(o) \neq o$, then*

$$\|C_\phi\|^p \geq (q+1)q^{|\phi(o)|-1}.$$

Proof. Let $w = \phi(o)$ and, as before, consider $f_w = \{W(w)\chi_w\}^{\frac{1}{p}}$. Now, we observe that

$$\|f_w\| = 1 \;\text{ and }\; M_p^p(0, C_\phi f_w) = |f_w(\phi(o))|^p = (q+1)q^{|w|-1}$$

which shows that $\|C_\phi(f_w)\|^p \geq (q+1)q^{|w|-1}$ and the desired conclusion follows. ☐

Next, we consider composition operators on $\mathbb{T}_p$ for $1 \leq p < \infty$ over $(q+1)-$homogeneous tree with $q \geq 2$. A self-map ϕ of T is called *bounded* if $\{|\phi(v)| : v \in T\}$ is a bounded set in $\mathbb{N}_0$. From Theorem 4.2.4, it is easy to see that every bounded self-map of T induces a bounded composition operator. Recall that $N_{m,n}$ denotes the maximum of $N_\phi(n,w)$ over $|w| = m$.

Theorem 4.2.9. *If T is a $(q+1)-$homogeneous tree with $q \geq 2$ and ϕ is a self-map of T such that $\sup_{v\in T} |\phi(v)| = M$, then $\|C_\phi\|^p \leq c_M$. Moreover, $\|C_\phi\|^p = c_M$ if and only if*

$$\sup_{n\in\mathbb{N}_0} \frac{N_{M,n}}{c_n} = 1.$$

Proof. For $n \in \mathbb{N}_0$ and $f \in \mathbb{T}_p$, by Lemma 2.6.1 on growth estimate, we have

$$M_p^p(n, C_\phi f) \leq c_M \|f\|^p.$$

Thus, ϕ induces a bounded C_ϕ with $\|C_\phi\|^p \leq c_M$.

Let us now prove the equality case. Suppose that

$$\sup_{n\in\mathbb{N}_0} \frac{N_{M,n}}{c_n} = 1.$$

Then there are two cases. First we consider the case $N_{M,k} = c_k$ for some $k \in \mathbb{N}_0$. This means that $\phi : D_k \to D_M$ is a constant, say, $\phi(v) = w \in D_M$ for all $v \in D_k$. For $f = (c_M)^{\frac{1}{p}}\chi_w$, we have

$$\|f\| = 1 \quad \text{and} \quad \|C_\phi(f)\|^p = c_M$$

which proves that $\|C_\phi\|^p = c_M$.

Next, we suppose that $N_{M,k} \neq c_k$ for all $k \in \mathbb{N}_0$. Then, there is a sequence $\{n_k\}$ such that

$$\frac{N_{M,n_k}}{c_{n_k}} \to 1 \ \text{ as } k \to \infty.$$

For each $k \in \mathbb{N}$, choose $w_k \in D_M$ such that $N_{M,n_k} = N_\phi(n_k, w_k)$. Take $f_k = (c_M)^{\frac{1}{p}}\chi_{w_k}$ so that $\|f_k\| = 1$ and

$$c_M \frac{N_{M,n_k}}{c_{n_k}} \leq M_p^p(n_k, C_\phi(f)) \leq \|C_\phi(f)\|^p \leq \|C_\phi\|^p.$$

By allowing $k \to \infty$, we get $c_M \leq \|C_\phi\|^p$, and thus, $\|C_\phi\|^p = c_M$ in either case.

For the converse part, we assume that $\|C_\phi\|^p = c_M$. Suppose on the contrary that

$$\sup_{n\in\mathbb{N}_0} \frac{N_{M,n}}{c_n} \leq \delta < 1.$$

Then, $N_{M,n} \leq \delta c_n$ for every n. Note that, there are at least $N_{M,n}$ vertices from D_n mapped into D_M and therefore, there are at most $c_n - N_{M,n}$ vertices of D_n mapped into $\{v : |v| < M\}$ for each n. For $f \in \mathbb{T}_p$, we obtain

$$\begin{aligned} M_p^p(n, C_\phi f) &= \frac{1}{c_n} \sum_{\substack{|\phi(v)|=M \\ |v|=n}} |f(\phi(v))|^p + \frac{1}{c_n} \sum_{\substack{|\phi(v)|<M \\ |v|=n}} |f(\phi(v))|^p \\ &\leq \frac{N_{M,n}}{c_n} c_M \|f\|^p + \frac{c_n - N_{M,n}}{c_n} c_{M-1} \|f\|^p \\ &\leq (1 + (q-1)\delta) c_{M-1} \|f\|^p . \end{aligned}$$

Therefore,

$$\|C_\phi\|^p \leq (1 + (q-1)\delta) c_{M-1} < c_M,$$

which is a contradiction. Hence, $\|C_\phi\|^p = c_M$ if and only if $\sup_{n\in\mathbb{N}_0} \frac{N_{M,n}}{c_n} = 1$. □

Now, we consider general self-maps on $(q+1)-$homogeneous trees.

Theorem 4.2.10. *Let T be a $(q+1)-$homogeneous tree and $1 \leq p < \infty$. Then C_ϕ is bounded on $\mathbb{T}_p$ if and only if*

$$\alpha = \sup_{n\in\mathbb{N}_0} \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\} < \infty.$$

Moreover, $\|C_\phi\|^p = \alpha$.

Proof. Assume that $\alpha < \infty$. First we show that C_ϕ is bounded on $\mathbb{T}_p$. To do this, for $n \in \mathbb{N}_0$ and $f \in \mathbb{T}_p$, we find that

$$\begin{aligned} M_p^p(n, C_\phi f) &= \frac{1}{c_n} \left\{ \sum_{m=0}^{\infty} \sum_{\substack{|\phi(v)|=m \\ |v|=n}} |f(\phi(v))|^p \right\} \\ &\leq \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\} \|f\|^p \\ &\leq \alpha \|f\|^p , \end{aligned}$$

which yields that C_ϕ is bounded on $\mathbb{T}_p$ and

$$\|C_\phi\|^p \leq \alpha = \sup_{n\in\mathbb{N}_0} \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\}. \tag{4.2.4}$$

Conversely, suppose that C_ϕ is bounded on $\mathbb{T}_p$. In order to show that α is finite, we fix $n \in \mathbb{N}_0$. For each $m \in \mathbb{N}_0$, choose $v_m \in D_m$ such that $N_\phi(n, v_m) = N_{m,n}$. Take

$$f = \sum_{m=0}^{\infty} (c_m)^{\frac{1}{p}} \chi_{v_m},$$

so that $\|f\| = 1$ and

$$M_p^p(n, C_\phi f) = \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m,$$

which gives that

$$\alpha = \sup_{n\in\mathbb{N}_0} \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\} \leq \|C_\phi\|^p, \tag{4.2.5}$$

and hence the desired result follows. Moreover, by (4.2.4) and (4.2.5), it follows that $\|C_\phi\|^p = \alpha$. The proof is complete. □

4.3 Norm of C_ϕ for automorphism symbol

A self-map ϕ of T is called an *automorphism* of T, denoted as $\phi \in \mathrm{Aut}(T)$, if ϕ is bijective and any two vertices v, w are neighbours ($v \sim w$) if and only if $\phi(v) \sim \phi(w)$. Now we will compute the norm of the composition operator C_ϕ when the inducing symbol ϕ is an automorphism of T.

Theorem 4.3.1. *Let T be a $(q+1)-$homogeneous tree and consider C_ϕ on $\mathbb{T}_p$, where $1 \leq p < \infty$, $q \geq 1$ and $\phi \in \mathrm{Aut}(T)$. Then we have*

(i) $\|C_\phi\| = 1$ *if* $\phi(o) = o$,

(ii) $\|C_\phi\|^p = (q+1)q^{|\phi(o)|-1}$ *if* $\phi(o) \neq o$.

In particular, every $\phi \in \mathrm{Aut}(T)$ induces bounded composition operator C_ϕ on $\mathbb{T}_p$.

Proof. First, we consider the case $\phi(o) = o$. Then, for each n, ϕ is a bijective map from D_n to D_n (since $\phi \in \mathrm{Aut}(T)$ and $\phi(o) = o$). For $n \in \mathbb{N}_0$ and $f \in \mathbb{T}_p$, we thus have

$$M_p^p(n, C_\phi f) \quad = \quad \frac{1}{c_n} \sum_{|\phi(v)|=n} |f(\phi(v))|^p = M_p^p(n, f).$$

Taking supremum on both sides, we get $\|C_\phi(f)\| = \|f\|$ which proves the first part.

Next, we consider the case $\phi(o) \neq o$. The result is obviously true for $q = 1$, by Theorem 4.2.2. Thus, it suffices to prove the theorem for $(q+1)-$homogeneous tree with $q \geq 2$. Let $k = |\phi(o)|$. Since $\phi \in \mathrm{Aut}(T)$, it is easy to see that

Domain	**Range of ϕ contained in**
D_0	D_k
D_m $(1 \leq m \leq k-1)$	$D_{k+m}, D_{k+m-2}, \ldots, D_{k-m}$
D_k	$D_{2k}, D_{2k-2}, \ldots, D_2, D_0$
D_{k+m+1} $(m \geq 0)$	$D_{2k+m+1}, D_{2k+m-1}, \ldots, D_{2m+1}$

$$M_p^p(0, C_\phi f) = |f(\phi(0))|^p \leq (q+1)q^{k-1} \|f\|^p.$$

For the remaining part of the proof, we need to deal with the cases $n = m$ $(1 \leq m \leq k-1)$, $n = k$, and $n \geq k+1$ separately. We begin with

$$\begin{aligned}
M_p^p(m, C_\phi f) &= \frac{1}{c_m} \sum_{|v|=m} |f(\phi(v)|^p \\
&\leq \frac{1}{c_m} \left\{ \sum_{|v|=k+m} |f(v)|^p + \sum_{|v|=k+m-2} |f(v)|^p + \cdots + \sum_{|v|=k-m} |f(v)|^p \right\} \\
&\leq \frac{1}{c_m} \left\{ (q+1)q^{k+m-1} + (q+1)q^{k+m-3} + \cdots + (q+1)q^{k-m-1} \right\} \|f\|^p \\
&= \left\{ q^k + q^{k-2} + \cdots + q^{k-2m} \right\} \|f\|^p \\
&\leq (q+1)q^{k-1} \|f\|^p
\end{aligned}$$

showing that $M_p^p(n, C_\phi f) \leq (q+1)q^{k-1} \|f\|^p$ for $n = 1, 2, \ldots, k-1$. Next, for $n = k$, we find that

$$M_p^p(k, C_\phi f) \quad = \quad \frac{1}{(q+1)q^{k-1}} \sum_{|v|=k} |f(\phi(v)|^p$$

$$\begin{aligned}
&\le \frac{1}{(q+1)q^{k-1}}\left\{\sum_{|v|=2k}|f(v)|^p+\sum_{|v|=2k-2}|f(v)|^p+\cdots+\sum_{|v|=2}|f(v)|^p+|f(o)|^p\right\}\\
&\le \frac{1}{(q+1)q^{k-1}}\left\{(q+1)q^{2k-1}+(q+1)q^{2k-3}+\cdots+(q+1)q+1\right\}\|f\|^p\\
&= \left\{q^k+q^{k-2}+\cdots+q^{2-k}+\frac{1}{(q+1)q^{k-1}}\right\}\|f\|^p\\
&\le \left\{q^k+q^{k-2}+\cdots+q^{2-k}+q^{1-k}\right\}\|f\|^p\\
&\le (q+1)q^{k-1}\|f\|^p.
\end{aligned}$$

Finally, for each $m\in\mathbb{N}_0$,

$M_p^p(m+k+1,C_\phi f)$

$$\begin{aligned}
&= \frac{1}{(q+1)q^{m+k}}\sum_{|v|=m+k+1}|f(\phi(v)|^p\\
&\le \frac{1}{(q+1)q^{m+k}}\left\{\sum_{|v|=m+2k+1}|f(v)|^p+\sum_{|v|=m+2k-1}|f(v)|^p+\cdots+\sum_{|v|=2m+1}|f(v)|^p\right\}\\
&\le \frac{1}{(q+1)q^{m+k}}\left\{(q+1)q^{m+2k}+(q+1)q^{m+2k-2}+\cdots+(q+1)q^{m}\right\}\|f\|^p\\
&= \left\{q^k+q^{k-2}+\cdots+q^{-k}\right\}\|f\|^p\\
&\le (q+1)q^{k-1}\|f\|^p.
\end{aligned}$$

The above discussion implies that

$$M_p^p(n,C_\phi f)\le (q+1)q^{k-1}\|f\|^p \quad \text{for all } n\in\mathbb{N}_0$$

and thus, $\|C_\phi\|^p\le (q+1)q^{|\phi(o)|-1}$. Other way inequality follows from Corollary 4.2.8 and the proof is complete. □

From Theorems 4.2.1 and 4.3.1, we have the following result:

Corollary 4.3.2. *Let C_ϕ be a composition operator on $\mathbb{T}_p$ induced by an automorphic symbol ϕ of T. Then we have the following:*

(i) $\|C_\phi\|=1$ *if* $p=\infty$.

(ii) *For* $q\ge 1$ *and* $1\le p<\infty$,

$$\|C_\phi\|^p=\begin{cases}(q+1)q^{|\phi(o)|-1} & \text{if } \phi(o)\ne o,\\ 1 & \text{if } \phi(o)=o.\end{cases}$$

4.4 Injective symbols

Every self-map ϕ of T induces a bounded composition operator C_ϕ on $\mathbb{T}_\infty$, or $\mathbb{T}_p$ spaces over $2-$homogeneous trees. Unlike the classical Hardy space settings, there are self-maps ϕ of T which do not induce bounded C_ϕ on $\mathbb{T}_p$ with $1 \leq p < \infty$ over $(q+1)-$homogeneous trees, $q \geq 2$.

Example 4.4.1. For each $n \in \mathbb{N}_0$, choose the vertex v_n such that $v_n \in D_n$. Define $\phi_1(v) = v_n$ if $|v| = n$. Consider the function f defined by

$$f(v) = \begin{cases} (c_n)^{\frac{1}{p}} & \text{if } v = v_n \text{ for some } n \in \mathbb{N}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f \in \mathbb{T}_p$ with $\|f\| = 1$ and for each $m \in \mathbb{N}$, we see that

$$M_p^p(m, C_{\phi_1} f) \quad = \quad \frac{1}{c_m} \sum_{|v|=m} |f(v_m)|^p = c_m$$

showing that $\|C_{\phi_1} f\| = \sup_{m \in \mathbb{N}_0} M_p(m, C_{\phi_1} f)$ which is not finite for $q \geq 2$. This example shows that there are self-maps of T which induce unbounded composition operators on $\mathbb{T}_p$

The following example shows that there are bijective self-maps of T which do not induce bounded composition operator C_ϕ for $(q+1)-$homogeneous trees with $q \geq 2$.

Example 4.4.2. For each $n \in \mathbb{N}$ which is not of the form $n = 4k$, $k \in \mathbb{N}_0$, choose $v_n \in T$ such that $|v_n| = n$. Define

$$\phi(v) = \begin{cases} v_{4k+2} & \text{if } \ v = v_{2k+1} \ \text{ for some } \ k \in \mathbb{N}_0, \\ v_{2k+1} & \text{if } \ v = v_{4k+2} \ \text{ for some } \ k \in \mathbb{N}, \\ v & \text{elsewhere.} \end{cases}$$

Clearly, ϕ is bijective on T. For $k \in \mathbb{N}$, let $f_k = (c_{4k+2})^{\frac{1}{p}} \chi_{v_{4k+2}}$. Then $\|f\| = 1$ and

$$\|C_\phi\|^p \geq \|C_\phi(f_k)\|^p \geq M_p^p(2k+1, C_\phi(f_k)) = q^{2k+1}.$$

Since $q \geq 2$, it follows that C_ϕ is an unbounded operator on $\mathbb{T}_p$.

Motivated by the above example, we wish to characterize all the bounded composition operators that are induced by univalent (injective) symbols (see Corollary 4.4.4).

Proposition 4.4.3. *Let ϕ be a self-map of $(q+1)-$homogeneous tree T with $q \geq 2$, and $1 \leq p < \infty$. If C_ϕ is bounded on $\mathbb{T}_p$, then there exists an $M > 0$ such that $|\phi(v)| \leq |v| + M$ for all $v \in T$.*

Proof. Suppose that C_ϕ is bounded on $\mathbb{T}_p$. Set $a_n = \max_{|v|=n} |\phi(v)|$ for $n \in \mathbb{N}_0$, and for each n, choose $v_n \in D_n$ such that $|\phi(v_n)| = a_n$. Furthermore, for each n, take $f_n = (c_{a_n})^{\frac{1}{p}} \chi_{\phi(v_n)}$. Then

$$M_p^p(n, C_\phi f_n) = q^{a_n - n} \leq \|C_\phi\|^p ,$$

which gives that $\{a_n - n\}$ is a bounded sequence. The desired result follows. □

Converse of Proposition 4.4.3 holds if, in addition, ϕ is injective or finite-valent.

Corollary 4.4.4. *If ϕ is an injective self-map of $(q+1)-$homogeneous tree T with $q \geq 2$ and $1 \leq p < \infty$, then C_ϕ is bounded on $\mathbb{T}_p$ if and only if there exists an $M > 0$ such that $|\phi(v)| \leq |v| + M$ for all $v \in T$.*

Proof. Suppose that there exists an $M > 0$ such that $|\phi(v)| \leq |v| + M$ for all $v \in T$. Therefore, $a_n \leq n + M$ for all n, where a_n is taken as in Proposition 4.4.3. For an arbitrary function f with $\|f\| = 1$, we have

$$\begin{aligned}
M_p^p(n, C_\phi f) &\leq \frac{1}{c_n} \sum_{m=0}^{a_n} \sum_{|w|=m} |f(w)|^p \quad \text{(since } \phi \text{ is injective)} \\
&\leq \frac{1}{c_n} \sum_{m=0}^{a_n} c_m = \frac{1}{c_n} + \frac{1}{(q+1)\, q^{n-1}} \sum_{m=1}^{a_n} (q+1)\, q^{m-1} \\
&= \frac{1}{c_n} + \frac{1}{q^{n-1}} \sum_{k=0}^{a_n - 1} q^k = \frac{1}{c_n} + \frac{q^{a_n} - 1}{q^{n-1}(q-1)} \\
&\leq \frac{1}{c_n} + \frac{q^{a_n}}{q^{n-1}(q-1)} = \frac{1}{c_n} + q^{a_n - n} \frac{q}{q-1} \\
&\leq 1 + q^M \left(\frac{q}{q-1} \right).
\end{aligned}$$

Thus, C_ϕ is bounded on $\mathbb{T}_p$. The converse part is a consequence of Proposition 4.4.3. □

Definition 4.4.5. Let ϕ be a self-map of T and $k \in \mathbb{N}$ be fixed. We say that ϕ is $k-$*valent map* if every vertex of T has at most k pre-images and there is a vertex of T which has exactly k pre-images. The map ϕ is said to be *finite-valent* if there exists a $k \in \mathbb{N}$ such that ϕ is $k-$valent.

Corollary 4.4.6. *Let ϕ be a finite-valent self-map of $(q+1)-$homogeneous tree T with $q \geq 2$, and $1 \leq p < \infty$. Then the operator C_ϕ is bounded on $\mathbb{T}_p$ if and only if there exists an $M > 0$ such that $|\phi(v)| \leq |v| + M$ for all $v \in T$.*

Proof. Consult the proof of Proposition 4.4.3 and Corollary 4.4.4. □

Remark 4.4.7. Finite-valentness cannot be removed in Corollary 4.4.6. To do this, for each n, fix $v_n \in D_n$ and $\phi(v) = v_n$ if $|v| = n$. For each n, choose $f_n = (c_n)^{\frac{1}{p}} \chi_{v_n}$ so that

$$M_p^p(n, C_\phi f_n) = c_n \leq \|C_\phi\|^p ,$$

which gives that C_ϕ cannot be a bounded operator.

4.5 Bounded composition operators on $\mathbb{T}_{p,0}$

Proposition 4.5.1. *Let ϕ be a self-map of T and $f \in \mathbb{T}_p$. If $|\phi(v)| \to \infty$ and $|f(v)| \to 0$ as $|v| \to \infty$, then $|C_\phi f(v)| \to 0$ as $|v| \to \infty$.*

Proof. Assume the hypothesis. Let $\epsilon > 0$ be given. Then there exists an $N_1 \in \mathbb{N}$ such that $|f(w)| < \epsilon$ for all $|w| \geq N_1$. Given $N_1 > 0$, there exists an $N \in \mathbb{N}$ such that $|\phi(v)| \geq N_1$ for all $|v| \geq N$. This gives that $|C_\phi f(v)| < \epsilon$ for all $|v| \geq N$, i.e., $|C_\phi f(v)| \to 0$ as $|v| \to \infty$. □

Lemma 4.5.2. *1. $h \in \mathbb{T}_{\infty,0}$ if and only if $|h(v)| \to 0$ as $|v| \to \infty$.*

2. Let T be a 2-homogeneous tree. For $1 \leq p < \infty$, $h \in \mathbb{T}_{p,0}$ if and only if $|h(v)| \to 0$ as $|v| \to \infty$.

Proof. For $n \in \mathbb{N}$ and $h \in \mathbb{T}_\infty$, we have

$$M_\infty(n,h) = \max_{|v|=n} |h(v)| = h(v_n) \text{ for some } v_n \text{ with } |v_n| = n.$$

In view of this, it is easy to see that $h \in \mathbb{T}_{\infty,0}$ if and only if $M_\infty(n,h) \to 0$ as $n \to \infty$ if and only if $|h(v)| \to 0$ as $|v| \to \infty$.
Let T be a 2-homogeneous tree and for $n \in \mathbb{N}$, take $D_n = \{a_n, b_n\}$. For $h \in \mathbb{T}_p$, we have

$$M_p^p(n,h) = \frac{1}{2}(|h(a_n)|^p + |h(b_n)|^p).$$

This yields that $h \in \mathbb{T}_{p,0}$ if and only if $M_p(n,h) \to 0$ as $n \to \infty$ if and only if $|h(v)| \to 0$ as $|v| \to \infty$. □

Theorem 4.5.3. *The composition operator C_ϕ is bounded on $\mathbb{T}_{\infty,0}$ if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$. Moreover, $\|C_\phi\| = 1$.*

Proof. Since $\|C_\phi(f)\|_\infty \leq \|f\|_\infty$ for all $f \in T_\infty$, it is enough to prove that $\mathbb{T}_{\infty,0}$ is invariant under C_ϕ if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$.

Suppose that $|\phi(v)| \to \infty$ as $|v| \to \infty$. Then, by Proposition 4.5.1 and Lemma 4.5.2, $|C_\phi f(v)| \to 0$ as $|v| \to \infty$ for all $f \in T_{\infty,0}$. That is, $\mathbb{T}_{\infty,0}$ is invariant under C_ϕ.

For the converse part, assume that $|\phi(v)| \not\to \infty$ as $|v| \to \infty$. Then there exist a sequence $\{v_k\}$ and an $M > 0$ such that $|v_k| \geq k$ and $|\phi(v_k)| \leq M$ for all $k \in \mathbb{N}$. Define

$$f(v) = \begin{cases} 1 & \text{if } |v| \leq M, \\ 1/|v_k| & \text{if } |v| > M \text{ and } v = v_k \text{ for some } k \in \mathbb{N}, \\ 0 & \text{elsewhere.} \end{cases}$$

Then $f \in \mathbb{T}_{\infty,0}$. But $M_\infty(f \circ \phi, |v_k|) = 1$ for all k and so, $f \circ \phi \notin \mathbb{T}_{\infty,0}$. Finally, it is easy to verify that $\|C_\phi\| = 1$. This completes the proof. □

Theorem 4.5.4. *Let T be a $2-$homogeneous tree with root o and let $D_n = \{a_n, b_n\}$ for each $n \in \mathbb{N}$ and ϕ be a self-map of T. Then, C_ϕ is a bounded operator on $\mathbb{T}_{p,0}$, $1 \leq p < \infty$, if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$. Moreover, we have the following norm estimates:*

(1) *If $\phi(o) \neq o$, then $\|C_\phi\|^p = 2$.*

(2) *If $\phi(o) = o$, then any one of the following distinct cases occurs:*

   (a) *For every $n \in \mathbb{N}$, if ϕ maps D_n bijectively onto D_m for some $m \in \mathbb{N}$, then $\|C_\phi\|^p = 1$.*

   (b) *Either there exists an $n \in \mathbb{N}$ such that $\phi(a_n) = \phi(b_n) \neq o$ or if there exists an $n \in \mathbb{N}$ such that $|\phi(a_n)|$ and $|\phi(b_n)|$ are not equal and different from 0 then $\|C_\phi\|^p = 2$.*

Proof. For 2−homogeneous trees, every self-map ϕ induces a bounded composition operator on $\mathbb{T}_p$. Therefore it suffices to prove that $\mathbb{T}_{p,0}$ is invariant under C_ϕ if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$.

Necessary part follows from Proposition 4.5.1 and Lemma 4.5.2. For the proof of the sufficiency part, assume on the contrary that $|\phi(v)| \not\to \infty$ as $|v| \to \infty$. Take f as in Theorem 4.5.3. Then $f \in \mathbb{T}_{p,0}$. But $f(\phi(v_k)) = 1$ for all k which gives $M_p^p(f \circ \phi, |v_k|) \geq 1/2$ for all k and so, $f \circ \phi \notin \mathbb{T}_{p,0}$.

The proof of norm estimates is similar to that of Theorem 4.2.2. This completes the proof. □

Theorem 4.5.5. *Let $q \geq 2$ and $1 \leq p < \infty$. If $\frac{1}{c_n} \sum\limits_{m=0}^{\infty} N_{m,n} c_m \to 0$ as $n \to \infty$, then C_ϕ is bounded on $\mathbb{T}_{p,0}$. Moreover, $\|C_\phi\|^p = \alpha$, where*

$$\alpha = \sup_{n \in \mathbb{N}_0} \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\}.$$

Proof. By Theorem 4.2.10, for the boundedness of C_ϕ on $\mathbb{T}_{p,0}$, it is enough to prove that C_ϕ maps $\mathbb{T}_{p,0}$ into $\mathbb{T}_{p,0}$. It follows from the proof of Theorem 4.2.10 that

$$M_p^p(n, C_\phi f) \leq \left\{ \frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \right\} \|f\|^p,$$

which forces that $\mathbb{T}_{p,0}$ is invariant under C_ϕ whenever

$$\frac{1}{c_n} \sum_{m=0}^{\infty} N_{m,n} c_m \to 0 \text{ as } n \to \infty.$$

Moreover, we have $\|C_\phi\|^p \leq \alpha$.

To prove that the equality holds in the last inequality, we fix $n \in \mathbb{N}_0$. For each $m \in \mathbb{N}_0$, choose $v_m \in D_m$ such that $N_\phi(n, v_m) = N_{m,n}$. Then,

$$f_k = \sum_{m=0}^{k} (c_m)^{\frac{1}{p}} \chi_{v_m} \in \mathbb{T}_{p,0} \text{ and } \|f_k\| = 1 \text{ for all } k.$$

Therefore,

$$\frac{1}{c_n} \sum_{m=0}^{k} N_{m,n} c_m = M_p^p(n, C_\phi f_k) \leq \|C_\phi\|^p \quad \text{for all } k,$$

which gives that $\alpha \leq \|C_\phi\|^p$, and this completes the proof. □

Proposition 4.5.6. *If C_ϕ is bounded on $\mathbb{T}_{p,0}$, $1 \leq p < \infty$, then*

$$\frac{c_{|v|}}{c_n} N_\phi(n, v) \to 0 \quad \textit{as} \quad n \to \infty \quad \textit{for every} \quad v \in T. \tag{4.5.1}$$

Proof. For each $v \in T$, define $f_v = (c_{|v|})^{\frac{1}{p}} \chi_v$. Then $f_v \in \mathbb{T}_{p,0}$ with $\|f_v\| = 1$ and

$$M_p^p(n, C_\phi f_v) = \frac{c_{|v|}}{c_n} N_\phi(n, v).$$

Since $C_\phi(\mathbb{T}_{p,0}) \subseteq \mathbb{T}_{p,0}$, we have

$$\frac{c_{|v|}}{c_n} N_\phi(n, v) \to 0 \quad \text{as} \quad n \to \infty \text{ for every } v \in T.$$

The proof is complete. □

Remark 4.5.7. The condition (4.5.1) is equivalent to $C_\phi(\chi_v) \in \mathbb{T}_{p,0}$ for every $v \in T$. This in turn is also equivalent to saying that $C_\phi(E) \subseteq \mathbb{T}_{p,0}$, where $E =$ Span$\{\chi_v : v \in T\}$ is a dense subspace of $\mathbb{T}_{p,0}$ under $\|\cdot\|_p$.

4.6 Invertible composition operators

In this section, we will discuss about invertible composition operators on $\mathbb{T}_p$ spaces.

Lemma 4.6.1. *If C_ϕ is an invertible operator on $\mathbb{T}_p$, $p \geq 1$, then ϕ is bijective on T. Moreover, $C_\phi^{-1} = C_{\phi^{-1}}$.*

Proof. Assume that C_ϕ is invertible.

Suppose on the contrary that ϕ is not onto. Pick a vertex $w \in T \setminus \phi(T)$, where $\phi(T)$ denotes the image of T under ϕ. Then for $f = \chi_w$, we find that $f \not\equiv 0$ and $C_\phi(f) = 0$. Therefore, C_ϕ is not injective which leads to a contradiction. Hence ϕ is onto.

Suppose on the contrary that ϕ is not injective on T. Then there exist $v_1, v_2 \in T$ such that $v_1 \neq v_2$ and $\phi(v_1) = \phi(v_2) = w$ (say). Take $g = \chi_{v_1} \in \mathbb{T}_p$. But there is no $f \in \mathbb{T}_p$ such that $C_\phi(f) = g$, because $0 = g(v_2) = f(w) = g(v_1) = 1$. Therefore, C_ϕ is not onto which is again a contradiction. Thus ϕ is injective and onto.

Since C_ϕ is invertible, ϕ is bijective and there is a bounded linear operator S on $\mathbb{T}_p$ such that $C_\phi \circ S = S \circ \phi = I$, where I is the identity operator on $\mathbb{T}_p$. Now, it is easy to see that $S = C_{\phi^{-1}}$ and thus $C_\phi^{-1} = C_{\phi^{-1}}$. The proof is complete. □

Theorem 4.6.2. *A bounded operator C_ϕ on $\mathbb{T}_p$ is invertible if and only if ϕ is bijective on T and $C_{\phi^{-1}}$ is a bounded operator on $\mathbb{T}_p$.*

Proof. Suppose C_ϕ is an invertible operator on $\mathbb{T}_p$. Then by Lemma 4.6.1, ϕ is bijective on T and $C_\phi^{-1} = C_{\phi^{-1}}$ is a bounded operator on $\mathbb{T}_p$. Converse holds trivially, since $C_{\phi^{-1}}$ will be an inverse of C_ϕ. □

Since every self-map ϕ of T induces a bounded operator C_ϕ on $\mathbb{T}_\infty$ (resp. on $\mathbb{T}_p$ spaces over 2−homogeneous trees), it is easy to obtain the following results.

Corollary 4.6.3. *The operator C_ϕ is invertible on $\mathbb{T}_\infty$ if and only if ϕ is bijective on T.*

Corollary 4.6.4. *Let T be a 2−homogeneous tree and let $1 \leq p < \infty$. The operator C_ϕ is invertible on $\mathbb{T}_p$ if and only if ϕ is bijective on T.*

Corollary 4.6.5.
1. *The operator C_ϕ is invertible on $\mathbb{T}_{\infty,0}$ if and only if ϕ is bijective on T and $|\phi(v)| \to \infty$ and $|\phi^{-1}(v)| \to \infty$ as $|v| \to \infty$.*
2. *The operator C_ϕ is invertible on $\mathbb{T}_{p,0}$ space over 2−homogeneous trees if and only if ϕ is bijective on T and $|\phi(v)| \to \infty$ and $|\phi^{-1}(v)| \to \infty$ as $|v| \to \infty$.*

Example 4.4.2 shows that there are bijective self-maps of T which do not induce bounded composition operator C_ϕ for the case of $(q+1)-$homogeneous trees with $q \geq 2$. Indeed, there are bijective self-maps ϕ of T which induce a bounded composition operator C_ϕ on $\mathbb{T}_p$ over $(q+1)-$homogeneous trees with $q \geq 2$, but ϕ^{-1} does not necessarily induce a bounded composition operator $C_{\phi^{-1}}$.

Example 4.6.6. For each $k \in \mathbb{N}$, choose a subset A_{2k-1} of $k-1$ vertices in D_{2k-1} and choose a subset A_{2k} of k vertices in D_{2k}. Label the elements of A_n as $A_n = \{v_{n,1}, v_{n,2}, v_{n,3}, \ldots\}$ for each $n \in \mathbb{N}$.

Define ϕ as follows: $\phi(o) = o$, $\phi(v) = v$ if $v \in D_k \setminus A_k$ and $\phi(v_{2k,1}) = v_{k,1}$. For each $k \in \mathbb{N}$, we see that A_{2k-1} and $A_{2k} \setminus \{v_{2k,1}\}$ have the same number of elements. We can thus define $\phi : A_{2k-1} \to A_{2k} \setminus \{v_{2k,1}\}$ bijectively and so does for defining $\phi : A_{2k} \setminus \{v_{2k,1}\} \to A_{2k+1} \setminus \{v_{2k+1,1}\}$ bijectively. Thus, $\phi : T \to T$ becomes a bijective self-map of T.

Take an arbitrary function $f \in \mathbb{T}_p$ with $\|f\| = 1$. Fix $n = 2k-1$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
M_p^p(n, C_\phi f) &= \frac{1}{c_n}\left(\sum_{v \in D_n \setminus A_n} |f(\phi(v))|^p + \sum_{v \in A_n} |f(\phi(v))|^p\right) \\
&= \frac{1}{c_n}\left(\sum_{w \in D_n \setminus A_n} |f(w)|^p + \sum_{w \in A_{n+1} \setminus \{v_{n+1,1}\}} |f(w)|^p\right) \\
&\leq \frac{1}{c_n}\left(c_n \|f\|^p + c_{n+1} \|f\|^p\right) \\
&= (1+q)\|f\|^p = 1+q.
\end{aligned}$$

Next, fix $n = 2k$ for some $k \in \mathbb{N}$. Then

$$\begin{aligned}
M_p^p(n, C_\phi f) &= \frac{1}{c_n}\left(\sum_{v \in D_n \setminus A_n} |f(\phi(v))|^p + \sum_{v \in A_n \setminus \{v_{n,1}\}} |f(\phi(v))|^p + |f(\phi(v_{n,1}))|^p\right) \\
&= \frac{1}{c_n}\left(\sum_{w \in D_n \setminus A_n} |f(w)|^p + \sum_{w \in A_{n+1} \setminus \{v_{n+1,1}\}} |f(w)|^p + |f(v_{k,1})|^p\right) \\
&\leq \frac{1}{c_n}\left(c_n \|f\|^p + c_{n+1} \|f\|^p + c_k \|f\|^p\right) \\
&\leq (2+q)\|f\|^p = 2+q.
\end{aligned}$$

Thus, ϕ induces a bounded composition operator with $\|C_\phi\|^p \leq 2+q$.

Finally, we consider the composition operator induced by ϕ^{-1}. Recall that $\phi^{-1}(v_{n,1}) = v_{2n,1}$ for each n. For $n \in \mathbb{N}$, take $f_n = (c_{2n})^{\frac{1}{p}} \chi_{v_{2n,1}}$. Then

$$M_p^p(n, C_{\phi^{-1}} f_n) = q^n \leq \left\| C_{\phi^{-1}} \right\|^p,$$

which gives that ϕ^{-1} cannot induce a bounded composition operator.

The following result characterizes invertible composition operators on $\mathbb{T}_p$ over $(q+1)-$homogeneous trees with $q \geq 2$.

Theorem 4.6.7. *Let T be a $(q+1)-$homogeneous tree with $q \geq 2$, and $1 \leq p < \infty$. The operator C_ϕ is invertible on $\mathbb{T}_p$, if and only if ϕ is invertible and there exists an $M > 0$ such that $\left| |\phi(v)| - |v| \right| \leq M$ for all $v \in T$.*

Proof. Suppose C_ϕ is an invertible operator on $\mathbb{T}_p$. Then, by Theorem 4.6.2, ϕ is bijective on T and both C_ϕ, $C_{\phi^{-1}}$ are bounded operators on $\mathbb{T}_p$. By Corollary 4.4.4, there exist $M_1, M_2 > 0$ such that for all $v \in T$, $|\phi(v)| \leq |v| + M_1$ and $|\phi^{-1}(v)| \leq |v| + M_2$. Since ϕ is bijective, we have, $|\phi(v)| \leq |v| + M_1$ and $|v| \leq |\phi(v)| + M_2$ for all $v \in T$. By taking $M = \max\{M_1, M_2\}$, we get $\left| |\phi(v)| - |v| \right| \leq M$ for all $v \in T$.
For the converse part, assume ϕ is bijective and there exists an $M > 0$ such that $\left| |\phi(v)| - |v| \right| \leq M$ for all $v \in T$. This gives that $|\phi(v)| \leq |v| + M$ and $|v| \leq |\phi(v)| + M$ for all $v \in T$. Equivalently, $|\phi(v)| \leq |v| + M$ and $|\phi^{-1}(v)| \leq |v| + M$ for all $v \in T$. Then, by Corollary 4.4.4 we get that both C_ϕ, $C_{\phi^{-1}}$ are bounded operators on $\mathbb{T}_p$. Thus, C_ϕ is an invertible operator on $\mathbb{T}_p$ with an inverse $C_{\phi^{-1}}$. □

4.7 Isometry

This section devoted to isometric composition operators on various $\mathbb{T}_p$ spaces.

Theorem 4.7.1. *The operator C_ϕ is an isometry on $\mathbb{T}_\infty$ if and only if $\phi : T \to T$ is onto.*

Proof. Suppose that ϕ is onto. Then, since $\phi(T) = T$, we have $\|f \circ \phi\|_\infty = \|f\|_\infty$ for all $f \in \mathbb{T}_\infty$, and hence C_ϕ is an isometry on $\mathbb{T}_\infty$.

Conversely, assume that C_ϕ is an isometry on $\mathbb{T}_\infty$. Now, suppose on the contrary that ϕ is not onto. Then, choose $w \notin \phi(T)$. If $f = \chi_w$, then $\|f \circ \phi\|_\infty \neq \|f\|_\infty$, which is a contradiction. The result follows. □

Corollary 4.7.2. *The operator C_ϕ is an isometry on $\mathbb{T}_{\infty,0}$ if and only if $\phi : T \to T$ is onto and $|\phi(v)| \to \infty$ as $|v| \to \infty$.*

Theorem 4.7.3. *Let T be a 2−homogeneous tree and let $1 \leq p < \infty$. Then C_ϕ is an isometry on $\mathbb{T}_p$ if and only if the following properties hold:*

(1) $\phi(o) = o$.

(2) *ϕ is onto.*

(3) *$|\phi(v)| = |\phi(w)|$ whenever $|v| = |w|$.*

(4) *If $\phi(w) \neq o$ for some $w \in T$, then ϕ is injective on $D_{|w|}$.*

Proof. Assume that C_ϕ is an isometry on $\mathbb{T}_p$.

First let us suppose that $\phi(o) \neq o$. If $f = \chi_o + (2)^{\frac{1}{p}} \chi_{\phi(o)}$, then $\|f \circ \phi\| \neq \|f\|$, which is a contradiction. Thus (1) holds.

Secondly, let us suppose that ϕ is not onto. Then pick a $w \notin \phi(T)$. If $f = \chi_w$, then $\|f \circ \phi\| \neq \|f\|$, which is again a contradiction. So, (2) holds.

Thirdly, let us assume that there exist $v_1, v_2 \in T$ such that $|v_1| = |v_2|$ and $|\phi(v_1)| \neq |\phi(v_2)|$. Let $w_1 = \phi(v_1)$ and $w_2 = \phi(v_2)$. Then take

$$f = (c_{|w_1|})^{\frac{1}{p}} \chi_{w_1} + (c_{|w_2|})^{\frac{1}{p}} \chi_{w_2},$$

and observe that $\|f\| = 1$. But,

$$\|C_\phi(f)\|^p \geq M_p^p(|v_1|, C_\phi(f)) \geq 3/2,$$

which is not possible. Thus property (3) holds.

Finally, let us suppose that there exists a $v_1 \in T$ such that $\phi(v_1) \neq o$ and ϕ is not injective on $D_{|v_1|}$. By property (3), $\phi \not\equiv o$ on $D_{|v_1|}$. Since ϕ is not injective on $D_{|v_1|}$, we have $\phi(v_1) = \phi(v_2) = w$ (say), where $|v_1| = |v_2|$. Now, we take $f = 2^{\frac{1}{p}}\chi_w$. Then, $\|f\| = 1$. But,

$$\|C_\phi(f)\|^p \geq M_p^p(|v_1|, C_\phi(f)) = 2,$$

which is a contradiction, and hence (4) holds.

Conversely, assume that all the four properties (1) – (4) hold. We need to show that C_ϕ is an isometry on $\mathbb{T}_p$. To do this, we fix $f \in \mathbb{T}_p$. Then, by the property (1), we have

$$|f(\phi(o))|^p = |f(o)|^p \leq \|f\|^p .$$

Fix $n \in \mathbb{N}$. By properties (3) and (4), it follows that either $\phi \equiv o$ on D_n or ϕ is bijective from D_n onto D_m for some $m \in \mathbb{N}$. In either case, $M_p^p(n, C_\phi(f)) \leq \|f\|^p$, and thus

$$\|C_\phi(f)\|^p \leq \|f\|^p .$$

Now, fix $m \in \mathbb{N}$. By properties (2) and (4), there exists an $n_m \in \mathbb{N}$ such that ϕ maps bijectively from D_{n_m} onto D_m. Therefore,

$$M_p^p(m, f) = M_p^p(n_m, C_\phi(f)) \leq \|f \circ \phi\|^p ,$$

and thus $\|f\|^p \leq \|C_\phi(f)\|^p$. Hence, C_ϕ is an isometry on $\mathbb{T}_p$. □

Corollary 4.7.4. *Let $1 \leq p < \infty$. The operator C_ϕ is an isometry on $\mathbb{T}_{p,0}$ over 2−homogeneous trees if and only if $|\phi(v)| \to \infty$ as $|v| \to \infty$ and all the properties* (1) *to* (4) *in Theorem 4.7.3 hold.*

Remark 4.7.5. If the operator C_ϕ is an isometry on $\mathbb{T}_p$ (or $\mathbb{T}_{p,0}$) over 2−homogeneous trees, then the properties (3) and (4) in Theorem 4.7.3 hold. However, this is not the case for $(q+1)$−homogeneous trees with $q \geq 2$. We will now provide an example to demonstrate this fact.

For $v \neq o$, let v^- denote the parent of v. Fix $k \in \mathbb{N}$ with $k \geq 2$. For each element of D_k, choose one of its child and call them as $v_1, v_2, \ldots, v_{c_k}$. Define,

$$\phi(v) = \begin{cases} v & \text{if } |v| < k, \\ o & \text{if } v \in D_k, v \neq v_1, v_2, \ldots, v_{c_k}, \\ v^- & \text{elsewhere.} \end{cases}$$

For $f \in \mathbb{T}_p$, we can easily see that

$$M_p^p(n, C_\phi(f)) = \begin{cases} M_p^p(n, f) & \text{if } \ n < k, \\ M_p^p(n-1, f) & \text{if } \ n > k, \end{cases}$$

and $M_p^p(k, C_\phi(f)) \leq \|f\|^p$. This gives that, C_ϕ is an isometry on $\mathbb{T}_p$ (or $\mathbb{T}_{p,0}$).

Note that some vertices of D_k are mapped into its parents, but all other vertices in D_k are mapped to o. Consequently, ϕ violates both the properties (3) and (4). The desired assertion follows.

It is natural to characterize isometric composition operators C_ϕ over $(q+1)-$homogeneous trees with $q \geq 2$.

Theorem 4.7.6. *Let T be a $(q+1)-$homogeneous tree with $q \geq 2$ and let $1 \leq p < \infty$. Denote $\frac{c_k N_{k,n}}{c_n}$ by $\lambda_{k,n}$. Then, C_ϕ is an isometry on $\mathbb{T}_p$ if and only if the following properties hold:*

(1) $|\phi(v)| \leq |v|$. *In particular,* $\phi(o) = o$.

(2) $\sum\limits_{k=0}^{n} \lambda_{k,n} = 1$ *for all* $n \in \mathbb{N}_0$.

(3) *For each* $k \in \mathbb{N}_0$, $N_\phi(n, w) = N_{k,n}$ *whenever* $|w| = k$.

(4) $\sup\limits_{n \in \mathbb{N}_0} \lambda_{k,n} = 1$ *for all* $k \in \mathbb{N}_0$. *In particular,* ϕ *is onto.*

Proof. Assume that C_ϕ is an isometry on $\mathbb{T}_p$.

Suppose that there exists a $v \in T$ such that $|v| < |\phi(v)|$. Let $w = \phi(v)$. Then the function $f = (c_{|w|})^{\frac{1}{p}} \chi_w$ contradicts the fact that C_ϕ is an isometry. Hence, property (1) holds.

Fix $n \in \mathbb{N}_0$. By property (1), $\phi(D_n) \subseteq \bigcup\limits_{m=0}^{n} D_m$. For each $k = 0, 1, 2, \ldots, n$, choose $v_k \in D_k$ such that $N_{k,n} = N_\phi(n, v_k)$. Take $f = \sum\limits_{k=0}^{n} (c_k)^{\frac{1}{p}} \chi_{v_k}$ so that

$$\frac{1}{c_n} \sum_{k=0}^{n} c_k N_{k,n} = M_p^p(n, f \circ \phi) \leq \|f \circ \phi\|^p = \|f\|^p = 1, \quad \text{i.e.,} \quad \sum_{k=0}^{n} c_k N_{k,n} \leq c_n.$$

By the definition of $N_{k,n}$, one can note that the number of vertices in D_n which are mapped into D_k under ϕ is less than or equal to $c_k N_{k,n}$. Again by (1), we get

$$c_n \leq \sum_{k=0}^{n} c_k N_{k,n}.$$

Therefore, $\sum\limits_{k=0}^{n} \lambda_{k,n} = 1$ for all $n \in \mathbb{N}_0$.

Suppose that there exist an $n_1 \in \mathbb{N}$, and a $w \in T$ such that $N_\phi(n_1, w) < N_{k,n_1}$. Then the number of vertices in D_{n_1} which are mapped into D_k is strictly less than $c_k N_{k,n_1}$. Then by (2), the total number of elements in D_{n_1} is strictly less than

$$\sum_{m=0}^{n_1} c_m N_{m,n_1} = c_{n_1},$$

which is a contradiction. Thus, the property (3) is verified.

Fix $k \in \mathbb{N}_0$ and $w \in D_k$. Take $f = (c_{|w|})^{\frac{1}{p}} \chi_w$. By (3), for each $n \in \mathbb{N}_0$, we see that

$$M_p^p(n, f \circ \phi) = \frac{c_{|w|}}{c_n} N_\phi(n, w) = \frac{c_k N_{k,n}}{c_n} = \lambda_{k,n}$$

and hence,

$$\sup_{n \in \mathbb{N}_0} \lambda_{k,n} = \|f \circ \phi\|^p = \|f\|^p = 1.$$

Conversely, assume that all the four properties (1) – (4) hold. In order to prove that C_ϕ is an isometry on $\mathbb{T}_p$, we fix $f \in \mathbb{T}_p$. Then, for $n \in \mathbb{N}_0$, we have

$$\begin{aligned} M_p^p(n, C_\phi f) &= \frac{1}{c_n} \sum_{k=0}^{n} \sum_{\substack{|\phi(v)|=k \\ |v|=n}} |f(\phi(v))|^p \quad \text{(by (1))} \\ &= \frac{1}{c_n} \sum_{k=0}^{n} c_k N_{k,n} M_p^p(k, f) \quad \text{(by (3))} \\ &= \sum_{k=0}^{n} \lambda_{k,n} M_p^p(k, f). \end{aligned}$$

Thus, $\|C_\phi f\|^p = \sup A$, where

$$A = \left\{ s_n = \sum_{k=0}^{n} \lambda_{k,n} M_p^p(k, f) : n \in \mathbb{N}_0 \right\}.$$

For each $n \in \mathbb{N}_0$, $s_n \leq \|f\|^p$ which implies $\sup A \leq \|f\|^p$. Fix $m \in \mathbb{N}_0$. Then, by (4), we have $\sup\limits_{n\in\mathbb{N}_0} \lambda_{m,n} = 1$ and that there exists an $n_1 \in \mathbb{N}_0$ such that $\lambda_{m,n_1} = 1$, or else, there exists a subsequence $\{\lambda_{m,n_k}\}$ converging to 1 as $k \to \infty$. In the first case, $s_{n_1} = M_p^p(m, f) \in A$ so that $M_p^p(m, f) \leq \sup A$. In the later case,

$$\begin{aligned} |M_p^p(m,f) - s_{n_k}| &\leq \sum_{\substack{k=0\\k\neq m}}^{n_k} \lambda_{k,n_k} M_p^p(k,f) + (1-\lambda_{m,n_k}) M_p^p(m,f) \\ &\leq 2(1-\lambda_{m,n_k})\,\|f\|^p\,, \end{aligned}$$

which implies that $M_p^p(m, f)$ is a limit point of A and therefore, $M_p^p(m, f) \leq \sup A$. Since m was arbitrary, we have $\|f\|^p \leq \sup A$. Hence, $\|f\|^p = \sup A = \|f \circ \phi\|^p$. The desired conclusion follows. □

Corollary 4.7.7. *Let $1 \leq p < \infty$ and C_ϕ be a bounded composition operator on $\mathbb{T}_{p,0}$ over $(q+1)-$homogeneous tree with $q \geq 2$. Then C_ϕ is an isometry on $\mathbb{T}_{p,0}$ if and only if all the four properties (1) – (4) of Theorem 4.7.6 hold.*

Corollary 4.7.8. *Let $1 \leq p < \infty$. Suppose that C_ϕ is an isometry either on $\mathbb{T}_p$ or on $\mathbb{T}_{p,0}$, and $|\phi(v)| = |v|$ for some $v \neq o$. Then ϕ is a permutation on $D_{|v|}$.*

Proof. The results holds easily for all $2-$homogeneous trees and thus, we assume that T to be a $(q+1)-$homogeneous tree with $q \geq 2$. Assume that there exist $v_1, v_2 \in T$ with $|v_1| = |v_2| \neq 0, |\phi(v_1)| = |v_1|$ and $|\phi(v_2)| \neq |v_2|$. Consider the function

$$f = (c_{|v_1|})^{\frac{1}{p}} \chi_{w_1} + (c_{|w_2|})^{\frac{1}{p}} \chi_{w_2},$$

where $w_1 = \phi(v_1)$ and $w_2 = \phi(v_2)$. Since $|w_1| \neq |w_2|$, we have $\|f\| = 1$. But

$$\|C_\phi(f)\|^p \geq M_p^p(|v_1|, C_\phi(f)) \geq 1,$$

which contradicts the hypothesis. Thus, $\phi(D_{|v_1|}) \subseteq D_{|v_1|}$.

Next, we claim that ϕ is a permutation on $D_{|v_1|}$. Suppose not. Then there exist $w_1, w_2 \in D_{|v_1|}$ with $\phi(w_1) = \phi(w_2) = w$ (say). The function $f = (c_{|v_1|})^{\frac{1}{p}} \chi_w$ leads to non-isometry of C_ϕ. Thus, ϕ is injective on $D_{|v_1|}$ and the result follows. □

Corollary 4.7.9. *Suppose $\phi \in \mathrm{Aut}(T)$. Then, C_ϕ is an isometry on $\mathbb{T}_p$ if and only if $\phi(o) = o$.*

Proof. Suppose that $\phi \in \mathrm{Aut}(T)$ and $\phi(o) = o$. Then, ϕ is a bijective map from D_n to D_n for each n. Therefore, for $n \in \mathbb{N}_0$ and $f \in \mathbb{T}_p$, we have $M_p^p(n, C_\phi f) = M_p^p(n, f)$. Hence, C_ϕ is an isometry on $\mathbb{T}_p$.

Converse part is already contained in Theorems 4.7.3 and 4.7.6. □

4.8 Compact composition operators

Before we move on to discuss the results on compact composition operators on $\mathbb{T}_p$ spaces, we recall certain well-known classical results on Hardy spaces.

In the classical case, for an analytic self-map ϕ of $\mathbb{D}$, the following statements are equivalent (see [68, Section 2.7 and Compactness Theorem, Chapter 10]):

(a) C_ϕ is compact on H^p for $1 \leq p < \infty$,

(b) C_ϕ is compact on H^2,

(c) $\displaystyle\lim_{|w|\to 1^-} \frac{N_\phi(w)}{\log \frac{1}{|w|}} = 0$, where N_ϕ is the Nevanlinna counting function of ϕ.

Also, C_ϕ is compact on H^∞ if and only if $\sup\{|\phi(z)| : z \in \mathbb{D}\} < 1$ (see [68, Problem 10, Chapter 2]).

For the discrete setting, we now consider the compactness of composition operators on $\mathbb{T}_p$ spaces.

Theorem 4.8.1. *Every bounded self-map ϕ of T induces compact composition operator on $\mathbb{T}_p$ for $1 \leq p \leq \infty$.*

Proof. Suppose ϕ is a bounded self-map of a $(q+1)-$homogeneous tree T. Then Range$\,(\phi)$ is finite set, say, Range$\,(\phi) = \{v_1, v_2, \ldots, v_k\}$. For each $1 \leq i \leq k$, denote by E_i for the pre-image of v_i under ϕ. If $\phi(v) = v_i$, then $f \circ \phi(v) = f(v_i)$ so that

$$f \circ \phi = f(v_1)\chi_{E_1} + f(v_2)\chi_{E_2} + \cdots + f(v_k)\chi_{E_k}$$

and Range (C_ϕ) = Span $\{\chi_{E_1}, \chi_{E_2}, \ldots, \chi_{E_k}\}$. Thus, C_ϕ is a finite rank operator and hence it is compact. □

Theorem 4.8.2. *If ϕ is a self-map of $(q+1)-$homogeneous tree T, then the following are equivalent:*

(a) *C_ϕ is compact on $\mathbb{T}_p$ for $1 \leq p \leq \infty$,*

(b) *$\|C_\phi f_n\| \to 0$ as $n \to \infty$ whenever bounded sequence of functions $\{f_n\}$ that converges to 0 pointwise.*

Proof. (a) ⇒ (b): Assume that C_ϕ is compact on $\mathbb{T}_p$ and $\{f_n\}$ is a bounded sequence in $\mathbb{T}_p$ that converges to 0 pointwise. Suppose on the contrary that $\|C_\phi(f_n)\| \not\to 0$ as $n \to \infty$. Then there exist a subsequence $\{f_{n_j}\}$ and an $\epsilon > 0$ such that $\left\|C_\phi(f_{n_j})\right\| \geq \epsilon$ for all j. Denote $\{f_{n_j}\}$ by $\{g_j\}$. Since C_ϕ is compact, there is a subsequence $\{g_{j_k}\}$ of $\{g_j\}$ such that $\{C_\phi(g_{j_k})\}$ converges to some function, say, g. It follows that $\{C_\phi(g_{j_k})\}$ converges to g pointwise and $g \equiv 0$ implying that $\{C_\phi(g_{j_k})\}$ converges to 0 which is a contradiction to $\|C_\phi(g_j)\| \geq \epsilon$ for all j. Hence, $\|C_\phi(f_n)\| \to 0$ as $n \to \infty$.

(b) ⇒ (a): Conversely, suppose that case (b) holds. First let us consider the case $1 \leq p < \infty$. Let $\{g_n\}$ be a sequence in the unit ball of $\mathbb{T}_p$. By Lemma 2.6.1, for each $v \in T$, the sequence $\{g_n(v)\}$ is bounded. By the diagonalization process, there is a subsequence $\{g_{nn}\}$ of $\{g_n\}$ that converges pointwise to g (say). We see that, for each $m \in \mathbb{N}_0$,

$$M_p^p(m,g) = \lim_{n\to\infty} \frac{1}{c_m} \sum_{|v|=m} |g_{nn}(v)|^p \leq \limsup \|g_{nn}\|^p \leq 1$$

showing that $g \in \mathbb{T}_p$ with $\|g\| \leq 1$. Consequently, if $f_n = g_{nn} - g$, then $\{f_n\}$ converges to 0 pointwise and $\|f_n\| \leq 2$. By the assumption (b), $\|C_\phi f_n\| \to 0$ as $n \to \infty$ and thus, $\{C_\phi(g_{nn})\}$ converges to $C_\phi(g)$. Hence C_ϕ is compact on $\mathbb{T}_p$.

The proof for the case $p = \infty$ is similar to the above. □

Remark 4.8.3. Since edge counting metric on T induces discrete topology, compact sets are only sets having finitely many elements. Thus uniform convergence on compact subsets of T is equivalent to pointwise convergence. In view of this observation, Theorem

4.8.2 is a discrete analog of weak convergence theorem (see [68, section 2.4, p. 29]) in the classical case.

Corollary 4.8.4. *Let ϕ be a self-map of T. Then C_ϕ is compact on $\mathbb{T}_\infty$ if and only if ϕ is a bounded self-map of T.*

Proof. If ϕ is a bounded self-map of T, then C_ϕ is compact, by Theorem 4.8.1. Conversely, suppose ϕ is not a bounded map. Then, there exists a sequence of vertices $\{v_k\}$ of T such that $\phi(v_k) = w_k$ and $|w_k| \to \infty$ as $k \to \infty$. Take $f_k = \chi_{w_k}$ for each $k \in \mathbb{N}$. Then, $\|f_k\|_\infty = 1$ for each k and $\{f_k\}$ converges to 0 pointwise. Since C_ϕ is compact, $\|C_\phi(f_k)\|_\infty \to 0$ as $k \to \infty$, by Theorem 4.8.2. This is not possible, because $\|C_\phi(f_k)\|_\infty = 1$ for each $k \in \mathbb{N}$, which can be observed from the definition of f_k. Hence ϕ should be a bounded map. ☐

Corollary 4.8.5. *Let T be a $(q+1)-$homogeneous tree and $1 \leq p < \infty$. If C_ϕ is compact on $\mathbb{T}_p$, then*

$$\sup_{n\in\mathbb{N}_0} \left\{ q^{|w|-n} N_\phi(n,w) \right\} \to 0 \quad \textit{as } |w| \to \infty.$$

Proof. As in the earlier situations, for each $w \in T \setminus \{o\}$, we let $f_w = \{W(w)\chi_w\}^{\frac{1}{p}}$. Then, $\|f_w\| = 1$ for all w and, since $f_w(v) = 0$ whenever $|w| > n = |v|$, it follows that $\{f_w\}$ converges to 0 ponitwise. Since C_ϕ is compact, we see that $\|C_\phi(f_w)\| \to 0$ as $|w| \to \infty$. However, we have already shown that

$$\|C_\phi f_w\|^p = \sup_{n\in\mathbb{N}_0} \left\{ \frac{W(w)}{c_n} N_\phi(n,w) \right\} = \sup_{n\in\mathbb{N}_0} \left\{ q^{|w|-n} N_\phi(n,w) \right\}$$

and the desired conclusion follows. ☐

Remark 4.8.6. For 2−homogeneous trees, Corollary 4.8.5 takes a simpler form: *If C_ϕ is compact on $\mathbb{T}_p$, then*

$$\sup_{n\in\mathbb{N}_0} \{N_\phi(n,w)\} \to 0 \quad \textit{as} \quad |w| \to \infty.$$

This remark is helpful in the proof of Corollary 4.8.8.

Corollary 4.8.7. *If C_ϕ is compact on $\mathbb{T}_p$, then $|v| - |\phi(v)| \to \infty$ as $|v| \to \infty$.*

Proof. We will prove this result by contradiction. Suppose that $|v|-|\phi(v)| \not\to \infty$ as $|v| \to \infty$. Then there exists a sequence of vertices $\{v_k\}$ and an $M > 0$ such that $|v_k|-|\phi(v_k)| \leq M$ for all k which implies that $|\phi(v_k)| \to \infty$ as $k \to \infty$. Since $N_\phi(|v_k|, \phi(v_k)) \geq 1$ for all k, we obtain that

$$N_\phi(|v_k|, \phi(v_k))q^{|\phi(v_k)|-|v_k|} \geq q^{-M}$$

which yields that

$$\sup_{n\in\mathbb{N}_0} \left\{N_\phi(n, \phi(v_k))q^{|\phi(v_k)|-n}\right\} \geq q^{-M} \text{ for all } k$$

and thus,

$$\sup_{n\in\mathbb{N}_0} \left\{q^{|w|-n}N_\phi(n, w)\right\} \not\to 0 \text{ as } |w| \to \infty$$

which gives that C_ϕ is not compact, by Corollary 4.8.5. This contradiction completes the proof. □

Corollary 4.8.8. *Let T be a $2-$homogeneous tree. Then C_ϕ is compact on $\mathbb{T}_p$ if and only if ϕ is a bounded self-map of T.*

Proof. Since every bounded self-map ϕ of T induces compact composition operator on $\mathbb{T}_p$, one way implication is true. For the proof of the converse part, we suppose that ϕ is not bounded. Then the range contains an infinite set, say $\{w_1, w_2, \ldots\}$. For each k, choose $v_k \in T$ such that $\phi(v_k) = w_k$. This gives $N_\phi(|v_k|, w_k) \geq 1$ and thus $\sup_{n\in\mathbb{N}_0} N_\phi(n, w_k) \geq 1$ for all k. It follows that $\sup_{n\in\mathbb{N}_0} \{N_\phi(n, w)\} \not\to 0$ as $|w| \to \infty$ and hence, C_ϕ cannot be compact. □

Remark 4.8.9. It is worth to recall from [68, Chapter 3, p. 37] that if a "big-oh" condition describes a class of bounded operators, then the corresponding "little-oh" condition picks out the subclass of compact operators". We have already shown that if $\sum_{|v|=n} q^{|\phi(v)|} = O(q^n)$, then C_ϕ is bounded on $\mathbb{T}_p$. So it is natural to ask whether $\sum_{|v|=n} q^{|\phi(v)|} = o(q^n)$ guarantees the compactness of C_ϕ on $\mathbb{T}_p$. Indeed, the answer is yes. Clearly the later observation is not useful because no self-map ϕ of T will satisfy this condition. This is because $\sum_{|v|=n} q^{|\phi(v)|} \geq \sum_{|v|=n} q^0 = (q + 1)q^{n-1}$ and thus, $\sum_{|v|=n} q^{|\phi(v)|} = o(q^n)$ cannot be possible.

Theorem 4.8.10. *Let T be a $(q+1)-$homogeneous tree with $q \geq 2$, and $1 \leq p < \infty$. Then C_ϕ is compact operator on $\mathbb{T}_p$ whenever*

$$\frac{1}{c_n}\sum_{m=0}^{\infty} N_{m,n}c_m \to 0 \quad as \quad n \to \infty.$$

Proof. Let $\{f_k\}$ be a bounded sequence such that $\{f_k\}$ converges to 0 pointwise. Without loss of generality, we may assume that $\|f_k\| \leq 1$ for all k. Then, by Theorem 4.8.2, it suffices to show that $\|C_\phi(f_k)\| \to 0$ for all $k \to \infty$.

Fix $\epsilon > 0$. Then, by the hypothesis, there exists an $N_1 \in \mathbb{N}$ such that

$$\frac{1}{c_n}\sum_{m=0}^{\infty} N_{m,n}c_m \leq \epsilon^p \text{ for all } n \geq N_1. \tag{4.8.1}$$

Set $S = \{\phi(v) : |v| < N_1\}$. Then, since $\{f_k\}$ converges to 0 pointwise and S is a finite set, it follows that $\{f_k\}$ converges to 0 uniformly on S and thus, there exists an $N \in \mathbb{N}$ such that

$$\sup_{w\in S}|f_k(w)| \leq \epsilon \text{ for all } n \geq N.$$

Fix $k \geq N$. Then, for $n \in \mathbb{N}_0$ with $n < N_1$, we have $M_p^p(n, C_\phi f_k) \leq \epsilon^p$. Next, for $n \geq N_1$, we have

$$\begin{aligned} M_p^p(n, C_\phi f_k) &= \frac{1}{c_n}\sum_{m=0}^{\infty}\sum_{\substack{|\phi(v)|=m\\|v|=n}} |f_k(\phi(v))|^p \\ &\leq \frac{1}{c_n}\sum_{m=0}^{\infty} c_m N_{m,n}\|f_k\|^p \\ &\leq \epsilon^p \quad \text{(by (4.8.1))} \end{aligned}$$

which shows that $\|C_\phi f_k\| \to 0$ as $k \to \infty$. Thus, C_ϕ is compact on $\mathbb{T}_p$. □

Following example shows that there are bounded composition operators on $\mathbb{T}_p$ which are not compact.

Example 4.8.11. Consider the following self-map ϕ_2 of T defined by

$$\phi_2(v) = \begin{cases} o & \text{if } v = o, \\ v^- & \text{otherwise,} \end{cases}$$

where v^- denotes the parent of v. Then it follows easily that

$$M_p^p(0, C_{\phi_2} f) = M_p^p(0, f) \text{ and } M_p^p(1, C_{\phi_2} f) = \frac{1}{(q+1)} \sum_{|v|=1} |f(o)|^p = M_p^p(0, f).$$

Finally, for $n \geq 2$, we have

$$\begin{aligned} M_p^p(n, C_{\phi_2} f) &= \frac{1}{c_n} \sum_{|v|=n} |f(v^-)|^p \\ &= \frac{q}{c_n} \sum_{|w|=n-1} |f(w)|^p \\ &= M_p^p(n-1, f) \end{aligned}$$

and thus,

$$\|C_{\phi_2} f\| = \sup_{m \in \mathbb{N}_0} M_p(m, C_{\phi_2} f) = \sup_{m \in \mathbb{N}_0} M_p(m, f) = \|f\|$$

showing that C_{ϕ_2} is bounded on $\mathbb{T}_p$. On the other hand, since $|v| - |\phi_2(v)| = 1$ for all $|v| \geq 1$, we have $|v| - |\phi_2(v)| \not\to \infty$ as $|v| \to \infty$. Hence C_{ϕ_2} is not compact, by Corollary 4.8.7.

Remark 4.8.12. Let ϕ_3 be a map on T such that ϕ_3 maps every vertex into any one of its child. Then, as in the case of C_{ϕ_2}, it is easy to see that C_{ϕ_3} is bounded but not compact. Moreover, it can be seen that, for each $n \in \mathbb{N}$, $(C_{\phi_2})^n$ and $(C_{\phi_3})^n$ are also bounded but not compact.

By Corollaries 4.8.4 and 4.8.8, we see that, only bounded self-maps of T induces compact composition operator for $2-$homogeneous trees or on $\mathbb{T}_\infty$. Thus, it is natural to ask whether only bounded self-maps of T induces compact composition operators on $\mathbb{T}_p$ spaces for the case of $(q+1)-$homogeneous trees with $q \geq 2$.

Example 4.8.13. For each $n \in \mathbb{N}_0$, choose a vertex v_n such that $|v_n| = n$. Define a self-map ϕ_4 by

$$\phi_4(v) = \begin{cases} v_k & \text{if } v = v_{2k} \text{ for some } k \in \mathbb{N}, \\ o & \text{otherwise.} \end{cases}$$

Then we obtain that

$$M_p^p(0, C_{\phi_4} f) = |f(\phi_4(o))|^p = |f(o)|^p = M_p^p(0, f).$$

Next, for an odd natural number n, we see that

$$M_p^p(n, C_{\phi_4} f) = \frac{1}{c_n} \sum_{|v|=n} |f(o)|^p = |f(o)|^p.$$

Finally, for an even natural number, say $n = 2k$, for some $k \in \mathbb{N}$, we find that

$$\begin{aligned} M_p^p(n, C_{\phi_4} f) &= \frac{1}{c_n} \left\{ \sum_{\substack{|v|=n \\ v \neq v_{2k}}} |f(\phi_4(v))|^p + |f(\phi_4(v_{2k}))|^p \right\} \\ &\leq |f(o)|^p + \frac{|f(v_k)|^p}{c_n}. \end{aligned}$$

Thus, by Lemma 2.6.1, we have

$$\|C_{\phi_4} f\|^p \leq |f(o)|^p + \sup_{k \in \mathbb{N}} \left\{ \frac{f(v_k)|^p}{(q+1)q^{2k-1}} \right\} \leq 2\,\|f\|^p$$

which shows that C_{ϕ_4} is bounded on $\mathbb{T}_p$.

Suppose now that T is a $(q+1)-$homogeneous tree with $q \geq 2$. Let $\{f_n\}$ be a sequence in the unit ball of $\mathbb{T}_p$ which converges to 0 pointwise. Note that

$$\frac{f_n(v_k)|^p}{(q+1)q^{2k-1}} \leq \frac{1}{q^k},$$

by Lemma 2.6.1. We now claim that $\|C_{\phi_4} f_n\|^p \to 0$ as $n \to \infty$.

Let $\epsilon > 0$ be given. Then there exists a natural number N_1 such that $q^{-k} < \epsilon/2$ for all $k \geq N_1$. Consider the set $S = \{v_1, v_2, \ldots, v_{N_1}\}$. Since $\{f_n\}$ converges to 0 pointwise, we can choose a natural number $N > N_1$ such that $|f_n(o)|^p < \epsilon/2$ and $|f_n(v)|^p < \epsilon/2$ for all $v \in S$ and for all $n \geq N$. Thus,

$$\|C_{\phi_4} f_n\|^p \leq |f_n(o)|^p + \sup\left\{\frac{\epsilon}{2}, \frac{1}{q^{N_1}}, \frac{1}{q^{N_1+1}}, \cdots\right\} \leq \epsilon \quad \text{for all} \quad n \geq N$$

which gives that $\|C_{\phi_4} f_n\|^p \to 0$ as $n \to \infty$ and hence, C_{ϕ_4} is compact on $\mathbb{T}_p$. This example shows that there are unbounded self-maps of T which induce compact composition operators on $\mathbb{T}_p$ for the case of $(q+1)-$homogeneous trees with $q \geq 2$.

Proposition 4.8.14. *If C_ϕ is compact on $\mathbb{T}_{p,0}$ for $1 \leq p \leq \infty$, then $\|C_\phi f_n\| \to 0$ as $n \to \infty$ whenever $\{\|f_n\| : n \in \mathbb{N}\}$ is bounded, and $\{f_n\}$ converges to 0 pointwise.*

Proof. Proof of this result is similar to the proof of the implication "(a) $\Rightarrow$ (b)" in Theorem 4.8.2. □

Theorem 4.8.15. *There are no compact composition operators on $\mathbb{T}_{\infty,0}$.*

Proof. Suppose that ϕ is a bounded self-map of T. Then, by Theorem 4.5.3, C_ϕ is not bounded and hence, it is not compact. Suppose that ϕ is an unbounded self-map of T. Then, there exists a sequence of vertices $\{v_n\}$ such that $\phi(v_n) = w_n$ and $|w_n| \to \infty$ as $n \to \infty$. Take $f_n = \chi_{\{w_n\}}$ for each $n \in \mathbb{N}$. Then it easy to see that $\{f_n\}$ converges to 0 pointwise and $\|C_\phi(f_n)\|_\infty = \|f_n\|_\infty = 1$ for each n. Therefore, C_ϕ cannot be a compact operator on $\mathbb{T}_{\infty,0}$ by Proposition 4.8.14. □

Theorem 4.8.16. *Let T be a $2-$homogeneous tree and $1 \leq p < \infty$. Then, there are no compact composition operators on $\mathbb{T}_{p,0}$.*

Proof. By Theorem 4.5.4, no bounded self-map of T can induce a bounded (in particular, compact) composition operator. Suppose that ϕ is an unbounded self-map of T. Then, choose a sequence of vertices $\{v_n\}$ such that $\{w_n\}$ is unbounded, where $\phi(v_n) = w_n$. Take $f_n = 2^{1/p}\chi_{\{w_n\}}$ so that $\{f_n\}$ converges to 0 pointwise and $\|f_n\| = 1$ for each n. Finally, since

$$\frac{1}{2} \leq M_p^p(|v_n|, f_n \circ \phi) \leq \|C_\phi(f_n)\|^p \quad \text{for all} \quad n \in \mathbb{N},$$

it follows that C_ϕ cannot be a compact operator on $\mathbb{T}_{p,0}$ by Proposition 4.8.14. □

Theorem 4.8.17. *Let T be a $(q+1)-$homogeneous tree with $q \geq 2$. Then the operator C_ϕ cannot be compact on $\mathbb{T}_{p,0}$, $1 \leq p < \infty$, for any self-map ϕ of T.*

Proof. Suppose C_ϕ is compact for a self-map ϕ of T. Consider the sequence of functions defined by

$$g_n(v) = \frac{n}{n+|v|} \quad \text{for} \quad v \in T,\, n \in \mathbb{N}.$$

It is easy to see that

$$M_p(m, g_n) = \frac{n}{n+m} \quad \text{for} \quad n \in \mathbb{N},\, m \in \mathbb{N}_0.$$

Therefore, $g_n \in \mathbb{T}_{p,0}$ with $\|g_n\| = 1$ for all $n \in \mathbb{N}$. For each fixed $v \in T$, $g_n(v) \to 1$ pointwise. Since C_ϕ is compact on $\mathbb{T}_{p,0}$, there exists a subsequence $\{g_{n_k}\}$ of $\{g_n\}$ and $g \in \mathbb{T}_{p,0}$ such that $C_\phi(g_{n_k}) \to g$ in $\|\cdot\|_p$. Then, by the growth estimate (see Lemma 2.6.1), we have $g_{n_k}(\phi(v)) \to g(v)$ pointwise for $v \in T$, which gives that $g \equiv 1$. Since $g \notin \mathbb{T}_{p,0}$, C_ϕ cannot be compact on $\mathbb{T}_{p,0}$. The desired conclusion follows. $\square$

Chapter 5

Composition operators on the Hardy space of Dirichlet series

5.1 Introduction

In this chapter, we consider composition operators on the Hardy-Dirichlet space $\mathcal{H}^2$, the space of Dirichlet series with square summable coefficients, which is a Dirichlet series analogue of the classical Hardy space. Necessary and sufficient conditions for the boundedness of a composition operator on $\mathcal{H}^2$ are given in [38], but good estimates for norm of such operators are not known. By using the Schur test, we give some upper and lower estimates on the norm of a composition operator on $\mathcal{H}^2$, for the affine-like inducing symbol $\varphi(s) = c_1 + c_q q^{-s}$, where $q \geq 2$ is a fixed integer. We also give an estimate for approximation numbers of a composition operator in our $\mathcal{H}^2$ setting.

Determining the value of the norm of composition operators is not an easy task and hence, not much is known on this problem even in the case of classical Hardy space except for some special cases. For example, the norm of a composition operator on H^2 induced by the simple affine mapping of $\mathbb{D}$ is complicated (see [28, Theorem 3]). Not to speak of the approximation numbers of C_ϕ, even though the latter were computed in [23]. In case of the space $\mathcal{H}^2$, there are no good lower and upper bounds even for the norm of such operators except for some special cases. As a first step, we give some upper and lower estimates on the norm of a composition operator on $\mathcal{H}^2$, for the inducing symbol $\phi(s) = c_1 + c_q q^{-s}$ with $q \in \mathbb{N}, \quad q \geq 2$. Without loss of generality, we will assume that

$q = 2$. One significant difference is that some properties of the Riemann zeta function, be it only in the half-plane $\mathbb{C}_1$, are required. For a real number θ, we denote the half plane $\{s \in \mathbb{C} : \operatorname{Re} s > \theta\}$ by $\mathbb{C}_\theta$.

One may refer to [71] for basic information about analytic function spaces of $\mathbb{D}$ and operators on them. Basic issues on composition operators on various function spaces on $\mathbb{D}$ may be obtained from [29]. See also [45] for results related to analytic number theory.

This chapter is based on our paper [57].

5.2 Hardy-Dirichlet space $\mathcal{H}^2$

The Hardy-Dirichlet space $\mathcal{H}^2$ is defined by

$$\mathcal{H}^2 = \left\{ f(s) = \sum_{n=1}^{\infty} a_n n^{-s} : \|f\|^2 = \sum_{n=1}^{\infty} |a_n|^2 < \infty \right\}. \tag{5.2.1}$$

The space $\mathcal{H}^2$ has been used in [44] for the study of completeness problems of a system of dilates of a given function. The following properties are obvious:

- If $f \in \mathcal{H}^2$, then the Dirichlet series in (5.2.1) converges absolutely in $\mathbb{C}_{1/2}$, and therefore $\mathcal{H}^2$ is a Hilbert space of analytic functions on $\mathbb{C}_{1/2}$.
- The functions $\{e_n\}$ defined on $\mathbb{C}_{1/2}$ by $e_n(s) = n^{-s}, n \geq 1$, form an orthonormal basis for $\mathcal{H}^2$.
- Accordingly, the reproducing kernel K_a of $\mathcal{H}^2$ ($f(a) = \langle f, K_a \rangle$ for all $f \in \mathcal{H}^2$) is given by
$$K_a(s) = \sum_{n=1}^{\infty} e_n(s)\overline{e_n(a)} = \zeta(s + \overline{a}), \text{ with } a, s \in \mathbb{C}_{1/2},$$
where ζ denotes the Riemann zeta function.
- $C_\phi^*(K_a) = K_{\phi(a)}$, where C_ϕ^* denotes the adjoint of an operator C_ϕ.

Let $\mathcal{H}(\Omega)$ denote the space of all analytic functions defined on Ω. If $\phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ is analytic, then the composition operator

$$C_\phi : \mathcal{H}^2 \to \mathcal{H}(\mathbb{C}_{1/2}), \quad C_\phi(f) = f \circ \phi,$$

is well defined and we wish to know for which "symbols" ϕ, the operator C_ϕ maps $\mathcal{H}^2$ to itself. Then, C_ϕ is a bounded linear operator on $\mathcal{H}^2$ by the closed graph theorem. A complete answer to this fairly delicate question was obtained in [38]. A slightly improved version of the same may be stated in the following form, as far as uniform convergence on all half-planes $\mathbb{C}_\epsilon$ is concerned. See [65] for details.

Theorem A. *The analytic function $\phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ induces a bounded composition operator on $\mathcal{H}^2$ if and only if*

$$\phi(s) = c_0 s + \sum_{n=1}^{\infty} c_n n^{-s} =: c_0 s + \psi(s), \tag{5.2.2}$$

where $c_0 \in \mathbb{N}_0$ and the Dirichlet series $\sum_{n=1}^{\infty} c_n n^{-s}$ converges uniformly in each half-plane $\mathbb{C}_\epsilon,\ \epsilon > 0$. Moreover, ψ has the following mapping properties:

1. *If $c_0 \geq 1$, then $\psi(\mathbb{C}_0) \subset \mathbb{C}_0$ and so $\phi(\mathbb{C}_0) \subset \mathbb{C}_0$.*

2. *If $c_0 = 0$, then $\psi(\mathbb{C}_0) = \phi(\mathbb{C}_0) \subset \mathbb{C}_{1/2}$.*

In addition to the above formulation, it is worth to mention that $\|C_\phi\| \geq 1$ and

$$\|C_\phi\| = 1 \Longleftrightarrow c_0 \geq 1.$$

This result follows easily from the fact that C_ϕ is contractive on $\mathcal{H}^2$ if $c_0 \geq 1$ (See [38]).

5.3 A special, but interesting case

To our knowledge, except the recent work of Brevig [17] in a slightly different context, no result has appeared in the literature on sharp evaluations of the norm of C_ϕ when $c_0 = 0$. The purpose of this work is to make some attempt, in the apparently simple-minded case

$$\phi(s) = c_1 + c_2 2^{-s} \text{ with } \operatorname{Re} c_1 \geq \frac{1}{2} + |c_2|. \tag{5.3.1}$$

The condition on c_1 and c_2 in (5.3.1) is the exact translation of the mapping conditions of "affine map" to be a map of $\mathbb{C}_0$ into $\mathbb{C}_{1/2}$.

We should point out the fact that, even though the symbol ϕ is very simple, the boundedness of C_ϕ, and its norm, are far from being clear. This is already the case for affine maps $\phi(z) = az + b$ from $\mathbb{D}$ into $\mathbb{D}$ whose exact norm has a complicated expression first obtained by Cowen [28] and then by Queffélec (see [64]) with a simpler approach based on an adequate use of the Schur test, which we recall in Lemma B below, under an adapted form.

Finally, we would like to mention the following: In [46], Hurst obtained the norm of C_ϕ on weighted Bergman spaces for the affine symbols whereas in [41], Hammond obtained a representation for the norm of C_ϕ on the Dirichlet space for such affine symbols.

Lemma B. *[40, page 24] Let $A = (a_{i,j})_{i\geq 0, j\geq 1}$ be a scalar matrix, formally defining a linear map $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}_0)$ by the formula $A(x) = y$ with $y_i = \sum_{j=1}^{\infty} a_{i,j}x_j$. Assume that there exist two positive numbers α and β and two sequences $(p_i)_{i\geq 0}$ and $(q_j)_{j\geq 1}$ of positive numbers such that*

$$\sum_{i=0}^{\infty} |a_{i,j}| q_i \leq \alpha p_j \quad \text{for all} \quad j \geq 1 \tag{5.3.2}$$

and

$$\sum_{j=1}^{\infty} |a_{i,j}| p_j \leq \beta q_i \quad \text{for all} \quad i \geq 0. \tag{5.3.3}$$

Then, A defines a bounded operator with $\|A\| \leq \sqrt{\alpha\beta}$.

Remark 5.3.1. Let ϕ be a map as in (5.3.1). Then C_ϕ is a compact operator on $\mathcal{H}^2$ if and only if $\operatorname{Re} c_1 > \frac{1}{2} + |c_2|$ (see [15, Corollary 3]). Also the spectrum of C_ϕ is

$$\sigma(C_\phi) = \{0, 1\} \cup \{(\phi'(\alpha))^k : k \in \mathbb{N}\},$$

where α is the fixed point of the map ϕ in $\mathbb{C}_{1/2}$ (see [15, Theorem 4]). Since the spectrum $\sigma(C_\phi)$ is compact, we have $|\phi'(\alpha)| < 1$ and thus the spectral radius

$$r(C_\phi) := \sup\{|\lambda| : \lambda \in \sigma(C_\phi)\}$$

is equal to 1.

In [42], Hedenmalm asked for estimate from above for the norm $\|C_\phi\|$ in terms of $\phi(+\infty)$, that is, c_1 for the map $\phi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$. We give a partial answer to his question at least for this special choice of $\phi(s) = c_1 + c_2 2^{-s}$. To do this, we list below some useful lemmas here.

Lemma 5.3.2. *Let $s > 1$. Then, we have*

$$\frac{1}{s-1} \leq \zeta(s) \leq \frac{s}{s-1}.$$

Proof. The result follows, by comparison with an integral, from the fact that $x \mapsto x^{-s}$ is decreasing for $s > 1$. See for instance, [62, p. 299]. Indeed for $f(x) = x^{-s} = e^{-s \ln x}$, we have

$$\int_1^{\infty} f(x)dx \leq \sum_{k=1}^{\infty} f(k) \leq f(1) + \int_1^{\infty} f(x)dx,$$

from which one can obtain the desired inequality, since $\int_1^{\infty} f(x)dx = \frac{1}{s-1}$. □

Lemma 5.3.3. *For all $s > 1$, we have*

$$\frac{1}{s-1} + \left(\frac{s-1}{s}\right)\frac{1}{\sqrt{2\pi}} \leq \zeta(s). \tag{5.3.4}$$

Proof. Let

$$h(s) = \frac{1}{s-1} + \left(\frac{s-1}{s}\right)\frac{1}{\sqrt{2\pi}}.$$

Then, we observe that both h and ζ are decreasing functions on $(1, \infty)$. Thus,

$$h(s) \leq h(3) = \frac{1}{2} + \frac{1}{3}\sqrt{\frac{2}{\pi}} < \frac{1}{2} + \frac{1}{3} < 1 < \zeta(s) \quad \text{for all } s \geq 3.$$

This shows that the inequality (5.3.4) is true for $s \geq 3$. Now we need to verify the inequality (5.3.4) only for $1 < s < 3$. By setting $s = x + 1$, it is enough to prove that

$$h(x+1) = \frac{1}{x} + f(x) \leq \zeta(x+1) \text{ for } 0 < x < 2,$$

where

$$f(x) = \frac{1}{\sqrt{2\pi}}\left(\frac{x}{x+1}\right).$$

Clearly, f is an increasing function on $x > 0$. From [17, Lemma 10], we have

$$\frac{1}{x} + g(x) \leq \zeta(1+x) \quad \text{for } x > 0,$$

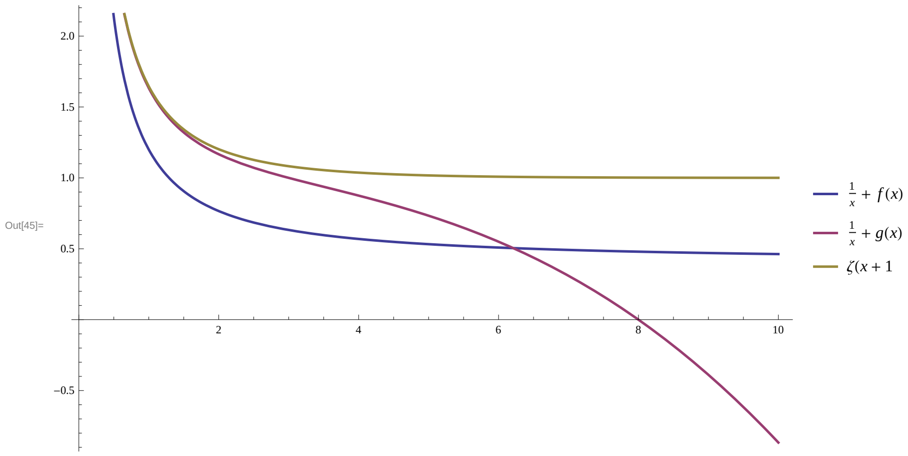

FIGURE 5.1: The range for x varies from 0.1 to 10

where

$$g(x) = \frac{1}{2} + \frac{x+1}{12} - \frac{(x+1)(x+2)(x+3)}{6!} = \frac{1}{6!}(414 + 49x - 6x^2 - x^3).$$

In view of [17, Lemma 10], it suffices to show that $f(x) \le g(x)$ on $(0, 2)$. For $0 < x < 2$,

$$g'(x) = \frac{1}{6!}(49 - 3x(x+4)) > 0,$$

which shows that g is increasing on $(0, 2)$. Since

$$f(2) = \frac{1}{3}\sqrt{\frac{2}{\pi}} < \frac{1}{3} < g(0) = \frac{23}{40},$$

we have $f(x) \le f(2) \le g(0) \le g(x)$ for all $0 < x < 2$. This proves the claim for $0 < x < 2$, i.e., $1 < s < 3$. In conclusion, the inequality (5.3.4) is verified for all $s > 1$. □

Remark 5.3.4. Consider the functions f and g as in Lemma 5.3.3. Thus, both $\frac{1}{x} + f(x)$ and $\frac{1}{x} + g(x)$ form a lower bound for $\zeta(1 + x)$ for $x > 0$. For $x > 3$, we have

$$g'(x) = -\frac{1}{6!}(3x(x+4) - 49) < 0,$$

which shows that g is decreasing on $(3, \infty)$ and therefore, $g(x) \le f(x)$ for all $x > s_2 \approx 6.2102$, where s_2 is the unique positive root of the equation given by $f(x) = g(x)$, i.e.,

$$\left(\frac{x}{x+1}\right) \frac{1}{\sqrt{2\pi}} = \frac{1}{2} + \frac{x+1}{12} - \frac{(x+1)(x+2)(x+3)}{6!}.$$

It follows that Lemma 5.3.3 is an improved version of [17, Lemma 10] for $x \geq s_2$. For a quick comparison with the zeta function, in Figure 5.1, we have drawn the graphs of $(1/x) + f(x)$, $(1/x) + g(x)$ and $\zeta(x+1)$.

Remark 5.3.5. Before seeing the work of [17], we made use of a result of Lavrik [50]: For $1 < s < 3$,

$$\zeta(s) - \frac{1}{s-1} - \gamma = \sum_{n=1}^{\infty} \frac{\gamma_n}{n!}(s-1)^n,$$

where γ is the Euler constant and $|\gamma_n| \leq \frac{n!}{2^{n+1}}$. We thus obtained an alternative proof of (5.3.4).

Lemma 5.3.6. *If $s > 1$, $i \geq 1$ is an integer, and $f(x) = \frac{(\log x)^i}{x^s}$, then one has*

$$\sum_{k=1}^{\infty} f(k) \leq \frac{i!}{(s-1)^i}\zeta(s).$$

Proof. The function f increases for $x \leq e^{i/s}$ and then decreases for $x \geq e^{i/s}$. By a simple change of variables, we have

$$I = \int_1^{\infty} f(x)dx = \frac{i!}{(s-1)^{i+1}}.$$

Let $N \geq 1$ be the integral part of $e^{i/s}$, so that $N \leq e^{i/s} < N+1$. Computations give, with help of Stirling's inequality $(i/e)^i \leq \frac{i!}{\sqrt{2\pi i}}$:

$$\sum_{k=1}^{N-1} f(k) \leq \int_1^N f(x)dx$$

and

$$\sum_{k=N+2}^{\infty} f(k) \leq \int_{N+1}^{\infty} f(x)dx.$$

It follows that

$$\int_N^{N+1} f(x)dx \geq \begin{cases} f(N) & \text{if } f(N) \leq f(N+1) \\ f(N+1) & \text{otherwise,} \end{cases}$$

and therefore,

$$f(N) + f(N+1) - \int_N^{N+1} f(x)dx \leq f(e^{i/s}) = \frac{(i/s)^i}{e^i} \leq \frac{i!}{\sqrt{2\pi i}s^i}.$$

From the above three inequalities, we get that

$$\begin{aligned}
\sum_{k=1}^{\infty} f(k) &\leq I + f(e^{i/s}) \\
&\leq i! \left[\frac{1}{(s-1)^{i+1}} + \frac{1}{\sqrt{2\pi i} s^i} \right] \\
&\leq \frac{i!}{(s-1)^i} \left[\frac{1}{s-1} + \frac{1}{\sqrt{2\pi}} \left(\frac{s-1}{s} \right) \right] \\
&\leq \frac{i!}{(s-1)^i} \zeta(s).
\end{aligned}$$

The third and the fourth inequalities follow from $\frac{s-1}{s} < 1$ and Lemma 5.3.3, respectively. This completes the proof of the lemma. □

Our next result provides bounds for the norm estimate of C_ϕ on both sides.

Theorem 5.3.7. *Let $\phi(s) = c_1 + c_2 2^{-s}$ with $\operatorname{Re} c_1 \geq \frac{1}{2} + |c_2|$ and $c_2 \neq 0$, thus inducing a bounded composition operator $C_\phi : \mathcal{H}^2 \to \mathcal{H}^2$. Then, we have*

$$\zeta(2\operatorname{Re} c_1) \leq \|C_\phi\|^2 \leq \zeta(2\operatorname{Re} c_1 - r|c_2|),$$

where $r \leq 1$ is the smallest positive root of the quadratic polynomial

$$P(r) = |c_2| r^2 + (1 - 2\operatorname{Re} c_1) r + |c_2|.$$

Remark 5.3.8. Observe that P has two positive roots with product 1, so one of them is less than or equal to 1 (because $P(0) > 0$ and $P(1) \leq 0$) and by our assumption $2\operatorname{Re} c_1 - r|c_2| \geq 2\operatorname{Re} c_1 - |c_2| \geq 1 + |c_2| > 1$, so that $\zeta(2\operatorname{Re} c_1 - r|c_2|)$ is well defined.

Proof of Theorem 5.3.7. Without loss of generality, we can assume that c_1 and c_2 are positive. Indeed, in the general case, for $\phi(s) = c_1 + c_2 2^{-s}$, we set $c_1 = \sigma_1 + it_1$ and $c_2 = |c_2| 2^{i\phi_2}$. Note that $\operatorname{Re} c_1 = \sigma_1 > 0$ by our assumption of the theorem. Consider the two vertical translations T_1 and T_2 defined respectively by $T_1(s) = s + it_1$ and $T_2(s) = s - i\phi_2$, and set $\psi(s) = \sigma_1 + |c_2| 2^{-s}$. Then, one has $\phi = T_1 \circ \psi \circ T_2$ whence

$$C_\phi = C_{T_2} \circ C_\psi \circ C_{T_1},$$

where C_{T_2} and C_{T_1} are unitary operators.

Note that $C_\phi(1) = 1$. Now for $j > 1$, we see that

$$C_\phi(j^{-s}) = j^{-c_1}\exp(-c_2 2^{-s}\log j) = j^{-c_1}\sum_{i=0}^{\infty}\frac{(-c_2\log j)^i}{i!}(2^i)^{-s}.$$

In other terms, considering the orthonormal system $\{(2^i)^{-s}\}_{i\geq 0}$ as the canonical basis of the range of C_ϕ and the orthonormal system $\{j^{-s}\}_{j\geq 1}$ as the canonical basis of $\mathcal{H}^2$, C_ϕ can be viewed as the matrix $A = (a_{i,j})_{i\geq 0, j\geq 1} : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}_0)$ with

$$a_{i1} = \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i > 0, \end{cases}$$

and

$$a_{i,j} = j^{-c_1}\frac{(-c_2\log j)^i}{i!} \quad \text{for} \quad i \geq 0,\ j > 1.$$

By Theorem A, we already know that C_ϕ is a bounded operator. We will give a direct proof of this fact, and moreover an upper and lower estimates of its norm. To that effect, we apply the Schur test with the following values of the parameters

$$\alpha = 1,\ \beta = \zeta(2c_1 - rc_2),\ p_j = j^{rc_2 - c_1} \text{ and } q_i = r^i.$$

Now, we can check the assumptions of Schur's lemma. Equality holds trivially in the inequality (5.3.2) for the case of $j = 1$. For $j > 1$,

$$\sum_{i=0}^{\infty}|a_{i,j}|q_i = \sum_{i=0}^{\infty} j^{-c_1}\frac{(c_2\log j)^i}{i!}r^i = j^{rc_2 - c_1} = \alpha p_j.$$

Thus, the inequality (5.3.2) is verified. Now, we verify the inequality (5.3.3). For the case $i = 0$, we have

$$\sum_{j=1}^{\infty}|a_{0,j}|p_j = \sum_{j=1}^{\infty} j^{-(2c_1 - rc_2)} = \zeta(2c_1 - rc_2) \leq \beta q_0.$$

Finally, for $i \geq 1$, with the help of Lemma 5.3.6, we have

$$\sum_{j=1}^{\infty}|a_{i,j}|p_j = \frac{c_2^i}{i!}\sum_{j=2}^{\infty}\frac{(\log j)^i}{j^{2c_1 - rc_2}} \leq \frac{c_2^i}{i!}\frac{i!}{(2c_1 - rc_2 - 1)^i}\zeta(2c_1 - rc_2) = \beta q_i,$$

where $\frac{c_2}{2c_1 - rc_2 - 1} = r$, that is, $P(r) = 0$. The assumptions of the Schur lemma with the claimed values are thus verified, and the upper bound ensues.

For the lower bound, we use reproducing kernels as usual (recall that $C_\phi^*(K_a) = K_{\phi(a)}$):

$$\|C_\phi\|^2 \geq (S_\phi^*)^2 := \sup_{a\in\mathbb{C}_{1/2}} \frac{\|K_{\phi(a)}\|^2}{\|K_a\|^2} = \sup_{a\in\mathbb{C}_{1/2}} \frac{\zeta(2\mathrm{Re}\,\phi(a))}{\zeta(2\mathrm{Re}\,a)} = \sup_{x>1/2} \frac{\zeta(2c_1 - 2c_2 2^{-x})}{\zeta(2x)}.$$

The last equality in the above is obtained from basic trigonometry and the fact that $\zeta(s)$ is a decreasing function on $(1,\infty)$. Now by letting $x \to \infty$, we get the lower bound for $\|C_\phi\|$. □

Corollary 1. *Let $\phi(s) = c_1 + c_2 2^{-s}$ with $\mathrm{Re}\,c_1 = \frac{1}{2} + |c_2|$ and $c_2 \neq 0$. Then, for the inducing composition operator $C_\phi : \mathcal{H}^2 \to \mathcal{H}^2$, we have*

$$\zeta(2\mathrm{Re}\,c_1) = \zeta(1 + 2|c_2|) \leq \|C_\phi\|^2 \leq \zeta(1 + |c_2|) = \zeta(2\mathrm{Re}\,c_1 - |c_2|).$$

Proof. It suffices to observe that $r = 1$ in Theorem 5.3.7 when $\mathrm{Re}\,c_1 = \frac{1}{2} + |c_2|$. □

Remark 5.3.9. From the proof of Theorem 5.3.7, it is evident that the lower bound of $\|C_\phi\|$ continues to hold for any composition operator C_ϕ with $c_0 = 0$ in (5.2.2), namely, for any $\phi(s) = \sum_{n=1}^{\infty} c_n n^{-s}$.

Remark 5.3.10. (a) Note that, if $c_2 = 0$, then ϕ becomes a constant map and the induced composition operator C_ϕ is the evaluation map at c_1. Also it is known that

$$\|C_\phi\|^2 = \zeta(2\mathrm{Re}\,c_1).$$

(b) Let ϕ be a map as in (5.3.1). Then C_ϕ cannot be a normal operator. More generally, it cannot be a normaloid operator because,

$$r(C_\phi) = 1 < \sqrt{\zeta(2\mathrm{Re}\,c_1)} \leq \|C_\phi\|.$$

(see Remark 5.3.1 and Theorem 5.3.7).

5.4 Approximation numbers

Recall that the N^{th} approximation number $a_N(A)$, $N = 1, 2, \ldots$, of an operator $A : H \to H$, where H is a Hilbert space, is the distance (for the operator norm) of A to operators of rank $< N$. We refer to [19] for the definition and basic properties of those numbers. In the case of C_ϕ on H^2, for $\phi(z) = az + b$ with $|a| + |b| \leq 1$, Clifford and Dabkowski [23] computed exactly the approximation numbers $a_N(C_\phi)$. In the compact case $|a| + |b| < 1$, they [23] showed in particular that

$$a_N(C_\phi) = |a|^{N-1} Q^{N-1/2} \quad \text{for all } N \geq 1,$$

where

$$Q = \frac{1 + |a|^2 - |b|^2 - \sqrt{\Delta}}{2|a|^2}$$

and where $\Delta > 0$ is a discriminant depending on a and b.

It is natural to ask whether we could get something similar for $\phi(s) = c_1 + c_2 2^{-s}$ and the associated C_ϕ acting on $\mathcal{H}^2$. We have here the following upper bound, in which $2 \operatorname{Re} c_1 - 2|c_2| - 1$ is assumed to be positive which is indeed a necessary and sufficient condition for the compactness of C_ϕ.

Theorem 5.4.1. *Assume that* $2 \operatorname{Re} c_1 - 2|c_2| - 1 > 0$. *Then the following exponential decay holds:*

$$a_{N+1}(C_\phi) \leq \sqrt{\frac{(2 \operatorname{Re} c_1 - 1)(2 \operatorname{Re} c_1)}{(2 \operatorname{Re} c_1 - 1)^2 - (2|c_2|)^2}} \left(\frac{2|c_2|}{2 \operatorname{Re} c_1 - 1} \right)^N .$$

Proof. Without loss of generality, we can assume that c_1 and c_2 are non-negative. Let $f(s) = \sum_{n=1}^{\infty} b_n n^{-s} \in \mathcal{H}^2$. Then

$$\begin{aligned} C_\phi f(s) &= \sum_{n=1}^{\infty} b_n n^{-c_1} \exp(-c_2 2^{-s} \log n) \\ &= \sum_{k=0}^{\infty} \frac{(-c_2)^k}{k!} \left(\sum_{n=1}^{\infty} b_n n^{-c_1} (\log n)^k \right) 2^{-ks}. \end{aligned}$$

Thus, designating by R the operator of rank $\leq N$ defined by

$$Rf(s) = \sum_{k=0}^{N-1} \frac{(-c_2)^k}{k!} \left(\sum_{n=1}^{\infty} b_n n^{-c_1} (\log n)^k \right) 2^{-ks},$$

we obtain via the classical Cauchy-Schwarz inequality that

$$\|C_\phi(f) - R(f)\|^2 \;=\; \sum_{k=N}^{\infty} \frac{c_2^{2k}}{k!^2} \left| \sum_{n=1}^{\infty} b_n n^{-c_1} (\log n)^k \right|^2$$

$$\leq \sum_{k=N}^{\infty} \frac{c_2^{2k}}{k!^2} \left(\sum_{n=1}^{\infty} |b_n|^2\right) \left(\sum_{n=1}^{\infty} \frac{(\log n)^{2k}}{n^{2c_1}}\right).$$

By Lemma 5.3.6, the latter sum is nothing but

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(\log n)^{2k}}{n^{2c_1}} &= \zeta^{(2k)}(2c_1) \\
&\leq \frac{(2k)!}{(2c_1-1)^{2k}} \zeta(2c_1) \\
&\leq \frac{(2k)!(2c_1)}{(2c_1-1)^{2k+1}}.
\end{aligned}$$

The last inequality follows by the simple fact that $\zeta(s) \leq \frac{s}{s-1}$ (see Lemma 5.3.2). Since

$$\sum_{n=1}^{\infty} |b_n|^2 = \|f\|^2 \quad \text{and} \quad \frac{(2k)!}{(k!)^2} \leq \sum_{j=0}^{2k} \binom{2k}{j} = 4^k,$$

we get the following:

$$\begin{aligned}
\|C_\phi - R\|^2 &\leq \sum_{k=N}^{\infty} \frac{c_2^{2k}}{(k!)^2} \frac{(2k)!(2c_1)}{(2c_1-1)^{2k+1}} \\
&\leq \sum_{k=N}^{\infty} \left(\frac{2c_2}{2c_1-1}\right)^{2k} \frac{2c_1}{2c_1-1} \\
&= \frac{2c_1(2c_1-1)}{(2c_1-1)^2-(2c_2)^2} \left(\frac{2c_2}{2c_1-1}\right)^{2N}.
\end{aligned}$$

Thus, we complete the proof. □

Question 2. 1. Is there a symbol ϕ for which the strict inequalities

$$\|C_\phi\| > S_\phi^* > S_\phi$$

hold for C_ϕ on $\mathcal{H}^2$? (refer to [12] for similar problem in the case of classical Hardy space H^2). In the case $\phi(s) = c_1 + c_2 2^{-s}$, we probably have

$$\|C_\phi\| = S_\phi^* = S_\phi,$$

but this still needs a proof. Also observe that this ϕ is not injective on $\mathbb{C}_{1/2}$.

2. What can be said about $\|C_\phi\|$ acting on $H^2(\Omega)$, where Ω is the ball $\mathbb{B}_d$, or the polydisk $\mathbb{D}^d$, when $\phi(z) = A(z)+b$ with $A : \mathbb{C}^d \to \mathbb{C}^d$ a linear operator, i.e. when ϕ is an affine map such that $\phi(\Omega) \subset \Omega$? This might be difficult [16], but interesting.

Chapter 6

Weighted composition operators on $\mathcal{P}_\alpha$

6.1 Introduction

In this chapter, we use the notion of topological vector space and few other related definitions. A *topological vector space* is a vector space together with a topology such that the vector space operations, namely, vector addition and scalar multiplication, are continuous with respect to this topology. A set C in a topological vector space X is said to be *convex* if $tx + (1-t)y \in C$ for all $x, y \in C$ and $0 \leq t \leq 1$.

Let E be a subset of a topological vector space X. Then the *convex hull* of E is defined to be the intersection of all convex sets that contains E. The *closed convex hull* of E, denoted by $\overline{co}(E)$, is defined to be the intersection of all closed convex sets that contains E. A topological vector space X is said to be *locally convex* if there is a collection of convex sets which forms a local base for the topology of X. A topological vector space X is said to be *metrizable* if the topology of X is induced by some metric d.

Let Ω be an open subset of $\mathbb{C}$ and let $\mathcal{H}(\Omega)$ be the set of all analytic functions defined on Ω. Consider the compact-open topology on $\mathcal{H}(\Omega)$, i.e., the topology of uniform convergence on compact subsets of Ω. Then $\mathcal{H}(\Omega)$ becomes a metrizable, locally convex topological vector space. On $\mathcal{H}(\Omega)$, f_n converges to f, denoted by $f_n \xrightarrow{\text{u.c}} f$, we mean that the convergence is locally uniformly on compact subsets of Ω.

Weighted composition operator is a combination of multiplication and composition operators. These operators are mainly studied in various Banach spaces or Hilbert spaces of $\mathcal{H}(\mathbb{D})$. Recently, Arévalo et al. [13] initiated the study of weighted composition operator restricted to the Carathéodory class $\mathcal{P}_1$, which consists of all $f \in \mathcal{H}(\mathbb{D})$ with positive real part and with the normalization $f(0) = 1$. Clearly the class $\mathcal{P}_1$ is not a linear space but it is helpful to solve some extremal problems in geometric function theory. See [39].

We generalize the recent work of Arévalo et al. [13] by considering weighted composition operators preserving the class $\mathcal{P}_\alpha$ of analytic functions subordinate to $\frac{1+\alpha z}{1-z}$ for $|\alpha| \leq 1, \alpha \neq -1$. This class is connected with various geometric subclasses of $\mathcal{H}(\mathbb{D})$ in the univalent function theory (see [34, 39, 59]). Since the class $\mathcal{P}_\alpha$ is not a linear space, for a given map on $\mathcal{P}_\alpha$, questions about operator theoretic properties are not meaningful. However, one can talk about, for example, special classes of self-maps of $\mathcal{P}_\alpha$ and fixed points of those maps.

In this chapter, we discuss the weighted composition operators preserving the class $\mathcal{P}_\alpha$. Some of its consequences and examples for some special cases are also presented. Furthermore, we discuss about the fixed points of weighted composition operators.

This chapter is based on our paper [55].

6.2 Preliminaries about the class $\mathcal{P}_\alpha$

For f and $g \in \mathcal{H}(\mathbb{D})$, we say that f *is subordinate to* g (denoted by $f(z) \prec g(z)$ or $f \prec g$) if there exists an analytic function $\omega : \mathbb{D} \to \mathbb{D}$ such that $\omega(0) = 0$ and $f = g \circ \omega$. If $f(z) \prec z$, then f is called *Schwarz function* (i.e., analytic function $f : \mathbb{D} \to \mathbb{D}$ with $f(0) = 0$). For $|\alpha| \leq 1, \alpha \neq -1$, define h_α on $\mathbb{D}$ by $h_\alpha(z) = \frac{1+\alpha z}{1-z}$ and the half plane $\mathbb{H}_\alpha$ is described by

$$\mathbb{H}_\alpha := h_\alpha(\mathbb{D}) = \{w \in \mathbb{C} : 2\mathrm{Re}\,\{(1+\overline{\alpha})w\} > 1 - |\alpha|^2\}.$$

In particular, if $\alpha \in \mathbb{R}$ and $-1 < \alpha \leq 1$, then

$$h_\alpha(\mathbb{D}) = \{w \in \mathbb{C} : \mathrm{Re}\, w > (1-\alpha)/2\}$$

so that $\operatorname{Re} h_\alpha(z) > (1-\alpha)/2$ in $\mathbb{D}$.

For $|\alpha| \leq 1, \alpha \neq -1$, it is natural to consider the class $\mathcal{P}_\alpha$ defined by

$$\mathcal{P}_\alpha := \{f \in \mathcal{H}(\mathbb{D}) : f(z) \prec h_\alpha(z)\}.$$

It is worth noting that for every $f \in \mathcal{P}_\alpha$, there is an unique Schwarz function ω such that

$$f(z) = \frac{1+\alpha\omega(z)}{1-\omega(z)}.$$

It is well-known [59, Lemma 2.1] that, if g is an univalent (injective) analytic function on $\mathbb{D}$, then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathbb{D}) \subseteq g(\mathbb{D})$. In view of this result, the class $\mathcal{P}_\alpha$ can be stated in an equivalent form as

$$\mathcal{P}_\alpha := \{f \in \mathcal{H}(\mathbb{D}) : f(0) = 1, f(\mathbb{D}) \subseteq \mathbb{H}_\alpha\}.$$

We continue the discussion by stating a few basic and useful properties of the class $\mathcal{P}_\alpha$.

Proposition 6.2.1. *Suppose $f \in \mathcal{P}_\alpha$ and $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$, then $|a_n| \leq |\alpha+1|$ for all $n \in \mathbb{N}$. The bound is sharp as the function $h_\alpha(z) = 1 + \sum_{n=1}^{\infty}(1+\alpha)z^n$ shows.*

Proof. This result is an immediate consequence of Rogosinski's result [66, Theorem X] (see also [34, Theorem 6.4(i), p. 195]) because $h_\alpha(z)$ (and hence, $(h_\alpha(z)-1)/(1+\alpha)$) is a convex function. □

Proposition 6.2.2. (Growth estimate) *Let $f \in \mathcal{P}_\alpha$. Then for all $z \in \mathbb{D}$, one has*

$$\frac{1-|\alpha z|}{1+|z|} \leq |f(z)| \leq \frac{1+|\alpha z|}{1-|z|}.$$

Proof. This result trivially follows from clever use of classical Schwarz lemma and the triangle inequality. □

From the 'growth estimate' and the familiar Montel's theorem on normal family, one can easily get the following result.

Proposition 6.2.3. *The class $\mathcal{P}_\alpha$ is a compact family in the compact-open topology (that is, topology of uniform convergence on compact subsets of $\mathbb{D}$).*

Because the half plane $\mathbb{H}_\alpha$ is convex, the following result is obvious.

Proposition 6.2.4. *The class $\mathcal{P}_\alpha$ is a convex family.*

For $p \in (0, \infty)$, the Hardy space H^p consists of analytic functions f on $\mathbb{D}$ with

$$\|f\|_p := \sup_{r\in[0,1)} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right\}^{\frac{1}{p}}$$

is finite and H^∞ denotes the set of all bounded analytic functions on $\mathbb{D}$. We refer to [33] for the theory of Hardy spaces. By Littlewood's subordination theorem [51, Theorem 2], it follows that if $f \prec g$ and $g \in H^p$ for some $0 < p \leq \infty$, then $f \in H^p$ for the same p. As a consequence we easily have the following.

Proposition 6.2.5. *The class $\mathcal{P}_\alpha$ is a subset of the Hardy space H^p for each $0 < p < 1$.*

Proof. Because $(1 - z)^{-1} \in H^p$ for each $0 < p < 1$, it follows easily that $h_\alpha \in H^p$ for each $0 < p < 1$ and for $|\alpha| \leq 1, \alpha \neq -1$. The desired conclusion follows. □

Remark 6.2.6. Although $\mathcal{P}_\alpha$ does not posses the linear structure, due to being part of H^p, the results on H^p space, such as results about boundary behavior, are valid for functions in the class $\mathcal{P}_\alpha$

6.3 Weighted composition on $\mathcal{P}_\alpha$

For an analytic self-map ϕ of $\mathbb{D}$, the composition operator C_ϕ is defined by

$$C_\phi(f) = f \circ \phi \text{ for } f \in \mathcal{H}(\mathbb{D}).$$

One can refer [29], for the study of composition operators on various function spaces on the unit disk. Throughout the chapter, unless otherwise stated explicitly, α denotes a complex number such that $|\alpha| \leq 1, \alpha \neq -1$, and ϕ denotes an analytic self-map of $\mathbb{D}$. The following result deals with composition operator when it is restricted to the class $\mathcal{P}_\alpha$.

Proposition 6.3.1. *The composition operator C_ϕ induced by the symbol ϕ, preserves the class $\mathcal{P}_\alpha$ if and only if ϕ is a Schwarz function.*

Proof. Suppose that C_ϕ preserves the class $\mathcal{P}_\alpha$. Then $C_\phi(h_\alpha) \in \mathcal{P}_\alpha$, and thus

$$\frac{1+\alpha\phi(0)}{1-\phi(0)} = 1.$$

This gives that $\phi(0) = 0$ which implies that ϕ is a Schwarz function. The converse part holds trivially. □

For a given analytic self-map ϕ of $\mathbb{D}$ and analytic map ψ of $\mathbb{D}$, the corresponding weighted composition operator $W_{\psi,\phi}$ is defined by

$$W_{\psi,\phi}(f) = \psi(f \circ \phi) \text{ for } f \in \mathcal{H}(\mathbb{D}).$$

If $\psi \equiv 1$, then $W_{\psi,\phi}$ reduced to a composition operator C_ϕ and if $\phi(z) = z$ for all $z \in \mathbb{D}$, then $W_{\psi,\phi}$ reduced to a multiplication operator M_ψ. For a given analytic map ψ of $\mathbb{D}$, the corresponding multiplication operator M_ψ is then defined by

$$M_\psi(f) = \psi f \text{ for } f \in \mathcal{H}(\mathbb{D}).$$

The characterization of M_ψ that preserves the class $\mathcal{P}_\alpha$ is given in Section 6.4.

In this section, we discuss weighted composition operator that preserves $\mathcal{P}_\alpha$. Before, we do this, let us recall some useful results from the theory of extreme points.

Lemma 6.3.2. ([39, Theorem 5.7]) *Extreme points of the class $\mathcal{P}_\alpha$ consists of functions given by*

$$f_\lambda(z) = \frac{1+\alpha\lambda z}{1-\lambda z}, \quad |\lambda| = 1.$$

A point p of a convex set E is called *extreme point* if p is not a interior point of any line segment which entirely lies in E. We denote, the set of all extreme points of the class $\mathcal{P}_\alpha$ by $\mathcal{E}_\alpha$. That is, $\mathcal{E}_\alpha = \{f_\lambda : |\lambda| = 1\}$. Now, we recall a well-known result by Krein and Milman [49].

Lemma 6.3.3. ([39, Theorem 4.4]) *Let X be a locally convex, topological vector space and A be a convex, compact subset of X. Then, the closed convex hull of extreme points of A is equal to A.*

The original version of it is proved in [49]. It is easy to see that $W_{\psi,\phi}(f_n) \xrightarrow{\text{u.c}} W_{\psi,\phi}(f)$ whenever $f_n \xrightarrow{\text{u.c}} f$. Thus, $W_{\psi,\phi}$ is continuous on $\mathcal{H}(\mathbb{D})$ (in particular on $\mathcal{P}_\alpha$).

Proposition 6.3.4. *Suppose that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Then ϕ is a Schwarz function and there exists a Schwarz function ω such that*

$$\psi = h_\alpha \circ \omega = \frac{1+\alpha\omega}{1-\omega}.$$

Proof. Suppose that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Take $f \equiv 1$ to be a constant function, which belongs to $\mathcal{P}_\alpha$. Thus, $W_{\psi,\phi}(f) = \psi \in \mathcal{P}_\alpha$ and hence, there exists a Schwarz function ω such that

$$\psi = h_\alpha \circ \omega = \frac{1+\alpha\omega}{1-\omega}.$$

In particular, $\psi(0) = 1$.

Since $h_\alpha \in \mathcal{P}_\alpha$, we have $\psi(0)(h_\alpha(\phi(0))) = 1$, which gives $\phi(0) = 0$. Hence ϕ will be a Schwarz function. □

In view of above result, from now on, we will assume that $\psi = h_\alpha \circ \omega = \frac{1+\alpha\omega}{1-\omega}$ and ϕ, ω are Schwarz functions.

Theorem 6.3.5. *Let ϕ, ω and ψ be as above. Then, $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ if and only if*

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \ \text{on } \mathbb{D}, \tag{6.3.1}$$

where, $P(\omega) = |\alpha\omega|^2 - |1+(\alpha-1)\omega|^2$ and $Q(\omega) = |(\alpha-1)|\omega|^2 + \overline{\omega} - \alpha\omega|$.

Proof. At first we prove that, $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ which is equivalent to the inclusion $W_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$. To do this, we suppose that $W_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$. Since $\mathcal{P}_\alpha$ is a convex family, we obtain

$$W_{\psi,\phi}(\text{ convex hull } (\mathcal{E}_\alpha)) \subset \mathcal{P}_\alpha.$$

Now, by Krein-Milman theorem and the fact that $W_{\psi,\phi}$ is continuous on the compact family $\mathcal{P}_\alpha$, we see that $W_{\psi,\phi}(\mathcal{P}_\alpha) \subset \mathcal{P}_\alpha$. The converse part is trivial.

Next, we prove that $W_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$ if and only if

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \text{ on } \mathbb{D}.$$

Assume that $W_{\psi,\phi}(\mathcal{E}_\alpha) \subset \mathcal{P}_\alpha$. This gives $\psi(f_\lambda \circ \phi) \in \mathcal{P}_\alpha$ for all $|\lambda| = 1$. Thus, for all $|\lambda| = 1$, there exists a Schwarz function ω_λ such that $\psi(f_\lambda \circ \phi) = h_\alpha \circ \omega_\lambda$. That is,

$$\frac{1+\alpha\omega}{1-\omega}\frac{1+\alpha\lambda\phi}{1-\lambda\phi} = \frac{1+\alpha\omega_\lambda}{1-\omega_\lambda}.$$

Solving this equation for ω_λ, we get that

$$\omega_\lambda = \frac{\omega+\lambda\phi+(\alpha-1)\lambda\omega\phi}{1+\alpha\lambda\phi\omega}.$$

For each $|\lambda| = 1$, ω_λ is a Schwarz function if and only if

$$|\omega+\lambda\phi+(\alpha-1)\lambda\omega\phi|^2 < |1+\alpha\lambda\phi\omega|^2 \quad \text{for all} \quad |\lambda| = 1,$$

which is equivalent to

$$2\text{Re}\,(\lambda\phi\{(\alpha-1)|\omega|^2+\overline{\omega}-\alpha\omega\}) < (1-|\omega|^2) + |\phi|^2(|\alpha\omega|^2 - |1+(\alpha-1)\omega|^2),$$

for all $|\lambda| = 1$. By taking supremum over λ on both sides, the last inequality gives (6.3.1). The converse part follows by repeating the above arguments in the reverse direction. □

Remark 6.3.6. Suppose that $\alpha = a+ib$ and $\omega(z) = u(z)+iv(z)$. Then,

$$\begin{aligned} P(\omega) &= |\alpha\omega|^2 - |1+(\alpha-1)\omega|^2 \\ &= a(|\omega|^2-1) + (a-1)|\omega-1|^2 + 2bv. \end{aligned}$$

Set $q(\omega) = (\alpha-1)|\omega|^2 + \overline{\omega} - \alpha\omega$ so that $Q(\omega) = |q(\omega)|$. Upon simplifying, we get that

$$q(\omega) = (\alpha-1)\,\overline{\omega}\,(\omega-1) - 2i\alpha\, v = (\alpha-1)(|\omega|^2-\omega) - 2iv$$

and thus

$$q(\omega) = [(a-1)(|\omega|^2-u)+bv] + i[b(|\omega|^2-u) - v(a+1)]. \tag{6.3.2}$$

Also, it is easy to see that

$$-q(\omega) = |1-\omega|^2\psi + (|\omega|^2 - 1) \text{ with } \psi = \frac{1+\alpha\omega}{1-\omega}. \tag{6.3.3}$$

6.4 Special cases

In this section, first we recall some familiar results on Hardy space H^p which will help the smooth traveling of this section. In what follows $\mathbb{T}$ denotes the unit circle $\{z \in \mathbb{C} : |z| = 1\}$.

Proposition 6.4.1. ([33, Theorem 1.3]) *For every bounded analytic function f on $\mathbb{D}$, the radial limit $\lim_{r\to 1} f(re^{i\theta})$ exists almost everywhere (abbreviated by a.e.).*

In view of Proposition 6.4.1, every Schwarz function has radial limit a.e. and using the fact that the function h_α has radial limit a.e., it is easy to see that, every function $f \in \mathcal{P}_\alpha$ has radial limit a.e. on $\mathbb{T}$. Also, it is well-known that (see [33, Section 2.3])

$$\sup_{|z|<1} |f(z)| = \operatorname*{ess\,sup}_{0\le\theta\le 2\pi} |f(e^{i\theta})|,$$

for every $f \in H^\infty$. Now, we will state a classical theorem of Nevanlinna.

Proposition 6.4.2. ([33, Theorem 2.2]) *If $f \in H^p$ for some $p > 0$ and its radial limit $f(e^{i\theta}) = 0$ on a set of positive measure, then $f \equiv 0$.*

Since every Schwarz function f belongs to H^∞ and every $f \in \mathcal{P}_\alpha$ belongs to H^p for $0 < p < 1$, the above result is valid for functions in the class $\mathcal{P}_\alpha$ and in the class of Schwarz functions.

An analytic function f on $\mathbb{D}$ is said to be an *inner function* if $|f(z)| \le 1$ for all $z \in \mathbb{D}$ and its radial limit $|f(\zeta)| = 1$ a.e. on $|\zeta| = 1$.

Theorem 6.4.3. *Suppose that ϕ and ω are Schwarz functions, ϕ is inner and $\psi = \frac{1+\alpha\omega}{1-\omega}$. Then, $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ if and only if $\psi \equiv 1$ (i.e., $\omega \equiv 0$).*

Proof. If $\psi \equiv 1$, then $W_{\psi,\phi}$ becomes a composition operator C_ϕ and thus, $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, because ϕ is a Schwarz function.

Conversely, suppose that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Then, by Theorem 6.3.5, one has the inequality

$$2Q(\omega)|\phi| < (1-|\omega|^2) + P(\omega)|\phi|^2 \text{ on } \mathbb{D}.$$

With abuse of notation, we denote the radial limits of ϕ, ω and ψ again by ϕ, ω and ψ, respectively. Also, let $\alpha = a+ib$ and $\omega(z) = u(z)+iv(z)$. By allowing $|z| \to 1$ in (6.3.1), we get that

$$2Q(\omega) \le (1-|\omega|^2) + P(\omega) \text{ a.e. on } \mathbb{T},$$

which after computation is equivalent to

$$Q(\omega) \le (a-1)(|\omega|^2 - u) + bv \text{ a.e. on } \mathbb{T}.$$

In view of (6.3.2) in Remark 6.3.6, the above inequality can rewritten as

$$|q(\omega)| \le \mathrm{Re}\,[q(\omega)] \text{ a.e. on } \mathbb{T}$$

which gives that $\mathrm{Im}\,[q(\omega)] = 0$ a.e. on $\mathbb{T}$. Again, by using (6.3.3) in Remark 6.3.6, we have

$$|1-\omega|^2 \mathrm{Im}(\psi) = 0 \text{ a.e. on } \mathbb{T}.$$

Analyzing the function ω through the classical theorem of Nevanlinna (see Proposition 6.4.2), one can get that $\mathrm{Im}\,\psi = 0$ a.e. on $\mathbb{T}$. Now the proof of $\psi \equiv 1$ is as follows:

Consider the analytic map $f = e^{-i(\psi-1)}$. Then, $|f| = e^{\mathrm{Im}\,\psi} = 1$ a.e. on $\mathbb{T}$ and

$$1 = f(0) \le \sup_{|z|<1} |f(z)| = \operatorname*{ess\,sup}_{0\le\theta\le 2\pi} |f(e^{i\theta})| = 1.$$

Hence, by the maximum modulus principle, we get that $f \equiv 1$ which gives $\psi \equiv 1$. □

Corollary 6.4.4. *M_ψ preserves the class $\mathcal{P}_\alpha$ if and only if $\psi \equiv 1$.*

Proof. The desired result follows if we set $\phi(z) \equiv z$ in Theorem 6.4.3. □

Theorem 6.4.5. *Suppose that α is a real number, ϕ and ω are Schwarz functions, ω is an inner function and $\psi = \frac{1+\alpha\omega}{1-\omega}$. Then, $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$ if and only if ϕ is identically zero.*

Proof. If $\phi \equiv 0$, then $W_{\psi,\phi}$ becomes a constant map ψ and hence it preserves $\mathcal{P}_\alpha$. Conversely, suppose that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. Then, by Theorem 6.3.5,

$$2Q(\omega)|\phi| < (1 - |\omega|^2) + P(\omega)|\phi|^2 \text{ on } \mathbb{D}.$$

By allowing $|z| \to 1$, we get that

$$2|1-\omega|^2|\psi|\,|\phi| \leq (2\mathrm{Im}\,\alpha\,\mathrm{Im}\,\omega + (\mathrm{Re}\,\alpha - 1)|1-\omega|^2)|\phi|^2 \text{ a.e. on } \mathbb{T},$$

from which we obtain that

$$|1-\omega|^2|\psi|\,|\phi| \leq 0 \text{ a.e. on } \mathbb{T}.$$

By the hypothesis on ω and ψ, and the classical theorem of Nevanlinna, we find that $\phi \equiv 0$. $\square$

Here is an easy consequence of Theorem 6.4.5.

Corollary 6.4.6. *Let α be a real number, ϕ and ω are Schwarz functions and that $\phi \not\equiv 0$. Suppose that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, and*

$$E = \{\zeta \in \mathbb{T} : |\omega(\zeta)| = 1\}.$$

Then the Lebesgue arc length measure of the set E is zero, i.e., $m(E) = 0$.

6.5 Examples for special cases

In this section, we give specific examples of ϕ and ψ so that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$. For a bounded analytic function on $\mathbb{D}$, we denote $\sup_{|z|<1} |f(z)|$ by $\|f\|$.

Example 6.5.1. *Suppose that $\|\phi\| < 1$. If $\|\omega\| < \frac{1-\|\phi\|}{1+\|\phi\|}$, then $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, for $\alpha \in [0,1]$.*

Proof. In view of Theorem 6.3.5, it suffices to verify the inequality (6.3.1). This inequality can be rewritten as

$$2|(1-\alpha)\,\overline{\omega}\,(\omega-1)+2i\alpha\,\mathrm{Im}\,\omega|\,|\phi|+(1-\alpha)(|\omega|^2+|1-\omega|^2|\phi|^2-1)<\alpha(1-|\omega|^2)(1-|\phi|^2).$$

We may set $\|\omega\|=A$ and $\|\phi\|=B$. Thus, it is enough to check that

$$2(1-\alpha)A(A+1)B+4\alpha AB+(1-\alpha)(A^2+(1+A)^2B^2-1)<\alpha(1-A^2)(1-B^2)$$

which is equivalent to

$$[A+B+AB-1][(1-\alpha)(A+B+AB+1)+\alpha(A+B-AB+1)]<0.$$

This yields the condition $A+B+AB-1<0$. This means that $A<\frac{1-B}{1+B}$ and the desired conclusion follows. ☐

Since the condition $A+B+AB-1<0$ gives $B<\frac{1-A}{1+A}$, we have the following result.

Example 6.5.2. *Suppose that $\|\omega\|<1$. If $\|\phi\|<\frac{1-\|\omega\|}{1+\|\omega\|}$, then $W_{\psi,\phi}$ preserves the class $\mathcal{P}_\alpha$, for $\alpha\in[0,1]$.*

Example 6.5.3. *Suppose that $\phi(z)=z(az+b)$, where a and b are non-zero real numbers such that $|a|+|b|=1$. Take $\omega(z)=z(cz+d)$ with*

$$c=-\frac{ab}{K}\quad\text{and}\quad d=\frac{1-(a^2+b^2)}{K}\quad\text{for } K>2+\sqrt{5}.$$

Then $W_{\psi,\phi}$ preserves the class $\mathcal{P}_1$.

Proof. Clearly $|\phi(z)|^2\leq a^2+b^2+2abx$ for $z=x+iy$ and thus,

$$0<1-(a^2+b^2)-2abx\leq(1-|\phi|^2).$$

Also note that

$$|\mathrm{Im}\,\omega|\leq|2cx+d|=\frac{1-(a^2+b^2)-2abx}{K}$$

and

$$|\omega(z)|\leq|c|+|d|=\frac{1-|ab|-(|a|-|b|)^2}{K}\leq\frac{1}{K}.$$

The inequality (6.3.1) for $\alpha = 1$ reduces to

$$4|\phi|\,|\mathrm{Im}\,\omega| < (1-|\omega|^2)(1-|\phi|^2).$$

Since $4|\phi|\,|\mathrm{Im}\,\omega| \le 4|\mathrm{Im}\,\omega| \le 4|2cx+d|$ and

$$\left(1-\frac{1}{K^2}\right)K|2cx+d| \le (1-|\omega|^2)(1-|\phi|^2),$$

to verify the inequality (6.3.1), it suffices to verify the inequality

$$\frac{4}{K} < 1-\frac{1}{K^2}, \quad \text{i.e.,} \quad K^2-4K-1>0.$$

This gives the condition $K > 2+\sqrt{5}$ and the proof is complete. □

Remark 6.5.4. By letting $\alpha = 0$ in Theorem 6.3.5, we see that $W_{\psi,\phi}$ preserves the class $\mathcal{P}_0$ if and only if $|1-\omega|\,|\phi| + |\omega| < 1$ on $\mathbb{D}$.

Example 6.5.5. *If $|\phi| \le |\omega| < \sqrt{2}-1$ on $\mathbb{D}$, then $W_{\psi,\phi}$ preserves $\mathcal{P}_0$.*

Proof. In view of Remark 6.5.4 and the assumption that $|\phi| \le |\omega|$, it is enough to show that $|\omega|\,|1-\omega| < 1-|\omega|$ which, by squaring and then simplifying, is seen to be equivalent to the inequality

$$|\omega|^4 - 2\mathrm{Re}\,\omega|\omega|^2 + 2|\omega| - 1 < 0.$$

In order to verify the last inequality, we observe that

$$\begin{aligned}|\omega|^4 - 2\mathrm{Re}\,\omega|\omega|^2 + 2|\omega| - 1 &\le |\omega|^4 + 2|\omega|^3 + 2|\omega| - 1\\ &= (|\omega|^2+1)(|\omega|^2+2|\omega|-1),\end{aligned}$$

which is negative whenever $|\omega|^2 + 2|\omega| - 1 < 0$, i.e., $|\omega| < \sqrt{2}-1$. The desired result follows. □

Example 6.5.6. *If either $|\phi| \le |\omega| < s_0$ or $|\omega| \le |\phi| < s_0$ on $\mathbb{D}$, then $W_{\psi,\phi}$ preserves $\mathcal{P}_\alpha$ for every α with $-1 < \alpha < 0$, where s_0 (≈ 0.2648) is the unique positive root of the polynomial $P(x) = 2x^4 + 8x^3 + 12x^2 - 1$.*

Proof. Without loss of generality, we assume that $|\phi| \leq |\omega|$. In view of Remark 6.3.6 and the assumption that $\alpha \in (-1,0)$, the inequality (6.3.1) can be rewritten as

$$2|(1-\alpha)\,\overline{\omega}\,(\omega-1)+2i\alpha\,\mathrm{Im}\,\omega|\,|\phi|+(1-\alpha)(|\omega|^2+|1-\omega|^2|\phi|^2-1)-\alpha(1-|\omega|^2)(1-|\phi|^2)<0.$$

By setting $\|\omega\| = A$ and $\|\phi\| = B$ (so that $B \leq A$), it suffices to check that

$$2(1-\alpha)A(A+1)B - 4\alpha AB + (1-\alpha)(A^2+(1+A)^2B^2-1) - \alpha < 0,$$

which is equivalent to

$$-\alpha[(A+B+AB)^2+4AB] + (A+B+AB)^2 - 1 < 0.$$

Since $B \leq A$ and $\alpha \in (-1,0)$, the last inequality holds if

$$(2A+A^2)^2+4A^2] + (2A+A^2)^2 - 1 = 2A^4+8A^3+12A^2-1 < 0.$$

Clearly, the function $P(x) = 2x^4+8x^3+12x^2-1$ is strictly increasing on $(0,\infty)$ and thus, $P(x) < 0$ for $0 \leq x < s_0$, where s_0 is the unique positive root of $P(x)$. The desired result follows. $\square$

6.6 Fixed points

In this section, we discuss the fixed points of weighted composition operators. It is time to recall a well known result [32, Theorem V.10.5]. The modern way of writing it is as follows:

Proposition 6.6.1. *Let X be a metrizable topological vector space and C be a convex compact subset of X. Then, every continuous mapping $T : C \to C$ has a fixed point in C.*

We set $X = \mathcal{H}(\mathbb{D})$, $C = \mathcal{P}_\alpha$, $T = W_{\psi,\phi}$ and observe that every weighted composition operator on $\mathcal{P}_\alpha$ has a fixed point. Indeed, one has something more to conclude than this as we can see below.

Theorem 6.6.2. *Let ϕ, ψ, ω be as before, and ϕ not a rotation. Suppose that $W_{\psi,\phi}$ is a self-map of $\mathcal{P}_\alpha$. Then, $W_{\psi,\phi}$ has a unique fixed point which can be obtained by iterating $W_{\psi,\phi}$ for any $f \in \mathcal{P}_\alpha$. Further more, if ϕ is an inner function, then the fixed point is the constant function 1.*

Theorem 6.6.3. *Let ϕ, ψ, ω be as before, and ϕ be a rotation. Suppose that $W_{\psi,\phi}$ is a self-map of $\mathcal{P}_\alpha$ and F denotes the set of all fixed points of $W_{\psi,\phi}$. Then, there are three distinct cases:*

1. *If $\phi(z) \equiv z$, then $F = \mathcal{P}_\alpha$.*
2. *If $\phi(z) \equiv \lambda z$ and $\lambda^n \neq 1$ for every $n \in \mathbb{N}$, then $F = \{1\}$.*
3. *If $\phi(z) \equiv \lambda z$ and $\lambda^n = 1$ for some $n > 1$, then*

$$F = \{f : f(z) = g(z^n) \;\; \textit{for some} \;\; g \in \mathcal{P}_\alpha\}.$$

The proofs of these two theorems follow from the lines of the proofs of the corresponding results of Section 4 of [13]. Moreover, the key tools for the proofs are from Section 6.1 of Shapiro's book [68]. So we omit the details.

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List of Publications

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