ON ONE-SIDED DISTRIBUTION FUNCTIONS

By B. RAMACHANDRAN
Indian Statistical Institute

SUMMARY. Necessary and sufficient conditions on the characteristic function are obtained for a probability distribution function on the real line to be one-sided (following the earlier work of G. Polya and E. Lukacs). The results are then specialized to infinitely divisible laws and, in particular, stable laws.

Let F be a distribution function (d.f.) on the real line. If a real a (respectively b) exists such that $F(a) = 0$ $[F(b) = 1]$, then F is said to be *bounded to the left* (*bounded to the right*) and its *left extremity* (*right extremity*) is defined in an obvious manner as $\sup(a : F(a) = 0)$ $[\inf(b : F(b) = 1)]$. In either case, F is said to be a *one-sided* d.f. If F is bounded on both sides, it is said to be a *finite* d.f.

Following the discussion of finite d.f.'s in Polya (1949), Lukacs (1957; pp. 229-230 and 1960; p.139) established that a necessary and sufficient condition (NASC) for a d.f. with *analytic* characteristic function (c.f.) to be bounded to the left is that the c.f. be also analytic and of exponential type in the upper half-plane; the left extremity of the d.f. is then given by $-\limsup_{y\to\infty}[\ln f(iy)/y]$. Of course, the dual of this result holds for d.f.'s bounded to the right. We show below that the stringent assumption of the analyticity of the c.f. is superfluous and (perhaps equalling this fact in importance) the 'lim sup' in the above statement can in fact be replaced by 'lim'. We apply the result so obtained to establish NASC's for an infinitely divisible (i.d.) c.f. to correspond to a one-sided d.f. The results are then specialized for the "stable" laws.

Theorem 1 : *A NASC for a c.f. f to correspond to a d.f. F bounded to the left is that f be extensible to the upper half-plane as a function continuous in the closed half-plane and analytic in the open half-plane and of exponential type there. The left extremity of F is then given by*

$$\text{l ext } F = -\lim_{y\to\infty} [\ln f(iy)/y] \quad (\textit{which exists}). \qquad \dots \quad (1)$$

The expression for the right extremity in the dual of the above result for d.f.'s bounded to the right is : $\lim_{y\to\infty} [\ln f(-iy)/y]$. It follows from these two results that a NASC for a d.f. to be finite is that its c.f. f be entire and of exponential type throughout the plane. It can be further asserted in this case that either $f \equiv 1$ or is of order one. The formulas above give the two extremities of the d.f.

Proof : If F is bounded to the left and l ext $F = a$, then $f(t) = \int_a^\infty e^{itx}\, dF(x)$ for real t. The integral on the right, however, defines a function of the complex variable t continuous in $\text{Im}\, t \geqslant 0$ and analytic in $\text{Im}\, t > 0$ and of exponential type $\leqslant |a|$ there. Hence the necessity part of the theorem.

Conversely, let f be extensible to the upper half-plane as described. Denote the extended function also by f. If $F(0-) = 0$, there is nothing to prove. Let then $F(0-) > 0$, and let the functions f^- and f^+ be defined by : $f^-(t) = \int_{-\infty}^{0-} e^{itx} dF(x)$ and $f^+(t) = \int_0^\infty e^{itx}\, dF(x)$. f^- is continuous in $\text{Im}\, t \leqslant 0$ and analytic in $\text{Im}\, t < 0$; f by assumption and f^+ by definition are continuous in $\operatorname{Im} t \geqslant 0$ and analytic in $\operatorname{Im} t > 0$, so that the same is true of $f - f^+$. Further, f^- and $f - f^+$ coincide on the real axis. Hence, by a well-known consequence of the "principle of reflection", they are analytic continuations of each other. f^- is thus an entire function which coincides on the real axis with a constant multiple of a c.f. It therefore follows from the Lévy-Raikov theorem that the representation : $f^-(t) = \int_{-\infty}^{0-} e^{itx}\, dF(x)$ holds for all complex t, and hence that, for $f = f^+ + f^-$, the integral representation $\int_{-\infty}^{\infty} e^{itx}\, dF(x)$ holds for $\operatorname{Im} t \geqslant 0$; in particular, $f(iy) > 0$ for all $y \geqslant 0$. Since further f is of exponential type there, there exists a $c > 0$ such that $f(iy) \leqslant e^{cy}$ for all $y > 0$; it then follows as in Lukacs (1960) that the d.f. is bounded to the left and that its left extremity is given by the relation

$$\text{l ext } F = -\limsup_{y \to \infty} [\ln f(iy)/y] = a, \text{ say.} \qquad \ldots \quad (2)$$

Let now, for $y \geqslant 0$, $\phi(y) = \ln f(iy)$. $\phi(y)$ is a convex function of y in $y \geqslant 0$, since it is continuous there and since its second derivative exists and is non-negative in $y > 0$ by the Cauchy-Schwarz inequality $\int_a^{\infty} x^2 e^{-yx}\, dF(x) . \int_a^{\infty} e^{-yx}\, dF(x) \geqslant \left[\int_a^{\infty} x\, e^{-yx}\, dF(x)\right]^2$. The convexity of ϕ and the fact that $\phi(0) = 0$ imply that $\ln f(iy)/y$ is a non-decreasing function of y in $y > 0$. (This fact plays a vital role in the author's proofs of certain results relating to finite and denumerable "α-decompositions" of probability laws.) In particular, therefore, formula (2) can be replaced by (1) and our theorem stands proved.

We proceed to the consideration of i.d. laws. Let us denote by $L(\beta, \gamma, M, N)$ the Lévy representation for the logarithm of an i.d.c.f. :

$$\ln f(t) = i\beta t - \gamma t^2 + \int_{-\infty}^{0-} h(t, u)\, dM(u) + \int_{0+}^{\infty} h(t, u)\, dN(u) \qquad \ldots \quad (3)$$

where $h(t, u) = e^{itu} - 1 - itu/(1+u^2)$, M and N are non-decreasing right-continuous functions defined on $(-\infty, 0)$ and $(0, \infty)$ respectively, with $M(-\infty) = N(\infty) = 0$ and $\int_{-a}^{0-} u^2 dM(u) + \int_{0+}^{a} u^2\, dN(u) < \infty$ for some (and so for all) $a > 0$. Suppose F, the d.f. corresponding to $L(\beta, \gamma, M, N)$, is bounded to the left. Noting the trivial fact that any component thereof is also bounded to the left (if F is the convolution of the d.f.'s G and H, each of the latter is bounded to the left), we see that F cannot have a Normal component and so we must have $\gamma = 0$. Also, if M does not vanish identically, there exist real numbers $a < b < 0$ such that $M(b) - M(a) > 0$; hence f has the i.d. factor $\exp\left[\int_a^b (e^{itu} - 1)\ dM(u)\right]$ which, though an entire function, is *not* of exponential type in $\operatorname{Im} t \geqslant 0$, so that the corresponding d.f. is not bounded to the left, by Theorem 1. (A non-function-theoretic argument for this fact, of some interest, would be : taking a and b to be continuity-points of M, without loss of generality, let G be the bounded non-decreasing right-continuous function defined by

$$G(x) = \begin{cases} 0 & \text{for } x \leqslant a \\ M(x) - M(a) & \text{for } a < x < b \\ M(b) - M(a) & \text{for } x \geqslant b \end{cases}$$

G^{*n} the n-fold convolution of G with itself, and $C = \exp[M(a)-M(b)]$. Then H defined by $H(x) = C \sum_{n=0}^{\infty} G^{*n}(x)/n!$ is a component of F, since its c.f. $\exp[\int_a^b (e^{itu}-1)\, dM(u)]$ is a factor of f. But, for every positive integer n, $H(nb)-H(na) \geqslant C[G^{*n}(nb)-G^{*n}(na)]/n! = C.[G(b)-G(a)]^n/n! > 0$, so that H (and so F) is not bounded to the left. Hence we must have $M \equiv 0$.

Thus, the Lévy representation of an i.d.c.f. f corresponding to a d.f. F bounded to the left is necessarily of the form (t real)

$$\ln f(t) = i\beta t + \int_{0+}^{\infty} h(t, u)\, dN(u). \qquad \dots \quad (4)$$

Denoting the expression on the right by $\psi(t)$, we see that ψ, and consequently e^{ψ}, is continuous in $\text{Im}\, t \geqslant 0$ and analytic in $\text{Im}\, t > 0$; since f, as the c.f. of a d.f. bounded to the left, has the same property, and further coincides with e^{ψ} on the real axis, it coincides with e^{ψ} in $\text{Im}\, t \geqslant 0$ and it follows that (4) holds for all such t.

It then follows from Theorem 1 that $\lim_{y\to\infty} \int_{0+}^{\infty} \left(\frac{e^{-yu}-1}{y} + \frac{u}{1+u^2} \right) dN(u)$ exists finitely (and is equal to $\beta-$l ext F). Since, for every fixed $u > 0$, $(e^{-yu}-1)/y = -\int_0^u e^{-xy}\, dx$ is an increasing function of y and tends to zero as $y \to \infty$, it follows from the monotone convergence theorem that $\int_{0+}^{\infty} (u/(1+u^2))\, dN(u)$ exists finitely and is equal to (the above limit and so to) $\beta-$l ext F.

Conversely, let $\gamma = 0$, $M \equiv 0$ and $\int_{0+}^{\infty} [u/(1+u^2)] dN(u) = \delta < \infty$ in the Lévy representation $L(\beta, \gamma, M, N)$ of an i.d.c.f. f. Then, setting $\beta' = \beta - \delta$, $\ln f(t) = i\beta' t + \int_{0+}^{\infty} (e^{itu}-1)\, dN(u)$ for real t. Denoting the expression on the right side of the last relation by $\psi(t)$, we see that e^{ψ} is continuous in $\text{Im}\, t \geqslant 0$, analytic in $\text{Im}\, t > 0$ and of exponential type there (noting that, for any $y > 0$, $\int_{0+}^{1} (1-e^{-yu})\, dN(u) \leqslant y \int_{0+}^{1} u\, dN(u) \leqslant 2\delta y$). Hence by Theorem 1, f corresponds to a d.f. bounded to the left and its left extremity is given by $\beta' = \beta - \delta$.

We have thus established the following result.

Theorem 2a : *A set of NASC's for an i.d.c.f. with the Lévy representation $L(\beta, \gamma, M, N)$ to belong to a d.f. F bounded to the left is that* : (i) $\gamma = 0$, (ii) $M \equiv 0$, *and* (iii) $\int_{0+}^{\infty} [u/(1+u^2)]\, dN(u)$ *exists finitely. If the value of this integral be denoted by δ, then we have l ext $F = \beta - \delta$.*

Theorem 2b : *A set NASC's for an i.d.c.f. with the Lévy representation $L(\beta, \gamma, M, N)$ to belong to a d.f. F bounded to the right is that* : (i) $\gamma = 0$, (ii) $N \equiv 0$, and (iii) $\int_{-\infty}^{0-} [u/(1+u^2)] dM(u)$ *exists finitely. If the value of this integral is denoted by δ, then we have r ext $F = \beta - \delta$.*

Theorem 2b can be obtained from Theorem 2a by considering the d.f. $\tilde{F}$ conjugate to F (given by $\tilde{F}(x) = 1 - F(-x-0)$ for all x).

Corollary 1 : *A finite d.f. with an i.d.c.f. is necessarily degenerate.*

For, in such a case, we must have $\gamma = 0$, $M \equiv 0$ and $N \equiv 0$.

(The function-theoretic proof of this fact is as follows : if an i.d.c.f. corresponds to a finite d.f., then it is a non-vanishing entire function of order one (and of exponential type) or identically equal to 1. In the former case, it is necessarily of the form $\exp(i\beta t)$ for some real $\beta \neq 0$, thus proving our assertion.)

A non-Normal stable law with exponent α is given by

$$\gamma = 0,\ dM(x) = (h/|x|^{1+\alpha})\,dx,\ dN(x) = (k/x^{1+\alpha})dx,$$

where $0 < \alpha < 2$, and h and k are constants. It follows from Theorems 2a and 2b that :

Corollary 2 : *No non-Normal stable law with exponent* $\alpha \geqslant 1$ *(and, trivially, no Normal law) can have a one-sided d.f. Among stable laws with exponent* $\alpha < 1$, *only those have one-sided d.f.'s for which either* $M \equiv 0$ *or* $N \equiv 0$, *i.e., either* h *or* k *above is zero* (cf. Theorem 5.7.7. of Lukacs, 1960, proved by a direct evaluation of the corresponding density function due to H. Bergström).

The above assertion follows from the fact that $\int_0^1 x^{-\alpha}\,dx$ exists finitely if and only if $\alpha < 1$.

REFERENCES

LUKACS, E. (1957) : Les fonctions caractéristiques analytiques. *Annales Inst. Henri Poincaré*, 15, 217-251.

——— (1960) : *Characteristic Functions*, Charles Griffin, London.

POLYA, G. (1949) : Remarks on characteristic functions. *Proc. First Berkeley Symp. Math. Statist. Prob.*, University of California Press, 115-123.

Paper received : June, 1966.