Infinite Mode Quantum Gaussian States

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Dedicated

to

Amma, Daddy, Vaava, Achachan and my teachers

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Contents

Introduction

A study of finite mode quantum Gaussian states were initiated back in the early 1970's [CH71, Hol75]. The subject is well studied both theoretically and experimentally [KLM02] in the literature. Recently, it has been getting more attention in the context of its importance in continuous variable quantum information theory. In a quantum physics laboratory, coherent states from a laser, thermal states from a black body source, vacuum state etc. are all Gaussian [ARL14]. The experimental realisations and successes in quantum communication protocols make it exciting for a physicist [FOP05, BvL05, WGC06, WHTH07, ARL14]. A review of this subject can be seen in [WPGP+12]. Apart from its physical relevance Gaussian states have got an elegant mathematical structure. It fits well into the fabric of quantum probability as quantum versions of classical Gaussian distributions [Par10]. An expository article by K. R. Parthasarathy [Par10] explains the beautiful mathematical structure of finite mode Gaussian states.

Due to the Stone-Von Neumannn theorem, any quantum state of a system of n-degrees of freedom(finite mode) can be considered as a state on $L^2(\mathbb{R}^n)$. Roughly speaking, finite mode (quantum) Gaussian states are a particular class of quantum states on $L^2(\mathbb{R}^n)$ in which every element in the position-momentum field has a normal distribution. We call n as the number of *modes* of the state in this case. $L^2(\mathbb{R}^n)$ can be interpreted as the Bosonic (symmetric) Fock space built over $\mathbb{C}^n$ and the Fock space properties of $L^2(\mathbb{R}^n)$ are what we exploit to study the Gaussian states. Hence it is natural to ask the question- *What is a Gaussian state on an arbitrary Bosonic Fock space?* In other words we are asking- *What is an infinite mode Gaussian state?* Apart from the mathematical aspects, we believe that this entity can have a physical meaning. The primary purpose of this thesis is to define, characterize, and study properties of infinite mode quantum Gaussian states.

In Chapter 1, we review the background materials needed for our work. This includes minor improvements and improvisations of various known results. Also, we state several results available in the literature without providing proofs. Further, Section 1.7 establishes some crucial notations and conventions which we will use throughout this thesis.

The main aim of Chapter 2 is to prove an appropriate generalization of a finite dimensional result called Williamson's normal form. Theorem 2.6.1 establishes this. Here we need to consider normal operators on separable real Hilbert spaces, and we find some interesting results about them. For example, Theorem 2.3.1 proves that any normal operator on a real Hilbert space is orthogonally equivalent to its transpose (adjoint). Although this result is known in the literature [Vis78], we provide an elementary proof which shows the symmetry we have in this situation. Corollary 2.5.1 proves a structure theorem for real skew-symmetric operators which is exactly like the finite dimensional situation. This result is elementary but new to best of our knowledge (in a recent paper [BP12], Böttcher et al. prove the same result for the special case compact skew-symmetric operators). Here also the symmetry of the situation is explicit in our proof. Further, we get some improvements and shortcuts in the spectral theory of real normal operators in this chapter.

A systematic study of the quantum Gaussian states in the infinite mode setting is initiated in Chapter 3. We define Gaussian states using their quantum Fourier transform (otherwise called as quantum characteristic function) and characterize them in two different ways. Theorem 3.2.1 characterizes the Gaussian states in terms of their covariance operators, and Theorem 3.4.4 identifies Gaussian states with a particular class of quasifree states on CCR-Algebra. The power of an apparently simple result, the Williamson's normal form, can be seen in this chapter.

In Chapter 4, we prove some results about convexity, symmetry associated with Gaussian states, we obtain a structure theorem also here. Results in this chapter are mostly the infinite mode extensions of those in [Par13b]. Two important results in this chapter are: (i) Theorem 4.2.1, which establishes the structure of a Gaussian state up to unitary equivalence, and (ii) Theorem 4.3.2, which identifies all Gaussian symmetries on Bosonic Fock space, where a unitary operator U is called a Gaussian symmetry if $U\rho U^*$ is a Gaussian state for every Gaussian state ρ.

To summarize, our work put infinite mode Gaussian states into a rigorous mathematical framework and extend results in [Par10] and [Par13b] to this general situation. The subsequent papers by Parthasarathy [Par13b, Par13a, Par15b, Par15a], Parthasarathy and Sengupta [PS15a, PS15b], and Bhat, Parthasarathy and Sengupta [RBPS17] show that this subject has good prospects from a mathematical perspective. Because of its importance in quantum information theory and the open problems asked in the above papers, we believe that Gaussian states open up a vast realm for doing some interesting mathematics which are relevant to the future of quantum communication theory.

CHAPTER 1

Preliminaries

1.1 Symmetric Fock space

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle \cdot, \cdot \rangle$, which is anti-linear in the first variable. All Hilbert spaces (over real or complex field) considered in this thesis are separable. For $n \in \mathbb{N}$, let S_n denote the group of all permutations of the the set $\{1, 2, \dots, n\}$. Thus any $\sigma \in S_n$ is a one-to-one map of $\{1, 2, \dots, n\}$ onto itself. For each $\sigma \in S_n$, let U_σ be defined on the product vectors in $\mathcal{H}^{\otimes^n}$ by

$$U_\sigma(f_1 \otimes \cdots \otimes f_n) = f_{\sigma^{-1}(1)} \otimes \cdots \otimes f_{\sigma^{-1}(n)},$$

where σ^{-1} is the inverse of σ. Then U_σ is a scalar product preserving map of the total set of product vectors in $\mathcal{H}^{\otimes^n}$ onto itself. Hence U_σ extends uniquely to a unitary operator on $\mathcal{H}^{\otimes^n}$, which we shall denote by U_σ itself. Clearly $\sigma \mapsto U_\sigma$ is a unitary representation of the group S_n. The closed subspace of fixed points,

$$\mathcal{H}^{\circledS^n} = \{f \in \mathcal{H}^{\otimes^n} | U_\sigma f = f, \forall \sigma \in S_n\} \tag{1.1.1}$$

of $\mathcal{H}^{\otimes^n}$ is called the n-fold symmetric tensor product of $\mathcal{H}$. The symmetric Fock space (also known as Boson Fock space) over $\mathcal{H}$ is defined as

$$\Gamma_s(\mathcal{H}) := \bigoplus_{n=0}^{\infty} \mathcal{H}^{\circledS^n},$$

where we take $\mathcal{H}^{\otimes^0} := \mathbb{C}$. The n-th direct summand is called the *n-particle subspace*. Any element in the n-particle subspace is called an *n-particle vector*. When $n = 0$ we call it as the vacuum space. The vector $\Phi := 1 \oplus 0 \oplus 0 \oplus \cdots$ is called the *vacuum vector*. We denote by $\Gamma_s^0(\mathcal{H})$ the dense linear subspace generated by all n-particle vectors, $n = 0, 1, 2, \dots$ and we call them as *finite particle spaces*. For $f \in \mathcal{H}$, define the exponential vector

$$e(f) = 1 \oplus f \oplus \frac{f^{\otimes^2}}{\sqrt{2!}} \oplus \cdots \oplus \frac{f^{\otimes^n}}{\sqrt{n!}} \oplus \cdots, \tag{1.1.2}$$

then $e(f) \in \Gamma_s(\mathcal{H})$. Notice that

$$\langle e(f), e(g)\rangle = \exp \langle f, g\rangle, \tag{1.1.3}$$

for all $f, g \in \mathcal{H}$. The set $E := \{e(f) | f \in \mathcal{H}\}$ of all exponential vectors is linearly independent and total in $\Gamma_s(\mathcal{H})$. Further if A is a dense set in $\mathcal{H}$ then the linear span of the set $\{e(f) | f \in A\}$ is dense in $\Gamma_s(\mathcal{H})$.

Example 1 (Example 19.8 and Exercise 20.20 in [Par92]). $\Gamma_s(\mathbb{C}) = L^2(\mathbb{R})$ by identifying $e(z) \in \Gamma_s(\mathbb{C})$ with the L^2 function $x \mapsto (2\pi)^{-1/4} \exp\{-4^{-1}x^2 + zx - 2^{-1}z^2\}$.

1.2 Basic operators in quantum theory

For any fixed $f \in \mathcal{H}$, consider the map defined on the set of exponential vectors $E = \{e(g) : g \in \mathcal{H}\}$, by $e(g) \mapsto \{\exp\left(-\frac{1}{2}\|f\|^2 - \langle f, g\rangle\right)\}e(f+g)$. This yields an inner product preserving map of E onto itself. As E is total, there exists a unique unitary operator $W(f) \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ satisfying

$$W(f)e(g) = \{\exp\Big(-\frac{1}{2}\|f\|^2 - \langle f, g\rangle\Big)\}e(f+g). \tag{1.2.1}$$

$W(f)$ is called the *Weyl* operator associated with $f \in \mathcal{H}$. The mapping $f \mapsto W(f)$ from $\mathcal{H}$ into $\mathcal{B}(\Gamma_s(\mathcal{H}))$ is known as the *Weyl representation*. The following proposition sayes that the Weyl representation is a *strongly continuous, projective, unitary* representation

Proposition 1.2.1. *The Weyl representation is strongly continuous. Further,*

$$W(-f) = W(f)^*, \forall f \in \mathcal{H}, \tag{1.2.2}$$

$$W(f)W(g) = \exp(-i \operatorname{Im} \langle f, g\rangle)W(f+g). \tag{1.2.3}$$

By Proposition 1.2.1, every $f \in \mathcal{H}$ yields a strongly continuous one parameter unitary group $\{W(tf) | t \in \mathbb{R}\}$. Let us denote by $p(f)$, the observable obtained as the Stone generator of this group. Then

$$W(tf) = e^{-itp(f)}, t \in \mathbb{R}, f \in \mathcal{H}. \tag{1.2.4}$$

Recall the fact that the exponential domain $\mathcal{E}$ (which is the dense subspace spanned by exponential vectors in $\Gamma_s(\mathcal{H})$) is a core for $p(f)$ for all $f \in \mathcal{H}$. The space of all finite particle vectors, $\Gamma_s^0(\mathcal{H})$ is also a core for $p(f)$ for all f. Let us fix a basis $\{e_j\}$ for $\mathcal{H}$ and let

$$p_j = 2^{-1/2}p(e_j), \qquad q_j = -2^{-1/2}p(ie_j), \tag{1.2.5}$$

$$a_j = 2^{-1/2}(q_j + ip_j), \qquad a_j^\dagger = 2^{-1/2}(q_j - ip_j), \tag{1.2.6}$$

for each $j \in \mathbb{N}$. Then we have the Lie brackets

$$[q_r, p_s] = i\delta_{rs} I, \quad [a_r, a_s^\dagger] = \delta_{rs}, \quad \forall r, s \in \mathbb{N}, \tag{1.2.7}$$

on their respective common domains which are also dense. Further $\{a_r, r \in \mathbb{N}\}$ and $\{a_r^\dagger, r \in \mathbb{N}\}$ commute among themselves. We call p_j and q_j as the j-th *momentum* and *position* operator, a_j and $a_j^\dagger$ as the j-th *annihilation* and *creation* operator for all $j \in \mathbb{N}$. We refer to Section 20 of [Par92] for more details on these operators.

Proposition 1.2.2. *Let $z \in \mathcal{H}$ be such that $z = \sum_{j=1}^{n} \alpha_j e_j$, where $\alpha_j = x_j + iy_j, x_j, y_j \in \mathbb{R}, \forall j$, then*

$$W(z) = e^{-i\sqrt{2}\sum_{j=1}^{n}(x_j p_j - y_j q_j)}; \quad p(z) = \sqrt{2}\sum_{j=1}^{n}(x_j p_j - y_j q_j).$$

If $T \in \mathcal{B}(\Gamma_s(\mathcal{H}))$, observe that $\mathcal{H}^{ⓢ n}$ is an invariant subspace for $T^{\otimes n}$ for all $n \in \mathbb{N}$. So for $n \in \mathbb{N}$, we can define an operator $T^{ⓢ n}$ on $\mathcal{H}^{ⓢ n}$ as the restriction of $T^{\otimes n}$ to $\mathcal{H}^{ⓢ n}$.

Definition 1.2.1. If T is any contraction on $\mathcal{H}$, define $\Gamma_s(T)$ on $\Gamma_s(\mathcal{H})$ by

$$\Gamma_s(T) = 1 \oplus T \oplus T^{ⓢ 2} \oplus \cdots \oplus T^{ⓢ n} \oplus \cdots . \tag{1.2.8}$$

Then $\Gamma_s(T)$ is a contraction and it satisfies

$$\Gamma_s(T)(e(f)) = e(Tf). \tag{1.2.9}$$

These are called the *second quantization maps.*

Note that if U is a unitary then $\Gamma_s(U)$ is also a unitary. Further we have

$$\Gamma_s(U)^{-1} = \Gamma_s(U^{-1}), \tag{1.2.10}$$

$$\Gamma_s(U)W(u)\Gamma_s(U)^{-1} = W(Uu). \tag{1.2.11}$$

Also, it is possible to define $\Gamma_s(U)$ via (1.2.9) even if U is a unitary mapping $\mathcal{H}$ to a different Hilbert space $\mathcal{K}$, where the exponential vector on the left is in $\Gamma_s(H)$ and that on the right is in $\Gamma_s(\mathcal{K})$.

Proposition 1.2.3 (Exercise 20.22 (iv) in [Par92])**.** *Let T be a positive operator of finite trace with eigenvalues $\{\lambda_j | j = 1, 2, \ldots\}$ inclusive of multiplicity and $\sup_j |\lambda_j| < 1$. Then $\Gamma_s(T)$ is trace class and*

$$\operatorname{Tr}\Gamma_s(T) = \Pi_j(1 - \lambda_j)^{-1}.$$

1.3 Factorizability and Irreducibility of the Weyl representation

Proposition 1.3.1 (Factorizability)**.** *If $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then there is a unique unitary isomorphism between $\Gamma_s(\mathcal{H})$ and $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ satisfying $e(f \oplus g) \mapsto e(f) \otimes e(g)$. Further, under this isomorphism we have $W(f \oplus g) = W(f) \otimes W(g)$.*

Proof. Exercise 20.21, [Par92] □

Recall the countable tensor product of Hilbert spaces (Exercise 15.10 in [Par92]). We summarise some properties of infinite tensor product of Fock spaces in the proposition below. Let $\Phi_n \in \Gamma_s(\mathcal{H}_n)$ denote the vacuum vector in $\Gamma_s(\mathcal{H}_n)$ for every n. By construction of $\otimes_{n=1}^{\infty} \Gamma_s(\mathcal{H}_n)$ using $\{\Phi_n\}$ as stabilising sequence , the set $\{\otimes_{j=1}^{N} e(x_j) \otimes e(0) \otimes e(0) \otimes \cdots | x_j \in \mathcal{H}_j, N \in \mathbb{N}\}$ is a total set. If we identify the exponential vectors in the natural way,

$$e(\oplus_{n=1}^{\infty} x_n) = \otimes_{n=1}^{\infty} e(x_n) := \lim_{N \to \infty} \otimes_{j=1}^{N} e(x_j) \otimes e(0) \otimes e(0) \otimes \cdots , \tag{1.3.1}$$

then this identification becomes an isomorphism, it should be noted that the limit in the right most term exists. We have

Proposition 1.3.2. *Let $\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n$, where $\mathcal{H}_n, n = 1, 2, 3 \ldots$ is a sequence of Hilbert spaces. Consider the infinite tensor product $\otimes_{n=1}^{\infty} \Gamma_s(\mathcal{H}_n)$ constructed using the stabilising sequence $\{\Phi_n\}$, where $\Phi_n \in \Gamma_s(\mathcal{H}_n)$ is the vacuum vector for every n. Then*

$$\Gamma_s(\mathcal{H}) = \otimes_{n=1}^{\infty} \Gamma_s(\mathcal{H}_n) \tag{1.3.2}$$

under the natural isomorphism (1.3.1). In this identification, for $\oplus_{n=1}^{\infty} x_n \in \mathcal{H}$, and contractions $A_n \in \mathcal{B}(\mathcal{H}_n), n \geq 1$,

$$W(\oplus_{n=1}^{\infty} x_n) = \otimes_{n=1}^{\infty} W(x_n) \qquad := \operatorname*{s-lim}_{N \to \infty} \otimes_{j=1}^{N} W(x_j) \otimes I \otimes I \otimes \cdots , \tag{1.3.3}$$

$$\Gamma_s(\oplus_{n=1}^{\infty} A_n) = \otimes_{n=1}^{\infty} \Gamma_s(A_n) \qquad := \operatorname*{s-lim}_{N \to \infty} \otimes_{j=1}^{N} \Gamma_s(A_j) \otimes I \otimes I \otimes \cdots . \tag{1.3.4}$$

It may also be noted that

$$\Gamma_s(\oplus_{n=1}^{\infty} A_n) = \operatorname*{s-lim}_{N \to \infty} \otimes_{j=1}^{N} \Gamma_s(A_j) \otimes |e(0)\rangle\langle e(0)| \otimes |e(0)\rangle\langle e(0)| \otimes \cdots . \tag{1.3.5}$$

where $|e(0)\rangle\langle e(0)|$ denote the rank one projection onto $e(0)$.

Proof. All the statements are easily verified on exponential vectors. □

Proposition 1.3.3. *Let $\mathcal{T}$ be any bounded operator in $\Gamma_s(\mathcal{H})$ such that $\mathcal{T}W(u) = W(u)\mathcal{T}$ for all u in $\mathcal{H}$. The $\mathcal{T}$ is a scalar multiple of the identity.*

Proof. Proposition 20.9 in [Par92]. □

Corollary 1.3.1. *[Irreducibility] There is no proper subspace of $\Gamma_s(\mathcal{H})$ which is invariant under $W(u)$ for all u.*

Proof. If a subspace is invariant under $W(u)$ for all u then it is a reducing subspace due to Proposition 1.2.1. Now irreducibility follows from Corollary 1.3.1. □

Remarks 1. (i) First equation in (1.2.7) is the most fundamental one to quantum mechanics. Relations in (1.2.7) are called called the canonical commutation relations(CCR). The equations (1.2.2) and (1.2.3) together is the exponentiated version of the CCR known as the Weyl commutation relations or as the Weyl form of the CCR.

(ii) The Weyl repesentation is a *strongly continuous, factorizable, irreducible* and *projective* unitary representation of the abelian group $\mathcal{H}$.

(iii) When the dimension of $\mathcal{H}$ is finite the Stone-von Neumann Theorem (refer Exercise 13.8 in [Par92]) states that the Weyl representation is the only strongly continuous, irreducible and projective unitary representation of the CCR up to unitary equivalence.

1.4 Algebra of canonical commutation relations

In the previous sections, we developed a representation of the CCR starting from a Hilbert space. This can be done more generally by starting from a real linear space and a bilinear form σ on it called the symplectic structure. In this section, we list some basic facts about symplectic spaces and quasifree states of CCR algebras. This is intended only to be a quick review of what is needed for our work in this thesis. For more on these notions see [Pet90, Hol71a, Hol71b, Hol75, vD71].

Definition 1.4.1. Let H be a real linear space. A bilinear form $\sigma : H \times H \to \mathbb{R}$ is called a *symplectic* form if $\sigma(f, g) = -\sigma(g, f)$, for every $f, g \in H$. The pair (H, σ) is called a *symplectic space*. A symplectic form σ on H is called *nondegenerate* if $\sigma(f, g) = 0, \forall g \in H$ implies $f = 0$. A symplectic space (H, σ) is called a *standard (symplectic) space* if H is a Hilbert space over $\mathbb{C}$ with respect to some inner product $\langle \cdot, \cdot \rangle$ and $\sigma(\cdot, \cdot) = \operatorname{Im} \langle \cdot, \cdot \rangle$. It is called *separable* if there exists a countable family of vectors $\{f_k\}$ in H such that $\sigma(f, f_k) = 0$ for all k implies $f = 0$. Note that standard symplectic spaces are separable when the Hilbert space under consideration is separable.

Definition 1.4.2. Let (H, σ) be a symplectic space. The C^*-algebra of the canonical commutation relation over (H, σ), written as $CCR(H, \sigma)$ is by definition a C^*-algebra generated by elements $\{W(f) : f \in H\}$ such that

$$W(-f) = W(f)^*, \forall f \in H, \tag{1.4.1}$$

$$W(f)W(g) = \exp(i\sigma(f,g))W(f+g). \tag{1.4.2}$$

Theorem 1.4.1. *[Pet90] For any nondegenerate symplectic space (H, σ), the C^*-algebra of commutation relations, $CCR(H, \sigma)$ exists and unique up to isomorphism. Further, the linear hull of $\{W(f) : f \in H\}$ is dense in $CCR(H, \sigma)$.*

Proof. First we shall prove the existence of the C^*- algebra CCR(H, σ). Later we show that such C^*- algebras CCR(H, σ) are unique upto ismorphism. Consider H as a discrete abelian group (with vector space addition).

$$l^2(H) = \left\{ F : H \to \mathbb{C} : \text{Supp}(F) \text{ is countable}, \sum_{x \in H} |F(x)|^2 < \infty \right\}$$

is a Hilbert space with the innerproduct $\langle F, G \rangle = \sum_{x \in H} \overline{F(x)} G(x)$. For each $x \in H$ define $R(x)$ by

$$(R(x)F)(y) = \exp(i\sigma(y,x))F(x+y) \qquad \forall x, y \in H \tag{1.4.3}$$

Then the following computation shows that it is inner product preserving and thus $R(x)$ is a unitary for each $x \in H$.

$$\begin{aligned}
\langle R(x)F, R(x)G \rangle &= \sum_{y \in H} \overline{(R(x)F)(y)} (R(x)G)(y) \\
&= \sum_{y \in H} \overline{\exp(i\sigma(y,x))F(x+y)} \exp(i\sigma(y,x))G(x+y) \\
&= \sum_{y \in H} \overline{F(x+y)} G(x+y) \\
&= \langle F, G \rangle
\end{aligned}$$

Further for any $F \in l^2(H)$,

$$\begin{aligned}
R(x_1)R(x_2)F(y) &= R(x_1)\Big(R(x_2)F\Big)(y) \\
&= \exp(i\sigma(y,x_1))R(x_2)F(x_1+y) \\
&= \exp(i\sigma(y,x_1))\exp(i\sigma(x_1+y,x_2))F(x_1+x_2+y) \\
&= \exp(i\sigma(x_1,x_2))\exp(i\sigma(y,x_1+x_2))F(x_1+x_2+y)
\end{aligned}$$

$$= \exp(i\sigma(x_1,x_2))R(x_1+x_2)F(y).$$

Thus we have

$$R(x_1)R(x_2) = \exp(i\sigma(x_1,x_2))R(x_1+x_2). \tag{1.4.4}$$

The fact that $R(x)$ is a unitary for each $x \in H$ along with (1.4.4) and shows that

$$R(x)^* = R(-x). \tag{1.4.5}$$

Let $\mathcal{A}$ be the norm closure of the set

$$\left\{ \sum_{j=1}^{n} \lambda_j R(x_j) : \lambda_j \in \mathbb{C}, 1 \leq j \leq n, n \in \mathbb{N}, x_j \in H \right\}. \tag{1.4.6}$$

in $\mathcal{B}(l^2(H))$. Then $\mathcal{A}$ is a C^*-algebra satisfying the conditions (i) and (ii) in the Definition 1.4.2. Thus the existence of $CCR(H,\sigma)$ is proved.

Now we discuss about the uniqueness of C^*- algebra CCR(H,σ). Assume that $\mathcal{B} \subset B(\mathcal{H})$ (for some Hilbert space $\mathcal{H}$) is another C^*- algebra generated by elements $W(x)$, $x \in H$ satisfying (i) and (ii), i.e., $\mathcal{B}$ is the norm closure of the set

$$\left\{ \sum_{j=1}^{n} \lambda_j W(x_j) : \lambda_j \in \mathbb{C}, 1 \leq j \leq n, n \in \mathbb{N}, x_j \in H \right\}$$

in $B(\mathcal{H})$. We show that $\mathcal{A}$ and $\mathcal{B}$ are isomorphic by using the following steps:

Step 1. Let us consider the Hilbert space

$$\ell^2(H,\mathcal{H}) := \left\{ A\colon H \to \mathcal{H} \ :\ \text{Supp(A) is countable, } \sum_{x\in H} \|A(x)\|^2 < +\infty \right\}.$$

Note that for every $x \in H$, the map $\delta_x(y)$ defined as 1 at $x = y$ and 0 elsewhere, is in $\ell^2(H)$. Let $x \in H$, $f \in \mathcal{H}$, and define $\delta_x \otimes f\colon H \to \mathcal{H}$ by

$$(\delta_x \otimes f)(y) = \begin{cases} f & \text{if } x = y; \\ 0 & \text{if } x \neq y. \end{cases}$$

This identification shows that $\ell^2(H,\mathcal{H})$ is isomorphic to $\ell^2(H) \otimes \mathcal{H}$. The application $y \mapsto \pi(y)$, where

$$\pi(y)(\delta_x \otimes f) = \delta_{x-y} \otimes W(y)f, \text{ for all } \delta_x \otimes f \in \ell^2(H,\mathcal{H})$$

is a representation of the CCR on the Hilbert space $\ell^2(H,\mathcal{H})$. In order to show that R is equivalent to π, we define a operator $U\colon \ell^2(H,\mathcal{H}) \to \ell^2(H,\mathcal{H})$ by

$$U(\delta_x \otimes f) = \delta_x \otimes W(x)f, \text{ forall } \delta_x \otimes f \in \ell^2(H,\mathcal{H}).$$

We claim that U is unitary operator, and $U\pi(y) = (R(y) \otimes id)U$, for all $y \in H$.

Let $\delta_x \otimes f \in \ell^2(H,\mathcal{H})$ and consider

$$\begin{aligned} U\pi(y)(\delta_x \otimes f) = U(\delta_{x-y} \otimes W(y)f) &= \delta_{x-y} \otimes W(x-y)W(y)f \\ &= \delta_{x-y} \otimes \exp(i\sigma(x-y,y))W(x)f \\ &= \exp(i\sigma(x-y,y))\delta_{x-y} \otimes W(x)f \\ &= R(y)\delta_x \otimes W(x)f \\ &= (R(y) \otimes id)(\delta_x \otimes W(x)f) \\ &= (R(y) \otimes id)U(\delta_x \otimes f). \end{aligned}$$

Since R and π are equivalent, it is enough to isomorphism between $\mathcal{B}$ and the C^*- algebra Π generated by $\{\pi(y) \;:\; y \in H\} \subset B(\ell^2(H,\mathcal{H}))$.

Step 2. Now we show that for any finite linear combination

$$\left\| \sum_i \lambda_i W(y_i) \right\| = \left\| \sum_i \lambda_i \pi(y_i) \right\| \tag{1.4.7}$$

holds true.

Let $\widehat{H}$ be the dual group of the discrete group H. Note that $\widehat{H}$ is a compact topological group by the endowed topology of point wise convergence. We consider the normalized Haar measure say μ on $\widehat{H}$. Let us consider the Hilbert space

$$L^2(\widehat{H},\mathcal{H}) := \left\{ \widehat{A}\colon \widehat{H} \to \mathcal{H} \;:\; \operatorname{Supp}(\widehat{A}) \text{ is countable, } \sum_{\chi \in \widehat{H}} \|\widehat{A}(\chi)\|^2 < +\infty \right\}.$$

For every $y \in H$, we define $\widehat{\pi}(y)\colon L^2(\widehat{H},\mathcal{H}) \to L^2(\widehat{H},\mathcal{H})$ by

$$\widehat{\pi}(y)(\widehat{A})(\chi) = \chi(y)W(y)\widehat{A}(\chi), \text{ for all } \widehat{A} \in L^2(\widehat{H},\mathcal{H}).$$

Here $\widehat{\pi}(y)$ is a multiplication operator on $L^2(\widehat{H},\mathcal{H})$ by $\chi(y)W(y)$, for each $y \in H$. So the norm of the finite linear combination is given by

$$\left\| \sum \lambda_i \widehat{\pi}(y_i) \right\| = \sup\left\{ \left\| \sum \lambda_i \; \chi(y_i)W(y_i) \right\| \;:\; \chi \in \widehat{H} \right\}. \tag{1.4.8}$$

The spaces $\ell^2(H)$ and $L^2(\widehat{H})$ are isomorphic by the Fourier transformation, which establishes unitary equivalence between π and $\widehat{\pi}$. This implies the following:

$$\left\| \sum \lambda_i \pi(y_i) \right\| = \left\| \sum \lambda_i \widehat{\pi}(y_i) \right\| \tag{1.4.9}$$

By Equations (1.4.8) and (1.4.9), we have that

$$\left\| \sum \lambda_i \pi(y_i) \right\| = \sup \left\{ \left\| \sum \lambda_i \ \chi(y_i) W(y_i) \right\| \ : \ \chi \in \widehat{H} \right\}. \tag{1.4.10}$$

Step 3. Let $G := \left\{ \exp(2i\sigma(x, \cdot)) \ : \ x \in H \right\}$. Then $G \subset \widehat{H}$ is a subgroup. We show that G is dense subset of $\widehat{H}$ by using the following result of harmonic analysis:

If $K \subset \widehat{H}$ is a proper closed subgroup, then there exist $0 \neq h \in H$ such that $k(h) = 1$ for every $k \in K$.

Suppose $\overline{G}$ is proper subgroup of $\widehat{H}$ (i.e., the closure of G, $\overline{G} \neq \widehat{H}$). By above result, there exist $0 \neq y \in H$ such that $\exp(2i\sigma(x, y)) = 1$, for every $x \in H$. Then for every $t \in \mathbb{R}$ there exist an integer l such that $t\sigma(x, y) = l \ 2\pi i$. This is possible on when $\sigma(x, y) = 0$, for every $x \in H$. Since σ is non degenerate symplectic form, we have $y = 0$. This is a contradiction. Thus $\overline{G} = \widehat{H}$.

Suppose $\chi \in G$ i.e.,

$$\chi(\cdot) := \exp(2i\sigma(x, \cdot)),$$

then we have the following:

$$\begin{aligned}
\left\| \sum \lambda_i \chi(y_i) W(y_i) \right\| &= \left\| \sum \lambda_i \exp(2i\sigma(x, y_i)) W(y_i) \right\| \\
&= \left\| \sum \lambda_i W(x) W(y_i) W(-x) \right\| \\
&= \left\| W(x) \sum \lambda_i W(x) W(y_i) W(x)^* \right\| \\
&= \left\| \sum \lambda_i W(y_i) \right\|.
\end{aligned}$$

This shows that

$$\sup \left\{ \left\| \sum \lambda_i \chi(y_i) W(y_i) \right\| \ : \ \chi \in G \right\} = \left\| \sum \lambda_i W(y_i) \right\|.$$

Since G is a dense in $\widehat{H}$ and $\sum \lambda_i \chi(y_i) W(y_i)$ is bounded, we conclude from equation (1.4.10) that

$$\left\| \sum \lambda_i \pi(y_i) \right\| = \left\| \sum \lambda_i W(y_i) \right\|.$$

Define $\Psi \colon \Pi \to \mathcal{B}$ by
$$\Psi\Big(\sum \lambda_i \pi(y_i)\Big) = \sum \lambda_i W(y_i).$$
Then Ψ is an isometric isomorphism. Hence C^* algebra $CCR(H,\sigma)$ is unique up to isomorphism. $\square$

Corollary 1.4.1. *$CCR(H,\sigma)$ has a representation in $\mathcal{B}(l^2(H))$.*

Corollary 1.4.2. *The set $\{W(f) : f \in H\}$ is a linearly independent set in $CCR(H,\sigma)$.*

Proof. Consider the representation of $CCR(H,\sigma)$ in $\mathcal{B}(l^2(H))$ given in the proof of Theorem 1.4.1. By the uniqueness of $CCR(H,\sigma)$ it is sufficient to show that $\{R(x) : x \in H\} \subseteq \mathcal{B}(l^2(H))$ is linearly independent. Let $\lambda_1, \lambda_2, \cdots, \lambda_n$ be complex numbers and $x_1, x_2, \cdots, x_n$ be distinct elements in H such that $\sum_{j=1}^{n} \lambda_j R(x_j) = 0$. For $y \in H$, consider $\delta_y \in l^2(H)$, where $\delta_y(x) = 1$ for $x = y$ and 0 everywhere else. Then $\sum_{j=1}^{n} \lambda_j R(x_j)\delta_y(x) = 0$, for all $x, y \in H$. Then we get
$$\sum_j \lambda_j \exp(i\sigma(x_j, x))\delta_y(x_j + x) = 0 \qquad \forall x, y \in H \tag{1.4.11}$$
Let $1 \leq k \leq n$, taking $x = 0$ and $y = x_k$ in (1.4.11) gives $\lambda_k = 0$. $\square$

Proposition 1.4.1. *If $f, g \in H$ are different then*
$$\|W(f) - W(g)\| \geq \sqrt{2}.$$

Proof. For $h_1 \neq h_2$, $\tau(W(h_1)W(-h_2)) = e^{-i\sigma(h_1,h_2)}W(h_1 - h_2) = 0$. Hence
$$\|W(f) - W(g)\|^2 \geq \tau((W(f) - W(g))^*(W(f) - W(g))) = 2.$$
$\square$

Remarks 2. (i) Strong continuity of the general representation of the CCR considered in this section is meaningless.

(ii) Norm continuity cannot be demanded because of Proposition 1.4.1.

(iii) Stone-von Neumann Uniqueness theorem does not hold in this general set up.

1.5 Quasifree States on CCR Algebra

We need a few preliminaries from operator algebras to proceed further. We discuss that first.

Preliminaries from Operator Algebras

Definition 1.5.1. A linear functional ϕ on a unital C^*-algebra $\mathcal{A}$ is a called *state* if $\phi(x^*x) \geq 0$, for every $x \in \mathcal{A}$, and $\phi(I) = 1$, where I is the identity in $\mathcal{A}$.

Definition 1.5.2. Let $\mathcal{A}$ be a C^*-algebra. If $\mathcal{S} \subseteq \mathcal{A}$ then we set $\mathcal{S}^* = \{a : a^* \in \mathcal{S}\}$ and we call $\mathcal{S}$ *self-adjoint* when $\mathcal{S} = \mathcal{S}^*$. If $\mathcal{A}$ has a unit 1 and $\mathcal{S}$ is a self-adjoint subspace of $\mathcal{A}$ containing 1, then we call $\mathcal{S}$ an *operator system.*

Definition 1.5.3. If $\mathcal{S}$ is an operator system, $\mathcal{B}$ is a C^*-algebra, and $\phi : \mathcal{S} \to \mathcal{B}$ is a linear map, then ϕ is called a positive map if it maps positive elements of $\mathcal{S}$ to positive elements of $\mathcal{B}$.

Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathbb{C}$ be a positive map. If $a \in \mathcal{S}$ is self-adjoint then $a = \frac{1}{2}(\|a\| \cdot 1 + a) - \frac{1}{2}(\|a\| \cdot 1 - a)$ is a difference of two positive elements in $\mathcal{S}$. Hence $\phi(a)$ is a real number for every self-adjoint element in $\mathcal{S}$. If $x \in \mathcal{S}$ is arbitrary, then $x = a + ib$, where $a = \frac{x+x^*}{2}$ and $b = \frac{x-x^*}{2i}$ are self adjoint elements in $\mathcal{S}$. We write $a = \operatorname{Re} x$ and $b = \operatorname{Im} x$. Hence if $x = a + ib$ is the cartesian decomposition of x then $\phi(x^*) = \phi(a - ib) = \phi(a) - i\phi(b) = \overline{\phi(x)}$. With these observations we have

Lemma 1.5.1. *Let $\mathcal{S}$ be an operator system and $\phi : \mathcal{S} \to \mathbb{C}$ be a positive map then*

(i) ϕ is bounded.
(ii) ϕ can be extended to a unique positive map (denoted by $\tilde{\phi}$) on the norm closure of $\mathcal{S}$ (denoted by $\bar{\mathcal{S}}$).

Proof. (i) Let $x \in \mathcal{S}$. Let $\lambda \in \mathbb{C}$ be such that $|\phi(x)| = \lambda\phi(x)$. Then $|\phi(x)| = \phi(\lambda x) = \overline{\phi(\lambda x)} = \phi\left((\lambda x)^*\right)$. Then

$$\begin{aligned}
|\phi(x)| &= \frac{1}{2}[\phi(\lambda x) + \phi\left((\lambda x)^*\right)] \\
&= \phi\left(\frac{\lambda x + (\lambda x)^*}{2}\right) \\
&= \phi\left(\operatorname{Re}(\lambda x)\right) \\
&\leq \phi(\|\operatorname{Re}(\lambda x)\| \cdot 1) \\
&\leq \phi(\|\lambda x\| \cdot 1) \\
&= \|x\|\phi(1)
\end{aligned}$$

Thus ϕ is bounded.

(ii) Since ϕ is bounded, it is uniformly continuous. Therefore, there exists a unique continuous extension $\tilde{\phi}$, of ϕ on $\bar{\mathcal{S}}$. We will prove that $\tilde{\phi}$ is positive. Let $p \in \bar{\mathcal{S}}$ and $p \geq 0$. There exists a sequence $\{x_n\}$ in $\mathcal{S}$ such that $x_n \to p$, ie. $\|x_n - p\| \to 0$. Since $p \geq 0$, $\|x_n^* - p\| \to 0$. This further implies $\|\frac{x_n + x_n^*}{2} - p\| \to 0$. Let $h_n = \frac{x_n + x_n^*}{2}$. Then h_n is self adjoint and $h_n \to p$.

Claim. Given $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $h_n + \epsilon \cdot 1 \geq 0$, for all $n \geq \mathbb{N}$.

Proof (of claim). Let $h_n \in \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Choose $N \in \mathbb{N}$ such that $\|h_n - p\| \leq \frac{\epsilon}{2}$, whenever $n \geq N$. Let $n \geq N$ and $x \in \mathcal{H}$ then,

$$\begin{aligned}
\langle (h_n + \epsilon \cdot 1)x, x\rangle &= \langle (p + \epsilon \cdot 1)x, x\rangle + \langle (h_n + \epsilon \cdot 1) - (p + \epsilon \cdot 1)x, x\rangle \\
&= \langle (p + \epsilon \cdot 1)x, x\rangle + \langle (h_n - p)x, x\rangle \\
&\geq |\, \langle (p + \epsilon \cdot 1)x, x\rangle \,| - |\, \langle (h_n - p)x, x\rangle \,| \\
&\geq \epsilon \|x\|^2 - |\, \langle (h_n - p)x, x\rangle \,|, \qquad \text{(since } p \geq 0) \\
&\geq \epsilon \|x\|^2 - \frac{\epsilon}{2}\|x\|^2, \qquad \text{(Cauchy-Schwartz)} \\
&\geq \frac{\epsilon}{2}\|x\|^2
\end{aligned}$$

Thus the claim is proved.

Now we have

$$\tilde{\phi}(p + \epsilon \cdot 1) = \lim_{n \to \infty} \phi(h_n + \epsilon \cdot 1) \geq 0$$

Thus we have proved that, given $\epsilon > 0$, $\tilde{\phi}(p) + \epsilon \cdot \tilde{\phi}(1) \geq 0$. Hence $\tilde{\phi}(p) \geq 0$. ☐

Definition 1.5.4. (i) A **representation** of a unital C^* -algebra $\mathcal{A}$ is a *-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ (where $\mathcal{H}$ is some Hilbert space), which will always be assumed to be a "unital homomorphism", meaning that $\pi(1) = 1$ - where the symbol 1 on the left (respectively, right) denotes the identity of the C^*-algebra $\mathcal{A}$ (respectively, $\mathcal{B}(\mathcal{H})$).

(ii) Two representations $\pi_i : \mathcal{A} \to \mathcal{B}(\mathcal{H}_i), i = 1, 2$, are said to be equivalent if there exists a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ with the property that $\pi_2(a) = U\pi_1(a)U^*, \forall a \in \mathcal{A}$.

(iii) A representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is said to be **cyclic** if there exists a vector $\Omega \in \mathcal{H}$ such that $\{\pi(a)\Omega : a \in \mathcal{A}\}$ is dense in $\mathcal{H}$. (In this case, the vector Ω is said to be a cyclic vector for the representation π).

Theorem 1.5.1 (GNS Construction). *Let $\mathcal{A}$ be a C^*-algebra and ϕ be a state on it, then there exists a triple consisting of $(H_\phi, \Pi_\phi, \Omega_\phi)$, where H_ϕ is a Hilbert space, Π_ϕ is* *the representation of $\mathcal{A}$ in $\mathcal{B}(H_\phi)$ and $\Omega_\phi \in H_\phi$ is the cyclic vector associated with the representation such that*

$$\phi(a) = \langle \Omega_\phi, \Pi_\phi(a)\Omega_\phi \rangle, \forall a \in \mathcal{A}.$$

The cyclic representation above is unique in the sense that if (H, Π, Ω) is another cyclic representation for ϕ such that $\phi(a) = \langle \Omega, \Pi(a)\Omega \rangle, \forall a \in \mathcal{A}$ then there exists a unique unitary $U : H \to H_\phi$ such that $U(\Omega) = \Omega_\phi$ and $U(\Pi(a))U^* = \Pi_\phi(a), \forall a \in \mathcal{A}$.

Definition 1.5.5. A state ϕ on $\mathcal{A}$ is called *primary* if the von Neumann algebra $(\Pi_\phi(\mathcal{A}))''$ corresponding to the GNS-representation is a factor. It is called *type I* if $(\Pi_\phi(\mathcal{A}))''$ is a type 1 factor.

Definition 1.5.6. Two states ϕ and ψ on $\mathcal{A}$ are called *quasiequivalent* if $(\Pi_\phi(\mathcal{A}))''$ and $(\Pi_\psi(\mathcal{A}))''$ are isomorphic von Neumann algebras.

Quasifree states

Definition 1.5.7. A representation Π of $CCR(H, \sigma)$ in $\mathcal{B}(\mathcal{H})$ is called *regular* if the mapping

$$t \mapsto \langle \Pi(W(tf))\zeta, \eta \rangle$$

is continuous for all $\zeta, \eta \in \mathcal{H}$, and for every $f \in H$.

All the representations of $CCR(H, \sigma)$ considered in this thesis will be regular.

Proposition 1.5.1. *Let (H, σ) be a symplectic space and $G : H \to \mathbb{C}$ a function. There exists a state ϕ on $CCR(H, \sigma)$ such that*

$$\phi(W(f)) = G(f), \quad \forall f \in H$$

if and only if $G(0) = 1$ and the kernel

$$(f, g) \mapsto G(f - g)\exp(-i\sigma(f, g)) \tag{1.5.1}$$

is positive definite.

Proof. For a state ϕ on $CCR(H, \sigma)$, $\phi(I) = \phi(W(0)) = 1$. Hence it is necessary to have $G(0) = 1$. For $x = \sum c_j W(f_j)$, we have $xx^* = \sum c_j \overline{c_k} W(f_j)W(-f_k) = \sum c_j \overline{c_k} W(f_j - f_k)\exp(-i\sigma(f, g))$. Since for a state ϕ, $\phi(xx^*) \geq 0$, we see that the positivity condition also is necessary. On the other hand, the positivity condition along with $G(0) = 1$, allows us to define a positive functional on the linear hull of the Weyl operators (Note that Corollary

1.4.2 guaranties the well definedness of this map). Because of Theorem 1.4.1, and Lemma 1.5.1 we can extend it continuously to $CCR(H, \sigma)$. This supplies a state with required properties. Hence the conditions are sufficient. $\square$

Lemma 1.5.2. *Let (H, σ) be a (nondegenerate) symplectic space. If $\alpha(\cdot, \cdot)$ is a positive symmetric bilinear form on H, then the following conditions are equivalent.*

(i) The kernel $(f, g) \mapsto \alpha(f, g) - i\sigma(f, g)$ is positive definite.
(ii) $\sigma(f, g)^2 \leq \alpha(f, f)\alpha(g, g)$ for every $f, g \in H$.

Proof. Note that α is almost an inner product except for strict positivity.

Step 1: α can be taken as an innerproduct

It is clear that If $\alpha(f, f) = 0$ for some $f \neq 0$, then the condition (ii) implies σ is degenerate

Claim. If $\alpha(f, f) = 0$ for some non zero f then $(i) \Rightarrow \sigma$ is degenerate.

Proof (of the Claim). Let the condition (i) hold and $\alpha(f_1, f_1) = 0$ for some $0 \neq f_1 \in H$. Then by Cauchy-Schwartz $\alpha(f_1, f_2) = 0$ for every $f_2 \in H$, since $\det\big((\alpha(f_k, f_l) - i\sigma(f_k, f_l))\big) \geq 0$ for the 2×2 matrix $\big((\alpha(f_k, f_l) - i\sigma(f_k, f_l))\big), k, l = 1, 2$, we get that $\sigma(f_1, f_2) = 0, \forall f_2 \in H$. Thus for proving the lemma we can assume that α is strictly positive and thus a real inner product on H.

Step 2: Reduction to finite dimensions, but we will lose non degeneracy.

Both the conditions (i) and (ii) hold on H if and only if they hold on all finite dimensional subspaces. H is a m-dimensional real inner product space, m finite. Therefore, using the bilinearity of σ there exists an operator Q such that

$$\sigma(f, g) = \alpha(Qf, g), \qquad \forall f, g \in H. \tag{1.5.2}$$

Since $\sigma(f, g) = -\sigma(g, f)$, we get $Q^* = -Q$. According to the spectral decomposition of skew-symmetric matrix, there exists a basis for H in which Q has a diagonal form $\mathrm{Diag}(0, 0, \ldots, 0, A_1, A_2, \ldots, A_n)$, where

$$A_j = \begin{bmatrix} 0 & a_j \\ -a_j & 0 \end{bmatrix}, \qquad 1 \leq j \leq n.$$

Step 3: Proof in finite dimensions.

We will show that both the condition (i) and (ii) are equivalent to having $|a_j| \leq 1, 1 \leq j \leq n$. Consider 3-dimensions and let $\{e_0, e_1, e_2\}$ be an orthonormal basis in which $Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & a \\ 0 & -a & 0 \end{bmatrix}$.

Step 3.1: (ii) $\Rightarrow$ $|\mathbf{a}| \leq \mathbf{1}$.

Since $Qe_1 = -ae_2$, by taking $f = e_1$ and $g = e_2$ in (ii), it becomes $\alpha(-ae_2, e_2)^2 \leq \alpha(e_1, e_1)\alpha(e_2, e_2)$, which trivially reduces to $|a|^2 \leq 1$.

Step 3.2: (i) $\Rightarrow$ $|\mathbf{a}| \leq \mathbf{1}$.

If we assume (i) then the 3×3 matrix $\big((\alpha(e_j, e_k) - i\sigma(e_j, e_k))\big) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & ia \\ 0 & -ia & 1 \end{bmatrix}$ must be positive. This implies $|a| \leq 1$.

Step 3.3:$|\mathbf{a}| \leq \mathbf{1} \Rightarrow$ (ii)

If $|a| \leq 1$, then $\|Q\| \leq 1$ and since α is an innerproduct, (ii) is obtained by applying Cauchy-Schwartz inequality to $\alpha(Qf, g)^2$.

Step 3.3:$|\mathbf{a}| \leq \mathbf{1} \Rightarrow$ (i)

Let $f_j = f_{j0}e_0 + f_{j1}e_1 + f_{j2}e_2$, $c_j \in \mathbb{C}$, $1 \leq j \leq m$ and put $x_l = \sum\limits_{j=1}^{m} c_j f_{jl}, l = 0, 1, 2$ then

$$\begin{aligned}
\sum_{j,k=1}^{m} \overline{c}_j c_k (\alpha(f_j, f_k) - i\sigma(f_j, f_k)) &= \sum_{j,k=1}^{m} \overline{c}_j c_k [(f_{j_0} f_{k_0} + f_{j1} f_{k1} + f_{j2} f_{k2}) \\
&\qquad - ia(f_{j2} f_{k1} - f_{j1} f_{k2})] \\
&= (\overline{x}_0 x_0 + \overline{x}_1 x_1 + \overline{x}_2 x_2) - ia(\overline{x}_2 x_1 - \overline{x}_1 x_2) \\
&= \begin{bmatrix} \overline{x}_0 & \overline{x}_1 & \overline{x}_2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & ia \\ 0 & -ia & 1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} \\
&\geq 0
\end{aligned}$$

whenever $|a| \leq 1$.

$\square$

Corollary 1.5.1. *Let $((a_{ij}))_{n \times n}$ be a positive definite then $((\exp\{a_{ij}\}))$ is also positive, i.e entry wise exponentiation preserves positivity.*

Proof. The fact that entry wise product and sum of positive matrices are positive proves this. It may also be noted that the $n \times n$ matrix with all the entries equal to 1 is a positive matrix (because it is of the form b^*b, where $b = [1, 1, \dots, 1]_{1\times n}$). ☐

Theorem 1.5.2. *[Pet90] Let (H, σ) be a symplectic space and $\alpha : H \times H \to \mathbb{R}$ be a real inner product on H. Then there exists a state ϕ on $CCR(H, \sigma)$ such that*

$$\phi(W(f)) = \exp\left(-\frac{1}{2}\alpha(f, f)\right) \qquad \forall f \in H. \tag{1.5.3}$$

if and only if

$$\sigma(f, g)^2 \leq \alpha(f, f)\alpha(g, g), \qquad \forall f, g \in H. \tag{1.5.4}$$

Proof. We will apply Proposition 1.5.1. Take $G(f) = \exp\left(-\frac{1}{2}\alpha(f, f)\right)$ and we want to prove the positivity of the kernel given in (1.5.1). Let $c_j \in \mathbb{C}, b_j = c_j \exp\left(-\frac{1}{2}\alpha(f_j, f_j)\right)$, $1 \leq j \leq m$.

$$\begin{aligned}
&\sum_{j,k=1}^{m} \bar{c}_j c_k \exp\left(-\frac{1}{2}\alpha(f_j - f_k, f_j - f_k) - i\sigma(f_j, f_k)\right) \\
&\quad = \sum_{j,k=1}^{m} \left(\bar{c}_j \exp\left(-\frac{1}{2}\alpha(f_j, f_j)\right)\right)\left(c_k \exp\left(-\frac{1}{2}\alpha(f_k, f_k)\right)\right) \\
&\qquad\qquad\qquad\qquad \times \exp(\alpha(f_j, f_k) - i\sigma(f_j, f_k)) \\
&\quad = \sum_{j,k=1}^{m} \bar{b}_j b_k \exp(\alpha(f_j, f_k) - i\sigma(f_j, f_k)) \\
&\quad \geq 0
\end{aligned}$$

because of (1.5.4), Lemma 1.5.2 and Corollary 1.5.1. ☐

Definition 1.5.8. A state ϕ on $CCR(H, \sigma)$ determined in the form of (1.5.3) is called a *quasifree state*. A CCR-algebra corresponding to a standard symplectic space (H, σ) will be called a *standard* C^*-algebra of the CCR or standard $CCR(H, \sigma)$.

Example 2. Let (H, σ) be a standard symplectic space. Take $\alpha(\cdot, \cdot) = \operatorname{Re}\langle \cdot, \cdot \rangle$, then (1.5.4) is satisfied and thus there exist a quasifree state ϕ on the $CCR(H, \sigma)$ such that

$$\phi(W(f)) = \exp\left(-\frac{1}{2}\langle f, f\rangle\right).$$

Notation. If A is a real linear operator on a real Hilbert space H we use the notation A^T to denote the transpose of the operator defined by the equation $\langle x, Ay\rangle = \langle A^T x, y\rangle$ for all $x, y \in H$.

Proposition 1.5.2 (Proposition 1 and 2 from [Hol71a]). *A quasifree state of a standard $CCR(H, \sigma)$, (Take $\sigma(\cdot,\cdot) = -\operatorname{Im}\langle\cdot,\cdot\rangle$ here) is primary (c.f Definition 1.5.5) if and only if α in equation (1.5.3) satisfies one of the following equivalent conditions.*

(i) The space H is complete with respect to the norm coming from α. In other words, $(H, \alpha(\cdot,\cdot))$ is a real Hilbert space.

(ii) There exists a bounded, invertible real linear operator A on (H, α) such that

$$\alpha(f, g) = \sigma(Af, g), \qquad \forall f, g \in H. \tag{1.5.5}$$

Further in this case,

$$A^T = -A; \qquad -A^2 - I \geq 0 \tag{1.5.6}$$

on $(H, \alpha(\cdot,\cdot))$.

Since (H, σ) is standard (1.5.5) can also be written as

$$\alpha(f, g) = -\operatorname{Im}\langle Af, g\rangle, \qquad \forall f, g \in H.$$

Sometimes we write ϕ_A to denote the primary quasifree state obtained by (1.5.5) on a standard $CCR(H, \sigma)$, also we write H_A to denote the real Hilbert space $(H, \alpha(\cdot,\cdot))$ in this case.

Theorem 1.5.3. *[Hol71a] Two primary quasifree states ϕ_A and ϕ_B on a standard CCR algebra, $CCR(H, \sigma)$, are quasiequivalent if and only if $A - B$ and $\sqrt{-A^2 - I} - \sqrt{-B^2 - I}$ are Hilbert-Schmidt operators on H_A.*

1.6 Representation of standard CCR algebra in Fock Space

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\bar{\mathcal{H}}$ denote $(\mathcal{H}, \overline{\langle\cdot,\cdot\rangle})$. Take $\sigma(\cdot,\cdot) = -\operatorname{Im}\langle\cdot,\cdot\rangle$, Since $\bar{\mathcal{H}}$ is a Hilbert space, $(\mathcal{H}, \sigma)$ is the standard symplectic space associated with $\bar{\mathcal{H}}$. Consider the symmetric Fock space $\Gamma_s(\mathcal{H})$ associated with $\mathcal{H}$, then Proposition 1.2.1 provides a regular representation of $CCR(\mathcal{H}, \sigma)$ in $\Gamma_s(\mathcal{H})$. It should be noted that this representation is also the GNS representation corresponding to the quasifree state ϕ_{-i} (also known as fock-vacuum state), where $-i$ denotes the operator of scalar multiplication by the complex number $-i$ considered as a real linear operator on $\mathcal{H}$. We may call ϕ_{-i} as vacuum state. The name vacuum state will have a precise meaning when we consider the "quantum characteristic function" (see Definition 3.2.1) of the vacuum state $|e(0)\rangle\langle e(0)|$ on $\Gamma_s(\mathcal{H})$.

1.7 Symplectic automorphisms and transformations

Our basic set up is as in Section 1.6. Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Often we consider $\mathcal{H}$ as real Hilbert space with $\langle\cdot,\cdot\rangle_{\mathbb{R}} = \operatorname{Re}\langle\cdot,\cdot\rangle$. Let $H \subset \mathcal{H}$ be a real subspace such that $\mathcal{H} = \{x+iy|x,y \in H\} = H+iH($ i.e., $\mathcal{H}$ is the complexification of the real Hilbert Space $(H, \langle\cdot,\cdot\rangle_{\mathbb{R}})$, where $\langle\cdot,\cdot\rangle_{\mathbb{R}} := \operatorname{Re}\langle\cdot,\cdot\rangle)$. Now consider $\mathcal{H}$ as real Hilbert space with the inner product $\operatorname{Re}\langle\cdot,\cdot\rangle$. Then

$$\operatorname{Re}\langle x+iy, u+iv\rangle = \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{\mathbb{R}}, \tag{1.7.1}$$

where $\langle\cdot,\cdot\rangle_{\mathbb{R}}$ on right is the canonical inner product on $H \oplus H$ inherited from H. Thus the real Hilbert space $\mathcal{H}$ is isomorphic to $H \oplus H$ via the map U which takes $x+iy \mapsto \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$.

For any real linear operator S on $\mathcal{H}$, define operators S_{ij} on H such that

$$S(x+iy) = S_{11}x + iS_{21}x + S_{12}y + iS_{22}y. \tag{1.7.2}$$

Define the operator S_0 on $H \oplus H$ by

$$S_0 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{1.7.3}$$

Then (1.7.1) and (1.7.2) implies

$$\begin{aligned} &\operatorname{Re}\langle S(x+iy), S(u+iv)\rangle \\ &= \operatorname{Re}\langle S_{11}x + S_{12}y + i(S_{21}x + S_{22}y), S_{11}u + S_{12}v + i(S_{21}u + S_{22}v)\rangle \\ &= \left\langle \begin{pmatrix} S_{11}x + S_{12}y \\ S_{21}x + S_{22}y \end{pmatrix}, \begin{pmatrix} S_{11}u + S_{12}v \\ S_{21}u + S_{22}v \end{pmatrix} \right\rangle_{\mathbb{R}} \\ &= \left\langle S_0 \begin{pmatrix} x \\ y \end{pmatrix}, S_0 \begin{pmatrix} u \\ v \end{pmatrix} \right\rangle_{\mathbb{R}}. \end{aligned} \tag{1.7.4}$$

Thus

$$S = U^T S_0 U. \tag{1.7.5}$$

Therefore, we identify S with S_0 as a real linear operartor and often switch between them freely. We also note here that if S is a complex linear operator then $S_{11} = S_{22}(= S_1$, say) and $S_{12} = -S_{21}(= S_2$, say), then we can write $S_0 = \left[\begin{smallmatrix} S_1 & S_2 \\ -S_2 & S_1 \end{smallmatrix}\right]$.

If $\mathcal{K}$ is another Hilbert space and $S : \mathcal{H} \to \mathcal{K}$ is real linear, then also the same analysis hold. When we talk about a real linear operator S on $\mathcal{H}$, we reserve the notation S_0 to mean the operator we constructed as above.

Let J be the operator of multiplication by $-i$ on $\mathcal{H}$ considered as a real linear map, then

$$J_0 = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}.$$

We have $J_0^T = J_0^{-1} = -J_0$, (same is true for J also) and thus J_0 (and J) are orthogonal transformations.

A real linear bijective map $L : \mathcal{H} \to \mathcal{H}$ is said to be a *symplectic automorphism* if it satisfies (i) L and L^{-1} are continuous (bounded) (ii) $\operatorname{Im}\langle Lz, Lw\rangle = \operatorname{Im}\langle z, w\rangle$ for all $z, w \in \mathcal{H}$. If L from $\mathcal{H}$ to $\mathcal{K}$ satisfies the same conditions then we say L is a *symplectic transformation*. Correspondingly L_0 will also be called as symplectic automorphism (or transformation).

Proposition 1.7.1 (Section 22 in [Par92])**.** *$L : \mathcal{H} \to \mathcal{K}$ is symplectic if and only if*

$$L_0^T J_0 L_0 = J_0,$$

where J_0 on left side is the involution operator on $K \oplus K$ and that on the right side is the involution operator on $H \oplus H$.

Example 3. Let $A \in \mathcal{B}(H)$ be any symmetric invertible operator on H, then the operator T defined on $\mathcal{H}$ by $T(u + iv) = Au + iA^{-1}v$ is a symplectic automorphism of $\mathcal{H}$. Further note that

$$T_0 = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix}.$$

A complex Hilbert space can be considered as a real Hilbert space if we define the real inner product as $\operatorname{Re}\langle \cdot, \cdot \rangle$. We will have occasions to deal with the complexification of real Hilbert spaces occurring in this manner. We saw above that $\mathcal{H}$ has a canonical isomorphism to $H \oplus H$ as a real Hilbert space. Let $\hat{\mathcal{H}}$ denote the complexification $(\mathcal{H}, \operatorname{Re}\langle \cdot, \cdot \rangle)$. If A is a real linear operator on $\mathcal{H}$, then let $\hat{A}$ denote the complexification of A defined by $\hat{A}(z + iw) = Az + iAw$. The following holds.

Proposition 1.7.2. *$\hat{\mathcal{H}}$ is canonically isomorphic to $\mathcal{H} \oplus \mathcal{H}$ as a complex Hilbert space. Further, if a real linear operator S on $\mathcal{H}$ corresponds to $S_0 = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$ on $H \oplus H$, then under this isomorphism $\hat{S} = \begin{bmatrix} \hat{S_{11}} & \hat{S_{12}} \\ \hat{S_{21}} & \hat{S_{22}} \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$.*

Proof. As an element of the real Hilbert space $\mathcal{H}$, the vector $x + iy$ is identified with $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)$. Consider the mapping $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) + i\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right) \mapsto \left(\begin{smallmatrix} x+iu \\ y+iv \end{smallmatrix}\right)$ from $\hat{\mathcal{H}}$ to $\mathcal{H} \oplus \mathcal{H}$. Let us denote the inner product in $\hat{\mathcal{H}}$ by $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ and that in $\mathcal{H} \oplus \mathcal{H}$ by $\langle \cdot, \cdot \rangle$. Then

$$\begin{aligned} \left\langle \begin{pmatrix} x_1 + iu_1 \\ y_1 + iv_1 \end{pmatrix}, \begin{pmatrix} x_2 + iu_2 \\ y_2 + iv_2 \end{pmatrix} \right\rangle &= \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} iu_1 \\ iv_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \begin{pmatrix} iu_2 \\ iv_2 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + i \begin{pmatrix} u_1 \\ v_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + i \begin{pmatrix} u_2 \\ v_2 \end{pmatrix} \right\rangle_{\mathbb{C}}. \end{aligned}$$

Therefore, the isomorphism is proved. Now we proceed to prove the second statement. We know that S and S_0 are identified.

$$\begin{aligned} \hat{S_0}\left(\begin{pmatrix} x \\ y \end{pmatrix} + i \begin{pmatrix} u \\ v \end{pmatrix} \right) &= S_0 \begin{pmatrix} x \\ y \end{pmatrix} + iS_0 \begin{pmatrix} u \\ v \end{pmatrix} \\ &= \begin{pmatrix} S_{11}x + S_{12}y \\ S_{21}x + S_{22}y \end{pmatrix} + i \begin{pmatrix} S_{11}u + S_{12}v \\ S_{21}u + S_{22}v \end{pmatrix} \\ &= \begin{pmatrix} S_{11}x + iS_{11}u + S_{12}y + +iS_{12}v \\ S_{21}x + iS_{21}u + S_{22}y + +iS_{22}v \end{pmatrix} \\ &= \begin{bmatrix} \hat{S_{11}} & \hat{S_{12}} \\ \hat{S_{21}} & \hat{S_{22}} \end{bmatrix} \begin{pmatrix} x + iu \\ y + iv \end{pmatrix}. \end{aligned}$$

□

Corollary 1.7.1. $\hat{J} = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$ *on $\mathcal{H} \oplus \mathcal{H}$.*

Proposition 1.7.3. *[Generalization of Proposition 22.1 in [Par92]] Let $\mathcal{H}, \mathcal{K}$ be complex Hilbert spaces and let $S : \mathcal{H} \to \mathcal{K}$ be symplectic. Then it admits a decomposition:*

$$S = UTV \tag{1.7.6}$$

where $U : \mathcal{H} \to \mathcal{K}$ and $V : \mathcal{H} \to \mathcal{H}$ are unitaries and $T : \mathcal{H} \to \mathcal{H}$ has the form

$$T(u + iv) = Au + iA^{-1}v,$$

where $A \in \mathcal{B}(H)$ is a positive and invertible operator.

Proof. Apply polar decomposition to S so that $S = U'P$ where $U' : \mathcal{H} \to \mathcal{K}$ and P is a positive operator on $\mathcal{K}$. Now we can do the do the same analysis as in Proposition 22.1 in [Par92] to the operator P to write it as $V^T T V$. □

1.8 Shale Unitaries

Shale's theorem was proved in [Sha62]. It was further generalized in [BS05] for the case of operators of the form T above (but between two different Hilbert spaces) in Proposition 1.7.3. In the work done later, we need a generalization of this (Theorem 2.1 in [BS05]) to the case of general symplectic operators. Let $\mathcal{H}, \mathcal{K}$ be two Hilbert spaces, define $\mathcal{S}(\mathcal{H}, \mathcal{K})$ by

$$\mathcal{S}(\mathcal{H}, \mathcal{K}) = \{L \in \mathcal{B}_{\mathbb{R}}(\mathcal{H}, \mathcal{K}) : L \text{ is symplectic and } L^T L - I \text{ is Hilbert-Schmidt.}\}$$

We denote $\mathcal{S}(\mathcal{H}) := \mathcal{S}(\mathcal{H}, \mathcal{H})$. Elements of $\mathcal{S}(\mathcal{H}, \mathcal{K})$ are called *Shale operators.*

Theorem 1.8.1. *(i) Let $L \in \mathcal{B}_{\mathbb{R}}(\mathcal{H}, \mathcal{K})$ be a symplectic operator. Then there exists a unitary operator $\Gamma_s(L) : \Gamma_s(\mathcal{H}) \to \Gamma_s(\mathcal{K})$ such that*

$$\Gamma_s(L) W(u) \Gamma_s(L)^* = W(Lu), \forall u \in \mathcal{H}, \tag{1.8.1}$$

if and only if $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$. In such a case, $\Gamma_s(L)$ is determined uniquely up to a scalar of modulus unity.

(ii) A unitary $\Gamma_s(L)$ satisfying (1.8.1) can be chosen such that it satisfies

$$\langle \Gamma_s(L) \Phi_{\mathcal{H}}, \Phi_{\mathcal{K}} \rangle \in \mathbb{R}^+, \tag{1.8.2}$$

where $\Phi_{\mathcal{H}}$ and $\Phi_{\mathcal{K}}$ are vacuum vectors in $\Gamma_s(\mathcal{H})$ and $\Gamma_s(\mathcal{K})$ respectively, this choice makes $\Gamma_s(L)$ unique. In this case,

$$\Gamma_s(L^{-1}) = \Gamma_s(L)^* \tag{1.8.3}$$

(iii) Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be three Hilbert spaces and $L_1 \in \mathcal{S}(\mathcal{H}_1, \mathcal{H}_2), L_2 \in \mathcal{S}(\mathcal{H}_2, \mathcal{H}_3)$. Then

$$\Gamma_s(L_2 L_1) = \sigma(L_2, L_1) \Gamma_s(L_2) \Gamma_s(L_1), \tag{1.8.4}$$

where $\sigma(L_2, L_1) \in \mathbb{C}, |\sigma(L_2, L_1)| = 1$.

Proof. (i). Assume that $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$. We will prove the existence of $\Gamma_s(L)$ based on the construction in [BS05]. By Proposition 1.7.3 there exist unitaries $U : \mathcal{H} \to \mathcal{K}$, $V : \mathcal{H} \to \mathcal{H}$ such that $L = UTV$ where T is a symplectic automorphism of H such that

$$T(u + iv) = Au + iA^{-1}v$$

where $A \in \mathcal{B}(H)$ is positive and invertible. It can be seen from the proof of Proposition 1.7.3 that

$$L_0 = U_0 \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} V_0$$

for some orthogonal transformations $U_0 \in \mathcal{B}(H, K)$ and $V_0 \in \mathcal{B}(H)$. Now it can be seen that

$$L_0^T L_0 = V_0^{-1} \begin{bmatrix} A^2 & 0 \\ 0 & A^{-2} \end{bmatrix} V_0.$$

Therefore $L_0^T L_0 - I = V_0^{-1} \left(\begin{bmatrix} A^2 & 0 \\ 0 & A^{-2} \end{bmatrix} - \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right) V_0$. Hence we get that $A^2 - I$ is Hilbert-Schmidt and since A is positive, Theorem 2.1 of [BS05] applies. Thus there exists $\Gamma_s(T)$ such that

$$\Gamma_s(T) W(u) \Gamma_s(T)^* = W(Tu), \forall u \in \mathcal{H}, \tag{1.8.5}$$
$$\langle \Gamma_s(T) \Phi_{\mathcal{H}}, \Phi_{\mathcal{K}} \rangle \in \mathbb{R}^+. \tag{1.8.6}$$

Define

$$\Gamma_s(L) := \Gamma_s(U) \Gamma_s(T) \Gamma_s(V), \tag{1.8.7}$$

where $\Gamma_s(U)$ and $\Gamma_s(V)$ are the second quantization associated with the unitary U and V. A direct computation shows that $\Gamma_s(L)$ satisfies the (1.8.1)(because of properties of $\Gamma_s(U), \Gamma_s(V)$ and equation 1.8.6) and (1.8.2) (because second quantizations $\Gamma_s(U_j)$ acts as identity on vacuum vector). We refer to Theorem 22.11 in [Par92] for the necessity part.

(ii). Equation (1.8.2) is automatically satisfied in our construction in (1) above because of (1.8.6). To see the uniqueness, let $\Gamma_s^1(L)$ and $\Gamma_s^2(L)$ satisfy (1.8.1) and (1.8.2). Therefore we get $\Gamma_s^2(L)^* \Gamma_s^1(L) W(u) = W(u) \Gamma_s^2(L)^* \Gamma_s^1(L), \forall u \in \mathcal{H}$. Therefore by irreducibility of Weyl operators (Proposition 20.9 in [Par92]), $\Gamma_s^2(L)^* \Gamma_s^1(L) = cI$ for some complex scalar of unit modulus. But now by (1.8.2) we get $\Gamma_s^2(L) = \Gamma_s^1(L)$.

To prove (1.8.3), note that $\langle \Gamma_s(L)^* \Phi_{\mathcal{K}}, \Phi_{\mathcal{H}} \rangle = \langle \Gamma_s(L) \Phi_{\mathcal{H}}, \Phi_{\mathcal{K}} \rangle \in \mathbb{R}^+$ therefore if we show that $\Gamma_s(L)^* W(u) \Gamma_s(L) = W(L^{-1}u)$ then by the uniqueness of $\Gamma_s(L^{-1})$ we get (1.8.3).

Recall from Theorem 2.1 of [BS05] that $\Gamma_s(T^{-1}) = \Gamma_s(T)^*$ and for second quantization (1.2.9) unitary we have, $\Gamma_s(U_j^*) = \Gamma_s(U_j)^*$. Further by (1.8.7), and (1.2.11) we have

$$\begin{aligned}\Gamma_s(L)^* W(u) \Gamma_s(L) &= \Gamma_s(U_2)^* \Gamma_s(T)^* \Gamma_s(U_1)^* W(u) \Gamma_s(U_1) \Gamma_s(T) \Gamma_s(U_2) \\ &= W(U_2^* T^{-1} U_1^* u) \\ &= W(L^{-1} u).\end{aligned}$$

This completes the proof of (ii).

(iii). Follows immediately from (i). □

Remarks 3. (i) When $\mathcal{H}$ is finite dimensional, it is known that there exists a choice of $\Gamma_s(L)$ such that the multiplier $\sigma(L_1, L_2) = \pm 1, \forall L_1, L_2 \in Sp_{2n}(\mathbb{R})$, where $Sp_{2n}(\mathbb{R}) \subseteq M_{2n}(\mathbb{R})$ is the subgroup of all $2n \times 2n$ symplectic matrices. This is called the metaplectic representation of the symplectic group. An elementary and self-contained presentation can be found in Chapter 4 of [Fol89], Theorem 4.37 there is of particular interest in this regard. In the infinite dimensional case, [MS04] and [Tve04] are of interest.

(ii) The map $W(u) \mapsto W(Lu)$ is known as the *Bogoliubov transformation* of the CCR algebra, induced by L. Whenever we write $\Gamma_s(L)$, we mean the unique unitary operator satisfying (1.8.2). It is called the *Shale unitary* corresponding to an $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$. It is to be noted that if L is a non-unitary contraction then $\Gamma(L)$ defined by (1.2.9) is not a unitary and hence in such a case $\Gamma_s(L) \neq \Gamma(L)$.

CHAPTER 2

Real Normal Operators and Williamson's Normal Form

2.1 Introduction

Symplectic formalism is an important tool in mathematical physics and in particular in the study of Quantum mechanics [dG11]. Williamson's normal form is a useful theorem in this subject. It states that for a natural number n, suppose A is a strictly positive (hence invertible) real matrix of order $2n \times 2n$. Then there exists a symplectic matrix L of order $2n \times 2n$ and a strictly positive diagonal matrix P of order $n \times n$, such that

$$A = L^T \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} L.$$

Moreover the matrix P is uniquely determined up to permutation and the diagonal entries are known as symplectic eigenvalues of A.

This theorem was first proved by J. Williamson in [Wil36]. It has been extensively used to understand the symplectic geometry and has applications in Harmonic Analysis and Physics (See [dG11]). In recent years there is somewhat renewed interest in the field [BJ15, ISGW17] in view of its relevance in quantum information theory and its usefulness to understand symmetries of finite mode quantum Gaussian states [Par13b]. Since this theorem is very useful in the finite mode case of the Gaussian states, it is natural only to expect that an appropriate generalisation of it will be useful in the infinite mode situation. We prove this result in infinite dimensions and Williamson's theorem becomes a crucial ingredient in our study of the quantum Gaussian states. In the process we find shortcuts and simplifications of some known results on real normal operators.

In this subject it is necessary to deal with real linear operators on real Hilbert spaces and their complexifications. We know that non-real eigenvalues of real matrices appear in conjugate pairs. We need appropriate generalization of this result in infinite dimensions. In Theorem 2.3.1 we show that every normal operator on a real Hilbert space is orthogonally equivalent to its adjoint. This result is known [Vis78], however we have an elementary direct proof of this fact. This is crucial for our understanding of spectral theorem of real normal operators, for which we refer to Wong [Won69], Goodrich [Goo72], Viswanath [Vis78], Agrawal and Kulkarni [AK94]. This acts as the main tool for obtaining infinite dimensional version of Williamson's theorem.

2.2 Preliminary definitions and observations

In this Section, we shall recall the basic definitions and observations relevant to our work on real Hilbert spaces. To be consistent with the existing literature, we keep them similar to what is seen in [Won69] and [Goo72].

Definition 2.2.1. Let A be a bounded operator on a real Hilbert space $(H, \langle\cdot,\cdot\rangle)$. Its transpose A^T, is defined by $\langle Ax, y\rangle = \left\langle x, A^T y\right\rangle, \forall x, y \in H$. Such an A^T exists uniquely as a bounded operator on H. A is said to be normal if $AA^T = A^T A$.

Often we use the term '*real normal (self-adjoint, positive, invertible) operator*' for a normal (self-adjoint, positive, invertible) operator defined on a real Hilbert space. Following standard notation, for complex linear operators on complex Hilbert spaces * would denote the adjoint.

Definition 2.2.2. Let H be a real Hilbert space, by the complexification of H we mean the complex Hilbert space $\mathcal{H} := H + iH := \{x + i\cdot y : x \in H, y \in H\}$ with addition, complex-scalar product and inner product defined in the obvious way.

For example, if $\langle\cdot,\cdot\rangle$ is the inner product on H then the inner product on $\mathcal{H}$ is given by $\langle x_1 + i\cdot y_1, x_2 + i\cdot y_2\rangle_{\mathbb{C}} := \langle x_1, x_2\rangle + \langle y_1, y_2\rangle + i\left(\langle x_1, y_2\rangle - \langle y_1, x_2\rangle\right)$. Also note that the mapping $x \mapsto x + i\cdot 0$ provides an embedding of H into $\mathcal{H}$ as a real Hilbert space.

Definition 2.2.3. Let A be a bounded operator on the real Hilbert space H. Define an operator $\hat{A}$ on the complexification $\mathcal{H}$ of H by $\hat{A}(x + iy) = Ax + iAy$. Then $\hat{A}$ is well defined, complex linear and bounded, with $(\hat{A})^*(x + iy) = A^T x + i\cdot A^T y = \widehat{(A^T)}(x + iy)$ and $\|\hat{A}\| = \|A\|$. If A is normal, then $\hat{A}$ is also normal. $\hat{A}$ is called the complexification of A. Define the spectrum of A, denoted by $\sigma(A)$, to be the spectrum of $\hat{A}$.

Note that the definition above of spectrum matches with the usual notion of eigenvalues of a finite dimensional real matrix.

The mapping $\mathcal{J}\colon \mathcal{H} \to \mathcal{H}$, defined by $\mathcal{J}(x + i \cdot y) = x - i \cdot y$ is such that $\mathcal{J}(a_1 z_1 + a_2 z_2) = \bar{a}_1 \mathcal{J}(z_1) + \bar{a}_2 \mathcal{J}(z_2)$, for $z_1, z_2 \in \mathcal{H}, a_1, a_2 \in \mathbb{C}$. In other words $\mathcal{J}$ is anti-linear. Moreover, $\mathcal{J}^2 = I$, the identity operator on $\mathcal{H}$ and $\langle \mathcal{J} z_1, \mathcal{J} z_2 \rangle = \langle z_2, z_1 \rangle$. We observe that $H = \{z \in \mathcal{H}\colon \mathcal{J} z = z\}$ and an operator B on $\mathcal{H}$ is the complexification of some operator on $\mathcal{H}$ if and only if $B\mathcal{J} = \mathcal{J} B$.

Definition 2.2.4. Let A be a bounded normal operator on a real Hilbert space H. Then a vector $x \in H$ is is said to be *transpose cyclic* for A, if the set $\{A^n (A^T)^m x : m, n \geq 0\}$ is total in H. It is said to be *cyclic* for A, if $\{A^n x : n \geq 0\}$ is total in H.

For the next two sections, H denotes a real Hilbert space, $\mathcal{H}$ its complexification, A denotes a bounded normal operator on H and $\hat{A}$ its complexification on $\mathcal{H}$, as described above.

2.3 Symmetry of a real normal operator

Here we prove that any normal operator on a real Hilbert space is orthogonally equivalent to its transpose (or adjoint). Our proof just exploits the geometry of real Hilbert space. This result is crucial to understand the spectral theory of real normal operators.

Theorem 2.3.1. *Let H be a real Hilbert space and let $A \in B(H)$ be a normal operator. Then there exists an orthogonal transformation $U \in B(H)$, such that*

$$UAU^T = A^T. \tag{2.3.1}$$

Further, U can be chosen such that $U^T = U$.

Proof. Let us assume first that A has a transpose cyclic vector *i.e.* there exists $x \in H$ such that $\mathcal{E} := \{A^n (A^T)^m x : n, m \geq 0\}$ is total in H. Define U on $\mathcal{E}$ by

$$U(A^n (A^T)^m x) = (A^T)^n A^m x, \text{ for } n, m \geq 0. \tag{2.3.2}$$

Then for $n, m, k, l \geq 0$,

$$\begin{aligned} \langle A^n (A^T)^m x, A^k (A^T)^l x \rangle &= \langle (A^T)^k A^l A^n (A^T)^m x, x \rangle \\ &= \langle A^n (A^T)^m (A^T)^k A^l x, x \rangle \\ &= \langle (A^T)^k A^l x, (A^T)^n A^m x \rangle \end{aligned}$$

$$\begin{aligned}
&= \langle (A^T)^n A^m x, (A^T)^k A^l x\rangle \qquad (2.3.3)\\
&= \langle U(A^n(A^T)^m x), U(A^k(A^T)^l x)\rangle,
\end{aligned}$$

where the second equality follows from normality of A and fourth equality because real inner product is symmetric. Since U preserves the inner product on a total set U can be extended as a bounded linear operator on $\overline{\operatorname{span}}\,\mathcal{E} = H$. Note that the extended operator also preserves the inner product. We use the same symbol U for the extended operator also. Thus U is a real orthogonal transformation on H. Further by using (2.3.3) note that

$$\langle U(A^m(A^T)^n x), A^k(A^T)^l x\rangle = \langle (A^T)^n A^m x, (A^T)^k A^l x\rangle = \langle A^m(A^T)^n x, U(A^k(A^T)^l x)\rangle. \tag{2.3.4}$$

Therefore,

$$U^T = U. \tag{2.3.5}$$

Also,

$$\begin{aligned}
UAU^T(A^n(A^T)^m x) &= UAU(A^n(A^T)^m x)\\
&= UA((A^T)^n A^m x)\\
&= U(A^{m+1}(A^T)^n x)\\
&= (A^T)^{m+1} A^n x\\
&= A^T(A^n(A^T)^m x).
\end{aligned}$$

Thus (2.3.1) is satisfied on a total set which in turn proves the required relation on H, in the special case when U has a transpose cyclic vector. The general case follows by a familiar application of Zorn's lemma. □

Corollary 2.3.1. *$\hat{A}$ is unitarily equivalent to $(\hat{A})^* = \widehat{(A^T)}$.*

Proof. Let U be as in Theorem 2.3.1, then $\hat{U}$ is a unitary which does the job. □

Corollary 2.3.2. *For any real normal operator A, $\sigma(A) = \sigma(A^T) = \overline{\sigma(A)}$ and thus the spectrum is symmetric about the real axis.*

Proof. Immediate from Definition 2.2.3 and Corollary 2.3.1. □

Note that Corollary 2.3.2 is analogous to the fact that complex eigenvalues of a real matrix occur in pairs.

2.4 The spectral theorem for real normal operators

In this section, we provide an expository note on the spectral theory of real normal operators which is more operator theoretic in nature and is parallel to the spectral theory of complex normal operators in its methods.

Let A be a bounded normal operator on a real Hilbert space H. Consider its complexification $\hat{A}$. By the spectral theorem in the complex case, there exists a spectral measure $\hat{E}$, such that $\hat{A} = \int\limits_{\sigma(\hat{A})} \lambda \, d\hat{E}(\lambda)$. Notice that there is no reason why $\hat{E}(e)$ is a complexification of a real operator. It is just a notation for the spectral measure of $\hat{A}$. However the following is true.

Lemma 2.4.1 (Wong's lemma, [Won69]). *$\mathcal{J}\hat{E}(e) = \hat{E}(\bar{e})\mathcal{J}$, for every Borel subset $e \subseteq \mathbb{C}$, where $\bar{e}$ denotes the set of all complex conjugates of elements of e.*

Proof. Here is a sketch of the proof presented as Lemma 3.1 in [Won69]. For $\lambda \in \mathbb{C}, \epsilon > 0$, let $\mathfrak{F}(\lambda, \epsilon)$ denote the subspace $\{x \in \mathcal{H} \colon \|(A-\lambda)^n x\| \leq \epsilon^n \|x\|, \forall n \in \mathbb{N}\}$. For a Borel set $M \subseteq \mathbb{C}$, let $\mathfrak{F}(M, \epsilon) = \bigvee\limits_{\lambda \in M} \mathfrak{F}(\lambda, \epsilon)$ and $\mathfrak{F}(M) = \bigcap\limits_{\epsilon > 0} \mathfrak{F}(M, \epsilon)$. We know that $\mathfrak{F}(e) = \operatorname{Ran} \hat{E}(e)$ for any compact subset e of $\mathbb{C}$ (Section 42 [Hal98]).

Notice that $\langle \mathcal{J}z, w \rangle = \langle z, -\mathcal{J}w \rangle$. Therefore $\mathcal{J}\hat{E}(e)\mathcal{J}$ is a projection for any Borel subset e of $\mathbb{C}$. Therefore, to prove the lemma it is enough to show that $\mathcal{J}\hat{E}(e)\mathcal{J}$ and $\hat{E}(\bar{e})$ have the same range for every Borel set e. To this end first notice that for $\lambda \in \mathbb{C}$ and $\epsilon > 0$, if $x \in \mathfrak{F}(\lambda, \epsilon)$ then $\mathcal{J}x \in \mathfrak{F}(\bar{\lambda}, \epsilon)$. From this it follows that $x \in \mathfrak{F}(e)$ implies $\mathcal{J}x \in \mathfrak{F}(\bar{e})$ for any Borel set. Hence we get $\mathcal{J}\mathfrak{F}(e) \subseteq \mathfrak{F}(\bar{e})$. Therefore for a compact Borel set e we have $\mathcal{J}\hat{E}(e)\mathcal{H} \subseteq \hat{E}(\bar{e})\mathcal{H}$. Now by regularity of the spectral measure, if e is a Borel set we have $\mathcal{J}\hat{E}(e)\mathcal{H} \subseteq \hat{E}(\bar{e})\mathcal{H}$. Therefore we also have $\mathcal{J}\hat{E}(\bar{e})\mathcal{H} \subseteq \hat{E}(e)\mathcal{H}$. Applying $\mathcal{J}$ on both side we have $\hat{E}(\bar{e})\mathcal{H} \subseteq \mathcal{J}\hat{E}(e)\mathcal{H}$. Thus we see that $\mathcal{J}\hat{E}(e)\mathcal{H} = \hat{E}(\bar{e})\mathcal{H}$. Since $\mathcal{H} = \mathcal{J}\mathcal{H}$, we get $\mathcal{J}\hat{E}(e)\mathcal{J}\mathcal{H} = \hat{E}(\bar{e})\mathcal{H}$. ☐

We use the notation $\mathcal{B}(D)$ to denote the Borel σ-algebra on the set D. Let $e \subseteq \sigma(\hat{A})$(which is same as $\sigma(A)$) be any Borel set, define $\hat{E}_1(e) := \frac{\hat{E}(e)+\hat{E}(\bar{e})}{2}$ and $\hat{E}_2(e) := \frac{\hat{E}(e)-\hat{E}(\bar{e})}{2i}$. Then

$$\hat{E}(e) = \hat{E}_1(e) + i\hat{E}_2(e). \tag{2.4.1}$$

By Lemma 2.4.1, $\hat{E}_1(e)$ and $\hat{E}_2(e)$ both commute with $\mathcal{J}$. Therefore, $\hat{E}_1(e)$ and $\hat{E}_2(e)$ are complexifications of some operators $E_1(e)$ and $E_2(e)$, respectively on H. $E_1(e)$ is symmetric and $E_2(e)$ is skew symmetric for every Borel set e, because their complexifications are so.

$$\begin{aligned} E_1(\bar{e}) &= \quad E_1(e) \\ E_2(\bar{e}) &= -E_2(e) \end{aligned} \tag{2.4.2}$$

for every $e \in \mathcal{B}(\mathbb{C})$. Also note that by Corollary 2.3.2 and (2.4.2) we have $E_1(\sigma(A)) = I$ and $E_2(\sigma(A)) = 0$.

Proposition 2.4.1. *For any fixed $x, y \in H$, define $\mu_j^{(x,y)}(e) := \langle x, E_j(e)y \rangle, j = 1,2$ for every Borel subset $e \subseteq \sigma(A)$. Then $\mu_j^{(x,y)}$ is a regular Borel (finite valued) signed measure on $\sigma(A)$, for $j = 1,2$. In particular, since $E_2(e)$ is skew symmetric we have $\mu_2^{(x,x)}(e) = \langle E_2(e)x, x\rangle = 0, \forall x$.*

Proof. Clearly $\mu_1^{(x,y)}$ and $\mu_2^{(x,y)}$ are real valued functions defined on the Borel subsets of $\sigma(A)$. Let $e = \cup_i e_i$ be an atmost countable disjoint union of Borel sets. By using the identification of H inside $\mathcal{H}$, (2.4.1) and properties of the spectral measure $\hat{E}$, we have

$$\begin{aligned} \langle x, E_1(e)y\rangle + i\langle x, E_2(e)y\rangle &= \langle x, \hat{E}(e)y\rangle_{\mathbb{C}} \\ &= \Sigma_i \langle x, \hat{E}(e_i)y\rangle_{\mathbb{C}} \\ &= \Sigma_i \langle x, (\hat{E}_1(e_i) + i\hat{E}_2(e_i))y\rangle_{\mathbb{C}} \\ &= \Sigma_i \langle x, E_1(e_i)y\rangle + i\Sigma_i\langle x, E_2(e_i)\rangle. \end{aligned} \tag{2.4.3}$$

This proves the countable additivity of $\mu_j^{(x,y)}, j = 1,2$. Regularity also follows by going to real and imaginary parts. ☐

Corollary 2.4.1. *$\mu_1^{(x,x)}$ is a positive measure.*

Remark 1. E_1 is a positive operator valued measure(POVM).

For a bounded Borel function f on $\sigma(A)$, define a bilinear functional $\hat{\phi}$ by

$$\hat{\phi}(z_1, z_2) := \int f(\lambda)\, d\langle z_1, \hat{E}(\lambda)z_2\rangle, \forall z_1, z_2 \in \mathcal{H}.$$

Then $\hat{\phi}$ is a bounded bilinear functional which provides the usual functional calculus for $\hat{A}$ (proof is easy and can be found in the Theorem 1, Section 37 of [Hal98]).

Theorem 2.4.1. *Let f be a complex valued bounded Borel measurable function defined on $\sigma(A)$, then for the values $j = 1,2$ there exists a unique bounded operator A_j on H such that*

$$\langle x, A_j y\rangle = \int f(\lambda)\, d\langle x, E_j(\lambda)y\rangle \tag{2.4.4}$$

for every pair of vectors x and y, and we write $A_j = \int f\, dE_j = \int f(\lambda)\, dE_j(\lambda)$.

Proof. Proposition 2.4.1 and the boundedness of f implies that the integral

$$\phi_j(x, y) = \int f(\lambda)\, d\langle x, E_j(\lambda)y\rangle$$

may be formed for every pair of vectors x and y. An easy computation shows that ϕ_j is a bilinear functional. Also, because of 2.4.3 we have $|\phi_j(x, y)| \leq |\hat{\phi}(x, y)|, \forall x, y \in H, j = 1, 2$. Therefore, ϕ_j is a bounded bilinear functional and so there exists a unique operator A_j (see Section 22, [Hal98]), satisfying (2.4.4). □

Corollary 2.4.2. *$\int \mathrm{Im}(\lambda)\, dE_1(\lambda) = 0$ and $\int \mathrm{Re}(\lambda)\, dE_2(\lambda) = 0$, where Im denotes the function $\lambda = \lambda_1 + i\lambda_2 \mapsto \lambda_2$ and Re denotes the function $\lambda_1 + i\lambda_2 \mapsto \lambda_1$ defined on $\sigma(A)$.*

Proof. Note that $\mathrm{Im}(\bar{\lambda}) = -\mathrm{Im}(\lambda)$ and $\mathrm{Re}(\bar{\lambda}) = \mathrm{Re}(\lambda)$. Now the result follows from (2.4.2). □

Let $\lambda = \lambda_1 + i\lambda_2$, be an arbitrary element in $\sigma(A)$. By going to the definitions and using the Theorem 2.4.1, for $x \in H$ we have,

$$\begin{aligned} A(x) = \hat{A}(x) &= \int \lambda\, d\hat{E}(\lambda)(x) \\ &= \int \lambda\, dE_1(\lambda)(x) + i \int \lambda\, dE_2(\lambda)(x) \\ &= \int \lambda_1\, dE_1(\lambda)(x) - \int \lambda_2\, dE_2(\lambda)(x), \end{aligned} \tag{2.4.5}$$

where (2.4.5) is obtained by using Corollary 2.4.2 and the fact that there is no "imaginary" part for elements of H. By considering λ_1 as the function $r\cos\theta$ and λ_2 as $r\sin\theta$, we have

$$A = \int r\cos\theta\, dE_1 - \int r\sin\theta\, dE_2. \tag{2.4.6}$$

Similar to (2.4.6), for $s, t \in \mathbb{N}$, we obtain for future reference

$$A^s (A^T)^t (x) = \int r^{s+t} \cos(s-t)\theta\, dE_1(x) - \int r^{s+t} \sin(s-t)\theta\, dE_2(x). \tag{2.4.7}$$

In particular, we have

$$\begin{aligned} \left\langle A^s (A^T)^t x, x \right\rangle &= \int_{\sigma(A)} r^{s+t} \cos(s-t)\theta\, d\mu_1^{(x,x)} - \int_{\sigma(A)} r^{s+t} \sin(s-t)\theta\, d\mu_2^{(x,x)} \\ &= \int_{\sigma(A)} r^{s+t} \cos(s-t)\theta\, d\mu_1^{(x,x)}, \end{aligned} \tag{2.4.8}$$

where (2.4.8) is obtained by using the fact that $\mu_2^{(x,x)}(e) = \langle E_2(e)x, x\rangle = 0, \forall e \in \mathcal{B}(\mathbb{C})$.

Definition 2.4.1. Let D be a compact subset of $\mathbb{C}$, which is symmetric about the real axis. A pair of operator valued functions (E_1, E_2) defined on the Borel σ-algebra of D is called a spectral pair if the following holds for every $e \in \mathcal{B}(D)$,

i) $E_1(e)$ is a bounded symmetric operator and $E_2(e)$ is a bounded skew symmetric operator.
ii) They satisfy (2.4.2).
iii) If $\hat{E}_1(e)$ and $\hat{E}_2(e)$ denote the complexification of the corresponding operator, then the operator valued function defined by $\hat{E}(e) := \hat{E}_1(e) + i\hat{E}_2(e)$, is a spectral measure on $\mathcal{H}$.

Now we obtain the spectral theorem for a real normal operator.

Theorem 2.4.2. *If A is a bounded normal operator on a real Hilbert space H, then there exists a unique spectral pair (E_1, E_2) such that (2.4.6) holds.*

Proof. We already proved everything except the uniqueness of the spectral pair. Suppose (F_1, F_2) is another spectral pair satisfying

$$A = \int r \cos\theta \, dF_1 - \int r \sin\theta \, dF_2. \tag{2.4.9}$$

Let $\hat{F}(e) = \hat{F}_1(e) + i\hat{F}_2(e)$, where $\hat{F}_1(e)$ and $\hat{F}_2(e)$ denotes the complexification of the corresponding operator. Then a direct computation of $\hat{A}(x + i \cdot y) = Ax + i \cdot Ay$ using (2.4.9) and Corollary 2.4.2 proves that $\hat{A} = \int \lambda \, d\hat{F}(\lambda)$. By uniqueness of spectral measure in the complex case we have $\hat{F} = \hat{E}$ this implies $F_1 = E_1$ and $F_2 = E_2$. $\square$

Observe that if A was originally a self adjoint operator then $E_2 = 0$ in (2.4.1) and we obtain the spectral theorem for a real bounded self adjoint operator exactly the same as that in complex case. The self adjoint case was already done in [RSN90].

Remark 2. For a definition of the spectral pair which is independent of complexification, we refer to [Goo72]. But we find going to the complexification easier. Further it should be noticed that our Corollory 2.3.2 which is a consequence of the fact that a real normal operator orthogonally equivalent to its adjoint enables us to confine the definition of spectral pair to subsets symmetric to the real axis. Also the definition of the spectral pair (and proof of Spectral Theorem) has been relatively simplified and made more operator theoretic in our approach.

Corollary 2.4.3. *If A is a bounded self adjoint operator on a real Hilbert space H, then there exists a unique spectral measure E on the real line such that $A = \int_{\sigma(A)} \lambda \, dE(\lambda)$.*

For future reference, let us note that it makes sense to talk about the positive operators and their square roots in the real field case also.

Definition 2.4.2. A bounded self adjoint operator A on a real Hilbert space H, is called a positive operator if $\sigma(A) \subseteq [0, \infty)$; it is called strictly positive if it is positive and $0 \notin \sigma(A)$.

Corollary 2.4.4. *If A is a positive operator on a real Hilbert space then there exists a unique positive square root operator for A, i.e. there exists a unique operator B such that $B^2 = A$. We denote the positive square root of A as $A^{1/2}$(or $\sqrt{A}$).*

Proof. Take $B = \int_{\sigma(A)} \lambda^{1/2}\, dE(\lambda)$, where E is the spectral measure associated with A and $A = \int \lambda\, dE(\lambda)$. The proof is same as that of complex case. □

Even though the polar decomposition exists for general bounded linear operators between real Hilbert spaces, we give the following special case which will be sufficient for our purpose.

Theorem 2.4.3. *Let H and K be two real Hilbert spaces and $A\colon H \to K$ be an invertible bounded linear operator, then there exists a unique orthogonal transformation $U\colon H \to K$ such that $A = U(A^T A)^{1/2}$.*

Proof. Similar to the complex situation. For example one can imitate the proof of Theorem VI.10 in [RS80]. □

2.5 Spectral Representation

In the finite dimensional situation we have, any real normal operator is orthogonally equivalent to an operator of the form

$$\begin{bmatrix} a_1 & & & & & & & \\ & a_2 & & & & & \text{\Huge 0} & \\ & & \ddots & & & & & \\ & & & a_k & & & & \\ & & & & B_1 & & & \\ & & & & & B_2 & & \\ & \text{\Huge 0} & & & & & \ddots & \\ & & & & & & & B_m \end{bmatrix}$$

where $a_1, a_2, \cdots, a_k$ are the real eigenvalues (counting multiplicity) of the operator and B_j is a 2×2 matrix of the form $B_j = \left[\begin{smallmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{smallmatrix}\right], \alpha_j, \beta_j \in \mathbb{R}, j = 1, 2, \ldots, m$. B_j corresponds to the complex eigenvalues $\alpha_j \pm i\beta_j$. Our next aim in this section is to obtain an analogous decomposition in the infinite dimensional situation. Note that in the finite dimensional situation a complex eigenvalue is prescribed by a 2-dimensional real subspace. A similar scenario happens in the infinite dimensional case also. We present it here for reader's convenience as we couldn't find it in the literature.

Let μ be a regular Borel measure on the Borel σ-algebra of a compact subset $E \subseteq \mathbb{C}$, which is symmetric about the real axis. Partition E into a union of three disjoint sets, $E = I \cup K \cup \bar{K}$ where $I = E \cap \mathbb{R}$, $K = E \cap \mathbb{H}^+$, where $\mathbb{H}^+ = \{\lambda \in \mathbb{C} \colon \operatorname{Im}(\lambda) > 0\}$. Let $L^2(E)$ denote the collection of all real valued, square integrable functions on E, with respect to the measure μ. Then $L^2(E)$ is a real Hilbert space and $L^2(E) = L^2(I) \oplus L^2(K) \oplus L^2(\bar{K})$. Further, if μ is symmetric about the real axis i.e. $\mu(e) = \mu(\bar{e})$, for every Borel set e, then there exists an orthogonal transformation between $L^2(\bar{K})$ and $L^2(K)$, which maps $f \mapsto \bar{f}$, where $\bar{f}(z) = f(\bar{z})$ for all $f \in L^2(\bar{K})$. Therefore, $L^2(E)$ is orthogonally equivalent to $L^2(I) \oplus L^2(K) \oplus L^2(K)$. Considering this orthogonal equivalence, we will not distinguish between $L^2(E)$ and this direct sum decomposition in the case where μ is symmetric about the real axis. Define an operator S on $L^2(I) \oplus L^2(K) \oplus L^2(K)$ by

$$S = \begin{bmatrix} M_\lambda & 0 & 0 \\ 0 & M_{\operatorname{Re}(\lambda)} & M_{\operatorname{Im}(\lambda)} \\ 0 & -M_{\operatorname{Im}(\lambda)} & M_{\operatorname{Re}(\lambda)} \end{bmatrix}, \tag{2.5.1}$$

where M_f for a bounded Borel measurable function f denotes the multiplication operator $g \mapsto fg$, $\operatorname{Re}(\lambda), \operatorname{Im}(\lambda)$ are as in Corollary 2.4.2, defined on K and λ denotes the function $\lambda(t) = t, \forall t$, on $I \subset \mathbb{R}$. S is a normal operator and we will prove that every normal operator is orthogonally equivalent to a direct sum of operators of this form. We will need the following elementary lemma,

Lemma 2.5.1. *Let μ be a finite and positive regular Borel measure on a compact set $E \subseteq \mathbb{C}$. Consider the real Hilbert space $L^2(E)$. By abuse of notation, for $n, m \in \mathbb{N} \cup \{0\}$, let $r^{n+m}\cos(n-m)\theta$ denote the polar coordinate function $(r, \theta) \mapsto r^{n+m}\cos(n-m)\theta$ and $r^{n+m}\sin(n-m)\theta$ denote a function defined similarly, such that at $(0,0)$ both functions take value 0. Then the set of functions*

$$\{r^{n+m}\cos(n-m)\theta : n, m \in \mathbb{N} \cup \{0\}\} \cup \{r^{n+m}\sin(n-m)\theta : n, m \in \mathbb{N} \cup \{0\}\}$$

is a total set in $L^2(E)$.

Proof. Assume $f \in L^2(E)$ is such that $\langle f, r^{n+m}\cos(n-m)\theta\rangle = \langle f, r^{n+m}\sin(n-m)\theta\rangle = 0$, for $n, m \in \mathbb{N}\cup\{0\}$. By considering f as an element of the complex Hilbert space $(L^2(\mu), \langle\cdot,\cdot\rangle_{\mathbb{C}})$ and by going to polar coordinates, we have $\langle f, z^n\rangle_{\mathbb{C}} = \langle f, \bar{z}^m\rangle_{\mathbb{C}} = 0$, for all $n, m \in \mathbb{N}\cup\{0\}$. Similarly by expanding trignometric identities we have $\langle f, z^n\bar{z}^m\rangle_{\mathbb{C}} = 0$. Therefore, $\langle f, P(z,\bar{z})\rangle_{\mathbb{C}}$, for every polynomial P in z and $\bar{z}$. Now by Stone-Weierstrass theorem we have $\langle f, g\rangle_{\mathbb{C}} = 0$, for every continuous function g on E. This in turn implies that $f = 0$, the zero element in $L^2(E)$. □

Recall from Corollary 2.3.2 that the spectrum of a bounded real normal operator is symmetric about the real axis.

Lemma 2.5.2. *Let H be a real Hilbert space and A be a bounded normal operator on H with $\sigma(A) = E$. Assume that A has a cyclic vector. Then there exists a positive measure μ defined on $\mathcal{B}(E)$, with the following properties*

i) μ is symmetric about the real axis.

ii) There exists an orthogonal transformation $U\colon H \to L^2(E,\mu)$ such that $UAU^T = S$, where S on $L^2(E,\mu)$ is given by (2.5.1), via the identification described there.

Proof. Define $\mu = \mu_1^{(x,x)}$ as in Proposition 2.4.1. We look at the polar coordinates for making the computations simpler. Let $\mathcal{E} := \{A^n(A^T)^m x : n, m \in \mathbb{N}\cup\{0\}\}$. Define an operator $U : \operatorname{span}\mathcal{E} \to L^2(I) \oplus L^2(K) \oplus L^2(K)$ by

$$\begin{aligned} U\Big(\sum a_{nm}A^n((A^T)^m x\Big) = \sum a_{nm}r^{n+m} &\oplus \sum a_{nm}r^{n+m}(\cos(n-m)\theta + \sin(n-m)\theta) \\ &\oplus \sum a_{nm}r^{n+m}(\cos(n-m)\theta - \sin(n-m)\theta). \end{aligned} \tag{2.5.2}$$

We set out to prove that U is inner product preserving and can be extended as an onto map from H. Strictly as an element of $L^2(\sigma(A))$, $U(A^n(A^T)^m x)$ is the element $r^{n+m}(\cos(n-m)\theta + \sin(n-m)\theta)$. Therefore, we have

$$\begin{aligned} &\langle U(A^k(A^T)^l x), U(A^n(A^T)^m x)\rangle \\ &= \langle r^{k+l}(\cos(k-l)\theta + \sin(k-l)\theta), r^{n+m}(\cos(n-m)\theta + \sin(n-m)\theta)\rangle \\ &= \int\limits_{\sigma(A)} r^{k+l}(\cos(k-l)\theta + \sin(k-l)\theta)r^{n+m}(\cos(n-m)\theta + \sin(n-m)\theta)\, d\mu. \end{aligned} \tag{2.5.3}$$

Note that since $\sin(-\theta) = -\sin\theta$ and μ is symmetric about the real axis we have

$$\int_{\sigma(A)} \sin q\theta \, d\mu = 0, \forall q \in \mathbb{Z}.$$

Therefore, by expanding (2.5.3) using trignometric identities, we have

$$\langle U(A^k(A^T)^l x), U(A^n(A^T)^m x)\rangle = \int\limits_{\sigma(A)} r^{k+l+n+m}\cos(k-l)-(n-m)\theta\, d\mu. \tag{2.5.4}$$

So for proving that U is inner product preserving, we just need to prove

$$\langle A^k(A^T)^l x, A^n(A^T)^m x\rangle = \int\limits_{\sigma(A)} r^{k+l+n+m}\cos(k-l)-(n-m)\theta\, d\mu. \tag{2.5.5}$$

Write $k+m=s$ and $l+n=t$. Then by using elementary properties of normal operators and (2.4.8) we have left hand side of (2.5.5) is same as

$$\begin{aligned}\left\langle A^s x, A^t x\right\rangle &= \left\langle A^s(A^T)^t x, x\right\rangle \\ &= \int\limits_{\sigma(A)} r^{s+t}\cos(s-t)\theta\, d\mu_1^{(x,x)},\end{aligned}$$

which is now same as the right hand side of 2.5.4 and thus U is inner product preserving. Now we will show that U extends as an orthogonal transformation of H onto $L^2(I)\oplus L^2(K)\oplus L^2(K)$. To this end we show that the range of U is dense. We have $U(\frac{1}{2}(A^n(A^T)^m+(A^T)^n A^m)) = r^{n+m}\oplus r^{n+m}\cos(n-m)\theta\oplus r^{n+m}\cos(n-m)\theta$, which is the element corresponding to $r^{n+m}\cos(n-m)\theta\in L^2(\sigma(A))$ and $U(\frac{1}{2}(A^n(A^T)^m-(A^T)^n A^m)) = r^{n+m}\oplus r^{n+m}\sin(n-m)\theta\oplus -r^{n+m}\sin(n-m)\theta$, which is the element corresponding to $r^{n+m}(\sin(n-m)\theta\in L^2(\sigma(A))$. We already know from Lemma 2.5.1 that this collection is total. Thus we have proved that U can be extended uniquely as an orthogonal transformation on H, we use the notation U for denoting this extended operator also. Further, a direct computation using trigonometric identities shows that $UAU^T=S$ on vectors of the form $r^{n+m}\oplus r^{n+m}\cos(n-m)\theta\oplus r^{n+m}\cos(n-m)\theta$ and $r^{n+m}\oplus r^{n+m}\sin(n-m)\theta\oplus -r^{n+m}\sin(n-m)\theta$. This proves the lemma. $\square$

Theorem 2.5.1. *Let A be a bounded normal operator on a real Hilbert space H. Then A is orthogonally equivalent to a countable direct sum of operators of the form (2.5.1). (* i.e., *there exists a countable family of compact sets $E_i\subseteq\mathbb{C}$, positive measures μ_i symmetric to the real axis, on $\mathcal{B}(E_i)$ and a real orthogonal transformation $U\colon H\to\oplus_i L^2(E_i,\mu_i)$ such that $UAU^T=\oplus_i S_i$, where S_i on $L^2(E_i,\mu_i)$ is as in (2.5.1)).*

Proof. Use the previous lemma and apply Zorn's lemma. $\square$

Corollary 2.5.1. *If A is a skew symmetric operator on a real Hilbert space H, then there exists a countable family of compact subsets $F_j\subset[0,\infty)$ and positive measures μ_j on $\mathcal{B}(F_j)$ and a real orthogonal transformation $V\colon H\to\left(\oplus_j L^2(\mu_j)\right)\oplus\left(\oplus_j L^2(\mu_j)\right)$ such that*

$$A = V^T \begin{bmatrix} 0 & \oplus_j - M_\lambda^j \\ \oplus_j M_\lambda^j & 0 \end{bmatrix} V \tag{2.5.6}$$

where $M_\lambda^j \colon L^2(\mu_j) \to L^2(\mu_j)$ *is such that* $f \mapsto \lambda f$ *with* $(\lambda f)(x) = xf(x), \forall x \in F_j$.

Proof. Since A is skew-symmetric $\sigma(A)$ is contained in the imaginary axis. Use Theorem 2.5.1 to get $L^2(E_j)$, where E_j subset of the positive imaginary axis. Take $F_j = -iE_j$. Use the obvious transformation to transfer measure to F_j. □

The following theorem is new to best of our knowledge in a recent paper [BP12], Böttcher et al. prove the same result for the special case compact skew-symmetric operators. Here also the symmetry of the situation is explicit in our proof.

Corollary 2.5.2. *If A is a skew-symmetric invertible operator on a real Hilbert space H, then there exists a real Hilbert space K, a positive invertible operator P on K and a real orthogonal transformation $V \colon H \to K \oplus K$ such that*

$$A = V^T \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix} V. \tag{2.5.7}$$

We give two proofs for this result. The second proof is very elementary and does not use the spectral measure.

Proof. (1) If A is invertible then each M_λ^j in Corollary 2.5.1 is invertible. □

Proof. (2) Assume first that A has a cyclic vector x. Note that if A is skew-symmetric, A^{2k+1} is skew-symmetric and A^{2k} is symmetric for $k \in \mathbb{N}$. Therefore

$$\langle A^{2n}x, A^{2m+1}x \rangle = 0, \quad \forall n, m \geq 0, \tag{2.5.8}$$

because the skew-symmetry of A^{2k+1} implies $\langle x, A^{2k+1}x \rangle = 0$ for all $k \geq 0$. Set

$$K = \overline{\operatorname{span}}\{x, A^2x, A^4x, \dots\}, N = \overline{\operatorname{span}}\{Ax, A^3x, A^5x, \dots\}.$$

Then $K \perp N$ by (2.5.8) and since x is cyclic $N = K^\perp$. Therefore, there exists an operator $R : K \to N$ such that

$$A = \begin{bmatrix} 0 & -R^T \\ R & 0 \end{bmatrix} \tag{2.5.9}$$

in the direct sum decomposition $H = K \oplus N$. Since $A(A^{2n}x) = A^{2n+1}x$, $R := A|_K$ maps K onto N, since A is invertible, R is an invertible operator. Now we apply polar decomposition to R. If $R = UP$ then $U : K \to N$ and $P : K \to K$ are such that U is orthogonal (because R is invertible) and P $(=\sqrt{R^T R})$ is positive definite and invertible.

Now we have

$$A = \begin{bmatrix} I_K & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix} \begin{bmatrix} I_K & 0 \\ 0 & U^T \end{bmatrix}, \tag{2.5.10}$$

where I_K is the identity operator on K and $\left[\begin{smallmatrix} I_K & 0 \\ 0 & U \end{smallmatrix}\right] : K \oplus K \to K \oplus N$ is orthogonal.

If A doesn't have a cyclic vector then a usual argument using Zorn's lemma along with a permutation proves the result. □

2.6 Williamson's Normal Form

All the work we have done till now was to obtain the right machinery for a proof of Williamson's normal form in the infinite dimensional set up. We refer to [Par13b] for an easy proof of the theorem in the finite dimensional setup. Let us recall the following definitions from Section 1.7, in view of Proposition 1.7.1.

Definition 2.6.1. Let H be a real Hilbert space and I be the identity operator on H. Define the involution operator J on $H \oplus H$ by $J = \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}$.

Definition 2.6.2. Let H and K be two real Hilbert spaces. A bounded invertible linear operator $Q\colon H \oplus H \to K \oplus K$ is called a symplectic transformation if $Q^T J Q = J$, where J on left side is the involution operator on $K \oplus K$ and that on the right side it is the involution operator on $H \oplus H$.

Remark 3. If Q is symplectic then Q^{-1} and Q^T are symplectic.

Proof. We have $QQ^{-1} = I$ therefore, $(QQ^{-1})^T JQQ^{-1} = J$. Since Q is symplectic, this is equivalent to $(Q^{-1})^T JQ^{-1} = J$. This proves that Q^{-1} is symplectic. It is easy to see that the product of symplectic operators is symplectic, therefore $Q^T = JQ^{-1}J^{-1}$ is also symplectic.

Theorem 2.6.1 (Williamson's normal form in infinite dimensions)**.** *Let H be a real Hilbert space and A be a strictly positive invertible operator on $H \oplus H$, then there exists a Hilbert space K, a positive invertible operator P on K and a symplectic transformation $L\colon H \oplus H \to K \oplus K$ such that*

$$A = L^T \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} L. \tag{2.6.1}$$

The decomposition is unique in the sense that if M is any strictly positive invertible operator on a Hilbert space $\tilde{H}$ and $\tilde{L}\colon H \oplus H \to \tilde{H} \oplus \tilde{H}$ is a symplectic transformation such that

$$A = \tilde{L}^T \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \tilde{L}, \tag{2.6.2}$$

then P and M are orthogonally equivalent.

Proof. Define $B = A^{1/2} J A^{1/2}$, where $A^{1/2}$ is as described in Corollary 2.4.4 and J is given by Definition 2.6.1. Then B is a skew symmetric invertible operator on $H \oplus H$. Hence by Corollary 2.5.2 there exists a real Hilbert space K, an invertible positive operator P and a real orthogonal transformation $\Gamma\colon K \oplus K \to H$ such that

$$\Gamma^T B \Gamma = \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix}. \tag{2.6.3}$$

Define $L\colon H \oplus H \to K \oplus K$, by

$$L = \begin{bmatrix} P^{-1/2} & 0 \\ 0 & P^{-1/2} \end{bmatrix} \Gamma^T A^{\frac{1}{2}}. \tag{2.6.4}$$

Then clearly (2.6.1) is satisfied. A direct computation using (2.6.3) shows that L is symplectic, that is $LJL^T = J$, where J on the left side is the involution operator on $H \oplus H$ and on the right side is the corresponding involution operator on $K \oplus K$.

To prove the uniqueness, let

$$A = L^T \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} L = \tilde{L}^T \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \tilde{L},$$

where P, M are two positive operators and L, $\tilde{L}$ are symplectic. Putting $N = L\tilde{L}^{-1}$ we get a symplectic N such that

$$N^T \begin{bmatrix} P & 0 \\ 0 & P \end{bmatrix} N = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}.$$

Substituting $N^T = JN^{-1}J^{-1}$ with appropriate J's we get

$$N^{-1}\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} N = \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix}. \tag{2.6.5}$$

Now we recall the fact that two similar normal operators are unitarily equivalent (this can be proved using Fuglede-Putnam theorem, see Theorem 12.36 in [Rud91] and real case follows by complexification). However, we continue with our proof without using this result. To this end, taking transpose on both sides of (2.6.5) we get

$$N^T\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} (N^T)^{-1} = \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix}.$$

Hence again by using (2.6.5), we get

$$N^T\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} (N^T)^{-1} = N^{-1}\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} N,$$

or

$$\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} (N^{-1})^T N^{-1} = (N^{-1})^T N^{-1}\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix}.$$

This implies

$$\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} \Big((N^{-1})^T N^{-1}\Big)^{1/2} = \Big((N^{-1})^T N^{-1}\Big)^{1/2}\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix}, \tag{2.6.6}$$

where the reasoning for (2.6.6) is same as that in the complex case. Let

$$N^{-1} = U\Big((N^{-1})^T N^{-1}\Big)^{1/2}$$

be the polar decomposition of N^{-1}. From (2.6.5) we get

$$U\Big((N^{-1})^T N^{-1}\Big)^{1/2}\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} \Big((N^{-1})^T N^{-1}\Big)^{-1/2} U^T = \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix}.$$

Hence by (2.6.6), we have

$$U\begin{bmatrix} 0 & P \\ -P & 0 \end{bmatrix} U^T = \begin{bmatrix} 0 & M \\ -M & 0 \end{bmatrix}. \tag{2.6.7}$$

Now we will prove that (2.6.7) implies that P and M are orthogonally equivalent. Note that by taking squares, and getting rid of the negative sign,

$$U \begin{bmatrix} P^2 & 0 \\ 0 & P^2 \end{bmatrix} U^T = \begin{bmatrix} M^2 & 0 \\ 0 & M^2 \end{bmatrix}.$$

It is true that if A and B are self adjoint operators such that $A \oplus A$ and $B \oplus B$ are orthogonally equivalent then A and B are orthogonally equivalent. We will give a proof of this as a Lemma below. But if we assume this fact our proof is complete because we see that for the positive operators P and M, P^2 and M^2 are orthogonally equivalent. Hence P and M are orthogonally equivalent. $\square$

Now we proceed to provide the proof of the lemma we promised. We depend on Hall [Hal13] for notations and results used below. We write the following in the framework of complex Hilbert spaces, but as it was observed after Theorem 2.4.2, the spectral theory of a self-adjoint operator is identical on both real and complex Hilbert spaces and hence what we write below works on separable real Hilbert spaces also.

By the direct integral version of spectral theorem, any bounded self-adjoint operator A on a separable Hilbert space is unitarily equivalent to the multiplication operator $s \mapsto xs$ where $xs(\lambda) := \lambda s(\lambda), \lambda \in \sigma(A)$ on $\int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda d\,\mu(\lambda)$ for some σ-finite measure μ with a measurable family of Hilbert spaces $\{\mathcal{H}_\lambda\}$, satisfying $\dim(\mathcal{H}_\lambda) > 0$ almost everywhere μ. It is understood that we work with the Borel subsets of the spectrum $\sigma(A)$. The function $\lambda \mapsto \dim(\mathcal{H}_\lambda)$ is called the multiplicity function associated with the direct integral representation of A. By Proposition 7.24 from [Hal13], two bounded self-adjoint operators expressed as direct integrals on their spectrum are unitarily equivalent if and only if (i) the spectrum are same; (ii) the associated measures are equivalent in the sense that they are mutually absolutely continuous and (iii) the multiplicity functions coincide almost everywhere.

Lemma 2.6.1. *Let A, B be self-adjoint operators on a separable Hilbert space such that $A \oplus A$ and $B \oplus B$ are unitarily equivalent. Then A and B are unitarily equivalent.*

Proof. Let A be unitarily equivalent to the multiplication operator for each section s, with respect to a measure μ on $\sigma(A)$ in the direct integral Hilbert space

$$\int_{\sigma(A)}^{\oplus} \mathcal{H}_\lambda d\,\mu(\lambda).$$

Then it can be seen that $A \oplus A$ is unitarily equivalent to the multiplication operator on the direct integral

$$\int_{\sigma(A)}^{\oplus} \mathcal{K}_\lambda d\,\mu(\lambda),$$

where $\mathcal{K}_\lambda = \mathcal{H}_\lambda \oplus \mathcal{H}_\lambda$. Since $A \oplus A$ and $B \oplus B$ are unitarily equivalent, by the uniqueness of integral representation mentioned above, by comparing spectrum, measures and multiplicity functions it is easy to see that A and B are unitarily equivalent. □

Remark 4. We observe that Lemma 2.6.1 can be proved using standard versions of the Hahn-Hellinger theorem also, for example Theorem 7.6 in [Par92] can also be used. We also note that if we take infinitely many copies of self-adjoint operators A, B and $\oplus_{i=1}^{\infty} A$ is unitarily equivalent to $\oplus_{i=1}^{\infty} B$, it does not mean that A and B are unitarily equivalent. So it is only natural that the multiplicity theory is required in the proof of the last Lemma.

Remark 5. Under the situation of Theorem 2.6.1, in view of the uniqueness part of the theorem, the spectrum of P, can be defined as the *symplectic spectrum* of the positive invertible operator A.

In the following corollary we rewrite Theorem 2.6.1 using the formalism developed in Section 1.7. It will be useful in the future chapters.

Corollary 2.6.1. *Let S be a real linear positive, invertible operator on a complex Hilbert space $\mathcal{H}$. Then there exists a complex Hilbert space $\mathcal{K}$, a complex linear positive invertible operator $\mathcal{P}$ and a symplectic transformation $L : \mathcal{H} \to \mathcal{K}$ such that*

$$S = L^{\tau} \mathcal{P} L. \tag{2.6.8}$$

Further, $\mathcal{P}$ has the property that $\mathcal{P}_0 = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$.

CHAPTER 3

Characterization of Quantum Gaussian states

3.1 Introduction

In this chapter, we begin our study of infinite mode quantum Gaussian states. Initially, we see some fundamental properties of the quantum Fourier transform and define Gaussian states on an arbitrary Fock space by using the quantum Fourier transform. Later we characterize the covariance matrices associated with them and identify the characteristic function of Gaussian states with a particular class of quasifree states on the CCR algebra. The formalism developed in Section 1.7 and the Williamson's theorem proved in the previous chapter play a major role here.

3.2 Quantum Gaussian States

By a *state (or density matrix)* ρ on a Hilbert space $\mathcal{K}$ we mean a positive operator of unit trace *i.e.* $\rho \geq 0$ and $\operatorname{Tr} \rho = 1$. Note that a density matrix ρ on $\mathcal{K}$ gives rise to a unique state on the C^*- algebra $\mathcal{B}(\mathcal{K})$ (in the sense of Definition 1.5.1) as the functional $Y \mapsto \operatorname{Tr} \rho Y$, $Y \in \mathcal{B}(\mathcal{K})$. This is why we use the word 'state' for both these and the meaning will be clear from the context. We take $\mathcal{K} = \Gamma_s(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ in the following definition.

Definition 3.2.1. Let $\rho \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ be a density matrix. Then a complex valued function $\hat{\rho}$ on $\mathcal{H}$ defined by

$$\hat{\rho}(z) = \operatorname{Tr} \rho W(z), \qquad z \in \mathcal{H} \tag{3.2.1}$$

is called the *quantum characteristic function*(or *quantum Fourier transform*) of ρ.

We want to observe at this point that the mapping $\rho \to \hat{\rho}$ is a one-one mapping; proof of this fact is essentially the same as that of Proposition 2.4 in [Par10]. We prove it below for the convenience of the reader. Further, it may be noted that $W(z) \mapsto \hat{\rho}(z)$ defines a state on the CCR-algebra generated by the Weyl operators; it is the restriction of the state $Y \mapsto \operatorname{Tr} \rho Y$, $Y \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ to the Weyl algebra. Therefore, sometimes we notate this state on $\mathcal{B}(\Gamma_s(\mathcal{H}))$ as $\hat{\rho}$.

Lemma 3.2.1. *The von-Neumann algebra generated by the Weyl operators is $\mathcal{B}(\Gamma_s(\mathcal{H}))$.*

Proof. Because of the Weyl form of the CCR (1.2.3), the linear span of $\{W(z)|z \in \mathcal{H}\}$ is a *-closed unital sub-algebra of $\mathcal{B}(\Gamma_s(\mathcal{H}))$. Therefore by the irreducibility of the Weyl representation (Proposition 1.3.3) and von-Neumann density theorem we see that the von Neumann algebra $\{W(z)|z \in \mathcal{H}\}'' = \mathcal{B}(\Gamma_s(\mathcal{H}))$. $\square$

Proposition 3.2.1. *The correspondence $\rho \mapsto \hat{\rho}$ is bijective.*

Proof. If there is another state ρ' such that the quantum characteristic functions of ρ and ρ' are same then $\operatorname{Tr}(\rho - \rho')W(z) = 0$ for all z. Therefore by continuity property of trace and Lemma 3.2.1, we see that $\operatorname{Tr}(\rho - \rho')X = 0$ for every $X \in \mathcal{B}(\mathcal{H})$. Hence $\rho = \rho'$. $\square$

Let us recall that if $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$, then $\Gamma_s(\mathcal{H})$ is canonically isomorphic to $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$, so we write $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ and identify the operators and vectors of these two spaces. Upon agreeing this, we have

Proposition 3.2.2. *If ρ_1 and ρ_2 are states on $\Gamma_s(\mathcal{H}_1)$ and $\Gamma_s(\mathcal{H}_2)$ respectively, then the quantum characteristic function of the state $\rho_1 \otimes \rho_2$ is given by*

$$(\rho_1 \otimes \rho_2)^\wedge(f \oplus g) = \hat{\rho}_1(f)\hat{\rho}_2(g). \tag{3.2.2}$$

Further, if ρ is any state on $\Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$ then the marginal state ρ_1 obtained by

$$\rho_1 = \operatorname{Tr}_2 \rho, \tag{3.2.3}$$

where Tr_2 denotes the relative trace (partial trace) over the second factor $\Gamma_s(\mathcal{H}_2)$,

$$\hat{\rho}_1(f) = \hat{\rho}(f \oplus 0). \tag{3.2.4}$$

Proof. We have $W(f \oplus g) = W(f) \otimes W(g)$ under the identification $\Gamma_s(\mathcal{H}) = \Gamma_s(\mathcal{H}_1) \otimes \Gamma_s(\mathcal{H}_2)$. Now

$$(\rho_1 \otimes \rho_2)^\wedge(f \oplus g) = \operatorname{Tr} \rho_1 \otimes \rho_2 W(f \oplus g)$$

$$= \operatorname{Tr} \rho_1 W(f) \otimes \rho_2 W(g)$$
$$= \operatorname{Tr} \rho_1 W(f) \operatorname{Tr} \rho_2 W(g)$$
$$= \hat{\rho}_1(f)\hat{\rho}_2(g).$$

To prove (3.2.4) note that by the fundamental property of partial trace

$$\hat{\rho}_1(f) = \operatorname{Tr}(\operatorname{Tr}_2 \rho W(f)) = \operatorname{Tr}(\rho W(f) \otimes I) = \operatorname{Tr} \rho W(f \oplus 0) = \hat{\rho}(f \oplus 0).$$

□

If ρ is a density matrix so is any unitary conjugation of it. It is important to understand how the quantum characteristic function changes when ρ conjugated with the fundamental unitaries, Weyl operators and second quantizations. We will explore this now. Recall Definition 1.2.1, by using Theorem 1.8.1, proof of the following proposition follows in the same way as that of Proposition 2.5 in [Par10].

Proposition 3.2.3. *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces. If ρ is a state on $\Gamma_s(\mathcal{K})$ and $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ then*

$$\{\Gamma_s(L)^* \rho \Gamma_s(L)\}^\wedge(f) = \hat{\rho}(Lf)$$

and,

$$\{W(f)\rho W(f)^*\}^\wedge(g) = \hat{\rho}(g)e^{2i \operatorname{Im}\langle f,g\rangle}$$

for every $f, g \in \mathcal{H}$.

Proof. By (1.8.1), $\Gamma_s(L)W(f)\Gamma_s(L)^* = W(Lf), \forall f \in \mathcal{H}$. Therefore,

$$\begin{aligned}
\{\Gamma_s(L)^* \rho \Gamma_s(L)\}^\wedge(f) &= \operatorname{Tr} \Gamma_s(L)^* \rho \Gamma_s(L) W(f) \\
&= \operatorname{Tr} \rho \Gamma_s(L) W(f) \Gamma_s(L)^* \\
&= \operatorname{Tr} \rho W(Lf) \\
&= \hat{\rho}(Lf).
\end{aligned}$$

To prove the second inequality recall that $W(g)W(f) = e^{-i \operatorname{Im}\langle g,f\rangle} W(g+f)$ and $W(f)^* = W(-f)$. Now

$$\begin{aligned}
\{W(f)\rho W(f)^*\}^\wedge(g) &= \operatorname{Tr} W(f)\rho W(f)^* W(g) \\
&= \operatorname{Tr} \rho W(f)^* W(g) W(f) \\
&= e^{2i \operatorname{Im}\langle f,g\rangle} \operatorname{Tr} \rho W(g).
\end{aligned}$$

□

Now we are ready to define our main object of study. Recall that $\mathcal{B}_{\mathbb{R}}(\mathcal{H})$ denotes the collection of all bounded real linear operators on $\mathcal{H}$.

Definition 3.2.2. Let $\rho \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ be a density matrix, ρ is said to be a quantum Gaussian state (or simply, a Gaussian state) if there exists $w \in \mathcal{H}$ and a symmetric, invertible $S \in \mathcal{B}_{\mathbb{R}}(\mathcal{H})$ such that

$$\hat{\rho}(z) = \exp\left\{-i \operatorname{Re}\langle w, z\rangle - \frac{1}{2}\operatorname{Re}\langle z, Sz\rangle\right\}, \forall z \in \mathcal{H}. \tag{3.2.5}$$

In such a case we write $\rho = \rho_g(w, S)$.

Note that this definition determines a real linear functional $z \mapsto \operatorname{Re}\langle w, z\rangle$ and a bounded quadratic form $z \mapsto \operatorname{Re}\langle z, Sz\rangle$ on the real Hilbert space $\mathcal{H}$. Hence w and S are uniquely determined by the definition.

We call w the *mean vector* and S the *covariance operator* associated with ρ. Suppose $\mathcal{H} = H + iH$, where H is a real subspace and let $w = \sqrt{2}(l - im)$, then we call l and m as *mean momentum vector* and *mean position vector* respectively. Further S_0 corresponding to S (Section 1.7) will be called as the *momentum-position covariance operator*. When $\mathcal{H}$ is infinite dimensional we call ρ as an *infinite mode quantum Gaussian state*.

Notation. Let $\mathcal{G}(\mathcal{H})$ denote the set of all Gaussian states on $\Gamma_s(\mathcal{H})$ and $\mathcal{C}(\mathcal{H})$ denote the set of all Gaussian covariance operators on $\mathcal{H}$.

We will characterize the elements of $\mathcal{C}(\mathcal{H})$ in Theorem 3.2.1.

Examples. (i) For $f \in \mathcal{H}$ consider the normalized exponential vector

$$\psi(f) := e^{-\frac{1}{2}\|f\|^2} e(f).$$

Let the pure state $|\psi(f)\rangle\langle\psi(f)|$ be called the *coherent state*.
We prove below that the coherent state is a pure Gaussian state on $\Gamma_s(\mathcal{H})$ with the identity operator as the covariance matrix and $-2if$ as the mean vector. Consider the quantum characteristic function,

$$\begin{aligned}
|\psi(f)\rangle\langle\psi(f)|)^\wedge(z) &= \operatorname{Tr}|\psi(f)\rangle\langle\psi(f)|\, W(z) \\
&= \langle\psi(f), W(z)\psi(f)\rangle \\
&= e^{-\|f\|^2}\langle e(f), W(z)e(f)\rangle \\
&= e^{-\|f\|^2} e^{\{-\frac{1}{2}\|z\|^2 - \langle z, f\rangle\}}\langle e(f), e(f+z)\rangle \\
&= \exp\left\{-\|f\|^2 - \frac{1}{2}\|z\|^2 - \langle z, f\rangle + \langle f, f+z\rangle\right\}
\end{aligned}$$

$$= \exp\left\{-\frac{1}{2}\|z\|^2 - \langle z, f\rangle + \overline{\langle z, f\rangle}\right\}$$
$$= \exp\left\{-2i\,\mathrm{Im}\,\langle z, f\rangle - \frac{1}{2}\|z\|^2\right\}.$$

But $-2i\,\mathrm{Im}\,\langle z, f\rangle = i\,\mathrm{Re}\,\langle 2if, z\rangle$. Thus we have proved

$$(|\psi(f)\rangle\langle\psi(f)|)^\wedge(z) = \exp\left\{-i\,\mathrm{Re}\,\langle -2if, z\rangle - \frac{1}{2}\|z\|^2\right\}. \tag{3.2.6}$$

In particular,

$$|e(0)\rangle\langle e(0)| = \rho_g(0, I). \tag{3.2.7}$$

Notice at this point that the quantum characteristic function of the density matrix $|e(0)\rangle\langle e(0)|$ corresponds to the vacuum state defined in Section 1.6 because $\mathrm{Im}\,\langle -iz, z\rangle = \mathrm{Re}\,\langle z, z\rangle$.

(ii) Let L be a symplectic automorphism on $\mathcal{H}$ such that $L^T L - I$ is Hilbert-Schmidt. Define $\psi_L = \Gamma_s(L)^* \,|e(0)\rangle$. Then

$$\begin{aligned}
(|\psi_L\rangle\langle\psi_L|)^\wedge(z) &= \mathrm{Tr}\,|\psi_L\rangle\langle\psi_L|\,W(z) \\
&= \mathrm{Tr}\,|\psi_L\rangle\langle W(z)^*\psi_L| \\
&= \langle\psi_L, W(z)\psi_L\rangle \\
&= \langle e(0), \Gamma_s(L)W(z)\Gamma_s(L)^* e(0)\rangle \\
&= \langle e(0), W(Lz)e(0)\rangle \\
&= e^{-\frac{1}{2}\langle z, L^T L z\rangle}.
\end{aligned}$$

Therefore, $|\psi_L\rangle\,\langle\psi_L| = \rho_g(0, L^T L)$.

(iii) Consider $\Gamma_s(\mathbb{C}) = L^2(\mathbb{R})$, by Example 1 in Chapter 1. If we write $e(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}}\psi_n$, then observe that ψ_n is an orthonormal basis for $L^2(\mathbb{R})$. Then for the number operator, $a^\dagger a\psi_n = n\psi_n, \forall n \in \mathbb{N}$. Therefore,

$$\mathrm{Tr}\,e^{-sa^\dagger a} = (1 - e^{-s})^{-1}, \quad s > 0.$$

Therefore the states

$$\rho_s = (1 - e^{-s})e^{-sa^\dagger a}, \quad s > 0 \tag{3.2.8}$$

are well defined. In this case, by Proposition 2.12 in [Par10] we have

$$\hat{\rho}_s(z) = \exp\left\{-\frac{1}{2}(\coth\frac{s}{2})|z|^2\right\}. \tag{3.2.9}$$

Therefore ρ_s is a Gaussian state. Since the spectrum of $a^\dagger a$ is $\{0, 1, 2, \dots\}$, it is not a pure state.

Proposition 3.2.4. *Let $f \in \mathcal{H}$. Then*

$$W(f)\rho_g(w, S)W(f)^{-1} = \rho_g(w - 2if, S).$$

In particular,

$$W(\frac{-i}{2}w)\rho_g(w, S)W(\frac{-i}{2}w)^{-1} = \rho_g(0, S).$$

Proof. This is a direct consequence of the definition of $\rho_g(\cdot,\cdot)$ and Proposition 3.2.3. □

Proposition 3.2.5. *Let $\rho_1 = \rho_g(w_1, S_1)$ and $\rho_2 = \rho_g(w_2, S_2)$ be Gaussian states on $\Gamma_s(\mathcal{H}_1)$ and $\Gamma_s(\mathcal{H}_2)$ respectively. Then $\rho_1 \otimes \rho_2 = \rho_g(w_1 \oplus w_2, S_1 \oplus S_2)$.*

Proof. This follows directly from Proposition 3.2.2. □

Proposition 3.2.6. *If $\rho = \rho_g(w, S)$ on $\Gamma_s(\mathcal{K})$ and $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ then*

$$\Gamma_s(L)^*\rho\Gamma_s(L) = \rho_g(L^T w, L^T SL).$$

Proof. This follows from Proposition 3.2.3. □

Our main theorem in this chapter is the following:

Theorem 3.2.1. *Let S be a real linear, bounded, symmetric and invertible operator on $\mathcal{H}$. Then S is the covariance operator of a quantum Gaussian state (i.e., $S \in \mathcal{C}(\mathcal{H})$) if and only if the following hold:*

(i) $\hat{S} - i\hat{J} \geq 0$ *on* $\hat{\mathcal{H}}$.
(ii) $S - I$ *is Hilbert-Schmidt on* $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$.
(iii) $(\sqrt{S}J\sqrt{S})^T(\sqrt{S}J\sqrt{S}) - I$ *is trace class on* $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$.

We prove this theorem in several steps in the next two sections.

3.3 Necessary conditions on the covariance operator

Lemma 3.3.1. *If ρ is any density matrix, then the kernel K_ρ on $\mathcal{H}$ defined by $K_\rho(z, w) = e^{i \operatorname{Im}\langle z,w\rangle}\hat{\rho}(w - z)$ is positive definite.*

Proof.

$$\begin{aligned}
\sum_{j,k=1}^{n} \overline{c_j} c_k K_\rho(z_j, z_k) &= \sum_{j,k=1}^{n} \overline{c_j} c_k e^{i \operatorname{Im} \langle z_j, z_k \rangle} \hat{\rho}(z_k - z_j) \\
&= \sum_{j,k=1}^{n} \overline{c_j} c_k e^{i \operatorname{Im} \langle z_j, z_k \rangle} \operatorname{Tr} \rho W(z_k - z_j) \\
&= \sum_{j,k=1}^{n} \overline{c_j} c_k \operatorname{Tr} \rho W(-z_j) W(z_k) \\
&= \operatorname{Tr} \rho X^* X \\
&\geq 0
\end{aligned}$$

where $X = \sum_{j=1}^{n} c_j W(z_j)$. $\square$

Recall from Section 1.6 that $CCR(\mathcal{H}, \sigma) \hookrightarrow \mathcal{B}(\Gamma_s(\mathcal{H}))$ as a standard space, if we take $\sigma(\cdot, \cdot) = -\operatorname{Im} \langle \cdot, \cdot \rangle$. Also we will use the work done in Section 1.7 in what follows.

Lemma 3.3.2. *Let S be a real linear, invertible operator on $\mathcal{H}$ and $\hat{S} - i\hat{J} \geq 0$ on $\hat{\mathcal{H}}$. Then*

(i) $S \geq 0$.

(ii) If $S = L^T \mathcal{P} L$ is the Williamson's normal form associated with S (as in Corollary 2.6.1), then $\mathcal{P} - I \geq 0$ on $\mathcal{K}$.

(iii) There exists a primary quasifree state ϕ on $CCR(\mathcal{H}, \sigma)$ such that

$$\phi(W(z)) = e^{-\frac{1}{2} \operatorname{Re} \langle z, Sz \rangle}. \tag{3.3.1}$$

Further, $\phi = \phi_A$, where $A = -JS$ (the notation ϕ_A is as in Section 1.5).

Proof. (i). Note that $\hat{S} - i\hat{J} \geq 0$ implies $\hat{S}$ is symmetric, hence we have S is also symmetric. Let us denote the complex inner product in both $\mathcal{H}$ and $\hat{\mathcal{H}}$ by $\langle \cdot, \cdot \rangle$. Let $z, w \in \mathcal{H}$, then $z + iw \in \hat{\mathcal{H}}$ and

$$\begin{aligned}
0 &\leq \left\langle z + iw, (\hat{S} - i\hat{J}) z + iw \right\rangle \\
&= \operatorname{Re} \langle z, Sz \rangle + i \operatorname{Re} \langle z, Sw \rangle - i \operatorname{Re} \langle w, Sz \rangle + \operatorname{Re} \langle w, Sw \rangle \\
&\quad - i \operatorname{Re} \langle z, Jz \rangle + \operatorname{Re} \langle z, Jw \rangle - \operatorname{Re} \langle w, Jz \rangle - i \operatorname{Re} \langle w, Jw \rangle \\
&= \operatorname{Re} \langle z, Sz \rangle + \operatorname{Re} \langle w, Sw \rangle + 2 \operatorname{Re} \langle z, Jw \rangle
\end{aligned} \tag{3.3.2}$$

where we used the facts that S is symmetric, real inner product is symmetric and $\mathrm{Re}\,\langle z, Jz\rangle = 0$ for all z to obtain (3.3.2). If we take $z = w$ in the above computation then we get $S \geq 0$, since it is already symmetric. Note that the invertibility of S is not used to prove this.

(ii). Let $\mathcal{P}_0 = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$. Then $\hat{\mathcal{P}} = \left[\begin{smallmatrix} \hat{P} & 0 \\ 0 & \hat{P} \end{smallmatrix}\right]$ and $\hat{J} = \left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right]$ on $\hat{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ by Proposition 1.7.2. Further, $\hat{S} - i\hat{J} \geq 0$ implies $\hat{L^T}\left[\begin{smallmatrix} \hat{P} & 0 \\ 0 & \hat{P} \end{smallmatrix}\right]\hat{L} - i\hat{J} \geq 0$. By a conjugation with $\hat{L}^{-1}$ and using the fact that L^{-1} is symplectic we get $\left[\begin{smallmatrix} \hat{P} & 0 \\ 0 & \hat{P} \end{smallmatrix}\right] - i\left[\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right] \geq 0$ on $\hat{\mathcal{K}} = \mathcal{K} \oplus \mathcal{K}$. Hence (by Proposition 1.7.2) we get $\left[\begin{smallmatrix} P & -iI \\ iI & P \end{smallmatrix}\right] \geq 0$ on $K \oplus K$. But this means $P \geq I$ on K and correspondingly $\mathcal{P} \geq I$ on $\mathcal{K}$.

(iii). Since the $CCR(\mathcal{H}, \sigma)$ is standard we will use (i) of Proposition 1.5.2. Since S is positive and invertible, $\alpha(z, w) := \mathrm{Re}\,\langle z, Sw\rangle$ defines a complete real inner product on $\mathcal{H}$. Therefore by Proposition 1.5.2, ϕ as in (3.3.1) exists if $\sigma(z, w)^2 \leq \alpha(z, z)\alpha(w, w)$, for all $f, g \in \mathcal{H}$. This is same as

$$\mathrm{Im}\,\langle z, w\rangle^2 \leq \mathrm{Re}\,\langle z, Sz\rangle\,\mathrm{Re}\,\langle w, Sw\rangle\,. \tag{3.3.3}$$

Thus it is enough to prove (3.3.3) to show the existence of ϕ. To keep track of the inner product in $\mathcal{H}$ and $\mathcal{K}$ we put a subscript, thus we write $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ to denote the inner product in $\mathcal{H}$ and similarly for $\mathcal{K}$. Now

$$\begin{aligned}
\mathrm{Im}\,\langle z, w\rangle_{\mathcal{H}}^2 &= \mathrm{Im}\,\langle Lz, Lw\rangle_{\mathcal{K}}^2 \\
&\leq |\langle Lz, Lw\rangle_{\mathcal{K}}|^2 \\
&\leq \langle Lz, Lz\rangle_{\mathcal{K}}\,\langle Lw, Lw\rangle_{\mathcal{K}} \\
&\leq \langle Lz, \mathcal{P}Lz\rangle_{\mathcal{K}}\,\langle Lw, \mathcal{P}Lw\rangle_{\mathcal{K}} \qquad (3.3.4) \\
&= \mathrm{Re}\,\langle Lz, \mathcal{P}Lz\rangle_{\mathcal{K}}\,\mathrm{Re}\,\langle Lw, \mathcal{P}Lw\rangle_{\mathcal{K}} \\
&= \left\langle z, L^T\mathcal{P}Lz\right\rangle_{\mathcal{H}}\left\langle w, L^T\mathcal{P}Lw\right\rangle_{\mathcal{H}},
\end{aligned}$$

where (3.3.4) follows from (ii). Thus we proved (3.3.3). Hence first part of (iii) is proved. Further, $\phi = \phi_A$ because $\mathrm{Re}\,\langle \cdot, S(\cdot)\rangle_{\mathcal{H}} = -\,\mathrm{Im}\,\langle A(\cdot), \cdot\rangle$. □

Lemma 3.3.3. *Let H be a real Hilbert space and $\mathcal{H} = H + iH$ be its complexification. Let $A \in \mathcal{B}(\mathcal{H})$ be self adjoint. Define a hermitian kernal, K on H by*

$$K(x, y) := \langle x, Ay\rangle \quad \text{for all } x, y \in H.$$

Then K is positive definite if and only if $A \geq 0$ in the sense of positive definiteness of operators in $\mathcal{B}(\mathcal{H})$.

Proof. Assume $A \geq 0$. Let $a_1, a_2, \ldots, a_n \in \mathbb{C}$ and $x_1, x_2, \ldots, x_n \in H$.

$$\begin{aligned}\sum_{j,k=1}^{n} \overline{a}_j a_k K(x_j, x_k) &= \sum_{j,k=1}^{n} \overline{a}_j a_k \langle x_j, Ax_k \rangle \\ &= \left\langle \sum_{n=1}^{n} a_j x_j, A(\sum_{k=1}^{n} a_k x_k) \right\rangle \\ &= \langle z, Az \rangle \\ &\geq 0\end{aligned}$$

where $z = \sum\limits_{n=1}^{n} a_j x_j \in H \subset \mathcal{H}$. Conversely, if K is positive definite then $\langle x, Ax \rangle \geq 0$ for all $x \in H$. Now if $z = x + iy \in \mathcal{H}$, then

$$\begin{aligned}\langle z, Az \rangle &= \langle x + iy, A(x + iy) \rangle \\ &= \langle x, Ax \rangle + i \langle x, Ay \rangle - i \langle y, Ax \rangle + \langle y, Ay \rangle \\ &= \langle x, Ax \rangle + i \langle x, Ay \rangle - i \langle x, Ay \rangle + \langle y, Ay \rangle \qquad (3.3.5) \\ &\geq 0,\end{aligned}$$

where (3.3.5) follows because the real innerproduct is symmetric and A is self-adjoint. Thus $A \geq 0$. $\square$

The following theorem proves the necessary conditions we have on the covariance operators in Theorem 3.2.1.

Theorem 3.3.1. *Let S be a real linear symmetric and invertible operator on $\mathcal{H}$, and let the function $f : \mathcal{H} \to \mathbb{R}$ defined by $f(z) = e^{-\frac{1}{2} \operatorname{Re}\langle z, Sz \rangle}$ be the quantum characteristic function of a density matrix ρ i.e., $S \in \mathcal{C}(\mathcal{H})$ then*

(i) On $\hat{\mathcal{H}}$ we have,

$$\hat{S} - i\hat{J} \geq 0. \tag{3.3.6}$$

(ii) $S - I$ is Hilbert-Schmidt on $(\mathcal{H}, \operatorname{Re}\langle \cdot, \cdot \rangle)$.
(iii) $(\sqrt{S}J\sqrt{S})^T(\sqrt{S}J\sqrt{S}) - I$ is trace class on $(\mathcal{H}, \operatorname{Re}\langle \cdot, \cdot \rangle)$.

Proof. (i). Proof of (3.3.6) will follow in similar lines to the proof of the corresponding theorem in [Par10] for the finite mode case, we will give a proof here because there are slight changes to be noticed in the infinite mode case. Define the kernel

$$K_\rho(\alpha, \beta) = e^{i \operatorname{Im}\langle \alpha, \beta \rangle} f(\beta - \alpha), \quad \alpha, \beta \in \mathcal{H}. \tag{3.3.7}$$

By Lemma 3.3.1, K_ρ is a positive definite kernel on $\mathcal{H}$. If $\alpha = x+iy, \beta = u+iv$ where $x, y, u, v \in H$, then $\operatorname{Im}\langle \alpha, \beta\rangle = \langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\rangle$ on $H \oplus H$ (Section 1.7). We can rewrite the definition of K_ρ as

$$K_\rho(\alpha,\beta) = \exp\left\{ i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle - \left\langle \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right)\right\rangle \right\}. \tag{3.3.8}$$

Now positive definiteness of K_ρ in $\mathcal{H}$ reduces to that of L in $H \oplus H$ where

$$L\left((x,y),(u,v)\right) = \exp\left\{ i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle - \left\langle \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right)\right\rangle \right\}. \tag{3.3.9}$$

This is equivalent to the positive definiteness of

$$L_t\left((x,y),(u,v)\right) = L\left(\sqrt{t}(x,y), \sqrt{t}(u,v)\right)$$

for all $t \geq 0$. But $\{L_t\}$ is a one parameter multiplicative semigroup of kernels on $H \oplus H$. By elementary properties of positive definite kernels as described in Section 1 of [PS72], positive definiteness of $L_t, t \geq 0$ is equivalent to the conditional positive definiteness of

$$N\left((x,y),(u,v)\right) = i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle - \left\langle \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right)\right\rangle$$

or equivalently (by the same Proposition), the positive definiteness of

$$\begin{aligned}
&N\left((x,y),(u,v)\right) - N\left((x,y),(0,0)\right) - N\left((0,0),(u,v)\right) - N\left((0,0),(0,0)\right) \\
&= i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle - \left\langle \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u-x \\ v-y \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle \\
&= i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle \\
&= i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), \frac{1}{2} S_0 \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle && (3.3.10) \\
&= i \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), J_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right), S_0\left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right)\right\rangle && (3.3.11) \\
&= i \left\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), -J_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), S_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle && (3.3.12)
\end{aligned}$$

where (3.3.10) follows because the real inner-product is symmetric and (3.3.11) because S_0 is symmetric, and (3.3.12) for the same reasons. But $H \oplus H \subset \hat{\mathcal{H}} = (H \oplus H) + i(H \oplus H)$, the positive definiteness of (3.3.12) lifts to the positive definiteness of

$$M(w,z) := \left\langle w, \left\{\hat{S} - i\hat{J}\right\} z\right\rangle = \left\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), -i\hat{J}_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle + \left\langle \left(\begin{smallmatrix} u \\ v \end{smallmatrix}\right), \hat{S}_0\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right)\right\rangle \tag{3.3.13}$$

where M is a kernel defined (as above) in $\mathcal{H} \subset \hat{\mathcal{H}}$. Now by Lemma 3.3.3, positive definiteness of M in (3.3.13) is equivalent to (3.3.6).

(ii). Now we set out to prove that $S - I$ is Hilbert-Schmidt on the real Hilbert space $\mathcal{H}$. We are given that there exists a density matrix ρ such that $\hat{\rho}(z) = e^{-\frac{1}{2}\operatorname{Re}\langle z, Sz\rangle}$. Since $\hat{S} - i\hat{J} \geq 0$, by Lemma 3.3.2 there exists a primary quasifree state ϕ on $CCR(\mathcal{H}, \sigma)$ such that

$$\phi(W(z)) = e^{-\frac{1}{2}\operatorname{Re}\langle z, Sz\rangle}.$$

Claim : ϕ_A and $\phi_{(-J)}$ are quasi equivalent, where $A = -JS$.

Proof (of Claim). Consider the state ψ on $\mathcal{B}(\Gamma_s(\mathcal{H}))$ given by $X \mapsto \operatorname{Tr}\rho X$. The quasifree state ϕ_A is the restriction of ψ to $\mathcal{A} := CCR(\mathcal{H}, \sigma) \hookrightarrow \mathcal{B}(\Gamma_s(\mathcal{H}))$.

Let $(H_\psi, \Pi_\psi, \Omega_\psi)$ be the GNS triple for $\mathcal{B}(\mathcal{H})$ with respect to ψ. Then $(H_\psi, \Pi_\psi|_{\mathcal{A}}, \Omega_\psi)$ is the GNS triple for $\mathcal{A}$ with respect to ϕ_A. To see this, only thing to be noticed is Ω_ψ is cyclic for $\Pi_\psi(\mathcal{A})$, which is clear since $\mathcal{A}$ is strongly dense in $\mathcal{B}(\Gamma_s(\mathcal{H}))$. We further note that the inclusion $\mathcal{A} \subseteq \mathcal{B}(\Gamma_s(\mathcal{H}))$ is the GNS representation with respect to the vacuum state which is the quasi-free state given by ϕ_{-J}. It can be seen that the association

$$W(x) \mapsto \Pi_\psi(W(x))$$

can be extended as an isomorphism between $\mathcal{B}(\Gamma_s(\mathcal{H})) = \mathcal{A}''$ and $\Pi_\psi(\mathcal{B}(\Gamma_s(\mathcal{H})))$. Thus the claim is proved.

Since $\phi_{(-J)}$ and ϕ_A are quasi equivalent, by Theorem 1.5.3 we get $A + J$ is Hilbert-Schmidt on $\mathcal{H}_{-J}$ which is the same as $\mathcal{H}$ with the real inner product $\operatorname{Re}\langle\cdot,\cdot\rangle_{\mathcal{H}}$.

(iii). This follows due to the same reason as that of (ii) because of Theorem 1.5.3 itself. We get $\sqrt{-A^2 - I}$ is Hilbert-Schmidt on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$. This is same as $-A^2 - I$ is trace class on the same Hilbert space. Hence we have $-JSJS - I$ is trace class. By multiplying with $\sqrt{S}$ on the left and $(\sqrt{S})^{-1}$ on the right we see that $-\sqrt{S}JSJ\sqrt{S} - I$ is trace class. The result follows because $J^T = -J$. $\square$

Note. It may be noted at this point that the operator $\sqrt{S}J\sqrt{S}$ in (iii) of the above theorem is the skew symmetric operator B appearing in the proof of Williamson's normal form in Theorem 2.6.1. Proof of Williamson's normal form was obtained there by applying Corollary 2.5.2 to B,

$$\Gamma^T B \Gamma = \begin{bmatrix} 0 & -P \\ P & 0 \end{bmatrix}$$

where Γ is an orthogonal transformation. L was obtained by taking

$$L = \begin{bmatrix} P^{-1/2} & 0 \\ 0 & P^{-1/2} \end{bmatrix} \Gamma^T S^{1/2}.$$

This choice of L provides $S = L^T \mathcal{P} L$, where $\mathcal{P}_0 = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$.

Corollary 3.3.1. *$(\sqrt{S}J\sqrt{S})^T(\sqrt{S}J\sqrt{S}) - I$ is trace class if and only if $-JSJS - I$ is trace class.*

Proof. This is the content of the proof of (iii) in Theorem 3.3.1. □

Corollary 3.3.2. *Assuming the hypothesis of Theorem 3.3.1 we have*

(i) If $S - I \geq 0$ then $S - I$ is trace class on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$
(ii) If S is complex linear then $S - I \geq 0$ and $S - I$ is trace class on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$.

Proof. (i). We have $-\sqrt{S}JSJ\sqrt{S} - I$ is trace class on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$. Hence by multiplying with $(\sqrt{S})^{-1}$ on both sides, $(-J)SJ - S^{-1}$ is trace class. Since $S - I \geq 0$, $(-J)SJ - I \geq 0$ and $S^{-1} \leq I$ therefore we have

$$0 \leq (-J)SJ - I \leq (-J)SJ - S^{-1}$$

and we conclude that $(-J)SJ - I$ is trace class on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$. Thus the proof is complete by taking a conjugation with J.

(ii). By (iii) in Lemma 3.3.2 and (ii) of Proposition 1.5.2 we have

$$-A^2 - I \geq 0 \tag{3.3.14}$$

with respect to the real inner product $\operatorname{Re}\langle\cdot, S(\cdot)\rangle$. We have $A^2 = JSJS$ but since S is complex linear it commutes with J, thus $A^2 = -S^2$ and we see that $S^2 - I \geq 0$, consequently $S \geq I$ on $(\mathcal{H}, \operatorname{Re}\langle\cdot, S(\cdot)\rangle_{\mathcal{H}})$. But this implies $S \geq I$ on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$ since S is positive. Since S commutes with J, by (iii)) of Theorem 3.3.1 we see that $S^2 - I$ is Hilbert-Schmidt on $(\mathcal{H}, \operatorname{Re}\langle\cdot,\cdot\rangle)$. Now the result follows because $0 \leq S - I \leq S^2 - I$. □

We observe the following Corollary which follows from the fact that ϕ_A and ϕ_{-J} are quasiequivalent.

Corollary 3.3.3. *ϕ_A is a Type 1 quasifree state.*

Note. By (ii) of Examples in Section 3.2 we have seen that for a symplectic automorphism L, $L^T L$ is a covariance operator whenever $L^T L - I$ is Hilbert-Schmidt. Now by Theorem 3.3.1 we get that $L^T L$ satisfies the conditions (i), (ii), and (iii) there. This is true also for any such symplectic transformation. But since $\sqrt{L^T L}$ is symplectic whenever L is so, the condition (iii) is just void. Also it can be proved independently that for any symplectic transformation the positivity condition (i) on L^TL is true. Therefore, $L^TL - I$ is Hilbert-Schmidt is the only non-trivial condition here.

What is the meaning of the condition $\hat{S} - i\hat{J} \geq 0$?

We will answer this question now.

Lemma 3.3.4. *Let S be a real linear operator on $\mathcal{H}$. Then $\hat{S} - i\hat{J} \geq 0$ if and only if there exists a state ϕ on $CCR(H, \sigma)$ such that $\phi(W(z)) = e^{-\frac{1}{2}\operatorname{Re}\langle z, Sz\rangle}$.*

Proof. We saw in the proof of Theorem 3.3.1 that the condition $\hat{S} - i\hat{J} \geq 0$ is equivalent to the positive definiteness of the kernel K_ρ in (3.3.7), where $f(z) = e^{-\frac{1}{2}\operatorname{Re}\langle z, Sz\rangle}$. Since $f(0) = 1$, by Proposition 1.5.1, we have K_ρ is positive definite if and only if there exists a state ϕ on $CCR(H, \sigma)$ such that $\phi(W(z)) = f(z)$. $\square$

By Lemma 3.3.2, if S is real linear, invertible and $\hat{S} - i\hat{J} \geq 0$ then there exists a primary quasifree state ϕ such that (3.3.1) holds. On the other hand if there is a primary quasifree state ϕ such that (3.3.1) holds, by Lemma 3.3.4, we have $\hat{S} - i\hat{J} \geq 0$. Thus, we have

Theorem 3.3.2. *Let S be a real linear, invertible operator on $\mathcal{H}$. Then $\hat{S} - i\hat{J} \geq 0$ on $\hat{\mathcal{H}}$ if and only if there exists a primary quasifree state ϕ on $CCR(\mathcal{H}, \sigma)$ such that*

$$\phi(W(z)) = e^{-\frac{1}{2}\operatorname{Re}\langle z, Sz\rangle}. \tag{3.3.15}$$

Corollary 3.3.4. *Let S be a real linear, invertible operator on $\mathcal{H}$. Then $\hat{S} - i\hat{J} \geq 0$ on $\hat{\mathcal{H}}$ if and only if $\operatorname{Im}\langle z, w\rangle^2 \leq \operatorname{Re}\langle z, Sz\rangle \operatorname{Re}\langle w, Sw\rangle$.*

3.4 Positivity and Trace class conditions imply Gaussian state

Now we proceed to prove the converse of Theorem 3.3.1.

Lemma 3.4.1. *If $s_j > 0$ then $\sum\limits_{j=1}^{\infty}\left(\frac{e^{-s_j}}{1-e^{-sj}}\right) < \infty$ if and only if $\sum\limits_{j=1}^{\infty} e^{-s_j}$ is convergent.*

Proof. Assume $\sum\limits_{j=1}^{\infty}\left(\frac{e^{-s_j}}{1-e^{-sj}}\right) < \infty$. Since $\frac{e^{-s_j}}{1-e^{-sj}} > 0$ and $\frac{1}{1-e^{-s_j}} > 0$, we have $0 < \sum\limits_{j=1}^{\infty} e^{-s_j} < \sum\limits_{j=1}^{\infty}\left(\frac{e^{-s_j}}{1-e^{-sj}}\right) < \infty$. Now assume that $\sum\limits_{j=1}^{\infty} e^{-s_j} < \infty$. Then $s_j \to \infty$ and hence $\frac{1}{1-e^{-s_j}} \to 1$. This means we have $0 < \frac{1}{1-e^{-s_j}} < M, \forall j$, for some $M > 1$. Therefore, $\sum\limits_{j=1}^{\infty}\left(\frac{e^{-s_j}}{1-e^{-sj}}\right) < \infty$. $\square$

Let $\mathcal{H} = H + iH$ and $\{e_1, e_2, e_3 \cdots\}$ be an orthonormal basis for H. Note that $\{e_j\}$ is also a basis for $\mathcal{H}$ as a complex Hilbert space. Let $D = \mathrm{Diag}(d_j)$ be a bounded diagonal operator on $\mathcal{H}$, with $d_j > 1$, $j = 1, 2, 3, \ldots$ in the given basis. Since $d_j > 1$ there exists $s_j > 0$ such that $d_j = \coth\left(\frac{s_j}{2}\right)$ for all j, where coth denotes the hyperbolic cotangent. If we consider D as a real linear operator on $\mathcal{H}$, then $D_0 = \left[\begin{smallmatrix} D & 0 \\ 0 & D \end{smallmatrix}\right]$ on $H \oplus H$.

Lemma 3.4.2. *Let $D = \mathrm{Diag}(d_j)$ be a bounded diagonal operator on $\mathcal{H}$, with $d_j > 1$, $j = 1, 2, 3, \ldots$ with respect to a basis. Write $d_j = \coth\left(\frac{s_j}{2}\right)$ for all j. Then $D - I$ is trace class if and only if $\sum\limits_{j=1}^{\infty} e^{-s_j}$ is convergent.*

Proof. Observe,

$$\begin{aligned}
D - I \ \text{ is in trace class} &\Leftrightarrow \sum_{j=1}^{\infty}(d_j - 1) < \infty \\
&\Leftrightarrow \sum_{j=1}^{\infty}\left(\coth\left(\frac{s_j}{2}\right) - 1\right) < \infty \\
&\Leftrightarrow \sum_{j=1}^{\infty}\left(\frac{1 + e^{-s_j}}{1 - e^{-s_j}} - 1\right) \\
&\Leftrightarrow \sum_{j=1}^{\infty}\left(\frac{e^{-s_j}}{1 - e^{-sj}}\right) < \infty \\
&\Leftrightarrow \sum_{j=1}^{\infty} e^{-s_j} < \infty \qquad (3.4.1)
\end{aligned}$$

where (3.4.1) follows from Lemma 3.4.1.

□

Proposition 3.4.1. *Let $D = \mathrm{Diag}(d_j)$ be a bounded diagonal operator on $\mathcal{H}$, with $d_j > 1$, $j = 1, 2, 3, \ldots$ with respect to a basis. Write $d_j = \coth\left(\frac{s_j}{2}\right)$ for all j. Then there exists a state ρ_D on $\Gamma_s(\mathcal{H})$ such that $\hat{\rho}_D(x) = e^{-\frac{1}{2}\langle x, Dx\rangle}$.*

Proof. Consider the diagonal operator $T = \mathrm{Diag}(e^{-s_j})$ with respect to the same basis in which D is diagonal then the second quantization $\Gamma_s(T)$ is a trace class operator on the symmetric Fock space, $\Gamma_s(\mathcal{H})$. This is because of the following reasoning. T is positive and by Lemma 3.4.2 it is a trace class operator. Thus we have $s_j > 0$ and $s_j \to \infty$, which implies $e^{-s_j} < 1$, for all j. Since e^{-s_j} is maximal when s_j is minimal we get $\sup_j(e^{-s_j}) < 1$. Now by Proposition 1.2.3, $\Gamma_s(T)$ exists and is trace class with

$$\mathrm{Tr}\,\Gamma_s(T) = \Pi_{j=1}^{\infty}(1 - e^{-s_j})^{-1}. \qquad (3.4.2)$$

Define $\rho_D = \Pi_{j=1}^{\infty}(1-e^{-s_j})\Gamma_s(T)$, then ρ is a density matrix on $\Gamma_s(\mathcal{H})$. We have $\mathcal{H} = \oplus_j \mathbb{C}e_j$. Since $\Gamma_s(e^{-s_j}) = e^{-s_j a_j^\dagger a_j}$ on $\Gamma_s(\mathbb{C}e_j)$, under the isomorphisms described in Proposition 1.3.2, $\rho_D = \Pi_{j=1}^{\infty}(1-e^{-s_j})\Gamma_s(\oplus_j e^{-s_j}) = \otimes_j^{\infty}\rho_j$, where $\rho_j = (1-e^{-s_j})e^{-s_j a_j^\dagger a_j}$ and $\otimes_{j=1}^{\infty}\rho_j$ is defined as the strong limit of the states $\rho^N \in \mathcal{B}(\otimes_j \Gamma_s(\mathbb{C}e_j))$ defined for each $N \in \mathbb{N}$ as

$$\rho^N := \rho_1 \otimes \rho_2 \otimes \cdots \rho_N \otimes |e(0)\rangle\langle e(0)| \otimes |e(0)\rangle\langle e(0)| \otimes \cdots . \tag{3.4.3}$$

Let $x = \oplus_j x_j e_j$, then define for each $N \in \mathbb{N}$,

$$W^N(x) = W(x_1) \otimes W(x_2) \otimes W(x_N) \otimes I \otimes I \otimes \cdots . \tag{3.4.4}$$

Now we can can compute the quantum Fourier transform of ρ_D.

$$\begin{aligned}
\hat{\rho}_D(x) &= \operatorname{Tr} \rho W(x) \\
&= \operatorname{Tr}\left(\Pi_{j=1}^{\infty}(1-e^{-s_j})\Gamma_s(\oplus_j e^{-s_j})W(\oplus_j x_j)\right) \\
&= \operatorname{Tr}\left(\otimes_j (1-e^{-s_j})\Gamma_s(e^{-s_j})W(x_j)\right) \\
&= \operatorname{Tr}\left(\otimes_j (1-e^{-s_j})e^{-s_j a_j^\dagger a_j}W(x_j)\right) \\
&= \operatorname{Tr}\left(\operatorname*{s-lim}_N \rho^N W^N(x)\right) \\
&= \lim_N \operatorname{Tr} \rho^N W^N(x) \\
&= \lim_N \Pi_{j=1}^{N} \operatorname{Tr}\left((1-e^{-s_j})e^{-s_j a_j^\dagger a_j}W(x_j)\right) \\
&= \Pi_{j=1}^{\infty} e^{-\left\langle x_j, \frac{1}{2}\coth\left(\frac{s_j}{2}\right)x_j\right\rangle} \qquad (3.4.5) \\
&= e^{-\frac{1}{2}\langle x, Dx\rangle}
\end{aligned}$$

where (3.4.5) follows from (iii) of Examples in Section 3.2. □

Recall from (i) of Examples in Section 3.2 that the vacuum state $|e(0)\rangle\langle e(0)|$ on $\Gamma_s(\mathcal{H})$ is a Gaussian state with covariance operator I.

Theorem 3.4.1. *If $\mathcal{P}$ is any complex linear operator on $\mathcal{H}$ such that $\mathcal{P} - I$ is positive and trace class, then there exists a state ρ on $\Gamma_s(\mathcal{H})$ such that the quantum characteristic function $\hat{\rho}$ associated with ρ is given by*

$$\hat{\rho}(x) = e^{-\frac{1}{2}\langle x, \mathcal{P}x\rangle}$$

for every $x \in \mathcal{H}$.

Proof. Let U be a unitary operator such that $\mathcal{P} = U^*DU$. Such a U exists by applying spectral theorem to the compact positive operator $\mathcal{P} - I$. Since $\mathcal{P} \geq I$ assume without loss of generality that $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ is such that $D = \left[\begin{smallmatrix} D_1 & 0 \\ 0 & I \end{smallmatrix}\right]$, where we seperated all the diagonal entries of D which are equal to one and not equal to one. Then D_1 satisfies the assumptions in Proposition 3.4.1 and ρ_{D_1} exists as a Gaussian state on $\Gamma_s(\mathcal{H}_1)$. Let ρ_0 denote the vacuum state $|e(0)\rangle\langle e(0)|$ on $\Gamma_s(\mathcal{H}_2)$, which is Gaussian by Example (i). Then by Proposition 3.2.5, $\rho_{D_1} \otimes \rho_0 = \rho_g(0, D)$. Define $\rho = \Gamma_s(U^*)\rho_{D_1} \otimes \rho_0 \Gamma_s(U)$ and the result follows from Proposition 3.2.3. □

Lemma 3.4.3. *Let $C - I$ be Hilbert-Schmidt (trace class), then*

(i) If $C \geq 0$ then $\sqrt{C} - I$ is Hilbert-Schmidt (trace class).
(ii) If C is invertible then $C^{-1} - I$ is Hilbert-Schmidt (trace class).

Lemma 3.4.4. *Let S be a real linear, positive and invertible operator on $\mathcal{H}$. Then L and $\mathcal{P}$ as in Corollary 2.6.1 can be chosen such that*

(i) If $S - I$ is Hilbert-Schmidt then $L^T L - I$ is Hilbert Schmidt, i.e $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$.
(ii) If $(\sqrt{S}J\sqrt{S})^T(\sqrt{S}J\sqrt{S}) - I$ is trace class then $\mathcal{P} - I$ is a trace class operator on $\mathcal{K}$.

Proof. (i). It can be seen from the proof of Williamson's normal form in Theorem 2.6.1 that L can be chosen as $L = \mathcal{P}^{-1/2}\Gamma^T S^{1/2}$, where $\Gamma_0 \colon K \oplus K \to H \oplus H$ is an orthogonal transformation such that the skew symmetric operator

$$B_0 := S_0^{1/2} J_0 S_0^{1/2} = \Gamma_0 \left[\begin{smallmatrix} 0 & -P \\ P & 0 \end{smallmatrix}\right] \Gamma_0^T \tag{3.4.6}$$

and $\mathcal{P}_0 = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$. Then

$$L^T L = S^{1/2}\Gamma\mathcal{P}^{-1}\Gamma^T S^{1/2}. \tag{3.4.7}$$

But

$$\left[\begin{smallmatrix} 0 & P \\ -P & 0 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 & -P \\ P & 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} P^2 & 0 \\ 0 & P^2 \end{smallmatrix}\right], \tag{3.4.8}$$

therefore if we write $\mathsf{P}_0 = \left[\begin{smallmatrix} 0 & -P \\ P & 0 \end{smallmatrix}\right]$, we see that

$$\mathcal{P}^{-1} = (\sqrt{\mathsf{P}^T\mathsf{P}})^{-1}. \tag{3.4.9}$$

Since Γ is orthogonal, by (3.4.6) and (3.4.9) we get $(\sqrt{B^T B})^{-1} = \Gamma\mathcal{P}^{-1}\Gamma^T$. Now by (3.4.7), we get

$$L^T L = S^{1/2}(\sqrt{B^T B})^{-1} S^{1/2}. \tag{3.4.10}$$

We have $S-I$ is Hilbert-Schmidt. Therefore, so is J^TSJ-I. Hence $S^{1/2}J^TSJS^{1/2}-S$ is Hilbert-Schmidt. By adding and subtracting I and using the fact the $S-I$ is Hilbert-Schmidt we get $S^{1/2}J^TSJS^{1/2}-I$ is also so. In other words, we just got B^TB-I is Hilbert-Schmidt. Now by Lemma 3.4.3 we get $(\sqrt{B^TB})^{-1}-I$ is Hilbert-Schmidt. This along with (3.4.10) finally allows us to conclude that L^TL-I is Hilbert-Schmidt.

(ii). By keeping the notations above and using Lemma 3.4.3, we have $(\sqrt{B^TB})^{-1}-I$ is trace class and thus $S^{1/2}(\sqrt{B^TB})^{-1}S^{1/2}-S=L^TL-S$ is trace class. Since $S=L^T\mathcal{P}L$ we get $L^T(\mathcal{P}-I)L$ is trace class. Since L is invertible we see that $\mathcal{P}-I$ is trace class. □

The next theorem shows that the necessary conditions we have on the covariance operator in Theorem 3.3.1 are sufficient for the existence of a Gaussian state with the given operator as the covariance operator.

Theorem 3.4.2. *Let S be a real linear invertible operator on $\mathcal{H}$ such that*

(i) $\hat{S}-i\hat{J}\geq 0$ *on* $\hat{\mathcal{H}}$.
(ii) $S-I$ *is Hilbert-Schmidt on* $(\mathcal{H},\operatorname{Re}\langle\cdot,\cdot\rangle)$.
(iii) $(\sqrt{S}J\sqrt{S})^T(\sqrt{S}J\sqrt{S})-I$ *is trace class on* $(\mathcal{H},\operatorname{Re}\langle\cdot,\cdot\rangle)$.

Then there exists a density matrix ρ on $\Gamma_s(\mathcal{H})$ such that the quantum characteristic function $\hat{\rho}(z)=e^{-\frac{1}{2}\operatorname{Re}\langle z,Sz\rangle}$ i.e., $S\in\mathcal{C}(\mathcal{H})$.

Proof. Since $\hat{S}-i\hat{J}\geq 0$, $S\geq 0$. Since S is invertible, we apply Williamson's normal form to it. Thus there exists a Hilbert space $\mathcal{K}$ and a symplectic transformation $L\colon\mathcal{H}\to\mathcal{K}$ such that $S=L^T\mathcal{P}L$ (Corollary 2.6.1). By Lemma 3.4.4 $\mathcal{P}-I$ is trace class and L^TL-I is Hilbert-Schmidt. Now by Theorem 1.8.1, there exists a unique unitary operator $\Gamma_s(L)\colon\Gamma_s(\mathcal{H})\to\Gamma_s(\mathcal{K})$ such that

$$\Gamma_s(L)W(u)\Gamma_s(L)^*=W(Lu).\tag{3.4.11}$$

It is understood that $W(\cdot)$ on either side of the above equality are considered in the corresponding Fock spaces.

Since $\mathcal{P}-I$ is trace class and positive, by Theorem 3.4.1 there exists a density matrix $\rho_{\mathcal{P}}$ such that $\hat{\rho}_{\mathcal{P}}(y)=e^{-\frac{1}{2}\langle y,\mathcal{P}y\rangle}$ for every $y\in\mathcal{K}$. Define

$$\rho=\Gamma_s(L)^*\rho_{\mathcal{P}}\Gamma_s(L).\tag{3.4.12}$$

Claim. $\hat{\rho}(z)=e^{-\frac{1}{2}\operatorname{Re}\langle z,Sz\rangle}$ for every $z\in\mathcal{H}$.

Proof (of Claim). By Proposition 3.2.3, we have

$$\begin{aligned}\hat{\rho}(z) &= \hat{\rho}_{\mathcal{P}}(Lz)\\ &= e^{-\frac{1}{2}\langle Lz,\mathcal{P}Lz\rangle}\\ &= e^{-\frac{1}{2}\operatorname{Re}\langle Lz,\mathcal{P}Lz\rangle}\\ &= e^{-\frac{1}{2}\operatorname{Re}\left\langle z,L^T\mathcal{P}Lz\right\rangle}\\ &= e^{-\frac{1}{2}\operatorname{Re}\langle z,Sz\rangle}.\end{aligned}$$

□

Thus by combining Theorem 3.3.1 and Theorem 3.4.2, we have Theorem 3.2.1.

Corollary 3.4.1. *Let S be a complex linear positive and invertible operator on $\mathcal{H}$, then $S \in \mathcal{C}(\mathcal{H})$ if and only if $\hat{S} - i\hat{J} \geq 0$ and $S - I$ is trace class.*

Note that by Theorem 3.3.2, the condition $\hat{S} - i\hat{J} \geq 0$ is equivalent to the existence of a primary quasifree state ϕ on $CCR(H,\sigma)$ such that $\phi(W(z)) = e^{-\frac{1}{2}\operatorname{Re}\langle z,Sz\rangle}$. Further, by Theorem 1.5.3, along with (ii) and (iii) we infer that this ϕ is quasiequivalent to the vacuum state. So a restatement of Theorem 3.4.2 is as follows.

Theorem 3.4.3. *A primary quasifree state which is quasi equivalent to the vacuum state extends uniquely to a normal state in the GNS representation corresponding to the vacuum state (Section 1.6) and the this extended state can be constructed explicitly on $\mathcal{B}(\Gamma_s(\mathcal{H}))$.*

Corollary 3.4.2. *Let S be a complex linear, self-adjoint and invertible operator on $\mathcal{H}$. Then S is the covariance operator of a quantum Gaussian state on $\Gamma_s(\mathcal{H})$ if and only if $\hat{S} - i\hat{J} \geq 0$ and $S - I$ is trace class.*

Corollary 3.4.3. *Let $S \geq I$ be real linear then S is the covariance operator of a quantum Gaussian state on $\Gamma_s(\mathcal{H})$ if and only if $S - I$ is trace class.*

The following theorem is another characterization of Gaussian states in terms of the quantum Fourier transform.

Theorem 3.4.4. *There exists a quantum Gaussian state ρ with covariance matrix S if and only if $\hat{\rho}_{|_{CCR(\mathcal{H},\sigma)}}$ is a primary quasifree state ϕ_A quasiequivalent to the vacuum state ϕ_{-J} on $CCR(\mathcal{H},\sigma)$, where $A = -JS$.*

Proof. If S is a covariance operator, by Theorem 3.3.2 and proofs of (ii) and (iii) in Theorem 3.3.1, we see the existence of the required quasifree state. It can be seen from Theorem 1.5.3, the same proofs mentioned above and Theorem 3.2.1, that the existence of a quasifree state as in the statement gives rise to the existence of the required quantum Gaussian state. □

Note. By Theorem 3.4.4, it is established that quantum Gaussian states are characterized by a subclass of quasifree states on $CCR(\mathcal{H}, \sigma)$. More clearly, quantum Gaussian states are precisely those density matrices whose quantum characteristic function define a quasifree states on $CCR(\mathcal{H}, \sigma)$ which are quasi equivalent to the vacuum state ϕ_{-J}.

CHAPTER 4

The Symmetry Group of Quantum Gaussian States

The previous chapter described and characterized infinite mode Gaussian states. In this chapter, we extend some beautiful convexity and symmetry properties of Gaussian states proved by Parthasarathy [Par13b] in the finite mode case, to this setting. We also present a structure theorem for Gaussian states. The methods similar to those of Parthasarathy also work in this situation because we have the right machinery at our disposal from the previous chapters. Recall that $\mathcal{H}, \mathcal{K}$ are separable complex Hilbert spaces, $\mathcal{C}(\mathcal{H})$ denotes the collection of covariance operators for Gaussian states on $\Gamma_s(\mathcal{H})$ and $\mathcal{S}(\mathcal{H}, \mathcal{K})$ are Shale operators from $\mathcal{H}$ to $\mathcal{K}$.

4.1 Convexity Properties of Covariance Operators

The following proposition helps us to prove that $\mathcal{C}(\mathcal{H})$ is a convex set.

Proposition 4.1.1. *Consider two mean zero Gaussian states*

$$\rho_i = \rho_g(0, S_i), i = 1, 2$$

on $\Gamma_s(\mathcal{H})$. For $\theta \in \mathbb{R}$, let U_θ be the unitary operator $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ on $\mathcal{H} \oplus \mathcal{H}$. Then

$$\mathrm{Tr}_2\left(\Gamma_s(U_\theta)(\rho_1 \otimes \rho_2)\Gamma_s(U_\theta)^*\right) = \rho_g(0, (\cos^2\theta)S_1 + (\sin^2\theta)S_2)$$

where Tr_2 denotes the relative trace over the second factor of $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$ and $\Gamma_s(U_\theta)$ is considered under the identification between $\Gamma_s(\mathcal{H} \oplus \mathcal{H})$ and $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$.

Proof. The proof is an easy consequence of Proposition 3.2.2 and Proposition 3.2.6 and the definition of Gaussian states. We note that $(\cos^2\theta)S_1 + (\sin^2\theta)S_2$ is a convex combination of two positive and invertible operators and hence it is positive and invertible. This last result can be seen by using the numerical range. If S is positive and invertible, the numerical range $W(S)$ is defined as $W(S) := \{\langle x, Sx\rangle : \|x\| = 1\}$. It is easy to prove that elements of $W(S)$ are strictly positive and away from zero in this case. Further, it can be proved that the spectrum $\sigma(S) \subseteq \overline{W(S)}$. It is also direct to see that the numerical range of convex combination of positive invertible operators lie bounded away from zero in the positive half of the real line. $\square$

As a consequence, we have the following result.

Corollary 4.1.1. *$\mathcal{C}(\mathcal{H})$ is a convex set.*

Now we proceed to describe the extreme points of $\mathcal{C}(\mathcal{H})$.

Lemma 4.1.1. *Let $P \geq I$ be a positive operator, then there exists invertible positive operators P_1 and P_2 such that*

$$P = \frac{1}{2}(P_1 + P_2) = \frac{1}{2}(P_1^{-1} + P_2^{-1}). \tag{4.1.1}$$

Proof. Take $P_1 = P + \sqrt{P^2 - I}$ and $P_2 = P - \sqrt{P^2 - I}$. Then $P_1 P_2 = P_2 P_1 = I$ and (4.1.1) is satisfied. $\square$

Recall the definition $\mathcal{S}(\mathcal{H})$ of Shale operators in Section 1.8 and the notations developed in Section 1.7, we have

Lemma 4.1.2. *Let $\mathcal{H} = H + iH$ and $\mathcal{P} \in \mathcal{B}(\mathcal{H})$ be such that $\mathcal{P} - I$ is positive and trace class, further let $\mathcal{P}_0 = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$ on $H \oplus H$ (Section 1.7). Then $\mathcal{P} = \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2)$, for some $\mathcal{P}_j \geq 0$, and $\mathcal{P}_j^{\frac{1}{2}} \in \mathcal{S}(\mathcal{H})$, $j = 1, 2$.*

Proof. Take $P_1 = P + \sqrt{P^2 - I}$ and $P_2 = P - \sqrt{P^2 - I}$, then by (4.1.1),

$$\mathcal{P}_0 = \frac{1}{2}\left\{\begin{bmatrix} P_1 & 0 \\ 0 & P_1^{-1} \end{bmatrix} + \begin{bmatrix} P_2 & 0 \\ 0 & P_2^{-1} \end{bmatrix}\right\}.$$

Define $\mathcal{P}_j$ such that $\mathcal{P}_j(x + iy) = P_j x + P_j^{-1} y, \forall x, y \in H, j = 1, 2$. Then $\mathcal{P}_j$ is symplectic and positive. To prove $\mathcal{P}_j^{\frac{1}{2}} \in \mathcal{S}(\mathcal{H})$, it is enough to show that $P_j - I$ is Hilbert-Schmidt, $j = 1, 2$. Since $P - I$ is trace class (and hence Hilbert-Schmidt) it is enough to show $\sqrt{P^2 - I}$ is Hilbert-Schmidt or equivalently $P^2 - I$ is is trace class. This is true because $P^2 - I = (P - I)^2 + 2(P - I)$. $\square$

Theorem 4.1.1. *$S \in \mathcal{C}(\mathcal{H})$ if and only if*

$$S = \frac{1}{2}(N^T N + M^T M) \tag{4.1.2}$$

for some $N, M \in \mathcal{S}(\mathcal{H}, \mathcal{K})$, for some Hilbert space $\mathcal{K}$. Further, S is an extreme point of $\mathcal{C}(\mathcal{H})$ if and only if $S = N^T N$ for some $N \in \mathcal{S}(\mathcal{H}, \mathcal{K})$.

Proof. Note that if $N \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ for some $\mathcal{K}$ then $N^T N$ is a covariance operator by taking $S = I$ and $L = N$ in Proposition 3.2.6. Therefore by convexity of $\mathcal{C}(\mathcal{H})$, if S is of the form (4.1.2) then $S \in \mathcal{C}(\mathcal{H})$.

Now let $S \in \mathcal{C}(\mathcal{H})$, let $S = L^T \mathcal{P} L$ be the Williamson's normal form as in Corollary 2.6.1. Then by Lemma 3.4.4 and Lemma 3.3.2 we have $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$ and $\mathcal{P} - I$ is trace class and positive. By Corollary 2.6.1 we have $\mathcal{P}_0 = \left[\begin{smallmatrix} P & 0 \\ 0 & P \end{smallmatrix}\right]$. By Lemma 4.1.2, $\mathcal{P} = \frac{1}{2}(\mathcal{P}_1 + \mathcal{P}_2)$ with $\mathcal{P}_j \geq 0, j = 1, 2$. Therefore we have

$$S = \frac{1}{2} L^T (\mathcal{P}_1 + \mathcal{P}_2) L.$$

By taking $N = \mathcal{P}_1^{1/2} L$ and $M = \mathcal{P}_2^{1/2} L$ we get (4.1.2). An easy computation shows $N, M \in \mathcal{S}(\mathcal{H}, \mathcal{K})$.

The proof of second part of the Theorem goes in similar lines to the proof of the similar statement in the finite mode case, Theorem 3 in [Par13b]. We give it here for completeness. The first part also shows that for an element S of $\mathcal{C}(\mathcal{H})$ to be extremal it is necessary that $S = L^T L$ for some Shale operator L. To prove sufficiency, suppose there exist a Shale operator L and $S_1, S_2 \in \mathcal{C}(\mathcal{H})$ such that

$$L^T L = \frac{1}{2}(S_1 + S_2).$$

By the first part of the theorem there exist Shale operators L_j such that

$$L^T L = \frac{1}{4} \sum_{j=1}^{4} L_j^T L_j \tag{4.1.3}$$

where $S_1 = \frac{1}{2}(L_1^T L_1 + L_2^T L_2)$, $S_2 = \frac{1}{2}(L_3^T L_3 + L_3^T L_3)$. Left multiplication by $(L^T)^{-1}$ and right multiplication by L^{-1} on both sides of (4.1.3) gives

$$I = \frac{1}{4} \sum_{j=1}^{4} M_j \tag{4.1.4}$$

where

$$M_j = (L^T)^{-1} L_j^T L_j L^{-1}.$$

Each M_j is a positive Shale operator. Multiplying by J on both sides of (4.1.4). We get

$$\begin{aligned} J &= \frac{1}{4}\sum_{j=1}^{4} M_j J \\ &= \frac{1}{4}\sum_{j=1}^{4} M_j J M_j M_j^{-1} \\ &= \frac{1}{4} J \sum_{j=1}^{4} M_j^{-1}. \end{aligned}$$

Thus

$$I = \frac{1}{4}\sum_{j=1}^{4} M_j = \frac{1}{4}\sum_{j=1}^{4} M_j^{-1} = \frac{1}{4}\sum_{j=1}^{4} \frac{1}{2}(M_j + M_j^{-1}),$$

which implies

$$\sum_{j=1}^{4} (M_j^{1/2} - M_j^{-1/2})^2 = 0,$$

or

$$M_j = I, \quad \forall 1 \leq j \leq 4.$$

Thus

$$L_j^T L_j = L^T L, \quad \forall j$$

and $S_1 = S_2$. □

Remark 6. The relation (4.1.2) is called the circle property because every point in the convex set $\mathcal{C}(\mathcal{H})$ is the midpoint of a line joining two extreme points.

Corollary 4.1.2. *Let S_1, S_2 be extreme points of $\mathcal{C}(\mathcal{H})$. If $S_1 \geq S_2$ then $S_1 = S_2$.*

Proof. By Theorem 4.1.1, let $S_1 = L_1^T L_1$ and $S_2 = L_2^T L_2$ for some $L_1 \in \mathcal{S}(\mathcal{H}, \mathcal{K}_1)$ and $L_2 \in \mathcal{S}(\mathcal{H}, \mathcal{K}_2)$. Without loss of generality we assume that $\mathcal{K}_1 = \mathcal{K}_2$. This can be done by going through the proof of previous theorem and identifying $\mathcal{K}_1$ and $\mathcal{K}_2$ in such a way that the Willliamson's normal form of both S_1 and S_2 are obtained in the same Hilbert space $\mathcal{K}$ (by a possibly different real and complex decomposition of $\mathcal{K}$). $L_1^T L_1 \geq L_2^T L_2$ implies that the symplectic transformation $M := L_2 L_1^{-1}$(well defined because $\mathcal{K}_1 = \mathcal{K}_2$) has the property $M^T M \leq I$. But since $M^T M$ is a positive symplectic automorphism $M^T M = VTV^*$ for some unitary V, where $\mathcal{H} = H + iH$ and $T(x + iy) = Ax + iA^{-1}y$ for some positive invertible operator A on H. But for such a T, $T \leq I$ if and only if $A = I$. This proves $M^T M = I$. But this implies $L_2^T L_2 = L_1^T L_1$ from the definition of M. □

4.2 Structure of Quantum Gaussian States

In this section, we prove a structure theorem for Gaussian states, we will see that any Gaussian state is a Weyl, Shale conjugation of the fundamental examples of Gaussian states we provided in Section 3.2.

Let S be a Gaussian covariance operator. It satisfies the properties listed in Theorem 3.2.1. Then by combining Lemma 3.3.2 and Lemma 3.4.4 we get a Williamson's normal form (Corollary 2.6.1), $S = L^T \mathcal{P} L$ such that $\mathcal{P} - I$ is positive and trace class. By applying spectral theorem to $\mathcal{P} - I$ we see that there exists a unitary U such that $\mathcal{P} = U^* D U$, where D is diagonal and positive. Recall that a unitary is already symplectic. Therefore, whenever S is a covariance operator we can assume without loss of generality that the $\mathcal{P} \in \mathcal{B}(\mathcal{K})$ occurring in the Williamson's normal form is of the form $\mathcal{P} = \left[\begin{smallmatrix} D & 0 \\ 0 & I \end{smallmatrix}\right]$ on a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, with $D = \mathrm{Diag}(d_1, d_2, \dots)$, $d_1 \geq d_2 \geq \cdots > 1$. Fix a basis of $\mathcal{K}$ in this way. Now consider the identification of $\Gamma_s(\mathbb{C})$ with $L^2(\mathbb{R})$, where $e(z) \in \Gamma_s(\mathbb{C})$ is identified with the L^2-function $x \mapsto (2\pi)^{-1/4} \exp\{-4^{-1}x^2 + zx - 2^{-1}z^2\}$ (Example 1 in Chapter 1). Therefore, we can assume without loss of generality that $\Gamma_s(K) = \otimes_j L^2(\mathbb{R})$, with respect to the stabilizing vector $e(0)$. It may be noted that making these identifications does not alter $\Gamma_s(L)$.

Theorem 4.2.1. *Let $\rho_g(w, S)$ be a Gaussian state on $\Gamma_s(\mathcal{H})$. Let $S = L^T \mathcal{P} L$ be a Williamson's normal form of S, where $L : \mathcal{H} \to \mathcal{K}$, with $L^T L - I$ is Hilbert-Schmidt (i.e., $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$) and $\mathcal{P} = \left[\begin{smallmatrix} D & 0 \\ 0 & I \end{smallmatrix}\right]$, on a decomposition $\mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2$, with $D = \mathrm{Diag}(d_1, d_2, \dots)$, $d_1 \geq d_2 \geq \cdots > 1$, $d_j = \coth\left(\frac{s_j}{2}\right)$, $\forall j$. Then*

$$\rho_g(w, S) = W\left(\frac{-i}{2}w\right)^* \Gamma_s(L)^* [\otimes_j (1 - e^{-s_j}) e^{-s_j a_j^\dagger a_j} \otimes \rho_0] \Gamma_s(L) W\left(\frac{-i}{2}w\right). \tag{4.2.1}$$

where $\rho_0 = |e(0)\rangle\langle e(0)|$ is the the vacuum state on $\Gamma_s(\mathcal{K}_2)$.

Proof. By Proposition 3.2.4, $\rho_g(w, S) = W(\frac{-i}{2}w)^{-1} \rho_g(0, S) W(\frac{-i}{2}w)$. Since $S = L^T \mathcal{P} L$, by Proposition 3.2.6, $\rho_g(0, S) = \Gamma_s(L)^* \rho_g(0, \mathcal{P}) \Gamma_s(L)$. Since $\mathcal{P} = D \oplus I$, by Proposition 3.2.5, $\rho_g(0, \mathcal{P}) = \rho_g(0, D) \otimes \rho_g(0, I)$. But $\rho_g(0, D) = \otimes_j (1 - e^{-s_j}) e^{-s_j a_j^\dagger a_j}$ since both on left and right hand sides have same quantum characteristic function by proof of Proposition 3.4.1 and it is obvious that $\rho_g(0, I) = \rho_0$. $\square$

Corollary 4.2.1. *If $\{e_j\}$ is a basis of $\mathcal{H}$, consider $\Gamma_s(\mathcal{H}) = \otimes_j L^2(\mathbb{R})$, then the wave function of a general pure quantum Gaussian state is of the form*

$$|\psi\rangle = W(\alpha)^{-1} \Gamma_s(U) (\otimes_j \left| e_{\lambda_j} \right\rangle) \tag{4.2.2}$$

where $e_\lambda \in L^2(\mathbb{R})$ *and*

$$e_\lambda(x) = (2\pi)^{-1/4}\lambda^{-1/2}\exp\Big\{-4^{-1}\lambda^{-2}x^2\Big\}, \quad x \in \mathbb{R}, \lambda > 0,$$

$\alpha \in \mathcal{H}$, *U is a unitary operator on* $\mathcal{H}$, $\Gamma_s(U)$ *is the second quantization unitary operator associated with* U *and* $\lambda_j, j \in \mathbb{N}$ *are positive scalars.*

Proof. This proof follows along similar lines to the proof of Corollary 2 in [Par13b]. We know that the spectrum of the number operator $a^\dagger a$ is the set $\{0, 1, 2, \dots\}$. Therefore it follows that, in Theorem 4.2.1 the part involving the number operator is not there. Thus any pure state is just a Shale and Weyl conjugation of a vacuum state. Therefore

$$|\psi\rangle = W(\alpha)^{-1}\Gamma_s(L)\,|e(0)\rangle \tag{4.2.3}$$

for some $L \in \mathcal{S}(\mathcal{H}, \mathcal{K})$. The covariance operator of this pure state is $L^T L$. By Proposition 1.7.3, L_0 (recall the notation introduced in Section 1.7) can be decomposed as

$$L_0 = U_0 \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} V_0,$$

where U_0 and V_0 are orthogonal transformations. But now

$$\begin{aligned} L_0^T L_0 &= V_0 \begin{bmatrix} A^2 & 0 \\ 0 & A^{-2} \end{bmatrix} V_0 \\ &= N_0^T N_0, \end{aligned}$$

where

$$N_0 = \begin{bmatrix} A & 0 \\ 0 & A^{-1} \end{bmatrix} V_0.$$

Since $L^T L - I$ is Hilbert-Schmidt we can choose A to be diagonal without loss of generality. Since the covariance matrix of $|\psi\rangle\langle\psi|$ can also be written as $N^T N$, ψ in (4.2.3) can also be written as

$$|\psi\rangle = W(\alpha)^{-1}\Gamma_s(V)\Gamma_s(T)\,|e(0)\rangle\,, \tag{4.2.4}$$

where T is such that $T_0 = \left[\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right]$. T is unitarily equivalent to $\oplus T^j$ where $T_0^j = \left[\begin{smallmatrix} \lambda_j & 0 \\ 0 & \lambda_j^{-1} \end{smallmatrix}\right]$. Finally $\Gamma_s(T) = \otimes\Gamma_s(T^j)$ upto a conjugation with a second quantization unitary. Equation (4.2.2) follows by taking $\left|e_{\lambda_j}\right\rangle = \Gamma_s(T_j)\,|e(0)\rangle$ via the identification of $\Gamma_s(\mathbb{C})$ with $L^2(\mathbb{R})$. $\square$

Now we show that all Gaussian states can be purified to get pure Gaussian states.

Theorem 4.2.2 (Purification). *Let ρ be a mixed Gaussian state in $\Gamma_s(\mathcal{H})$. Then there exists a pure Gaussian state $|\psi\rangle$ in $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$ such that*

$$\rho = \mathrm{Tr}_2\, U\, |\psi\rangle\langle\psi|\, U^*$$

where U is a unitary and Tr_2 is the relative trace over the second factor.

Proof. Proof is similar to that of Theorem 5 in [Par13b]. Let $\rho = \rho_g(w, S)$. By Theorem 4.1.1, $S = \frac{1}{2}(L_1^T L_1 + L_2^T L_2)$, $L_j \in \mathcal{S}(\mathcal{H}, \mathcal{K}), j = 1, 2$. Now consider the pure Gaussian states $\Gamma_s(L_j)^* |e(0)\rangle, j = 1, 2$ in $\Gamma_s(\mathcal{H})$. Let Γ_0 be the second quantization unitary satisfying

$$\Gamma_0 e(u \oplus v) = e\left(\frac{u+v}{\sqrt{2}} \oplus \frac{u-v}{\sqrt{2}}\right), \forall u, v \in \mathcal{H}$$

in $\Gamma_s(\mathcal{H} \oplus \mathcal{H})$ identified with $\Gamma_s(\mathcal{H}) \otimes \Gamma_s(\mathcal{H})$, so that $e(u \oplus v) = e(u) \otimes e(v)$. Then by Proposition 4.1.1, we have

$$\mathrm{Tr}_2\, \Gamma_0(|\psi_{L_1}\rangle\langle\psi_{L_1}| \otimes |\psi_{L_2}\rangle\langle\psi_{L_2}|) = \rho_g(0, S).$$

Further by Proposition 3.2.4, there exists $\alpha \in \mathcal{H}$ such that

$$W(\alpha)\rho_g(0, S)W(\alpha)^{-1} = \rho_g(w, S).$$

Putting $U = (W(\alpha) \otimes I)\Gamma_0(\Gamma_s(L_1)^{-1} \otimes \Gamma_s(L_2)^{-1})$, we get

$$\rho_g(w, S) = \mathrm{Tr}_2\, U\, |e(0) \otimes e(0)\rangle\langle e(0) \otimes e(0)|\, U^*,$$

where $e(0)$ is the exponential vector in $\Gamma_s(\mathcal{H})$. □

4.3 Symmetry group of Gaussian states

Let $\mathcal{H}$ be a complex separable infinite dimensional Hilbert space and let $\mathcal{G}(\mathcal{H})$ denote the set of all Gaussian states on $\Gamma_s(\mathcal{H})$.

Definition 4.3.1. A unitary operator U on $\Gamma_s(\mathcal{H})$ is called a *Gaussian symmetry* if $U\rho U^* \in \mathcal{G}(\mathcal{H})$ for every $\rho \in \mathcal{G}(\mathcal{H})$.

We will give a complete characterization of Gaussian symmetries. This will be achieved in Theorem 4.3.2. We need some preliminary results for proving this. Towards this end, let $\mathbb{Z}_+$ denote the set $\{0, 1, 2, 3, \dots\}$ and take

$$\mathbb{Z}_+^\infty := \{(k_1, k_2, \dots, k_n, 0, 0, \dots)^T | k_j \in \mathbb{Z}_+, j, n \in \mathbb{N}\}.$$

Let $\{e_j\}_{j\in\mathbb{N}}$ denote the standard orthonormal basis for $\ell^2(\mathbb{N})$, where e_j is the column vector with 1 at the j^{th} position and zero elsewhere. An infinite order matrix A is said to be a permutation matrix if A is the matrix with respect to the standard orthonormal basis, corresponding to a unitary operator which maps $\{e_j\}$ to itself.

Lemma 4.3.1. *Let $\{s_j\}_{j\in\mathbb{N}}$ and $\{t_j\}_{j\in\mathbb{N}}$ be two sets consisting of positive numbers such that*

$$\left\{\sum_{j=1}^{n} s_j k_j | k_j \in \mathbb{Z}_+ \forall j, n \in \mathbb{N}\right\} = \left\{\sum_{j=1}^{n} t_j k_j | k_j \in \mathbb{Z}_+ \forall j, n \in \mathbb{N}\right\}. \tag{4.3.1}$$

If $\{s_j\}$ and $\{t_j\}$ are linearly independent over the field $\mathbb{Q}$, then $\{s_j\} = \{t_j\}$.

Proof. Consider $s = (s_1, s_2, s_3, \dots)^T$ and $t = (t_1, t_2, t_3, \dots)^T$. These vectors need not be in ℓ^2, we consider them as formal vectors only. Now it is enough to prove that there exists an infinite permutation matrix A such that $As = t$. Fix $i \in \mathbb{N}$. From (4.3.1) there exists $n_i, m_i \in \mathbb{N}$ and $a_{ij}, b_{ij} \in \mathbb{Z}_+$ with $b_{ik} = 0 = a_{il}$ for all $k > n_i, l > m_i$ such that

$$s_i = \sum_{j=1}^{\infty} t_j b_{ij} \quad \text{and} \quad t_i = \sum_{j=1}^{\infty} s_j a_{ij}.$$

Set $A = ((a_{ij}))$ and $B = ((b_{ij}))$. Note that, by construction, each row of A, B, AB and BA has only finitely many non zero entries. Clearly $As = t, Bt = s$ and hence $BAs = s$. Since each row of BA has only finitely many non zero entries, by the rational linear independence of $\{s_j\}$, we get $BA = I$. Similarly since $ABt = t$, we get $AB = I$. Now $BA = I$ implies that $\sum\limits_{j=1}^{\infty} b_{1j}a_{j1} = 1$ and $\sum\limits_{j=1}^{\infty} b_{ij}a_{j1} = 0$ for all $i \neq 1$. Now since $a_{ij}, b_{ij} \in \mathbb{Z}_+$, there exists $k_1 \in \mathbb{N}$ such that $b_{1k_1} = a_{k_11} = 1$. Since $b_{ik_1}a_{k_11} = 0$ we have $b_{ik_1} = 0$ for all $i \neq 1$. Thus k_1-th column of B is e_1. Similarly, if $k \neq 1$, since $\sum\limits_{j=1}^{\infty} b_{1j}a_{jk} = 0$ we get $a_{jk} = 0, \forall j \neq 1$ or row k_1 of A is e_1.

Suppose $k_1, k_2, \dots k_{n-1} \in \mathbb{N}$ are obtained such that $k_i \neq k_j$ for $i \neq j$, column k_i of B (and row k_i of A) is e_i, $1 \leq i \leq n-1$. We prove that there exist $k_n \in \mathbb{N}$ such that $k_n \neq k_i$ for $i < n$ and column k_n of B (and row k_n of A) is e_n. Since $\sum\limits_{j=1}^{\infty} b_{nj}a_{jn} = 1$ there exists $k_n \in \mathbb{N}$ such that $b_{nk_n} = a_{k_nn} = 1$. If $k_n = k_i$ for some $i < n$ then column k_i of B cannot be e_i thus $k_n \neq k_i$ for $i < n$. Now a similar argument as above concludes that column k_n of B (and row k_n of A) is e_n. Thus every e_n occurs at least once in the columns of B and rows of A. Similarly, by considering $AB = I$ we see that every e_n occurs at least once in the columns of A and rows of B. Now to see that A and B are permutation matrices, first note that because $AB = BA = I$, none of the rows (or columns) of B or A can be zero. Further if there exists a row of B where there are two non zero entries, say at the positions l and m then because of the presence of e_l and e_m in the columns of A we see that the product BA cannot be I. Continuing similar arguments it is seen that A and B are permutation matrices. $\square$

Let us fix some notations and conventions before we proceed further. Recall from Exercise 20.18(b) in [Par92] that on $\Gamma_s(\mathbb{C})$, the spectrum of the number operator, $\sigma(a^\dagger a) = \mathbb{Z}_+$, where each $k \in \mathbb{Z}_+$ is an eigenvalue with multiplicity one. Let us denote by $|k\rangle$ the eigenvector corresponding to the eigenvalue k. It is also true that $|0\rangle = e(0)$, the vacuum vector. Further, $\{|k\rangle \,|k \in \mathbb{Z}_+\}$ forms an orthonormal basis for $\Gamma_s(\mathbb{C})$. Now consider $\Gamma_s(\mathcal{H}) = \otimes_{j=1}^{\infty}\Gamma_s(\mathbb{C}e_j)$, where $\{e_j\}$ is an orthonormal basis for $\mathcal{H}$ (recall Proposition 1.3.2). If E denote the orthogonal projection of $\Gamma_{fr}(\mathcal{H})$ (the free Fock space which we didn't define but a standard object in the literature) onto $\Gamma_s(\mathcal{H})$, we define

$$|\mathbf{k}\rangle = E(|k_1\rangle \otimes |k_2\rangle \otimes \cdots \otimes |k_N\rangle \otimes |0\rangle \otimes |0\rangle \otimes \cdots) =: |k_1\rangle\, |k_2\rangle \cdots |k_N\rangle \tag{4.3.2}$$

corresponding to an element $\mathbf{k} = (k_1, k_2, k_3, \dots)^T \in \mathbb{Z}_+^\infty$ where $k_j = 0, \forall j > N$. It can be seen that $\{|\mathbf{k}\rangle \,|\mathbf{k} \in \mathbb{Z}_+^\infty\}$ forms an orthonormal basis for $\Gamma_s(\mathcal{H})$. We have

$$(I \otimes I \otimes \cdots \otimes I \otimes a_j^\dagger a_j \otimes I \otimes I \otimes \cdots)(|\mathbf{k}\rangle) = \begin{cases} k_j\,|\mathbf{k}\rangle\,, & \text{if } j \le N \\ 0, & \text{otherwise} \end{cases}, \tag{4.3.3}$$

where $a_j^\dagger a_j$ is the number operator on $\Gamma_s(\mathbb{C}e_j), j \in \mathbb{N}$.

Consider $\mathcal{H} = \oplus_{j=1}^{\infty}\mathcal{H}_j$ where $\mathcal{H}_j$'s are all one dimensional. For a sequence of positive numbers $\{s_j\}_{j\in\mathbb{N}}$ such that $d_j = \coth\left(\frac{s_j}{2}\right) > 1$ and $\sum\limits_j (d_j - 1)$ is finite, we know from Theorem 4.2.1 that, there exists a Gaussian state

$$\rho_s = \Pi_{j=1}^{\infty}(1 - e^{-s_j}) \otimes_{j=1}^{\infty} e^{-s_j a_j^\dagger a_j} \in \mathcal{B}(\Gamma_s(\mathcal{H})). \tag{4.3.4}$$

Then we have

Lemma 4.3.2. *The spectrum of the Gaussian state ρ_s is the closure of the set,*

$$\sigma_{\mathrm{p}}(\rho_s) = \left\{ p e^{-\sum\limits_{j=1}^{N} s_j k_j} \,\middle|\, k_j \in \mathbb{Z}_+, N \in \mathbb{N} \right\}, \tag{4.3.5}$$

where $p := \Pi_{j=1}^{\infty}(1 - e^{-s_j})$. Further, if $\{s_j\}_{j\in\mathbb{N}}$ is a sequence of (distinct) irrational numbers which are linearly independent over the field $\mathbb{Q}$ then each number $p e^{-\sum\limits_{j=1}^{N} s_j k_j}$ is an eigenvalue with multiplicity one.

Proof. Without loss of generality we assume $\mathcal{H} = \ell^2(\mathbb{N})$ and $\mathcal{H}_j = \mathbb{C}e_j$, where $\{e_j\}$ is the standard orthonormal basis of $\ell^2(\mathbb{N})$. We have $\otimes_{j=1}^{\infty} e^{-s_j a_j^\dagger a_j} :=$ s-$\lim_{N\to\infty} \otimes_{j=1}^{N} e^{-s_j a_j^\dagger a_j} \otimes I \otimes I \otimes \cdots$. Therefore,

$$\otimes_{j=1}^{\infty} e^{-s_j a_j^\dagger a_j}(e(u)\otimes e(0)\otimes e(0)\otimes\cdots) = \left(\otimes_{j=1}^{N} e^{-s_j a_j^\dagger a_j} e(u)\right)\otimes e(0)\otimes e(0)\otimes\cdots, \forall u \in \mathbb{C}^N.$$

Thus $\Gamma_s(\mathbb{C}^N)$ is a reducing subspace for ρ_s and $\rho_{s|_{\Gamma_s(\mathbb{C}^N)}} = \otimes_{j=1}^{N} e^{-s_j a_j^\dagger a_j}$, $\forall N$. Therefore,

$$\rho_s(|\mathbf{k}\rangle) = (p \otimes_{j=1}^{\infty} e^{-s_j a_j^\dagger a_j}) |\mathbf{k}\rangle = p e^{-\sum\limits_{j=1}^{\infty} s_j k_j} |\mathbf{k}\rangle, \forall \mathbf{k} \in \mathbb{Z}_+^\infty. \tag{4.3.6}$$

Since $\{|\mathbf{k}\rangle \,|\mathbf{k} \in \mathbb{Z}_+^\infty\}$ forms a complete orthonormal basis for $\Gamma_s(\mathcal{H})$, $\{pe^{-\sum\limits_{j=1}^{N} s_j k_j} \big| k_j \in \mathbb{Z}_+, N \in \mathbb{N}\}$ is the complete set of eigen values for ρ_s. If $\{s_j\}$ is linearly independent over $\mathbb{Q}$, then we see that the eigenvalues corresponding to $|\mathbf{k}_1\rangle \neq |\mathbf{k}_2\rangle$ are not same. Thus the multiplicity of each of these eigenvalues is one. □

The following theorem characterizes all unitaries which transform the particular Gaussian state described in (4.3.4) to a Gaussian state. This will help us to prove our general theorem on Gaussian symmetries.

Theorem 4.3.1. *Let ρ_s be as in (4.3.4) where $\{s_j\}_{j\in\mathbb{N}}$ is a sequence of (distinct) irrational numbers which are linearly independent over the field $\mathbb{Q}$. Then a unitary operator U in $\Gamma_s(\mathcal{H})$ is such that $U\rho_s U^*$ is a Gaussian state if and only if for some $\alpha \in \mathcal{H}$, $L \in \mathcal{S}(\mathcal{H})$ and a complex valued function β of modulus one on $\mathbb{Z}_+^\infty$*

$$U = W(\alpha)\Gamma_s(L)\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots),$$

where $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)$ is the unique unitary which satisfies

$$\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)|\mathbf{k}\rangle = \beta(\mathbf{k})|\mathbf{k}\rangle, \forall \mathbf{k} \in \mathbb{Z}_+^\infty.$$

Proof. Since $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)$ commutes with ρ_s the sufficiency is immediate from Proposition 3.2.6 and Proposition 3.2.4. To prove the necessity let

$$U\rho_s U^* = \rho_g(w, \mathcal{P}'). \tag{4.3.7}$$

The eigenvalues and multiplicities of ρ_s and $U\rho_s U^*$ are same. Therefore by Theorem 4.2.1 there exists $z \in \mathcal{H}$, a Hilbert space $\mathcal{K}$, $M \in \mathcal{S}(\mathcal{H},\mathcal{K})$ and $\rho_t := \Pi_{j=1}^{\infty}(1-e^{-t_j}) \otimes_{j=1}^{\infty} e^{-t_j a_j^\dagger a_j} \in \mathcal{B}(\Gamma_s(\mathcal{K}))$ such that

$$U\rho_s U^* = W(z)^*\Gamma_s(M)^*\rho_t\Gamma_s(M)W(z). \tag{4.3.8}$$

By Lemma 4.3.2, ρ_s has a complete orthonormal eigenbasis with corresponding eigenvalues distinct. By (4.3.8), ρ_s and ρ_t are unitarily equivalent and therefore their eigenvalues and multiplicities are same. In particular, the maximum eigenvalue of ρ_s and that of ρ_t are the same. Therefore by applying Lemma 4.3.2 to ρ_t, we get $\Pi_{j=1}^{\infty}(1-e^{-t_j}) = p$ and ρ_t has a set of distinct eigenvalues $pe^{-\sum\limits_{j=1}^{N} t_j k_j}$ corresponding to the eigenvectors $|\mathbf{k}\rangle$, where $\mathbf{k} = (k_1, k_2, \dots, k_N, 0, 0, \dots)^T \in \mathbb{Z}_+^{\infty}$, $N \in \mathbb{N}$.

Claim. The sequence $\{t_j\}_{j\in\mathbb{N}}$ consists of (distinct) numbers which are linearly independent over the field $\mathbb{Q}$.

Proof (of Claim). If $t_i = t_k$ for some $i \neq k$ then it is possible to choose distinct $\mathbf{k}, \mathbf{k}' \in \mathbb{Z}_{+,0}^{\infty}$ such that the eigenvalues of ρ_t corresponding to $|\mathbf{k}\rangle$ and $|\mathbf{k}'\rangle$ are same. This will imply that the corresponding eigenspace is at least two dimensional which is not possible. To see the rational independence note that for any two finite subsets $I, J \subset \mathbb{N}$, $\sum\limits_{j\in I} t_j k_j \neq \sum\limits_{j\in J} t_j k_j'$ where $k_j, k_j' \in \mathbb{Z}_+, \forall j$. Now if

$$\sum_{j=1}^{N} t_j q_j = 0 \tag{4.3.9}$$

for a finite collection of rational numbers q_j's, since $t_j > 0, \forall j$ then there must be negative rational numbers in the set $\{q_1, q_2, \dots, q_N\}$ (unless $q_j = 0, \forall j$). Then 4.3.9 can be written in the form $\sum\limits_{j\in I} t_j k_j = \sum\limits_{j\in J} t_j k_j'$ for two finite sets I, J, which is not possible. Thus the claim is proved.

We have $\{pe^{-\sum\limits_{j=1}^{N} s_j k_j} \big| k_j \in \mathbb{Z}_+, N \in \mathbb{N}\} = \{pe^{-\sum\limits_{j=1}^{N} t_j k_j} \big| k_j \in \mathbb{Z}_+, N \in \mathbb{N}\}$. Therefore $\{\sum\limits_{j=1}^{n} s_j k_j | k_j \in \mathbb{Z}_+ \forall j, n \in \mathbb{N}\} = \{\sum\limits_{j=1}^{n} t_j k_j | k_j \in \mathbb{Z}_+ \forall j, n \in \mathbb{N}\}$. Now by the proof of Lemma 4.3.1, there is a bijection $\sigma : \mathbb{N} \to \mathbb{N}$ such that $s_j = t_{\sigma(j)}$ for all $j \in \mathbb{N}$.

By (4.3.8) there exists a unitary V such that

$$V\rho_s V^* = \rho_t. \tag{4.3.10}$$

where $V = \Gamma_s(M)W(z)U$. Let $\mathbf{k} = (k_1, k_2, \dots, k_N, 0, 0, \dots)^{\tau} \in \mathbb{Z}_{+,0}^{\infty}$ be arbitrary. By (4.3.10) if $|\mathbf{k}\rangle$ is an eigenvector for ρ_s then $V|\mathbf{k}\rangle$ is an eigenvector for ρ_t with the same eigenvalue. Therefore, $V|\mathbf{k}\rangle$ is an eigenvector for ρ_t with eigenvalue $pe^{-\sum\limits_{j=1}^{N} s_j k_j} = pe^{-\sum_{j\in\mathbb{N}} t_{\sigma j} k_j}$. But defining the unitary operator A on $\mathcal{H}$, $A_\sigma(e_j) = e_{\sigma(j)}$, its second quantization, $\Gamma(A)\mathbf{k}$ is an eigenvector for ρ_t with same eigenvalue. Since the multiplicity for each eigenvalue is one, there exists a complex number $\beta(\mathbf{k})$ of unit modulus such that,

$$\begin{aligned} V\left|\mathbf{k}\right\rangle &= \beta(\mathbf{k})\left|A\mathbf{k}\right\rangle \\ &= \Gamma_s(A)\beta(\mathbf{k})\left|\mathbf{k}\right\rangle \\ &= \Gamma_s(A)\beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots)\left|\mathbf{k}\right\rangle \end{aligned}$$

Then by (4.3.10) $U = W(z)^*\Gamma_s(M)^*\Gamma_s(A)\beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots)$. Now the proof is complete due to Theorem 1.8.1. It should be noted that we may need to redefine β if the multiplier $\sigma(M^{-1}, A) \neq 1$ (refer Theorem 1.8.1). □

Now we are ready to prove the main theorem on Gaussian symmetries.

Theorem 4.3.2. *A unitary operator $U \in \mathcal{B}(\Gamma_s(\mathcal{H}))$ is a Gaussian symmetry if and only if*

$$U = \lambda W(\alpha)\Gamma_s(L),$$

for some $\lambda \in \mathbb{C}$ with $|\lambda| = 1$, $\alpha \in \mathcal{H}$, and L is a Shale operator ($L \in \mathcal{S}(\mathcal{H})$).

Proof. The sufficiency is immediate from Proposition 3.2.6 and Proposition 3.2.4. To prove the necessity, let us consider $\mathcal{H} = \oplus_j \mathbb{C}e_j$ with respect to some orthonormal basis $\{e_j\}$, if U is a Gaussian symmetry then in particular $U\rho_s U^*$ is a Gaussian state for ρ_s as in Theorem 4.3.1. Therefore we can assume without loss of generality that $U = \beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots)$. We will show that $U = \Gamma_s(D)$ for some unitary operator D and this will prove the theorem because of (iii) of Theorem 1.8.1.

Let $\psi \in \Gamma_s(\mathcal{H})$ be such that $|\psi\rangle\langle\psi|$ is a pure Gaussian state. Then by assumption $|U\psi\rangle\langle U\psi|$ is also a Gaussian state. It is pure state because it is obtained from the wave function $|U\psi\rangle$. We choose the coherent state ((i) of Examples in Section 3.2)

$$\psi = e^{-\frac{1}{2}\|u\|^2}\left|e(u)\right\rangle = W(u)\left|e(0)\right\rangle,$$

where $u = (u_1, u_2, \dots)^T \in \oplus_j \mathbb{C}e_j$. Now

$$\left|U\psi\right\rangle = e^{-\frac{1}{2}\|u\|^2}\beta(a_1^\dagger a_1, a_2^\dagger a_2, \dots)\left|e(u)\right\rangle. \tag{4.3.11}$$

By Corollary 4.2.1, there exists a unitary A and an $\alpha \in \mathcal{H}$ such that

$$\left|U\psi\right\rangle = W(\alpha)\Gamma_s(A) \otimes_j \left|e_{\lambda_j}\right\rangle, \tag{4.3.12}$$

where $e_\lambda(x) = (2\pi)^{-1/4}\lambda^{-1/2}\exp\{-4^{-1}\lambda^{-2}x^2\}, \quad x \in \mathbb{R}, \lambda > 0$ on $L^2(\mathbb{R})$.

We have by Proposition 1.3.2, $e(u) = \lim_{M\to\infty} \otimes_{j=1}^{M} e(u_j)\otimes e(0)\otimes e(0)\otimes\cdots$. Let $\mathbf{k}\in\mathbb{Z}_+^\infty$, with $\mathbf{k}=(k_1,k_2,\ldots,k_n,0,0\ldots)^T$. Since $\langle e(u_j)|e(0)\rangle = 1$,

$$\begin{aligned}
\langle e(u)|\mathbf{k}\rangle &= \lim_{M\to\infty} \left\langle \otimes_{j=1}^{M} e(u_j)\otimes e(0)\otimes e(0)\otimes\cdots \middle| \mathbf{k}\right\rangle \\
&= \Pi_{j=1}^{n} \langle e(u_j)|k_j\rangle \\
&= \Pi_{j=1}^{n} \left\langle \sum_{m=0}^{\infty} \frac{u_j^m}{\sqrt{m!}} |m\rangle \middle| k_j \right\rangle \\
&= \Pi_{j=1}^{n} \frac{u_j^{k_j}}{\sqrt{k_j!}} := \frac{u^{\mathbf{k}}}{\sqrt{\mathbf{k}!}},
\end{aligned}$$

where the last line defines the multi index notation and here we assume $0^0 = 1$. Therefore we write,

$$e(u) = \sum_{\mathbf{k}\in\mathbb{Z}_+^\infty} \frac{u^{\mathbf{k}}}{\sqrt{\mathbf{k}!}} |\mathbf{k}\rangle . \tag{4.3.13}$$

Now for each finite vector $z=(z_1,z_2,\ldots,z_N,0,0,\ldots)^T \in \oplus_j \mathbb{C}e_j, N\in\mathbb{N}$,

$$e(z) = \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \frac{z^{\mathbf{k}}}{\sqrt{\mathbf{k}!}} |\mathbf{m}\rangle , \tag{4.3.14}$$

where $\mathbf{m}\in\mathbb{Z}_+^N$ is considered as the vector $(m_1,m_2,\ldots,m_N,0,0,\ldots)^t\in\mathbb{Z}_+^\infty$ and $|\mathbf{m}\rangle = |m_1\rangle |m_2\rangle\cdots|m_n\rangle \in \Gamma_s(\mathcal{H})$ as in the notation of (4.3.2).

We will evaluate the function $f(z) = \langle U\psi, e(z)\rangle$ using (4.3.11) and (4.3.12). From (4.3.11), (4.3.13), (4.3.14) and continuity of $\beta(a_1^\dagger a_1, a_2^\dagger a_2,\ldots)$, we have

$$\begin{aligned}
f(z) &= e^{-\frac{1}{2}\|u\|^2} \left\langle \beta(a_1^\dagger a_1, a_2^\dagger a_2,\ldots)e(u), e(z)\right\rangle \\
&= e^{-\frac{1}{2}\|u\|^2} \left\langle \beta(a_1^\dagger a_1, a_2^\dagger a_2,\ldots) \sum_{\mathbf{k}\in\mathbb{Z}_+^\infty} \frac{u^{\mathbf{k}}}{\sqrt{\mathbf{k}!}} |\mathbf{k}\rangle , \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \frac{z^{\mathbf{k}}}{\sqrt{\mathbf{k}!}} |\mathbf{m}\rangle \right\rangle \\
&= e^{-\frac{1}{2}\|u\|^2} \left\langle \sum_{\mathbf{k}\in\mathbb{Z}_+^\infty} \frac{u^{\mathbf{k}}}{\sqrt{\mathbf{k}!}} \beta(\mathbf{k}) |\mathbf{k}\rangle , \sum_{\mathbf{m}\in\mathbb{Z}_+^N} \frac{z^{\mathbf{k}}}{\sqrt{\mathbf{k}!}} |\mathbf{m}\rangle \right\rangle \\
&= e^{-\frac{1}{2}\|u\|^2} \sum_{\mathbf{k}\in\mathbb{Z}_+^N} \frac{(\bar{u}z)^{\mathbf{k}}}{\mathbf{k}!}
\end{aligned} \tag{4.3.15}$$

where $\bar{u} := (\bar{u}_1,\bar{u}_2,\ldots)^T, \bar{u}z := (\bar{u}_1 z_1)(\bar{u}_2 z_2)\cdots(\bar{u}_N z_N)$ and the last line follows because the second term in the innerproduct of (4.3.15) has summation over $\mathbf{m}\in\mathbb{Z}_+^N$. Thus

$$f(z) = e^{-\frac{1}{2}\|u\|^2} \sum_{\mathbf{k}\in\mathbb{Z}_+^N} \frac{(\bar{u}_1 z_1)^{k_1}(\bar{u}_2 z_2)^{k_2}\cdots(\bar{u}_N z_N)^{k_N}}{k_1!k_2!\cdots k_N!}\beta(\mathbf{k}). \tag{4.3.16}$$

Since $|\beta(\mathbf{k})| = 1$, from (4.3.16) we see that

$$|f(z)| \leq \exp\left\{-\frac{1}{2}\|u\|^2 + \sum_{j=1}^{N} |u_j||z_j|\right\}. \tag{4.3.17}$$

From the definition of $e(w)$ and e_λ in $L^2(\mathbb{R})$ we have

$$\langle e_\lambda, e(w)\rangle = \sqrt{\frac{2\lambda}{1+\lambda^2}} \exp\frac{1}{2}\left(\frac{\lambda^2-1}{\lambda^2+1}\right) w^2, \lambda > 0, w \in \mathbb{C}. \tag{4.3.18}$$

Using (4.3.12),

$$\begin{aligned} f(z) &= \Big\langle W(\alpha)\Gamma_s(A) \otimes_j e_{\lambda_j}, e(z)\Big\rangle \\ &= \Big\langle \otimes_j e_{\lambda_j}, \Gamma_s(A^*)W(-\alpha)e(z)\Big\rangle \\ &= e^{\langle\alpha,z\rangle - \frac{1}{2}\|\alpha\|^2}\left\langle \otimes_j e_{\lambda_j}, e\Big(A^*(z-\alpha)\Big)\right\rangle. \end{aligned} \tag{4.3.19}$$

Since z is a finite vector and α is fixed, each coordinate of $A^*(z-\alpha)$ is a first degree polynomial in the z_j's. Therefore $e\Big(A^*(z-\alpha)\Big) = \otimes_j e(w_j)$ where each w_j is a first degree polynomial in the z_j's. Therefore from (4.3.18) and the property of infinite tensor products

$$f(z) = e^{\langle\alpha,z\rangle - \frac{1}{2}\|\alpha\|^2} \lim_{n\to\infty} \Pi_{j=1}^{n} \sqrt{\frac{2\lambda_j}{1+\lambda_j^2}} \exp\sum_{j=1}^{n}\frac{1}{2}\left(\frac{\lambda_j^2-1}{\lambda_j^2+1}\right) w_j^2.$$

Since each w_j^2 is a second degree polynomial in $z_1, z_2, \ldots, z_N$ on its own. This contradicts (4.3.17) unless $\lambda_j = 1$ for all j. Now (4.3.12) implies

$$\begin{aligned} |U\psi\rangle &= W(\alpha)\Gamma_s(A)\,|e(0)\rangle \\ &= e^{-\frac{1}{2}\|\alpha\|^2}\,|e(\alpha)\rangle\,. \end{aligned}$$

Now from (4.3.11) we get

$$e^{-\frac{1}{2}\|u\|^2}\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)\,|e(u)\rangle = e^{-\frac{1}{2}\|\alpha\|^2}\,|e(\alpha)\rangle\,. \tag{4.3.20}$$

Thus $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)$ is a unitary with the following properties:

(i) $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)\,|\mathbf{k}\rangle = \beta(\mathbf{k})\,|\mathbf{k}\rangle$ for every $\mathbf{k} \in \mathbb{Z}_+^\infty$.

(ii) It maps coherent vectors to coherent vectors.

We will prove that $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots) = \Gamma_s(D)$ for a diagonal unitary D. To this end we fix a $u = (u_1, u_2, \ldots)^T \in \oplus_j \mathbb{C}e_j$ with $u_j \neq 0, \forall j$. We have $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)\,|e(u)\rangle = e^{\frac{1}{2}(\|u\|^2 - \|\alpha\|^2)}\,|e(\alpha)\rangle$. Therefore if $\alpha = (\alpha_1, \alpha_2, \ldots)^T$ from (4.3.20) and (4.3.13) we get,

$$\sum_{\mathbf{k}\in\mathbb{Z}_+^\infty} \frac{u^{\mathbf{k}}}{\sqrt{\mathbf{k}!}}\beta(\mathbf{k})\,|\mathbf{k}\rangle = e^{\frac{1}{2}(\|u\|^2 - \|\alpha\|^2)} \sum_{\mathbf{k}\in\mathbb{Z}_+^\infty} \frac{\alpha^{\mathbf{k}}}{\sqrt{\mathbf{k}!}}\,|\mathbf{k}\rangle\,.$$

Therefore,

$$u^{\mathbf{k}}\beta(\mathbf{k}) = e^{\frac{1}{2}(\|u\|^2-\|\alpha\|^2)}\alpha^{\mathbf{k}}, \forall \mathbf{k} \in \mathbb{Z}_+^{\infty}.$$

Since $u_j \neq 0$ for all j, we see that if $\mathbf{k} = (k_1, k_2, \ldots, k_m, 0, 0, \ldots) \in \mathbb{Z}_+^{\infty}$,

$$\beta(\mathbf{k}) = e^{\frac{1}{2}(\|u\|^2-\|\alpha\|^2)}\left(\frac{\alpha_1}{u_1}\right)^{k_1}\left(\frac{\alpha_2}{u_2}\right)^{k_2}\cdots\left(\frac{\alpha_m}{u_m}\right)^{k_m}, \forall \mathbf{k} \in \mathbb{Z}_+^{\infty}.$$

Since $|\beta(\mathbf{k})| = 1$, we get $\left|\frac{\alpha_j}{u_j}\right| = 1$ for all j. If we write $\frac{\alpha_j}{u_j} = e^{i\theta_j}$, then from (4.3.20) we get

$$\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots)\left|e(u)\right\rangle = \left|e(Du)\right\rangle, \tag{4.3.21}$$

where D is the unitary $\mathrm{Diag}(e^{i\theta_1}, e^{i\theta_2}, \ldots)$, for every $u = (u_1, u_2, \ldots)^T \in \oplus_j \mathbb{C}e_j$ with $u_j \neq 0, \forall j$. Now it is easy to see that (4.3.21) holds for all $u \in \mathcal{H}$. We conclude that $\beta(a_1^\dagger a_1, a_2^\dagger a_2, \ldots) = \Gamma_s(D)$. □

Publications

This thesis is based on the following articles.

(1) B. V. Rajarama Bhat and Tiju Cherian John Real Normal Operators and Williamson's Normal form *arxiv 1804.03921* (accepted for publication at **Acta Sci Math (Szeged)**, December, 2018).

(2) B. V. Rajarama Bhat, Tiju Cherian John and R. Srinivasan Infinite Mode Quantum Gaussian States *arxiv 1804.05049* (accepted for publication at **Reviews in Mathematical Physics**, Vol 31, 2019,
DOI: https://doi.org/10.1142/S0129055X19500302).

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Index