Levi-Civita connections in noncommutative geometry

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Contents

Chapter 0

Introduction

A fundamental result of Riemannian geometry states that if (M, g) is a Riemannian manifold, then there exists a unique connection ∇ on the space of one-forms $\Omega^1(M)$ of M which is torsionless and compatible with g. The connection ∇ is called the Levi-Civita connection for the pair (M, g). The goal of this thesis is to study analogues of this theorem in the context of noncommutative geometry. The noncommutative geometry of a unital (possibly noncommutative) algebra $\mathcal{A}$ is dictated by the choice of a differential calculus on $\mathcal{A}$. Thus, given a differential calculus on $\mathcal{A}$, the task of making sense of the question of existence of a Levi-Civita connection includes the following steps: Firstly, one needs to define a notion of pseudo-Riemannian metrics on a differential calculus. Second, one needs to make sense of the torsion of connections and that of the compatibility of a connection with a pseudo-Riemannian metric. Then, we need to verify whether there indeed exists any connection on the space of one-forms of the differential calculus which is both torsionless and compatible with the given pseudo-Riemannian metric. Finally, there is the question of uniqueness of such a connection. As we will shortly discuss, there are already a number of articles available in literature which have addressed the question of existence of Levi-Civita connections on some particular noncommutative manifolds. In many of these articles, the technique to prove the existence of Levi-Civita connections is example specific. Moreover, the definitions of metric as well as the metric compatibility conditions vary from example to example. There are also some works (see [22], as well as Appendix B of [51]) where the existence or uniqueness of a Levi-Civita connection fails.

Our goal in this thesis is to derive some sufficient conditions on the differential calculus which will guarantee the existence and uniqueness of Levi-Civita connections for some class of noncommutative manifolds. We have focussed our attention to a class of differential calculi constructed via spectral triples and a class of bicovariant differential calculi on Hopf algebras. The notion of spectral triples was introduced by Connes ([25]). A spectral triple on an algebra $\mathcal{A}$ is the data $(\mathcal{A}, \mathcal{H}, D)$ where $\mathcal{A}$ is a *-subalgebra of the bounded linear operators on a Hilbert space $\mathcal{H}$ and D is a (typically unbounded) self-adjoint operator on $\mathcal{H}$ satisfying some conditions. Starting from a spectral triple, there is a canonical construction of the space of forms. Other than Connes' seminal text [25], some of the books which provide introductory as well as extensive overview of spectral triple-based noncommutative geometry are [28], [64], [60] and [89]. On the other hand, bicovariant differential calculi are a class of differential calculi on Hopf algebras and were introduced by Woronowicz ([93]). For generalities on Hopf algebras and bicovariant differential calculi, we refer to [1], [23], [71], [78] and references therein.

Now we explain the set-up under which we will work. If $(\Omega^\bullet(\mathcal{A}), d)$ is a differential calculus on an algebra $\mathcal{A}$, our connections will be maps $\nabla : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ satisfying a Leibniz rule. Thus, unlike the articles [83], [3], [4], [5] and [80], we will not be working with covariant derivatives on the level of vector fields of a differential calculus. Our pseudo-Riemannian metric are right $\mathcal{A}$-linear maps $g : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \mathcal{A}$ satisfying a symmetry condition and a nondegeneracy condition. Since in most occasions, we will not be using the *-structure on the algebra and the space of forms, we do not assume g to be sesquilinear or positive definite. In the case of spectral triples (Chapters 2 and 3), we will in addition assume g to be left $\mathcal{A}$-linear while in the case of bicovariant differential calculi, we concentrate on those pseudo-Riemannian metric which are left-invariant with respect to the coaction of the Hopf algebra $\mathcal{A}$. For making sense of the symmetry in g, we need a braiding-like operator $\sigma : \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$. For spectral triples, we postulate the existence of this map σ while for bicovariant differential calculi, σ is the canonical braiding map discovered by Woronowicz.

In order to formulate the metric-compatibility conditions, we need some assumptions. For spectral triples, these are conditions which appear in the definition of tame spectral triples. Moreover, our proof of existence and uniqueness of Levi-Civita connections works for pseudo-Riemannian bilinear metrics, i.e. those which are both left and right $\mathcal{A}$-linear. In the case of bicovariant differential calculi, we assume that the braiding map σ as discussed above is diagonalisable in a certain sense. Moreover, we restrict our attention to left-invariant pseudo-Riemannian metrics.

Let us mention a few relevant topics and questions which have appeared in literature but which will not be addressed in this thesis. Firstly, in the last decade, there has been a lot of research around the scalar curvature coming from the asymptotic expansion of a Laplace-type operator associated to a spectral triple. This approach was pioneered by the landmark paper of Connes and Tretkoff ([30]) which proves a noncommutative analogue of the Gauss-Bonnet theorem. We refer to the papers [38], [39], [29], [65], [67] and references therein for the subsequent developments around this topic. For a different treatment of curvature on Hilbert modules, we refer to [77]. For computation of curvature of a noncommutative manifold via the Levi-Civita connection, we refer to [70], [72], [73] and references therein. A comprehensive account of the work of Beggs, Majid and their collaborators in this regard can be found in [11].

Many interesting examples of differential calculi on *-algebras are equipped with *-structures in the sense of [9]. The *-compatibility of a connection was also studied in the same paper. For examples of *-compatible Levi-Civita connections, we refer to [9] and [19]. A weaker notion of metric compatibility called cotorsion free has been studied by Beggs, Majid and their collaborators (see [73]). In [46], spin geometry on quantum groups have been studied. Very recently, the author of [76] proved the existence of a torsion and cotorsion free connection for the Fubini-Study metric on quantum projective spaces. For existence of Chern connections on quantum complex manifolds, we refer to [10]. The article [44] deals with a notion of strong connections to introduce a definition of a global curvature form. The article [34] considers metric compatibility of pairs of left and right connections.

Let us give a brief overview of the contents of this thesis.

In Chapter 1, we collect some initial notions and results needed in later chapters, to make the text reasonably self-contained. In Sections 1.1 and 1.2, we recall the concepts of algebras, in particular C*-algebras and Hopf algebras, and modules and comodules over them. Section 1.3 will introduce the notion of noncommutative calculi on noncommutative spaces. This section contains two subsections. The first one is on spectral triples due to Connes ([25]) and the second on bicovariant differential calculi due to Woronowicz [93]. The contents of the rest of the thesis also fall under these two broad headings. Indeed, whereas Chapters 2 and 3 are devoted to spectral triples, Chapters 4, 5 and 6 are devoted to bicovariant differential calculi. In Section 1.4 of Chapter 1, we briefly discuss the Levi-Civita connection problem on classical (pseudo-)Riemannian geometry and some equivalent formulations. The section ends with some basic definitions regarding connections in noncommutative differential calculi.

Chapter 2 deals with the existence and uniqueness of Levi-Civita connections on the class of tame spectral triples as given in [15, 16]. The chapter begins with a brief discussion on centered bimodules over algebras, on which the space of one-forms will be modelled. In Section 2.2, a more general class called quasi-tame spectral triples is introduced. In Section 2.3 pseudo-Riemannian metrics are defined on quasi-tame spectral triples. Following [41], we introduce a canonical candidate for a pseudo-Riemannian bilinear metric on a spectral triple and discuss some regularity conditions. From Section 2.4 onwards, we restrict our focus to the class of tame spectral triples. A definition of compatibility of connections on the space of one-forms on tame spectral triples with pseudo-Riemannian metrics is discussed. This, in particular, provides the definition of Levi-Civita connections on tame spectral triples. In Section 2.5, we prove that given a bilinear pseudo-Riemannian metric on a tame spectral triple, there exists a unique Levi-Civita connection on the space of one-forms.

Chapter 3 continues the discussion on tame spectral triples, and provides some concrete examples. In Section 3.1, the example of fuzzy 3-sphere as given in [41] is recalled and is shown to be a tame spectral triple. In Section 3.2, we discuss the spectral triple on the quantum Heisenberg manifold as defined in [22] and show that it is an example of a tame spectral triple. Spectral triples on Rieffel deformations ([82], [26]) of compact Riemannian manifolds were defined in [27]. In Section 3.3, we show that under some technical assumptions, these turn out to be tame spectral triples. In particular, the last section shows that our formulation of Levi-Civita connections is well-behaved with respect to Rieffel deformations of compact Riemannian manifolds.

In Chapter 4, we concentrate on the existence and uniqueness of Levi-Civita connections on bicovariant differential calculi over Hopf algebras. We begin by collecting some preliminary material on bicovariant bimodules over Hopf algebras and their relationship with Yetter-Drinfeld modules. In Section 4.2, we discuss a mild constraint on Woronowicz's braiding map given in [93], for bicovariant bimodules. In Section 4.3, we define and discuss the notion of invariant pseudo-Riemannian metrics on bicovariant differential calculi. In Section 4.4, we define the compatibility of left-covariant connections with left-invariant pseudo-Riemannian metrics as per [17]. In Section 4.5, we discuss a metric-independent sufficient condition for the existence of a unique left-covariant Levi-Civita connection compatible with a bi-invariant pseudo-Riemannian metric. In this section, we also show that subject to the Hopf algebra being cosemisimple, the unique left-covariant connection is also right covariant.

Chapters 5 and 6 are devoted to providing examples of bicovariant differential calculi which satisfy the criterion of existence and uniqueness of Levi-Civita connections as derived in Chapter 4. In Chapter 5, the concrete example is that of cocycle deformations, as given in [74], of differential calculi over Hopf algebras of regular functions on linear algebraic groups. On the way we discuss the cocycle deformation of bicovariant differential calculi and bi-invariant pseudo-Riemannian metrics on the differential calculi of (not necessarily commutative) Hopf algebras. We show that our formulation of bicovariant Levi-Civita connections is well-behaved with cocycle deformations, i.e., Levi-Civita connections associated to bicovariant differential calculi are in one-to-one correspondence with those on their cocycle deformations. This in particular proves the existence of a unique bicovariant Levi-Civita connection for every bi-invariant pseudo-Riemannian metric on the Hopf algebra of regular functions of a linear algebraic group. Chapter 6 deals with the example of example of $4D_{\pm}$ calculi on the Hopf algebra $SU_q(2)$ as introduced in [93]. We recall results in [73] and [20] to show that the corresponding Woronowicz braiding maps satisfy the requisite assumptions made in Chapter 4. Regarding the metric-independent sufficient condition for the existence of a unique bi-covariant Levi-Civita connection, the complexity of the $4D_{\pm}$ calculi required us to use brute-force to verify this, rather than as part of any axiomatic framework.

Chapter 1

Preliminaries

In this chapter, we collect preparatory material for this thesis. In Sections 1.1 and 1.2, we recall the notions of algebras, modules and comodules. As examples of interest, we introduce various noncommutative spaces. In Section 1.3, we introduce noncommutative differential calculi on noncommutative spaces, which are one of the basic objects of study for our purpose. By way of examples, we give a couple of constructions of differential calculi. Section 1.4 is devoted to a brief discussion of the Levi-Civita connection problem in classical Riemannian geometry, and then to some inital notions for investigating the problem in the noncommutative set-up.
Throughout this thesis, we will work over the field of complex numbers. Thus, unless mentioned otherwise, all vector spaces, algebras and modules will be over $\mathbb{C}$.

1.1 Algebras and Modules

Definition 1.1.1. *An algebra is a triple* $(\mathcal{A}, \mu, u)$ *with* $\mathcal{A}$ *a vector space,* $\mu : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{A}$ *a linear map called the multiplication map and* $u : \mathbb{C} \to \mathcal{A}$ *a linear map called the unit, such that*

$$\mu(\mathrm{id} \otimes_{\mathbb{C}} \mu) = \mu(\mu \otimes_{\mathbb{C}} \mathrm{id}), \qquad \mu(u \otimes_{\mathcal{A}} \mathrm{id}) = \mu(\mathrm{id} \otimes_{\mathbb{C}} u) = \mathrm{id}.$$

Let $(\mathcal{A}, \mu_{\mathcal{A}}, u_{\mathcal{A}})$ *and* $(\mathcal{B}, \mu_{\mathcal{B}}, u_{\mathcal{B}})$ *be two algebras. A linear map* $T : \mathcal{A} \to \mathcal{B}$ *is called an algebra map if* $T \circ \mu_{\mathcal{A}} = \mu_{\mathcal{B}} \circ (T \otimes_{\mathbb{C}} T)$ *and* $T \circ u_{\mathcal{A}} = u_{\mathcal{B}}$.

We will for the most part replace $\mu(a \otimes_{\mathbb{C}} b)$ with the usual ab to imply multiplication of elements a and b of an algebra. Next we introduce the definition of left and right modules over an algebra.

Definition 1.1.2. *Given an algebra $\mathcal{A}$, a left $\mathcal{A}$-module is a pair $(M, \triangleright)$ with a vector space M and a linear map $\triangleright : \mathcal{A} \otimes_{\mathbb{C}} M \to M$ such that*

$$\triangleright(u \otimes_{\mathbb{C}} \mathrm{id}) = \mathrm{id}, \qquad \triangleright(\mu \otimes_{\mathbb{C}} \mathrm{id}) = \triangleright(\mathrm{id} \otimes_{\mathbb{C}} \triangleright).$$

Similarly, a right $\mathcal{A}$-module is a pair $(M, \triangleleft)$ with a vector space M and a linear map $\triangleleft : M \otimes_{\mathbb{C}} \mathcal{A} \to M$ such that

$$\triangleleft(\mathrm{id} \otimes_{\mathbb{C}} u) = \mathrm{id}, \qquad \triangleleft(\mathrm{id} \otimes_{\mathbb{C}} \mu) = \triangleleft(\triangleleft \otimes_{\mathbb{C}} \mathrm{id}).$$

Lastly, an $\mathcal{A}$-bimodule is a triple $(M, \triangleright, \triangleleft)$ such that $(M, \triangleright)$ is a left $\mathcal{A}$-module, $(M, \triangleleft)$ is a right $\mathcal{A}$-module and

$$\triangleright(\mathrm{id} \otimes_{\mathbb{C}} \triangleleft) = \triangleleft(\triangleright \otimes_{\mathbb{C}} \mathrm{id}).$$

From now on, we are going to dispense of the symbols $\triangleleft$ and $\triangleright$ whenever the implied algebra actions are unambiguous.

Definition 1.1.3. *Given two left $\mathcal{A}$-modules M and N, a linear map $T : M \to N$ is called a left A-linear map if for all a in $\mathcal{A}$ and m in M, $T(am) = aT(m)$.*
If M and N are two right $\mathcal{A}$-modules, a linear map $T : M \to N$ is called a right $\mathcal{A}$-linear map if $T(ma) = T(m)a$.
If M and N are $\mathcal{A}$-bimodules, then a map $T : M \to N$ is called an $\mathcal{A}$-bimodule map or $\mathcal{A}$-bilinear map if it is both left and right $\mathcal{A}$-linear.
The set of all right $\mathcal{A}$-linear maps from M to N will be denoted by $\mathrm{Hom}_{\mathcal{A}}(M, N)$, the set of all left $\mathcal{A}$-linear maps from M to N will be denoted by ${}_{\mathcal{A}}\mathrm{Hom}(M, N)$ and the set of all $\mathcal{A}$-bilinear maps from M to N will be denoted by ${}_{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M, N)$.

Definition 1.1.4. *Given two $\mathcal{A}$-bimodules M and N, one can give an $\mathcal{A}$-bimodule structure on $\mathrm{Hom}_{\mathcal{A}}(M, N)$. If T is a map in $\mathrm{Hom}_{\mathcal{A}}(M, N)$, the left and right module actions are defined respectively by*

$$(aT)(m) = a(T(m)) \quad \text{and} \quad (Ta)(m) = T(am),$$

for all a in $\mathcal{A}$ and m in M.

Now we recall the notion of a right (respectively, left) $\mathcal{A}$-total set in a right (respectively, left) $\mathcal{A}$-module.

Definition 1.1.5. *A subset S of a right $\mathcal{A}$-module M is called right $\mathcal{A}$-total in M if*

$$M = \mathrm{Span}_{\mathbb{C}}\{sa : s \in S, a \in \mathcal{A}\}.$$

Similarly, a subset S of a left $\mathcal{A}$-module M is called left $\mathcal{A}$-total in M if $M = \mathrm{Span}_{\mathbb{C}}\{as : s \in S, a \in \mathcal{A}\}$.

Then we have the following lemma which will be used repeatedly in the thesis.

Lemma 1.1.6. *Let S be a right $\mathcal{A}$-total subset of a right $\mathcal{A}$-module M. If T_1 and T_2 are two right $\mathcal{A}$-linear maps from M to another right $\mathcal{A}$-module N such that they agree on S, then they agree everywhere on M.*

Proof. If m is an element of M, there exist elements s_i in S and a_i in $\mathcal{A}$ such that $m = \sum_i s_i a_i$. Then,

$$T_1(m) = \sum_i T_1(s_i)a_i = \sum_i T_2(s_i)a_i = T_2(m).$$

$\square$

We record the following lemma for future use.

Lemma 1.1.7. *Let M be an $\mathcal{A}$-bimodule and $h : M \otimes_{\mathcal{A}} M \to \mathcal{A}$ be a right $\mathcal{A}$-linear map such that the map*

$$V_h : M \to M^* := \mathrm{Hom}_{\mathcal{A}}(M, \mathcal{A}),\ V_h(m)(m') = h(m \otimes_{\mathcal{A}} m')$$

for all m, m' in M is a right $\mathcal{A}$-linear isomorphism.
Then for all T in ${}_{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M, M)$, there exists a unique element T^* in $\mathrm{Hom}_{\mathcal{A}}(M, M)$ such that for all m, m' in M,

$$h(T^*(m) \otimes_{\mathcal{A}} m') = h(m \otimes_{\mathcal{A}} T(m')).$$

Proof. Suppose m is an element in M. We define an element $z(m)$ in M by the equation

$$V_h(z(m))(m') = h(m \otimes_{\mathcal{A}} T(m')) \text{ for all } m' \text{ in } M.$$

The above definition is well-defined since $V_h : M \to M^*$ is an isomorphism. Clearly, the element $z(m)$ is the unique choice for $T^*(m)$.
For proving that the map $m \mapsto T^*(m) := z(m)$ is right $\mathcal{A}$-linear, we compute

$$\begin{aligned} & V_h(T^*(ma))(m') = V_h(ma)(T(m')) = h(ma \otimes_{\mathcal{A}} T(m')) \\ = \; & h(m \otimes_{\mathcal{A}} aT(m')) = h(m \otimes_{\mathcal{A}} T(am')) \text{ (since } T \text{ is left } \mathcal{A}\text{-linear)} \\ = \; & V_h(T^*(m))(am') = V_h(T^*(m)a)(m'). \end{aligned}$$

Since V_h is an isomorphism, we have $T^*(ma) = T^*(m)a$. □

The following well-known fact is known to experts (see [55]), but we provide a proof for the sake of completeness.

Proposition 1.1.8. *Let M and N be $\mathcal{A}$-bimodules which are finitely generated and projective as right $\mathcal{A}$-modules. Then for elements m_i in M, n in N and ϕ_i in $N^* := \mathrm{Hom}_{\mathcal{A}}(N, \mathcal{A})$, the map*

$$\zeta_{M,N} : M \otimes_{\mathcal{A}} N^* \to \mathrm{Hom}_{\mathcal{A}}(N, M), \; \zeta_{M,N}(\sum_i m_i \otimes_{\mathcal{A}} \phi_i)(f) = \sum_i m_i \phi_i(n)$$

defines an isomorphism of $\mathcal{A}$-bimodules.

Proof. Since M is finitely generated projective, there exists an integer d and an idempotent P in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^d, \mathcal{A}^d)$ such that $M = P(\mathcal{A}^d)$. Let $\{a_1, a_2, ..., a_d\}$ be a basis of $\mathcal{A}^d$ so that M is generated by $\{P(a_i)\}_i$. If T be an element of $\mathrm{Hom}_{\mathcal{A}}(N, M)$, then there exists elements ϕ_j in N^* such that

$$T(n) = \sum_j P(a_j)\phi_j(n), \text{ for all } n.$$

Clearly, $T = \zeta_{M,N}(\sum_j P(a_j) \otimes_{\mathcal{A}} \phi_j)$, proving that $\zeta_{M,N}$ is onto. For proving that $\zeta_{M,N}$ is one-to-one, we observe that it can be easily verified that the map $\zeta_{M,N}$ is a restriction of $\zeta_{\mathcal{A}^d,N}$ and therefore also one-to-one. This completes the proof. □

We will be using the notions of left and right duals in certain monoidal categories and so we recall the relevant definitions and results here.

Definition 1.1.9. *([37]) Suppose $(\mathcal{C}, \otimes, 1)$ is a (strict) monoidal category. An object X in $\mathcal{C}$ is said to have a left dual if there exists an object $\widetilde{X}$ in $\mathcal{C}$ and morphisms*

$$\mathrm{ev}_X : \widetilde{X} \otimes X \to 1 \text{ and } \mathrm{coev}_X : 1 \to X \otimes \widetilde{X}$$

such that the following two equations hold:

$$(\mathrm{id}_X \otimes \mathrm{ev}_X)(\mathrm{coev}_X \otimes \mathrm{id}_X) = \mathrm{id}_X, \ (\mathrm{ev}_X \otimes \mathrm{id}_{\widetilde{X}})(\mathrm{id}_{\widetilde{X}} \otimes \mathrm{coev}_X) = \mathrm{id}_{\widetilde{X}}.$$

X is said to have a right dual if there exists an object $^*X$ in $\mathcal{C}$ and morphisms

$$\mathrm{ev}'_X : X \otimes {}^*X \to 1 \text{ and } \mathrm{coev}'_X : 1 \to {}^*X \otimes X$$

such that

$$(\mathrm{ev}'_X \otimes \mathrm{id}_X)(\mathrm{id}_X \otimes \mathrm{coev}'_X) = \mathrm{id}_X \text{ and } (\mathrm{id}_{^*X} \otimes \mathrm{ev}'_X)(\mathrm{coev}'_X \otimes \mathrm{id}_{^*X}) = \mathrm{id}_{^*X}.$$

We collect some well-known facts about left duals in a monoidal category in the next proposition.

Proposition 1.1.10. *(Subsection 2.10 of [37]) Suppose $(\mathcal{C}, \otimes, 1)$ is a monoidal category and X be an object in $\mathcal{C}$. We have the following:*

(i) If X admits a left dual, then it is unique upto isomorphism. In fact, if $(\mathrm{ev}_1, \mathrm{coev}_1, \widetilde{X})$ and $(\mathrm{ev}_2, \mathrm{coev}_2, Y)$ are two left duals of the object X, then the morphism

$$(\mathrm{ev}_1 \otimes \mathrm{id}_Y)(\mathrm{id}_{\widetilde{X}} \otimes \mathrm{coev}_2) : \widetilde{X} \to Y$$

is actually an isomorphism.

(ii) Now suppose $\mathcal{D}$ is another monoidal category and F a monoidal functor from $\mathcal{C}$ to $\mathcal{D}$. If $\widetilde{X}$ is a left dual of the object X, then $F(\widetilde{X})$ is a left dual of the object $F(X)$ in $\mathcal{D}$.

Proof. We refer to Proposition 2.10.5 and Exercise 2.10.6 of [37] for the proofs. □

We recall that an object V in an abelian category $\mathcal{C}$ is said to be a direct sum of objects V_i, $i = 1, 2, \cdots n$, if there exist morphisms α_i, $i = 1, 2, \cdots n$, in $\mathrm{Hom}(V_i, V)$ and β_i, $i = 1, 2, \cdots n$, in $\mathrm{Hom}(V, V_i)$ such that

$$\beta_i \alpha_i = 1_{V_i} \ \forall\, i = 1, 2, \cdots n, \quad \sum_{i=1}^{n} \alpha_i \beta_i = 1_V. \tag{1.1.1}$$

The following result is well-known to the experts. Nevertheless, we prove it for the sake of completeness.

Proposition 1.1.11. *If V and W are two objects in a semisimple monoidal category $\mathcal{C}$, then*

$$\dim(\mathrm{Hom}(V,W)) = \dim(\mathrm{Hom}(W,V)).$$

Proof. We consider the decomposition $V \cong \oplus_{i=1}^{n}(Z_i)^{\oplus k_i}$ and $W \cong \oplus_{i=1}^{n}(Z_i)^{\oplus l_i}$ into mutually non-isomorphic simple objects Z_i, where some of the k_i's and l_i's could be zero. Thus, there exist morphisms $\alpha_{i,s}$, $s = 1, 2, \cdots k_i$, $i = 1, 2, \cdots n$, in $\mathrm{Hom}(Z_i, V)$ and $\beta_{i,s}$, $s = 1, 2, \cdots k_i$, $i = 1, 2, \cdots n$, in $\mathrm{Hom}(V, Z_i)$ such that

$$\beta_{i,s}\alpha_{i,s} = 1_{Z_{i,s}} \text{ and } \sum_{i,s}\alpha_{i,s}\beta_{i,s} = 1_V. \tag{1.1.2}$$

Similarly, from the decomposition $W \cong \oplus_{i=1}^{n}(Z_i)^{\oplus l_i}$, we have morphisms $\sigma_{i,t}$, $t = 1, 2, \cdots l_i$, $i = 1, 2, \cdots n$, in $\mathrm{Hom}(Z_i, W)$ and morphisms $\mu_{i,t}$, $t = 1, 2, \cdots l_i$, $i = 1, 2, \cdots n$, in $\mathrm{Hom}(W, Z_i)$.

The morphism $\beta_{i,s}\alpha_{i,t}$ belongs to $\mathrm{Hom}(Z_{i,t}, Z_{i,s})$ and so by Schur's Lemma, $\beta_{i,s}\alpha_{i,t} = c^i_{s,t}$ for some scalars $c^i_{s,t}$. Let c_i denote the matrix whose (s,t)-th element is $c^i_{s,t}$. We claim that the matrix c_i is the identity matrix. Indeed,

$$\begin{aligned}
(c_i^2)_{(s,t)} &= \sum_k c^i_{s,k}c^i_{k,t} \\
&= \beta_{i,s}(\sum_k \alpha_{i,k}\beta_{i,k})\alpha_{i,t} \\
&= \beta_{i,s}(\sum_{j,k} \alpha_{j,k}\beta_{j,k})\alpha_{i,t} \\
&\quad \text{(as Schur's Lemma implies } \beta_{i,s}\alpha_{j,k} \in \mathrm{Hom}(Z_j, Z_i) \text{ equals 0 if } i \neq j) \\
&= \beta_{i,s}\alpha_{i,t} \text{ (by (1.1.2))} \\
&= c^i_{s,t}.
\end{aligned}$$

Moreover, (1.1.2) implies that $c^i_{s,s} = 1$ for all s.

Thus, c_i is an idempotent matrix such that its trace is equal to n. Hence, c_i is equal to the identity matrix. This proves our claim.

We note that for all i, t, j, s, $\mu_{i,t}f\alpha_{j,s}$ is in $\mathrm{Hom}(Z_j, Z_i)$ and since Z_i, Z_j are simple objects, Schur's Lemma implies that

$$\mu_{i,t}f\alpha_{j,s} = \delta_{ij}\lambda^i_{t,s}1_{Z_i} \text{ for some scalar } \lambda^i_{t,s}. \tag{1.1.3}$$

Now suppose f is an element in $\mathrm{Hom}(V, W)$. Then we can write

$$\begin{aligned} f &= 1_W f 1_V \\ &= \sum_{i,t,j,s} \sigma_{i,t}\mu_{i,t} f \alpha_{j,s}\beta_{j,s} \\ &= \sum_{i,t,j,s} \sigma_{i,t}\delta_{ij}\lambda^i_{t,s} 1_{Z_i}\beta_{j,s} \text{ (by (1.1.3))} \\ &= \sum_{i,s,t} \lambda^i_{t,s}\sigma_{i,t}\beta_{i,s}. \end{aligned}$$

Thus, any element f in $\mathrm{Hom}(V, W)$ is a linear combination of elements in the set $S = \{\sigma_{i,t}\beta_{i,s} : i = 1, \cdots n, s = 1, \cdots k_i, t = 1, \cdots l_i\}$. If we can prove that S is a linearly independent set, then it follows that $\dim(\mathrm{Hom}(V, W)) = \sum_{i=1}^n |\{k : k_i \neq 0\}| . |\{l : l_i \neq 0\}|$.

Similarly, $\dim(\mathrm{Hom}(W, V)) = \sum_{i=1}^n |\{l : l_i \neq 0\}| . |\{k : k_i \neq 0\}|$. Hence, the proof is complete once we can deduce that S is a linearly independent set.

To that end, let $d^i_{t,s}$ be scalars so that

$$\sum_{i,t,s} d^i_{t,s}\sigma_{i,t}\beta_{i,s} = 0. \tag{1.1.4}$$

We fix indices i_0, s_0, t_0. Then we have

$$\mu_{i_0,t_0}(\sum_{i,t,s} d^i_{t,s}\sigma_{i,t}\beta_{i,s})\alpha_{i_0,s_0} = 0.$$

However,

$$\begin{aligned} \mu_{i_0,t_0}(\sum_{i,t,s} d^i_{t,s}\sigma_{i,t}\beta_{i,s})\alpha_{i_0,s_0} &= \sum_{i,t,s} d^i_{t,s}\mu_{i_0,t_0}\sigma_{i,t}\beta_{i,s}\alpha_{i_0,s_0} \\ &= \sum_{t,s} d^{i_0}_{t,s}\mu_{i_0,t_0}\sigma_{i_0,t}\beta_{i_0,s}\alpha_{i_0,s_0} \text{ (as } \beta_{i,s}\alpha_{i_0,s_0} \in \mathrm{Hom}(Z_{i_0}, Z_i)) \\ &= \sum_{t,s} d^{i_0}_{t,s}\mu_{i_0,t_0}\sigma_{i_0,t}c^{i_0}_{s,s_0} \\ &= \sum_{t} d^{i_0}_{t,s_0}\mu_{i_0,t_0}\sigma_{i_0,t} \end{aligned}$$

since c_{i_0} is the identity matrix. Due to similar reasons, $\sum_t d^{i_0}_{t,s_0}\mu_{i_0,t_0}\sigma_{i_0,t} = d^{i_0}_{t_0,s_0}$.

Therefore, (1.1.4) reduces to the equation $d^{i_0}_{t_0,s_0} = 0$. Since i_0, s_0, t_0 are arbitrary, this proves that S is a linearly independent set completing the proof of the proposition. $\square$

For more details on monoidal categories and duality therein, we refer to the books [37] and [59].

We will be using the notions of duality in monoidal categories for the category of $\mathcal{A}$-bimodules in this section and categories of covariant bimodules (see Section 1.2) later.

Definition 1.1.12. *We will denote the category of all $\mathcal{A}$-bimodules by the symbol ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Thus, the objects of ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ are $\mathcal{A}$-bimodules. If $\mathcal{F}$ and $\mathcal{G}$ are objects of ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, then a morphism from $\mathcal{F}$ to $\mathcal{G}$ is nothing but an $\mathcal{A}$-bimodule map from $\mathcal{F}$ to $\mathcal{G}$.*

The category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ is a monoidal category, i.e., if $\mathcal{F}, \mathcal{G}$ are objects of ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$, then the balanced tensor product $\mathcal{F} \otimes_{\mathcal{A}} \mathcal{G}$ is an object in ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. The following proposition gives a necessary and sufficient condition for the existence of a left dual in the monoidal category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$.

Proposition 1.1.13. *(Exercise 2.10.16 of [37]) Suppose M is an object in ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. Then M has a left dual if and only if M is finitely generated and projective as a right $\mathcal{A}$-module.*

If $\mathcal{E}$ is an $\mathcal{A}$-bimodule, we will continue to denote $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ by the symbol $\mathcal{E}^*$. It is well-known (see, for example, [7]) that $\mathcal{E}$ is finitely generated and projective as a right $\mathcal{A}$-module if and only if there exists a natural number n and elements $e_1, \cdots e_n$ in $\mathcal{E}$ and $e_1^*, \cdots e_n^*$ in $\mathcal{E}^* = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ such that for all e in $\mathcal{E}$ and for all ϕ in $\mathcal{E}^*$, the following equations hold:

$$e = \sum_i e_i e_i^*(e), \ \phi = \sum_i \phi(e_i) e_i^*.$$

The pair $\{e_i, e_i^*\}$ is called a pair of dual bases for $\mathcal{E}$.

If $\mathcal{E}$ is an $\mathcal{A}$-bimodule, then from Definition 1.1.4, we know that $\mathcal{E}^* = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ is also an $\mathcal{A}$-bimodule. The following proposition holds.

Proposition 1.1.14. *([7]) If $\mathcal{E}$ and $\mathcal{F}$ are $\mathcal{A}$-bimodules such that they are finitely generated and projective as right $\mathcal{A}$-modules, then the following statements hold:*

(i) The left dual $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ (in the category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$) is isomorphic to $\mathcal{E}^* = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$.

(ii) The left dual of the object $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ (in the category ${}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$) is isomorphic to $\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{E}^*$. In particular, $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})^* \cong \mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{E}^*$ as right $\mathcal{A}$-modules.

Proof. Since both the assertions have been proved in [7], we provide the sketch of the proof. We will let $\{e_i, e_i^* : i = 1, \cdots n\}$ and $\{f_j, f_j^* : j = 1, 2, \cdots m\}$ denote pairs of dual bases for $\mathcal{E}$ and $\mathcal{F}$ respectively.

For the first assertion, consider the $\mathcal{A}$-bimodule maps

$$\mathrm{ev}_{\mathcal{E}} : \mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A},\ \phi \otimes_{\mathcal{A}} e \mapsto \phi(e),\ \mathrm{coev}_{\mathcal{E}} : \mathcal{A} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^*,\ \mathrm{coev}_{\mathcal{E}}(a) = a \sum_i e_i \otimes_{\mathcal{A}} e_i^*.$$

Then it can be easily checked that

$$(\mathrm{id}_{\mathcal{E}} \otimes_{\mathcal{A}} \mathrm{ev}_{\mathcal{E}})(\mathrm{coev}_{\mathcal{E}} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}}) = \mathrm{id}_{\mathcal{E}},\ (\mathrm{ev}_{\mathcal{E}} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}^*})(\mathrm{id}_{\mathcal{E}^*} \otimes_{\mathcal{A}} \mathrm{coev}_{\mathcal{E}}) = \mathrm{id}_{\mathcal{E}^*}.$$

Since the left dual of $\mathcal{E}$ is unique upto isomorphism, this proves that $\widetilde{\mathcal{E}}$ is isomorphic to $\mathcal{E}^*$.

Now, for the second assertion, the evaluation and the coevaluation maps can be defined as:

$$\mathrm{ev}_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}} : (\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{E}^*) \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}) \to \mathcal{A};\ (\psi \otimes_{\mathcal{A}} \phi) \otimes_{\mathcal{A}} (e \otimes_{\mathcal{A}} f) \mapsto \psi(\phi(e) f),$$

$$\mathrm{coev}_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}} : \mathcal{A} \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}) \otimes_{\mathcal{A}} (\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{E}^*);\ \mathrm{coev}_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}}(a) = a \sum_{i,j} (e_i \otimes_{\mathcal{A}} f_j) \otimes_{\mathcal{A}} (f_j^* \otimes_{\mathcal{A}} e_i^*).$$

Thus, $\widetilde{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}}$ is isomorphic to $\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{E}^*$. However, by the first assertion, $\widetilde{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}}$ is isomorphic to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})^*$ and so $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F})^*$ is isomorphic to $\mathcal{F}^* \otimes_{\mathcal{A}} \mathcal{E}^*$. □

1.1.1 C^*-algebras and Hilbert C^*-modules

In this subsection, we briefly recall the notion of C^*-algebras. The discussion that follows illustrates why C^*-algebras are good analogues for classical topological spaces.

Definition 1.1.15. *Let $\mathcal{A}$ be an algebra over $\mathbb{C}$. A norm $\|\cdot\|$ on $\mathcal{A}$ is said to be submultiplicative if*

$$\|ab\| \leq \|a\|\|b\|$$

for all a, b in $\mathcal{A}$. If $\|\cdot\|$ is a submultiplicative norm on an algebra $\mathcal{A}$, then the pair $(\mathcal{A}, \|\cdot\|)$ is called a normed algebra. If the algebra is unital and $\|1\| = 1$ then $\mathcal{A}$ is a unital normed algebra. A complete normed algebra is called a Banach algebra. Unital Banach algebras are defined in the obvious way.

Definition 1.1.16. *A Banach $*$-algebra is a triplet $(\mathcal{A}, \|\cdot\|, *)$ where $(\mathcal{A}, \|\cdot\|)$ is a Banach algebra and $* : \mathcal{A} \to \mathcal{A}$ is an involution such that for all a, b in $\mathcal{A}$ and for all λ in $\mathbb{C}$,*

$$(\lambda a + b)^* = \overline{\lambda} a^* + b^*, \|a^*\| = \|a\|.$$

A C^*-algebra is a Banach $*$-algebra such that the "C^*-identity" holds:

$$\|a^* a\| = \|a\|^2$$

for all a in $\mathcal{A}$

The Gelfand-Naimark theorem states that any unital commutative C^*-algebra is isometrically isomorphic to $C(X)$ for some compact Hausdorff space X. On the other hand, any C^*-algebra is isometrically isomorphic to a norm closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$.

Definition 1.1.17. *Given a C^*-algebra $\mathcal{A}$, a representation $(\mathcal{H}, \pi)$ on it consists of a Hilbert space $\mathcal{H}$ and a $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$. The representation is called faithful if π is one-to-one.*

Let us give some examples of some noncommutative C^*-algebras that we are going to use in this thesis.

Example 1.1.18. *The noncommutative 2-torus $C(\mathbb{T}^2_\theta)$, defined for θ in $[0, 1]$ is the universal C^*-algebra generated by two unitary elements U and V satisfying the relation $UV = e^{2\pi i\theta} VU$. The C^*-algebra $C(\mathbb{T}^2_\theta)$ has a representation $(L^2(S^1), \pi)$, with π defined on the generators U and V by*

$$\pi(U)(f)(z) = f(e^{2\pi i\theta} z), \quad \pi(V)(f)(z) = z f(z),$$

where f is an elements in $L^2(S^1)$ and z is in S^1.

Now we come to the examples of reduced and twisted group C^*-algebras.

Example 1.1.19. *Suppose Γ is a discrete group and for g in Γ, let λ be the left-regular representation of Γ on $l^2(\Gamma)$. Thus, $\lambda_g : l^2(\Gamma) \to l^2(\Gamma)$ will denote the operator defined by*

$$\lambda_g \xi(h) = \xi(g^{-1} h) \text{ for all } \xi \text{ in } l^2(\Gamma) \text{ and } h \text{ in } \Gamma.$$

The reduced reduced C^*-algebra of the group Γ, denoted by $C_r^*(\Gamma)$ is defined to be the C^*-subalgebra of $\mathcal{B}(l^2(\Gamma)$ generated by $\{\lambda_g : g \in \Gamma\}$.

A slight variation of the above example produces twisted group C^*-algebras associated to cocycles. In this thesis, we will only need the θ-twisted reduced group C^*-algebra for $\Gamma = \mathbb{Z}^n$.

Example 1.1.20. *Consider the group $\mathbb{Z}^n$. We will denote an element $(m_1, \cdots m_n)$ by the symbol $\underline{m}$. Fix an $n \times n$ real skew-symmetric matrix θ.*

For $\underline{n}$ in $\mathbb{Z}^n$, we define a bounded operator $\lambda_{\underline{n}}^{\theta}$ on $l^2(\mathbb{Z}^n)$ by the formula

$$\lambda_{\underline{n}}^{\theta}\xi(\underline{m}) = e^{-\pi i \langle \underline{m}, \theta \underline{n} \rangle}\xi(\underline{m} - \underline{n}).$$

The θ-twisted reduced C^*-algebra of the group $\mathbb{Z}^n$, denoted by $C_r^*(\mathbb{Z}^n, \theta)$ is defined to be the C^*-subalgebra of $\mathcal{B}(l^2(\Gamma)$ generated by $\{\lambda_{\underline{n}}^{\theta} : \underline{n} \in \mathbb{Z}^n\}$.

For $n = 2$, this construction gives us back the noncommutative 2-torus defined above. So, in general, this C^*-algebra is called the noncommutative n-torus and denoted by $C(\mathbb{T}_\theta^n)$

The following recipe produces interesting examples of noncommutative C^*-algebras from commutative ones.

Example 1.1.21. *(Rieffel deformation, [82]) We will continue to use the notations introduced in Example 1.1.20. Suppose the group $\mathbb{T}^n$ has a strongly continuous action α on a unital C^*-algebra $\mathcal{A}$. The spectral subspaces are parametrized by the dual group $\mathbb{Z}^n$ and are defined by*

$$\mathcal{A}_{\underline{n}} := \{a \in \mathcal{A} : \alpha_g(a) = \chi_{\underline{n}}(g)a\},$$

where $\chi_{\underline{m}}(g_1, \cdots, g_n) = g_1^{m_1} \cdots g_n^{m_n}$. Consider the set

$$\mathcal{A}_\theta^{\mathrm{alg}} = \mathrm{Span}_{\mathbb{C}}\{a \otimes \lambda_{\underline{n}}^{\theta} : a \in \mathcal{A}_{\underline{n}}, \}.$$

Then $\mathcal{A}_\theta^{\mathrm{alg}}$ is a $*$-closed subalgebra of $\mathcal{A} \otimes C(\mathbb{T}_\theta^n)$.

The norm-closure of $\mathcal{A}_\theta^{\mathrm{alg}}$ in $\mathcal{A} \otimes C(\mathbb{T}_\theta^n)$ is a unital C^*-algebra called the Rieffel deformation of $\mathcal{A}$ under the action α and denoted by $\mathcal{A}_\theta$.

Rieffel deformation for actions of the groups $\mathbb{R}^n$ and $\mathbb{T}^n$ were introduced by Rieffel in [82]. Our reference is the paper [95]. For the generalization of this construction to the case of actions of locally compact abelian groups, we refer to [58].

Before we can define what is called a GNS triple on C^*-algebra, we need the following definition.

Definition 1.1.22. *An element x in a C^*-algebra $\mathcal{A}$ is said to be positive if there exists some y in $\mathcal{A}$ such that $x = y^*y$. The set of positive elements on $\mathcal{A}$ is denoted by $\mathcal{A}_+$*
A linear functional $\phi : \mathcal{A} \to \mathbb{C}$ is said to be positive if $\phi(x) \geq 0$ for all x in $\mathcal{A}_+$. A positive linear functional ϕ that satisfies $\phi(1) = 1$ is called a state.

We have the following:

Definition 1.1.23. *(G.N.S construction) Given a C^*-algebra $\mathcal{A}$ equipped with a state ϕ, one can construct a Gelfand-Naimark-Segal (GNS) triple $(L^2(\mathcal{A}, \phi), \pi_\phi, \xi_\phi)$, where $L^2(\mathcal{A}, \phi)$ is the Hilbert space completion of $\mathcal{A}$ with respect to the semi-inner product $\langle\langle a, b\rangle\rangle = \phi(a^*b)$ on $\mathcal{A}$, $\pi_\phi : \mathcal{A} \to \mathcal{B}(L^2(\mathcal{A}, \phi))$ is a $*$-representation and ξ_ϕ is a cyclic vector in $L^2(\mathcal{A}, \phi)$, i.e., the set $\mathrm{Span}\{\pi_\phi(x)\xi_\phi : x \in \mathcal{A}\}$ is dense in $L^2(\mathcal{A}, \phi)$. This GNS triple satisfies the relation*

$$\phi(x) = \langle \xi_\phi, \pi_\phi(x)\xi_\phi\rangle.$$

For a Hilbert space $\mathcal{H}$, the weak operator topology (WOT) is a locally convex topology on $\mathcal{B}(\mathcal{H})$ given by a family of seminorms

$$\mathcal{F}_1 := \{p_{\xi,\eta} : \xi, \eta \in \mathcal{H}\} \text{ where } p_{\xi,\eta}(X) = |\langle x\xi, \eta\rangle|.$$

This places us in a position to define von Neumann algebras.

Definition 1.1.24. *A unital $*$-subalgebra $\mathcal{A}$ of the space of bounded operators $\mathcal{B}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ which is closed under WOT is said to be a von Neumann algebra.*

We need to recall the following two notions before we can give the statement of von Neumann's Double Commutant Theorem. Firstly, the strong operator topology (SOT) on the space of bounded operator $\mathcal{B}(\mathcal{H})$ of a Hilbert space $\mathcal{H}$ is given by a family of seminorms

$$\mathcal{F}_2 := \{p_\xi : \xi \in \mathcal{H}\} \text{ where } p_\xi(x) = ||x\xi||.$$

Also, for any subset $\mathcal{B}$ of $\mathcal{B}(\mathcal{H})$, the commutant $\mathcal{B}'$ is defined by

$$\mathcal{B}' := \{x \in \mathcal{B}(\mathcal{H}) : xb = bx \ \forall b \in B\}.$$

Moreover, we use the symbol B'' to denote the set $(B')'$. Then we can state the following:

Theorem 1.1.25. *(von Neumann) Given a Hilbert space $\mathcal{H}$ and a unital $*$-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$, the closures of $\mathcal{A}$ under WOT and SOT are equal, and they are further equal to the double commutant A'' of $\mathcal{A}$.*

Note that, as a consequence of the above theorem, a C^*-algebra which is equal to its double commutant is a von Neumann algebra.

Definition 1.1.26. *A linear functional ϕ on a von Neumann algebra $\mathcal{A}$ is said to be normal if whenever a net of positive elements x_α in $\mathcal{A}$ increases to an element x, $\phi(x_\alpha)$ goes to $\phi(x)$.*

For more details on C^* and von Neumann algebras, we refer to the books [31, 88].

Now we come to the definition and examples of Hilbert C^*-modules.

Definition 1.1.27. *Let $\mathcal{A}$ be a C^*-algebra with norm $\|\cdot\|$. A pre-Hilbert $\mathcal{A}$-module is a right $\mathcal{A}$-module M together with a map $\langle \cdot, \cdot \rangle : M \times M \to \mathcal{A}$ which is linear in the second variable and satisfies the following conditions:*

(i) $\langle x, ya \rangle = \langle x, y \rangle a$;

(ii) $\langle x, y \rangle^* = \langle y, x \rangle$;

(iii) $\langle x, x \rangle \geq 0$;

(iv) $x \neq 0$ *implies* $\langle x, x \rangle \neq 0$;

for all x, y in M and a in $\mathcal{A}$.

A pre-Hilbert module $(M, \langle \cdot, \cdot \rangle)$ is called a Hilbert C^*-module (or simply, a Hilbert module) if M is complete under the norm defined by $\|x\| := \|\langle x, x \rangle\|^{\frac{1}{2}}$.

We have the following analogue of Cauchy-Schwarz inequality for Hilbert modules:

Lemma 1.1.28. *Let M be a Hilbert $\mathcal{A}$-module. Then for all x, y in M, the following inequality holds: $\|\langle x, y \rangle\| \leq \|x\| \|y\|$.*

A Hilbert $\mathcal{A}$-module M is said to be countably generated if there exists a set $S = \{x_n : n \in \mathbb{N}, x_n \in M\}$ such that the set $\{x_n a : x_n \in S, a \in \mathcal{A}\}$ is dense in M.

Given a C^*-algebra $\mathcal{A}$, let $\mathcal{H}_{\mathcal{A}}$ denote the set of all $\mathcal{A}$-valued sequences $(a_n)_{n\in\mathbb{N}}$ such that $\sum_n \|a_n^* a_n\| < \infty$. Then the $\mathcal{A}$-valued inner product

$$\langle a, b\rangle = \sum_n a_n^* b_n$$

turns $\mathcal{H}_{\mathcal{A}}$ into a Hilbert module. We have the following theorem by Kasparov.

Theorem 1.1.29. *(Kasparov's stabilization theorem) A countably generated Hilbert $\mathcal{A}$-module E is isomorphic to a complemented Hilbert submodule of $\mathcal{H}_{\mathcal{A}}$. Moreover,*

$$E \cong E \oplus \mathcal{H}_{\mathcal{A}}.$$

For more details on Hilbert C^*-modules, we refer to the book [63].

1.2 Hopf algebras and covariant bimodules

In this section, we introduce the notion of Hopf algebras. We start by recalling the definitions of coalgebras and bialgebras.

A coalgebra over $\mathbb{C}$ is a triple $(\mathcal{C}, \Delta, \epsilon)$ with $\mathcal{C}$ a vector space, Δ a linear map called the comultiplication map and $\epsilon : \mathcal{C} \to \mathbb{C}$ a linear map called the counit, such that

$$(\Delta \otimes_{\mathbb{C}} \mathrm{id})\Delta = (\mathrm{id} \otimes_{\mathbb{C}} \Delta)\Delta, \qquad (\mathrm{id} \otimes_{\mathcal{A}} \epsilon)\Delta = (\epsilon \otimes_{\mathcal{A}} \mathrm{id})\Delta = \mathrm{id}.$$

Here, we have identified $\mathcal{C}\otimes_{\mathbb{C}}\mathbb{C}$ and $\mathbb{C}\otimes_{\mathbb{C}}\mathcal{C}$ with $\mathcal{C}$.

Sweedler notation

Let $(\mathcal{C}, \Delta, \epsilon)$ be a coalgebra. Let c be in C with $\Delta(c) = \sum_i c_{1i} \otimes_{\mathbb{C}} c_{2i}$, where c_{ji} are in $\mathcal{C}$. We indicate such an expression by the form

$$\Delta(c) = c_{(1)} \otimes_{\mathbb{C}} c_{(2)},$$

suppressing the summation $\sum$ and the index i. Thus coassociativity of Δ yields

$$c_{(1)}\otimes_{\mathbb{C}}(c_{(2)})_{(1)}\otimes_{\mathbb{C}}(c_{(2)})_{(2)} = (c_{(1)})_{(1)}\otimes_{\mathbb{C}}(c_{(1)})_{(2)}\otimes_{\mathbb{C}}c_{(2)}.$$

So we write $(\Delta\otimes_{\mathbb{C}}\mathrm{id})\Delta(c) = (\mathrm{id}\otimes_{\mathbb{C}}\Delta)\Delta(c)$ in the form

$$c_{(1)}\otimes_{\mathbb{C}}c_{(2)}\otimes_{\mathbb{C}}c_{(3)}.$$

Using the coassociativity on the higher order tensor products, we can similarly define, without ambiguity, a map $\mathcal{C} \to \mathcal{C}^{\otimes(n)}$:

$$c \mapsto c_{(1)}\otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}}c_{(n)}.$$

Definition 1.2.1. *Let $(\mathcal{C}, \Delta_{\mathcal{C}}, \epsilon_{\mathcal{C}})$ and $(\mathcal{D}, \Delta_{\mathcal{D}}, \epsilon_D)$ be coalgebras. A linear map $T : \mathcal{C} \to \mathcal{D}$ is called a coalgebra map if $(T\otimes_{\mathbb{C}}T)\Delta_{\mathcal{C}} = \Delta_{\mathcal{D}}T$ and $\epsilon_{\mathcal{D}}T = \epsilon_{\mathcal{C}}$. In Sweedler notation, the first equation can be written as*

$$T(c)_{(1)}\otimes_{\mathbb{C}}T(c)_{(2)} = T(c_{(1)})\otimes_{\mathbb{C}}T(c_{(2)}).$$

Definition 1.2.2. *Given a coalgebra $\mathcal{C}$, a left $\mathcal{C}$-comodule is a pair (N, Δ_N) where N is a vector space N and $\Delta_N : N \to \mathcal{C}\otimes_{\mathbb{C}}N$ is a $\mathbb{C}$-linear map such that*

$$(\epsilon\otimes_{\mathbb{C}}\mathrm{id})\Delta_N = \mathrm{id}, \qquad (\mathrm{id}\otimes_{\mathbb{C}}\Delta_N)\Delta_N = (\Delta\otimes_{\mathbb{C}}\mathrm{id})\Delta_N.$$

Similarly, a right $\mathcal{C}$-module is a pair $(N, {}_N\Delta)$ with a vector space N and a linear map ${}_N\Delta$: $N \to N\otimes_{\mathbb{C}}\mathcal{C}$ such that

$$(\mathrm{id}\otimes_{\mathbb{C}}\epsilon)_N\Delta = \mathrm{id}, \qquad ({}_N\Delta\otimes_{\mathbb{C}}\mathrm{id})_N\Delta = (\mathrm{id}\otimes_{\mathbb{C}}\Delta)_N\Delta.$$

Lastly, a $\mathcal{C}$-bicomodule is a triple $(N, \Delta_N, {}_N\Delta)$ such that (N, Δ_N) is a left $\mathcal{C}$-comodule, $(N, {}_N\Delta)$ is a right $\mathcal{C}$-comodule and

$$(\mathrm{id}\otimes_{\mathbb{C}N}\Delta)\Delta_N = (\Delta_N\otimes_{\mathbb{C}}\mathrm{id})_N\Delta$$

Sweedler notation

The Sweedler notation for coalgebras can be extended to the setting of comodules. For n in N, we write

$$\Delta_N(n) = n_{(-1)} \otimes_{\mathbb{C}} n_{(0)}, \qquad {}_N\Delta(n) = n_{(0)} \otimes_{\mathbb{C}} n_{(1)}.$$

Note that the index (0) indicates the comodule tensorand and non-zero indices indicate the coalgebra tensorand.

Let $\mathcal{C}$ and $\mathcal{D}$ be coalgebras. We define $\Delta_{\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D}} : \mathcal{C}\otimes_{\mathbb{C}}\mathcal{D} \to (\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D})\otimes_{\mathbb{C}}(\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D})$ by

$$\Delta_{\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D}} = (\mathrm{id}\otimes_{\mathbb{C}}\mathrm{flip}\otimes_{\mathbb{C}}\mathrm{id})(\Delta_{\mathcal{C}}\otimes_{\mathbb{C}}\Delta_{\mathcal{D}}),$$

where flip : $\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D} \to \mathcal{D}\otimes_{\mathbb{C}}\mathcal{C}$ is the permutation of the two factors. Then $(\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D}, \Delta_{\mathcal{C}\otimes_{\mathbb{C}}\mathcal{D}})$ is a coalgebra called the tensor product of $\mathcal{C}$ and $\mathcal{D}$. Explicitly,

$$\Delta(c\otimes_{\mathbb{C}}d) = c_{(1)}\otimes_{\mathbb{C}}d_{(1)}\otimes_{\mathbb{C}}c_{(2)}\otimes_{\mathbb{C}}d_{(2)} \quad \text{and} \quad \epsilon(c\otimes_{\mathbb{C}}d) = \epsilon(c)\epsilon(d).$$

Now suppose that $(\mathcal{A}, m, u)$ is an algebra and $(\mathcal{A}, \Delta, \epsilon)$ is a coalgebra. Thus $\mathcal{A}\otimes_{\mathbb{C}}\mathcal{A}$ is an algebra as well as a coalgebra. One has the following proposition.

Proposition 1.2.3. *The following are equivalent:*

(i) m and u are coalgebra maps;

(ii) Δ and ϵ are algebra maps.

Definition 1.2.4. *A pentuple $(\mathcal{A}, m, u, \Delta, \epsilon)$ satisfying any of the equivalent conditions of the above proposition is called a bialgebra.*

Let us recall the convolution product. Let $(\mathcal{C}, \Delta)$ be a coalgebra and $(\mathcal{A}, m)$ an algebra. We put an algebra structure on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{C}, \mathcal{A})$, called the convolution algebra as follows:

$$T_1 * T_2 = m(T_1\otimes_{\mathbb{C}}T_2)\Delta.$$

Explicitly, $(T_1 * T_2)(c) = T_1(c_{(1)})T_2(c_{(2)})$.

Now suppose $(\mathcal{A}, m, u, \Delta, \epsilon)$ is a bialgebra. We write $\mathcal{A}^{\mathcal{C}}$ for the underlying coalgebra and for the algebra, again $\mathcal{A}$. Then $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\mathcal{C}}, \mathcal{A})$ is an algebra under the convolution product. We note that the identity operator $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ is an element of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\mathcal{C}}, \mathcal{A})$.

A convolution inverse S in $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\mathcal{C}}, \mathcal{A})$ of $\mathrm{id} : \mathcal{A} \to \mathcal{A}$ is called an antipode of the bialgebra $\mathcal{A}$. Explicitly, $S(a_{(1)})a_{(2)} = a_{(1)}S(a_{(2)}) = \epsilon(a)1$ for all a in $\mathcal{A}$. Note that by definition, an antipode if exists, is unique.

Definition 1.2.5. *A bialgebra with an antipode is called a Hopf algebra.*

Definition 1.2.6. *([61]) A (left) 2-cocycle γ on a Hopf algebra $(\mathcal{A}, \Delta)$ is a $\mathbb{C}$-linear map $\gamma : \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathbb{C}$ such that it is convolution invertible, unital, i.e,*

$$\gamma(a \otimes_{\mathbb{C}} 1) = \epsilon(a) = \gamma(1 \otimes_{\mathbb{C}} a)$$

and for all a, b, c in $\mathcal{A}$,

$$\gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)})\gamma(a_{(2)}b_{(2)} \otimes_{\mathbb{C}} c) = \gamma(b_{(1)} \otimes_{\mathbb{C}} c_{(1)})\gamma(a \otimes_{\mathbb{C}} b_{(2)}c_{(2)}). \tag{1.2.1}$$

The convolution inverse $\overline{\gamma}$ of γ is a right 2-cocyle on the Hopf algebra $(\mathcal{A}, \Delta)$, i.e. a $\mathbb{C}$-linear map from $\mathcal{A} \otimes_{\mathbb{C}} \mathcal{A}$ to $\mathbb{C}$ which is convolution invertible with convolution inverse γ, unital and satsifies, for all a, b, c in $\mathcal{A}$,

$$\overline{\gamma}(a_{(1)}b_{(1)} \otimes_{\mathbb{C}} c)\overline{\gamma}(a_{(2)} \otimes_{\mathbb{C}} b_{(2)}) = \overline{\gamma}(a \otimes_{\mathbb{C}} b_{(1)}c_{(1)})\overline{\gamma}(b_{(2)} \otimes_{\mathbb{C}} c_{(2)}).$$

Given a Hopf algebra $(\mathcal{A}, \Delta)$ and such a 2-cocycle γ as above, we have a new Hopf algebra $(\mathcal{A}_\gamma, \Delta_\gamma)$ as given by the following definition.

Definition 1.2.7. *([32]) If $(\mathcal{A}, \Delta)$ is a Hopf algebra and γ is a 2-cocycle as above, the cocycle deformed Hopf algebra is given by the pair $(\mathcal{A}_\gamma, \Delta_\gamma)$ where $\mathcal{A}_\gamma$ is equal to $\mathcal{A}$ as a vector space and the coproduct $\Delta_\gamma = \Delta$. The algebra structure $*_\gamma$ on $\mathcal{A}_\gamma$ is defined by the following equation:*

$$a *_\gamma b = \gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)})a_{(2)}b_{(2)}\overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} b_{(3)}). \tag{1.2.2}$$

Remark 1.2.8. *The deformation of $\mathcal{A}_\gamma$ by $\overline{\gamma}$ gives back $\mathcal{A}$, i.e. $(\mathcal{A}_\gamma)_{\overline{\gamma}} \cong \mathcal{A}$.*

As an example of Hopf algberas, we introduce the Hopf algebra $SU_q(2)$.

Definition 1.2.9. *For $q \in [-1,1]\backslash 0$, $SU_q(2)$ is the $*$-algebra generated by the two elements α, γ, and their adjoints, satisfying the following relations:*

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,$$
$$\gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication map Δ is given on the generators by

$$\Delta(\alpha) = \alpha\otimes_{\mathbb{C}}\alpha - q\gamma^*\otimes_{\mathbb{C}}\gamma, \quad \Delta(\gamma) = \gamma\otimes_{\mathbb{C}}\alpha + \alpha^*\otimes_{\mathbb{C}}\gamma.$$

For more details on Hopf algebras, we refer to the books [1, 23, 59, 61, 78, 87]. In this thesis, we will only deal with the algebraic aspects of Hopf algebras. For the analytic aspects, we refer to [62, 68, 75, 90, 91, 94].

Now we recall the notion of covariant bimodules over a Hopf algebra $\mathcal{A}$. Covariant bimodules have been studied (under the name Hopf-bimodules) by many algebraists including Abe ([1]) and Sweedler ([87]). They were introduced independently by Woronowicz ([93]) for studying differential calculi over Hopf algebras.

Definition 1.2.10. *Suppose M is an $\mathcal{A}$-bimodule such that (M, Δ_M) is a left $\mathcal{A}$-comodule. Then (M, Δ_M) is called a left-covariant bimodule if for all a in $\mathcal{A}$ and m in M, the following equations hold:*

$$\Delta_M(am) = \Delta(a)\Delta_M(m), \qquad \Delta_M(ma) = \Delta_M(m)\Delta(a).$$

Similarly, if ${}_M\Delta$ is a right comodule coaction on M, then $(M, {}_M\Delta)$ is called a right covariant bimodule if for any a in $\mathcal{A}$ and m in M,

$${}_M\Delta(am) = \Delta(a){}_M\Delta(m), \qquad {}_M\Delta(ma) = {}_M\Delta(m)\Delta(a).$$

Finally, let M be a bimodule over $\mathcal{A}$ and $\Delta_M : M \to \mathcal{A}\otimes_{\mathbb{C}}M$ and ${}_M\Delta : M \to M\otimes_{\mathbb{C}}\mathcal{A}$ be $\mathbb{C}$-linear maps. Then we say that $(M, \Delta_M, {}_M\Delta)$ is a bicovariant $\mathcal{A}$-bimodule if the following conditions are satisfied:

(i) (M, Δ_M) *is left-covariant bimodule,*

(ii) $(M, {}_M\Delta)$ *is a right-covariant bimodule,*

(iii) $(\mathrm{id}\otimes_{\mathbb{C}}{}_M\Delta)\Delta_M = (\Delta_M\otimes_{\mathbb{C}}\mathrm{id}){}_M\Delta$.

Note that, by definition, a bicovariant $\mathcal{A}$-bimodule is in particular an $\mathcal{A}$-bicomodule.

The vector space of left (respectively, right) invariant elements of a left (respectively, right) covariant bimodules will play a crucial role in Chapter 4, and we introduce notations for them here.

Definition 1.2.11. *Let (M, Δ_M) be a left-covariant bimodule over $\mathcal{A}$. The subspace of left-invariant elements of M is defined to be the vector space*

$$_0M := \{m \in M : \Delta_M(m) = 1 \otimes_{\mathbb{C}} m\}.$$

Similarly, if $(M, {}_M\Delta)$ is a right-covariant bimodule over $\mathcal{A}$, the subspace of right-invariant elements of M is the vector space

$$M_0 := \{m \in M : {}_M\Delta(m) = m \otimes_{\mathbb{C}} 1\}.$$

Let us note the immediate consequences of the above definitions.

Lemma 1.2.12. *(Theorem 2.4 of [93]) Suppose M is a bicovariant $\mathcal{A}$-bimodule. Then*

$$_M\Delta({}_0M) \subseteq {}_0M \otimes_{\mathbb{C}} \mathcal{A}, \qquad \Delta_M(M_0) \subseteq \mathcal{A} \otimes_{\mathbb{C}} M_0. \tag{1.2.3}$$

Explicitly, if $\{m_i\}_i$ is a (finite) basis of $_0M$, then there exist elements $\{a_{ji}\}_{ij}$ in $\mathcal{A}$ such that

$$_M\Delta(m_i) = \sum_j m_j \otimes_{\mathbb{C}} a_{ji}. \tag{1.2.4}$$

Proof. This is a simple consequence of the fact that $_M\Delta$ intertwines with Δ_M. □

Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant $\mathcal{A}$-bimodules. Then $(M \otimes_{\mathcal{A}} N, \Delta_{M \otimes_{\mathcal{A}} N}, {}_{M \otimes_{\mathcal{A}} N}\Delta)$ forms a bicovariant $\mathcal{A}$-bimodule, called the tensor product of M and N where the coactions $\Delta_{M \otimes_{\mathcal{A}} N} : M \otimes_{\mathcal{A}} N \to \mathcal{A} \otimes_{\mathbb{C}} M \otimes_{\mathcal{A}} N$ and ${}_{M \otimes_{\mathcal{A}} N}\Delta : M \otimes_{\mathcal{A}} N \to M \otimes_{\mathcal{A}} N \otimes_{\mathbb{C}} \mathcal{A}$ are defined by

$$\Delta_{M \otimes_{\mathcal{A}} N}(m \otimes_{\mathcal{A}} n) = m_{(-1)} n_{(-1)} \otimes_{\mathbb{C}} m_{(0)} \otimes_{\mathcal{A}} n_{(0)}$$
$${}_{M \otimes_{\mathcal{A}} N}\Delta(m \otimes_{\mathcal{A}} n) = m_{(0)} \otimes_{\mathcal{A}} n_{(0)} \otimes_{\mathbb{C}} m_{(1)} n_{(1)}.$$

We recall now the definition of covariant maps between bimodules.

Definition 1.2.13. *Let (M, Δ_M) and (N, Δ_N) be left-covariant $\mathcal{A}$-bimodules and T be a $\mathbb{C}$-linear map from M to N.*

T is called left-covariant if for all m in M, n in N, a in A,

$$(\mathrm{id} \otimes_{\mathbb{C}} T)(\Delta_M(m)) = \Delta_N(T(m)).$$

T is called a right-covariant map between right-covariant $\mathcal{A}$-bimodules $(M, {}_M\Delta)$ and $(N, {}_N\Delta)$ if for all m in M, n in N, a in A,

$$(T \otimes_{\mathbb{C}} \mathrm{id})({}_M\Delta(m)) = {}_N\Delta(T(m)).$$

Finally, a map which is both left and right covariant $\mathcal{A}$-bilinear map will be called a bicovariant map.

1.3 Noncommutative differential calculus

In this section, we shall recall the notion of differential calculi on algebras. In Subsections 1.3.1 and 1.3.2, we shall give two examples of constructions of differential calculi, given by Connes and Woronowicz respectively. These will be used extensively in the thesis.

Definition 1.3.1. *A differential calculus on an algebra $\mathcal{A}$ is a triple $(\Omega(\mathcal{A}), \wedge, d)$, where*

(i) $\Omega(\mathcal{A}) := \bigoplus_{k \geq 0} \Omega^k(\mathcal{A})$ is a graded $\mathcal{A}$-bimodule, i.e., each $\Omega^k(\mathcal{A})$ is an $\mathcal{A}$-bimodule, with $\Omega^0(\mathcal{A}) = \mathcal{A}$,

(iii) $\wedge : \Omega(\mathcal{A}) \otimes_{\mathcal{A}} \Omega(\mathcal{A}) \to \Omega(\mathcal{A})$ is a graded $\mathcal{A}$-bilinear map, i.e., it restricts to $\mathcal{A}$-bilinear maps

$$\wedge|_{\Omega^k(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^l(\mathcal{A})} : \Omega^k(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^l(\mathcal{A}) \to \Omega^{k+l}(\mathcal{A}),$$

(iii) $d : \Omega(\mathcal{A}) \to \Omega(\mathcal{A})$ is a graded exterior derivative with degree one, i.e. d restricts to maps

$$d|_{\Omega^k} : \Omega^k(\mathcal{A}) \to \Omega^{k+1}(\mathcal{A})$$

and for any ω, ω' *in* $\Omega(\mathcal{A})$,

$$d(\omega \wedge \omega') = d(\omega) \wedge \omega' + (-1)^{\deg \omega} \omega \wedge d(\omega'),$$

(iv) $\Omega^k(\mathcal{A}) = \mathrm{Span}_{\mathbb{C}}\{\sum d(a_0) \wedge d(a_1) \wedge \ldots d(a_{k-1}) a_k : a_l \in \mathcal{A}\}$.

The above definition is the noncommutative analogue of the following example:

Example 1.3.2. *Let* M *be a smooth manifold of dimension* n *and* $\mathcal{A}$ *be the algebra* $C^\infty(M)$ *of smooth functions on* M. *Let* $\Omega^k(M)$ *denote the* $\mathcal{A}$*-bimodule of* k*-forms on* M *and* $\Omega(M) = \oplus_{k=0}^n \Omega^k(M)$. *We have the de-Rham differential* $d : \Omega^k(M) \to \Omega^{k+1}(M)$ *and the classical wedge map* $\wedge : \Omega^k(M) \otimes_{\mathcal{A}} \Omega^l(M) \to \Omega^{k+l}(M)$. *The triplet* $(\Omega(M), \wedge, d)$ *is a differential calculus over the algebra* $\mathcal{A}$. *We will call this calculus the classical differential calculus on* M.

Definition 1.3.3. *A first order differential calculus on* $\mathcal{A}$ *is a pair* $(\Omega^1(\mathcal{A}), d)$ *where* $\Omega^1(\mathcal{A})$ *is an* $\mathcal{A}$*-bimodule and* $d : \mathcal{A} \to \Omega^1(\mathcal{A})$ *is a derivation such that* $\Omega^1(\mathcal{A}) = \mathrm{Span}_{\mathbb{C}}\{d(a)b : a, b \in \mathcal{A}\}$.

1.3.1 Spectral triples

In this subsection, we recall the notion of spectral triples due to Connes ([25]) and the construction of a differential calculus out of it. Spectral triples will be the basic objects of study in Chapters 2 and 3. Our references for this subsection are [25], [64].

Definition 1.3.4. *A spectral triple on a unital* $*$*-algebra* $\mathcal{A}$ *is a triple* $(\mathcal{A}, \mathcal{H}, D)$ *where* $\mathcal{H}$ *is a separable Hilbert space and* D *is a (possibly unbounded) self-adjoint operator on* $\mathcal{H}$ *such that the following conditions are satisfied:*

(i) there exists a faithful representation π *of* $\mathcal{A}$ *on* $\mathcal{H}$,

(i) for all a *in* $\mathcal{A}$, *the operator* $[D, \pi(a)]$ *has a bounded extension.*

In addition, if the operator $(1 + D^2)^{-\frac{1}{2}}$ *is compact, we say that the spectral triple* $(\mathcal{A}, \mathcal{H}, D)$ *is of compact type. A spectral triple of compact type is called finitely summable if there exists some* $p' > 0$ *such that* $\mathrm{Tr}(|D|^{-p'}) < \infty$. *The infimum* p *of all such admissible* p' *is called the dimension of the spectral triple and the spectral triple is called* p*-summable.*

Note that in the above definition, the functional Tr denotes the usual trace on $\mathcal{B}(\mathcal{H})$.

All the examples of spectral triples in this thesis will be of compact type. It is clear from the definition that the algebra $\mathcal{A}$ sits inside $\mathcal{B}(\mathcal{H})$ as a $*$-closed subalgebra $\{\pi(a) : a \in \mathcal{A}\}$. However, in order to simplify notations, we will often omit the representation π. Thus, we will simply write $[D, a]b$ to denote the bounded operator $[D, \pi(a)]\pi(b)$.

There is a canonical spectral triple of compact type associated to any compact Riemannian manifold. We explain this in the next example.

Example 1.3.5. *Let (M, g) be a compact Riemannian manifold of dimension m and $\Omega^k(M)$ be the space of smooth k-forms on M. The space of all differential forms $\Omega(M) = \bigoplus_{k=0}^m \Omega^k(M)$ can be made into a pre-Hilbert space via the pre-inner product given by*

$$\langle\langle \omega, \eta \rangle\rangle = \int_M \star(\omega) \wedge \eta d\mathrm{vol},$$

where $\star$ denotes the Hodge $\star$-map and dvol *denotes the volume form. Let $\mathcal{H}$ denote the Hilbert space completion of $\Omega(M)$. The $C^\infty(M)$ left-module structure on $\Omega^k(M)$ extends to define a representation π of $C^\infty(M)$ on $\mathcal{H}$.*

If d denotes the de-Rham differential and d^* its adjoint, then the Hodge-Dirac operator $d+d^*$ is a self-adjoint (densely defined) operator on $\mathcal{H}$. The triplet $(C^\infty(M), \mathcal{H}, d+d^*)$ forms a spectral triple of compact type which is m-summable.

Next we give an example of a compact spectral triple over a genuinely noncommutative algebra.

Example 1.3.6. *Recall the example (Example 1.1.18) of the noncommutative 2-torus $C(\mathbb{T}^2_\theta)$ generated by two unitary elements U and V. Consider the dense $*$-algebra $C^\infty(\mathbb{T}^2_\theta)$ of $C(\mathbb{T}^2_\theta)$ defined as:*

$$C^\infty(\mathbb{T}^2_\theta) := \{ \sum_{m,n \in \mathbb{Z}} a_{mn} U^m V^n : \mathrm{sup_{m,n}} |m^k n^l a_{mn}| < \infty \ \forall k, l \in \mathbb{N} \}.$$

We define two derivations d_1 and d_2 on $C^\infty(\mathbb{T}^2_\theta)$ defined on the generators U and V by

$$d_1(U) = U, \quad d_1(V) = 0,$$
$$d_2(U) = 0, \quad d_2(V) = V.$$

A faithful trace τ can be defined on $C^\infty(\mathbb{T}^2_\theta)$ by:

$$\tau(\sum_{m,n} a_{mn}U^mV^n) = a_{00}.$$

Define $\mathcal{H} := L^2(\tau) \oplus L^2(\tau)$ where $L^2(\tau)$ denotes the GNS Hilbert space of $C(\mathbb{T}^2_\theta)$ with respect to the state τ. The representation of $C(\mathbb{T}^2_\theta)$ on $\mathcal{H}$ is given by the diagonal embedding $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ into $\mathcal{B}(\mathcal{H})$. Finally, define

$$D := \begin{pmatrix} 0 & d_1 + id_2 \\ d_1 - id_2 & 0 \end{pmatrix}.$$

Then, $(C^\infty(\mathbb{T}^2_\theta), \mathcal{H}, D)$ gives a spectral triple of compact type on $C(\mathbb{T}^2_\theta)$.

For examples of spectral triples on $SU_q(2)$, we refer to [21] and [35]. We refer to [33] for an examples of a spectral triple on the Podles' spheres $S^2_{q,c}$. For spectral triples on q-deformations of compact semisimple Lie groups, we refer to [79].

Now we spell out the construction of the first order differential calculus (Definition 1.3.3) arising out of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$.

Definition 1.3.7. *The first order differential calculus on the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the pair $(\Omega^1_D(\mathcal{A}), d_D)$, where*

(i) $d_D := \sqrt{-1}[D, \cdot] : \mathcal{A} \to \mathcal{B}(\mathcal{H})$,

(ii) $\Omega^1_D(\mathcal{A}) = \mathrm{Span}_{\mathbb{C}}(d_D(a)b : a, b \in \mathcal{A})$.

Remark 1.3.8. *By the above definition, $\Omega^1_D(\mathcal{A})$ is a subset of $\mathcal{B}(\mathcal{H})$, and its $\mathcal{A}$-bimodule structure is inherited from $\mathcal{B}(\mathcal{H})$.*

For the rest of this subsection, we are going to dispense of the notation d_D and denote the derivative by d.

The space of two forms

The definition of the space of the higher order forms on $(\mathcal{A}, \mathcal{H}, D)$ requires a little more work. Even though $\mathcal{B}(\mathcal{H})$ comes with a natural multiplication map, which let us for the moment denote by m_0, this is not a good candidate for the $\wedge$ map of $\Omega_D(\mathcal{A})$. This is because there can possibly be a finite set $a_{i_0}, a_{i_1}, \ldots, a_{i_k}$ such that $\sum a_{i_0} d(a_{i_1}) \ldots d(a_{i_k}) = 0$ in $\mathcal{B}(\mathcal{H})$ (multiplication understood to be as between elements of $\mathcal{B}(\mathcal{H})$), but $\sum d(a_{i_0}) d(a_{i_1}) \ldots d(a_{i_k}) \neq 0$. This would in turn mean that d is not a well-defined exterior derivative. Since for the purpose of this thesis, we require only one-forms and two-forms of a calculus, we need only define the $\wedge$ and d maps onto the space of two-forms $\Omega^2_D(\mathcal{A})$. For this reason, we introduce the following definition.

Definition 1.3.9. *We denote by $m_0 : \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ the natural multiplication on the space of one-forms as a subspace of $\mathcal{B}(\mathcal{H})$. Moreover, we define $\mathcal{J}$, called the space of junk forms, to be the right $\mathcal{A}$-submodule of the range of m_0, to be denoted by* $\mathrm{Ran}(m_0)$, *spanned by elements of the set $\{\sum_i d(a_i)d(b_i) : a_i, b_i \in \mathcal{A}, \sum_i a_i d(b_i) = 0\}$.*

Lemma 1.3.10. *The space of junk forms $\mathcal{J}$ is closed under left and right $\mathcal{A}$-action. Thus, the quotient* $\mathrm{Ran}(m_0)/\mathcal{J}$ *is a well-defined $\mathcal{A}$-bimodule, and the composition of the quotient map with the multiplication map m_0 given by*

$$\Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \to \mathrm{Ran}(m_0) \to \mathrm{Ran}(m_0)/\mathcal{J}$$

is a well-defined $\mathcal{A}$-bilinear map.

Proof. $\mathcal{J}$ is by definition a right $\mathcal{A}$-submodule of $\mathrm{Ran}(m_0)$, so we have to verify that it is also closed under left $\mathcal{A}$-action. To this end, let $\sum_i a_i d(b_i) = 0$ be a finite sum, where a_i, b_i are in $\mathcal{A}$. For an arbitrary elements c in $\mathcal{A}$,

$$c \sum_i d(a_i) d(b_i) = \sum_i d(ca_i) d(b_i) - d(c) \sum_i a_i d(b_i) = \sum_I d(ca_i) d(b_i),$$

as $\sum_i a_i d(b_i) = 0$.

But since $\sum_i ca_i d(b_i) = 0$, we have that $\sum_i d(ca_i) d(b_i)$ is in $\mathcal{J}$, which implies that $\mathcal{J}$ is closed under left $\mathcal{A}$-action.

Since we have proved that $\mathcal{J}$ is an $\mathcal{A}$-sub-bimodule of $\mathrm{Ran}(m_0)$, the quotient $\mathrm{Ran}(m_0)/\mathcal{J}$ is a well-defined $\mathcal{A}$-bimodule and the quotient map is an $\mathcal{A}$-bilinear map. Since m_0 is the multiplication in $\mathcal{B}(\mathcal{H})$ and $\mathcal{A}$ is contained in $\mathcal{B}(\mathcal{H})$, m_0 is an $\mathcal{A}$-bilinear map, thus the composition of the quotient map and m_0 is also an $\mathcal{A}$-bilinear map. Thus we have our results. $\square$

Now we can satisfactorily make the following definition.

Definition 1.3.11. *Given a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we define the space of two-forms as $\Omega^2_D(\mathcal{A}) := \mathrm{Ran}(m_0)/\mathcal{J}$ as in Lemma 1.3.10. The map $\wedge : \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \to \Omega^2_D(\mathcal{A})$ is defined as the composition of the quotient map and the map m_0 as in the same lemma.*

For a compact Riemannian manifold M, consider the spectral triple $(C^\infty(M), \mathcal{H}, D)$ of Example 1.3.5. The operator

$$d : \mathcal{H}_0 := \overline{\Omega^0(M)} = L^2(M, d\mathrm{vol}) \to \mathcal{H}_1 := \overline{\Omega^1(M)}, \ d = \sqrt{-1}[D, \cdot]$$

is a densely defined operator which is closable. The operator

$$\mathcal{L} := -d^* d : \mathcal{H}_0 \to \mathcal{H}_0$$

is a self-adjoint (unbounded) operator whose domain contains $C^\infty(M)$. The operator $\mathcal{L}$ is called the Hodge-Laplacian and contains a lot of geometric information on the manifold.

In Chapter 2, we will need a noncommutative analogue of the Hodge Laplacian introduced by Goswami in [43]. So we record it here for later use. First we need to introduce an analogue of the volume form $d$vol for p-summable spectral triples. Here, and elsewhere, the domain of an unbounded operator will be denoted by $\mathrm{Dom}(T)$.

Definition 1.3.12. *For a p-summable spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the Dixmier trace on $\mathcal{B}(\mathcal{H})$ is given by the positive linear functional τ,*

$$\tau(X) = \mathrm{Lim}_\omega \frac{Tr(X|D|^{-p})}{Tr(|D|^{-p})},$$

where Lim_ω is as in Chapter 4 of [25].

For the spectral triple $(C^\infty(M), \mathcal{H}, D)$ of Example 1.3.5, the functional τ of the above definition gives back the volume form $d$vol. Then we have the following.

Proposition 1.3.13. *(Lemma 3.1, Lemma 3.2 and Lemma 5.1 of [43]) Let $(\mathcal{A}, \mathcal{H}, D)$ be a p-summable spectral triple of compact type and τ be as in Definition 1.3.12. We assume that the formula $\langle \eta, \eta' \rangle = \tau(\eta^* \eta')$ is a semi-inner product on the vector space $\Omega^1_D(\mathcal{A})$. We will denote the Hilbert space completion of $\Omega^1_D(\mathcal{A})$ under this semi-inner product by the symbol $\mathcal{H}^1_D$.*

We further assume that for all X in the $*$-algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$, the map

$$\mathbb{R} \to \mathcal{B}(\mathcal{H}),\ t \mapsto e^{itD} X e^{-itD}$$

is differentiable at $t = 0$ in the norm topology of $\mathcal{B}(\mathcal{H})$.

Consider the densely defined operator $d := \sqrt{-1}[D, \cdot] : L^2(\mathcal{A}, \tau) \to \mathcal{H}^1_D$. We have the following:

(i) d is closable. If $\mathcal{L}$ is defined to be the operator $-d^* d$, then $\mathcal{A} \subseteq \mathrm{Dom}(\mathcal{L})$ and $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}'' \subseteq \mathcal{B}(\mathcal{H})$.

(ii) If moreover, $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, then

(i) For all x in $\mathcal{A}$, we have

$$\mathcal{L}(x^*) = (\mathcal{L}(x))^*. \tag{1.3.1}$$

(ii) For all b, c in $\mathcal{A}$, the following equation holds:

$$d^*(d(b)c) = -\frac{1}{2}(b\mathcal{L}(c) - \mathcal{L}(b)c - \mathcal{L}(bc)). \tag{1.3.2}$$

1.3.2 Bicovariant differential calculi

In Section 1.2, we had recalled the notions of Hopf algebras and covariant bimodules. In this subsection, we recall bicovariant differential calculi on Hopf algebras.

Definition 1.3.14. *(Definitions 1.2, 1.3 of [93]) Let $(\mathcal{E}, d)$ be a first order differential calculus on a Hopf algebra $\mathcal{A}$.*
We say that $(\mathcal{E}, d)$ is left-covariant if for any a_k, b_k in $\mathcal{A}$, $k = 1, \ldots, K$,

$$(\sum_k a_k db_k = 0) \text{ implies that } (\sum_k \Delta(a_k)(\mathrm{id} \otimes_{\mathbb{C}} d)\Delta(b_k) = 0).$$

We say that $(\mathcal{E}, d)$ is right-covariant if for any a_k, b_k in $\mathcal{A}$, $k = 1, \ldots, K$,

$$(\sum_k a_k db_k = 0) \text{ implies that } (\sum_k \Delta(a_k)(d \otimes_{\mathbb{C}} \mathrm{id})\Delta(b_k) = 0).$$

We say $(\mathcal{E}, d)$ is bicovariant if it is both left-covariant and right-covariant.

Bicovariant differential calculi on Hopf algebras have been studied by many mathematicians (including [20, 42, 45, 52, 53, 69, 92, 93] and references therein). Majid and Oeckl ([74]) proved that if $(\mathcal{E}, d)$ is a bicovariant differential calculus on a Hopf algebra $\mathcal{A}$ and γ is a 2-cocycle (in the sense of Definition 1.2.6), then $(\mathcal{E}, d)$ can be twisted to a bicovariant differential calculus $(\mathcal{E}_\gamma, d_\gamma)$ over the twisted Hopf algebra $\mathcal{A}_\gamma$ (as in Definition 1.2.7). This particular example will be studied in Chapter 5. The bicovariant $4D_\pm$ calculi on the Hopf algebra $SU_q(2)$ will be studied in Chapter 6. For more details on bicovariant differential calculi and their classifications, we refer to the books [61], [71] and references therein. For examples of covariant differential calculi on quantum homogeneous spaces, we refer to [47–50] and references therein.

Woronowicz ([93]) proved that a bicovariant differential calculus is automatically endowed with a left as well as a right comodule coaction.

Proposition 1.3.15. *(Propositions 1.2, 1.3 and 1.4 of [93]) Let $(\mathcal{E}, d)$ be a bicovariant first order differential calculus on $\mathcal{A}$. Then there exists linear mappings*

$$\Delta_{\mathcal{E}} : \mathcal{E} \to \mathcal{A} \otimes_{\mathbb{C}} \mathcal{E}, \quad {}_{\mathcal{E}}\Delta : \mathcal{E} \to \mathcal{E} \otimes_{\mathbb{C}} \mathcal{A}$$

such that

(i) $(\mathcal{E}, \Delta_{\mathcal{E}}, {}_{\mathcal{E}}\Delta)$ *is a bicovariant $\mathcal{A}$-bimodule, i.e. $(\mathcal{E}, \Delta_{\mathcal{E}})$ is a left $\mathcal{A}$-comodule, $(\mathcal{E}, {}_{\mathcal{E}}\Delta)$ is a right $\mathcal{A}$-comodule and the following equations hold for all e in $\mathcal{E}$ and a in $\mathcal{A}$:*

$$\Delta_{\mathcal{E}}(ae) = \Delta(a)\Delta_{\mathcal{E}}(e), \quad \Delta_{\mathcal{E}}(ea) = \Delta_{\mathcal{E}}(e)\Delta(a) \tag{1.3.3}$$

$${}_{\mathcal{E}}\Delta(ae) = \Delta(a)\,{}_{\mathcal{E}}\Delta(e), \quad {}_{\mathcal{E}}\Delta(ea) = {}_{\mathcal{E}}\Delta(e)\Delta(a) \tag{1.3.4}$$

$$(\mathrm{id} \otimes_{\mathbb{C}} {}_{\mathcal{E}}\Delta)\Delta_{\mathcal{E}} = (\Delta_{\mathcal{E}} \otimes_{\mathbb{C}} \mathrm{id})\,{}_{\mathcal{E}}\Delta \tag{1.3.5}$$

(ii) d is bicovariant, i.e.

$$\Delta_{\mathcal{E}} \circ d = (\mathrm{id} \otimes_{\mathbb{C}} d)\Delta \quad {}_{\mathcal{E}}\Delta \circ d = (d \otimes_{\mathbb{C}} \mathrm{id})\Delta. \tag{1.3.6}$$

We note the following consequence of Proposition 1.3.15.

Lemma 1.3.16. *For any $a \in \mathcal{A}$, the following equations holds:*

(i) $a_{(1)} \otimes_{\mathbb{C}} d(a_{(2)}) = (da)_{(-1)} \otimes_{\mathbb{C}} (da)_{(0)}$

(ii) $d(a_{(1)})\otimes_{\mathbb{C}}a_{(2)} = (da)_{(0)}\otimes_{\mathbb{C}}(da)_{(1)}$

(iii) $a_{(1)}\otimes_{\mathbb{C}}d(a_{(2)})\otimes_{\mathbb{C}}a_{(3)} = (da)_{(-1)}\otimes_{\mathbb{C}}(da)_{(0)}\otimes_{\mathbb{C}}(da)_{(1)}$

Proof. Part (i) and part (ii) follow from (1.3.6). For Part (iii), we have

$$
\begin{aligned}
& a_{(1)}\otimes_{\mathbb{C}}d(a_{(2)})\otimes_{\mathbb{C}}a_{(3)} = (\mathrm{id}\otimes_{\mathbb{C}}d\otimes_{\mathbb{C}}\mathrm{id})(\mathrm{id}\otimes_{\mathbb{C}}\Delta)\Delta(a) \\
= \ & (\mathrm{id}\otimes_{\mathbb{C}\mathcal{E}}\Delta)(\mathrm{id}\otimes_{\mathbb{C}}d)\Delta(a)\ (\mathrm{by}(1.3.6)) = (\mathrm{id}\otimes_{\mathbb{C}\mathcal{E}}\Delta)\Delta_{\mathcal{E}}(da)\ (\mathrm{by}(1.3.6)) \\
= \ & (\mathrm{id}\otimes_{\mathbb{C}\mathcal{E}}\Delta)((da)_{(-1)}\otimes_{\mathbb{C}}(da)_{(0)}) = (da)_{(-1)}\otimes_{\mathbb{C}\mathcal{E}}\Delta((da)_{(0)}) \\
= \ & (da)_{(-1)}\otimes_{\mathbb{C}}(da)_{(0)}\otimes_{\mathbb{C}}(da)_{(1)}.
\end{aligned}
$$

This proves the lemma. □

The space of two-forms

Following Woronowicz ([93]), let us define the space of two forms associated to a bicovariant different calculus. For this we need to recall the following fundamental result concerning bicovariant bimodules.

Proposition 1.3.17. *(Proposition 3.1 of [93]) Given two bicovariant $\mathcal{A}$-bimodules $\mathcal{E}$ and $\mathcal{F}$, there exists a unique bimodule homomorphism $\sigma : \mathcal{E}\otimes_{\mathcal{A}}\mathcal{F} \to \mathcal{F}\otimes_{\mathcal{A}}\mathcal{E}$ such that $\sigma(\omega\otimes_{\mathcal{A}}\eta) = \eta\otimes_{\mathcal{A}}\omega$ for any left-invariant element ω in $\mathcal{E}$ and right-invariant element η in $\mathcal{F}$.*
In particular, taking $\mathcal{F} = \mathcal{E}$, there exists a unique bimodule homomorphism

$$\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \text{ such that}$$

$$\sigma(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega \tag{1.3.7}$$

for any left-invariant element ω and right-invariant element η in $\mathcal{E}$. σ is invertible and is a bicovariant $\mathcal{A}$-bimodule map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself, i.e

$$(\mathrm{id}_{\mathcal{A}} \otimes_{\mathcal{A}} \sigma)\Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} = \Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} \circ \sigma, \qquad (\sigma \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{A}})_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta = {}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta \circ \sigma. \tag{1.3.8}$$

Moreover, σ satisfies the following braid equation on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$:

$$(\mathrm{id} \otimes_{\mathcal{A}} \sigma)(\sigma \otimes_{\mathcal{A}} \mathrm{id})(\mathrm{id} \otimes_{\mathcal{A}} \sigma) = (\sigma \otimes_{\mathcal{A}} \mathrm{id})(\mathrm{id} \otimes_{\mathcal{A}} \sigma)(\sigma \otimes_{\mathcal{A}} \mathrm{id}).$$

Then, the space of two-forms is defined as follows:

Definition 1.3.18. *Let $(\mathcal{E}, d)$ be a bicovariant first order differential calculus and σ be the map as in Proposition 1.3.17. We define*

$$\Omega^2(\mathcal{A}) := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})\big/\mathrm{Ker}(\sigma - 1).$$

The symbol $\wedge$ will denote the quotient map

$$\wedge : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^2(\mathcal{A}).$$

Finally, we will denote $\mathrm{Ker}(\wedge)$ *by the symbol* $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$*. Thus,*

$$\mathrm{Ker}(\wedge) = \mathrm{Ker}(\sigma - 1) = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} \tag{1.3.9}$$

From now on, we will use the notation $\rho \wedge \rho' := \wedge(\rho \otimes_{\mathcal{A}} \rho')$, for elements ρ, ρ' in $\mathcal{E}$.

Proposition 1.3.19. *The subspace $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} :=$ $\mathrm{Ker}(\wedge)$ is a bicovariant sub-bimodule of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Moreover, the space $\Omega^2(\mathcal{A})$ is a bicovariant bimodule, and the quotient map $\wedge$ is a bicovariant bimodule map.*

Proof. Since the map σ is a bicovariant bimodule map, $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\sigma - 1)$ is a sub-bimodule of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ invariant under the left and right coactions of $\mathcal{A}$. The quotient $\Omega^2(\mathcal{A}) = (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})\big/\mathrm{Ker}(\sigma - 1)$ is the cokernel of the inclusion map $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} \hookrightarrow \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Hence, $\Omega^2(\mathcal{A})$ is a bicovariant $\mathcal{A}$-bimodule, and the quotient map $\wedge$ is a bicovariant bimodule map. □

The other order forms are defined similarly. We refer to [93] for the details. In particular, we have $\Omega^0(\mathcal{A}) = \mathcal{A}$ and $\Omega^1(\mathcal{A}) = \mathcal{E}$. Then $\wedge$ extends to a map

$$\wedge : \Omega^k(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^l(\mathcal{A}) \to \Omega^{k+l}(\mathcal{A}).$$

Collecting all the notations and results, we have the following proposition.

Proposition 1.3.20. *([93]) Suppose $(\mathcal{E}, d)$ is a first order bicovariant differential calculus on $\mathcal{A}$ and $\Omega^2(\mathcal{A})$ is the space of two-forms. The left and right comodule coactions $\Delta_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ and ${}_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}\Delta$ of $\mathcal{A}$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ descend to comodule coactions of $\mathcal{A}$ on $\Omega^2(\mathcal{A})$ as* $\mathrm{Ker}(\sigma - 1)$ *is left and* *right-invariant. This makes $\Omega^2(\mathcal{A})$ a bicovariant $\mathcal{A}$-bimodule. The same is true for $\Omega^k(\mathcal{A})$ for all $k \geq 0$.*

Moreover, the map d extends uniquely to a bicovariant map from $\oplus_{k\geq 0}\Omega^k(\mathcal{A})$ to itself and satisfies $d^2 = 0$ and

$$d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta'$$

if θ is in $\Omega^k(\mathcal{A})$.

Our definition of two-forms is in general different than that considered in [51].

Remark 1.3.21. *Suppose $\mathcal{A}$ is a q-deformation of a classical compact semisimple Lie group and $\mathcal{E}$ be a bicovariant bimodule over $\mathcal{A}$. Then typically, the (q-dependent) eigenvalues of σ consist of real numbers other than ± 1. Let I be the set of eigenvalues of σ which tend (in limit) to 1 as q tends to 1.*

The authors of [20] define

$$\Omega^2(\mathcal{A}) := \frac{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}{\Pi_{\lambda \in I}(\sigma - \lambda\ \ )}.$$

It is this definition of $\Omega^2(\mathcal{A})$ which was taken in [51]. Thus, the definition of two-forms considered in this chapter are different than that in [51] unless the only eigenvalues of σ are ± 1.

1.4 Connections in classical and noncommutative geometry

Connections on noncommutative differential calculi are the main focus of study in this thesis. The rest of the thesis will then be devoted to giving a coherent definition of Levi-Civita connections on some classes of noncommutative differential calculi, and investigating their existence and uniqueness.

1.4.1 Levi-Civita connections in Riemannian geometry

Throughout this subsection, M will denote a smooth manifold and $\mathcal{A}$ will stand for the unital algebra $C^\infty(M)$ of smooth functions on a smooth manifold M. The symbols $\mathfrak{X}(\mathcal{A})$ and $\Omega^k(M)$ will denote the Lie algebra of smooth vector fields and the set of all smooth k-forms on M respectively. In particular, $\Omega^1(M) = \mathrm{Hom}_{\mathcal{A}}(\mathfrak{X}(\mathcal{A}), \mathcal{A})$. Since $\mathcal{A}$ is commutative, the right $\mathcal{A}$-modules $\mathfrak{X}(\mathcal{A})$ and $\Omega^k(M)$ are $\mathcal{A}$-bimodules in a natural way.

Vector fields on M are in one to one correspondence with derivations of the algebra $\mathcal{A} = C^\infty(M)$. This correspondence maps X in $\mathfrak{X}(\mathcal{A})$ to the map $\delta_X : \mathcal{A} \to \mathcal{A}$ defined by

$$\delta_X(a) = da(X).$$

Since δ_X is a derivation, it satisfies the identity

$$\delta_X(ab) = \delta_X(a)b + a\delta_X(b)$$

for all a, b in $\mathcal{A}$.

We have an $\mathcal{A}$-bilinear map

$$\wedge : \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M) \to \Omega^2(M),\ \omega \otimes_{\mathcal{A}} \eta \mapsto \omega \wedge \eta.$$

The space of k-forms $\Omega^k(M)$ are spanned (as right $\mathcal{A}$-modules) by elements of the form $da_1 \wedge \cdots \wedge da_k$ where $a_1, \cdots, a_k$ are elements in $\mathcal{A}$. The next proposition collects some well-known facts about the spaces of one and two-forms on a manifold. We will see that any tame differential calculi on a (possibly) noncommutative algebra (Chapter 2) and certain bicovariant differential calculi on a Hopf algebra (Chapter 4) satisfy some of these properties.

Proposition 1.4.1. *The following short-exact sequence of right $\mathcal{A}$-modules splits:*

$$0 \to \mathrm{Ker}(\wedge) \to \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M) \to \Omega^2(M) \to 0.$$

We have

$$\Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M) = \mathrm{Ker}(\wedge) \oplus \mathcal{F} \tag{1.4.1}$$

where $\mathcal{F}$ is isomorphic to $\Omega^2(M)$ as right $\mathcal{A}$-modules. Concretely, we have

$$\mathrm{Ker}(\wedge) = \{\omega \otimes_{\mathcal{A}} \eta + \eta \otimes_{\mathcal{A}} \omega : \omega, \eta \in \Omega^1(M)\},\ \mathcal{F} = \{\omega \otimes_{\mathcal{A}} \eta - \eta \otimes_{\mathcal{A}} \omega : \omega, \eta \in \Omega^1(M)\}.$$

Let $P_{\mathrm{sym}} : \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M) \to \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M)$ be the idempotent with range $\mathrm{Ker}(\wedge)$ *and kernel equal to $\mathcal{F}$. Then the map P_{sym} is $\mathcal{A}$-bilinear and given by the formula:*

$$P_{\mathrm{sym}}(\omega \otimes_{\mathcal{A}} \eta) = \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta + \eta \otimes_{\mathcal{A}} \omega)$$

for all ω, η in $\Omega^1(M)$.

Definition 1.4.2. *A connection on a manifold M is a $\mathbb{C}$-linear map $\nabla : \Omega^1(M) \to \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M)$ such that*

$$\nabla(\omega a) = \nabla(\omega)a + \omega \otimes_{\mathcal{A}} da$$

for all ω in $\Omega^1(M)$ and for all a in $\mathcal{A}$.

A covariant derivative on M is a map

$$\mathfrak{X}(\mathcal{A}) \times \mathfrak{X}(\mathcal{A}) \to \mathfrak{X}(\mathcal{A}), (X, Y) \mapsto \nabla_Y X$$

such that for all a in $\mathcal{A}$ and for all X, Y, X', Y' in $\mathfrak{X}(\mathcal{A})$, the following equations hold:

$$\nabla_Y(X + X') = \nabla_Y X + \nabla_Y X', \ \nabla_{Y+Y'}(X) = \nabla_Y X + \nabla_{Y'} X,$$
$$\nabla_{Ya} X = (\nabla_Y X)a, \ \nabla_Y(Xa) = (\nabla_Y X)a + X\delta_Y(a).$$

The notions of connections and covariant derivatives are equivalent. Indeed, given a connection ∇ on $\Omega^1(M)$, a covariant derivative on the level of vector fields is uniquely defined by the following equation for all ω in $\Omega^1(\mathcal{A})$:

$$\omega(\nabla_Y X) = \delta_Y(\omega(X)) - (\nabla(\omega))(X \otimes_{\mathcal{A}} Y). \tag{1.4.2}$$

Conversely, given a covariant derivative, a connection ∇ can be recovered from (1.4.2).

Now we recall the notion of torsionless-ness of a connection.

Definition 1.4.3. *A connection ∇ on M is called torsionless if the covariant derivative defined by* (1.4.2) *satisfies the following equation for all X, Y in $\mathfrak{X}(\mathcal{A})$:*

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0.$$

Let us state the definition of a pseudo-Riemannian metric on a manifold. In this section, the symbol flip : $\Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M) \to \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M)$ will denote the flip map, i.e, $\mathrm{flip}(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega$.

Definition 1.4.4. *A pseudo-Riemannian metric on a manifold M is a right $\mathcal{A}$-linear map*

$$g : \Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M) \to \mathcal{A} \text{ such that}$$

(i) $g \circ \mathrm{flip} = g$,

(ii) the map

$$V_g : \Omega^1(M) \to \mathrm{Hom}_{\mathcal{A}}(\Omega^1(M), \mathcal{A}), \; V_g(\omega)(\eta) = g(\omega \otimes_{\mathcal{A}} \eta)$$

is an isomorphism of right $\mathcal{A}$-modules.

Remark 1.4.5. *Since $\mathcal{A} = C^\infty(M)$ is commutative and $a\omega = \omega a$ for all a in $\mathcal{A}$ and ω in $\Omega^1(M)$, a pseudo-Riemannian metric g is automatically left $\mathcal{A}$-linear. Thus, g is $\mathcal{A}$-bilinear. The same is true for the map V_g.*

Throughout this thesis, we will not postulate the condition of positive definiteness in the definition of a metric. The next proposition shows that pseudo-Riemannian metrics as in Definition 1.4.4 are in one to one correspondence with the usual notion of pseudo Riemannian metrics on the level of vector fields.

Proposition 1.4.6. *Let $\widetilde{\mathrm{flip}} : \mathfrak{X}(\mathcal{A}) \otimes_{\mathcal{A}} \mathfrak{X}(\mathcal{A}) \to \mathfrak{X}(\mathcal{A}) \otimes_{\mathcal{A}} \mathfrak{X}(\mathcal{A})$ denote the flip map on the level of vector fields and g be a pseudo-Riemannian metric on M as in Definition 1.4.4. Then*

$$\widetilde{g} : \mathfrak{X}(\mathcal{A}) \otimes_{\mathcal{A}} \mathfrak{X}(\mathcal{A}) \to \mathcal{A}, \; \widetilde{g}(X \otimes_{\mathcal{A}} Y) = g(V_g^{-1}(X) \otimes_{\mathcal{A}} V_g^{-1}(Y))$$

coincides with the usual notion of a smooth pseudo-Riemannian metric in differential geometry, i.e, $\widetilde{g} \circ \widetilde{\mathrm{flip}} = \widetilde{g}$ and on a co-ordinate neighborhood (U, x) of M, the matrix $((\widetilde{g}_{ij}))_{ij} = ((\widetilde{g}(\frac{\partial}{\partial x_i} \otimes_{\mathcal{A}} \frac{\partial}{\partial x_j})))_{ij}$ is an invertible matrix with entries in $C^\infty(U)$.

Proof. We begin by observing that as g maps $\Omega^1(M) \otimes_{\mathcal{A}} \Omega^1(M)$ to $\mathcal{A}$, the range of $\widetilde{g}$ also lies in $\mathcal{A}$. If X, Y belong to $\mathfrak{X}(\mathcal{A})$, we have

$$\begin{aligned} \widetilde{g}(\widetilde{\mathrm{flip}}(X \otimes_{\mathcal{A}} Y)) &= \widetilde{g}(Y \otimes_{\mathcal{A}} X) = g(V_g^{-1}(Y) \otimes_{\mathcal{A}} V_g^{-1}(X)) = g(\mathrm{flip}(V_g^{-1}(X) \otimes_{\mathcal{A}} V_g^{-1}(Y))) \\ &= g(V_g^{-1}(X) \otimes_{\mathcal{A}} V_g^{-1}(Y)) = \widetilde{g}(X \otimes_{\mathcal{A}} Y). \end{aligned}$$

Next, if (U, x) is a co-ordinate of M, then by an usual partition of unity argument, we can extend the local vector fields $\frac{\partial}{\partial x_i}$ smoothly to the whole of M. Then since $\widetilde{g}(\mathfrak{X}(\mathcal{A}) \otimes_{\mathcal{A}} \mathfrak{X}(\mathcal{A})) \subseteq \mathcal{A}$, $\widetilde{g}_{ij} \in \mathcal{A}$ and in particular, the map $U \to \mathbb{C}, \; m \mapsto \widetilde{g}_{ij}(m)$ belongs to $C^\infty(U)$.

Finally, we need to check the invertibility of the matrix $(\widetilde{g}_{ij})$. We view the local vector fields $\frac{\partial}{\partial x_i}$ as elements of $\mathrm{Hom}_{\mathcal{A}}(\Omega^1(M), \mathcal{A})$. Since g is a pseudo-Riemannian metric, the condition (ii) of Definition 1.4.4 implies that the matrix $((g_{ij}))_{ij}$ is invertible, where $g_{ij} = g(dx_i \otimes_{\mathcal{A}} dx_j)$. Denote the inverse of the matrix by the notation $((g^{ij}))_{ij}$. Then the following computation shows that the matrix $g^t = (((g_{ij}))_{ij})^t$ is the inverse to the matrix $((\widetilde{g}_{ij}))_{ij}$ and in particular, $((\widetilde{g}_{ij}))_{ij}$ is an invertible matrix.

$$
\begin{aligned}
\sum_j \widetilde{g}_{ij}(g^t)_{jk} = & \sum_j \widetilde{g}_{ij} g_{kj} = \sum_j \widetilde{g}(\frac{\partial}{\partial x_i} \otimes_{\mathcal{A}} g_{kj} \frac{\partial}{\partial x_j}) \\
= & \widetilde{g}(\frac{\partial}{\partial x_i} \otimes_{\mathcal{A}} V_g(dx_k)) \text{ (since } V_g(dx_k) = \sum_j g_{kj} \frac{\partial}{\partial x_j}) \\
= & \widetilde{g}(\sum_l g^{il} V_g(dx_l) \otimes_{\mathcal{A}} V_g(dx_k)) \text{ (since } \frac{\partial}{\partial x_i} = \sum_l g^{il} V_g(dx_l)) \\
= & \sum_l g^{il} g_{lk} \text{ (by the definition of } \widetilde{g}) \\
= & \delta_{ik}.
\end{aligned}
$$

□

The following definition is regarding the compatibility of a connection with respect to a pseudo-Riemannian metric in terms of the associated covariant derivative.

Definition 1.4.7. *Suppose g is a pseudo-Riemannian metric on a manifold M. A connection ∇ on M is said to be compatible with g if*

$$\delta_Y(\widetilde{g}(Z \otimes_{\mathcal{A}} X)) = \widetilde{g}(\nabla_Y Z \otimes_{\mathcal{A}} X) + \widetilde{g}(\nabla_Y X \otimes_{\mathcal{A}} Z)$$

for all X, Y, Z in $\mathfrak{X}(\mathcal{A})$.

Now we state some equivalent criteria for a connection to be torsionless and compatiblity with a pseudo-Riemannian metric. These criteria are well-known but since we were unable to find a reference which states these criteria exactly as we need them, let us refer to Proposition 5.1 and Proposition 5.4 of [14] for the proofs.

Proposition 1.4.8. *A connection ∇ on a manifold M is torsionless if and only if*

$$\wedge \circ \nabla(\omega) = -d\omega \text{ for all } \omega \text{ in } \Omega^1(M).$$

If g is a pseudo-Riemannian metric on M, then a connection ∇ is compatible with g if and only if

$$(g \otimes_{\mathcal{A}} \mathrm{id})[\mathrm{flip}_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta) + (\omega \otimes_{\mathcal{A}} \nabla(\eta))] = d(g(\omega \otimes_{\mathcal{A}} \eta))$$

for all ω, η in $\Omega^1(M)$.

If g is a pseudo-Riemannian metric on a manifold M and ∇ is a connection on M which is torsionless and compatible with g, then ∇ is called a Levi-Civita connection. The fundamental theorem of differential geometry states that such a connection exists uniquely.

Theorem 1.4.9. *(Levi-Civita's theorem) If g is a pseudo-Riemannian metric on a manifold M, there exists a unique connection on M which is torsionless and compatible with g.*

The goal of this thesis is to prove this theorem for some bimodules over a (possibly) noncommutative algebra.

1.4.2 Connections on a noncommutative differential calculus

In Subsection 1.4.1, we discussed connections on the module of one-forms of manifolds (Definition 1.4.2), and their torsion (Proposition 1.4.8). This motivates the following definitions.

Definition 1.4.10. *Given a first order differential calculus $(\Omega^1(\mathcal{A}), d)$ on an algebra $\mathcal{A}$, a (right) connection ∇ on the space of one-forms $\Omega^1(\mathcal{A})$ is a $\mathbb{C}$-linear map $\nabla : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that, for any ω in $\Omega^1(\mathcal{A})$ and a in $\mathcal{A}$,*

$$\nabla(\omega a) = \nabla(\omega)a + \omega \otimes_{\mathcal{A}} d(a).$$

Definition 1.4.11. *Given a differential calculus $(\Omega(\mathcal{A}), \wedge, d)$ on an algebra $\mathcal{A}$, the torsion T_∇ of a connection ∇ is a right $\mathcal{A}$-linear map $T_\nabla : \wedge \circ \nabla + d : \Omega^1(\mathcal{A}) \to \Omega^2(\mathcal{A})$. A connection ∇ is said to be torsionless if $T_\nabla = 0$.*

Remark 1.4.12. *There are articles in literature which work with left connections (for example [41], [51]). A left connection is a $\mathbb{C}$-linear map $\nabla : \Omega^1(\mathcal{A}) \to \Omega^1(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1(\mathcal{A})$ such that*

$\nabla(a\omega) = a\nabla(\omega) + d(a) \otimes_{\mathcal{A}} \omega$. *In this case, the torsion is defined to be the map* $\wedge \circ \nabla - d$. *In this thesis, we will only work with right connections.*

Chapter 2

Levi-Civita connections on tame spectral triples

In this chapter, we deal with the question of existence of Levi-Civita connections for a class of spectral triples which we call tame spectral triples. As explained before, the formulation of the question of existence and uniqueness of Levi-Civita connection for a bimodule $\mathcal{E}$ over a (possibly) noncommutative algebra $\mathcal{A}$ needs two ingredients: firstly, an analogue of the flip map and a metric compatibility condition. We start with a class of spectral triples called quasi-tame spectral triples (see Section 2.2) with the bimodule of one-forms $\mathcal{E}$ which postulates a decomposition of the $\mathcal{A}$-bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ analogous to Proposition 1.4.1. This gives rise to a canonical $\mathcal{A}$-bilinear map $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ (satisfying $\sigma^2 = \mathrm{id}$) which plays the role of the flip map in this chapter. As a bonus, we also demonstrate that if $\mathcal{E}$ is the bimodule of one-forms on a quasi-tame spectral triple, then $\mathcal{E}$ always admits a torsionless connection. In Section 2.3 we define and study the notion of pseudo-Riemannian metrics on a quasi-tame spectral triple. We also give a candidate of a canonical Riemannian bilinear metric for a spectral triple. This solves the first problem.

Next, in order to formulate the metric-compatibility condition, we will work with a smaller class of spectral triples which are the tame spectral triples introduced in Section 2.4. In the same section, we also introduce the notion of compatibility of connections associated to tame spectral triple with pseudo-Riemannian metrics. In Theorem 2.5.1, we prove that if g is a pseudo-Riemannian bilinear metric on the space of one-forms of a tame spectral triple, then there exists a unique connection on the space of one-forms of the spectral triple which is torsionless and compatible with g. Examples of tame spectral triples will be given in the next chapter. At the end of this chapter, we have compared our approach with some of the existing works in the literature.

All algebras discussed here will be unital and all spectral triples will be of compact type. The contents of this chapter are from [16].

2.1 Centered Bimodules

As mentioned above, the existence and uniqueness theorem for Levi-Civita connections that we are going to prove works for tame spectral triples. We will soon see (Remark 2.4.2) that the space of one-forms of a tame spectral triple is a centered bimodule. In this section, we recall the definition of centered bimodules and discuss some of their properties which will be useful for us.

Definition 2.1.1. *The center of a bimodule $\mathcal{E}$ over an algebra $\mathcal{A}$ is defined to be the set*

$$\mathcal{Z}(\mathcal{E}) = \{e \in \mathcal{E} : ea = ae \ \forall \ a \ \in \mathcal{A}\}.$$

The bimodule $\mathcal{E}$ is called centered if $\mathcal{Z}(\mathcal{E})$ is right $\mathcal{A}$-total in $\mathcal{E}$, i.e, the right $\mathcal{A}$-linear span of $\mathcal{Z}(\mathcal{E})$ is equal to $\mathcal{E}$.

From the above definition, it is easy to see that $\mathcal{Z}(\mathcal{E})$ is a $\mathcal{Z}(\mathcal{A})$-bimodule. Indeed, if e is an element of $\mathcal{Z}(\mathcal{E})$, a belongs to $\mathcal{Z}(\mathcal{A})$ and b belongs to $\mathcal{A}$, then

$$b(ea) = eba = (ea)b.$$

Remark 2.1.2. *In [36], a related notion called central bimodules is defined. An $\mathcal{A}$-bimodule $\mathcal{E}$ is called central if every element e in $\mathcal{E}$ commutes with every element of $\mathcal{Z}(\mathcal{A})$.*
If $\mathcal{E}$ is a centered module in the sense of Definition 2.1.1, then it is also central. Indeed, if e is an element of the centered bimodule $\mathcal{E}$, then there exists some elements e_i in $\mathcal{Z}(\mathcal{E})$ and a_i in $\mathcal{A}$, for a finite number i, such that $e = \sum_i e_i a_i$. Then, for an arbitrary element a' in $\mathcal{Z}(\mathcal{A})$,

$$ea' = \sum_i e_i a_i a' = a' \sum_i e_i a_i = a'e.$$

The converse however, does not hold in general as the notion of centered bimodules is more stringent.

The following example motivates our interest in centered bimodules.

Example 2.1.3. *If $\mathcal{A} = C^\infty(M)$ for some compact manifold M, and $\Gamma(E)$ is the $\mathcal{A}$-bimodule of sections of some smooth vector bundle E on M, then since $\mathcal{A}$ is commutative, the right $\mathcal{A}$-action on $\mathcal{E}$ can be defined to be the left $\mathcal{A}$-action and so $\Gamma(E)$ is centered. In particular, the $\mathcal{A}$-bimodule $\Omega^k(M)$ of k-forms on M is centered.*

The following is an example of a centered bimodule over a noncommutative algebra.

Example 2.1.4. *Suppose $\mathcal{A}$ is a (possibly) noncommutative unital algebra. Then $\mathcal{E} := \mathcal{A} \oplus \cdots \oplus \mathcal{A}$, i.e. the direct sum of finitely many copies of $\mathcal{A}$, is a centered $\mathcal{A}$-bimodule. The center $\mathcal{Z}(\mathcal{E})$ is given by $\mathcal{Z}(\mathcal{A}) \oplus \cdots \oplus \mathcal{Z}(\mathcal{A})$. It is easy to see that $\mathcal{Z}(\mathcal{E})$ is right $\mathcal{A}$-total in $\mathcal{E}$.*

As an immediate corollary to the definition of centered bimodules, we have the following lemma:

Lemma 2.1.5. *Suppose $\mathcal{E}$ is a centered bimodule over $\mathcal{A}$. Then the following statements hold:*

(i) *$\mathcal{Z}(\mathcal{E})$ is also left $\mathcal{A}$-total in $\mathcal{E}$.*

(ii) *The set $\{\omega \otimes_{\mathcal{A}} \eta : \omega, \eta \in \mathcal{Z}(\mathcal{E})\}$ is both left and right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$.*

(iii) *If X is an element of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, there exist ω_i in $\mathcal{E}$, η_i in $\mathcal{Z}(\mathcal{E})$ and a_i in $\mathcal{A}$ such that*

$$X = \sum_i \omega_i \otimes_{\mathcal{A}} \eta_i a_i.$$

(iv) *If in addition, $\mathcal{E}$ is a free right $\mathcal{A}$-module with a basis $\{e_1, e_2, \cdots, e_n\} \subseteq \mathcal{Z}(\mathcal{E})$, then any element X in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ can be written as a unique linear combination $\sum_{ij} e_i \otimes_{\mathcal{A}} e_j a_{ij}$ for some elements a_{ij} in $\mathcal{A}$.*

Proof. Let e be an element of $\mathcal{E}$. Since $\mathcal{Z}(\mathcal{E})$ is right $\mathcal{A}$-total in $\mathcal{E}$, there exist $\omega_i \in \mathcal{Z}(\mathcal{E})$ and a_i in $\mathcal{A}$ such that $e = \sum_i \omega_i a_i$. But since ω_i belongs to $\mathcal{Z}(\mathcal{E})$, we have $\omega_i a_i = a_i \omega_i$ for all i. Thus,

$$e = \sum_i a_i \omega_i,$$

proving that $\mathcal{Z}(\mathcal{E})$ is left $\mathcal{A}$-total in $\mathcal{E}$.

Now we prove part (ii). Let e and f belong to $\mathcal{E}$. It is enough to prove that $e \otimes_{\mathcal{A}} f$ belongs to the complex linear span of the set $\{\omega \otimes_{\mathcal{A}} \eta a : \omega, \eta \in \mathcal{Z}(\mathcal{E}), a \in \mathcal{A}\}$. Indeed, if there exist $\omega_i, \eta_i \in \mathcal{Z}(\mathcal{E})$ and a_i in $\mathcal{A}$ such that

$$e \otimes_{\mathcal{A}} f = \sum_i \omega_i \otimes_{\mathcal{A}} \eta_i a_i, \tag{2.1.1}$$

then

$$e \otimes_{\mathcal{A}} f = \sum_i \omega_i \otimes_{\mathcal{A}} a_i \eta_i = \sum_i \omega_i a_i \otimes_{\mathcal{A}} \eta_i = \sum_i a_i \omega_i \otimes_{\mathcal{A}} \eta_i$$

proving that $\{\omega \otimes_{\mathcal{A}} \eta : \omega, \eta \in \mathcal{Z}(\mathcal{E})\}$ is left $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$.

The fact that $e \otimes_{\mathcal{A}} f$ is of the form (2.1.1) is also easily proved. Since $\mathcal{Z}(\mathcal{E})$ is right $\mathcal{A}$-total in $\mathcal{E}$, there exist ω_k, η_l in $\mathcal{Z}(\mathcal{E})$ and a_k, b_l in $\mathcal{A}$ such that

$$e = \sum_k \omega_k a_k, f = \sum_l \eta_l b_l$$

and so

$$\begin{aligned} e \otimes_{\mathcal{A}} f &= (\sum_k \omega_k a_k) \otimes_{\mathcal{A}} (\sum_l \eta_l b_l) = \sum_{k,l} \omega_k \otimes_{\mathcal{A}} a_k \eta_l b_l \\ &= \sum_{k,l} \omega_k \otimes_{\mathcal{A}} \eta_l a_k b_l. \end{aligned}$$

This finishes the proof of part (ii).

The third assertion directly follows from the second.

Finally, we prove the last assertion. By part (iii), $X = \sum_k \omega_k \otimes_{\mathcal{A}} \eta_k a_k$ where ω_k are in $\mathcal{E}$, η_k are in $\mathcal{Z}(\mathcal{E})$ and a_k are in $\mathcal{A}$. Since $\mathcal{E}$ is a free right $\mathcal{A}$-module with a basis $\{e_1, \cdots e_n\}$, there exist elements c_{ik}, d_{jk} in $\mathcal{A}$ such that

$$\omega_k = \sum_i e_i c_{ik}, \ \eta_l = \sum_j e_j d_{jl}.$$

Hence,

$$\begin{aligned} X &= \sum_k (\sum_i e_i c_{ik}) \otimes_{\mathcal{A}} (\sum_j e_j d_{jk}) a_k = \sum_{i,j,k} e_i \otimes_{\mathcal{A}} c_{ik} e_j d_{jk} a_k \\ &= \sum_{i,j,k} e_i \otimes_{\mathcal{A}} e_j c_{ik} d_{jk} a_k = \sum_{i,j} e_i \otimes_{\mathcal{A}} e_j (\sum_k c_{ik} d_{jk} a_k) \end{aligned}$$

where we have used the fact that $ae_i = e_i a$ for all i since e_i belongs to $\mathcal{Z}(\mathcal{E})$. Finally, uniqueness of the expression follows from the fact that $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is a free right $\mathcal{A}$-module with a basis $\{e_i \otimes_{\mathcal{A}} e_j\}_{ij}$. $\square$

We will use the lemma above repeatedly in the chapter, sometimes, without mentioning. For the purposes of this chapter, we will be dealing with a specific class of centered bimodules. The next proposition is motivated towards the same.

Proposition 2.1.6. *Suppose $\mathcal{E}$ is an $\mathcal{A}$-bimodule such that the map $u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$ given by*

$$u^{\mathcal{E}} \big(\sum_i e'_i \otimes_{\mathcal{Z}(\mathcal{A})} a_i \big) = \sum_i e'_i a_i$$

is an isomorphism of vector spaces. Then we have the following isomorphism of $\mathcal{A}$-bimodules:

$$\mathcal{E} \cong \mathcal{A} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \cong \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}.$$

The $\mathcal{A}$-bimodule structure of $\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}$ is given by

$$b(e \otimes_{\mathcal{Z}(\mathcal{A})} a)c = e \otimes_{\mathcal{Z}(\mathcal{A})} bac,$$

where e is in $\mathcal{Z}(\mathcal{E})$, a, b, c are in $\mathcal{A}$. The $\mathcal{A}$-bimodule structure of $\mathcal{A} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$ is similarly given. In particular, the set $\mathcal{Z}(\mathcal{E})$ is right $\mathcal{A}$-total in $\mathcal{E}$, i.e, $\mathcal{E}$ is centered.

Proof. Since $u^{\mathcal{E}}$ is an isomorphism, any element e can be written as $\sum_i e_i a_i$, where e_i are in $\mathcal{Z}(\mathcal{E})$ and a_i in $\mathcal{A}$. Let us make a small observation at this point. We claim that if b is in $\mathcal{Z}(\mathcal{A})$, $be = eb$ for all e in $\mathcal{E}$. Let $e = \sum_i e_i a_i$ as above. Then $be = b \sum_i e_i a_i = \sum_i e_i b a_i = \sum_i e_i a_i b = eb$ as e_i are in $\mathcal{Z}(\mathcal{E})$ and b is in $\mathcal{Z}(\mathcal{A})$. This proves the claim. It is clear from the definitions that the map $u^{\mathcal{E}}$ is left $\mathcal{Z}(\mathcal{A})$-linear and right $\mathcal{A}$-linear. Let us define a left $\mathcal{A}$-linear, right $\mathcal{Z}(\mathcal{A})$-linear map $v^{\mathcal{E}} : \mathcal{A} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \to \mathcal{E}$ by

$$v^{\mathcal{E}}\Big(\sum_i a_i \otimes_{\mathcal{Z}(\mathcal{A})} e'_i\Big) = \sum_i a_i e'_i.$$

Consider the map $p : \mathcal{Z}(\mathcal{E}) \times \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$ given by $(e, a) \mapsto (a \otimes_{\mathcal{Z}(\mathcal{A})} e)$. Using the claim made above, it is clear that $p(ea', a) = p(e, a'a)$, so that we get a well-defined map

$$\overline{p} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{A} \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}), \text{ given by } (e \otimes_{\mathcal{Z}(\mathcal{A})} a) \mapsto (a \otimes_{\mathcal{Z}(\mathcal{A})} e).$$

It is in fact an isomorphism, with the inverse map, say q, given by

$$q(a \otimes_{\mathcal{Z}(\mathcal{A})} e) = (e \otimes_{\mathcal{Z}(\mathcal{A})} a).$$

Observe that $v^{\mathcal{E}} = u^{\mathcal{E}} \circ q$, hence $v^{\mathcal{E}}$ is an isomorphism as well. Thus, the map $v^{\mathcal{E}}$ is also a vector space isomorphism as well.

Next, we endow $\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}$ with an $\mathcal{A}$-bimodule structure defined by

$$b(e \otimes_{\mathcal{Z}(\mathcal{A})} a)c = e \otimes_{\mathcal{Z}(\mathcal{A})} bac,$$

where e is in $\mathcal{Z}(\mathcal{E})$, a, b, c are in $\mathcal{A}$. Then it is easy to see that $u^{\mathcal{E}}$ defines an $\mathcal{A}$-bimodule isomorphism. The other isomorphism follows by using the map $v^{\mathcal{E}}$. $\square$

The following theorem is crucial for this chapter.

Theorem 2.1.7. (Theorem 6.10 of [85]) *Let $\mathcal{E}$ be an $\mathcal{A}$-bimodule which is centered. Then there exists a unique $\mathcal{A}$-bimodule isomorphism $\sigma^{\mathrm{can}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ such that*

$$\sigma^{\mathrm{can}}(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega$$

for all ω, η in $\mathcal{Z}(\mathcal{E})$. Moreover, $(\sigma^{\mathrm{can}})^2 = \mathrm{id}$ so that $P^{\mathrm{can}}_{\mathrm{sym}} := \frac{1}{2}(1 + \sigma^{\mathrm{can}}) : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is an $\mathcal{A}$-bilinear idempotent map.

Proof. We only need to remark that the equation $(\sigma^{\mathrm{can}})^2 = \mathrm{id}$ is derived in the proof of Theorem 6.10 of [85]. Indeed, since $\mathcal{E}$ is centered, $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Span}_{\mathbb{C}}\{e \otimes_{\mathcal{A}} fa : e, f \in \mathcal{Z}(\mathcal{E}), a \in \mathcal{A}\}$, by part

(iii) of Lemma 2.1.5, it is enough to observe that

$$(\sigma^{\mathrm{can}})^2(e \otimes_{\mathcal{A}} fa) = \sigma^{\mathrm{can}}(\sigma^{\mathrm{can}}(e \otimes_{\mathcal{A}} fa)) = \sigma^{\mathrm{can}}(f \otimes_{\mathcal{A}} ea) = e \otimes_{\mathcal{A}} fa.$$

□

Let us make the following observation:

Lemma 2.1.8. *For a centered $\mathcal{A}$-bimodule $\mathcal{E}$, we have $\sigma^{\mathrm{can}}(\omega \otimes_{\mathcal{A}} e) = e \otimes_{\mathcal{A}} \omega$ and $\sigma^{\mathrm{can}}(e \otimes_{\mathcal{A}} \omega) = \omega \otimes_{\mathcal{A}} e$ for all ω in $\mathcal{Z}(\mathcal{E})$ and e in $\mathcal{E}$.*

Proof. Since $\mathcal{E}$ is centered and σ is right $\mathcal{A}$-linear, it is enough to prove the lemma for elements e of the form ηb where η is in $\mathcal{Z}(\mathcal{E})$ and b is in $\mathcal{A}$.

We compute $\sigma^{\mathrm{can}}(\omega \otimes_{\mathcal{A}} \eta b) = \sigma^{\mathrm{can}}(\omega \otimes_{\mathcal{A}} \eta)b = (\eta \otimes_{\mathcal{A}} \omega)b = \eta \otimes_{\mathcal{A}} \omega b = \eta \otimes_{\mathcal{A}} b\omega = \eta b \otimes_{\mathcal{A}} \omega = e \otimes_{\mathcal{A}} \omega$.

The other equality follows similarly. □

We will end this subsection with Lemma 2.1.10. But before that, we want to state and prove Proposition 2.1.9 whose proof is basically a reformulation of the proof of the existence and uniqueness of Levi-Civita connections for pseudo-Riemannian manifolds.

Let V be a complex vector space and flip denotes the map from $V \otimes_{\mathbb{C}} V \to V \otimes_{\mathbb{C}} V$ defined on simple tensors by the formula $\mathrm{flip}(v \otimes_{\mathbb{C}} w) = w \otimes_{\mathbb{C}} v$. We will use the maps $\mathrm{flip}_{12} := \mathrm{flip} \otimes_{\mathbb{C}} \mathrm{id}_V$, $\mathrm{flip}_{23} := \mathrm{id}_V \otimes_{\mathbb{C}} \mathrm{flip}$ and $\mathrm{flip}_{13} := \mathrm{flip}_{12}\mathrm{flip}_{23}\mathrm{flip}_{12}$.

Then the map $P^{\mathbb{C}} := \frac{\mathrm{flip}+1}{2}$ is an idempotent. We will denote $\mathrm{Ran}(P^{\mathbb{C}})$ by $V \otimes^{\mathrm{sym}}_{\mathbb{C}} V$. We will need the maps $P^{\mathbb{C}}_{12} := P^{\mathbb{C}} \otimes_{\mathbb{C}} \mathrm{id}_V$ and $P^{\mathbb{C}}_{23} := \mathrm{id}_V \otimes_{\mathbb{C}} P^{\mathbb{C}}$. Thus, for elements v_1, v_2, v_3 in V,

$$\begin{aligned} P^{\mathbb{C}}_{12}(v_1 \otimes_{\mathbb{C}} v_2 \otimes_{\mathbb{C}} v_3) &= \frac{1}{2}(v_1 \otimes_{\mathbb{C}} v_2 + v_2 \otimes_{\mathbb{C}} v_1) \otimes_{\mathbb{C}} v_3 \\ \text{and } P^{\mathbb{C}}_{23}(v_1 \otimes_{\mathbb{C}} v_2 \otimes_{\mathbb{C}} v_3) &= v_1 \otimes_{\mathbb{C}} \frac{1}{2}(v_2 \otimes_{\mathbb{C}} v_3 + v_3 \otimes_{\mathbb{C}} v_2). \end{aligned}$$

Proposition 2.1.9. *If V is a vector space, then each of the following maps is an isomorphism of vector spaces.*

$$P^{\mathbb{C}}_{12}|_{\mathrm{Ran}(P^{\mathbb{C}}_{23})} : \mathrm{Ran}(P^{\mathbb{C}}_{23}) = V \otimes_{\mathbb{C}} (V \otimes^{\mathrm{sym}}_{\mathbb{C}} V) \to \mathrm{Ran}(P^{\mathbb{C}}_{12}) = (V \otimes^{\mathrm{sym}}_{\mathbb{C}} V) \otimes_{\mathbb{C}} V$$

$$P_{23}^{\mathbb{C}}|_{\mathrm{Ran}(P_{12}^{\mathbb{C}})} : \mathrm{Ran}(P_{12}^{\mathbb{C}}) = (V \otimes_{\mathbb{C}}{}^{\mathrm{sym}} V) \otimes_{\mathbb{C}} V \to \mathrm{Ran}(P_{23}^{\mathbb{C}}) = V \otimes_{\mathbb{C}} (V \otimes_{\mathbb{C}}^{\mathrm{sym}} V)$$

Proof. We prove the statement about the first of the two maps since the proof for the other map is similar. Let us begin by proving that the first map is one-to-one. Let X be in $\mathrm{Ran}(P_{23}^{\mathbb{C}})$ such that $P_{12}^{\mathbb{C}}(X) = 0$. That is, $\mathrm{flip}_{23}(X) = X$ and $\mathrm{flip}_{12}(X) = -X$. Now, it is easy to verify the following braid relations:

$$\mathrm{flip}_{12}\mathrm{flip}_{23}\mathrm{flip}_{12} = \mathrm{flip}_{23}\mathrm{flip}_{12}\mathrm{flip}_{23}. \tag{2.1.2}$$

But we have $\mathrm{flip}_{12}\mathrm{flip}_{23}\mathrm{flip}_{12}(X) = -\mathrm{flip}_{12}\mathrm{flip}_{23}(X) = -\mathrm{flip}_{12}(X) = X$. On the other hand,

$$\mathrm{flip}_{23}\mathrm{flip}_{12}\mathrm{flip}_{23}(X) = \mathrm{flip}_{23}\mathrm{flip}_{12}(X) = -\mathrm{flip}_{23}(X) = -X.$$

This implies, $X = -X$, i.e. $X = 0$. Thus, the map $P_{12}^{\mathbb{C}}|_{\mathrm{Ran}(P_{23}^{\mathbb{C}})}$ is injective.

Now we come to surjectivity. If V is finite dimensional, surjectivity follows since $\mathrm{Ran}(P_{23}^{\mathbb{C}})$ and $\mathrm{Ran}(P_{12}^{\mathbb{C}})$ are of the same dimension and $P_{12}^{\mathbb{C}}|_{\mathrm{Ran}(P_{23}^{\mathbb{C}})}$ is injective. In the general case, given any ξ in $(V \otimes_{\mathbb{C}}^{\mathrm{sym}} V) \otimes_{\mathbb{C}} V$ such that $\mathrm{flip}_{23}(\xi) = \xi$, there exists a natural number n and linearly independent elements $e_1, e_2, ..., e_n$ of V such that ξ belongs to $(K \otimes_{\mathbb{C}}^{\mathrm{sym}} K) \otimes_{\mathbb{C}} K$, where $K := \mathrm{span}\{e_1, e_2, ..., e_n\}$. If $P_{K,12}^{\mathbb{C}}$ denotes the map $P_{12}^{\mathbb{C}}|_{K \otimes_{\mathbb{C}} K \otimes_{\mathbb{C}} K}$, then by the surjectivity of $P_{12}^{\mathbb{C}}|_{\mathrm{Ran}(P_{23}^{\mathbb{C}})}$ for finite dimensional vector spaces, there exists η in $K \otimes_{\mathbb{C}} (K \otimes_{\mathbb{C}}^{\mathrm{sym}} K)$ such that $P_{K,12}^{\mathbb{C}}(\eta) = \xi$. Since ξ is arbitrary, the proof of surjectivity is complete. □

Lemma 2.1.10. *Let $\mathcal{E}$ be a centered $\mathcal{A}$-bimodule and $\sigma^{\mathrm{can}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ be as in Theorem 2.1.7. Define $P_{ij}^{\mathrm{can}} := \frac{1}{2}(1 + \sigma_{ij}^{\mathrm{can}}) : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, $(i,j) = (12), (13), (23)$. Then the following maps are bimodule isomorphisms:*

$$P_{12}^{\mathrm{can}}|_{\mathrm{Ran}(P_{23}^{\mathrm{can}})} : \mathrm{Ran}(P_{23}^{\mathrm{can}}) \to \mathrm{Ran}(P_{12}^{\mathrm{can}}), \quad P_{23}^{\mathrm{can}}|_{\mathrm{Ran}(P_{12}^{\mathrm{can}})} : \mathrm{Ran}(P_{12}^{\mathrm{can}}) \to \mathrm{Ran}(P_{23}^{\mathrm{can}}).$$

Proof. We begin by noting that since σ^{can} is a bimodule map, the maps $\sigma_{12}^{\mathrm{can}}, \sigma_{23}^{\mathrm{can}}, \sigma_{13}^{\mathrm{can}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ defined as $\sigma^{\mathrm{can}} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}}$, $\mathrm{id}_{\mathcal{E}} \otimes_{\mathcal{A}} \sigma^{\mathrm{can}}$, $\sigma_{12}^{\mathrm{can}}\sigma_{23}^{\mathrm{can}}\sigma_{12}^{\mathrm{can}}$ respectively are well defined bimodule morphisms. The proof of injectivity follows by a verbatim adaptation of the arguments of Proposition 2.1.9, as the braid relations (2.1.2) do hold for the maps $\sigma_{ij}^{\mathrm{can}}$ as well.

For surjectivity, we also use Proposition 2.1.9. Consider the vector space $\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})$. By Proposition 2.1.9, taking $V = \mathcal{Z}(\mathcal{E})$, we have that

$$P_{12}^{\mathbb{C}}P_{23}^{\mathbb{C}}(\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})) = P_{12}^{\mathbb{C}}(\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})). \tag{2.1.3}$$

Let $q : \mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E}) \to \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$ be the canonical quotient map. Let us also define a map $\widetilde{P^{\mathrm{can}}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \to \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$ given by

$$\widetilde{P^{\mathrm{can}}}(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta) = \frac{1}{2}(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta + \eta \otimes_{\mathcal{Z}(\mathcal{A})} \omega).$$

To see that this map is well defined it is enough to note that $\mathcal{Z}(\mathcal{E})$ is a centered $\mathcal{Z}(\mathcal{A})$-bimodule with $\mathcal{Z}(\mathcal{Z}(\mathcal{E})) = \mathcal{Z}(\mathcal{E})$. Hence, by Theorem 2.1.7, there exists a well defined $\mathcal{Z}(\mathcal{A})$-bilinear idempotent map $\widetilde{P^{\mathrm{can}}} = (P^{\mathrm{can}}_{\mathrm{sym}})_{\mathcal{Z}(\mathcal{E})} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \to \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$. It is easy to check that for all ω, η, ξ in $\mathcal{Z}(\mathcal{E})$,

$$qP_{12}^{\mathbb{C}}(\omega\otimes_{\mathbb{C}}\eta\otimes_{\mathbb{C}}\xi) = \widetilde{P_{12}^{\mathrm{can}}}q(\omega\otimes_{\mathbb{C}}\eta\otimes_{\mathbb{C}}\xi), \tag{2.1.4}$$

$$qP_{12}^{\mathbb{C}}P_{23}^{\mathbb{C}}(\omega\otimes_{\mathbb{C}}\eta\otimes_{\mathbb{C}}\xi) = \widetilde{P_{12}^{\mathrm{can}}}\,\widetilde{P_{23}^{\mathrm{can}}}q(\omega\otimes_{\mathbb{C}}\eta\otimes_{\mathbb{C}}\xi), \tag{2.1.5}$$

where $\widetilde{P_{12}^{\mathrm{can}}} = \widetilde{P^{\mathrm{can}}} \otimes_{\mathcal{Z}(\mathcal{A})} \mathrm{id}_{\mathcal{Z}(\mathcal{E})}$ and $P_{23}^{\mathrm{can}} = \mathrm{id}_{\mathcal{Z}(\mathcal{E})} \otimes_{\mathcal{Z}(\mathcal{A})} \widetilde{P^{\mathrm{can}}}$. This implies that

$$\begin{aligned}
&\widetilde{P_{12}^{\mathrm{can}}}\,\widetilde{P_{23}^{\mathrm{can}}}(\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})) = \widetilde{P_{12}^{\mathrm{can}}}\,\widetilde{P_{23}^{\mathrm{can}}}q(\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E}))\\
=&qP_{12}^{\mathbb{C}}P_{23}^{\mathbb{C}}(\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})) \text{ (by (2.1.5))}\\
=&qP_{12}^{\mathbb{C}}(\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})) \text{ (by (2.1.3))}\\
=&\widetilde{P_{12}^{\mathrm{can}}}q(\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})\otimes_{\mathbb{C}}\mathcal{Z}(\mathcal{E})) \text{ (by (2.1.4))} = \widetilde{P_{12}^{\mathrm{can}}}(\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})).
\end{aligned}$$

Now, let us define the map

$$\mu : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$$

given by $\mu(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta \otimes_{\mathcal{Z}(\mathcal{A})} \xi \otimes_{\mathcal{Z}(\mathcal{A})} a) = \omega \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} \xi a$. Since $\{\omega \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} \xi : \omega, \eta, \xi \in \mathcal{Z}(\mathcal{E})\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, μ is an onto map. Moreover, by simple computation, it follows that $P_{12}^{\mathrm{can}} \circ \mu = \mu \circ (\widetilde{P_{12}^{\mathrm{can}}} \otimes_{\mathcal{Z}(\mathcal{A})} \mathrm{id}_{\mathcal{A}})$ and $P_{12}^{\mathrm{can}}P_{23}^{\mathrm{can}} \circ \mu = \mu \circ (\widetilde{P_{12}^{\mathrm{can}}}\,\widetilde{P_{23}^{\mathrm{can}}} \otimes_{\mathcal{Z}(\mathcal{A})} \mathrm{id}_{\mathcal{A}})$. Since

$\mathrm{Ran}(\widetilde{P_{12}^{\mathrm{can}}}\widetilde{P_{23}^{\mathrm{can}}}) = \mathrm{Ran}(\widetilde{P_{12}^{\mathrm{can}}})$, we can thus compute

$$\begin{aligned} &\mathrm{Ran}(P_{12}^{\mathrm{can}}P_{23}^{\mathrm{can}}) = \mathrm{Ran}(P_{12}^{\mathrm{can}}P_{23}^{\mathrm{can}} \circ \mu) \text{ (since } \mu \text{ is onto)} \\ =&\mathrm{Ran}(\mu \circ (\widetilde{P_{12}^{\mathrm{can}}}\widetilde{P_{23}^{\mathrm{can}}} \otimes_{\mathcal{Z}(A)} \mathrm{id}_{\mathcal{A}})) = \mathrm{Ran}(\mu \circ (\widetilde{P_{12}^{\mathrm{can}}} \otimes_{\mathcal{Z}(A)} \mathrm{id}_{\mathcal{A}})) \\ =&\mathrm{Ran}(P_{12}^{\mathrm{can}} \circ \mu) = \mathrm{Ran}(P_{12}^{\mathrm{can}}) \text{ (since } \mu \text{ is onto).} \end{aligned}$$

Thus, we have that $P_{12}^{\mathrm{can}}|_{\mathrm{Ran}(P_{23}^{\mathrm{can}})} : \mathrm{Ran}(P_{23}^{\mathrm{can}}) \to \mathrm{Ran}(P_{12}^{\mathrm{can}})$ is an onto map.
The surjectivity of the other map follows in a similar way. $\square$

2.2 Quasi-tame spectral triples

Recall that in Subsection 1.3.1 we defined Connes' space of forms for a spectral triple and in Subsection 1.4.2, the notion of torsionless connections on the space of one-forms. In this section, we define a certain class of spectral triples which we call quasi-tame spectral triples and prove that the bimodule of one-forms $\Omega^1_D(\mathcal{A})$ of any quasi-tame spectral triple admits a canonical torsionless connection. Moreover, in the next subsection, we will use the canonical $\mathcal{A}$-bilinear map σ of a quasi-tame spectral triple (see Definition 2.2.1) to define the notion of a pseudo-Riemannian metric.

From Proposition 1.4.1, we know that if M is a manifold with $\Omega^1(M)$ as the space of one-forms, we have the following decomposition of $C^\infty(M)$-bimodules:

$$\Omega^1(M) \otimes_{C^\infty(M)} \Omega^1(M) = \mathrm{Ker}(\wedge) \oplus \mathcal{F}.$$

Here, $\mathrm{Ker}(\wedge)$ is the space of all symmetric two-tensors and $\mathcal{F}$ is the space of all anti-symmetric 2-tensors which is isomorphic to $\Omega^2(M)$. This motivates the following definition.

Definition 2.2.1. *We say that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is quasi-tame if the following conditions hold:*

(i) The bimodule $\Omega^1_D(\mathcal{A})$ is finitely generated and projective as a right $\mathcal{A}$-module.

(ii) There exists a right $\mathcal{A}$-module $\mathcal{F}$ such that the following equality holds as right $\mathcal{A}$-modules:

$$\Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) = \mathrm{Ker}(\wedge) \oplus \mathcal{F}. \tag{2.2.1}$$

(iii) The idempotent $P_{\text{sym}} \in \text{Hom}_{\mathcal{A}}(\Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}), \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}))$ *mapping onto* $\text{Ker}(\wedge)$ *and with kernel* $\mathcal{F}$ *is an* $\mathcal{A}$*-bimodule map.*

Then we will denote $\text{Ker}(\wedge)$ *by the symbol* $\Omega^1_D(\mathcal{A}) \otimes^{\text{sym}}_{\mathcal{A}} \Omega^1_D(\mathcal{A})$*.*
Moreover, σ *will denote the map* $2P_{\text{sym}} - 1$*.*

The following lemma collects some consequences of the above definition.

Lemma 2.2.2. *Let* $(\mathcal{A}, \mathcal{H}, D)$ *be a quasi-tame spectral triple. Then we have the following*

(i) $\Omega^1_D(\mathcal{A}) \otimes^{\text{sym}}_{\mathcal{A}} \Omega^1_D(\mathcal{A}) := \text{Ker}(\wedge)$ *and* $\text{Ran}(\wedge)$ *are* $\mathcal{A}$*-bimodules.*

(ii) σ *is an* $\mathcal{A}$*-bimodule map.*

(iii) $P^2_{\text{sym}} = P_{\text{sym}}$ *and* $\sigma^2 = \text{id}$.

Proof. By Lemma 1.3.10 and Definition 1.3.11, the map $\wedge$ is $\mathcal{A}$-bilinear, hence $\text{Ker}(\wedge)$ and $\text{Ran}(\wedge)$ are $\mathcal{A}$-bimodules. This gives us the first claim. The second claim, i.e the $\mathcal{A}$-bilinearity of σ follows from the $\mathcal{A}$-bilinearity of P_{sym}. The third claim follows from the fact that P_{sym} is an idempotent. □

Let us recall (Definition 1.4.11) that a connection ∇ on $\Omega^1_D(\mathcal{A})$ is said to be torsionless if $T_\nabla = \wedge \circ \nabla + d = 0$. We have the following result as a consequence of the assumptions made in Definition 2.2.1.

Theorem 2.2.3. *If* $(\mathcal{A}, \mathcal{H}, D)$ *is a quasi-tame spectral triple, there exists a torsionless connection on* $\Omega^1_D(\mathcal{A})$*.*

Proof. We have a sub-bimodule $\mathcal{F} = \text{Ran}(1 - P_{\text{sym}})$ of $\Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ and a bimodule isomorphism, say Q, from $\mathcal{F}$ to $\text{Ran}(\wedge) = \Omega^2_D(\mathcal{A})$, satisfying

$$Q((1 - P_{\text{sym}})(\beta)) = \wedge(\beta) \text{ for all } \beta \in \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}). \tag{2.2.2}$$

Moreover, as $\Omega^1_D(\mathcal{A})$ is finitely generated and projective, we can find a free rank n right $\mathcal{A}$-module $\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^n$ containing $\Omega^1_D(\mathcal{A})$ as a complemented right submodule. Let p be an idempotent in $M_n(\mathcal{A}) \cong \text{Hom}_{\mathcal{A}}(\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^n, \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^n)$ such that $\Omega^1_D(\mathcal{A}) = p(\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^n)$. Let $e_i, i = 1, \ldots n$, be the standard basis of $\mathbb{C}^n$ (viewed as $1_{\mathcal{A}} \otimes_{\mathbb{C}} \mathbb{C}^n$, where $1_{\mathcal{A}}$ is the identity element in $\mathcal{A}$) and define $\tilde{\nabla}_0 : \mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^n \to \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ by

$$\widetilde{\nabla}_0(e_i a) := -Q^{-1}(d(p(e_i)))a + p(e_i) \otimes_{\mathcal{A}} da, \quad i = 1, \ldots, n, a \in \mathcal{A}. \tag{2.2.3}$$

$$\text{Then } \nabla_0 = \widetilde{\nabla}_0|_{\Omega^1_D(\mathcal{A})} \tag{2.2.4}$$

defines a connection on $\Omega^1_D(\mathcal{A})$.

Since $\mathrm{Ran}(P_{\mathrm{sym}}) = \Omega^1_D(\mathcal{A}) \otimes^{\mathrm{sym}}_{\mathcal{A}} \Omega^1_D(\mathcal{A}) = \mathrm{Ker}(\wedge)$, we observe that

$$\wedge \circ P_{\mathrm{sym}} = 0.$$

Hence,

$$\wedge \circ Q^{-1}(\wedge(\beta)) = \wedge \circ Q^{-1}(Q(1 - P_{\mathrm{sym}})\beta) = \wedge((1 - P_{\mathrm{sym}})\beta) = \wedge(\beta) \ \forall \ \beta \in \ \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$$

by using (2.2.2).

Thus, $\wedge \circ Q^{-1} : \mathrm{Ran}(\wedge) \to \mathrm{Ran}(\wedge)$ is the identity map. Since $d(p(e_i)a)$ belongs to the image of the map $\wedge$, we can write

$$\begin{aligned} \wedge \circ \nabla_0(p(e_i)a) &= - \wedge (Q^{-1}(d(p(e_i))a)) + p(e_i) \wedge da \ (\text{by } (2.2.3)) \\ &= -d(p(e_i))a + p(e_i) \wedge da \\ &= -d(p(e_i)a). \end{aligned}$$

Therefore, ∇_0 is a torsionless connection on $\Omega^1_D(\mathcal{A})$. □

2.3 Pseudo-Riemannian metrics on quasi-tame spectral triples

In this section, we want to introduce a noncommutative analogue of pseudo-Riemannian metrics. Recall that in Definition 1.4.4, we had defined pseudo-Riemannian metrics on manifolds. In the classical case there is no difference between right module maps or bimodule maps, as the left and right $C^\infty(M)$-actions on the module of forms coincide. This is no longer true in the noncommutative framework. In fact, as we will see, requiring a pseudo-metric to be a bimodule map restricts the choice of metrics. It is reasonable to require one-sided (right/left) $\mathcal{A}$-linearity only. For this reason, we give the following definition:

Definition 2.3.1. *Let $\mathcal{E} := \Omega^1_D(\mathcal{A})$ be the $\mathcal{A}$-bimodule of one-forms of a quasi-tame spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and σ be the $\mathcal{A}$-bilinear map of Definition 2.2.1. A pseudo-Riemannian metric g on $\mathcal{E}$ is an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ such that*

(i) g is symmetric, i.e. $g \circ \sigma = g$,

(ii) g is non-degenerate, i.e, the right $\mathcal{A}$-linear map $V_g : \mathcal{E} \to \mathcal{E}^*$ defined by $V_g(\omega)(\eta) = g(\omega \otimes_{\mathcal{A}} \eta)$ is an isomorphism of right $\mathcal{A}$ modules.

We will say that a pseudo-Riemannian metric g is a pseudo-Riemannian bilinear metric if g is an $\mathcal{A}$-bimodule map. It is called a Riemannian metric if for all $\omega_1, \omega_2, \ldots, \omega_n$ in $\mathcal{E}$, the matrix $((g(\omega_i^* \otimes_{\mathcal{A}} \omega_j)))_{i,j}$ is a positive element of $M_n(\mathcal{A})$ for all n.

As an immediate consequence of the definition, we have the following important proposition.

Proposition 2.3.2. *Suppose g is a pseudo-Riemannian bilinear metric on the space of one-forms $\mathcal{E} := \Omega^1_D(\mathcal{A})$ of a quasi-tame spectral triple. Then*

$$g(\omega \otimes_{\mathcal{A}} \eta) \in \mathcal{Z}(\mathcal{A}) \text{ if both } \omega \text{ and } \eta \text{ belong to } \mathcal{Z}(\mathcal{E}). \tag{2.3.1}$$

In particular, if $\mathcal{E}$ is a free right $\mathcal{A}$-module of rank n admitting a central basis $\{\omega_i\}_i \subseteq \mathcal{Z}(\mathcal{E})$, then the components of the metric $g_{ij} := g(\omega_i \otimes_{\mathcal{A}} \omega_j)$ belong to $\mathcal{Z}(\mathcal{A})$.

Proof. The proof is a trivial consequence of the fact that g is an $\mathcal{A}$-bimodule map. Indeed, since ω, η are in $\mathcal{Z}(\mathcal{E})$,

$$g(\omega \otimes_{\mathcal{A}} \eta)a = g(\omega \otimes_{\mathcal{A}} \eta a) = g(\omega \otimes_{\mathcal{A}} a\eta) = g(\omega a \otimes_{\mathcal{A}} \eta) = ag(\omega \otimes_{\mathcal{A}} \eta).$$

□

We record a remark at this point as clarification to the above result.

Remark 2.3.3. *If $\mathcal{A}$ is noncommutative, the metric need not take values in the center on the whole of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. For example, if ω, η is in $\mathcal{Z}(\mathcal{E})$ and a in $\mathcal{A}$, then $g(\omega \otimes_{\mathcal{A}} \eta a)$ typically does not belong to $\mathcal{Z}(\mathcal{A})$ unless a is in $\mathcal{Z}(\mathcal{A})$.*

Moreover, our definition of nondegeneracy of g is stronger than the definition given by most authors who require only the injectivity of V_g. However, in the classical situation, i.e, when $\mathcal{A} = C^\infty(M)$, these two definitions are equivalent as V_g is a bundle map from T^*M to $(T^*M)^* \cong TM$ in that case and the fibers are finite dimensional.

To compare our definition of a pseudo-Riemannian metric with that of [41], [83] and [6], let us consider the case when $\mathcal{E}$ is free (of rank n) as a right $\mathcal{A}$-module, i.e, $\mathcal{E}$ is isomorphic to $\mathbb{C}^n \otimes_{\mathbb{C}} \mathcal{A}$ as a right $\mathcal{A}$-module. Let $e_i, i = 1, \ldots, n$ be the standard basis of $\mathbb{C}^n$. A pseudo-Riemannian metric in our sense is determined by an invertible element $A := ((g_{ij}))_{ij}$ of $M_n(\mathcal{A})$, where

$$g_{ij} = g((e_i \otimes_{\mathbb{C}} 1) \otimes_{\mathcal{A}} (e_j \otimes_{\mathbb{C}} 1)) \text{ and } g((e_i \otimes_{\mathbb{C}} a) \otimes_{\mathcal{A}} (e_j \otimes_{\mathbb{C}} b)) = g_{ij}ab$$

for all a, b in $\mathcal{A}$. On the other hand, a pseudo-metric in the sense of [6] will be given by the sesquilinear pairing

$$\langle\langle e_i \otimes_{\mathbb{C}} a, e_j \otimes_{\mathbb{C}} b \rangle\rangle = a^* g_{ij} b.$$

Thus, there is a one-to-one correspondence between these two notions of pseudo-metric at least for the case when $\mathcal{E}$ is free as a right $\mathcal{A}$ module. In fact, they do agree in a sense on the basis elements. But their extensions are quite different as maps.

Throughout this section, we will assume that $(\mathcal{A}, \mathcal{H}, D)$ is a quasi-tame spectral triple, so that we can freely use the notation σ introduced in Definition 2.2.1 and the results in Lemma 2.2.2.

Definition 2.3.4. *Suppose g is a pseudo-Riemannian bilinear metric on $\mathcal{E}$. We define*

$$g^{(2)} : (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \to \mathcal{A}, \ \textit{by}$$

$$g^{(2)}((e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f')) = g(e \otimes_{\mathcal{A}} g(f \otimes_{\mathcal{A}} e') f')$$

We spell out the relationship between $g^{(2)}$ and the inner product on the internal tensor product of Hilbert modules. Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, $\mathcal{E}$ the bimodule of one-forms. We will need to make explicit use of the $*$-structure on $\mathcal{A}$ and $\mathcal{E} := \Omega^1_D(\mathcal{A})$ inherited from $B(\mathcal{H})$. Let us recall the conjugate bimodule $\overline{\mathcal{E}}$ (see [55], [9] and references therein) which is equal to $\mathcal{E}$ as a set but with the $\mathcal{A}$-bimodule structures defined by the following equations:

$$a\bar{e} = \overline{ea^*}, \ \bar{e}a = \overline{a^*e}.$$

Here, $\overline{e}$ is an element of $\mathcal{E}$ viewed in $\overline{\mathcal{E}}$.

We have a well-defined map $S : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, defined by

$$S(e \otimes_{\mathcal{A}} f) = \overline{f} \otimes_{\mathcal{A}} \overline{e}.$$

Now suppose g is a pseudo-Riemannian bilinear metric on $\mathcal{E}$. Then the following map makes $\mathcal{E}$ into a right $\mathcal{A}$-pre-Hilbert module:

$$\langle\langle e, f\rangle\rangle_g = g(\overline{e} \otimes_{\mathcal{A}} f).$$

On the right hand side of this equation, we have used the obvious identification between $\mathcal{E}$ and $\overline{\mathcal{E}}$.

Consequently, the $\mathcal{A}$-valued inner product on the internal tensor product $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is given by

$$\langle\langle e \otimes_{\mathcal{A}} f, e' \otimes_{\mathcal{A}} f'\rangle\rangle_{g^{(2)}} = \langle\langle f, \langle\langle e, e'\rangle\rangle_g f'\rangle\rangle_g.$$

We refer to [55] for the details.

We claim that $\langle\langle e \otimes_{\mathcal{A}} f, e' \otimes_{\mathcal{A}} f'\rangle\rangle_{g^{(2)}} = g^{(2)}(S(e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f'))$. Indeed,

$$\begin{aligned}
& \langle\langle e \otimes_{\mathcal{A}} f, e' \otimes_{\mathcal{A}} f'\rangle\rangle_{g^{(2)}} = \langle\langle f, g(\overline{e} \otimes_{\mathcal{A}} e') f'\rangle\rangle_g \\
= \quad & g(\overline{f} \otimes_{\mathcal{A}} g(\overline{e} \otimes_{\mathcal{A}} e') f') = g^{(2)}((\overline{f} \otimes_{\mathcal{A}} \overline{e}) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f')) \\
= \quad & g^{(2)}(S(e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f')).
\end{aligned}$$

We end this subsection by showing that the map $g^{(2)}$ is nondegenerate in a suitable sense.

Proposition 2.3.5. *Suppose $\mathcal{E}$ is the bimodule of one-forms of a quasi-tame spectral triple. We assume that $\mathcal{E}$ is centered as an $\mathcal{A}$-bimodule and also that $\mathcal{E}$ is finitely generated and projective as a right $\mathcal{A}$-module. Let g be a pseudo-Riemannian bilinear metric on $\mathcal{E}$. Then the map $V_{g^{(2)}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})^*$ defined by*

$$V_{g^{(2)}}(e \otimes_{\mathcal{A}} f)(e' \otimes_{\mathcal{A}} f') = g^{(2)}((e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f'))$$

is an isomorphism of right $\mathcal{A}$-modules. Moreover, the maps $g^{(2)}$ and $V_{g^{(2)}}$ are both left $\mathcal{A}$-bilinear.

Proof. Throughout the proof, we will repeatedly use Lemma 2.1.5.

Let us start by proving that the map $V_{g^{(2)}}$ is onto. Since $\mathcal{E}$ is a finitely generated projective module over $\mathcal{A}$, we can use the isomorphism of $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})^*$ with $\mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E}^*$ (Proposition 1.1.14). Thus, it is enough to show that $V_g(e) \otimes_{\mathcal{A}} V_g(f)$ belongs to the range of $V_{g^{(2)}}$ for arbitrary elements e, f of $\mathcal{Z}(\mathcal{E})$. Indeed, if x_{ij} in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is such that $V_{g^{(2)}}(x_{ij}) = V_g(e_i) \otimes_{\mathcal{A}} V_g(f_j)$ for some elements e_i, f_j in $\mathcal{Z}(\mathcal{E})$, then for elements a_i, b_j in $\mathcal{A}$ and $\omega = \sum e_i a_i$, $\eta = \sum f_j b_j$, we have

$$\begin{aligned} & V_g(\omega) \otimes_{\mathcal{A}} V_g(\eta) = \sum_{i,j} V_g(e_i) a_i \otimes_{\mathcal{A}} V_g(f_j) b_j = \sum_{i,j} V_g(e_i) \otimes_{\mathcal{A}} V_g(a_i f_j) b_j \\ = \quad & \sum_{i,j} V_g(e_i) \otimes_{\mathcal{A}} V_g(f_j) a_i b_j = \sum_{i,j} V_{g^{(2)}}(x_{ij}) a_i b_j = \sum_{i,j} V_{g^{(2)}}(x_{ij} a_i b_j), \end{aligned}$$

where we have used the fact that V_g is $\mathcal{A}$-bilinear as g is a bilinear pseudo-Riemannian metric. Now, for e, f in $\mathcal{Z}(\mathcal{E})$ and ω, η in $\mathcal{E}$, we compute

$$\begin{aligned} & V_{g^{(2)}}(f \otimes_{\mathcal{A}} e)(\omega \otimes_{\mathcal{A}} \eta) = g^{(2)}((f \otimes_{\mathcal{A}} e) \otimes_{\mathcal{A}} (\omega \otimes_{\mathcal{A}} \eta)) = g(f \otimes_{\mathcal{A}} g(e \otimes_{\mathcal{A}} \omega)\eta) \\ = \quad & g(g(e \otimes_{\mathcal{A}} \omega) f \otimes_{\mathcal{A}} \eta) = g(e \otimes_{\mathcal{A}} \omega) g(f \otimes_{\mathcal{A}} \eta) = (V_g(e) \otimes_{\mathcal{A}} V_g(f))(\omega \otimes_{\mathcal{A}} \eta). \end{aligned}$$

Hence, we have $V_g(e) \otimes_{\mathcal{A}} V_g(f) = V_{g^{(2)}}(f \otimes_{\mathcal{A}} e)$.

For proving that $V_{g^{(2)}}$ is one-to-one, let us suppose that for $i = 1, 2, \cdots n$, there exist ω_i, η_i in $\mathcal{E}$ such that for all ω', η' in $\mathcal{E}$,

$$g^{(2)}((\sum_i \omega_i \otimes_{\mathcal{A}} \eta_i) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) = 0.$$

Then by the definition of $g^{(2)}$, we see that

$$V_g(\sum_i \omega_i g(\eta_i \otimes_{\mathcal{A}} \omega')) = 0.$$

By nondegeneracy of g, we conclude that

$$\sum_i \omega_i g(\eta_i \otimes_{\mathcal{A}} \omega') = 0.$$

Thus, if $\zeta_{\mathcal{E},\mathcal{E}}$ is the map introduced in Proposition 1.1.8, then we have:

$$\zeta_{\mathcal{E},\mathcal{E}}(\sum_i \omega_i \otimes_{\mathcal{A}} \eta_i)(\omega') = 0 \text{ for all } \omega' \in \mathcal{E},$$

implying that $\sum_i \omega_i \otimes_{\mathcal{A}} \eta_i = 0$.

The left $\mathcal{A}$-linearity of $V_{g^{(2)}}$ comes from the left $\mathcal{A}$-linearity of g. The right $\mathcal{A}$-linearity of $V_{g^{(2)}}$ comes from the fact that $g^{(2)}$ is a well-defined map on $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$. □

2.3.1 The canonical Riemannian (bilinear) metric for a spectral triple

Let $(\mathcal{A}, \mathcal{H}, D)$ be a p-summable spectral triple (Definition 1.3.4) of compact type. Before we end this section, we want to derive some sufficient regularity conditions for obtaining a canonical bilinear form (candidate of a pseudo-Riemannian bilinear metric) on the module $\mathcal{E} := \Omega^1_D(\mathcal{A})$ of one-forms.

Consider the positive linear functional τ on $\mathcal{B}(\mathcal{H})$ given by

$$\tau(X) = \mathrm{Lim}_{\omega} \frac{\mathrm{Tr}(X\,|D|^{-p})}{\mathrm{Tr}(|D|^{-p})},$$

where Lim_{ω} is as in Chapter 4 of [25]. We will denote the $*$-subalgebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$ in $\mathcal{B}(\mathcal{H})$ by $\mathcal{S}_0$. We will assume that τ is a faithful normal trace on the von Neumann algebra generated by $\mathcal{S}_0$.

Let us recall from [41] the construction of an $\mathcal{A}''$-valued inner product $\langle\langle \cdot\ ,\ \cdot \rangle\rangle$ on $\mathcal{E} = \Omega^1_D(\mathcal{A})$ defined by the following equation:

$$\tau(\langle\langle \omega, \eta \rangle\rangle\, a) = \tau(\omega^* \eta a)\ \forall\ a\ \in \mathcal{A}'' \text{ and } \omega, \eta \in \mathcal{E} \subseteq \mathcal{B}(\mathcal{H}).$$

Here, ω^* denotes the usual adjoint of ω in $\mathcal{B}(\mathcal{H})$.

As seen in Theorem 2.9 of [41], it can be proved that $\langle\langle \omega, \eta \rangle\rangle$ takes values in $\mathcal{A}'' \subseteq L^2(\mathcal{A}'', \tau)$.

Now define a natural $\mathcal{A}''$-valued bilinear form g as follows:

Lemma 2.3.6. *Let $g : \mathcal{E} \otimes_{\mathbb{C}} \mathcal{E} \to \mathcal{A}''$ be given by*

$$g(\omega \otimes_{\mathbb{C}} \eta) = \langle\langle \omega^*, \eta \rangle\rangle\,.$$

Then for all ω, η in $\mathcal{E}$ and a in $\mathcal{A}$, we have:

$$g(\omega a \otimes_{\mathbb{C}} \eta) = g(\omega \otimes_{\mathbb{C}} a\eta),\ \ g(a\omega \otimes_{\mathbb{C}} \eta) = a g(\omega \otimes_{\mathbb{C}} \eta),\ \ g(\omega \otimes_{\mathbb{C}} \eta a) = g(\omega \otimes_{\mathbb{C}} \eta) a.$$

Proof. The proof of the above statements are straightforward consequences of the properties of an inner product and the fact that $(Xa)^* = a^* X^*$ for all a, X in $\mathcal{B}(\mathcal{H})$. □

Thus, g descends to an $\mathcal{A}$-bilinear, $\mathcal{A}''$-valued map, to be denoted by g again. The restriction of g to $\Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A})$ is the candidate of a Riemannian bilinear metric in our sense, provided $g(\omega \otimes_{\mathcal{A}} \eta)$ is in $\mathcal{A}$ for all ω, η in $\Omega^1_D(\mathcal{A})$.

Let us recall the definition of a quasi-tame spectral triple as well as the notation σ from Definition 2.2.1. Then we have the following definition:

Definition 2.3.7. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a quasi-tame spectral triple. Suppose the $\mathcal{A}$-bilinear map g as in Lemma 2.3.6 is $\mathcal{A}$-valued, $V_g : \mathcal{E} \to \mathcal{E}^*$ is nondegenerate and $g \circ \sigma = g$, i.e., it gives a bilinear metric. Then we call g a canonical Riemannian bilinear metric for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$.*

When $A = C^\infty(M)$ for a compact Riemannian manifold M, then this construction recovers the usual Riemannian metric (see page 128-129 of [41] and Subsection 2.1.3 of [40]). However, in the general noncommutative set-up, one usually needs additional regularity assumptions to ensure that g takes values in $\mathcal{A}$ (as opposed to $\mathcal{A}''$). This is the content of the next proposition for which we will make use of the noncommutative Laplacian introduced in Proposition 1.3.13 and its properties.

Proposition 2.3.8. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a p-summable spectral triple and τ is faithful on the von-Neumann algebra generated by $\mathcal{S}_0$. Let $\mathcal{H}^1_D$ be the Hilbert space of one-forms and $\mathcal{L} = -d^* d$ as in Proposition 1.3.13.*

Suppose that for all X in the $*$-algebra generated by $\mathcal{A}$ and $[D, \mathcal{A}]$, the map

$$\mathbb{R} \to \mathcal{B}(\mathcal{H}) \text{ defined by } t \mapsto e^{itD} X e^{-itD}$$

is differentiable at $t = 0$ in the norm topology of $\mathcal{B}(\mathcal{H})$. If we moreover assume that $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, then

$$g(\omega \otimes_{\mathcal{A}} \eta) \in \mathcal{A} \text{ for all } \omega, \eta \in \Omega^1_D(\mathcal{A}).$$

Proof. In this proof, we will denote the domain of the unbounded operator T by $\mathrm{Dom}(T)$. We begin by noting that since τ is faithful on the von-Neumann algebra generated by $\mathcal{S}_0$, the vector space $\Omega^1_D(\mathcal{A})$ can be equipped with a semi-inner product defined by the equation:

$$\langle \eta, \eta' \rangle = \tau(\eta^* \eta').$$

Moreover, as $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, all the hypotheses of Proposition 1.3.13 are satisfied.

We claim that

$$g(da \otimes_{\mathcal{A}} db) = -\frac{1}{2}(\mathcal{L}(b^* a^*) - \mathcal{L}(b^*) a^* - b^* \mathcal{L}(a^*)) \ \forall\ a, b \in \mathcal{A},$$

where $d(a) = \sqrt{-1}[D, a]$ as in Subsection 1.3.1.

Indeed, for all c in $\mathcal{A}$, by using the self-adjointness of $\mathcal{L}$, $\mathcal{L}(x^*) = (\mathcal{L}(x))^*$ (Lemma 3.2, [43] and Lemma 5.1 of [43]), we have

$$\begin{aligned}
\tau(\langle\langle (da)^*, db \rangle\rangle\, c) &= \tau(\langle\langle (da)^*, db.c \rangle\rangle) \\
&= \langle d(a^*), db.c \rangle \ \text{ (as } (da)^* = d(a^*)\text{)} \\
&= \langle a^*, d^*(db.c) \rangle \\
&= -\frac{1}{2} \langle a^*, (b\mathcal{L}(c) - \mathcal{L}(b)c - \mathcal{L}(bc)) \rangle \ \text{ (by (1.3.2))} \\
&= -\frac{1}{2} \langle \mathcal{L}(b^* a^*) - \mathcal{L}(b^*) a^* - b^* \mathcal{L}(a^*), c \rangle \ \text{ (as } \mathcal{L} \text{ is self-adjoint and by (1.3.1))} \\
&= -\frac{1}{2} \tau(\langle\langle \mathcal{L}(b^* a^*) - \mathcal{L}(b^*) a^* - b^* \mathcal{L}(a^*), c \rangle\rangle).
\end{aligned}$$

Thus, by the normality and faithfulness of τ on $\mathcal{A}''$, we conclude that

$$g(da \otimes_{\mathcal{A}} db) = \langle\langle (da)^*, db \rangle\rangle = \langle\langle da^*, db \rangle\rangle = -(\frac{1}{2}\mathcal{L}(b^* a^*) - \mathcal{L}(b^*) a^* - b^* \mathcal{L}(a^*)).$$

This proves the claim. Since $\mathcal{L}(\mathcal{A}) \subseteq \mathcal{A}$, the proof of the proposition is complete. □

Remark 2.3.9. *Our $\mathcal{H}^1_D$ and d are the same as the bimodule and derivation respectively constructed by Cipriani and Sauvagoet ([24]) from the Dirichlet form*

$$(a, b) \mapsto -\langle \mathcal{L}(a), b \rangle, \ \ a, b \in \mathrm{Dom}((-\mathcal{L})^{\frac{1}{2}}).$$

It also follows from the definition of inner product that the map V_g is one-to-one. However, the invertibility of V_g, which is the nondegeneracy in our sense, has to be verified case by case.

2.4 Tame spectral triples and metric compatibility of connections

In this section we finally define the notion of 'tame spectral triples', the class of spectral triples for which our main result, Theorem 2.5.1 holds. In Subsection 1.4.1, we had recalled the notion of metric-compatibility of connections in pseudo-Riemannian geometry. We use the tameness of the differential calculus to define a suitable notion of metric-compatibility of connections. In Subsection 2.4.1, we define and study tame spectral triples. We observe that the bimodule of one-forms of a tame spectral triple is a centered bimodule in the sense of Section 2.1. In Subsection 2.4.2, we prove a technical result which will be used in the next chapter. Subsection 2.4.3 is devoted mainly to defining compatibility of a connection on the space of one-forms of a tame spectral triple with a pseudo-Riemannian bilinear metric. In that subsection, we also show why our definition is compatible with the usual notion in the classical case.

2.4.1 Tame spectral triples

Let us recall the maps $P_{\rm sym}$ and σ from Definition 2.2.1 and the map $\sigma^{\rm can}$ from Theorem 2.1.7.

Definition 2.4.1. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple such that the following conditions hold:*

(i) $\mathcal{E} := \Omega^1_D(\mathcal{A})$ *is a finitely generated projective right $\mathcal{A}$-module,*

(ii) The map $u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$ defined by

$$u^{\mathcal{E}}(\sum_i e'_i \otimes_{\mathcal{Z}(\mathcal{A})} a_i) = \sum_i e'_i a_i$$

is an isomorphism of vector spaces,

(iii) Suppose that there exists a right $\mathcal{A}$-module $\mathcal{F}$ such that $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$ as right $\mathcal{A}$-modules,

(iv) $\sigma = \sigma^{\rm can}$.

Then, we say that $(\mathcal{A}, \mathcal{H}, D)$ is a tame spectral triple.

Here, the existence of the map σ follows from the decomposition $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$ as in Definition 2.2.1. Moreover, the map $\sigma^{\rm can}$ is as in Theorem 2.1.7.

Remark 2.4.2. *By virtue of Proposition 2.1.6, the condition (ii) of Definition 2.4.1 implies that the bimodule $\mathcal{E}$ of one-forms of a tame spectral triple is centered. So the statement $\sigma = \sigma^{\text{can}}$ makes sense. Secondly, we are allowed to use all the results of Subsection 2.1 on centered bimodules for tame spectral triples.*

It is worthwhile to explain the significance of the equality $\sigma = \sigma^{\text{can}}$. This is what we record in the following two propositions:

Proposition 2.4.3. *If $(\mathcal{A}, \mathcal{H}, D)$ is a tame spectral triple and g is a pseudo-Riemannian metric on $\mathcal{E} = \Omega^1_D(\mathcal{A})$, then we have*

$$g(\omega \otimes_{\mathcal{A}} \eta) = g(\eta \otimes_{\mathcal{A}} \omega)$$

if either ω or η belongs to $\mathcal{Z}(\mathcal{E})$.

Proof. Let ω be in $\mathcal{Z}(\mathcal{E})$ and η be in $\mathcal{E}$. As $\sigma = \sigma^{\text{can}}$, Lemma 2.1.8 implies that

$$g(\omega \otimes_{\mathcal{A}} \eta) = g \circ \sigma(\omega \otimes_{\mathcal{A}} \eta) = g(\sigma^{\text{can}}(\omega \otimes_{\mathcal{A}} \eta)) = g(\eta \otimes_{\mathcal{A}} \omega).$$

□

The next proposition should be compared with the classical results in Proposition 1.4.1.

Proposition 2.4.4. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a tame spectral triple*

(i) Let $\mathcal{E} = \Omega^1_D(\mathcal{A})$. Then the decomposition $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \text{Ker}(\wedge) \oplus \mathcal{F}$ on simple tensors is explicitly given by

$$\omega \otimes_{\mathcal{A}} \eta a = \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta a + \eta \otimes_{\mathcal{A}} \omega a) + \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta a - \eta \otimes_{\mathcal{A}} \omega a),$$

for all ω, η in $\mathcal{Z}(\mathcal{E})$ and for all a in $\mathcal{A}$.

(ii) If $\mathcal{E}$ is a free right $\mathcal{A}$-module with a central basis $\{e_1, e_2, ..., e_n\}$ and g is a pseudo-Riemannian metric on $\mathcal{E}$, then the components $g_{ij} = g(e_i \otimes_{\mathcal{A}} e_j)$ of g are symmetric in i and j.

Proof. The second assertion of Lemma 2.1.5 implies that any element of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is a $\mathbb{C}$-linear sum of elements of the form $\omega \otimes_{\mathcal{A}} \eta a$, where ω, η are in $\mathcal{Z}(\mathcal{E})$ and for a in $\mathcal{A}$. Since $\sigma = \sigma^{\text{can}}$,

$$P_{\text{sym}}(\omega \otimes_{\mathcal{A}} \eta a) = \frac{1}{2}(1 + \sigma^{\text{can}})(\omega \otimes_{\mathcal{A}} \eta a) = \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta a + \eta \otimes_{\mathcal{A}} \omega a)$$

$$\text{and} \quad (1-P_{\rm sym})(\omega \otimes_{\mathcal{A}} \eta a) = \frac{1}{2}(1-\sigma^{\rm can})(\omega \otimes_{\mathcal{A}} \eta a) = \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta a - \eta \otimes_{\mathcal{A}} \omega a).$$

Since $P_{\rm sym}$ is an idempotent, this implies that $\frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta a + \eta \otimes_{\mathcal{A}} \omega a)$ is in $\mathrm{Ran}(P_{\rm sym}) = \mathrm{Ker}(\wedge)$ and $\frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta a - \eta \otimes_{\mathcal{A}} \omega a)$ is in $\mathrm{Ker}(P_{\rm sym}) = \mathcal{F}$.

Now we prove the second assertion. Since g is a pseudo-Riemannian metric, and $\sigma = \sigma^{\rm can}$, we have

$$g_{ij} = g(e_i \otimes_{\mathcal{A}} e_j) = g \circ \sigma(e_i \otimes_{\mathcal{A}} e_j) = g(e_j \otimes_{\mathcal{A}} e_i) = g_{ji}.$$

This finished the proof. □

Let us make the following observation at this point:

Lemma 2.4.5. *Suppose that $(\mathcal{A}, \mathcal{H}, D)$ is a tame spectral triple. Then $P_{\rm sym}$ is an $\mathcal{A}$-bimodule map. In particular, a tame spectral triple is a quasi-tame spectral triple.*

Proof. Since equation (2.2.1) is satisfied, $P_{\rm sym}$ is a right $\mathcal{A}$-linear map by definition. But as $\sigma = \sigma^{\rm can}$ and $\sigma^{\rm can}$ is $\mathcal{A}$ bilinear by Theorem 2.1.7, σ is $\mathcal{A}$ bilinear. Therefore $P_{\rm sym} = \frac{1+\sigma}{2}$ is also $\mathcal{A}$ bilinear. □

2.4.2 A remark on the isomorphism of the map $u^{\mathcal{E}}$

In this subsection, we derive a sufficient condition which ensures the isomorphism of the map $u^{\mathcal{E}}$. The following result will be crucially used in Section 3.3, where we prove the existence of the Levi-Civita connection on a class of Connes-Landi isospectral deformations of classical spectral triples.

Proposition 2.4.6. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple. Suppose that there exists a unital subalgebra $\mathcal{A}'$ of $\mathcal{Z}(\mathcal{A})$ and an $\mathcal{A}'$-submodule $\mathcal{E}'$ of $\mathcal{Z}(\mathcal{E})$ such that $\mathcal{E}'$ is projective and finitely generated as a right $\mathcal{A}'$-module. If the map*

$$u^{\mathcal{E}}_{\mathcal{E}'} : \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \to \mathcal{E},$$

defined by

$$u^{\mathcal{E}}_{\mathcal{E}'}\big(\sum_i e'_i \otimes_{\mathcal{A}'} a_i\big) = \sum_i e'_i a_i$$

is an isomorphism of vector spaces, then $u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} A \to \mathcal{E}$ *is an isomorphism. Moreover, if* $\mathcal{Z}(\mathcal{E})$ *is a finitely generated projective module over* $\mathcal{Z}(\mathcal{A})$*, then* $u^{\mathcal{E}}$ *is an isomorphism if and only if there exists* $\mathcal{E}'$ *and* $\mathcal{A}'$ *such that* $u^{\mathcal{E}}_{\mathcal{E}'}$ *is an isomorphism.*

Proof. If $u^{\mathcal{E}}_{\mathcal{E}'}$ is an isomorphism, we claim that $\mathcal{Z}(\mathcal{E}) \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A})$. If our claim is true, then we have

$$\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} = \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \cong \mathcal{E},$$

so that $u^{\mathcal{E}}$ is an isomorphism. Thus, it is enough to prove our claim.
By a verbatim adaptation of the proof of Proposition 2.1.6, we have that $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \cong \mathcal{E}$ as bimodules where the bimodule structure of $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}$ is defined by $b_1(e' \otimes_{\mathcal{A}'} a)b_2 = e' \otimes_{\mathcal{A}'} b_1 a b_2$. Since $\mathcal{E}$ is a centered $\mathcal{A}$-bimodule, this implies that $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}$ is also a centered $\mathcal{A}$-bimodule and $\mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}) \cong \mathcal{Z}(\mathcal{E})$. Since $\mathcal{Z}(\mathcal{A}) \subseteq \mathcal{A}$, we have that $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}$. Now, let $\sum_i e_i \otimes_{\mathcal{A}'} a_i$ be an arbitrary element of $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A})$. For any element b in $\mathcal{A}$, since a_i are all in $\mathcal{Z}(\mathcal{A})$, we have that $b(\sum_i e_i \otimes_{\mathcal{A}'} a_i) = \sum_i e_i \otimes_{\mathcal{A}'} b a_i = \sum_i e_i \otimes_{\mathcal{A}'} a_i b = (\sum_i e_i \otimes_{\mathcal{A}'} a_i)b$. Thus, we have that $\sum_i e_i \otimes_{\mathcal{A}'} a_i$ is in $\mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A})$ and that $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A}) \subseteq \mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}) \cong \mathcal{Z}(\mathcal{E})$.

For the reverse inclusion, we use the fact that $\mathcal{E}'$ is finitely generated and projective as a right $\mathcal{A}'$-module. Thus, there exists a free $\mathcal{A}'$-module $\mathcal{G}$ and an idempotent P on $\mathcal{G}$ such that $P(\mathcal{G}) = \mathcal{E}'$. Let $m_1, m_2, \cdots m_n$ be a basis of $\mathcal{G}$. Therefore,

$$\mathcal{E} \cong \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} = P(\mathcal{G}) \otimes_{\mathcal{A}'} \mathcal{A} = (P \otimes_{\mathcal{A}'} \mathrm{id}_{\mathcal{A}})(\mathcal{G} \otimes_{\mathcal{A}'} \mathcal{A}).$$

Clearly, $P \otimes_{\mathcal{A}'} \mathrm{id}_{\mathcal{A}}$ is an idempotent on $\mathcal{G} \otimes_{\mathcal{A}'} \mathcal{A}$ and thus for all y in $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \subseteq \mathcal{G} \otimes_{\mathcal{A}'} \mathcal{A}$, we have

$$(P \otimes_{\mathcal{A}'} \mathrm{id}_{\mathcal{A}})(y) = y. \tag{2.4.1}$$

On the other hand, $\mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A})$ is also a submodule of $\mathcal{G} \otimes_{\mathcal{A}'} \mathcal{A}$ and if x is an element of $\mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}) \cong \mathcal{Z}(\mathcal{E})$, there exist unique elements a_i in $\mathcal{A}$ such that $x = \sum_i m_i \otimes_{\mathcal{A}'} a_i$. Since $xb = bx$ for all b in $\mathcal{A}$, we see that a_i in $\mathcal{Z}(\mathcal{A})$ for all i. Hence,

$$(P \otimes_{\mathcal{A}'} \mathrm{id}_{\mathcal{A}})(x) = \sum_i (P \otimes_{\mathcal{A}'} \mathrm{id}_{\mathcal{A}})(m_i \otimes_{\mathcal{A}'} a_i) = \sum_i P(m_i) \otimes_{\mathcal{A}'} a_i \in \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A}).$$

But by (2.4.1), $(P \otimes_{\mathcal{A}'} \mathrm{id}_{\mathcal{A}})(x) = x$ so that x is in $\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{Z}(\mathcal{A})$. Since x is an arbitrary element of $\mathcal{Z}(\mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A}) \cong \mathcal{Z}(\mathcal{E})$, this completes the proof. $\square$

2.4.3 The metric compatibility of a connection on $\Omega^1_D(\mathcal{A})$

In this subsection, we formulate a notion of metric compatibility of a connection on the space of one-forms of a tame spectral triple. Recall that in Proposition 1.4.8, we had given an equivalent definition for the compatibility of a connection with a pseudo-Riemannian metric on a manifold. The definition of metric compatibility (Definition 2.4.11) in this section is motivated by that equivalent formulation. However, since our algebra $\mathcal{A}$ is in general not commutative, and the left and right-actions of $\mathcal{A}$ on $\Omega^1_D(\mathcal{A})$ do not coincide, we require some preparation.

Throughout the rest of this section, we will work with tame spectral triples and continue to denote $\Omega^1_D(\mathcal{A})$ by the symbol $\mathcal{E}$. By Lemma 2.4.5, we are allowed to use all results concerning a quasi-tame spectral triple proved before and also the $\mathcal{A}$-bilinearity of the map $P_{\rm sym}$. Moreover, g will denote any pseudo-Riemannian bilinear metric (not necessarily the canonical one) on the bimodule $\mathcal{E}$ of one-forms.

Definition 2.4.7. *Let ∇ be a connection on $\mathcal{E}$. Then we define $\Pi^0_g(\nabla) : \mathcal{Z}(\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{Z}(\mathcal{E}) \to \mathcal{E}$ by the map given by*

$$\Pi^0_g(\nabla)(\omega \otimes_{\mathbb{C}} \eta) = (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega).$$

Then, we have the following:

Lemma 2.4.8. *$\Pi^0_g(\nabla)$ descends to a map from $\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$ to $\mathcal{E}$, to be denoted by the same notation. Moreover, for all a' in $\mathcal{Z}(\mathcal{A})$ and ω, η in $\mathcal{Z}(\mathcal{E})$*

$$\Pi^0_g(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta a') = \Pi^0_g(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a' + g(\omega \otimes_{\mathcal{A}} \eta)da'. \tag{2.4.2}$$

Proof. We write $\nabla(\eta) = \sum_i \eta_i^{(1)} \otimes_{\mathcal{A}} \eta_i^{(2)}$, where $\eta_i^{(1)}, \eta_i^{(2)}$ are in $\mathcal{E}$ and the sum has finitely many terms. Since ω, η are in $\mathcal{Z}(\mathcal{E})$, Lemma 2.1.8 implies that

$$\sigma_{23}(\omega \otimes_{\mathcal{A}} da' \otimes_{\mathcal{A}} \eta) = \omega \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} da' \text{ and } \sigma_{23}(\nabla(\eta)a' \otimes_{\mathcal{A}} \omega) = \sum_i \eta_i^{(1)} \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} \eta_i^{(2)} a'.$$

Using these equations and the Leibniz rule for the connection ∇, we get

$$\begin{aligned}
& \Pi^0_g(\nabla)(\omega a' \otimes_{\mathbb{C}} \eta) \\
= \ & (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega)a' \otimes_{\mathcal{A}} \eta + \omega \otimes_{\mathcal{A}} da' \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega a') \\
= \ & (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta a') + g(\omega \otimes_{\mathcal{A}} \eta)da' + \sum_i g(\eta_i^{(1)} \otimes_{\mathcal{A}} \omega)\eta_i^{(2)} a' \\
= \ & (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta a') + g(\eta \otimes_{\mathcal{A}} \omega)da' + \sum_i g(\eta_i^{(1)} \otimes_{\mathcal{A}} \omega)\eta_i^{(2)} a' \ \text{(by Proposition 2.4.3)} \\
= \ & (g \otimes_{\mathcal{A}} id)\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta a' + \eta \otimes_{\mathcal{A}} da' \otimes_{\mathcal{A}} \omega + \nabla(\eta)a' \otimes_{\mathcal{A}} \omega) \\
= \ & (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta a' + \nabla(\eta a') \otimes_{\mathcal{A}} \omega) \\
= \ & \Pi^0_g(\nabla)(\omega \otimes_{\mathbb{C}} \eta a') \\
= \ & \Pi^0_g(\nabla)(\omega \otimes_{\mathbb{C}} a' \eta).
\end{aligned}$$

This proves the first assertion. To prove the second assertion we make the following computation: for a' in $\mathcal{Z}(\mathcal{A})$ and ω, η in $\mathcal{Z}(\mathcal{E})$, we have:

$$\begin{aligned}
\Pi^{(0)}_g(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta a') &= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta a' + \nabla(\eta a') \otimes_{\mathcal{A}} \omega) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta a' + \nabla(\eta)a' \otimes_{\mathcal{A}} \omega + \eta \otimes_{\mathcal{A}} da' \otimes_{\mathcal{A}} \omega) \\
&\text{(since } \nabla \text{ is a connection)} \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega)a' + (g \otimes_{\mathcal{A}} \mathrm{id})(\eta \otimes_{\mathcal{A}} \omega \otimes_{\mathcal{A}} da') \\
&\text{(using Lemma 2.1.8)} \\
&= \Pi^0_g(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a' + g(\omega \otimes_{\mathcal{A}} \eta)da',
\end{aligned}$$

where we have used Proposition 2.4.3. □

For the next definition, recall that $u^{\mathcal{E}}$ is left $\mathcal{Z}(\mathcal{A})$-linear so that the map $\mathrm{id}_{\mathcal{Z}(\mathcal{E})} \otimes_{\mathcal{Z}(\mathcal{A})} u^{\mathcal{E}}$ is well-defined.

Definition 2.4.9. *We define a map from $\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A}$ to $\to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ by the formula:*

$$u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} := (u^{\mathcal{E}} \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}}) \circ (\mathrm{id}_{\mathcal{Z}(\mathcal{E})} \otimes_{\mathcal{Z}(\mathcal{A})} u^{\mathcal{E}}).$$

We note that $u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ is an isomorphism since $u^{\mathcal{E}}$ is so.

For all ω, η in $\mathcal{Z}(\mathcal{E})$ and a in $\mathcal{A}$, define $\Pi_g(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ by

$$\Pi_g(\nabla) \circ u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta \otimes_{\mathcal{Z}(\mathcal{A})} a) = \Pi_g^0(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a + g(\omega \otimes_{\mathcal{A}} \eta)da$$

Therefore, for ω, η in $\mathcal{Z}(\mathcal{E})$ and a in $\mathcal{A}$, we have

$$\Pi_g(\nabla)(\omega \otimes_{\mathcal{A}} \eta a) = \Pi_g^0(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a + g(\omega \otimes_{\mathcal{A}} \eta)da. \tag{2.4.3}$$

Proposition 2.4.10. *Let $dg : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ be the map defined by*

$$dg(e \otimes_{\mathcal{A}} f) = d(g(e \otimes_{\mathcal{A}} f)).$$

The map $\Pi_g(\nabla)$ defined in Definition 2.4.9 is a well defined map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{E}$. Moreover, $\Pi_g(\nabla) - dg : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ is right $\mathcal{A}$-linear.

Proof. Since the map $u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ is an isomorphism, it is enough to check that the map

$$\Pi_g(\nabla) \circ u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$$

is well defined. For ω, η in $\mathcal{Z}(\mathcal{E})$, a in $\mathcal{Z}(\mathcal{A})$, b in $\mathcal{A}$, the equalities

$$\Pi_g(\nabla) \circ u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}(\omega a \otimes_{\mathbb{C}} \eta \otimes_{\mathbb{C}} b) = \Pi_g(\nabla) \circ u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}(\omega \otimes_{\mathbb{C}} a\eta \otimes_{\mathbb{C}} b) \text{ and}$$

$$\Pi_g(\nabla) \circ u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}(\omega \otimes_{\mathbb{C}} \eta a \otimes_{\mathbb{C}} b) = \Pi_g(\nabla) \circ u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}(\omega \otimes_{\mathbb{C}} \eta \otimes_{\mathbb{C}} ab)$$

follow from Lemma 2.4.8 and Equation (2.4.2) respectively.

Lemma 2.1.5 implies that $\{\omega \otimes_{\mathcal{A}} \eta : \omega, \eta \in \mathcal{Z}(\mathcal{E})\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Therefore, for proving the right $\mathcal{A}$-linearity of the map $\Pi_g(\nabla) - dg$ it is sufficient to evaluate it on $\omega \otimes_{\mathcal{A}} \eta ab$, where $\omega, \eta \in \mathcal{Z}(\mathcal{E}),\ a, b \in \mathcal{A}$, since $u^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ is an isomorphism.

$$\begin{aligned}
(\Pi_g(\nabla) - dg)(\omega \otimes_{\mathcal{A}} \eta ab) =& \Pi_g^0(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)ab + g(\omega \otimes_{\mathcal{A}} \eta)d(ab) \\
& - d(g(\omega \otimes_{\mathcal{A}} \eta ab)) \text{ (by (2.4.3))} \\
=& \Pi_g^0(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)ab + g(\omega \otimes_{\mathcal{A}} \eta)(da.b + a.db) \\
& - d(g(\omega \otimes_{\mathcal{A}} \eta a))b - g(\omega \otimes_{\mathcal{A}} \eta a)db \\
=& (\Pi_g^0(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a + g(\omega \otimes_{\mathcal{A}} \eta)d(a) - dg(\omega \otimes_{\mathcal{A}} \eta a))b \\
=& (\Pi_g(\nabla) - dg)(\omega \otimes_{\mathcal{A}} \eta a)b
\end{aligned}$$

by another application of (2.4.3). □

Now we are in a position to suitably define compatibility of a connection with a pseudo-Riemannian bilinear metric.

Definition 2.4.11. *Let $dg : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ be as defined in Proposition 2.4.10. We say that a connection ∇ on $\mathcal{E}$ is compatible with a pseudo-Riemannian metric g if for all e, f in $\mathcal{E}$,*

$$\Pi_g(\nabla)(e \otimes_{\mathcal{A}} f) = dg(e \otimes_{\mathcal{A}} f).$$

Proposition 2.4.12. *The above definition of metric compatibility coincides with that in the classical case.*

Proof. Let (M, g) be a pseudo-Riemannian manifold and $\mathcal{A}$ be the algebra $C^\infty(M)$ of smooth functions on M. Thus, in this case, we have $\mathcal{A} = \mathcal{Z}(\mathcal{A}) = C^\infty(M)$, $\mathcal{E} = \mathcal{Z}(\mathcal{E}) = \Omega^1(M)$ and $\sigma = \text{flip}$.

By Proposition 1.4.8, a connection ∇ on $\Omega^1(M)$ is compatible in the classical sense with g if and only if for all ω, η in $\mathcal{E}$,

$$(g \otimes_{\mathcal{A}} \text{id})[\text{flip}_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta) + (\omega \otimes_{\mathcal{A}} \nabla(\eta))] = dg(\omega \otimes_{\mathcal{A}} \eta).$$

As $\sigma = \text{flip}$ and $g(e \otimes_{\mathcal{A}} f) = g(f \otimes_{\mathcal{A}} e)$ for all e, f in $\mathcal{E}$, it can be easily checked that

$$(g \otimes_{\mathcal{A}} \text{id})[\text{flip}_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta) + (\omega \otimes_{\mathcal{A}} \nabla(\eta))] = (g \otimes_{\mathcal{A}} \text{id})\sigma_{23}(\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega) = \Pi_g(\nabla)(\omega \otimes_{\mathcal{A}} \eta).$$

Thus, our definition of metric compatibility coincides with that in the classical case. □

2.5 Existence and uniqueness of Levi-Civita connections for tame spectral triples

The goal of this section is to prove the following theorem:

Theorem 2.5.1. *Suppose $(\mathcal{A}, \mathcal{H}, D)$ is a tame spectral triple and $\mathcal{E}$ is the space of one-forms on it. If g is any pseudo-Riemannian bilinear metric on $\mathcal{E}$, then there exists a unique connection on $\mathcal{E}$ which is torsionless and compatible with g (in the sense of Definition 2.4.11). In particular, this applies to the candidate of a Riemannian bilinear map in Definition 2.3.7.*

The theorem will be proved in two steps. In the first step, we construct a right $\mathcal{A}$-linear map $\Phi_g : \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E})$ (see Definition 2.5.3) and prove that (Theorem 2.5.5) the isomorphism of Φ_g is a sufficient condition for the existence and uniqueness of Levi-Civita connections for tame spectral triples. Then we show that for tame spectral triples, Φ_g is indeed an isomorphism.

Since we will be working with tame spectral triples, the isomorphism of the map $u^{\mathcal{E}}$ implies that $\mathcal{E}$ is centered. Therefore, we will freely use the fact that $\mathcal{E}$ is centered throughout this section, sometimes without mentioning.

We collect some results in a preparatory lemma.

Lemma 2.5.2. *(i) The map $\Pi_g(\nabla) - dg \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{E})$ is determined by its restriction on $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ for any connection ∇ and can be viewed as an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E})$*

(ii) For any two torsionless connections ∇_1 and ∇_2, $\nabla_1 - \nabla_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$

Proof. By the definition of $\Pi^0_g(\nabla)$ and the equality $g \circ \sigma = g$, it follows that $\Pi^0_g(\nabla) \circ \sigma = \Pi^0_g(\nabla)$ on $\mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{Z}(\mathcal{E})$. Now for $\omega, \eta \in \mathcal{Z}(\mathcal{E})$ and $a \in \mathcal{A}$, we have

$$\begin{aligned}(\Pi_g(\nabla) - dg) \circ \sigma(\omega \otimes_{\mathcal{A}} \eta a) &= (\Pi_g(\nabla) - dg)(\sigma(\omega \otimes_{\mathcal{A}} \eta)a) = (\Pi_g(\nabla) - dg) \circ \sigma(\omega \otimes_{\mathcal{A}} \eta)a \\ &= (\Pi_g(\nabla) - dg)(\omega \otimes_{\mathcal{A}} \eta)a = (\Pi_g(\nabla) - dg)(\omega \otimes_{\mathcal{A}} \eta a),\end{aligned}$$

since $\Pi_g(\nabla) - dg$ is right $\mathcal{A}$-linear by Proposition 2.4.10. Therefore, $\Pi_g(\nabla) - dg = (\Pi_g(\nabla) - dg) \circ \sigma$ on the whole of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Since $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} = \mathrm{Ran}(P_{\mathrm{sym}}) = \mathrm{Ran}(\frac{1+\sigma}{2})$, this proves (i).
Now we prove (ii). If ∇_1 and ∇_2 are two torsionless connections, $\wedge \circ \nabla_1 = -d = \wedge \circ \nabla_2$.

Therefore, $\mathrm{Ran}(\nabla_1 - \nabla_2) \subseteq \mathrm{Ker}(\wedge) = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$. Moreover $(\nabla_1 - \nabla_2)(\omega a) = \nabla_1(\omega)a - \nabla_2(\omega)a$ for ω in $\mathcal{E}$ and for a in $\mathcal{A}$. Hence, $\nabla_1 - \nabla_2 \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$. □

Definition 2.5.3. *We define a map*

$$\Phi_g : \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E}) \text{ by}$$

$$\Phi_g(L) = (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L \otimes_{\mathcal{A}} \mathrm{id})(1 + \sigma).$$

Proposition 2.5.4. *Φ_g is a right $\mathcal{A}$-linear map.*

Proof. Let ω, η be in $\mathcal{Z}(\mathcal{E})$, and a, b in $\mathcal{A}$ and L in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$. Then by using Proposition 2.4.3, the $\mathcal{A}$-bilinearity of σ (Lemma 2.4.5) and the equality $\sigma = \sigma^{\mathrm{can}}$, we obtain

$$\begin{aligned}
\Phi_g(La)(\omega \otimes_{\mathcal{A}} \eta b) &= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(La \otimes_{\mathcal{A}} \mathrm{id})(1 + \sigma)(\omega \otimes_{\mathcal{A}} \eta b) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(La \otimes_{\mathcal{A}} \mathrm{id})(\omega \otimes_{\mathcal{A}} \eta b + \eta \otimes_{\mathcal{A}} \omega b) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L(a\omega) \otimes_{\mathcal{A}} \eta b + L(a\eta) \otimes_{\mathcal{A}} \omega b) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L \otimes_{\mathcal{A}} \mathrm{id})(a\omega \otimes_{\mathcal{A}} \eta b + a\eta \otimes_{\mathcal{A}} \omega b) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L \otimes_{\mathcal{A}} \mathrm{id})(1 + \sigma)(a(\omega \otimes_{\mathcal{A}} \eta b)) \\
&= (\Phi_g(L)a)(\omega \otimes_{\mathcal{A}} \eta b).
\end{aligned}$$

Hence we have that $\Phi_g(La) = \Phi_g(L)a$. □

Now we are in a position to prove the following result which gives a sufficient condition for the existence and uniqueness of Levi-Civita connections.

Theorem 2.5.5. *If $\Phi_g : \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E})$ is an isomorphism of right $\mathcal{A}$-modules, then there exists a unique connection on $\mathcal{E}$ which is torsionless and compatible with g.*

Proof. We recall the torsionless connection ∇_0 constructed in Lemma 2.2.3. By (i) of Lemma 2.5.2, $dg - \Pi_g(\nabla_0) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E})$. Since Φ_g is an isomorphism from $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$ to $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E})$ there exists a unique element $\Phi_g^{-1}(dg - \Pi_g(\nabla_0)) \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$. Define the $\mathbb{C}$-linear map

$$\nabla := \nabla_0 + \Phi_g^{-1}(dg - \Pi_g(\nabla_0)).$$

We claim that ∇ is a torsionless connection on $\mathcal{E}$ which is compatible with g. Indeed, if ω is in $\mathcal{E}$ and a in $\mathcal{A}$, we have

$$\begin{aligned}\nabla(\omega a) &= \nabla_0(\omega)a + \omega \otimes_{\mathcal{A}} da + \Phi_g^{-1}(dg - \Pi_g(\nabla_0))(\omega)a \\ &= \nabla(\omega)a + \omega \otimes_{\mathcal{A}} da.\end{aligned}$$

so that ∇ is a connection. That ∇ is a torsionless connection is verified from the following:

$$\begin{aligned}\wedge \circ \nabla &= \wedge \circ \nabla_0 + \wedge \circ \Phi_g^{-1}(dg - \Pi_g(\nabla_0)) \\ &= \wedge \circ \nabla_0 \text{ (since } \mathrm{Ran}(\Phi_g^{-1})(dg - \Pi_g(\nabla_0) \subseteq \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E} = \mathrm{Ker}(\wedge)) \\ &= -d.\end{aligned}$$

By virtue of (ii) of Lemma 2.5.2, this in particular implies that $\nabla - \nabla_0$ in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E})$ so that $\Phi_g(\nabla - \nabla_0)$ is well-defined. Moreover, for ω, η in $\mathcal{Z}(\mathcal{E})$ and a in $\mathcal{A}$, we have

$$\begin{aligned}&(\Pi_g(\nabla) - \Pi_g(\nabla_0))(\omega \otimes_{\mathcal{A}} \eta a) \\ =&\Pi_g^0(\nabla)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a - \Pi_g^0(\nabla_0)(\omega \otimes_{\mathcal{Z}(\mathcal{A})} \eta)a \text{ (by (2.4.3))} \\ =&(g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}\Big((\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega) - (\nabla_0(\omega) \otimes_{\mathcal{A}} \eta + \nabla_0(\eta) \otimes_{\mathcal{A}} \omega)\Big)a \\ =&(g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}((\nabla - \nabla_0) \otimes_{\mathcal{A}} \mathrm{id})(1 + \sigma)(\omega \otimes_{\mathcal{A}} \eta a) \\ =&\Phi_g(\nabla - \nabla_0)(\omega \otimes_{\mathcal{A}} \eta a).\end{aligned}$$

Therefore, $\Phi_g(\nabla - \nabla_0) = \Pi_g(\nabla) - \Pi_g(\nabla_0)$. Since $\Phi_g(\nabla - \nabla_0) = dg - \Pi_g(\nabla_0)$ by the definition of ∇, we have $\Pi_g(\nabla) = dg$. Therefore, ∇ is compatible with g.

To show uniqueness, suppose ∇' is another torsionless connection compatible with the metric g. Then exactly as above, $\nabla - \nabla' \in \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E})$ and

$$\Phi_g(\nabla - \nabla') = \Pi_g(\nabla) - \Pi_g(\nabla') = dg - dg = 0,$$

where we have used the fact that ∇ and ∇' are compatible with g. Hence, $\nabla = \nabla'$, as Φ_g is an isomorphism. $\square$

The rest of this section will be devoted to proving that for tame spectral triples, Φ_g is indeed an isomorphism, which will then prove Theorem 2.5.1. We will make use of the isomorphism

$$\zeta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^* \to \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$$

as introduced in Proposition 1.1.8.

Lemma 2.5.6. *Let g be a pseudo-Riemannian bilinear metric on $\mathcal{E}$ and L be an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$ such that $\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L) = \xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega)$ for some ξ, η, ω in $\mathcal{E}$.*

(i) Then for all e in $\mathcal{E}$, we have

$$L(e) = \xi \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} e).$$

(ii) Let us define then an element L' in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$ by the equation

$$\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L') = \eta \otimes_{\mathcal{A}} \xi \otimes_{\mathcal{A}} V_g(\omega).$$

If L in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$ and ξ, η, ω are in $\mathcal{Z}(\mathcal{E})$, then $L = L'$ as elements of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$. Moreover,

$$\xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega) = \eta \otimes_{\mathcal{A}} \xi \otimes_{\mathcal{A}} V_g(\omega).$$

(iii) The set $\{\zeta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(\xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega) : \xi, \eta, \omega \in \mathcal{Z}(\mathcal{E})\}$ is right $\mathcal{A}$-total in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$.

Proof. Throughout the proof, we will repeatedly use Lemma 2.1.8 and the equation $\sigma = \sigma^{\mathrm{can}}$. Let e denote an element of $\mathcal{E}$. By the definition of $\zeta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}$, it follows that

$$L(e) = \xi \otimes_{\mathcal{A}} \eta V_g(\omega)(e) = \xi \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} e).$$

Now we prove part (ii). By part (i), we have

$$P_{\mathrm{sym}}(L(e)) = \frac{1}{2}(\xi \otimes_{\mathcal{A}} \eta + \eta \otimes_{\mathcal{A}} \xi) g(\omega \otimes_{\mathcal{A}} e).$$

Since $L(e)$ is in $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$, we have $P_{\mathrm{sym}} L(e) = L(e)$. Therefore, $\frac{1}{2}(\xi \otimes_{\mathcal{A}} \eta + \eta \otimes_{\mathcal{A}} \xi) g(\omega \otimes_{\mathcal{A}} e) = \xi \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} e)$ which implies that $\xi \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} e) = \eta \otimes_{\mathcal{A}} \xi g(\omega \otimes_{\mathcal{A}} e)$. This proves that

$L(e) = L'(e)$. Hence,

$$\xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega) = \zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L) = \zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L') = \eta \otimes_{\mathcal{A}} \xi \otimes_{\mathcal{A}} V_g(\omega).$$

Finally, for part (iii), we note that since g is bilinear, the set $S = \{\xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega) : \xi, \eta, \omega \in \mathcal{Z}(\mathcal{E})\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^*$ and therefore $\zeta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(S)$ is right $\mathcal{A}$-total in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$. □

Now we are going to make an use of Lemma 1.1.7. In the notation of Lemma 1.1.7, we define $\mathcal{F} = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, $h = g^{(2)}$and $T = P_{\mathrm{sym}}$. Since P_{sym} is $\mathcal{A}$-bilinear (Lemma 2.4.5) and Proposition 2.3.5 implies that $V_{g^{(2)}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})^*$ is an isomorphism, Lemma 1.1.7 implies that the adjoint P^*_{sym} of P_{sym} exists.

Lemma 2.5.7. *For all ω, η in $\mathcal{E}$, $V_{g^{(2)}}\sigma(\omega \otimes_{\mathcal{A}} \eta) = V_{g^{(2)}}(\omega \otimes_{\mathcal{A}} \eta)\sigma$. In particular, $P_{\mathrm{sym}} = P^*_{\mathrm{sym}}$ in the notation of Lemma 1.1.7.*

Proof. As $V_{g^{(2)}}$ is right $\mathcal{A}$-linear by Proposition 2.3.5, $\sigma = 2P_{\mathrm{sym}} - 1$ is $\mathcal{A}$-bilinear and $\{\omega \otimes_{\mathcal{A}} \eta : \omega, \eta \in \mathcal{Z}(\mathcal{E})\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ (Lemma 2.1.5), it is enough to prove that for all $\omega, \eta, \omega', \eta'$ in $\mathcal{Z}(\mathcal{E})$,

$$V_{g^{(2)}}(\sigma(\omega \otimes_{\mathcal{A}} \eta))(\omega' \otimes_{\mathcal{A}} \eta') = V_{g^{(2)}}(\omega \otimes_{\mathcal{A}} \eta)\sigma(\omega' \otimes_{\mathcal{A}} \eta').$$

But this follows from the following computation:

$$\begin{aligned} V_{g^{(2)}}(\sigma(\omega \otimes_{\mathcal{A}} \eta))(\omega' \otimes_{\mathcal{A}} \eta') &= g^{(2)}((\eta \otimes_{\mathcal{A}} \omega) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) \\ &= g(\eta \otimes_{\mathcal{A}} \eta') g(\omega \otimes_{\mathcal{A}} \omega') \\ &= g(\omega \otimes_{\mathcal{A}} \omega') g(\eta \otimes_{\mathcal{A}} \eta') \text{ (by Proposition 2.3.2)} \\ &= V_{g^{(2)}}(\omega \otimes_{\mathcal{A}} \eta)\sigma(\omega' \otimes_{\mathcal{A}} \eta'). \end{aligned}$$

This finishes the proof. □

Lemma 2.5.8. *Let L be in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$ be such that $\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L) = \xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega)$ for some ξ, η, ω in $\mathcal{Z}(\mathcal{E})$. Then*

$$\Phi_g(L) = \zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\eta \otimes_{\mathcal{A}} V_{g^{(2)}}(\xi \otimes_{\mathcal{A}} \omega + \omega \otimes_{\mathcal{A}} \xi)). \tag{2.5.1}$$

Proof. The set $\{\omega \otimes_{\mathcal{A}} \eta : \omega, \eta \in \mathcal{Z}(\mathcal{E})\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ by Lemma 2.1.5 and the map $\Phi_g(L)$ is right $\mathcal{A}$-linear. Therefore, it is enough to prove that for all ω', η' in $\mathcal{Z}(\mathcal{E})$,

$$\Phi_g(L)(\omega' \otimes_{\mathcal{A}} \eta') = \eta V_{g^{(2)}}(\xi \otimes_{\mathcal{A}} \omega + \omega \otimes_{\mathcal{A}} \xi)(\omega' \otimes_{\mathcal{A}} \eta').$$

By using part (i) of Lemma 2.5.6, we compute

$$\begin{aligned}
\Phi_g(L)(\omega' \otimes_{\mathcal{A}} \eta') &= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(L(\omega') \otimes_{\mathcal{A}} \eta' + L(\eta') \otimes_{\mathcal{A}} \omega') \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})\sigma_{23}(\xi \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} \omega') \otimes_{\mathcal{A}} \eta' + \xi \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} \eta') \otimes_{\mathcal{A}} \omega') \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})(\xi \otimes_{\mathcal{A}} \eta' \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} \omega') + \xi \otimes_{\mathcal{A}} \omega' \otimes_{\mathcal{A}} \eta g(\omega \otimes_{\mathcal{A}} \eta')) \\
&= g(\xi \otimes_{\mathcal{A}} \eta')\eta g(\omega \otimes_{\mathcal{A}} \omega') + g(\xi \otimes_{\mathcal{A}} \omega')\eta g(\omega \otimes_{\mathcal{A}} \eta') \\
&= \eta g(\xi \otimes_{\mathcal{A}} \eta')g(\omega \otimes_{\mathcal{A}} \omega') + \eta g(g(\xi \otimes_{\mathcal{A}} \omega')\omega \otimes_{\mathcal{A}} \eta') \text{ (since } g \text{ is bilinear)} \\
&= \eta V_{g^{(2)}}(\xi \otimes_{\mathcal{A}} \omega + \omega \otimes_{\mathcal{A}} \xi)(\omega' \otimes_{\mathcal{A}} \eta').
\end{aligned}$$

□

We have assumed that our pseudo-Riemannian metric g is bilinear and so in particular, left $\mathcal{A}$-linear. This implies that the map V_g (and hence V_g^{-1}) is left $\mathcal{A}$-linear. Hence, the map $\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1}$ in the following proposition makes sense.

Proposition 2.5.9. *Let L be in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E})$. Then*

$$\frac{1}{2}\Phi_g(L) = \zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\mathrm{id} \otimes_{\mathcal{A}} V_{g^{(2)}})(P_{\mathrm{sym}})_{23}(\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1})\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L). \tag{2.5.2}$$

Proof. Let L be in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E})$ be such that $\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L) = \xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega)$ for some ξ, η, ω in $\mathcal{Z}(\mathcal{E})$. Then by part 2. of Lemma 2.5.6, we have $\xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega) = \eta \otimes_{\mathcal{A}} \xi \otimes_{\mathcal{A}} V_g(\omega)$. Therefore,

$$\begin{aligned}
&\zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}((\mathrm{id} \otimes_{\mathcal{A}} V_{g^{(2)}})(P_{\mathrm{sym}})_{23}(\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1})\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}(L)) \\
=\; &\zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}((\mathrm{id} \otimes_{\mathcal{A}} V_{g^{(2)}})(P_{\mathrm{sym}})_{23}(\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1})(\xi \otimes_{\mathcal{A}} \eta \otimes_{\mathcal{A}} V_g(\omega))) \\
=\; &\zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}((\mathrm{id} \otimes_{\mathcal{A}} V_{g^{(2)}})(P_{\mathrm{sym}})_{23}(\eta \otimes_{\mathcal{A}} \xi \otimes_{\mathcal{A}} \omega)) \\
=\; &\frac{1}{2}\zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\eta \otimes_{\mathcal{A}} V_{g^{(2)}}(\xi \otimes_{\mathcal{A}} \omega + \omega \otimes_{\mathcal{A}} \xi)) \text{ (since } \xi, \omega \in \mathcal{Z}(\mathcal{E})) \\
=\; &\frac{1}{2}\Phi_g(L) \text{ (by Lemma 2.5.8)}.
\end{aligned}$$

Thus, we have proved (2.5.2) for all L of the above form. But since the maps $\zeta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}$, $\Phi_g, V_{g^{(2)}}$ and $P_{\rm sym}$ are all right $\mathcal{A}$-linear, we can conclude that (2.5.2) holds for all L in ${\rm Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})$ by appealing to part (iii) of Lemma 2.5.6. $\square$

Lemma 2.5.10. *$V_{g^{(2)}}$ is an isomorphism from $\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E}$ onto $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$.*

Proof. Let us start by claiming that $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$ can be identified with the bimodule $\{\phi\in(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})^*:\phi\circ(1-P_{\rm sym})=0\}$. Indeed, if ψ is in $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$, then ψ can be uniquely extended to an element ϕ in $(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})^*$ by using the decomposition $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}={\rm Ran}(P_{\rm sym})\oplus{\rm Ran}(1-P_{\rm sym})$. Clearly, $\psi=\phi\circ P_{\rm sym}$. Conversely, if ϕ is in $(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})^*$ then $\phi\circ P_{\rm sym}$ defines an element of $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$. This proves our claim.

Now we use our claim to prove that $V_{g^{(2)}}$ is one-to-one and onto as a map from $\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E}$ to $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$. Let ϕ in $(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})^*$ be such that $\phi\circ(1-P_{\rm sym})=0$. Since $V_{g^{(2)}}:\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}\to(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})^*$ is an isomorphism by Proposition 2.3.5, there exists ψ in $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ such that $V_{g^{(2)}}(\psi)=\phi$. We claim that $P_{\rm sym}\psi=\psi$. Indeed,

$$\begin{aligned}V_{g^{(2)}}(P_{\rm sym}\psi) &= V_{g^{(2)}}(\psi)\circ P_{\rm sym}=\phi\circ P_{\rm sym}\ (\text{since, by Lemma 2.5.7 } P^*_{\rm sym}=P_{\rm sym})\\ &= \phi\circ P_{\rm sym}+\phi\circ(1-P_{\rm sym})=\phi\\ &= V_{g^{(2)}}(\psi).\end{aligned}$$

By using Proposition 2.3.5, we conclude that $P_{\rm sym}\psi=\psi$. This proves that $V_{g^{(2)}}$ maps onto $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$.

To prove that $V_{g^{(2)}}$ is one-to-one as a map from $\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E}$ to $(\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E})^*$, let ψ in $\mathcal{E}\otimes^{\rm sym}_{\mathcal{A}}\mathcal{E}$ be such that $V_{g^{(2)}}(\psi)\circ P_{\rm sym}=0$. Therefore, by Lemma 2.5.7, we have

$$V_{g^{(2)}}(\psi)=V_{g^{(2)}}P_{\rm sym}(\psi)=V_{g^{(2)}}(\psi)\circ P_{\rm sym}=0, \tag{2.5.3}$$

so that by Proposition 2.3.5, we have $\psi=0$. $\square$

Finally, we give a proof of our main theorem for this chapter using the results just proved.

Proof of Theorem 2.5.1. We need to prove that the map Φ_g is an isomorphism from $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$ to $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}, \mathcal{E})$. By Lemma 2.1.10, the map

$$(P_{\mathrm{sym}})_{23} : (\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} (\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$$

is an isomorphism of right $\mathcal{A}$ modules. Since $(\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1})\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}$ is an isomorphism from $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$ to $(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{E}$ and $V_{g^{(2)}}$ is an isomorphism from $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ to $(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})^*$ by Lemma 2.5.10, we see that $\zeta_{\mathcal{E},\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\mathrm{id} \otimes_{\mathcal{A}} V_{g^{(2)}})(P_{\mathrm{sym}})_{23}(\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1})\zeta^{-1}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E},\mathcal{E}}$ is an isomorphism from $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$ to $\mathrm{Hom}_{\mathcal{A}}((\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} \mathcal{E})$. Finally, the equation (2.5.2) implies that Φ_g is an isomorphism. □

We end this chapter by comparing some of the related results in the literature. We will need the terminology of tame differential calculus whose definition is a verbatim adaptation of the definition of tame spectral triples in the context of differential calculi. Thus, a differential calculus is called tame if the bimodule of one-forms of the differential calculus satisfies conditions (i)-(iv) of Definition 2.4.1. For a precise definition of a tame differential calculus, we refer to Definition 2.2 of [14]. We continue to have an $\mathcal{A}$-bilinear map $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ as in Definition 2.2.1.

In [15] and [14], Theorem 2.5.1 was proved for an arbitrary tame differential calculus by adapting the classical Koszul-formula proof of existence and uniqueness of Levi-Civita connections. Indeed, Proposition 5.6 of [14] deduces a Koszul-formula for the Levi-Civita connection.

Now, we discuss the relevance of bimodule connections for tame spectral triples.

Definition 2.5.11. *Suppose $\mathcal{E}$ be the bimodule of one-forms of a differential calculus and $\sigma' : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ be a bimodule map. A right connection ∇_1 on $\mathcal{E}$ is said to be a bimodule connection for the pair $(\mathcal{E}, \sigma')$ if for all a in $\mathcal{A}$ and for all e in $\mathcal{E}$, the following equation holds:*

$$\nabla_1(ae) = a\nabla_1(e) + \sigma'(da \otimes_{\mathcal{A}} e).$$

The following theorem was proved in [15].

Theorem 2.5.12. *(Theorem 7.3, [15]) If g is a pseudo-Riemannian bilinear metric on a tame differential calculus $(\mathcal{E}, d)$, then the unique Levi-Civita connection for $(\mathcal{E}, g)$ is a bimodule connection for the pair $(\mathcal{E}, \sigma)$.*

Now we come to the issue of existence and uniqueness of Levi-Civita connections for pseudo-Riemannian metrics which are not necessarily $\mathcal{A}$-bilinear. Let us make the following definition:

Definition 2.5.13. *A pseudo-Riemannian metric (not necessarily $\mathcal{A}$-bilinear) g on a tame spectral triple is called strongly σ-compatible if for all e, f, e', f' in $\mathcal{E}$,*

$$g^{(2)}(\sigma(e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f')) = g^{(2)}((e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} \sigma(e' \otimes_{\mathcal{A}} f')).$$

Our Lemma 2.5.7 implies that any pseudo-Riemannian bilinear metric on a tame spectral triple is automatically strongly σ-compatible.

The main result of [13] states that if g is a strongly σ-compatible pseudo-Riemannian metric on any tame differential calculus $(\mathcal{E}, d)$ (see Definition 2.2 of [14]), then there exists a unique Levi-Civita connection for the triplet $(\mathcal{E}, d, g)$.

Since tame spectral triples are examples of tame differential calculus, the two results mentioned above also hold for tame spectral triples.

Chapter 3

Examples of Tame Spectral Triples

This chapter illustrates examples of tame spectral triples. By Theorem 2.5.1 these admit a unique torsionless connection compatible with a pseudo-Riemannian bilinear metric.

In Section 3.1, we discuss the example of the fuzzy 3-spheres. The question of existence and uniqueness of Levi-Civita connections on fuzzy 3-spheres was addressed in [41], albeit with a different formulation of metric compatibility. We will see (Proposition 3.1.4) that the candidate of a pseudo-Riemannian bilinear metric proposed in Lemma 2.3.6 is actually a Riemannian bilinear metric. Let us denote this by g. The authors of [41] proved that a family of Levi-Civita connections in the sense of that paper exist for the triple $(\mathcal{E}, d, g)$. However, if in addition, one demands the Levi-Civita connection to be real, then there exists a unique connection. In Theorem 3.1.5, we show that the spectral triple in [41] are tame. Thus, by Theorem 2.5.1, for each pseudo-Riemannian bilinear metric, there exists a unique Levi-Civita connection. In particular, if we take the Riemannian bilinear metric g as above, then the Levi-Civita connection from Theorem 2.5.1 coincides with the unique real Levi-Civita connection obtained in [41]. In [15], a spectral triple is defined on the fuzzy 3-spheres which is a truncated version of the spectral triple discussed here and in [41]. That spectral triple was also shown to be a tame spectral triple (Theorem 8.5 of [15]). In particular, after obtaining the unique Levi-Civita connection, that article also computes the Ricci and scalar curvatures associated to the spectral triple. We will not be addressing the issue of curvature in this thesis.

In Section 3.2, we discuss the example of the quantum Heisenberg manifolds introduced in [81]. In [22], a family of spectral triples and the corresponding space of forms were studied. However, it turned out that with a particular choice of a metric and the definition of the metric compatibility of the connection in the sense of [41], there exists no connection on the space of one-forms which is both torsionless and compatible with the metric. In Theorem 3.2.6, we show that the spectral triples of [22] are tame spectral triples, and hence each pseudo-Riemannian bilinear metric admits a unique Levi-Civita connection. We would like to mention that in [56] and [57] compatible connections for Hermitian metrics and Yang-Mills theory on the quantum Heisenberg manifolds have been studied.

In Section 3.3, we discuss the example of Connes Dubois-Violette Rieffel deformations $C^\infty(M)_\theta$ ([26, 82]) of a compact Riemannian manifold M and a spectral triples on it given in [27]. In Theorem 3.3.1, we show that this spectral triple is a tame spectral triple under some technical assumptions, thus admitting a unique Levi-Civita connection (as per our formulation) for each pseudo-Riemannian bilinear metric. In Corollary 3.3.38, we show that this unique Levi-Civita connection is the θ-deformation of the classical Levi-Civita connection on a compact Riemannian manifold. This also demonstrates that our formulation of Levi-Civita connections respects θ-deformations.

We would like to mention that in the recent paper [54], a spectral triple on the Cuntz algebra with three generators was given. In Theorem 3.4 of the same paper, it was shown that this spectral triple is a tame spectral triple, and thus admits a unique Levi-Civita connection for each pseudo-Riemannian bilinear metric.

The contents of this chapter are from [16]. As in the previous chapter, if $(\mathcal{A}, \mathcal{H}, D)$ is a spectral triple, we will often denote the space of one-forms $\Omega^1_D(\mathcal{A})$ of this spectral triple by the symbol $\mathcal{E}$.

3.1 Levi-Civita connection for fuzzy 3-spheres

We start by giving a brief description of the spectral triple on the fuzzy 3-sphere. Let G denote the compact Lie group $SU(2)$ and V_j, $j \in \frac{1}{2}\mathbb{N}_0$, denote the $(2j+1)$ dimensional irreducible representation of $SU(2)$. Let k be a positive integer and $\mathcal{H}_0 := \bigoplus_{j=0,\frac{1}{2},\ldots,\frac{k}{2}} V_j^* \otimes_{\mathbb{C}} V_j$ and $\mathcal{A} := \mathcal{B}(\mathcal{H}_0)$. Let W be the carrier vector space of the irreducible representation of the Clifford algebra generated by the vector space T_eG with respect to the Killing form on G as defined by equations (3.4) and (3.5) of [41]. There exists a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{H} := \mathcal{H}_0 \otimes_{\mathbb{C}} W$, called the "fuzzy" or non-commutative 3-sphere. We refer to [41] for the details.

In what follows, we will denote the elements $1 \otimes_{\mathbb{C}} \psi_i$ in the center of $\mathcal{E} := \Omega^1_D(\mathcal{A})$ as in [41] by the symbol e_i, so that

$$e_j \wedge e_k = -e_k \wedge e_j$$

and $\{e_i \wedge e_j : i \leq j\}$ is linearly independent. Let $\mathcal{E} := \Omega^1_D(\mathcal{A})$. One has the following result.

Theorem 3.1.1. *(Equation (3.19) and Theorem 3.2 of [41]) The space of forms for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ has the following description:*

(i) The module $\mathcal{E}$ is isomorphic to $\mathrm{Span}_{\mathbb{C}}\{e_i a_i : i = 1, 2, 3\}$ and thus is a free right $\mathcal{A}$ module of rank three.

(ii) The bimodule $\Omega^2_D(\mathcal{A})$ of two-forms is isomorphic to $\mathrm{Span}_{\mathbb{C}}\{e_i \wedge e_j a_{ij} : a_{ij} = -a_{ji}\}$ is a free right $\mathcal{A}$ module of rank three.

Moreover, it was also proven in [41] that the space of three-forms is a free rank one module and all the higher forms are zero. The bimodule structure for $\mathcal{E} := \Omega^1_D(\mathcal{A})$ (and similarly, for the higher forms) is given by

$$a(b \otimes_{\mathbb{C}} \psi_i)c = abc \otimes_{\mathbb{C}} \psi_i = e_i abc.$$

We note that this implies that $\mathcal{E}$ is centered. In fact, $\mathcal{Z}(\mathcal{E})$ is a complex linear span of $\{e_1, e_2, e_3\}$. We also note that we can identify $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ with $\mathrm{Span}_{\mathbb{C}}\{e_i \otimes_{\mathcal{A}} e_j a : i, j = 1, 2, 3\}$.

Lemma 3.1.2. *The space $\mathrm{Ker}(\wedge)$ is generated (as a right $\mathcal{A}$ module) by the set*

$$\{e_i \otimes_{\mathcal{A}} e_i,\ e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i : i, j = 1, 2, 3, i \neq j\}.$$

Proof. Throughout this proof, we will be using the fact that the elements e_i are in $\mathcal{Z}(\mathcal{E})$. Let $\omega = \sum_j e_j a_j, \eta = \sum_k e_k b_k$ be elements of $\mathcal{E}$. If ϵ_{ijk} denotes the Levi-Civita tensor, i.e,

$$\epsilon_{ijk} = \begin{cases} 0, \text{ if any two indices are repeated} \\ 1, \text{ if } (ijk) \text{ is an even permutation} \\ -1, \text{ if } (ijk) \text{ is an odd permutation,} \end{cases}$$

then by equation (3.29) of [41], we have

$$\begin{aligned}
\omega \wedge \eta &= \sum_{ijk} (\epsilon_{ijk})^2 e_j a_j \wedge e_k b_k = \sum_{ijk} (\epsilon_{ijk})^2 e_j \wedge e_k a_j b_k \\
&= \sum_{jk=2,3} (\epsilon_{1jk})^2 e_j \wedge e_k a_j b_k + \sum_{jk=1,3} (\epsilon_{2jk})^2 e_j \wedge e_k a_j b_k + \sum_{jk=1,2} (\epsilon_{3jk})^2 e_j \wedge e_k a_j b_k \\
&= \sum_{j \neq k} e_j \wedge e_k a_j b_k \\
&= \sum_{j<k} e_j \wedge e_k (a_j b_k - a_k b_j).
\end{aligned}$$

Therefore, we have

$$e_i \wedge e_i = 0 = e_i \wedge e_j + e_j \wedge e_i.$$

Hence, $\{e_i \otimes_{\mathcal{A}} e_i, e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i : 1 \leq i \leq j \leq 3\} \subseteq \mathrm{Ker}(\wedge)$.

Conversely, if a_{ij} in $\mathcal{A}$ is such that $\wedge(\sum_{i,j} e_i \otimes_{\mathcal{A}} e_j a_{ij}) = 0$, then by the above computation, we can conclude that $\sum_{i<j} e_i \wedge e_j (a_{ij} - a_{ji}) = 0$. Since $\{e_i \wedge e_j : i < j\}$ is linearly independent, we have $a_{ij} = a_{ji}$. Therefore,

$$\mathrm{Ker}(\wedge) \subseteq \mathrm{Span}_{\mathcal{A}}\{e_i \otimes_{\mathcal{A}} e_i, e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i : i, j = 1, 2, 3\}.$$

This finishes the proof. $\square$

Proposition 3.1.3. *Let $\mathcal{F}$ denote the right $\mathcal{A}$-linear span of the set $\{e_i \otimes_{\mathcal{A}} e_j - e_j \otimes_{\mathcal{A}} e_i : 1 \leq i < j \leq 3\}$. Then, the bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ admits a decomposition $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$ as right $\mathcal{A}$-modules. Moreover, the map $\sigma = 2P_{\mathrm{sym}} - 1$ is equal to the map σ^{can} as in Theorem 2.1.7, i.e. for all ω, η in $\mathcal{Z}(\mathcal{E})$, and a in $\mathcal{A}$,*

$$\sigma(\omega \otimes_{\mathcal{A}} \eta a) = \eta \otimes_{\mathcal{A}} \omega a.$$

Proof. From the description of $\mathrm{Ker}(\wedge)$ in Lemma 3.1.2 and the isomorphism $\Omega^2_D(\mathcal{A}) \cong \mathrm{Span}_{\mathbb{C}}\{e_i \wedge e_j a_{ij} : a_{ij} = -a_{ji}\}$ ((ii) of Theorem 3.1.1), it is clear that we have a right $\mathcal{A}$-linear splitting: $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$ where $\mathcal{F} = \mathrm{Span}_{\mathbb{C}}\{e_i \otimes_{\mathcal{A}} e_j a_{ij} : a_{ij} = -a_{ji}\}$ is satisfied. Moreover, it is easy to verify that for all ω, η in $\mathcal{Z}(\mathcal{E})$, the map

$$\omega \otimes_{\mathcal{A}} \eta \mapsto \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta + \eta \otimes_{\mathcal{A}} \omega)$$

extends to a bilinear idempotent map on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ with range equal to $\mathrm{Ker}(\wedge)$ and kernel equal to $\mathcal{F}$. Thus, for all ω, η in $\mathcal{Z}(\mathcal{E})$, we have

$$P_{\mathrm{sym}}(\omega \otimes_{\mathcal{A}} \eta) = \frac{1}{2}(\omega \otimes_{\mathcal{A}} \eta + \eta \otimes_{\mathcal{A}} \omega),$$

where P_{sym} is as in Definition 2.2.1. Therefore, $\sigma = 2P_{\mathrm{sym}} - 1 = \sigma^{\mathrm{can}}$. □

The following result concerns the canonical Riemannian bilinear metric of spectral triples discussed in Lemma 2.3.6.

Proposition 3.1.4. *The bilinear form g constructed in Lemma 2.3.6, given by $g(e_i \otimes_{\mathcal{A}} e_j) = \delta_{ij} 1_{\mathcal{A}}$ in the case of the fuzzy 3-spheres, is a Riemannian bilinear metric.*

Proof. From equation (3.49) of [41], we see that $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ is defined by

$$g(\omega \otimes_{\mathcal{A}} \eta) = \sum_{i=1,2,3} a_i b_i,$$

where $\omega = \sum_{i=1,2,3} e_i a_i$, $\eta = \sum_{i=1,2,3} e_i b_i$.

We need to check the conditions of Definition 2.3.7. From the definition of g, it is clear that g is an $\mathcal{A}$-valued map. Next, we check that the map V_g is nondegenerate. Let ω in $\mathcal{E}$ be such that $V_g(\omega)(\eta) = 0$ for all η in $\mathcal{E}$. In particular, $g(\omega \otimes_{\mathcal{A}} e_j) = 0$ for all $j = 1, 2, 3$. If $\omega = \sum_{i=1,2,3} e_i a_i$, we conclude that $a_i = 0$ for all i. Therefore, $\omega = 0$, proving that V_g is one-to-one.

Now we prove that V_g is onto. Suppose $\omega \in \mathcal{E}$ is of the form $\sum_i e_i a_i$. Then we define ϕ_ω in $\mathcal{E}^*$ by

$$\phi_\omega(e_i b) = a_i b.$$

It is clear that any ϕ in $\mathcal{E}^*$ is of the form ϕ_ω for some ω in $\mathcal{E}$. Since $V_g(\sum_{i=1,2,3} e_i \omega_i) = \phi_\omega$, V_g is onto.

Now we prove that g satisfies the equation $g \circ \sigma = g$. We have

$$g \circ \sigma(e_i \otimes_{\mathcal{A}} e_j) = g(e_j \otimes_{\mathcal{A}} e_i) = \delta_{ij} 1_{\mathcal{A}} = g(e_i \otimes_{\mathcal{A}} e_j).$$

Since $\mathcal{Z}(\mathcal{E}) = \mathrm{Span}\{e_i : i = 1, 2, 3\}$ and $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Span}_{\mathbb{C}} = \{\omega \otimes_{\mathcal{A}} \eta : \omega, \eta \in \mathcal{Z}(\mathcal{E})\}$ by Lemma 2.1.5, $g \circ \sigma(e \otimes_{\mathcal{A}} f) = g(e \otimes_{\mathcal{A}} f)$ for all e, f in $\mathcal{E}$. □

Finally, we have first of the two main results of this section.

Theorem 3.1.5. *The spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is a tame spectral triple. Hence, for each pseudo-Riemannian bilinear metric g on $\mathcal{E} := \Omega^1_D(\mathcal{A})$, there exists a unique torsionless connection which is compatible with g.*

Proof. We need to show that the spectral triple satisfies the hypotheses of Definition 2.4.1. By virtue of Theorem 3.1.1 and Proposition 3.1.3, we are left to verify that $u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$ is an isomorphism. But this is clear, since $\mathcal{Z}(\mathcal{A}) = \mathbb{C}.1$ and $\mathcal{Z}(\mathcal{E})$ is the $\mathbb{C}$-linear span of e_1, e_2, e_3. Therefore, by Theorem 2.5.1, for each pseudo-Riemannian bilinear metric g on $\mathcal{E}$ there exists a unique torsionless connection which is compatible with g. □

The authors of [41] investigated the existence of torsionless and unitary connections on $\mathcal{E}$. While the definition of torsion of a connection discussed in their paper is the same as that in ours, the definitions of "metric compatibility" of a connection are different, since the paper [41] views a Riemannian metric as a sesquilinear form as opposed to a bilinear form as in this thesis. In Proposition 3.7 of [41], it is proven that there exists a nontrivial family of torsionless connections which are also unitary. However, once the additional condition of the connection to be real is imposed, then Corollary 3.8 of [41] proves that such a connection is unique. We have the following result:

Theorem 3.1.6. *Consider the Riemannian bilinear metric g of Proposition 3.1.4. Then the Levi-Civita connection of Theorem 3.1.5 for the triple $(\mathcal{E}, d, g)$ coincides with the unique real unitary torsionless connection in Corollary 3.8 of [41].*

Proof. We take basis elements e_i in $\mathcal{E}$ and use the fact that e_i are elements of $\mathcal{Z}(\mathcal{E})$. We denote by Γ^i_{jk} the Christoffel symbols given by $\nabla(e_i) = \sum_{jk} e_j \otimes_{\mathcal{A}} e_k \Gamma^i_{jk}$. Then, we explicitly compute the metric compatibility criterion for the fuzzy 3-sphere by our definition:

$$
\begin{aligned}
0 &= d(\delta_{ij}) = d(g(e_i \otimes_{\mathcal{A}} e_j)) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})(\mathrm{id} \otimes_{\mathcal{A}} \sigma)(\nabla(e_i) \otimes_{\mathcal{A}} e_j + \nabla(e_j) \otimes_{\mathcal{A}} e_i) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})(\mathrm{id} \otimes_{\mathcal{A}} \sigma)(\sum_{k,l} e_k \otimes_{\mathcal{A}} e_l \otimes_{\mathcal{A}} e_j \Gamma^i_{kl} + \sum_{k,l} e_k \otimes_{\mathcal{A}} e_l \otimes_{\mathcal{A}} e_i \Gamma^j_{kl}) \\
&= (g \otimes_{\mathcal{A}} \mathrm{id})(\sum_{k,l} e_k \otimes_{\mathcal{A}} e_j \otimes_{\mathcal{A}} e_l \Gamma^i_{kl} + \sum_{k,l} e_k \otimes_{\mathcal{A}} e_i \otimes_{\mathcal{A}} e_l \Gamma^j_{kl}) \\
&= \sum_l e_l (\Gamma^i_{jl} + \Gamma^j_{il}), \text{ for all } l \text{ and for all } i \neq j.
\end{aligned}
$$

Hence, the metric compatibility criterion gives us that $\Gamma^i_{jl} = -\Gamma^j_{il}$. In [41], combining the necessary and sufficient condition for a connection to be unitary (Equation (3.51) of [41]) and to be a real connection, i.e. the connection coefficients must be anti-Hermitian, we get that the connection coefficients must satisfy $\Gamma^i_{jk} = -\Gamma^j_{ik}$. We see that this is the same condition that we arrive at for a metric compatible connection in our sense.

The torsionless criterion gives us that for all basis elements e_i in $\mathcal{E}$,

$$0 = (\wedge \circ \nabla + d)(e_i) = \sum_{jk} e_j \wedge e_k \Gamma^i_{jk} - \sqrt{-1} \sum_{jk} \epsilon^{ijk} e_j \wedge e_k,$$

where we obtain the expression for $d(e_i)$ from Equation (3.31) of [41]. From Proposition 6.6 and Proposition 3.7 of [41], we know that this is equivalent to the criterion $\Gamma^i_{jk} - \Gamma^i_{kj} = \sqrt{-1}\epsilon^{ijk}$ We see that the solution $\Gamma^i_{jk} = \frac{\sqrt{-1}}{2}\epsilon^{ijk}$ satisfies both the metric compatibility as well as the torsionless criteria. Hence these are the Christoffel symbols of our unique Levi-Civita connection.

Hence, the unique real unitary torsionless connection in Corollary 3.8 of [41] and the unique Levi-Civita connection for the fuzzy 3-sphere obtained by Theorem 3.1.5 coincide. □

3.2 Levi-Civita connection for quantum Heisenberg manifold

In this section, we consider the spectral triple on the quantum Heisenberg manifold as defined and studied in [22]. The definition of the Dirac operator and the space of one-forms require the Pauli spin matrices denoted by $\sigma_1, \sigma_2, \sigma_3$ in [22]. In particular, the σ_i's satisfy the following relations:

$$\sigma_j^2 = 1,\ \sigma_j\sigma_k = -\sigma_k\sigma_j,\ \sigma_1\sigma_2 = \sqrt{-1}\sigma_3,\ \sigma_2\sigma_3 = \sqrt{-1}\sigma_1,\ \sigma_1\sigma_3 = \sqrt{-1}\sigma_2. \tag{3.2.1}$$

Moreover, we are going to work with right connections instead of left connections as had been done in [22].

The description of the algebra of quantum Heisenberg manifold in [81] is as follows.

Definition 3.2.1. *For any positive integer c, let S^c denote the space of infinitely differentiable functions $\Phi : \mathbb{R} \times \mathbb{T} \times \mathbb{Z} \to \mathbb{C}$ that satisfy the following two conditions:*

(i) $\Phi(x+k,y,p) = e^{2\pi ickpy}\Phi(x,y,p)$ *for all* $k \in \mathbb{Z}$,

(ii) *for every partial differential operator* $\tilde{X} = \frac{\partial^{m+n}}{\partial x^m \partial y^n}$ *on* $\mathbb{R}\times\mathbb{T}$ *and every polynomial* p *on* $\mathbb{Z}$, *the function* $P(p)(\tilde{X}\Phi)(x,y,p)$ *is bounded on* $K\times\mathbb{Z}$ *for any compact set* K *of* $\mathbb{R}\times\mathbb{T}$.

For each $\hbar,\mu,\nu \in \mathbb{R}$ *with* $\mu^2+\nu^2 \neq 0$, *let* $\mathcal{A}_\hbar^\infty$ *denote the space* S^c *equipped with product and involution defined, respectively, by*

$$(\Phi * \Psi)(x,y,p) = \sum_q \Phi(x-\hbar(q-p)\mu, y-\hbar(q-p)\nu, q)\Psi(x-\hbar q\mu, y-\hbar q,\nu, p-q),$$

$$\Phi^*(x,y,p) = \Phi(x,y,-p).$$

Let π *be the representation of* $\mathcal{A}_\hbar^\infty$ *on* $L^2(\mathbb{R}\times\mathbb{T}\times\mathbb{Z})$ *given by*

$$(\pi(\Phi)\xi)(x,y,p) = \sum_q \Phi(x-\hbar(q-2p)\mu, y-\hbar(q-2p)\nu, q)\xi(x,y,p-q).$$

Then π *gives a faithful representation of the* $*$*-algebra* $\mathcal{A}_\hbar^\infty$. *The norm closure of* $\pi(\mathcal{A}_\hbar^\infty)$, *denoted by* $\mathcal{A}_{\mu,\nu}^{c,\hbar}$ *is called the quantum Heisenberg manifold.*

For the rest of this section, we will denote the $*$-algebra $\mathcal{A}_\hbar^\infty$ by $\mathcal{A}$. The algebra $\mathcal{A}$ admits an action of the Heisenberg group. The symbol τ will denote a certain state on $\mathcal{A}$ invariant under the action of the Heisenberg group. Let X_1, X_2, X_3 denote the canonical basis of the Lie algebra of the Heisenberg group so that we have associated self-adjoint operators d_{X_i} on $L^2(\mathcal{A},\tau)$ in the natural way. Then the triple $(\mathcal{A}, L^2(\mathcal{A},\tau)\otimes_{\mathbb{C}}\mathbb{C}^2, D)$ defines a spectral triple on $\mathcal{A}$ where $\mathcal{A}$ is represented on $L^2(\mathcal{A},\tau)\otimes_{\mathbb{C}}\mathbb{C}^2$ diagonally and the Dirac operator D is defined as

$$D = \sum_j d_{X_j}\otimes_{\mathbb{C}}\sigma_j,$$

where σ_j, $j=1,2,3$ are the self-adjoint Pauli spin matrices satisfying (3.2.1).

Let us denote the operator $1\otimes_{\mathbb{C}}\sigma_i$ by the symbol e_i. Then, the following lemma is a direct consequence of the proof of Proposition 9 of [22].

Lemma 3.2.2. *For all* a *in* $\mathcal{A}$,

$$d(a) = \sum_{j=1}^3 e_j\partial_j(a),$$

$$\text{where } \partial_1(a) = \frac{\partial a}{\partial x},\ \partial_2(a) = -2\pi\sqrt{-1}cpxa + \frac{\partial a}{\partial y},\ \partial_3(a) = -2\pi\sqrt{-1}cp\alpha a$$

for some α greater than 1. The derivations $\partial_1, \partial_2, \partial_3$ satisfy the following relation:

$$[\partial_1, \partial_3] = [\partial_2, \partial_3] = 0, \ [\partial_1, \partial_2] = \partial_3. \tag{3.2.2}$$

The space of one-forms and two-forms for the spectral triple $(\mathcal{A}, L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2, D)$ are as follows:

Proposition 3.2.3. *For $i = 1, 2, 3$, let e_i denote the element $1 \otimes_{\mathbb{C}} \sigma_i$. The module of one-forms $\mathcal{E} := \Omega^1_D(\mathcal{A})$ is a free module generated by e_1, e_2, e_3. Moreover, e_1, e_2, e_3 are central elements. As a subset of $B(L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2)$, $\mathcal{E}$ can be described as follows:*

$$\mathcal{E} = \{\sum_i a_i \otimes_{\mathbb{C}} \sigma_i : a_i \in \mathcal{A}\} = \{\sum_i a_i e_i : a_i \in \mathcal{A}\}.$$

The set of junk forms (see Subsection 1.3.1) is equal to $\{a \otimes_{\mathbb{C}} 1 : a \in \mathcal{A}\}$, and therefore is isomorphic to $\mathcal{A}$. Finally, the space of two forms $\Omega^2_D(\mathcal{A})$ is isomorphic to $\mathcal{A} \oplus \mathcal{A} \oplus \mathcal{A}$. Explicitly,

$$\Omega^2_D(\mathcal{A}) = \{a_1 \otimes_{\mathbb{C}} \sigma_1\sigma_2 + a_2 \otimes_{\mathbb{C}} \sigma_2\sigma_3 + a_3 \otimes_{\mathbb{C}} \sigma_1\sigma_3 : a_1, a_2, a_3 \in \mathcal{A}\} \subseteq B(L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2).$$

Proof. The space of one-forms is described in Proposition 21 of [22]. The fact that e_1, e_2, e_3 are central can be easily seen from the definition of the representation of $\mathcal{A}$ on $L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2$. The statement about the two forms follow from Proposition 22 of the same paper. □

Proposition 3.2.4. *The bilinear form g constructed in Lemma 2.3.6 satisfies the conditions of Definition 2.3.7, i.e, it is the canonical Riemannian bilinear metric for the spectral triple.*

Proof. We need to check the conditions of Definition 2.3.7. This essentially follows from the results of [22]. We will use Proposition 3.2.3 to identify $\mathcal{E}$ with $\mathcal{A} \otimes_{\mathbb{C}} \mathbb{C}^3$, the bimodule structure being defined as:

$$a(e_i b)c = e_i abc.$$

We will let τ denote the functional on $\mathcal{B}(\mathcal{H})$ as in Subsection 2.3.1. Let $\psi : \mathcal{A} \to \mathbb{C}$ be the faithful normal tracial state on $\mathcal{A}''$ as in Section 2 of [22] (denoted by τ in [22]). By Proposition 14 of [22],

$$\tau(X) = (\frac{1}{2}\psi \otimes_{\mathbb{C}} \mathrm{Tr})(X) \text{ for all } X \in \mathcal{E}.$$

since ψ is faithful on $\mathcal{A}''$, we can conclude that τ is faithful on the $*$-algebra generated by $\mathcal{A}$ and $\{[D,a] : a \in \mathcal{A}\}$. Moreover, by identifying $\mathcal{A} \subseteq \mathcal{E} = \mathcal{A}\otimes_{\mathbb{C}}\mathbb{C}^3$ via $a \mapsto a \otimes I_2$, $\tau(a) = \psi(a)$ for all a in $\mathcal{A}$.

If $\omega = \sum_{i=1}^3 e_i a_i$ and $\eta = \sum_{i=1}^3 e_i b_i$ are two one-forms, then

$$\frac{1}{2}(I\otimes_{\mathbb{C}}\mathrm{Tr})(\omega\eta) = \sum_{i=1}^3 a_i b_i.$$

Therefore, for all c in $\mathcal{A}$, the formula $g(\omega \otimes_{\mathcal{A}} \eta) = \langle\langle \omega^*,\ \eta\rangle\rangle$ (Lemma 2.3.6) implies that

$$\begin{aligned} & \tau(g(\omega \otimes_{\mathcal{A}} \eta)c) = \tau(\langle\langle \omega^*,\ \eta\rangle\rangle\, c) = \tau(\omega\eta c) \\ = \quad & (\frac{1}{2}\psi\otimes_{\mathbb{C}}\mathrm{Tr}))(\omega\eta c) = \sum_i \psi(a_i b_i c) = \tau((\sum_{i=1}^3 a_i b_i)c). \end{aligned}$$

Therefore, $g(\omega \otimes_{\mathcal{A}} \eta) = \sum_{i=1}^3 a_i b_i$ is in $\mathcal{A}$. The nondegeneracy of the map V_g follows just as in Proposition 3.1.4. □

Proposition 3.2.5. *Let $\mathcal{F}$ denote the right $\mathcal{A}$-linear span of the set $\{e_i \otimes_{\mathcal{A}} e_j - e_j \otimes_{\mathcal{A}} e_i : 1 \leq i < j \leq 3\}$. Then, the bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ admits a decomposition $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$ as right $\mathcal{A}$-modules. Moreover, the map $\sigma = 2P_{\mathrm{sym}} - 1$ is equal to the map σ^{can} as in Theorem 2.1.7, i.e. for all e, f in $\mathcal{Z}(\mathcal{E})$, and a in $\mathcal{A}$,*

$$\sigma(e \otimes_{\mathcal{A}} fa) = f \otimes_{\mathcal{A}} ea.$$

Proof. We will use the fact that e_i are central elements throughout the proof. Moreover, let $\wedge, m_0, \mathcal{J}$, be as in Subsection 1.3.1 while P_{sym} will be as in Definition 2.2.1. By the description of $\mathcal{J}$ and that of $\Omega^2_D(\mathcal{A})$ in Proposition 3.2.3, it is easy to see that $\mathrm{Ker}(\wedge)$ is spanned by $\{e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i : 1 \leq i \leq j \leq 3\}$ as a right $\mathcal{A}$-module and $\mathcal{F} = \mathrm{Span}_{\mathcal{A}}\{e_i \otimes_{\mathcal{A}} e_j - e_j \otimes_{\mathcal{A}} e_i : 1 \leq i < j \leq 3\}$. Clearly, $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$ as right $\mathcal{A}$ modules.

Since $e_1, e_2, e_3 \in \mathcal{Z}(\mathcal{E})$, it can be easily checked that $u^{\mathcal{E}}$ is an isomorphism. In particular, $\mathcal{E}$ is centered. Moreover, by the description of $\mathrm{Ker}(\wedge)$ as above, we have

$$P_{\mathrm{sym}}(e_i \otimes_{\mathcal{A}} e_j - e_j \otimes_{\mathcal{A}} e_i) = 0,\ P_{\mathrm{sym}}(e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i) = e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i$$

$$\text{and thus } 2P_{\mathrm{sym}}(e_i \otimes_{\mathcal{A}} e_j) = e_i \otimes_{\mathcal{A}} e_j + e_j \otimes_{\mathcal{A}} e_i.$$

$$\text{Therefore, } \sigma(e_i \otimes_{\mathcal{A}} e_j) = (2P_{\text{sym}} - 1)(e_i \otimes_{\mathcal{A}} e_j) = e_j \otimes_{\mathcal{A}} e_i.$$

Therefore, $\sigma = \sigma^{\text{can}}$. □

Finally, we have the main result of this section.

Theorem 3.2.6. *The spectral triple $(\mathcal{A}, L^2(\mathcal{A}, \tau) \otimes_{\mathbb{C}} \mathbb{C}^2, D)$ is a tame spectral triple. Hence, for any pseudo-Riemannian bilinear metric g on $\mathcal{E}$, there exists a unique Levi-Civita connection on the module $\mathcal{E}$ compatible with g.*

Proof. In Proposition 3.2.3 and Proposition 3.2.5, we have verified that the conditions of Definition 2.4.1, so the spectral triple is a tame spectral triple. By Theorem 2.5.1, we have the second part of our result. □

3.3 Levi-Civita connection for Connes-Landi deformed spectral triples

Suppose M is a compact Riemannian manifold such that the maximal torus of the isometry group of M has rank greater than or equal to 2. Then the action of the maximal torus on $C^\infty(M)$ allows us to define a deformed algebra $C^\infty(M)_\theta$ ([82], [26]). Moreover, the torus equivariant spectral triple $(C^\infty(M), \mathcal{H}, d+d^*)$ on M (as in Example 1.3.5) can be deformed to a new spectral triple on $C^\infty(M)_\theta$([27]).

The goal of this section is to prove the following theorem:

Theorem 3.3.1. *Suppose M is a compact Riemannian manifold equipped with a free isometric action of $\mathbb{T}^n$. Let $\mathcal{E} := \Omega^1(M)$ denote the space of one-forms of the spectral triple $(C^\infty(M), \mathcal{H}, d+d^*)$ discussed in Example 1.3.5. Then the deformed spectral triple $(C^\infty(M)_\theta, \mathcal{H}, d + d^*)$ as in Theorem 3.3.25 is a tame spectral triple and the metric g deforms to a Riemannian metric g_θ on the bimodule of one-forms $\mathcal{E}_\theta$ of the spectral triple $(C^\infty(M)_\theta, \mathcal{H}, d + d^*)$. Hence, there exists a unique Levi-Civita connection on $\mathcal{E}_\theta$ for g_θ.*

In the first subsection, we prove some preparatory results on the fixed point algebra under the action of a compact abelian Lie group. In Subsection 3.3.2 we prove some results on generalities of Rieffel deformations. In Subsection 3.3.3 we prove that there exists a Riemannian bilinear metric on $\mathcal{E}_\theta$ and that it is the deformation of the canonical metric on $\mathcal{E}$. In Subsection 3.3.4, we prove that under our assumptions, the deformed module of one-forms on the Rieffel deformed manifold satisfies the conditions of Definition 2.4.1.

Now we recall the concepts of spectral subspaces and spectral subalgebras (or spectral submodules) for actions of the group $\mathbb{T}^n$ on algebras and modules.

Definition 3.3.2. *Suppose β is an action of $\mathbb{T}^n$ on a module $\mathcal{G}$ (or an algebra $\mathcal{D}$). Then the spectral subspace corresponding to a character $\underline{m} \equiv (m_1, ..., m_n)$ in $\widehat{\mathbb{T}^n} \cong \mathbb{Z}^n$, denoted by $\mathcal{G}_{\underline{m}}$ (respectively $\mathcal{D}_{\underline{m}}$), consists of all ξ such that $\beta_t(\xi) = \chi_{\underline{m}}(t)\xi$ for all $t = (t_1, \ldots, t_n)$ in $\mathbb{T}^n$, where $\chi_{\underline{m}}(t) := t_1^{m_1} \ldots t_n^{m_n}$.*

It is easily seen that $\mathcal{D}_{\underline{m}}\mathcal{D}_{\underline{n}} \subseteq \mathcal{D}_{\underline{m}+\underline{n}}$.

Suppose that $\mathcal{G}$ is a $\mathcal{D}$-bimodule. Moreover, let us assume that both $\mathcal{D}$ and $\mathcal{G}$ are equipped with actions of $\mathbb{T}^n$ in such a way that $\mathcal{G}$ becomes a $\mathbb{T}^n$-equivariant $\mathcal{D}$-bimodule. This means that for all e in $\mathcal{G}$ and for all a, b in $\mathcal{D}$, we have:

$$\beta_t(aeb) = \alpha_t(a)\beta_t(e)\alpha_t(b).$$

In this case, we have

$$\mathcal{G}_{\underline{m}}\mathcal{D}_{\underline{n}} \subseteq \mathcal{G}_{\underline{m}+\underline{n}} \text{ and } \mathcal{D}_{\underline{n}}\mathcal{G}_{\underline{m}} \subseteq \mathcal{G}_{\underline{m}+\underline{n}}. \tag{3.3.1}$$

The subspace $\mathrm{Span}_{\mathbb{C}}\{\mathcal{D}_{\underline{m}} : \underline{m} \in \mathbb{Z}^n\}$ comprises the so-called 'spectral subalgebra' for the action. Similarly, $\mathrm{Span}_{\mathbb{C}}\{\mathcal{G}_{\underline{m}} : \underline{m} \in \mathbb{Z}^n\}$ is called the spectral submodule of the action.

Let G be a group. Let us recall that a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is called G-equivariant if there exists a unitary representation β of G on $\mathcal{H}$ such that $\beta_g D = D\beta_g$. Moreover, we recall the following well known fact (see [26] for the details).

Proposition 3.3.3. *Suppose that M is a compact Riemannian manifold with an isometric action of the n-torus $\mathbb{T}^n$ on M. Consider the spectral triple $(C^\infty(M), \mathcal{H}, D)$ of Example 1.3.5, i.e, $\mathcal{H}$ is the Hilbert space of all forms, d is the de-Rham differential on $\mathcal{H}$ and $D = d + d^*$. The $\mathbb{T}^n$ action on smooth forms extends to a unitary representation β on $\mathcal{H}$ and the spectral triple is equivariant w.r.t this representation of $\mathbb{T}^n$. In particular, if α denotes the action of $\mathbb{T}^n$ on $C^\infty(M)$ and $\delta(\cdot) = \sqrt{-1}[D, \cdot]$, then for all t in $\mathbb{T}^n$ and for all f, g in $C^\infty(M)$,*

$$\beta_t(f\delta(g)) = \alpha_t(f)\beta_t(\delta(g)) = \alpha_t(f)\delta(\alpha_t(g)).$$

In this set-up, it is easy to see the following result:

Lemma 3.3.4. *If $\mathcal{D}$ is a subalgebra of $C^\infty(M)$ kept invariant by the action of a compact group G acting by isometries on M and $\Omega^1(\mathcal{D}) := \mathrm{Span}_{\mathbb{C}}\{fdg : f, g \in \mathcal{D}\}$, then the map $\wedge : \Omega^1(\mathcal{D}) \otimes_{\mathcal{D}} \Omega^1(\mathcal{D}) \to \Omega^2(\mathcal{D})$ is G-equivariant.*

As an immediate corollary, we have

Corollary 3.3.5. *With the notations of Lemma 3.3.4, we have a decomposition*

$$\Omega^1(\mathcal{D}) \otimes_D \Omega^1(\mathcal{D}) = \mathrm{Ker}(\wedge) \oplus \mathcal{G},$$

where $\mathrm{Ker}(\wedge) = \mathrm{Span}_{\mathbb{C}}\{e \otimes_{\mathcal{D}} f + f \otimes_{\mathcal{D}} e : e, f \in \Omega^1(\mathcal{D})\}$ *and* $\mathcal{G} = \mathrm{Span}_{\mathbb{C}}\{e \otimes_{\mathcal{D}} f - f \otimes_{\mathcal{D}} e : e, f \in \Omega^1(\mathcal{D})\}$.

Moreover, $\mathrm{Ker}(\wedge)$ *and $\mathcal{G}$ are also kept invariant by G.*

Proof. The decomposition $\Omega^1(\mathcal{D}) \otimes_D \Omega^1(\mathcal{D}) = \mathrm{Ker}(\wedge) \oplus \mathcal{G}$ follows exactly as in the classical case.

The G-invariance of $\mathrm{Ker}(\wedge)$ follows from the G-equivariance of $\wedge$. Moreover, we have $\mathcal{G} = \mathrm{Ker}(1 - \mathrm{flip})$. Since flip is G-equivariant, $\mathcal{G}$ is G-invariant. $\square$

3.3.1 Some results on the fixed point algebra

Let us consider a compact Riemannian manifold M with the $\mathbb{T}^n$-equivariant spectral triple $(C^\infty(M), \mathcal{H}, d+d^*)$ as in Proposition 3.3.3. Throughout this section, we will follow the notations introduced in the following definition.

Definition 3.3.6. *Let $\mathcal{E} := \Omega^1(M)$ and $\mathcal{A} := C^\infty(M)$. $\mathcal{F}$ will denote the $\mathbb{T}^n$-equivariant spectral submodule of $\mathcal{E}$. The symbol $\mathcal{F}_{\underline{k}}$ will denote the $\underline{k}$-th spectral subspace of $\mathcal{F}$. Thus, $\mathcal{F} =$ $\mathrm{Span}_{\mathbb{C}}\{\mathcal{F}_{\underline{k}} : \underline{k} \in \mathbb{Z}^n\}$. Similarly, we define $\mathcal{C}$ to be the spectral subalgebra $\mathrm{Span}_{\mathbb{C}}\{\mathcal{C}_{\underline{k}} : \underline{k} \in \mathbb{Z}^n\}$ of $\mathcal{A}$ where $\mathcal{C}_{\underline{k}}$ is the $\underline{k}$-th spectral subspace of $\mathcal{C}$. In particular, $\mathcal{C}_{\underline{0}}$ and $\mathcal{F}_{\underline{0}}$ denote the $\mathbb{T}^n$-invariant spectral subalgebra and the $\mathbb{T}^n$-invariant spectral submodule respectively.*

Remark 3.3.7. *It is clear from the definition of spectral subspaces of algebras and modules that if $\mathcal{A}_{\underline{k}}$ and $\mathcal{E}_{\underline{k}}$ denote the spectral subspaces of $\mathcal{A}$ and $\mathcal{E}$ respectively, then $\mathcal{A}_{\underline{k}} = \mathcal{C}_{\underline{k}}$ and $\mathcal{E}_{\underline{k}} = \mathcal{F}_{\underline{k}}$. We will from now on use this fact, often without mentioning.*

Remark 3.3.8. *Since the representation β as in Proposition 3.3.3 commutes with $d+d^*$, it is easy to see that $\beta_t(\mathcal{F}) \subseteq \mathcal{F}$ for all t in $\mathbb{T}^n$. Moreover, it is easy to see that the space of one-forms for the spectral triple $(\mathcal{C}, \mathcal{H}, d+d^*)$ is precisely $\mathcal{F}$.*

Recall that one of the conditions of Definition 2.4.1 requires the map

$$u^{\mathcal{E}} : \mathcal{Z}(\mathcal{E}) \otimes_{\mathcal{Z}(\mathcal{A})} \mathcal{A} \to \mathcal{E}$$

to be an isomorphism for the underlying spectral triple to be a tame one. In the example of Connes-Landi deformed spectral triples of compact Riemannian manifolds, it proves difficult to show this directly. In Subsection 2.4.2, we discussed an auxiliary map

$$u^{\mathcal{E}}_{\mathcal{E}'} : \mathcal{E}' \otimes_{\mathcal{A}'} \mathcal{A} \to \mathcal{E},$$

where $\mathcal{A}'$ is a unital subalgebra of $\mathcal{Z}(\mathcal{A})$ and $\mathcal{E}'$ is an $\mathcal{A}'$-submodule of $\mathcal{Z}(\mathcal{E})$.

In particular, Proposition 2.4.6 shows that if $\mathcal{Z}(\mathcal{E})$ is a finitely generated projective module over $\mathcal{Z}(\mathcal{A})$, then $u^{\mathcal{E}}$ is an isomorphism if and only if $u^{\mathcal{E}}_{\mathcal{E}'}$ is one. We will employ that proposition in this section to obtain our desired result. The aim of this subsection is to prove that if the action of $\mathbb{T}^n$ on M is free, then the spectral subalgebra $\mathcal{C}_{\underline{0}}$ and the spectral submodule $\mathcal{F}_{\underline{0}}$ satisfy the hypotheses of Proposition 2.4.6.

Lemma 3.3.9. *Suppose that the $\mathbb{T}^n$ action on M is free. Then $\mathcal{F}_{\underline{0}}$ is a finitely generated projective right module over $\mathcal{C}_{\underline{0}}$.*

Proof. For a module $\mathcal{G}$ equipped with an action of $\mathbb{T}^n$, let us denote the $\mathbb{T}^n$-invariant submodule of $\mathcal{G}$ by the symbol $\mathcal{G}^{\mathbb{T}^n}$. Since the $\mathbb{T}^n$-action on M is free, $M/\mathbb{T}^n$ is a smooth compact manifold and M is a principal $\mathbb{T}^n$-bundle over $M/\mathbb{T}^n$. Let π denote the projection map from M onto $M/\mathbb{T}^n$. Given any point in M, we can find a $\mathbb{T}^n$-invariant open neighborhood U which is $\mathbb{T}^n$-equivariantly diffeomorphic with $U/\mathbb{T}^n \times \mathbb{T}^n$. Moreover, we can choose U in such a way that $U/\mathbb{T}^n$ is the domain of a local coordinate chart for the manifold $M/\mathbb{T}^n$. Thus, without loss in generality, we can assume that $U = \pi^{-1}(V)$, where V is the domain of some local chart for $M/\mathbb{T}^n$.

This gives the following isomorphism:

$$\Omega^1(U)^{\mathbb{T}^n} \cong \Omega^1(U/\mathbb{T}^n) \otimes_{\mathbb{C}} \Omega^1(\mathbb{T}^n)^{\mathbb{T}^n} \cong \Omega^1(U/\mathbb{T}^n) \otimes_{\mathbb{C}} \mathfrak{L},$$

$\mathfrak{L}$ being the complexified Lie algebra of $\mathbb{T}^n$ which is nothing but $\mathbb{C}^n$. As $U/\mathbb{T}^n$ is the domain of a local coordinate chart, the module of one-forms is a free $C^\infty(U/\mathbb{T}^n)$ module, say $C^\infty(U/\mathbb{T}^n)\otimes_{\mathbb{C}}\mathbb{C}^k$, hence $\Omega^1(U)^{\mathbb{T}^n}$ is isomorphic with $C^\infty(U/\mathbb{T}^n)\otimes_{\mathbb{C}}\mathbb{C}^{n+k}$, i.e. $\Omega^1(U/\mathbb{T}^n)$ is free. By covering M with finitely many such neighbourhoods, one proves that $\Omega^1(M)^{\mathbb{T}^n}$ is finitely generated projective over $C^\infty(M/\mathbb{T}^n)$. $\square$

We observe that $\mathcal{C}_{\underline{0}}$ is a unital subalgebra of $\mathcal{Z}(\mathcal{C}) = \mathcal{C}$ and $\mathcal{F}_{\underline{0}}$ is a $\mathcal{C}_{\underline{0}}$-submodule of $\mathcal{Z}(\mathcal{F}) = \mathcal{F}$. So we can make use of the notation

$$u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}} : \mathcal{F}_{\underline{0}} \otimes_{\mathcal{C}_{\underline{0}}} \mathcal{C} \to \mathcal{F}$$

introduced in Subsection 2.4.2.

Lemma 3.3.10. *If for each $\underline{m}$ in $\mathbb{Z}^n$, we can find $a_1, \ldots, a_k$ in $\mathcal{C}_{\underline{m}}$ and $b_1, \ldots, b_k$ in $\mathcal{C}_{-\underline{m}}$ (k depends on $\underline{m}$) such that $\sum_i b_i a_i = 1$, then the map $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}}$ is an isomorphism.*

Proof. We need to prove that under the above assumption, the map $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}}$ has a right $\mathcal{C}$-linear inverse. However, since $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}}$ is right $\mathcal{C}$-linear to start with, it suffices to prove that $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}}$ defines an isomorphism of vector spaces. Hence, it is sufficient to prove that for all $\underline{m}$, the restriction $p^{\mathcal{F}}_{\underline{m}}$ of $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}}$ to $\mathcal{F}_{\underline{0}} \otimes_{\mathcal{C}_{\underline{0}}} \mathcal{C}_{\underline{m}}$ is a vector space isomorphism onto its image $\mathcal{F}_{\underline{m}}$.

To this end, consider the map

$$q^{\mathcal{F}}_{\underline{m}} : \mathcal{F}_{\underline{m}} \to \mathcal{F}_{\underline{0}} \otimes_{\mathcal{C}_{\underline{0}}} \mathcal{C}_{\underline{m}} \text{ defined by } q^{\mathcal{F}}_{\underline{m}}(e) := \sum_i e b_i \otimes_{\mathcal{C}_{\underline{0}}} a_i.$$

Then $p^{\mathcal{F}}_{\underline{m}} \circ q^{\mathcal{F}}_{\underline{m}} = \mathrm{id}$.

On the other hand, as ab_i is in $\mathcal{C}_{\underline{0}}$ if a is in $\mathcal{C}_{\underline{m}}$, we have

$$q^{\mathcal{F}}_{\underline{m}} \circ p^{\mathcal{F}}_{\underline{m}}(e \otimes_{\mathcal{C}_{\underline{0}}} a) = \sum_i e a b_i \otimes_{\mathcal{C}_{\underline{0}}} a_i = e \otimes_{\mathcal{C}_{\underline{0}}} \sum_i a b_i a_i = e \otimes_{\mathcal{C}_{\underline{0}}} a.$$

This finishes the proof of the lemma. $\square$

Now we shall identify $\mathcal{C}_{\underline{m}}$ with the bimodule of sections of a certain vector bundle over $M/\mathbb{T}^n$.

Lemma 3.3.11. *Let M be a smooth compact Riemannian manifold equipped with a smooth and free right action of a compact connected abelian Lie group K. Let $M \times_{\chi_{-\underline{m}}} \mathbb{C} \to M/K$ denote the associated vector bundle (of $M \to M/K$) corresponding to the character $\chi_{-\underline{m}}$.*

Then the elements in $\mathcal{C}_{\underline{m}} = \{f \in \mathcal{A} : f(x.t) = \chi_{\underline{m}}(t)f(x)\}$ *is in one to one correspondence with the set of all smooth sections of the vector bundle* $M \times_{\chi_{-\underline{m}}} \mathbb{C} \to M/K$.

Proof. The elements of the total space of the associated vector bundle $M \times_{\chi_{-\underline{m}}} \mathbb{C}$ are given by the equivalence class $[y, \lambda]$ of (y, λ) in $M \times \mathbb{C}$ such that $(y, \lambda) \sim (y.t, \chi_{-\underline{m}}(t^{-1})\lambda)$ for all $t \in K$. Now, for f in $\mathcal{C}_{\underline{m}}$, we can define a section of the above vector bundle s_f by

$$s_f([x]) = [x, f(x)],$$

where $[x]$ denotes the class of the point x in M/K. We need to show that this is well defined. But for any t in K,

$$s_f([x.t]) = [x.t, f(x.t)] = [x.t, \chi_{\underline{m}}(t)f(x)] = [x.t, \chi_{-\underline{m}}(t^{-1})f(x)] = [x, f(x)].$$

This proves that s_f is well defined.

Conversely, given a section s of the above vector bundle we can define a function f_s on M by $f_s(x) = \lambda_x$ where $\lambda_x \in \mathbb{C}$ is such that $s([x]) = [x, \lambda_x]$. Clearly, λ_x is uniquely determined, because the K action is free. Moreover,

$$[x, \lambda_x] = s([x]) = s([x.t]) = [x.t, \lambda_{x.t}] = [x.t, \chi_{-\underline{m}}(t^{-1})\chi_{-\underline{m}}(t)\lambda_{x.t}] = [x, \chi_{-\underline{m}}(t)\lambda_{x.t}].$$

Therefore, $\lambda_x = \chi_{-\underline{m}}(t)\lambda_{x.t}$, i.e, $\lambda_{x.t} = \chi_{\underline{m}}(t)\lambda_x$.

Thus, f_s belongs to $\mathcal{C}_{\underline{m}}$.

Finally, it is easy to verify that the maps $f \mapsto s_f$ and $s \mapsto f_s$ are inverses of one another, completing the proof. □

The following lemma is well-known. However, we give a proof for it since we could not find any appropriate references.

Lemma 3.3.12. *For a complex smooth Hermitian vector bundle over a compact manifold* M, *there are finitely many smooth sections* s_i*'s such that* $\sum_i \langle\langle s_i, s_i\rangle\rangle = 1$ *where* $\langle\langle \cdot, \cdot\rangle\rangle$ *denotes the* $C^\infty(M)$*-valued inner product coming from the Hermitian structure.*

Proof. Corresponding to a finite open cover $\{U_i,\ i = 1, \ldots, l\}$ choose finitely many smooth sections γ_i which are non zero on U_i. Then choosing a smooth partition of unity $\psi_i, i = 1, \ldots, l$, we can construct $t_i = \psi_i \gamma_i$'s so that $t = \sum_i \langle\langle t_i, t_i \rangle\rangle$ is nowhere zero. The sections $s_i = \frac{t_i}{t^{\frac{1}{2}}}$ satisfy the conditions of the lemma. □

This gives us the following:

Lemma 3.3.13. *Suppose M is a compact Riemannian manifold equipped with a free and isometric action of $\mathbb{T}^n$. Then the map $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}} : \mathcal{F}_{\underline{0}} \otimes_{\mathcal{C}_{\underline{0}}} \mathcal{C} \to \mathcal{F}$ is an isomorphism.*

Moreover, the map $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}} : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_{\underline{m}} \to \mathcal{E}_{\underline{m}}$ is one-to-one.

Proof. Without loss of generality, we can assume M to be connected. In general, if M has k connected components $M_1, M_2, \cdots M_k$, the module $\mathcal{F}$ decomposes as $\mathcal{F}_1 \oplus \cdots \mathcal{F}_k$, where $\mathcal{F}_i$ is the linear span of spectral subspaces of $\Omega^1(M_i)$, and it is suffices to prove that for all i, $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}}$ is an isomorphism from $(\mathcal{F}_i)_{\underline{0}} \otimes_{(\mathcal{C}_i)_{\underline{0}}} \mathcal{C}_i$ onto $\mathcal{F}_i$.

Since the action of $\mathbb{T}^n$ on M is free, $M \to M/\mathbb{T}^n$ is a principal $\mathbb{T}^n$-bundle. Consider the associated vector bundle $M \times_{\chi_{-\underline{m}}} \mathbb{C} \to M/K$ as in Lemma 3.3.11. Then Lemma 3.3.12 gives us finitely many smooth sections $\{s_i\}_i$ of this vector bundle such that $\sum_i \langle\langle s_i, s_i \rangle\rangle = 1$. From Lemma 3.3.11, for each i, we have an element f_{s_i} (belonging to $\mathcal{C}_{\underline{m}}$) corresponding to the section s_i.

The relation $\sum \langle\langle s_i, s_i \rangle\rangle = 1$ implies that

$$\sum_i \overline{f_{s_i}} f_{s_i} = 1.$$

Since f_{s_i} belongs to $\mathcal{C}_{\underline{m}}$, the function $\overline{f_{s_i}}$ belongs to $\mathcal{C}_{-\underline{m}}$. Thus, we can apply Lemma 3.3.10 to deduce the first assertion of the theorem.

Now we prove the second assertion. The fact that $u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}} : \mathcal{F}_{\underline{0}} \otimes_{\mathcal{C}_{\underline{0}}} \mathcal{C} \to \mathcal{F}$ is an isomorphism implies that for all $\underline{m}$, the restriction

$$u^{\mathcal{F}}_{\mathcal{F}_{\underline{0}}} : \mathcal{F}_{\underline{0}} \otimes_{\mathcal{C}_{\underline{0}}} \mathcal{C}_{\underline{m}} \to \mathcal{C}_{\underline{m}}$$

is a one-to-one map. Since $\mathcal{E}_{\underline{0}} = \mathcal{F}_{\underline{0}}$ and $\mathcal{A}_{\underline{m}} = \mathcal{C}_{\underline{m}}$, this means that $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}} : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_{\underline{m}} \to \mathcal{E}_{\underline{m}}$ is a one-to-one map. □

Next we prove that with the hypothesis of Lemma 3.3.13, the map $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}}$ is also an isomorphism.

Lemma 3.3.14. *The map $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}} : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A} \to \mathcal{E}$ is an isomorphism.*

Proof. Let us start by proving that the map is one-to-one. Let $e'_i \in \mathcal{E}_{\underline{0}}$ and $f_i \in \mathcal{A}$ be elements such that

$$u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}}(\sum_i e'_i \otimes_{\mathcal{A}_{\underline{0}}} f_i) = 0, \text{ i.e, } \sum_i e'_i f_i = 0.$$

Then each spectral projection $\mathcal{P}_{\underline{m}}(\sum_i e'_i f_i) = 0$ i.e, $\sum_i e'_i \mathcal{P}_{\underline{m}}(f_i) = 0$. So for all $\underline{m}$, we obtain

$$u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}}(\sum_i e'_i \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{P}_{\underline{m}}(f_i)) = 0.$$

But from Lemma 3.3.13, we know that the map $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}} : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_{\underline{m}} \to \mathcal{E}_{\underline{m}}$ is one-to-one. Hence, for all $\underline{m}$,

$$\sum_i e'_i \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{P}_{\underline{m}}(f_i) = 0$$

which implies that

$$\sum_i e'_i \otimes_{\mathcal{A}_{\underline{0}}} f_i = \lim_N (\sum_{i,|m|\leq N} e'_i \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{P}_{\underline{m}}(f_i)) = 0,$$

where lim denotes the limit in the Frećhet topology. Therefore, the map is one-to-one.

Now we show that the map is onto. Since the map $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}}$ is right $\mathcal{A}$-linear, it suffices to check that for all f in $\mathcal{A}$, df has a pre-image in $\mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}$. Consider the principal $T = \mathbb{T}^n$ bundle $\pi : M \to M/T$. Since M/T is compact, we can take a finite atlas (U_i, ϕ_i) on it such that the bundle $\pi^{-1}(U_i) \to U_i$ is T-equivariantly diffeomorphic with the canonical bundle $U_i \times T \to U_i$. Let $\{\psi_i\}_i$ be a partition of unity on M subordinate to (U_i, ϕ_i). Then $f = \sum_i f\psi_i$ and $df = \sum_i d(f\psi_i)$. Thus in particular we can assume that f is supported in $\pi^{-1}(U_i)$ or equivalently in $U \times T$.

Let $\{dx_i\}$ be a basis for differential forms along the direction of U i.e. the horizontal direction of the bundle $U \times T \to U$ and $\{\omega_j\}$ be a basis of right invariant 1-forms in the vertical direction corresponding to the basis $\{\chi_j\}$ of right invariant vector fields along the direction of T. Then

$$df = \sum_i dx_i . \frac{\partial}{\partial x_i}(f) + \sum_j \omega_j . \chi_j(f).$$

The right action of T on $U \times T$ acts trivially in the direction of U, hence dx_i is in $\mathcal{E}_{\underline{0}}$. Since ω_j is invariant under the action induced by the right action of T on $U \times T$, so ω_j is in $\mathcal{E}_{\underline{0}}$. Hence

df has a pre-image $\sum_i dx_i \otimes_{\mathcal{A}_{\underline{0}}} \frac{\partial}{\partial x_i}(f) + \sum_j \omega_j \otimes_{\mathcal{A}_{\underline{0}}} \chi_j(f)$ in $\mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}$. Therefore, we have that $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}}$ is an onto map. This completes the proof. $\square$

3.3.2 Some generalities on Rieffel-deformation

Our main reference for Rieffel deformation of a C^*-algebra endowed with a strongly continuous action by $\mathbb{T}^n$ is [82]. However, we will also need to use equivalent descriptions of this deformation given in [26], [27], [66] and [67].

We begin with the definitions of $\mathbb{T}^n$ smooth modules and $\mathbb{T}^n$ smooth algebras from [67] for which we recall that the action α of a locally compact group G on a Fréchet space V is said to be strongly continuous if the map

$$G \to V, \ g \mapsto \alpha_g(v)$$

is continuous for all v in V.

Definition 3.3.15. *A Fréchet space V, whose topology is defined by a family of semi-norms is said to be a $\mathbb{T}^n$ smooth module if V admits a strongly continuous $\mathbb{T}^n$ action $\alpha_t : V \to V$ such that the function $t \mapsto \alpha_t(v)$ belongs to $C^\infty(\mathbb{T}^n, V)$ for all v in V.*

An algebra $\mathcal{D}$ is said to be a $\mathbb{T}^n$ smooth algebra if it is a $\mathbb{T}^n$ smooth module and the multiplication map $m : \mathcal{D} \otimes_{\mathbb{C}} \mathcal{D} \to \mathcal{D}$ is $\mathbb{T}^n$-equivariant and jointly continuous.

A $\mathcal{D}$-bimodule $\mathcal{G}$ is said to be a $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodule if it is a $\mathbb{T}^n$ smooth module such that the left and right $\mathcal{D}$-module structures are $\mathbb{T}^n$-equivariant and are jointly continuous.

The following is our motivating example of $\mathbb{T}^n$ smooth modules and algebras for this section.

Example 3.3.16. *Let M be a Riemannian manifold equipped with a smooth action of $\mathbb{T}^n$. Then the natural action of $\mathbb{T}^n$ on $C^\infty(M)$ makes the latter a $\mathbb{T}^n$ smooth algebra.*
Moreover, the space of one-forms $\Omega^1(M)$ which is a $C^\infty(M)$-bimodule admits an induced smooth action of $\mathbb{T}^n$ and forms a $\mathbb{T}^n$ smooth $C^\infty(M)$-bimodule.

We next define the deformation of a $\mathbb{T}^n$ smooth algebra $\mathcal{D}$. We refer to [67] for details.

Definition 3.3.17 (Definition 2.4 of [67]). *Let $\mathcal{D}$ be a $\mathbb{T}^n$ smooth algebra as in Definition 3.3.15. For a skew symmetric $n \times n$ matrix θ, consider the bicharacter χ_θ defined by*

$$\chi_\theta(\underline{k}, \underline{l}) = e^{\pi i \langle \underline{k}, \theta \underline{l} \rangle}, \ \underline{k}, \underline{l} \in \mathbb{Z}^n,$$

where the pairing $\langle ., . \rangle$ is the usual inner product in $\mathbb{R}^n$. The deformation of $\mathcal{D}$ is the algebra $\mathcal{D}_\theta$ whose underlying vector space is equal to $\mathcal{D}$ while the multiplication $\times_\theta$ is deformed as follows:

$$a \times_\theta b = \sum_{\underline{k}, \underline{l} \in \mathbb{Z}^n} \chi_\theta(\underline{k}, \underline{l}) a_{\underline{k}} b_{\underline{l}}, \ \forall \ a, b \in \mathcal{D}, \tag{3.3.2}$$

where $a = \sum_{\underline{k}} a_{\underline{k}}, \ b = \sum_{\underline{l}} b_{\underline{k}}$ are the isotypical decompositions.

The bicharacter χ_θ satisfies the following cocycle identity:

$$\chi_\theta(\underline{m}, \underline{k}) \chi_\theta(\underline{m} + \underline{k}, \underline{l}) = \chi_\theta(\underline{m}, \underline{k} + \underline{l}) \chi_\theta(\underline{k}, \underline{l}). \tag{3.3.3}$$

Remark 3.3.18. *By Proposition 2.2 of [67], the isotypical decompositions converge absolutely to the element.*

$\mathcal{D}_\theta$ turns out to be a $\mathbb{T}^n$ smooth algebra and the deformed product is associative.

One can similarly deform $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodules (see Definition 3.3.15) as follows:

Definition 3.3.19. *Let $\mathcal{G}$ be a $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodule. Then the deformed bimodule $\mathcal{G}_\theta$ is a $\mathcal{D}_\theta$-bimodule whose underlying vector space is equal to $\mathcal{G}$ while the deformed left and right module actions are as follows:*

$$e \times_\theta a = \sum_{\underline{k}, \underline{l} \in \mathbb{Z}^n} \chi_\theta(\underline{k}, \underline{l}) e_{\underline{k}} a_{\underline{l}}, \ a \times_\theta e = \sum_{\underline{k}, \underline{l} \in \mathbb{Z}^n} \chi_\theta(\underline{k}, \underline{l}) a_{\underline{k}} e_{\underline{l}} \ \forall \ e \in \mathcal{G}, \ \forall \ a \in \mathcal{D}, \tag{3.3.4}$$

where $e = \sum_{\underline{k}} e_{\underline{k}}$ and $a = \sum_{\underline{l}} a_{\underline{l}}$ are the isotypical decompositions.

If $\mathcal{G}$ is a $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodule, the equations (3.3.2) and (3.3.4) imply that

$$a \times_\theta b = ab, \ a \times_\theta e \times_\theta b = aeb \text{ for all } e \in \mathcal{G}_{\underline{0}} \text{ and } a, b \in \mathcal{D}_{\underline{0}}. \tag{3.3.5}$$

Remark 3.3.20. *Using the fact that $\mathcal{G}_\theta$ is isomorphic as a vector space to $\mathcal{G}$, for e in $\mathcal{G}$, we will denote its image under this isomorphism in $\mathcal{G}_\theta$ by e_θ from now on.*

As in the case of deformed algebras, $\mathcal{G}_\theta$ turns out to be a $\mathbb{T}^n$ smooth bimodule. In fact, if β is the $\mathbb{T}^n$-action on the $\mathbb{T}^n$-smooth bimodule $\mathcal{G}$, then we have a deformed $\mathbb{T}^n$-action β^θ on $\mathcal{G}_\theta$ defined by the following formula:

$$\beta^\theta_t(e_\theta) = \sum_{\underline{k}} \chi_{\underline{k}}(t) e_{\underline{k}} \ \forall\ t \in \mathbb{T}^n. \tag{3.3.6}$$

Thus, if $\mathcal{D}$ and $\mathcal{G}$ are as above, the spectral subspaces $\mathcal{D}_{\underline{k}}$ and $\mathcal{G}_{\underline{k}}$ make sense. In fact, by virtue of (3.3.5), we have the following remark:

Remark 3.3.21. *$(\mathcal{D}_\theta)_{\underline{0}}$ is isomorphic to $\mathcal{D}_{\underline{0}}$ as algebras. Moreover, $(\mathcal{G}_\theta)_{\underline{0}} \cong \mathcal{G}_{\underline{0}}$ as $\mathcal{D}_{\underline{0}}$-bimodules. We also note that by (3.3.4), when the right and left $\mathcal{D}$-module actions of $\mathcal{G}$ are symmetric, $(\mathcal{G}_\theta)_{\underline{0}} \subseteq \mathcal{Z}(\mathcal{G}_\theta)$ and in particular $(\mathcal{D}_\theta)_{\underline{0}} \subseteq \mathcal{Z}(\mathcal{D}_\theta)$.*

We have the following easy consequence of the definitions above:

Lemma 3.3.22. *Let $\mathcal{D}$ be a $\mathbb{T}^n$ smooth algebra and $\mathcal{G}_1, \mathcal{G}_2$ be $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodules, in the sense discussed above. Let $L : \mathcal{G}_1 \to \mathcal{G}_2$ be a $\mathbb{T}^n$-equivariant continuous $\mathcal{D}$-bimodule map. Then the underlying vector space map L from $\mathcal{G}_1$ to $\mathcal{G}_2$ becomes a $\mathbb{T}^n$-equivariant continuous $\mathcal{D}_\theta$-bimodule map, denoted by*

$$L_\theta : (\mathcal{G}_1)_\theta \to (\mathcal{G}_2)_\theta$$

defined by the equation

$$L_\theta(e_\theta) = (L(e))_\theta \ \forall\ e \in \mathcal{G}_1 \tag{3.3.7}$$

If L is a $\mathcal{D}$-bimodule isomorphism, then L_θ will be a $\mathcal{D}_\theta$-bimodule isomorphism. If $\mathcal{G}_1$ and $\mathcal{G}_2$ are algebras in particular, then L_θ is an algebra homomorphism.

Now suppose that $\mathrm{Ker}(L)$ is complemented as a $\mathcal{D}$-bimodule in $\mathcal{G}_1$, i.e, there exists a $\mathcal{D}$-bimodule $\mathcal{M} \subseteq \mathcal{G}_1$ such that $\mathcal{G}_1 \cong \mathrm{Ker}(L) \oplus \mathcal{M}$. Then

(i) $\mathrm{Ker}(L)$ is invariant under the action of $\mathbb{T}^n$.

(ii) $\mathcal{M} \cong \mathrm{Ran}(L)$.

(iii) If $\mathcal{M}$ is $\mathbb{T}^n$-invariant, then $(\mathcal{G}_1)_\theta \cong \mathrm{Ker}(L_\theta) \oplus \mathcal{M}_\theta$ and $\mathcal{M}_\theta \cong \mathrm{Ran}(L_\theta)$.

(iv) If $\mathcal{G}_2 = \mathcal{G}_1$ and L is an idempotent, then L_θ is also idempotent.

Proof. By 3.3.7, L_θ is equal to L as a vector space map, hence the $\mathbb{T}^n$-equivariance of L implies the IT^n-equivariance of L_θ. To check that L_θ is a $\mathcal{D}_\theta$-bimodule map, we first note that since $\mathcal{L}_\theta$ is $\mathbb{T}^n$-equivariant, for all e_θ in $((\mathcal{G}_1)_\theta)_{\underline{k}}$, $L_\theta(e_\theta)$ is in $((\mathcal{G}_2)_\theta)_{\underline{k}}$. Then, for e_θ in $(\mathcal{G}_1)_\theta$ and a_θ in $\mathcal{D}_\theta$, we compute the following:

$$L_\theta(e_\theta \times_\theta a_\theta) = L_\theta((\sum_{\underline{k},\underline{l}} \chi(\underline{k},\underline{l}) e_{\underline{k}} a_{\underline{l}})_\theta) = (L(\sum_{\underline{k},\underline{l}} \chi(\underline{k},\underline{l}) e_{\underline{k}} a_{\underline{l}}))_\theta$$

$$=(\sum_{\underline{k},\underline{l}} \chi(\underline{k},\underline{l}) L(e_{\underline{k}}) a_{\underline{l}})_\theta = (L(e))_\theta \times_\theta a_\theta = L_\theta(e_\theta) \times_\theta a_\theta.$$

This proves that L_θ is a right $\mathcal{D}_\theta$-module map. That L_θ is a left $\mathcal{D}_\theta$-module map can be proved similarly. Since L and L_θ are equal as vector space maps, if L is an isomorphism, then L_θ is also a $\mathcal{D}_\theta$-bimodule isomorphism. Similarly, if L happens to be an algebra homomorphism, then L_θ is also an algebra homomorphism.

Suppose e is an elements in $\mathrm{Ker}(L)$. Since L is $\mathbb{T}^n$-equivariant, $L(e) = \sum_{\underline{k}} L(e_{\underline{k}}) = 0$. Projecting onto $(\mathcal{G}_2)_{\underline{l}}$, we see that $L(e_{\underline{l}}) = 0$ for all $\underline{l}$. Suppose β is the $\mathbb{T}^n$ action on G_1. Then, $L(\beta_t(e)) = \sum_{\underline{k}} \chi_{\underline{k}}(t) L(e_{\underline{k}}) = 0$. Hence, $\beta_t(e)$ is also in $\mathrm{Ker}(L)$ and $\mathrm{Ker}(L)$ is invariant under the action of $\mathbb{T}^n$. This proves assertion (i).
Since $\mathcal{G}_1 \cong \mathrm{Ker}(L) \oplus \mathcal{M}$, $\mathrm{Ran}(L) \cong \big(\mathrm{Ker}(L) \oplus \mathcal{M}\big)/\mathrm{Ker}(L) \cong \mathcal{M}$ which gives us assertion (ii).
Since L and L_θ are equal as vector space maps and $\mathrm{Ker}(L)$ is $\mathbb{T}^n$-equivariant, $\mathrm{Ker}(L)$ can be deformed and $\mathrm{Ker}(L) = (\mathrm{Ker}(L))_\theta = \mathrm{Ker}(L_\theta)$. By assertion (ii), $\mathcal{M}$ is a $\mathbb{T}^n$-invariant $\mathcal{D}$-bimodule, hence $\mathcal{M}$ can also be deformed. Therefore, $(\mathcal{G}_1)_\theta \cong (\mathrm{Ker}(L) \oplus \mathcal{M})_\theta \cong (\mathrm{Ker}(L))_\theta \oplus \mathcal{M}_\theta \cong \mathrm{Ker}(L_\theta) \oplus \mathcal{M}_\theta$. The proof of the fact that $\mathcal{M}_\theta \cong \mathrm{Ran}(L_\theta)$ follows along the lines of the proof of assertion (ii). Hence, assertion (iii) is proved.
Assertion (iv) again follows from the fact that L and L_θ are equal as vector space maps. □

The following lemma will also be of use to us.

Lemma 3.3.23. *Let $\mathcal{D}$ be an algebra equipped with $\mathbb{T}^n$-action and $\mathcal{G}_1, \mathcal{G}_2$ be equivariant $\mathcal{D}$-bimodules. Then $(\mathcal{G}_1)_\theta \otimes_{\mathcal{D}_\theta} (\mathcal{G}_2)_\theta \cong (\mathcal{G}_1 \otimes_{\mathcal{D}} \mathcal{G}_2)_\theta$ as $\mathcal{D}_\theta$-bimodules, with the canonical isomorphism given by*

$$e_\theta \otimes_{\mathcal{D}_\theta} f_\theta \mapsto (\sum_{\underline{k},\underline{l} \in \mathbb{Z}^n} \chi_\theta(\underline{k},\underline{l}) e_{\underline{k}} \otimes_{\mathcal{D}} f_{\underline{l}})_\theta, \tag{3.3.8}$$

where $e = \sum_{\underline{k}} e_{\underline{k}}$ and $f = \sum_{\underline{l}} f_{\underline{l}}$ are isospectral deformations of elements in $\mathcal{G}_1$ and $\mathcal{G}_2$ respectively.

Proof. Using the notation adopted in Remark 3.3.20, define a map from $(\mathcal{G}_1 \otimes_{\mathcal{D}} \mathcal{G}_2)_\theta$ to $(\mathcal{G}_1)_\theta \otimes_{\mathcal{D}_\theta} (\mathcal{G}_2)_\theta$ given by

$$(e \otimes_{\mathcal{D}} f)_\theta \mapsto \sum_{\underline{k},\underline{l}} \chi_{-\theta}(\underline{k},\underline{l})(e_\theta)_{\underline{k}} \otimes_{\mathcal{D}_\theta} (f_\theta)_{\underline{l}}.$$

It can be easily checked that this map is an inverse of the map defined in (3.3.8) and that the map defined in (3.3.8) is a $\mathcal{D}_\theta$-bimodule map. □

Now we recall the Connes-Landi deformation ([27]) of a spectral triple and its associated space of forms. We will work in the set-up of Proposition 3.3.3. In particular, $\mathcal{A} = C^\infty(M)$ and $\mathcal{E} = \Omega^1_D(\mathcal{A})$ where $D = d + d^*$. As we recalled in Example 3.3.16, $\mathcal{A}$ is a $\mathbb{T}^n$ smooth algebra and $\mathcal{E}$ is a $\mathbb{T}^n$ smooth $\mathcal{A}$-bimodule. Hence by Definition 3.3.17 and Definition 3.3.19, $\mathcal{A}$ and $\mathcal{E}$ can be deformed to the algebra $\mathcal{A}_\theta$ and the $\mathcal{A}_\theta$-bimodule $\mathcal{E}_\theta$ respectively. In fact, the following lemma shows that the space of two-forms $\Omega^2_D(\mathcal{A})$ can also be deformed.

Lemma 3.3.24. *In the set-up of Proposition 3.3.3 and with $\mathcal{A} = C^\infty(M)$ and $D = d + d^*$, the bimodules of one-form $\mathcal{E} := \Omega^1_D(\mathcal{A})$ and two-forms $\Omega^2_D(\mathcal{A})$ can be deformed into $\mathbb{T}^n$ smooth $\mathcal{A}_\theta$-bimodules $\mathcal{E}_\theta$ and $(\Omega^2_D(\mathcal{A}))_\theta$ respectively.*

Proof. The lemma easily follows by verifying that $\Omega^1_D(\mathcal{A})$ and $\Omega^2_D(\mathcal{A})$ are $\mathbb{T}^n$-smooth bimodules. The case of $\Omega^1_D(\mathcal{A})$ follows from Example 3.3.16.

Now we come to the case of $\Omega^2_D(\mathcal{A})$. By Lemma 3.3.4, the quotient map $\wedge : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^2_D(\mathcal{A})$ is a $\mathbb{T}^n$-equivariant $\mathcal{A}$-bimodule map, and the $\mathbb{T}^n$ action on $\Omega^2_D(\mathcal{A})$ descends from the diagonal $\mathbb{T}^n$ action $\beta \times \beta$ on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Moreover, the $\mathcal{A}$-bimodule structure of $\Omega^2_D(\mathcal{A})$ also descends from that of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Hence, $\Omega^2_D(\mathcal{A})$ is a $\mathbb{T}^n$ smooth $\mathcal{A}$-bimodule. Then by Definition 3.3.19, $\Omega^2_D(\mathcal{A})$ deforms to a $\mathbb{T}^n$ smooth $\mathcal{A}_\theta$-bimodule $(\Omega^2_D(\mathcal{A}))_\theta$. □

Moreover, we have the following:

Theorem 3.3.25. *With the algebra structure of $\mathcal{A}_\theta$ as in (3.3.2), $(\mathcal{A}_\theta, \mathcal{H}, d + d^*)$ defines a spectral triple.*

If $\delta : \mathcal{A} \to \mathcal{E}$ denotes the map which sends a to $[d + d^*, a]$, then we have a deformed map δ_θ from $\mathcal{A}_\theta$ to $\mathcal{E}_\theta$. Moreover, $\Omega^1_D(\mathcal{A}_\theta)$ and $\Omega^2_D(\mathcal{A}_\theta)$ are canonically isomorphic as $\mathcal{A}_\theta$-bimodules with $\mathcal{E}_\theta$ and $(\Omega^2_D(\mathcal{A}))_\theta$ respectively.

Proof. For the proof that $(\mathcal{A}_\theta, \mathcal{H}, d+d^*)$ is a spectral triple, we refer to [26].

Since the map $\delta : \mathcal{A} \to \Omega^1_D(\mathcal{A})$ given by $a \mapsto [d+d^*, a]$ is a $\mathbb{T}^n$-equivariant map, it can be deformed to the map δ_θ, which gives us the second assertion.

The isomorphism of $\Omega^1_D(\mathcal{A}_\theta)$ and $(\Omega^1_D(\mathcal{A}))_\theta$ is an easy consequence of Proposition 2.12 of [67]. Indeed, this result implies that the map $\pi_\theta : \Omega^1_D(\mathcal{A}_\theta) \to (\Omega^1_D(\mathcal{A}))_\theta$ defined by

$$\pi_\theta(\omega_\theta)(f) = \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega_\theta)_{\underline{k}}(f_{\underline{l}})$$

defines an isomorphism from $\Omega^1_D(\mathcal{A}_\theta)$ to $(\Omega^1_D(\mathcal{A}))_\theta$. Here, we have viewed ω_θ in $\Omega^1_D(\mathcal{A}_\theta)$ as an operator acting on $\mathcal{H}$. Then it can be easily checked that

$$\pi_\theta(a_\theta \times_\theta \delta_\theta(b_\theta)) = \sum_{\underline{k},\underline{l}} (\chi_\theta(\underline{k},\underline{l}) a_{\underline{k}} \delta(b_{\underline{l}}))_\theta,$$

for all a, b in $\mathcal{A}$. To prove the isomorphism of $\Omega^2_D(\mathcal{A}_\theta)$ and $(\Omega^2_D(\mathcal{A}))_\theta$ requires some work. We start by adopting some notations.

The maps $(m_0)_{(\mathcal{A},\mathcal{H},D)} : \Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}) \to \mathcal{B}(\mathcal{H})$ and $(m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)} : \Omega^1_D(\mathcal{A}_\theta) \otimes_{\mathcal{A}_\theta} \Omega^1_D(\mathcal{A}_\theta) \to (\mathcal{B}(\mathcal{H}))_\theta$ will denote the appropriate multiplication maps.

The spaces $\mathcal{J}_{(\mathcal{A},\mathcal{H},D)}$ and $\mathcal{J}_{(\mathcal{A}_\theta,\mathcal{H},D)}$ will denote the junk-forms associated to the respective spectral triples.

The maps $q_{(\mathcal{A},\mathcal{H},D)} : \mathrm{Ran}((m_0)_{(\mathcal{A},\mathcal{H},D)}) \to \mathrm{Ran}((m_0)_{(\mathcal{A},\mathcal{H},D)})/\mathcal{J}_{(\mathcal{A},\mathcal{H},D)}$ and $q_{(\mathcal{A}_\theta,\mathcal{H},D)} : \mathrm{Ran}((m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}) \to \mathrm{Ran}((m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)})/\mathcal{J}_{(\mathcal{A}_\theta,\mathcal{H},D)}$ will denote the respective quotient maps.

Finally, $\wedge_{(\mathcal{A},\mathcal{H},D)} = q_{(\mathcal{A},\mathcal{H},D)} \circ (m_0)_{(\mathcal{A},\mathcal{H},D)}$ and $\wedge_{(\mathcal{A}_\theta,\mathcal{H},D)} = q_{(\mathcal{A}_\theta,\mathcal{H},D)} \circ (m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}$ denotes the wedge maps associated to each spectral triple. By abuse of notation we will often use $\wedge$ in both cases, when the context is unambiguous. Then we look at the composition of maps

$$\begin{tikzcd}
\Omega^1_D(\mathcal{A}_\theta) \otimes_{\mathcal{A}_\theta} \Omega^1_D(\mathcal{A}_\theta) \arrow[r, "\pi_\theta \otimes_{\mathcal{A}_\theta} \pi_\theta"] \arrow[d, "T"] & (\Omega^1_D(\mathcal{A}))_\theta \otimes_{\mathcal{A}_\theta} (\Omega^1_D(\mathcal{A}))_\theta \arrow[d, "\cong"] \\
(\Omega^2_D(\mathcal{A}))_\theta & (\Omega^1_D(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1_D(\mathcal{A}))_\theta \arrow[l, "(\wedge_{(\mathcal{A},\mathcal{H},D)})_\theta"']
\end{tikzcd}$$

where the second map is the isomorphism as in Lemma 3.3.23. We denote this composition of maps by T. Explicitly, for ω_θ and η_θ in $\Omega^1_D(\mathcal{A}_\theta)$, we have that

$$T(\omega_\theta \otimes_{\mathcal{A}_\theta} \eta_\theta) = (\sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega)_{\underline{k}} \wedge (\eta)_{\underline{l}})_\theta.$$

We claim that $\mathrm{Ker}(\wedge_{(\mathcal{A}_\theta,\mathcal{H},D)}) \subseteq \mathrm{Ker}(T)$. If our claim is true, then we have a map $\widetilde{T} : \Omega^2_D(\mathcal{A}_\theta) \to (\Omega^2_D(\mathcal{A}))_\theta$. So, suppose that there exist $(\omega_i)_\theta$, $(\eta_i)_\theta$ in $\Omega^1_D(\mathcal{A}_\theta)$ such that $\sum_i (\omega_i)_\theta \otimes_{\mathcal{A}_\theta} (\eta_i)_\theta$ is in $\mathrm{Ker}(\wedge_{(\mathcal{A}_\theta,\mathcal{H},D)})$. Hence, $(m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}(\sum_i (\omega_i)_\theta \otimes_{\mathcal{A}_\theta} (\eta_i)_\theta)$ is in $\mathrm{Ker}(q_{(\mathcal{A}_\theta,\mathcal{H},D)}) = \mathcal{J}_{(\mathcal{A}_\theta,\mathcal{H},D)}$. Since $(m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}(\sum_i (\omega_i)_\theta \otimes_{\mathcal{A}_\theta} (\eta_i)_\theta)$ is in $\mathcal{J}_{(\mathcal{A}_\theta,\mathcal{H},D)}$, there exist elements a_j, b_j in $\mathcal{A}$ such that

$$\sum_j (a_j)_\theta \times_\theta \delta_\theta((b_j)_\theta) = 0. \tag{3.3.9}$$

and

$$\sum_j (m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}(\delta_\theta((a_j)_\theta) \otimes_{\mathcal{A}_\theta} \delta_\theta((b_j)_\theta)) = (m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}(\sum_i (\omega_i)_\theta \otimes_{\mathcal{A}_\theta} (\eta_i)_\theta) \tag{3.3.10}$$

Applying the isomorphism $\pi_\theta : \Omega^1_D(\mathcal{A}_\theta) \to (\Omega^1_D(\mathcal{A}))_\theta$ on (3.3.9), we get

$$\sum_j \sum_{\underline{m},\underline{n}} \chi_\theta(\underline{m},\underline{n})(a_j)_{\underline{m}} \delta((b_j)_{\underline{n}}) = 0. \tag{3.3.11}$$

This implies that

$$(m_0)_{(\mathcal{A},\mathcal{H},D)}(\sum_j \sum_{\underline{m},\underline{n}} \chi_\theta(\underline{m},\underline{n}) \delta((a_j)_{\underline{m}}) \otimes_{\mathcal{A}} \delta((b_j)_{\underline{n}})) \in \mathcal{J}_{(\mathcal{A},\mathcal{H},D)} = \mathrm{Ker}(q_{(\mathcal{A},\mathcal{H},D)}). \tag{3.3.12}$$

The multiplication $(m_0)_{(\mathcal{A}_\theta,\mathcal{H},D)}$ is the deformed multiplication $\times_\theta$ as given in (3.3.2). Hence (3.3.10) implies that

$$(m_0)_{(\mathcal{A},\mathcal{H},D)}(\sum_j \sum_{\underline{m},\underline{n}} \chi_\theta(\underline{m},\underline{n}) \delta((a_j)_{\underline{m}}) \otimes_{\mathcal{A}} \delta((b_j)_{\underline{n}})) = (m_0)_{(\mathcal{A},\mathcal{H},D)}(\sum_i \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega_i)_{\underline{k}} \otimes_{\mathcal{A}} (\eta_i)_{\underline{l}}) \tag{3.3.13}$$

Using all of the above, we compute

$$\begin{aligned}
&\sum_i \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega_i)_{\underline{k}}) \wedge ((\eta_i)_{\underline{l}} \\
&= \wedge_{(\mathcal{A},\mathcal{H},D)} (\sum_i \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega_i)_{\underline{k}} \otimes_{\mathcal{A}} (\eta_i)_{\underline{l}}) \\
&= q_{(\mathcal{A},\mathcal{H},D)} \circ (m_0)_{(\mathcal{A},\mathcal{H},D)}(\sum_i \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega_i)_{\underline{k}} \otimes_{\mathcal{A}} (\eta_i)_{\underline{l}}) \\
&= q_{(\mathcal{A},\mathcal{H},D)} \circ (m_0)_{(\mathcal{A},\mathcal{H},D)}(\sum_j \sum_{\underline{m},\underline{n}} \chi_\theta(\underline{m},\underline{n}) \delta((a_j)_{\underline{m}}) \otimes_{\mathcal{A}} \delta((b_j)_{\underline{n}})) \text{ (applying (3.3.13))} \\
&= 0 \text{ (using (3.3.12))}.
\end{aligned}$$

This proves that the map $\widetilde{T} : \Omega^2_D(\mathcal{A}_\theta) \to (\Omega^2_D(\mathcal{A}))_\theta$ given by

$$\widetilde{T}(\omega_\theta \wedge \eta_\theta) = (\sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(\omega)_{\underline{k}} \wedge (\eta)_{\underline{l}})_\theta$$

is well defined. The inverse of $\widetilde{T}$ is given by the map

$$(\omega \wedge \eta)_\theta \mapsto \sum_{\underline{k},\underline{l}} \chi_{-\theta}(\underline{k},\underline{l})(\omega_\theta)_{\underline{k}} \wedge (\eta_\theta)_{\underline{l}}.$$

The proof of the fact that this map is well defined is the same as before. It is easy to check that $\widetilde{T}$ and this map are inverses of each other and that $\widetilde{T}$ is an $\mathcal{A}_\theta$-bimodule map. $\square$

Henceforth we will make the identifications $\mathcal{E}_\theta \cong \Omega^1_D(\mathcal{A}_\theta)$, $\Omega^2_D(\mathcal{A}_\theta) \cong (\Omega^2_D(\mathcal{A}))_\theta$ without explicitly mentioning.

3.3.3 The canonical Riemannian bilinear metric on $\mathcal{E}_\theta$

In this subsection, we prove that the prescription of Subsection 2.3.1 is indeed a Riemannian bilinear metric on $\mathcal{E}_\theta$. We prove this in two steps. In the first step, we deform a Riemannian $\mathcal{A}$-bilinear metric g to an $\mathcal{A}_\theta$-bilinear map g_θ and show that g_θ is a pseudo-Riemannian bilinear metric. In the second step, we show that the $\mathcal{A}$-bilinear map obtained from Lemma 2.3.6 (for the spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$) coincides with the deformation g_θ of g.

Let us recall the following definition:

Definition 3.3.26. *Let $\mathcal{G}_1$ and $\mathcal{G}_2$ be two $\mathcal{D}$-bimodules admitting actions by $\mathbb{T}^n$ and denoted by β_1 and β_2 respectively. Then $\mathrm{Hom}_{\mathcal{A}}(\mathcal{G}_1, \mathcal{G}_2)$ admits a natural $\mathbb{T}^n$ action γ defined by*

$$(\gamma_t.T)(e) = (\beta_2)_t.(T((\beta_1)_t{}^{-1}(e))).$$

Here, t, T and e belong to $\mathbb{T}^n, \mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \mathcal{G}_2)$ and $\mathcal{G}$ respectively.

Lemma 3.3.27. *In the set-up of Definition 3.3.26, assume furthermore that $\mathcal{D}$ admits an action α of $\mathbb{T}^n$ and β_1, β_2 are both α-equivariant. Then $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \mathcal{G}_2)$ is an $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodule, i.e, for a in $\mathcal{D}$, ω in $\mathcal{G}_1$ and T in $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \mathcal{G}_2)$, we have*

$$\gamma_t(Ta)(\omega) = (\gamma_t(T)\alpha_t(a))(\omega) \quad \text{and} \quad \gamma_t(aT)(\omega) = \alpha_t(a)(\gamma_t(T)(\omega)).$$

Proof. By using the right $\mathcal{D}$-module structure on $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \mathcal{G}_2)$ from Definition 1.1.4, we compute

$$\begin{aligned}\gamma_t(Ta)(\omega) &= (\beta_2)_t((Ta)((\beta_1)_t^{-1}(\omega))) = (\beta_2)_t T(a(\beta_1)_t^{-1}(\omega))\\ &= (\beta_2)_t T((\beta_1)_{t^{-1}}(\alpha_t(a)\omega)) = \gamma_t(T)(\alpha_t(a)\omega)\\ &= (\gamma_t(T)\alpha_t(a))(\omega).\end{aligned}$$

The other equality follows similarly. □

As a consequence of the fact that $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \mathcal{G}_2)$ is an $\mathbb{T}^n$ smooth $\mathcal{D}$-bimodule, we have the following remark.

Remark 3.3.28. *If $\mathcal{D}$, $\mathcal{G}_1$, $\mathcal{G}_2$ are as in Lemma 3.3.27, T an element in the $\underline{k}$-th spectral subspace of $\mathrm{Hom}_{\mathcal{D}}(\mathcal{G}_1, \mathcal{G}_2)$ and e belongs to $(\mathcal{G}_1)_{\underline{l}}$, then $T(e)$ belongs to $(\mathcal{G}_2)_{\underline{k}+\underline{l}}$.*

Now we are in a position to prove the following proposition:

Proposition 3.3.29. *Suppose M is a compact Riemannian manifold as in Proposition 3.3.3. If $\mathcal{A} = C^\infty(M)$ and $\mathcal{E}$ is the bimodule of one-forms as before, then the $\mathcal{A}$-bimodule $\mathcal{E}^*$ admits a deformation $(\mathcal{E}^*)_\theta$.*

Moreover, there exists a $\mathbb{T}^n$-equivariant $\mathcal{A}_\theta$-bimodule isomorphism from $(\mathcal{E}^*)_\theta$ to $(\mathcal{E}_\theta)^*$.

Proof. The bimodule $\mathcal{E}^*$ is isomorphic to the cotangent bundle of M and hence the left and right $\mathcal{A}$-module structures are jointly continuous. Moreover, by Lemma 3.3.27, $\mathcal{E}^*$ is a $\mathbb{T}^n$ smooth module. Thus, the bimodule $\mathcal{E}^*$ can be deformed.

Next, in order to prove the isomorphism, we define a map $T_\theta^{\mathcal{E}} : (\mathcal{E}^*)_\theta \to (\mathcal{E}_\theta)^*$ by

$$(T_\theta^{\mathcal{E}}(\phi_\theta))(e_\theta) = \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k}, \underline{l})\phi_{\underline{k}}(e_{\underline{l}}),$$

where $\phi = \sum_{\underline{k}} \phi_{\underline{k}}$ and $e = \sum_{\underline{l}} e_{\underline{l}}$ are the isospectral decompositions in $\mathcal{E}^*$ and $\mathcal{E}$ respectively. In particular, if e belongs to $\mathcal{E}_{\underline{l}}$, then

$$(T_\theta^{\mathcal{E}}(\phi_\theta))(e_\theta) = \sum_{\underline{k}} \chi_\theta(\underline{k}, \underline{l})\phi_{\underline{k}}(e). \tag{3.3.14}$$

Let $a = \sum_{\underline{l}} a_{\underline{l}}$ be the isospectral decomposition in $\mathcal{A}$. Then we have that

$$(T_\theta^{\mathcal{E}}(\phi_\theta))(e_\theta \times_\theta a_\theta) = (T_\theta^{\mathcal{E}}(\phi_\theta))((\sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l}) e_{\underline{k}} a_{\underline{l}})_\theta)$$

$$=\big(\sum_{\underline{k},\underline{l},\underline{m}} \chi_\theta(\underline{m},\underline{k}+\underline{l})\chi_\theta(\underline{k},\underline{l})\phi_{\underline{m}}(e_{\underline{k}} a_{\underline{l}})\big)_\theta$$

(since $e_{\underline{k}} a_{\underline{l}} \in \mathcal{E}_{\underline{k}+\underline{l}}$ by (3.3.1) and applying (3.3.14))

$$=\big(\sum_{\underline{k},\underline{l},\underline{m}} \chi_\theta(\underline{m},\underline{k})\chi_\theta(\underline{m}+\underline{k},\underline{l})\phi_{\underline{m}}(e_{\underline{k}} a_{\underline{l}})\big)_\theta \text{ (by virtue of (3.3.3))}$$

$$=\big(\sum_{\underline{k},\underline{m}} \chi_\theta(\underline{m},\underline{k})\phi_{\underline{m}}(e_{\underline{k}})\big)_\theta \times_\theta a_\theta$$

(since by Remark 3.3.28, $\phi_{\underline{m}}(e_{\underline{k}}) \in \mathcal{A}_{\underline{k}+\underline{l}}$)

$$=(T_\theta^{\mathcal{E}}(\phi_\theta))(e_\theta) \times_\theta a_\theta.$$

Hence, $(T_\theta^{\mathcal{E}}(\phi_\theta))$ is in $(\mathcal{E}_\theta)^*$. That $T_\theta^{\mathcal{E}}$ is right $\mathcal{A}_\theta$-linear can be seen from the following.

$$(T_\theta^{\mathcal{E}}(\phi_\theta \times_\theta a_\theta))(e_\theta) = (T_\theta^{\mathcal{E}}(\sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})\phi_{\underline{k}} a_{\underline{l}})_\theta)(e_\theta)$$

$$= \sum_{\underline{k},\underline{l},\underline{m}} \chi_\theta(\underline{k},\underline{l})\chi_\theta(\underline{k}+\underline{l},\underline{m})\phi_{\underline{k}} a_{\underline{l}}(e_{\underline{m}}) \text{ (since by (3.3.1), } \phi_{\underline{k}} a_{\underline{l}} \in \mathcal{E}^*_{\underline{k}+\underline{l}})$$

$$= \sum_{\underline{k},\underline{l},\underline{m}} \chi_\theta(\underline{k},\underline{l}+\underline{m})\chi_\theta(\underline{l},\underline{m})\phi_{\underline{k}}(a_{\underline{l}} e_{\underline{m}}) \text{ (by (3.3.3))}$$

$$=(T^{\mathcal{E}}(\phi_\theta))(a_\theta \times_\theta e_\theta) \text{ (since by (3.3.1), } a_\theta \times_\theta e_\theta \text{ is in } (\mathcal{E}_\theta)_{\underline{l}+\underline{m}}).$$

Let γ denote the action of $\mathbb{T}^n$ on $\mathcal{E}^* := \mathrm{Hom}_{\mathcal{A}}(\mathcal{E},\mathcal{A})$ defined by Definition 3.3.26. The $\mathbb{T}^n$ actions on $\mathcal{E}_\theta$ and $(\mathcal{E}^*)_\theta$ will be denoted by β^θ and γ^θ respectively as in (3.3.6). Moreover, the $\mathbb{T}^n$ action on $(\mathcal{E}_\theta)^* := \mathrm{Hom}_{\mathcal{A}_\theta}(\mathcal{E}_\theta,\mathcal{A}_\theta)$ as obtained from Definition 3.3.26 will be denoted by γ'. We claim that the map $T_\theta^{\mathcal{E}}$ is equivariant w.r.t the $\mathbb{T}^n$ actions on $(\mathcal{E}^*)_\theta$ and $(\mathcal{E}_\theta)^*$, i.e,

$$T_\theta^{\mathcal{E}}(\gamma_t^\theta(\phi_\theta)) = \gamma'_t(T_\theta^{\mathcal{E}}(\phi_\theta)). \tag{3.3.15}$$

Indeed, using the fact that $\mathbb{T}^n$ actions preserves spectral subspaces, we have:

$$T_\theta^{\mathcal{E}}(\gamma_t^\theta(\phi_\theta))(e_\theta) = T_\theta^{\mathcal{E}}(\sum_{\underline{k}}(\gamma_t(\phi_{\underline{k}}))_\theta)(e_\theta)$$
$$=(\sum_{\underline{k},\underline{l}}\chi_\theta(\underline{k},\underline{l})(\gamma_t(\phi_{\underline{k}}))(e_{\underline{l}}))_\theta = \big(\sum_{\underline{k},\underline{l}}\chi_\theta(\underline{k},\underline{l})\alpha_t(\phi_{\underline{k}}(\beta_{t^{-1}}(e_{\underline{l}})))\big)_\theta$$
$$=\alpha_t^\theta(\sum_{\underline{k},\underline{l}}\chi_\theta(\underline{k},\underline{l})\phi_{\underline{k}}(\beta_{t^{-1}}(e_{\underline{l}})))_\theta = \alpha_t^\theta(T_\theta^{\mathcal{E}}(\phi_\theta)(\beta_{t^{-1}}^\theta(e_\theta)))$$
$$=(\gamma_t'(T_\theta^{\mathcal{E}}(\phi_\theta)))(e_\theta).$$

This proves (3.3.15).

Thus, we have a well defined equivariant morphism

$$T_{-\theta}^{\mathcal{E}_\theta} : ((\mathcal{E}_\theta)^*)_{-\theta} \to ((\mathcal{E}_\theta)_{-\theta})^* \cong \mathcal{E}^*,$$

and subsequently, a morphism

$$(T_{-\theta}^{\mathcal{E}_\theta})_\theta : (\mathcal{E}_\theta)^* \cong (((\mathcal{E}^*)_\theta)_{-\theta})_\theta \to (\mathcal{E}^*)_\theta.$$

Finally, it is easy to check that the maps $T_\theta^{\mathcal{E}}$ and $(T_{-\theta}^{\mathcal{E}_\theta})_\theta$ are inverses of one another. This finishes the proof. □

Recall that the action of $\mathbb{T}^n$ on $C^\infty(M)$ and $\Omega^1(M) \cong \mathcal{E}$ are given by α and β respectively. Since $\mathbb{T}^n$ acts on M by isometries, the Riemannian metric g is equivariant under the $\mathbb{T}^n$ action i.e, for all ω, η in $\mathcal{E}$, we have

$$g(\beta_t(\omega) \otimes_{\mathcal{A}} \beta_t(\eta)) = \alpha_t(g(\omega \otimes_{\mathcal{A}} \eta)). \tag{3.3.16}$$

Let γ denote the $\mathbb{T}^n$-action on $\mathcal{E}^* = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$. Then by the $\mathbb{T}^n$-equivariance of g, it is easy to see that V_g is a $\mathbb{T}^n$-equivariant map from $\mathcal{E}$ to $\mathcal{E}^*$. Indeed,

$$(\gamma_t V_g(e))(f) = \alpha_t(V_g(e)(\beta_t^{-1}(f)))$$
$$=\alpha_t(g(e \otimes_{\mathcal{A}} \beta_t^{-1}(f))) = g(\beta_t(e) \otimes_{\mathcal{A}} \beta_t\beta_t^{-1}(f)) \text{ (by (3.3.16))}$$
$$=V_g(\beta_t(e))(f).$$

Hence, the map V_g is equivariant. Thus, by virtue of Lemma 3.3.22, we have a $\mathcal{A}_\theta$-bimodule isomorphism $(V_g)_\theta$ from $\mathcal{E}_\theta$ to $(\mathcal{E}^*)_\theta$.

Now we come to the deformation of the map g which is an element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$. The bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is equipped with the natural diagonal action $\beta \times \beta$ of $\mathbb{T}^n$. Therefore, by Definition 3.3.26, we have an action of $\mathbb{T}^n$ on $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$. Since by (3.3.16) g is $\mathbb{T}^n$-equivariant, by Lemma 3.3.22 we have a deformed map g_θ in $\mathrm{Hom}_{\mathcal{A}_\theta}((\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\theta, \mathcal{A}_\theta)$ by yet another application of Lemma 3.3.22. However, by Lemma 3.3.23, $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\theta \cong \mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$. Thus, we have a map in $\mathrm{Hom}_{\mathcal{A}_\theta}(\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta, \mathcal{A}_\theta)$ to be denoted again by g_θ which is the candidate for the Riemannian metric on $\mathcal{E}_\theta = \Omega^1_D(\mathcal{A}_\theta)$.

Our next result connects $(V_g)_\theta$ with V_{g_θ}.

Proposition 3.3.30. *If $T^{\mathcal{E}}_\theta : (\mathcal{E}^*)_\theta \to (\mathcal{E}_\theta)^*$ is the isomorphism appearing in the proof of Proposition 3.3.29. Then*

$$T^{\mathcal{E}}_\theta \circ (V_g)_\theta = V_{g_\theta} \tag{3.3.17}$$

and hence the map $V_{g_\theta} : \mathcal{E}_\theta \to (\mathcal{E}_\theta)^*$ is an isomorphism.

Proof. Since the map V_g is $\mathbb{T}^n$-equivariant, by Lemma 3.3.22 it can be deformed, and the map $(V_g)_\theta$ is an element of $\mathrm{Hom}_{\mathcal{A}_\theta}(\mathcal{E}_\theta, (\mathcal{E}^*)_\theta)$. Moreover, the $\mathbb{T}^n$-equivariance of V_g implies that $(V_g(e))_{\underline{k}} = V_g(e_{\underline{k}})$ for all e in $\mathcal{E}$. Using the notation adopted in Remark 3.3.20, by Proposition 3.3.29, for all e_θ, f_θ in $\mathcal{E}_\theta$,

$$\begin{aligned}
&(T^{\mathcal{E}}_\theta \circ (V_g)_\theta(e_\theta))(f_\theta) = T^{\mathcal{E}}_\theta((V_g(e))_\theta)(f_\theta)\\
=&\sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l})(V_g(e))_{\underline{k}} f_{\underline{l}} = \sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l}) g(e_{\underline{k}} \otimes_{\mathcal{A}} f_{\underline{l}}) \text{ (as } (V_g(e))_{\underline{k}} = V_g(e_{\underline{k}}))\\
=&g(\sum_{\underline{k},\underline{l}} \chi_\theta(\underline{k},\underline{l}) e_{\underline{k}} \otimes_{\mathcal{A}} f_{\underline{l}}) = g_\theta(e_\theta \otimes_{\mathcal{A}_\theta} f_\theta) \text{ (by Lemma 3.3.23 and Proposition 3.3.29)}\\
=&V_{g_\theta}(e_\theta)(f_\theta).
\end{aligned}$$

Moreover, since V_g is an isomorphism from $\mathcal{E}$ to $\mathcal{E}^*$, Lemma 3.3.22 implies that $(V_g)_\theta$ is an isomorphism from $\mathcal{E}_\theta$ to $(\mathcal{E}^*)_\theta$. As $T^{\mathcal{E}}_\theta$ is an isomorphism from $(\mathcal{E}^*)_\theta$ to $(\mathcal{E}_\theta)^*$, the isomorphism of V_{g_θ} follows from (3.3.17). □

Proposition 3.3.31. *g_θ is a Riemannian bilinear metric on $\mathcal{E}_\theta$.*

Proof. Clearly, σ $(= 2P_{\rm sym} - 1)$ is $\mathbb{T}^n$-equivariant, and as $g \circ \sigma = g$, we have $g_\theta \circ \sigma_\theta = g_\theta$ too, i.e. g_θ is symmetric. It is also clear that g_θ is a bilinear map. Finally, by Proposition 3.3.30, V_{g_θ} is nondegenerate. □

Proposition 3.3.32. *Let $g'_\theta : \mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta \to \mathcal{A}''_\theta$ be the $\mathcal{A}_\theta$-bilinear map from Lemma 2.3.6. Then $g'_\theta = g_\theta$ and hence g'_θ is a Riemannian bilinear metric on $\mathcal{E}_\theta$.*

Proof. Let $\omega = [D, a_1]a_2$ and $\eta = [D, b_1]b_2$ be elements in $\mathcal{E}$ to be viewed as elements of $\mathcal{B}(\mathcal{H})$. Let us denote the images of ω and η in $\mathcal{E}_\theta$ by ω_θ and η_θ respectively. Similarly, the representation of $\mathcal{A}_\theta$ in $\mathcal{B}(\mathcal{H})$ will be denoted by π_θ. Finally, recall from Subsection 2.3.1 that τ denotes the state $\mathrm{Lim}_\omega \frac{\mathrm{Tr}(X|D|^{-p})}{\mathrm{Tr}(|D|^{-p})}$ on $\mathcal{B}(\mathcal{H})$ for the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and so τ_θ will denote the state on $\mathcal{B}(\mathcal{H})$ for the deformed spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$. Then, if p is the dimension of the manifold M, we compute

$$\begin{aligned}
\tau_\theta(\omega_\theta\eta_\theta \times {}_\theta a_\theta) &= \mathrm{Lim}_\omega \frac{\mathrm{Tr}([D, \pi_\theta(a_1)]\pi_\theta(a_2)[D, \pi_\theta(b_1)]\pi_\theta(b_2)\pi_\theta(a)|D|^{-p})}{\mathrm{Tr}(|D|^{-p})} \\
&= \mathrm{Lim}_\omega \frac{\mathrm{Tr}([D, a_1]a_2[D, b_1]b_2 a|D|^{-p})}{\mathrm{Tr}(|D|^{-p})} \text{ (by Proposition 4.4.2 of [12])} \\
&= \tau(\omega\eta a) \\
&= \tau(g(\omega \otimes_{\mathcal{A}} \eta)a) \\
&= \tau_\theta(g(\omega \otimes_{\mathcal{A}} \eta)_\theta \pi_\theta(a)) \text{ (by Proposition 4.4.2 of [12])} \\
&= \tau_\theta(g_\theta(\omega_\theta \otimes_{\mathcal{A}_\theta} \eta_\theta) \times {}_\theta a_\theta)
\end{aligned}$$

This proves that the bilinear form of Lemma 2.3.6 for the spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ is equal to g_θ and hence it satisfies all the conditions of Definition 2.3.7. □

3.3.4 Existence and uniqueness of Levi-Civita connections

We will continue to use the notations introduced in Definition 3.3.6. The goal of this subsection is to apply the results deduced in the last two subsections for proving Theorem 3.3.1.

Lemma 3.3.33. *$\mathcal{E}_\theta$ is a finitely generated projective right module over $\mathcal{A}_\theta$.*

Proof. By Lemma 3.3.9, $\mathcal{E}_{\underline{0}}$ is a finitely generated projective right $\mathcal{A}_{\underline{0}}$ module. Then $\mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}$ is a finitely generated projective right $\mathcal{A}$ module. Since the isomorphism $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}} : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A} \to \mathcal{E}$ as given by Lemma 3.3.14 is $\mathbb{T}^n$-equivariant,

$$\mathcal{E}_\theta \cong (\mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A})_\theta \cong (\mathcal{E}_{\underline{0}})_\theta \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_\theta \cong \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_\theta$$

is finitely generated as a right $\mathcal{A}_\theta$ module. Here, we have used the facts that $(\mathcal{A}_\theta)_{\underline{0}} \cong \mathcal{A}_{\underline{0}}$ and $(\mathcal{E}_{\underline{0}})_\theta \cong \mathcal{E}_{\underline{0}}$ as right $\mathcal{A}_{\underline{0}}$ modules since $\mathcal{E}_{\underline{0}}$ is the fixed point submodule for the action of $\mathbb{T}^n$.

The projectivity of $\mathcal{E}_\theta \cong \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_\theta$ follows easily from the fact that $\mathcal{E}_{\underline{0}}$ is finitely generated and projective as a right $\mathcal{A}_{\underline{0}}$ module. $\square$

Lemma 3.3.34. *The map* $u^{\mathcal{E}_\theta}_{\mathcal{E}_{\underline{0}}} = (u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}})_\theta : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_\theta \to \mathcal{E}_\theta$ *is an isomorphism. Moreover, the map* $u^{\mathcal{E}_\theta} : \mathcal{Z}(\mathcal{E}_\theta) \otimes_{\mathcal{Z}(\mathcal{A}_\theta)} \mathcal{A}_\theta \to \mathcal{E}_\theta$ *is an isomorphism.*

Proof. By Lemma 3.3.14, the $\mathbb{T}^n$-equivariant map $u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}} : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A} \to \mathcal{E}$ is an isomorphism. Hence, by Lemma 3.3.22 and Lemma 3.3.23 the map $(u^{\mathcal{E}}_{\mathcal{E}_{\underline{0}}})_\theta : \mathcal{E}_{\underline{0}} \otimes_{\mathcal{A}_{\underline{0}}} \mathcal{A}_\theta \to \mathcal{E}_\theta$ is an isomorphism.

For the second assertion, we note that by Lemma 3.3.9, $\mathcal{F}_{\underline{0}} = \mathcal{E}_{\underline{0}} = (\mathcal{E}_\theta)_{\underline{0}}$ is finitely generated projective over $\mathcal{C}_{\underline{0}} = \mathcal{A}_{\underline{0}} = (\mathcal{A}_\theta)_{\underline{0}}$. But by Remark 3.3.7, $\mathcal{E}_{\underline{0}} = \mathcal{F}_{\underline{0}}$ and $\mathcal{C}_{\underline{0}} = \mathcal{A}_{\underline{0}}$ while by Remark 3.3.21, $\mathcal{E}_{\underline{0}} = (\mathcal{E}_\theta)_{\underline{0}}$ and $\mathcal{A}_{\underline{0}} = (\mathcal{A}_\theta)_{\underline{0}}$. Hence, $(\mathcal{E}_\theta)_{\underline{0}}$ is finitely generated and projective as a right $(\mathcal{A}_\theta)_{\underline{0}}$ module.

By Remark 3.3.21, $(\mathcal{A}_\theta)_{\underline{0}} \subseteq \mathcal{Z}(\mathcal{A}_\theta)$ and $(\mathcal{E}_\theta)_{\underline{0}} \subseteq \mathcal{Z}(\mathcal{E}_\theta)$. Therefore, by Proposition 2.4.6, we conclude that the map $u^{\mathcal{E}_\theta} : \mathcal{Z}(\mathcal{E}_\theta) \otimes_{\mathcal{Z}(\mathcal{A}_\theta)} \mathcal{A}_\theta \to \mathcal{E}_\theta$ is an isomorphism. $\square$

Lemma 3.3.35. *The bimodule* $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ *admits a decomposition* $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta = \mathrm{Ker}(\wedge_\theta) \oplus \mathcal{M}_\theta$ *of right* $\mathcal{A}_\theta$ *modules, where* $\mathcal{M}_\theta \cong \Omega^2(\mathcal{A}_\theta)$ *is satisfied.*

Proof. This follows by applying Lemma 3.3.22, Lemma 3.3.23 and Corollary 3.3.5 applied to the $\mathbb{T}^n$-equivariant map $\wedge$. $\square$

Lemma 3.3.36. *The map* $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ *deforms to a map* $\sigma_\theta : \mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta \to \mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$. *Moreover,*

$$\sigma_\theta(\omega \otimes_{\mathcal{A}_\theta} \eta) = \eta \otimes_{\mathcal{A}_\theta} \omega$$

for all ω, η *in* $\mathcal{Z}(\mathcal{E}_\theta)$.

Proof. The map σ is a $\mathbb{T}^n$-equivariant map and so by Lemma 3.3.22 can be deformed to a map from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\theta$ to $\mathcal{A}_\theta$. Using the isomorphism from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\theta$ to $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ as in Lemma 3.3.23, we can view σ_θ as a map from $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ to $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$.

Let us observe that by Lemma 3.3.34, the map $u^{\mathcal{E}_\theta}$ is an isomorphism, hence $\mathcal{E}_\theta$ is a centered $\mathcal{A}_\theta$-bimodule. Thus, by Theorem 2.1.7, there indeed exists a unique $\mathcal{A}_\theta$-bimodule map from $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ to itself which maps $\omega \otimes_{\mathcal{A}_\theta} \eta$ to $\eta \otimes_{\mathcal{A}_\theta} \omega$ for all ω and η in $\mathcal{Z}(\mathcal{E}_\theta)$. We need to show that this map is equal to σ_θ. For this, let us take e_θ, f_θ in $(\mathcal{E}_\theta)_{\underline{0}}$. Then, using Lemma 3.3.22, we get that

$$
\begin{aligned}
&\sigma_\theta(e_\theta \otimes_{\mathcal{A}_\theta} f_\theta) = \sigma_\theta((\chi_\theta(\underline{0},\underline{0})e \otimes_{\mathcal{A}} f)_\theta) \text{ (by Lemma 3.3.23)}\\
=&(\sigma(e \otimes_{\mathcal{A}} f))_\theta = (f \otimes_{\mathcal{A}} e)_\theta \text{ (since } \sigma \text{ is the classical flip map)}\\
=&f_\theta \otimes_{\mathcal{A}} e_\theta.
\end{aligned}
$$

Now, by Lemma 3.3.34, $(\mathcal{E}_\theta)_{\underline{0}}$ is right $\mathcal{A}_\theta$-total in $\mathcal{E}_\theta$ and hence $\{e_\theta \otimes_{\mathcal{A}_\theta} f_\theta : e_\theta, f_\theta \in (\mathcal{E}_\theta)_{\underline{0}}\}$ is right $\mathcal{A}_\theta$-total in $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$. Thus, by Lemma 1.1.6 the map σ_θ is equal to the unique $\mathcal{A}_\theta$-bimodule map on $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ as in Theorem 2.1.7.

□

Collecting the above results, we get the following:

Proof of Theorem 3.3.1. We start by recalling that we have already proved (Lemma 3.3.33) that $\mathcal{E}_\theta$ is a finitely generated projective right module over $\mathcal{A}_\theta$. By Lemma 3.3.34, the map $u^{\mathcal{E}_\theta} : \mathcal{Z}(\mathcal{E}_\theta) \otimes_{\mathcal{Z}(\mathcal{A}_\theta)} \mathcal{A}_\theta \to \mathcal{E}_\theta$ is an isomorphism.
Next, $\mathrm{Ker}(\wedge_\theta)$ is complemented in $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ by Lemma 3.3.35.
Lastly, the equality $\sigma_\theta(\omega \otimes_{\mathcal{A}_\theta} \eta) = \eta \otimes_{\mathcal{A}_\theta} \omega$ for all $\omega, \eta \in \mathcal{Z}(\mathcal{E}_\theta)$ follows from Lemma 3.3.36.

Thus we have shown that the spectral triple $(\mathcal{A}_\theta, \mathcal{H}, D)$ is a tame spectral triple. Moreover, Proposition 3.3.32 asserts that g_θ is a Riemannian metric on $\mathcal{E}_\theta$. By Theorem 2.5.1, the space of one-forms $\mathcal{E}_\theta$ admits a unique Levi-Civita connection for the Riemannian bilinear metric g. This completes the proof. □

Remark 3.3.37. *Let $\mathcal{F}$ be the spectral submodule of $\mathcal{E}$ as in Definition 3.3.6. Then for the deformed spectral submodule $\mathcal{F}_\theta$ of $\mathcal{E}_\theta$, analogues of the results Lemma 3.3.33, Lemma 3.3.34, Lemma 3.3.35 and Lemma 3.3.36 are proved the same way. Hence the analogous result of Theorem 3.3.1 also holds for the deformed submodule.*

Corollary 3.3.38. *Under the assumptions of Theorem 3.3.1, the Levi-Civita connection ∇ on the bimodule $\mathcal{E}$ deforms to the Levi-Civita connection ∇_θ on the bimodule $\mathcal{E}_\theta$.*

Proof. Since the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is $\mathbb{T}^n$-equivariant, it can be easily checked that the maps $d : \mathcal{A} \to \mathcal{E}$ and $d : \mathcal{E} \to \Omega^2(\mathcal{A})$ are $\mathbb{T}^n$-equivariant. It is easy to see that the map $\mathcal{A}_\theta \to \mathcal{E}_\theta$ given by $a_\theta \mapsto [D, \pi_\theta(a_\theta)]$ is nothing but the deformation of the map $d : \mathcal{A} \to \mathcal{E}$. By Lemma 3.3.22, the maps $d_\theta : \mathcal{A}_\theta \to \mathcal{E}_\theta$ and $d_\theta : \mathcal{E}_\theta \to \Omega^2_D(\mathcal{A}_\theta)$ are $\mathbb{T}^n$-equivariant.

Since the map ∇ is the Levi-Civita connection, ∇ is $\mathbb{T}^n$-equivariant. Thus, we have a $\mathbb{C}$-linear map $\nabla_\theta : \mathcal{E}_\theta \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\theta \cong \mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$ and it can be easily checked that ∇_θ is a connection.

By Lemma 3.3.4, $\wedge : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^2(\mathcal{A})$ is a $\mathbb{T}^n$-equivariant $\mathcal{A}$-bimodule map. Hence, $\wedge_\theta :$ $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta \to \Omega^2_D(\mathcal{A}_\theta)$ is defined, and $\wedge_\theta \circ \nabla_\theta = (\wedge \circ \nabla)_\theta = -d_\theta$. Therefore, ∇_θ is a torsionless connection.

Lastly we show that ∇_θ is compatible with the metric g_θ. We need to show that $\Pi_{g_\theta}(\nabla_\theta) = d_\theta g_\theta$. However, by Lemma 2.4.10, the map $\Pi_{g_\theta}(\nabla_\theta) - d_\theta g_\theta$ is right $\mathcal{A}_\theta$-linear. Since $\{\omega_\theta \otimes_{\mathcal{A}_\theta} \eta_\theta : \omega_\theta, \eta_\theta \in \mathcal{Z}(\mathcal{E}_\theta)\}$ is right $\mathcal{A}_\theta$-total in $\mathcal{E}_\theta \otimes_{\mathcal{A}_\theta} \mathcal{E}_\theta$, it is enough to show that for all $\omega_\theta, \eta_\theta$ in $\mathcal{Z}(\mathcal{E}_\theta)$, we have $(\Pi_{g_\theta}(\nabla_\theta) - d_\theta g_\theta)(\omega_\theta \otimes_{\mathcal{A}_\theta} \eta_\theta) = 0$ for all $\omega_\theta, \eta_\theta$ in $\mathcal{Z}(\mathcal{E}_\theta)$. Let $\omega_\theta, \eta_\theta$ in $\mathcal{Z}(\mathcal{E}_\theta)$. Then,

$$\begin{aligned}
(\Pi_{g_\theta}(\nabla_\theta))(\omega_\theta \otimes_{\mathcal{A}_\theta} \eta_\theta) &= (g_\theta \otimes_{\mathcal{A}_\theta} \mathrm{id}_{\mathcal{E}_\theta})(\sigma_\theta)_{23}(\nabla_\theta(\omega_\theta) \otimes_{\mathcal{A}_\theta} \eta_\theta + \nabla_\theta(\eta_\theta) \otimes_{\mathcal{A}_\theta} \omega_\theta) \\
&= ((g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}}) \circ \sigma_{23})_\theta (\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega)_\theta \\
&= \big(((g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}}) \circ \sigma_{23})(\nabla(\omega) \otimes_{\mathcal{A}} \eta + \nabla(\eta) \otimes_{\mathcal{A}} \omega)\big)_\theta \\
&= (\Pi_g(\nabla)(\omega \otimes_{\mathcal{A}} \eta))_\theta \\
&= (-dg(\omega \otimes_{\mathcal{A}} \eta))_\theta \\
&= -d_\theta g_\theta(\omega_\theta \otimes_{\mathcal{A}_\theta} \eta_\theta).
\end{aligned}$$

Therefore, ∇_θ is compatible with the metric g_θ.

Since the Levi-Civita connection of Theorem 3.3.1 is unique, this completes the proof of the Corollary. $\square$

Chapter 4

Covariant connections on bicovariant differential calculi

In this chapter, we study the problem of Levi-Civita connections on bicovariant differential calculi over Hopf algebras. As explained in Section 1.4 and as seen in Chapter 2, the formulation of the question of existence and uniqueness of Levi-Civita connection for a differential calculus over a (possibly) noncommutative algebra $\mathcal{A}$ needs two ingredients: an analogue of the flip map and a metric compatibility condition. Let us recall that by virtue of Proposition 1.3.15, we know that if $(\mathcal{E}, d)$ is a bicovariant differential calculus on a Hopf algebra $\mathcal{A}$, then $\mathcal{E}$ is in fact a bicovariant $\mathcal{A}$-bimodule. Hence, Proposition 1.3.17 establishes the existence of a unique bicovariant $\mathcal{A}$-bimodule map $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ satisfying some properties. This map σ will play the role of the flip map. In order to make sense of the metric-compatibility condition, we restrict our attention to left-covariant connections and left-invariant pseudo-Riemannian metrics. In Proposition 4.5.3, we prove a sufficient condition for the existence of a unique left-covariant Levi-Civita connection for any bi-invariant pseudo-Riemannian metric on a bicovariant differential calculus satisfying a mild condition. In Theorem 4.5.8, we prove that if the Hopf-algebra is cosemisimple, then the left-covariant Levi-Civita connection obtained in Proposition 4.5.3 is actually bicovariant. Our assumptions for these theorems are satisfied for cocycle deformations of bicovariant differential calculi over algebraic groups as well as the $4D_{\pm}$ calculi on $SU_q(2)$. These examples will be discussed in the next two chapters. For alternative approaches to the proof of existence of Levi-Civita connections on some quantum groups and their homogeneous spaces, we refer to [2], [7] and [76].

We will discuss bicovariant bimodules and associated notions in Section 4.1. In Section 4.2, we will impose a mild condition on the braiding map σ as in Proposition 1.3.17 which will lead to a decomposition of the bicovariant $\mathcal{A}$-bimodule $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ analogous to Proposition 1.4.1. In Section 4.3, we will define and study the notion of invariant pseudo-Riemannian metrics on bicovariant differential calculi. In Section 4.4, as a direct consequence of the assumptions on the braiding map σ, Theorem 4.4.4 will give us the construction of a canonical torsionless connection on $\mathcal{E}$. In the same section, we also introduce the notion of compatibility of left-covariant connections on $\mathcal{E}$ with left-invariant pseudo-Riemannian metrics. A comparison with existing notions of metric compatibility in literature will also be given.

In Section 4.5, we will discuss a metric-independent sufficient condition for the existence of a unique left-covariant connection on the space of one-forms of a Hopf algebra, which is torsionless and compatible with a bi-invariant pseudo-Riemannian metric. For the Hopf algebras of classical Lie groups, Levi-Civita connections compatible with a bi-invariant metric are automatically bicovariant. As an analogous result, in this section we will also show that if the Hopf algebra is cosemisimple, the unique left-covariant connection is also right-covariant. The contents of Section 4.3 are from [18] and that of the rest of this chapter are from [17].

Throughout this chapter, $\mathcal{A}$ will stand for a Hopf algebra. Moreover, the bicovariant differential calculus over $\mathcal{A}$ will always be assumed to be finite in the sense of Definition 4.1.3.

4.1 Bicovariant bimodules and Yetter-Drinfeld modules

We begin by recalling the definitions of certain categories which we will deal with in this chapter for which we will need the definitions and notations developed in Section 1.2 and Subsection 1.3.2.

Definition 4.1.1. *The category ${}^{\mathcal{A}}\mathcal{M}$ of left comodules over a Hopf algebra $\mathcal{A}$ consists of objects (V, Δ_V) which are left $\mathcal{A}$-comodules as in Definition 1.2.2, and morphisms $T : V_1 \to V_2$ which are $\mathbb{C}$-linear maps satisfying*

$$\Delta_{V_2} \circ T = (\mathrm{id} \otimes_{\mathbb{C}} T) \circ \Delta_{V_1}.$$

The category $\mathcal{M}^{\mathcal{A}}$ of right comodules over a Hopf algebra $\mathcal{A}$ consists of objects $(V, {}_V\Delta)$ which are right $\mathcal{A}$-comodules as in Definition 1.2.2, and morphisms $T : V_1 \to V_2$ which are $\mathbb{C}$-linear *maps satisfying*

$$_{V_2}\Delta \circ T = (T \otimes_{\mathbb{C}} \mathrm{id}) \circ {}_{V_1}\Delta.$$

The category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ of left covariant bimodules over a Hopf algebra $\mathcal{A}$ consists of objects (M, Δ_M) which are left covariant $\mathcal{A}$-bimodules as in Definition 1.2.10, and for all m in M and a in $\mathcal{A}$, satisfy

$$\Delta_M(am) = \Delta(a)\Delta_M(m), \quad \Delta_M(ma) = \Delta_M(m)\Delta(a).$$

Morphisms in this category are $\mathbb{C}$-linear maps $T : M_1 \to M_2$ satisfying

$$\Delta_{M_2} \circ T = (\mathrm{id} \otimes_{\mathbb{C}} T) \circ \Delta_{M_1},$$

The category ${}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ of right covariant bimodules over a Hopf algebra $\mathcal{A}$ consists of objects $(M, {}_M\Delta)$ which are right covariant $\mathcal{A}$-bimodules as in Definition 1.2.10, and for all m in M and a in $\mathcal{A}$, satisfy

$${}_M\Delta(am) = \Delta(a){}_M\Delta(m), \quad {}_M\Delta(ma) = {}_M\Delta(m)\Delta(a).$$

Morphisms in the category are $\mathbb{C}$-linear maps $T : M_1 \to M_2$ satisfying

$${}_{M_2}\Delta \circ T = (T \otimes_{\mathbb{C}} \mathrm{id}) \circ {}_{M_1}\Delta,$$

The category ${}^{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ of bicovariant right modules over a Hopf algebra $\mathcal{A}$ consists of objects $(M, \Delta_M, {}_M\Delta)$ which are $\mathcal{A}$-bicomodules as in Definition 1.2.2 as well as right $\mathcal{A}$-modules, satisfying for all m in M and a in $\mathcal{A}$

$$\Delta_M(ma) = \Delta_M(m)\Delta(a), \quad {}_M\Delta(ma) = {}_M\Delta(m)\Delta(a),$$

Morphisms in this category are $\mathbb{C}$-linear maps $T : M_1 \to M_2$ satisfying

$$\Delta_{M_2} \circ T = (\mathrm{id} \otimes_{\mathbb{C}} T) \circ \Delta_{M_1}, \quad {}_{M_2}\Delta \circ T = (T \otimes_{\mathbb{C}} \mathrm{id}) \circ {}_{M_1}\Delta.$$

The category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ of bicovariant bimodules over a Hopf algebra $\mathcal{A}$ consists of objects $(M, \Delta_M, {}_M\Delta)$ which are bicovariant $\mathcal{A}$-bimodules as in Definition 1.2.10, and for all m in

M and a in $\mathcal{A}$ satisfy

$$\Delta_M(am) = \Delta(a)\Delta_M(m), \qquad {}_M\Delta(am) = \Delta(a)\,{}_M\Delta(m),$$
$$\Delta_M(ma) = \Delta_M(m)\Delta(a), \qquad {}_M\Delta(ma) = {}_M\Delta(m)\Delta(a).$$

Morphisms in this category are $\mathbb{C}$-linear maps $T : M_1 \to M_2$ satisfying

$$\Delta_{M_2} \circ T = (\mathrm{id} \otimes_{\mathbb{C}} T) \circ \Delta_{M_1}, \quad {}_{M_2}\Delta \circ T = (T \otimes_{\mathbb{C}} \mathrm{id}) \circ {}_{M_1}\Delta$$

We refer to [78] for more details.

Thus, comparing with Definition 1.2.10 and Definition 1.2.13 with Definition 4.1.1, we have the following:

Proposition 4.1.2. *Suppose M is an $\mathcal{A}$-bimodule.*

(i) A left $\mathcal{A}$-comodule (M, Δ_M) is a left-covariant bimodule if and only if it is an object of the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$. A left-covariant $\mathcal{A}$-bimodule map between two left-covariant bimodules over $\mathcal{A}$ is nothing but a morphism of the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$.

(ii) A right $\mathcal{A}$-comodule $(M, {}_M\Delta)$ is a right-covariant bimodule if and only if it is an object of the category ${}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$. A right-covariant $\mathcal{A}$-bimodule map between two right-covariant bimodules over $\mathcal{A}$ is nothing but a morphism of the category ${}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$.

(iii) A bicomodule $(M, \Delta_M, {}_M\Delta)$ is a bicovariant bimodule if and only if it is an object of the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$. A bicovariant $\mathcal{A}$-bimodule map between two bicovariant bimodules is nothing but a morphism in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$.

We will be using the notations introduced in Definition 1.2.11. Thus, for a left $\mathcal{A}$-comodule M, the symbol ${}_0M$ will denote the set of all left-invariant elements in M. Similarly, if M is right $\mathcal{A}$-comodule, then M_0 will denote the set of all right-invariant elements in M.

Definition 4.1.3. *We will say that a bicovariant bimodule $(M, \Delta_M, {}_M\Delta)$ is finite if ${}_0M$ is a finite dimensional vector space.*

Remark 4.1.4. *Throughout this thesis, we will work only with bicovariant bimodules which are finite. Thus, if M is a bicovariant bimodule under consideration in this thesis, the vector space ${}_0M$ is finite dimensional.*

We also need the following notation.

Definition 4.1.5. *Let M and N be left-covariant $\mathcal{A}$-bimodules. The set of all right $\mathcal{A}$-linear left covariant maps from M to N will be denoted by the symbol ${}^{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M,N)$.*

The category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}}$ has a natural monoidal structure. Indeed, if (M,Δ_M) and (N,Δ_N) are left-covariant bimodules over $\mathcal{A}$, then we have a left coaction $\Delta_{M\otimes_{\mathcal{A}}N}$ of $\mathcal{A}$ on $M\otimes_{\mathcal{A}}N$ defined by the following formula:

$$\Delta_{M\otimes_{\mathcal{A}}N}(m\otimes_{\mathcal{A}}n)=m_{(-1)}n_{(-1)}\otimes_{\mathbb{C}}m_{(0)}\otimes_{\mathcal{A}}n_{(0)}. \tag{4.1.1}$$

Here we have made use of the Sweedler notation introduced in Subsection 1.2. This makes $M\otimes_{\mathcal{A}}N$ into a left covariant $\mathcal{A}$-bimodule. Similarly, ${}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ also has a natural monoidal structure. In particular, the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ is monoidal. Moreover, we have the following:

Theorem 4.1.6. *([93], Theorem 6.3 of [84]) For any two objects M, N in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$, the unique bicovariant bimodule morphism $\sigma: M\otimes_{\mathcal{A}}N\to N\otimes_{\mathcal{A}}M$ satisfying $\sigma(m\otimes_{\mathcal{A}}n)=n\otimes_{\mathcal{A}}m$ whenever m is in ${}_0M$ and n is in N_0 (as in Proposition 1.3.17) is a braiding. Along with the monoidal structure $\otimes_{\mathcal{A}}$ as defined in (4.1.1), this makes $({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}},\otimes_{\mathcal{A}},\sigma)$ into a braided monoidal category.*

The fundamental theorem of Hopf modules (Theorem 1.9.4 of [78]) states that if M is a left-covariant bimodule over $\mathcal{A}$, then M is free as a left (as well as a right) $\mathcal{A}$-module This was reproved by Woronowicz in [93]. The following statement rephrases the same in our notational formalism:

Proposition 4.1.7 (Theorem 2.1 and Theorem 2.3 of [93])**.** *Let (M,Δ_M) be a left-covariant bimodule over $\mathcal{A}$. Then the following multiplication maps are isomorphisms:*

$$\widetilde{u}^M: {}_0M\otimes_{\mathbb{C}}\mathcal{A}\to M,\qquad \widetilde{v}^M:\mathcal{A}\otimes_{\mathbb{C}}{}_0M\to M. \tag{4.1.2}$$

Similarly, if $(M,{}_M\Delta)$ is a right-covariant bimodule over $\mathcal{A}$, then the multiplication maps

$$M_0\otimes_{\mathbb{C}}\mathcal{A}\to M,\qquad \mathcal{A}\otimes_{\mathbb{C}}M_0\to M, \tag{4.1.3}$$

are also isomorphisms.

Then, the following is an immediate corollary.

Corollary 4.1.8. *Let (M, Δ_M) and (N, Δ_N) be left-covariant bimodules over $\mathcal{A}$ and $\{m_i\}_i$ and $\{n_j\}_j$ be vector space bases of ${}_0M$ and ${}_0N$ respectively. Then each element of $M \otimes_{\mathcal{A}} N$ can be written as $\sum_{ij} a_{ij} m_i \otimes_{\mathcal{A}} n_j$ and $\sum_{ij} m_i \otimes_{\mathcal{A}} n_j b_{ij}$, where a_{ij} and b_{ij} are uniquely determined. A similar result holds for right-covariant bimodules $(M, {}_M\Delta)$ and $(N, {}_N\Delta)$ over $\mathcal{A}$. Finally, if (M, Δ_M) is a left-covariant bimodule over $\mathcal{A}$ with basis $\{m_i\}_i$ of ${}_0M$, and $(N, {}_N\Delta)$ is a right-covariant bimodule over $\mathcal{A}$ with basis $\{n_j\}_j$ of N_0, then any element of $M \otimes_{\mathcal{A}} N$ can be written uniquely as $\sum_{ij} a_{ij} m_i \otimes_{\mathcal{A}} n_j$ as well as $\sum_{ij} m_i \otimes_{\mathcal{A}} n_j b_{ij}$.*

Proof. The proof of this result is an adaptation of Lemma 3.2 of [93] and we omit it. □

Now we recall the notion of right Yetter-Drinfeld modules and Schauenburg's result which showed that the category of bicovariant bimodules is braided monoidally equivalent to the category of right Yetter-Drinfeld modules.

Definition 4.1.9. *(Definition 4.1 of [84]) Suppose that $\mathcal{A}$ is a Hopf algebra. A right Yetter-Drinfeld module over $\mathcal{A}$ is a triplet $(M, \leftharpoonup, \delta)$ where $(M, \leftharpoonup)$ is a right $\mathcal{A}$-module, (M, δ) is a right $\mathcal{A}$-comodule such that for all a in $\mathcal{A}$ and for all m in M, the following compatibility condition holds:*

$$m_{(0)} \leftharpoonup a_{(1)} \otimes_{\mathbb{C}} m_{(1)} a_{(2)} = (m \leftharpoonup a_{(2)})_{(0)} \otimes_{\mathbb{C}} a_{(1)} (m \leftharpoonup a_{(2)})_{(1)}.$$

We will let $\mathcal{YD}_{\mathcal{A}}^{\mathcal{A}}$ denote the category of all right Yetter-Drinfeld modules. Here, the morphisms between two objects M and N in $\mathcal{YD}_{\mathcal{A}}^{\mathcal{A}}$ are $\mathbb{C}$-linear maps $T : M \to N$ which are right $\mathcal{A}$-linear and right $\mathcal{A}$-comodule maps.

Theorem 4.1.10. *([84]) Suppose $\mathcal{A}$ is a Hopf algebra with a bijective antipode. Then the category $\mathcal{YD}_{\mathcal{A}}^{\mathcal{A}}$ has a braided monoidal structure. Indeed, if M and N are objects of $\mathcal{YD}_{\mathcal{A}}^{\mathcal{A}}$, the following right $\mathcal{A}$-module and right $\mathcal{A}$-comodule structure makes $M \otimes_{\mathbb{C}} N$ an object of $\mathcal{YD}_{\mathcal{A}}^{\mathcal{A}}$:*

$$(m \otimes_{\mathbb{C}} n) a = m a_{(1)} \otimes_{\mathbb{C}} n a_{(2)}, \quad {}_{M \otimes_{\mathbb{C}} N}\Delta(m \otimes_{\mathbb{C}} n) = m_{(0)} \otimes_{\mathbb{C}} n_{(0)} \otimes_{\mathbb{C}} m_{(1)} n_{(1)}.$$

The braiding $\sigma_{\mathcal{YD}}$ is given by:

$$\sigma_{\mathcal{YD}} : M \otimes_{\mathbb{C}} N \to N \otimes_{\mathbb{C}} M, \quad \sigma_{\mathcal{YD}}(m \otimes_{\mathbb{C}} n) = n_{(0)} \otimes_{\mathbb{C}} m \leftharpoonup n_{(1)}.$$

The bijectivity of the antipode is needed only to guarantee that $\sigma_{\mathcal{YD}}$ is a braiding on $\mathcal{YD}^{\mathcal{A}}_{\mathcal{A}}$. In general, it is only a pre-braiding. Now we are in a position to state Schauenburg's results:

Theorem 4.1.11. *(Theorem 5.7 of [84]) The following statements hold:*

(i) The functor ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}_{\mathcal{A}} \to \mathcal{M}_{\mathcal{A}},\ M \mapsto {}_0M$ defines a monoidal equivalence of categories. The inverse functor is given by $V \mapsto \mathcal{A} \otimes_{\mathbb{C}} V$.

(ii) The functor ${}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}} \to {}_{\mathcal{A}}\mathcal{M},\ M \mapsto M_0$ defines a monoidal equivalence of categories. The inverse functor is given by $V \mapsto V \otimes_{\mathbb{C}} \mathcal{A}$.

(iii) The functor ${}^{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}} \to {}^{\mathcal{A}}\mathcal{M},\ M \mapsto M_0$ defines an equivalence of categories.

(iv) Suppose $\mathcal{A}$ is a Hopf algebra with a bijective antipode and consider the braided monoidal categories $({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}, \otimes_{\mathcal{A}}, \sigma)$ and $(\mathcal{YD}^{\mathcal{A}}_{\mathcal{A}}, \otimes, \sigma_{\mathcal{YD}})$ as in Theorem 4.1.6 and Theorem 4.1.10 respectively. The functor

$$({}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}, \otimes_{\mathcal{A}}, \sigma) \to (\mathcal{YD}^{\mathcal{A}}_{\mathcal{A}}, \otimes, \sigma_{\mathcal{YD}}),\ M \to {}_0M$$

defines a braided monoidal equivalence of categories.

For more details on Yetter-Drinfeld modules, we refer to [96] and [84].

Proposition 4.1.12. *(Theorem 5.7 of [84]) Let (M, Δ_M) and (N, Δ_N) be left-covariant bimodules over $\mathcal{A}$. Following Definition 1.2.11, we denote the left-invariant elements (with respect to the coaction $\Delta_{M \otimes_{\mathcal{A}} N}$) of $M \otimes_{\mathcal{A}} N$ by ${}_0(M \otimes_{\mathcal{A}} N)$. Similarly, the right-invariant elements of $M \otimes_{\mathcal{A}} N$ (with respect to the coaction ${}_{M \otimes_{\mathcal{A}} N}\Delta$) will be denoted by $(M \otimes_{\mathcal{A}} N)_0$. Then we have that*

$$ {}_0(M \otimes_{\mathcal{A}} N) = \mathrm{Span}_{\mathbb{C}}\{m \otimes_{\mathcal{A}} n : m \in {}_0M, n \in {}_0N\}. \tag{4.1.4}$$

Similarly, if $(M, {}_M\Delta)$ and $(N, {}_N\Delta)$ are right-covariant bimodules over $\mathcal{A}$, then we have that

$$(M \otimes_{\mathcal{A}} N)_0 = \mathrm{Span}_{\mathbb{C}}\{m \otimes_{\mathcal{A}} n : m \in M_0, n \in N_0\}.$$

Thus, ${}_0(M \otimes_{\mathcal{A}} N) = {}_0M \otimes_{\mathbb{C}} {}_0N$ and $(M \otimes_{\mathcal{A}} N)_0 = M_0 \otimes_{\mathbb{C}} N_0$.

Proof. This follows directly from the first two monoidal equivalences in Theorem 4.1.11. □

Remark 4.1.13. *In the light of Proposition 4.1.12, we are allowed to use the notations ${}_0M \otimes_{\mathbb{C}} {}_0N$ and ${}_0(M \otimes_{\mathcal{A}} N)$ interchangeably.*

4.1.1 A characterisation of left covariant maps and some consequences

In this subsection, we collect some results on left covariant maps which we will exploit throughout the chapter. For the rest of this subsection, we will use the notations introduced in Proposition 4.1.7 freely.

Proposition 4.1.14. *Let (M, Δ_M) and (N, Δ_N) be left-covariant bimodules over $\mathcal{A}$ and T be a left-covariant right $\mathcal{A}$-linear map from M to N. Then $T({}_0M) \subseteq {}_0N$. Moreover, there exists a unique $\mathbb{C}$-linear map ${}_0T$ in $\mathrm{Hom}_{\mathbb{C}}({}_0M, {}_0N)$ such that*

$$(\widetilde{u}^N)^{-1} \circ T = ({}_0T \otimes_{\mathbb{C}} \mathrm{id})(\widetilde{u}^M)^{-1}. \tag{4.1.5}$$

In particular, a left covariant right $\mathcal{A}$-linear map T from M to N is determined by its action on ${}_0M$.

Proof. Let $\{m_i\}_i$ be a vector space basis for ${}_0M$ and $\{n_j\}_j$ be a vector space basis for ${}_0N$. Since T is a left-covariant right $\mathcal{A}$-linear map from M to N, we have that

$$\Delta_N(T(m_i)) = (\mathrm{id} \otimes_{\mathbb{C}} T)\Delta_M(m_i) = (\mathrm{id} \otimes_{\mathbb{C}} T)(1 \otimes_{\mathbb{C}} m_i) = 1 \otimes_{\mathbb{C}} (T(m_i)).$$

Therefore, $T(m_i)$ is in ${}_0N$. This proves the first assertion.
Define ${}_0T$ to be the restriction of T on ${}_0M$. Let $m = \widetilde{u}^M(\sum_i m_i \otimes_{\mathbb{C}} a_i)$, where $\widetilde{u}^M$ is as defined in Proposition 4.1.7. Then

$$({}_0T \otimes_{\mathbb{C}} \mathrm{id})(\widetilde{u}^M)^{-1}(m) = \sum_i {}_0T(m_i) \otimes_{\mathbb{C}} a_i = (\widetilde{u}^N)^{-1} \circ T(\sum_i m_i a_i) = (\widetilde{u}^N)^{-1} \circ T(m)$$

and thus (4.1.5) follows. The uniqueness follows from the fact that the equation (4.1.5) implies that ${}_0T(m_i) = T(m_i)$ for all i. □

Corollary 4.1.15. *Let (M, Δ_M) be a left-covariant bimodule over $\mathcal{A}$ and T be a left-covariant right $\mathcal{A}$-linear map from M to $\mathcal{A}$. Then there exists a unique $\mathbb{C}$-linear map ${}_0T$ in $\mathrm{Hom}_{\mathbb{C}}({}_0M, \mathbb{C})$ such that*

$$T = ({}_0T \otimes_{\mathbb{C}} \mathrm{id})(\widetilde{u}^M)^{-1}.$$

Proof. The proof follows by taking $(N, \Delta_N) = (\mathcal{A}, \Delta)$ in Proposition 4.1.14. □

Proposition 4.1.16. *Let (M, Δ_M) and (N, Δ_N) be left-covariant bimodules over $\mathcal{A}$. Then ${}^{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M, N)$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}({}_0M, {}_0N)$ as complex vector spaces. Moreover a left-covariant right $\mathcal{A}$-linear map T from M to N is invertible if and only if ${}_0T$ is invertible. More generally, λ is an eigenvalue of T if and only if λ is an eigenvalue of ${}_0T$.*

Proof. Let us recall (Definition 4.1.5) that ${}^{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M, N)$ denotes the set of all right $\mathcal{A}$-linear left-covariant maps from M to N. We define a map

$$ {}^{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M, N) \to \mathrm{Hom}_{\mathbb{C}}({}_0M, {}_0N);\ T \mapsto {}_0T $$

as in Proposition 4.1.14. As T is left-covariant, by Proposition 4.1.14, $T({}_0M) \subseteq {}_0N$. Since T is determined by its action on ${}_0(M)$, this map is one-to-one. Given an element ${}_0T$ in $\mathrm{Hom}_{\mathbb{C}}({}_0M, {}_0N)$, the map $\widetilde{u}^N({}_0T \otimes_{\mathbb{C}} \mathrm{id}_{\mathcal{A}})(\widetilde{u}^M)^{-1}$ defines an element, say T, in $\mathrm{Hom}_{\mathcal{A}}(M, N)$ which can be easily checked to be left-covariant and whose image under the above map is ${}_0T$. Thus, the map is a bijection.

The equation (4.1.5) implies that T is invertible if and only if ${}_0T$ is invertible. Finally, λ is an eigenvalue of ${}_0T$ if and only if ${}_0(T - \lambda.1) = {}_0T - \lambda.1$ is not invertible and ${}_0(T - \lambda.1)$ is not invertible if and only if $T - \lambda.1$ is not invertible by the above argument. Hence, λ is an eigenvalue of T if and only if it is an eigenvalue of ${}_0T$. $\square$

Proposition 4.1.17. *Let (M, Δ_M) and (N, Δ_N) be left-covariant $\mathcal{A}$-bimodules. Then a right $\mathcal{A}$-linear map $T : M \to N$ is left-covariant if and only if $T({}_0M) \subseteq {}_0N$.*

In particular, if $S : M \otimes_{\mathcal{A}} N \to M \otimes_{\mathcal{A}} N$ is a right $\mathcal{A}$-linear map, then Proposition 4.1.12 implies that S is left-covariant if and only if $S({}_0M \otimes_{\mathbb{C}} {}_0N) \subseteq {}_0M \otimes_{\mathbb{C}} {}_0N$.

Proof. If the map T is left-covariant, then by Proposition 4.1.14, $T({}_0M) \subseteq {}_0N$. Conversely, suppose T is a right $\mathcal{A}$-linear map and $T({}_0M) \subseteq {}_0N$. Let $\{m_i\}_i$ be a vector space basis of ${}_0M$ and $\sum_i m_i a_i$ be an element of M. Then we have that

$$ \begin{aligned} &\Delta_N(T(\sum_i m_i a_i)) = \sum_i \Delta_N(T(m_i)a_i) = \sum_i \Delta_N(T(m_i))\Delta(a_i) \\ =&\sum_i (1 \otimes_{\mathbb{C}} T(m_i))(a_{i(1)} \otimes_{\mathbb{C}} a_{i(2)}) = \sum_i (a_{i(1)} \otimes_{\mathbb{C}} T(m_i)a_{i(2)}) \\ =&(\mathrm{id} \otimes_{\mathbb{C}} T)(\sum_i a_{i(1)} \otimes_{\mathbb{C}} m_i a_{i(2)}) = (\mathrm{id} \otimes_{\mathbb{C}} T)(\Delta_M(\sum_i m_i a_i)). \end{aligned} $$

Hence T is a left-covariant map. $\square$

Remark 4.1.18. *Analogues of Proposition 4.1.14, Corollary 4.1.15, Proposition 4.1.16 and Proposition 4.1.17 also hold for right-covariant right $\mathcal{A}$-linear maps from $(M, {}_M\Delta)$ to $(N, {}_N\Delta)$.*

We end this subsection by proving two results related to bicovariant right $\mathcal{A}$-linear maps. We need to recall (Lemma 1.2.12) that if M is a bicovariant $\mathcal{A}$-bimodule, then ${}_0M$ is a right $\mathcal{A}$-comodule.

Proposition 4.1.19. *Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant $\mathcal{A}$-bimodules and T be a left-covariant right $\mathcal{A}$-linear map from M to N. If the map ${}_0T = T|_{{}_0M} : {}_0M \to {}_0N$ as in Proposition 4.1.14 is right-covariant, i.e, ${}_N\Delta \circ {}_0T = ({}_0T \otimes_{\mathbb{C}} \mathrm{id})_M\Delta$, then the map T is also right-covariant.*

Proof. Let m be an element of ${}_0M$ and a an element of $\mathcal{A}$. Then by right $\mathcal{A}$-linearity of T and right-covariance of ${}_0T$, we get

$$\begin{aligned}
& {}_N\Delta(T(ma)) = {}_N\Delta(T(m)a) = {}_N\Delta(T(m))\Delta(a) \\
= \;& {}_N\Delta({}_0T(m))\Delta(a) = ({}_0T \otimes_{\mathbb{C}} \mathrm{id})({}_M\Delta(m))\Delta(a) \\
= \;& (({}_0T \otimes_{\mathbb{C}} \mathrm{id})(m_{(0)} \otimes_{\mathbb{C}} m_{(1)}))(a_{(1)} \otimes_{\mathbb{C}} a_{(2)}) \\
= \;& ({}_0T)(m_{(0)})a_{(1)} \otimes_{\mathbb{C}} m_{(1)}a_{(2)} = T(m_{(0)}a_{(1)}) \otimes_{\mathbb{C}} m_{(1)}a_{(2)} \\
= \;& (T \otimes_{\mathbb{C}} \mathrm{id})((m_{(0)} \otimes_{\mathbb{C}} m_{(1)})(a_{(1)} \otimes_{\mathbb{C}} a_{(2)})) = (T \otimes_{\mathbb{C}} \mathrm{id})_M\Delta(ma).
\end{aligned}$$

Since ${}_0M$ is right $\mathcal{A}$-total in M, this proves that T is a right covariant map. $\square$

Before stating the next result, let us note that if M and N are bicovariant $\mathcal{A}$-bimodules and $\{m_i\}_i$ and $\{n_j\}_j$ are vector space bases for ${}_0M$ and ${}_0N$ respectively, then by Lemma 1.2.12, we get

$${}_{{}_0M}\Delta(m_i) = \sum_k m_k \otimes_{\mathbb{C}} a_{ki} \text{ and } {}_{{}_0N}\Delta(n_j) = \sum_l n_l \otimes_{\mathbb{C}} b_{lj},$$

for some elements $\{a_{ki}\}_{ki}$ and $\{b_{lj}\}_{lj}$ in $\mathcal{A}$.

Lemma 4.1.20. *If an element T of $\mathrm{Hom}_{\mathbb{C}}({}_0M, {}_0N)$ is such that for all m, $T(m_i) = \sum_j n_j T^i_j$ for some elements T^i_j in $\mathbb{C}$, then T is a right-covariant map from ${}_0M$ to ${}_0N$ if and only if*

$$\sum_{il} n_l \otimes_{\mathbb{C}} b_{lj} T^i_j = \sum_{jk} n_j \otimes_{\mathbb{C}} T^k_j a_{ki}. \tag{4.1.6}$$

Proof. If T is a right-covariant complex linear map from ${}_0M$ to ${}_0N$, then ${}_{{}_0N}\Delta \circ T = (T \otimes_{\mathbb{C}} \mathrm{id})_{{}_0M}\Delta$. Now:

$$ {}_{{}_0N}\Delta(T(m_i)) = {}_{{}_0N}\Delta(\sum_j n_j T^i_j) = \sum_l n_l \otimes_{\mathbb{C}} \sum_i b_{lj} T^i_j. \tag{4.1.7} $$

On the other hand,

$$ (T \otimes_{\mathbb{C}} \mathrm{id})_{{}_0M}\Delta(m_i) = (T \otimes_{\mathbb{C}} \mathrm{id})(\sum_k m_k \otimes_{\mathbb{C}} a_{ki}) = \sum_j n_j \otimes_{\mathbb{C}} \sum_k T^k_j a_{ki} \tag{4.1.8} $$

Comparing equations (4.1.7) and (4.1.8), we get that T is an element of $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0M, {}_0N)$ if and only if (4.1.6) holds. □

4.2 The diagonalisability of ${}_0\sigma$

Recall that in Subsection 1.3.2, we defined bicovariant differential calculi and the space of one-forms and two-forms for Hopf algebras. The aim of this section is to prove a noncommutative analogue of the decomposition (1.4.1) under a mild assumption (Theorem 4.2.5) on the bicovariant differential calculi of a Hopf algebra $\mathcal{A}$. This decomposition will help us to construct a canonical bicovariant torsionless connection on a bicovariant differential calculus (see Theorem 4.4.4). The Woronowicz braiding map σ (see Proposition 1.3.17) will play the role of the classical flip map. By Proposition 1.3.15, the space of one-forms of a bicovariant differential calculus over $\mathcal{A}$ is a bicovariant bimodule. Hence all the results on bicovariant bimodules derived in Section 4.1 can be applied. In the sequel, the symbol $\mathcal{E}$ will always stand for the bimodule of one-forms of a bicovariant differential calculus $(\mathcal{E}, d)$.

Let $(\mathcal{E}, d)$ be a bicovariant differential calculus on a Hopf algebra $\mathcal{A}$. Proposition 4.1.7 guarantees the isomorphism of the multiplication map

$$ \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} : ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{A} = {}_0(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \tag{4.2.1} $$

Moreover, by Proposition 1.3.17, we have a canonical bicovariant $\mathcal{A}$-bimodule map σ from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Then Proposition 4.1.14 and Proposition 4.1.12 imply that there exists a unique map

$$ {}_0\sigma : {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} = {}_0(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \to {}_0(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) = {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} \tag{4.2.2} $$

such that

$$(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}})^{-1}\sigma = ({}_0\sigma\otimes_{\mathbb{C}}\mathrm{id})(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}})^{-1}. \tag{4.2.3}$$

For the rest of this chapter, we will make the assumption that the map ${}_0\sigma : {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} \to {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$ is diagonalisable. This assumption holds for a large class of Hopf algebras as indicated in the next proposition.

Proposition 4.2.1. *Let $\mathcal{E}$ be the space of one-forms of a first order differential calculus over a Hopf algebra and ${}_0\sigma : {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} \to {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$ be the map as in (4.2.2). Then*

(i) For the classical bicovariant differential calculus on a Lie group, the map ${}_0\sigma$ is diagonalisable.

(ii) Let $(\mathcal{E}, \mathrm{d})$ be the bicovariant differential calculus on the algebra $\mathcal{A}$ of regular functions on a linear algebraic group G such that the category of finite dimensional representations of G is semisimple. Suppose $\mathcal{A}_\gamma$ is the cocycle deformation of $\mathcal{A}$ with respect to a 2-cocycle γ (see Definition 1.2.7). Then we have a canonical bicovariant differential calculus $(\mathcal{E}_\gamma, \mathrm{d}_\gamma)$ on $\mathcal{A}_\gamma$ obtained by deforming the usual bicovariant differential calculus on $\mathcal{A}$ (see Proposition 5.3.1). Let σ_γ be the braiding map of Proposition 1.3.17 applied to the calculus $(\mathcal{E}_\gamma, \mathrm{d}_\gamma)$. Then ${}_0(\sigma_\gamma) : {}_0(\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma) \to {}_0(\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma)$ is diagonalisable.

(iii) The assumption holds for the bicovariant differential calculi on $SL_q(N)$, $O_q(N)$, $Sp_q(N)$ studied in [51]. More generally, if the map σ satisfies a Hecke-type relation $\Pi_i(\sigma - \lambda_i) = 0$ for distinct scalars λ_i, then ${}_0\sigma$ is diagonalisable.

Proof. Suppose the map σ satisfies a relation $\Pi_i(\sigma - \lambda_i) = 0$ for distinct scalars λ_i. Since ${}_0\sigma({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}) \subseteq {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$, we have the equality $\Pi_i({}_0\sigma - \lambda_i) = 0$ as maps from ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$ to itself. Therefore, the minimal polynomial of the map ${}_0\sigma$ is a product of distinct linear factors and so ${}_0\sigma$ is diagonalisable. Since the bicovariant differential calculi on $SL_q(N)$, $O_q(N)$ and $Sp_q(N)$ studied in [51] satisfy Hecke-type relations as above, this completes the proof of part (iii). The classical case follows similarly, since here $\sigma(e\otimes_{\mathcal{A}} f) = f\otimes_{\mathcal{A}} e$ for all e, f in $\mathcal{E}$, so that $\sigma^2 - 1 = 0$ and therefore, the above reasoning applies. Finally, cocycle deformations of bicovariant differential calculi are dealt with in Chapter 5 and we refer to Theorem 5.3.1 for the proof of part (ii). □

The sub-bimodule

$$\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} := \mathrm{Ker}(\wedge) = \mathrm{Ker}(\sigma - 1)$$

was introduced in (1.3.9). This bimodule is going to play an important role in this chapter. Moreover, let us introduce the following notations.

Definition 4.2.2. *Suppose the map ${}_0\sigma$ is diagonalisable. The eigenspace decomposition of ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$ will be denoted by ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} = \bigoplus_{\lambda\in\Lambda} V_\lambda$, where Λ is the set of distinct eigenvalues of ${}_0\sigma$ and V_λ is the eigenspace of ${}_0\sigma$ corresponding to the eigenvalue λ. Thus, V_1 will denote the eigenspace of ${}_0\sigma$ for the eigenvalue $\lambda = 1$.*

Moreover, we define ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ to be the set of all left-invariant elements of $\mathcal{E}\otimes^{\mathrm{sym}}_{\mathcal{A}}\mathcal{E}$, i.e,

$$ {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E} := \Big\{\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k \in \mathcal{E}\otimes_{\mathcal{A}}\mathcal{E} : \Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k) = 1\otimes_{\mathbb{C}}\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k, \sum_k \rho_k\wedge\nu_k = 0\Big\}. $$

We also define ${}_0\mathcal{F} := \bigoplus_{\lambda\in\Lambda\setminus\{1\}} V_\lambda$.

The assumption that ${}_0\sigma$ is diagonalisable is enough to prove Theorem 4.2.5 As a first step to prove that theorem, we make the following observation:

Lemma 4.2.3. *Let ${}_0\sigma$ be the map in (4.2.2). Then we have*

$$ \text{We have } {}_0\mathcal{E}\otimes^{\mathrm{sym}}_{\mathbb{C}}{}_0\mathcal{E} = \mathrm{Ker}({}_0\sigma - 1). $$

Proof. The result follows by a simple computation. Indeed,

$$
\begin{aligned}
& {}_0\mathcal{E}\otimes^{\mathrm{sym}}_{\mathbb{C}}{}_0\mathcal{E} \\
= \ & \{\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k \in \mathcal{E}\otimes_{\mathcal{A}}\mathcal{E} : \Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k) = 1\otimes_{\mathbb{C}}\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k, \sum_k \rho_k\wedge\nu_k = 0\} \\
= \ & \{\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k \in \mathcal{E}\otimes_{\mathcal{A}}\mathcal{E} : \Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k) \\
= \ & 1\otimes_{\mathbb{C}}\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k, (\sigma-1)(\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k) = 0\} \\
& \ (\text{since } \mathrm{Ker}(\wedge) = \ \mathrm{Ker}(\sigma-1) \text{ by } (1.3.9)) \\
= \ & \{\sum_k \rho_k\otimes_{\mathcal{A}}\nu_k \in {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} : ({}_0\sigma-1)(\sum_k \rho_k\otimes_{\mathbb{C}}\nu_k) = 0\} \\
& (\text{ as } {}_0(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}) = {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} \text{ by Proposition 4.1.12 }) \\
= \ & \mathrm{Ker}({}_0\sigma-1).
\end{aligned}
$$

□

Remark 4.2.4. *Let ${}_0\mathcal{E}\otimes^{\mathrm{sym}}_{\mathbb{C}}{}_0\mathcal{E}$ and ${}_0\mathcal{F}$ be as in Definition 4.2.2. We have already noted that ${}_0\mathcal{E}\otimes^{\mathrm{sym}}_{\mathbb{C}}{}_0\mathcal{E} = V_1$, where V_λ is as in Definition 4.2.2. Further note that since ${}_0\sigma$ is diagonalisable,* *we have the following decomposition:*

$$
{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} = {}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E} \oplus {}_0\mathcal{F}. \tag{4.2.4}
$$

In the sequel, $\Omega^2(\mathcal{A})$ will denote the space of two-forms as defined in Definition 1.3.18.

Theorem 4.2.5. *Suppose that the map ${}_0\sigma$ is diagonalisable. Let $\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}$ be the isomorphism of (4.2.1). We define $\mathcal{F} := \widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}({}_0\mathcal{F}\otimes_{\mathbb{C}}\mathcal{A})$, where ${}_0\mathcal{F}$ is as in Definition 4.2.2. Then $\wedge|_{\mathcal{F}} : \mathcal{F} \to \Omega^2(\mathcal{A})$ defines an isomorphism of right $\mathcal{A}$-modules. Moreover,*

$$
\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F} = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} \oplus \mathcal{F}.
$$

Proof. The proof easily follows by a computation and the following observation:

$$
{}_0(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) = {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E} \text{ and so } \widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}\otimes_{\mathbb{C}}\mathcal{A}) = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}. \tag{4.2.5}
$$

The equation ${}_0(\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}) = {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ follows directly from the definitions of ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ and $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge)$. Then the second equation of (4.2.5) follows from Proposition 4.1.7, since by Proposition 1.3.19, $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ is a bicovariant bimodule.

Now we can compute:

$$
\begin{aligned}
&\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}})^{-1}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})\\
=&\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})\otimes_{\mathbb{C}}\mathcal{A}) = \widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}((({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \oplus {}_0\mathcal{F})\otimes_{\mathbb{C}}\mathcal{A}) \ (\text{by}(4.2.4))\\
=&\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}((({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}\mathcal{A}) \oplus ({}_0\mathcal{F}\otimes_{\mathbb{C}}\mathcal{A})) = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} \oplus \mathcal{F}\\
&\quad(\text{by } (4.2.5) \text{ and the definition of } \mathcal{F})\\
=&\mathrm{Ker}(\wedge) \oplus \mathcal{F} \ (\text{by the definition of } \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}).
\end{aligned}
$$

Finally, since $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$, we have that

$$
\mathcal{F} \cong (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})/\mathrm{Ker}(\wedge) = (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})/\mathrm{Ker}(\sigma - 1) = \Omega^2(\mathcal{A}),
$$

by (1.3.9) and the definition of $\Omega^2(\mathcal{A})$ as in Definition 1.3.18. Hence, $\wedge|_{\mathcal{F}} : \mathcal{F} \to \Omega^2(\mathcal{A})$ is an isomorphism of right $\mathcal{A}$-modules. $\square$

4.2.1 The idempotent P_{sym} and its properties

In this subsection, we study the idempotent element of $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$ with range $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ and kernel $\mathcal{F}$.

Definition 4.2.6. *We will denote by ${}_0(P_{\mathrm{sym}})$ the idempotent element in $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})$ with range ${}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}} {}_0\mathcal{E}$ and kernel ${}_0\mathcal{F}$. By Proposition 4.1.16, ${}_0(P_{\mathrm{sym}})$ extends to a right $\mathcal{A}$-linear left-covariant map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. We are going to denote the extension by the symbol P_{sym}. More concretely,*

$$P_{\mathrm{sym}} := \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} ({}_0(P_{\mathrm{sym}}) \otimes_{\mathbb{C}} \mathrm{id}) (\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}})^{-1}.$$

Proposition 4.2.7. *The map P_{sym} is the idempotent map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself, with range onto $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ and with kernel $\mathcal{F}$. In fact, P_{sym} is also a left $\mathcal{A}$-linear and bicovariant map. Thus P_{sym} is a bicovariant $\mathcal{A}$-bimodule map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself.*

Proof. By Definition 4.2.6, P_{sym} is a left-covariant right $\mathcal{A}$-linear map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself. Since ${}_0(P_{\mathrm{sym}})$ is an idempotent, $P_{\mathrm{sym}} = \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} ({}_0(P_{\mathrm{sym}}) \otimes_{\mathbb{C}} \mathrm{id}) (\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}})^{-1}$ is also idempotent. We have that

$$\begin{aligned}
&\mathrm{Ran}(P_{\mathrm{sym}}) = \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} ({}_0(P_{\mathrm{sym}}) \otimes_{\mathbb{C}} \mathrm{id}) (\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}})^{-1} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \\
=&\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} ({}_0(P_{\mathrm{sym}}) \otimes_{\mathbb{C}} \mathrm{id}) (({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{A}) = \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} (({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{A}) \\
&(\text{by the definition of } {}_0(P_{\mathrm{sym}})) \\
=&\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} \ (\text{by } (4.2.5)).
\end{aligned}$$

Now we prove that $\mathrm{Ker}(P_{\mathrm{sym}}) = \mathcal{F}$. We note that P_{sym} is an idempotent from the complex vector space $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself with range $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ and kernel containing the subspace $\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}} ({}_0\mathcal{F} \otimes_{\mathbb{C}} \mathcal{A}) = \mathcal{F}$. Since $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} \oplus \mathcal{F}$ (Theorem 4.2.5), this proves that $\mathrm{Ker}(P_{\mathrm{sym}}) = \mathcal{F}$.
Finally, we prove that P_{sym} is a bicovariant $\mathcal{A}$-bimodule map. this follows from the observation that ${}_0(P_{\mathrm{sym}})$ is a polynomial in ${}_0\sigma$. Indeed, in the notation of Definition 4.2.2, ${}_0(P_{\mathrm{sym}})$ is the idempotent with range V_1 and kernel $\oplus_{\lambda \in \Lambda \setminus \{1\}} V_\lambda$ and so

$${}_0(P_{\mathrm{sym}}) = \Pi_{\lambda \in \Lambda \setminus \{1\}} \frac{{}_0\sigma - \lambda}{1 - \lambda}. \tag{4.2.6}$$

Therefore,

$$\begin{aligned}P_{\rm sym} &= \widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\big({}_0(P_{\rm sym})\otimes_{\mathbb{C}}\mathrm{id}\big)(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}})^{-1}\\ &=\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Big(\big(\Pi_{\lambda\in\Lambda\setminus\{1\}}\frac{1}{1-\lambda}({}_0\sigma-\lambda)\big)\otimes_{\mathbb{C}}\mathrm{id}\Big)(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}})^{-1}\\ &=\Pi_{\lambda\in\Lambda\setminus\{1\}}\Big(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\big((\frac{1}{1-\lambda}({}_0\sigma-\lambda))\otimes_{\mathbb{C}}\mathrm{id}\big)(\widetilde{u}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}})^{-1}\Big) = \Pi_{\lambda\in\Lambda\setminus\{1\}}\big(\frac{1}{1-\lambda}(\sigma-\lambda)\big)\end{aligned}$$

by (4.2.3). Hence,

$$P_{\rm sym} = \Pi_{\lambda\in\Lambda\setminus\{1\}}\big(\frac{1}{1-\lambda}(\sigma-\lambda)\big). \tag{4.2.7}$$

Now σ is a bicovariant $\mathcal{A}$-bimodule map from $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ to itself and so $P_{\rm sym}$, being a composition of bicovariant $\mathcal{A}$-bimodule maps from $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ to $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ is itself a bicovariant $\mathcal{A}$-bimodule map from $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ to $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$. □

In the classical case, we have $\Lambda = \pm 1$ and so in this case, we recover the formula $P_{\rm sym} = \frac{1}{2}(1+\sigma)$ from (4.2.7). Let us collect two facts in the following remark which will be used later.

Remark 4.2.8. *Since $P_{\rm sym}$ is a bicovarant $\mathcal{A}$-bimodule map, the right $\mathcal{A}$-module $\mathcal{F} = \mathrm{Ran}(1 - P_{\rm sym})$ is actually a bicovariant sub-bimodule of $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$.*

Definition 4.2.9. *Let $\mathcal{F}$ be the sub-bimodule of $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ as in Theorem 4.2.5. By Theorem 4.2.5, we have a right $\mathcal{A}$-linear isomorphism $\wedge|_{\mathcal{F}}:\mathcal{F}\to\Omega^2(\mathcal{A})$ which we will denote by Q.*

The following result is a corollary to Proposition 4.2.7.

Corollary 4.2.10. *If $(\mathcal{E}, d)$ is a bicovariant first order differential calculus, then $d\omega$ is in ${}_0(\Omega^2(\mathcal{A}))$ for all ω in ${}_0\mathcal{E} = {}_0(\Omega^1(\mathcal{A}))$. Moreover, Q is a bicovariant $\mathcal{A}$-bimodule map.*

Proof. From Proposition 1.3.20, we know that $d:\mathcal{E}\to\Omega^2(\mathcal{A})$ is bicovariant. Therefore, if ω is in ${}_0\mathcal{E}$, we have

$$\Delta_{\Omega^2(\mathcal{A})}(d\omega) = (\mathrm{id}_{\mathcal{A}}\otimes_{\mathbb{C}} d)\Delta_{\mathcal{E}}(\omega) = 1\otimes_{\mathbb{C}} d\omega.$$

For the second statement, we note that by Remark 4.2.8, $\mathcal{F}$ a bicovariant sub-bimodule of $\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$. By Proposition 1.3.19, the quotient map $\wedge$ is a bicovariant bimodule map. Hence, the restriction $Q := \wedge|_{\mathcal{F}}$ is also a bicovariant bimodule map from $\mathcal{F}$ to $\mathrm{Ran}(Q) = \Omega^2(\mathcal{A})$. In particular, this implies that

$$Q^{-1}({}_0(\Omega^2(\mathcal{A}))) \subseteq {}_0\mathcal{F} \text{ (Proposition 4.1.17)}. \tag{4.2.8}$$

□

We end this section with one more lemma which will be needed in the proofs of Lemma 4.2.11 and Theorem 4.5.9. In what follows, the set of all linear functionals on a finite dimensional complex vector space W will be denoted by the symbol W^*.

Lemma 4.2.11. *The following maps, defined in Proposition 1.1.8, are vector space isomorphisms:*

$$\zeta_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}} : ({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}({}_0\mathcal{E})^* \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}),$$

$$\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}} : {}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^* \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E}).$$

Proof. By the definition of the map $\zeta_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}$,

$$\zeta_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}({}_0\mathcal{E})^*) \subseteq \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}).$$

Since $\zeta_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}$ is an isomorphism from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})\otimes_{\mathbb{C}}({}_0\mathcal{E})^*$ onto $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})$ and

$$\dim(({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}({}_0\mathcal{E})^*) = \dim(\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})),$$

we have proved the first assertion.

Now we prove the second assertion. By the definition of ${}_0(P_{\mathrm{sym}})$ (Definition 4.2.6),

$${}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} = \mathrm{Ran}({}_0(P_{\mathrm{sym}})) \oplus \mathrm{Ran}(1 - {}_0(P_{\mathrm{sym}}))$$

and hence an element ψ of $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^* = (\mathrm{Ran}({}_0(P_{\mathrm{sym}})))^*$ extends to an element $\widetilde{\psi}$ of $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})^*$ by the formula

$$\widetilde{\psi}(X) = \psi({}_0(P_{\mathrm{sym}})(X)) \; \forall X \; \in {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}.$$

More generally,

$$\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, \mathbb{C}) = \{\psi \in \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}, \mathbb{C}) : \psi((1 - {}_0(P_{\mathrm{sym}}))(X)) = 0 \; \forall X \in {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}\}. \tag{4.2.9}$$

This allows us to view $\psi \in ({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^*$ as an element $\widetilde{\psi}$ in $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})^*$ such that $\widetilde{\psi}((1 - {}_0(P_{\mathrm{sym}}))(X)) = 0$.

Thus, for e in ${}_0\mathcal{E}$, $\widetilde{\psi}$ as above and for all X in ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$, we have

$$(\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}})(e\otimes_{\mathbb{C}}\widetilde{\psi})((1-{}_0(P_{\mathrm{sym}}))(X))=e\widetilde{\psi}((1-{}_0(P_{\mathrm{sym}}))(X))=0.$$

This implies that

$$\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}({}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^*)\subseteq\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E},{}_0\mathcal{E}).$$

As $\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}$ is an isomorphism from ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})^*$ onto $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E})$ and

$$\dim({}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^*)=\dim(\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E},{}_0\mathcal{E})),$$

$\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}$ maps ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^*$ isomorphically onto $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E},{}_0\mathcal{E})$. This finishes the proof of the lemma. $\square$

4.3 Pseudo-Riemannian metrics on bicovariant bimodules

In this section, we study pseudo-Riemannian metrics on bicovariant differential calculi over Hopf algebras. After defining pseudo-Riemannian metrics, we recall the definitions of left and right invariance of a pseudo-Riemannian metrics from [51]. We prove that a pseudo-Riemannian metric is left (respectively, right) invariant if and only if it is left (respectively, right) covariant. We will see that the coefficients of a left-invariant pseudo-Riemannian metric with respect to a left-invariant basis of $\mathcal{E}$ are scalars. We use this fact to clarify some properties of pseudo-Riemannian invariant metrics. We end the section by comparing our definition with those by Heckenberger and Schmüdgen ([51]) as well as by Beggs and Majid.

Definition 4.3.1. *([51]) Suppose $\mathcal{E}$ is a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$ and $\sigma:\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}\to\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}$ be the map as in Proposition 1.3.17. A pseudo-Riemannian metric for the pair $(\mathcal{E},\sigma)$ is a right $\mathcal{A}$-linear map $g:\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}\to\mathcal{A}$ such that the following conditions hold:*

(i) $g\circ\sigma=g$.

(ii) If $g(\rho\otimes_{\mathcal{A}}\nu)=0$ for all ν in $\mathcal{E}$, then $\rho=0$.

For other notions of metrics on covariant differential calculus, we refer to [11] and references therein.

Definition 4.3.2. *([51]) A pseudo-Riemannian metric g on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$ is said to be left-invariant if for all ρ, ν in $\mathcal{E}$,*

$$(\mathrm{id} \otimes_{\mathbb{C}} \epsilon g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)) = g(\rho \otimes_{\mathcal{A}} \nu).$$

Similarly, a pseudo-Riemannian metric g on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$ is said to be right-invariant if for all ρ, ν in $\mathcal{E}$,

$$(\epsilon g \otimes_{\mathbb{C}} \mathrm{id})({}_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}\Delta(\rho \otimes_{\mathcal{A}} \nu)) = g(\rho \otimes_{\mathcal{A}} \nu).$$

Finally, a pseudo-Riemannian metric g on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$ is said to be bi-invariant if it is both left-invariant as well as right-invariant.

We observe that a pseudo-Riemannian metric is invariant if and only if it is covariant.

Proposition 4.3.3. *Let g be a pseudo-Riemannian metric on the bicovariant bimodule $\mathcal{E}$. Then g is left-invariant if and only if $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ is a left-covariant map. Similarly, g is right-invariant if and only if $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ is a right-covariant map.*

Proof. Let g be a left-invariant metric on $\mathcal{E}$, and ρ, ν be elements of $\mathcal{E}$. Then the following computation shows that g is a left-covariant map.

$$\begin{aligned}
&\Delta(g(\rho \otimes_{\mathcal{A}} \nu)) = \Delta((\mathrm{id} \otimes_{\mathbb{C}} \epsilon g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)))\\
=&\Delta((\mathrm{id} \otimes_{\mathbb{C}} \epsilon g)(\rho_{(-1)}\nu_{(-1)} \otimes_{\mathbb{C}} \rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}))\\
=&\Delta(\rho_{(-1)}\nu_{(-1)})\epsilon(g(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}))\\
=&(\rho_{(-1)})_{(1)}(\nu_{(-1)})_{(1)} \otimes_{\mathbb{C}} (\rho_{(-1)})_{(2)}(\nu_{(-1)})_{(2)}\epsilon(g(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}))\\
=&(\rho_{(-1)})_{(1)}(\nu_{(-1)})_{(1)} \otimes_{\mathbb{C}} ((\mathrm{id} \otimes_{\mathbb{C}} \epsilon g)((\rho_{(-1)})_{(2)}(\nu_{(-1)})_{(2)} \otimes_{\mathbb{C}} \rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)}))\\
=&\rho_{(-1)}\nu_{(-1)} \otimes_{\mathbb{C}} ((\mathrm{id} \otimes_{\mathbb{C}} \epsilon g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)})))\\
&\quad\text{(where we have used coassociativity of comodule coactions)}\\
=&\rho_{(-1)}\nu_{(-1)} \otimes_{\mathbb{C}} g(\rho_{(0)} \otimes_{\mathcal{A}} \nu_{(0)})\\
=&(\mathrm{id} \otimes_{\mathbb{C}} g)(\Delta_{(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)).
\end{aligned}$$

On the other hand, suppose $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ is a left-covariant map. Then we have

$$(\mathrm{id} \otimes_{\mathbb{C}} \epsilon g)\Delta_{(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu) = (\mathrm{id} \otimes_{\mathbb{C}} \epsilon)(\mathrm{id} \otimes_{\mathbb{C}} g)\Delta_{(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})}(\rho \otimes_{\mathcal{A}} \nu)$$
$$= (\mathrm{id} \otimes_{\mathbb{C}} \epsilon)\Delta(g(\rho \otimes_{\mathcal{A}} \nu)) = g(\rho \otimes_{\mathcal{A}} \nu).$$

The proof of the right-covariant case is similar. □

The following key result will be used throughout the article.

Lemma 4.3.4. *([51]) If g is a pseudo-Riemannian metric which is left-invariant on a left-covariant $\mathcal{A}$-bimodule $\mathcal{E}$, then $g(\omega_1 \otimes_{\mathcal{A}} \omega_2) \in \mathbb{C}.1$ for all ω_1, ω_2 in ${}_0\mathcal{E}$. Similarly, if g is a right-invariant pseudo-Riemannian metric on a right-covariant $\mathcal{A}$-bimodule, then $g(\eta_1 \otimes_{\mathcal{A}} \eta_2) \in \mathbb{C}.1$ for all η_1, η_2 in $\mathcal{E}_0$.*

Let us clarify some of the properties of a left-invariant and right-invariant pseudo-Riemannian metrics. To that end, we make the next definition which makes sense as we always work with finite bicovariant bimodules (see Definition 4.1.3). The notations used in the next definition will be used throughout the chapter.

Definition 4.3.5. *Let $\mathcal{E}$ and g be as above. For a fixed basis $\{\omega_1, \cdots, \omega_n\}$ of ${}_0\mathcal{E}$, we define $g_{ij} = g(\omega_i \otimes_{\mathcal{A}} \omega_j)$. Moreover, we define $V_g : \mathcal{E} \to \mathcal{E}^* = \mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{A})$ to be the map defined by the formula*

$$V_g(e)(f) = g(e \otimes_{\mathcal{A}} f).$$

Proposition 4.3.6. *Let g be a left-invariant pseudo-Riemannian metric for $\mathcal{E}$ as in Definition 4.3.1. Then the following statements hold:*

(i) The map V_g is a one-to-one right $\mathcal{A}$-linear map from $\mathcal{E}$ to $\mathcal{E}^*$.

(ii) If $e \in \mathcal{E}$ is such that $g(e \otimes_{\mathcal{A}} f) = 0$ for all $f \in {}_0\mathcal{E}$, then $e = 0$. In particular, the map $V_g|_{{}_0\mathcal{E}}$ is one-to-one and hence an isomorphism from ${}_0\mathcal{E}$ to $({}_0\mathcal{E})^*$.

(iii) The matrix $((g_{ij}))_{ij}$ is invertible.

(iv) Let g^{ij} denote the (i, j)-th entry of the inverse of the matrix $((g_{ij}))_{ij}$. Then g^{ij} is an element of $\mathbb{C}.1$ for all i, j.

(v) If $g(e \otimes_{\mathcal{A}} f) = 0$ for all e in ${}_0\mathcal{E}$, then $f = 0$.

Proof. The right $\mathcal{A}$-linearity of V_g follows from the fact that g is a well-defined map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{A}$. The condition (2) of Definition 4.3.1 forces V_g to be one-to-one. This proves (i).

For proving (ii), note that $V_g|_{{}_0\mathcal{E}}$ is the restriction of a one-to-one map to a subspace. Hence, it is a one-to-one $\mathbb{C}$-linear map. Since g is left-invariant, by Lemma 4.3.4, for any e in ${}_0\mathcal{E}$, $V_g(e)({}_0\mathcal{E})$ is contained in $\mathbb{C}$. Therefore, V_g maps ${}_0\mathcal{E}$ into $({}_0\mathcal{E})^*$. Since, ${}_0\mathcal{E}$ and $({}_0\mathcal{E})^*$ have the same finite dimension as vector spaces, $V_g|_{{}_0\mathcal{E}} : {}_0\mathcal{E} \to ({}_0\mathcal{E})^*$ is an isomorphism. This proves (ii).

Now we prove (iii). Let $\{\omega_i\}_i$ be a basis of ${}_0\mathcal{E}$ and $\{\omega_i^*\}_i$ be a dual basis, i.e, $\omega_i^*(\omega_j) = \delta_{ij}$. Since $V_g|_{{}_0\mathcal{E}}$ is a vector space isomorphism from ${}_0\mathcal{E}$ to $({}_0\mathcal{E})^*$ by part (ii), there exist complex numbers a_{ij} such that

$$(V_g)^{-1}(\omega_i^*) = \sum_j a_{ij}\omega_j$$

. This yields

$$\delta_{ik} = \omega_i^*(\omega_k) = g(\sum_j a_{ij}\omega_j \otimes_{\mathcal{A}} \omega_k) = \sum_j a_{ij} g_{jk}.$$

Therefore, $((a_{ij}))_{ij}$ is the left-inverse and hence the inverse of the matrix $((g_{ij}))_{ij}$. This proves (iii).

For proving (iv), we use the fact that g_{ij} is an element of $\mathbb{C}.1$ for all i, j. Since

$$\sum_k g(\omega_i \otimes_{\mathcal{A}} \omega_k) g^{kj} = \delta_{ij}.1 = \sum_k g^{ik} g(\omega_k \otimes_{\mathcal{A}} \omega_j) = \delta_{ij},$$

we have

$$\sum_k g(\omega_i \otimes_{\mathcal{A}} \omega_k)\epsilon(g^{kj}) = \delta_{ij} = \sum_k \epsilon(g^{ik}) g(\omega_k \otimes_{\mathcal{A}} \omega_j).$$

So, the matrix $((\epsilon(g^{ij})))_{ij}$ is also an inverse to the matrix $((g(\omega_i \otimes_{\mathcal{A}} \omega_j)))_{ij}$ and hence $g^{ij} = \epsilon(g^{ij})$ and g^{ij} is in $\mathbb{C}.1$.

Finally, we prove (v) using (iv). Suppose f be an element in $\mathcal{E}$ such that $g(e \otimes_{\mathcal{A}} f) = 0$ for all e in ${}_0\mathcal{E}$. Let $f = \sum_k \omega_k a_k$ for some elements a_k in $\mathcal{A}$. Then for any fixed index i_0, we obtain

$$0 = g(\sum_j g^{i_0 j}\omega_j \otimes_{\mathcal{A}} \sum_k \omega_k a_k) = \sum_k \sum_j g^{i_0 j} g_{jk} a_k = \sum_k \delta_{i_0 k} a_k = a_{i_0}.$$

Hence, we have that $f = 0$. This finishes the proof. $\square$

We apply the results in Proposition 4.3.6 to exhibit a basis of the free right $\mathcal{A}$-module $V_g(\mathcal{E})$. This will be used in the next chapter to make Definition 5.2.1 which will be needed to prove Theorem 5.2.5.

Lemma 4.3.7. *Suppose $\{\omega_i\}_i$ is a basis of ${}_0\mathcal{E}$ and $\{\omega_i^*\}_i$ be the dual basis as in the proof of Proposition 4.3.6. If g is a pseudo-Riemannian left-invariant metric on $\mathcal{E}$, then $V_g(\mathcal{E})$ is a free right $\mathcal{A}$-module with basis $\{\omega_i^*\}_i$.*

Proof. We will use the notations $(g_{ij})_{ij}$ and g^{ij} from Proposition 4.3.6. Since V_g is a right $\mathcal{A}$-linear map, $V_g(\mathcal{E})$ is a right $\mathcal{A}$-module. Since

$$V_g(\omega_i) = \sum_j g_{ij}\omega_j^* \tag{4.3.1}$$

and the inverse matrix $(g^{ij})_{ij}$ has scalar entries (Proposition 4.3.6), we get

$$\omega_k^* = \sum_i g^{ki} V_g(\omega_i)$$

and so ω_k^* belongs to $V_g(\mathcal{E})$ for all k. By the right $\mathcal{A}$-linearity of the map V_g, we conclude that the set $\{\omega_i^*\}_i$ is right $\mathcal{A}$-total in $V_g(\mathcal{E})$.

Finally, if a_i are elements in $\mathcal{A}$ such that $\sum_k \omega_k^* a_k = 0$, then by (4.3.1), we have

$$0 = \sum_{i,k} g^{ki} V_g(\omega_i) a_k = V_g(\sum_i \omega_i (\sum_k g^{ki} a_k)).$$

As V_g is one-to-one and $\{\omega_i\}_i$ is a basis of the free module $\mathcal{E}$, we get

$$\sum_k g^{ki} a_k = 0 \ \forall \ i.$$

Multiplying by g_{ij} and summing over i yields $a_j = 0$. This proves that $\{\omega_i^*\}_i$ is a basis of $V_g(\mathcal{E})$ and finishes the proof. □

Remark 4.3.8. *Let us note that for all $e \in \mathcal{E}$, the following equation holds:*

$$e = \sum_i \omega_i \omega_i^*(e\ ). \tag{4.3.2}$$

For the next proposition, we will need the notion of a left dual of an object in a monoidal category as defined in Definition 1.1.9. We observe that if g is a pseudo-Riemannian metric **billinear** metric on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$, then $\mathcal{E}$ is self-dual.

Proposition 4.3.9. *Suppose g is a pseudo-Riemannian $\mathcal{A}$-**bilinear** pseudo-Riemannian metric on a finite bicovariant $\mathcal{A}$-bimodule. Let $\widetilde{\mathcal{E}}$ denote the left dual of the object $\mathcal{E}$ in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$. Then $\widetilde{\mathcal{E}}$ is isomorphic to $\mathcal{E}$ as objects in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ via the morphism V_g.*

Proof. Let $\{\omega_i\}_i$ be a basis of ${}_0\mathcal{E}$. It is well-known that $\widetilde{\mathcal{E}}$ and $\mathcal{E}^*$ are isomorphic objects in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$. This follows by using the bicovariant $\mathcal{A}$-bilinear maps

$$\mathrm{ev} : \mathcal{E}^* \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}; \quad \phi \otimes_{\mathcal{A}} e \mapsto \phi(e), \ \mathrm{coev} : \mathcal{A} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^*; \quad 1 \mapsto \sum_i \omega_i \otimes_{\mathcal{A}} \omega_i^*$$

We define $\mathrm{ev}_g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ and $\mathrm{coev}_g : \mathcal{A} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ by the following formulas:

$$\mathrm{ev}_g(e \otimes_{\mathcal{A}} f) = g(e \otimes_{\mathcal{A}} f), \quad \mathrm{coev}_g(1) = \sum_i \omega_i \otimes_{\mathcal{A}} V_g^{-1}(\omega_i^*).$$

Then since g is both left and right $\mathcal{A}$-linear, ev_g and coev_g are $\mathcal{A}$-bimodule maps. The bicovariance of g implies the bicovariance of ev_g while the bicovariance of $\mathrm{coev}_g = (\mathrm{id} \otimes_{\mathcal{A}} V_g^{-1}) \circ \mathrm{coev}$ follows from the bicovariance of V_g and coev.

Since the left dual of an object is unique upto isomorphism, we need to check the following identities for all e in $\mathcal{E}$:

$$(\mathrm{ev}_g \otimes_{\mathcal{A}} \mathrm{id})(\mathrm{id} \otimes_{\mathcal{A}} \mathrm{coev}_g)(e) = e, \ (\mathrm{id} \otimes_{\mathcal{A}} \mathrm{ev}_g)(\mathrm{coev}_g \otimes_{\mathcal{A}} \mathrm{id})(e) = e.$$

But these follow by a simple computation using the fact that ${}_0\mathcal{E}$ is right $\mathcal{A}$-total in $\mathcal{E}$ and the identity (4.3.2).

From the above discussion, we have that $\mathcal{E}$ and $\mathcal{E}^*$ are two left duals of the object $\mathcal{E}$ in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$. Then by Proposition 1.1.10, we know that $(\mathrm{ev}_g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}^*})(\mathrm{id}_{\mathcal{E}} \otimes_{\mathcal{A}} \mathrm{coev})$ is an isomorphism from $\mathcal{E}$ to $\mathcal{E}^*$. But it can be easily checked that $(\mathrm{ev}_g \otimes_{\mathcal{A}} \mathrm{id}_{\mathcal{E}^*})(\mathrm{id}_{\mathcal{E}} \otimes_{\mathcal{A}} \mathrm{coev}) = V_g$. This completes the proof. ☐

Now we state a result on bi-invariant pseudo-Riemannian metric.

Proposition 4.3.10. *Let g be a pseudo-Riemannian metric on $\mathcal{E}$ and the symbols $\{\omega_i\}_i$, $\{g_{ij}\}_{ij}$ be as above. If*

$$_{\mathcal{E}}\Delta(\omega_i) = \sum_j \omega_j \otimes_{\mathbb{C}} R_{ji} \tag{4.3.3}$$

(see (1.2.4)), then g is bi-invariant if and only if the elements g_{ij} are scalar and

$$g_{ij} = \sum_{kl} g_{kl} R_{ki} R_{lj}. \tag{4.3.4}$$

Proof. Since g is left-invariant, the elements g_{ij} are in $\mathbb{C}.1$. Moreover, the right-invariance of g implies that g is right-covariant (Proposition 4.3.3), i.e.

$$\begin{aligned} 1 \otimes_{\mathbb{C}} g_{ij} = \Delta(g_{ij}) &= (g \otimes_{\mathcal{A}} \mathrm{id})_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega_i \otimes_{\mathbb{C}} \omega_j) \\ =&(g \otimes_{\mathcal{A}} \mathrm{id})(\sum_{kl} \omega_k \otimes_{\mathcal{A}} \omega_l \otimes_{\mathbb{C}} R_{ki} R_{lj}) = 1 \otimes_{\mathbb{C}} \sum_{kl} g_{kl} R_{ki} R_{lj}, \end{aligned}$$

so that

$$g_{ij} = \sum_{kl} g_{kl} R_{ki} R_{lj}. \tag{4.3.5}$$

Conversely, if $g_{ij} = g(\omega_i \otimes_{\mathcal{A}} \omega_j)$ are scalars and (4.3.4) is satisfied, then g is left-invariant and right-covariant. By Proposition 4.3.3, g is right-invariant. $\square$

Next we compare our definition of pseudo-Riemannian metrics with some of the other definitions available in the literature.

Proposition 4.3.6 shows that our notion of pseudo-Riemannian metric coincides with the right $\mathcal{A}$-linear version of a “symmetric metric” introduced in Definition 2.1 of [51] if we impose the condition of left-invariance.

Next, we compare our definition with the one used by Beggs and Majid in Proposition 4.2 of [70] (also see [11] and references therein). To that end, we briefly recall the construction of the two forms as in Subsection 1.3.2.

If $\mathcal{E}$ is a bicovariant $\mathcal{A}$-bimodule and σ be the map as in Proposition 1.3.17, Woronowicz defined the space of two forms as:

$$\Omega^2(\mathcal{A}) := (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) / \mathrm{Ker}(\sigma - 1).$$

The symbol $\wedge$ will denote the quotient map

$$\wedge : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \Omega^2(\mathcal{A}).$$

Thus,

$$\mathrm{Ker}(\wedge) = \mathrm{Ker}(\sigma - 1).$$

In Proposition 4.2 of [70], the authors define a metric on a bimodule $\mathcal{E}$ over a (possibly) noncommutative algebra $\mathcal{A}$ as an element h of $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ such that $\wedge(h) = 0$. We claim that metrics in the sense of Beggs and Majid are in one to one correspondence with elements g in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ (not necessarily left-invariant) such that $g \circ \sigma = g$. Thus, modulo the nondegeneracy condition (ii) of Definition 4.3.1, our notion of pseudo-Riemannian metric matches with the definition of metric by Beggs and Majid.

Indeed, if g is in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ as above and $\{\omega_i\}_i$, is a basis of ${}_0\mathcal{E}$, then the equation $g \circ \sigma = g$ implies that

$$g \circ \sigma(\omega_i \otimes_{\mathcal{A}} \omega_j) = g(\omega_i \otimes_{\mathcal{A}} \omega_j).$$

Moreover, since σ is a bicovariant bimodule map, by Proposition 4.1.14, $\sigma({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})$ is contained in ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. Hence, by Proposition 4.1.12 we know that

$$\sigma(\omega_i \otimes_{\mathcal{A}} \omega_j) = \sum_{k,l} \sigma_{ij}^{kl} \omega_k \otimes_{\mathcal{A}} \omega_l$$

for some scalars σ_{ij}^{kl}. Therefore, we have

$$\sum_{k,l} \sigma_{ij}^{kl} g(\omega_k \otimes_{\mathcal{A}} \omega_l) = g(\omega_i \otimes_{\mathcal{A}} \omega_j). \tag{4.3.6}$$

We claim that the element $h = \sum_{i,j} g(\omega_i \otimes_{\mathcal{A}} \omega_j) \omega_i \otimes_{\mathcal{A}} \omega_j$ satisfies $\wedge(h) = 0$. Indeed, by virtue of (1.3.9), it is enough to prove that $(\sigma - 1)(h) = 0$. But this directly follows from (4.3.6) using the left $\mathcal{A}$-linearity of σ.

This argument is reversible and hence starting from h in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ satisfying $\wedge(h) = 0$, we get an element g in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}, \mathcal{A})$ such that for all i, j,

$$g \circ \sigma(\omega_i \otimes_{\mathcal{A}} \omega_j) = g(\omega_i \otimes_{\mathcal{A}} \omega_j).$$

Since $\{\omega_i \otimes_{\mathcal{A}} \omega_j\}_{ij}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ (Corollary 4.1.8) and the maps g, σ are right $\mathcal{A}$-linear, we get that $g \circ \sigma = g$. This proves our claim. Let us note that since we did not assume g to be left invariant, the quantities $g(\omega_i \otimes_{\mathcal{A}} \omega_j)$ need not be scalars. However, the proof goes through since the elements σ^{ij}_{kl} are scalars.

4.3.1 The $g^{(2)}$-adjoint of a left-covariant map

Suppose $\mathcal{E}$ is a bicovariant bimodule and g a pseudo-Riemannian metric. Then following the lines of Definition 2.3.4, it is straightforward to define (Definition 4.3.11) a complex valued map $g^{(2)}$ on ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. The goal of this subsection is to show that any complex linear map from ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ to itself admits an adjoint with respect to $g^{(2)}$. Moreover, in Lemma 4.3.13 and Proposition 4.3.14, we show that the maps ${}_0\sigma$ and ${}_0(P_{\mathrm{sym}})$ are actually self-adjoint. These results will be used in Lemma 4.5.4 and Theorem 4.5.9 for deriving a sufficient condition for the existence of a Levi-Civita connection.

Let $\mathcal{E}$ be a bicovariant bimodule over $\mathcal{A}$ and $\{\omega_i\}_i$ a basis of ${}_0\mathcal{E}$. Then the set $\{\omega_i \otimes_{\mathbb{C}} \omega_j\}_{ij}$ is a basis for the finite dimensional vector space ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. Thus, we are allowed to make the following definition.

Definition 4.3.11. *Suppose g is a left-covariant pseudo-Riemannian metric on $\mathcal{E}$. We define a map*

$$g^{(2)} : ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \to \mathbb{C} \text{ by the formula}$$

$$g^{(2)}((\omega_1 \otimes_{\mathbb{C}} \omega_2) \otimes_{\mathbb{C}} (\omega_3 \otimes_{\mathbb{C}} \omega_4)) = g(\omega_1 \otimes_{\mathcal{A}} g(\omega_2 \otimes_{\mathcal{A}} \omega_3) \otimes_{\mathcal{A}} \omega_4)$$

for all $\omega_1, \omega_2, \omega_3, \omega_4 \in {}_0\mathcal{E}$.

We also define a map $V_{g^{(2)}} : ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \to ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})^* := \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, \mathbb{C})$ by the formula

$$V_{g^{(2)}}(\omega_1 \otimes_{\mathbb{C}} \omega_2)(\omega_3 \otimes_{\mathbb{C}} \omega_4) = g^{(2)}((\omega_1 \otimes_{\mathbb{C}} \omega_2) \otimes_{\mathbb{C}} (\omega_3 \otimes_{\mathbb{C}} \omega_4)).$$

Since, by the second assertion of Lemma 4.3.4, $g(\omega_1 \otimes_{\mathcal{A}} \omega_2)$ belongs to $\mathbb{C}$, it is clear that the element $g^{(2)}((\omega_1 \otimes_{\mathcal{A}} \omega_2) \otimes_{\mathcal{A}} (\omega_3 \otimes_{\mathbb{C}} \omega_4))$ indeed belongs to $\mathbb{C}$.

Let us note that the maps $g^{(2)}$ and $V_{g^{(2)}}$ are both right $\mathcal{A}$-linear. The following non-degeneracy property is going to be crucial in the sequel.

Proposition 4.3.12. *Let Y be an element of ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$. If $g^{(2)}(X\otimes_{\mathbb{C}}Y)=0$ for all X in ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$, then $Y=0$. Similarly, if $g^{(2)}(Y\otimes_{\mathbb{C}}X)=0$ for all X in ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$, then $Y=0$. In particular, the map $V_{g^{(2)}}$ defined in Definition 4.3.11 is a vector space isomorphism from ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$ to $({}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E})^*$.*

Proof. Let $\{\omega_i\}_i$ be a basis for ${}_0\mathcal{E}$ so that $\{\omega_i\otimes_{\mathbb{C}}\omega_j\}_{ij}$ is a basis for ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$. By Proposition 4.3.6, the matrix whose i,j-th element is $g_{ij}=g(\omega_i\otimes_{\mathbb{C}}\omega_j)$ is invertible in $M_n(\mathbb{C})$. We will denote by g^{ij} the i,j-th entry of the inverse of the matrix $((g_{ij}))_{ij}$.

Suppose $\{b_{ij}\}_{ij}$ are complex numbers such that

$$Y=\sum_{ij}\omega_i\otimes_{\mathbb{C}}\omega_j b_{ij}.$$

Let us fix the indices i_0,j_0 and define

$$X=\sum_{kl}g^{i_0l}g^{j_0k}\omega_k\otimes_{\mathbb{C}}\omega_l.$$

Then we get

$$\begin{aligned}0=g^{(2)}(X\otimes_{\mathbb{C}}Y)&=g^{(2)}(\sum_{ijkl}g^{i_0l}g^{j_0k}(\omega_k\otimes_{\mathbb{C}}\omega_l)\otimes_{\mathbb{C}}(\omega_i\otimes_{\mathbb{C}}\omega_j)b_{ij})\\ =\sum_{ijkl}g^{i_0l}g^{j_0k}g(\omega_k\otimes_{\mathcal{A}}g_{li}\omega_j)b_{ij}&=\sum_{ijkl}g^{i_0l}g_{li}g^{j_0k}g_{kj}b_{ij}=\sum_{ij}\delta_{i_0i}\delta_{j_0j}b_{ij}=b_{i_0j_0}.\end{aligned}$$

Hence, if $g^{(2)}(X\otimes_{\mathbb{C}}Y)=0$ for all X, then $Y=0$.
To prove the second statement, fix indices i_0,j_0 and define $X=\sum_{kl}g^{li_0}g^{kj_0}\omega_k\otimes_{\mathbb{C}}\omega_l$. Then, we compute the following.

$$\begin{aligned}g^{(2)}(Y\otimes_{\mathbb{C}}X)=&g^{(2)}(\sum_{ijkl}(\omega_i\otimes_{\mathbb{C}}\omega_j b_{ij})\otimes_{\mathbb{C}}(\omega_k\otimes_{\mathbb{C}}\omega_l g^{li_0}g^{kj_0}))\\ &=\sum_{ijkl}g_{il}g^{li_0}g_{jk}g^{kj_0}b_{ij}=\sum_{ij}\delta_{i_0i}\delta_{j_0j}b_{ij}=b_{i_0j_0}.\end{aligned}$$

Hence, if $g^{(2)}(Y\otimes_{\mathbb{C}}X)=0$ for all X, then $Y=0$. $\square$

Before stating the next lemma, we note that the $g^{(2)}$-adjoint of the maps ${}_0\sigma$ and ${}_0(P_{\rm sym})$ make sense. Indeed, ${}_0\sigma$ and ${}_0(P_{\rm sym})$ are linear maps from the complex vector space ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$ to itself. By virtue of Proposition 4.3.12, we can apply Lemma 1.1.7 to $h=g^{(2)}$ and $T={}_0\sigma$ or ${}_0(P_{\rm sym})$. Thus, $({}_0\sigma)^*$ and $({}_0(P_{\rm sym}))^*$ exist.

Lemma 4.3.13. *Let $\mathcal{E}$ be a bicovariant $\mathcal{A}$-bimodule, σ the braiding map of Proposition 1.3.17 and g be a bi-invariant pseudo-Riemannian metric on $\mathcal{E}$, then $({}_0\sigma)^* = {}_0\sigma$.*

Proof. We will actually prove a stronger statement. Since $g^{(2)}$ is a map from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})$ to $\mathbb{C}$, it extends uniquely to a right $\mathcal{A}$-linear left-covariant map (to be denoted by $g^{(2)}$ again, by an abuse of notation) from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}) \otimes_{\mathcal{A}} (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$ to $\mathcal{A}$ by Proposition 4.1.16. We will prove that for e, f, e', f' in $\mathcal{E}$,

$$g^{(2)}(\sigma(e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} (e' \otimes_{\mathcal{A}} f')) = g^{(2)}((e \otimes_{\mathcal{A}} f) \otimes_{\mathcal{A}} \sigma(e' \otimes_{\mathcal{A}} f')). \tag{4.3.7}$$

To this end, we claim that it is enough to prove that for all ω, ω' in ${}_0\mathcal{E}$ and η, η' in $\mathcal{E}_0$,

$$g^{(2)}(\sigma(\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) = g^{(2)}((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} \sigma(\omega' \otimes_{\mathcal{A}} \eta')). \tag{4.3.8}$$

Indeed, by Corollary 4.1.8, for every element a in $\mathcal{A}$, there exist elements x_i in ${}_0\mathcal{E}$, y_i in $\mathcal{E}_0$ and a_i in $\mathcal{A}$ such that

$$a(\omega' \otimes_{\mathcal{A}} \eta') = \sum_i x_i \otimes_{\mathcal{A}} y_i a_i.$$

Hence, if (4.3.8) is true, the right $\mathcal{A}$-linearity of the map $g^{(2)}$ implies that

$$\begin{aligned}
&g^{(2)}(\sigma(\omega \otimes_{\mathcal{A}} \eta a) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta' b)) = g^{(2)}(\sigma(\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} a(\omega' \otimes_{\mathcal{A}} \eta'))b \\
=&\sum_i g^{(2)}(\sigma(\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} (x_i \otimes_{\mathcal{A}} y_i))a_i b = \sum_i g^{(2)}((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} \sigma(x_i \otimes_{\mathcal{A}} y_i))a_i b \\
=&\sum_i g^{(2)}((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} \sigma(x_i \otimes_{\mathcal{A}} y_i a_i))b = g^{(2)}((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} a\sigma(\omega' \otimes_{\mathcal{A}} \eta'))b \\
&= g^{(2)}((\omega \otimes_{\mathcal{A}} \eta a) \otimes_{\mathcal{A}} \sigma(\omega' \otimes_{\mathcal{A}} \eta' b)).
\end{aligned}$$

Here we have used the bilinearity of the map σ. Since ${}_0\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}_0$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ (by Corollary 4.1.8), this proves (4.3.7) provided we prove (4.3.8). This proves our claim.

Thus, we are left with proving (4.3.8) which follows from the following computation:

$$\begin{aligned}
&g^{(2)}(\sigma(\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) = g^{(2)}((\eta \otimes_{\mathcal{A}} \omega) \otimes_{\mathcal{A}} (\omega' \otimes_{\mathcal{A}} \eta')) \\
=&g(\eta \otimes_{\mathcal{A}} \eta')g(\omega \otimes_{\mathcal{A}} \omega') = g^{(2)}((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} (\eta' \otimes_{\mathcal{A}} \omega')) \\
=&g^{(2)}((\omega \otimes_{\mathcal{A}} \eta) \otimes_{\mathcal{A}} \sigma(\omega' \otimes_{\mathcal{A}} \eta')),
\end{aligned}$$

where we have used $\sigma(\omega \otimes_{\mathcal{A}} \eta) = \eta \otimes_{\mathcal{A}} \omega$ (see (1.3.7)) twice and the facts that $g(\omega \otimes_{\mathcal{A}} \omega')$ and $g(\eta \otimes_{\mathcal{A}} \eta')$ take values in $\mathbb{C}.1$ (second assertion of Lemma 4.3.4). This completes the proof of the lemma. $\square$

Proposition 4.3.14. *We have $({}_0(P_{\text{sym}}))^* = {}_0(P_{\text{sym}})$. Moreover, if $V_{g^{(2)}} : {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} \to ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})^*$ is the map defined in Definition 4.3.11, then*

$$V_{g^{(2)}}({}_0(P_{\text{sym}})(X))(Y) = V_{g^{(2)}}(X) \circ {}_0(P_{\text{sym}})(Y) \ \forall X, Y \in {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}. \tag{4.3.9}$$

In particular, $V_{g^{(2)}}$ is a vector space isomorphism from ${}_0\mathcal{E} \otimes_{\mathbb{C}}^{\text{sym}} {}_0\mathcal{E}$ onto $({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\text{sym}} {}_0\mathcal{E})^*$.

Proof. Since $({}_0\sigma)^* = {}_0\sigma$ by Lemma 4.3.13 and ${}_0(P_{\text{sym}})$ is a polynomial in ${}_0\sigma$ by (4.2.6), we have $({}_0(P_{\text{sym}}))^* = {}_0(P_{\text{sym}})$. Then (4.3.9) follows from the definition of $V_{g^{(2)}}$. Finally, for the last assertion, let us recall the identification

$$({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\text{sym}} {}_0\mathcal{E})^* = \{\phi \in ({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})^* : \phi(X) = \phi({}_0(P_{\text{sym}})(X)) \ \forall X \in {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}\} \tag{4.3.10}$$

from (4.2.9). Now, if X is in ${}_0\mathcal{E} \otimes_{\mathbb{C}}^{\text{sym}} {}_0\mathcal{E} = \text{Ran}({}_0(P_{\text{sym}}))$, then for all Y in ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$, we have

$$V_{g^{(2)}}(X)(Y) = V_{g^{(2)}}({}_0(P_{\text{sym}})(X))(Y) = V_{g^{(2)}}(X)({}_0(P_{\text{sym}})(Y)).$$

by (4.3.9). Therefore, $V_{g^{(2)}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\text{sym}} {}_0\mathcal{E})$ is a subspace of $({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\text{sym}} {}_0\mathcal{E})^*$ by (4.3.10). Now by Proposition 4.3.12, the map $V_{g^{(2)}}$ is one-to-one and so we reach our our desired conclusion by a dimension argument. $\square$

4.4 Bicovariant connections and metric compatibility

Recall that in Definition 1.4.10 and Definition 1.4.11, we had already defined connections on a first order differential calculus and their torsion. In this section, we define covariant connections on bicovariant differential calculi. As a consequence of the assumption of diagonalisability of ${}_0\sigma$ made in Section 4.2, in Subsection 4.4.1 we construct a canonical torsionless connection on a bicovariant differential calculus. In Subsection 4.4.2, we introduce the notion of compatibility of a left covariant connection with a bi-invariant pseudo-Riemannian metric. In that section we also make a comparison of our notion of metric compatibility with that of [51] in a limited setting.

Definition 4.4.1. *([51]) Let $(\mathcal{E}, d)$ be a bicovariant differential calculus on $\mathcal{A}$. A (right) connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ such that, for all a in $\mathcal{A}$ and ρ in $\mathcal{E}$, the following equation holds:*

$$\nabla(\rho a) = \nabla(\rho)a + \rho \otimes_{\mathcal{A}} da.$$

The connection ∇ is said to be left (right) covariant if it is a left (right) covariant linear map from $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. It is called a bicovariant connection if it is bicovariant as a linear map.

Lemma 4.4.2. *([51]) If ∇ is a left-covariant connection on a bicovariant differential calculus $(\mathcal{E}, d)$, then $\nabla({}_0\mathcal{E}) \subseteq {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$.*

Proof. This follows by combining Proposition 4.1.17 and Proposition 4.1.12. □

Our notion of torsion T_∇ (see Definition 1.4.11) of a connection is the same as that of [51], with the only difference being that they work with left connections.

The following result which will be needed in the proof of Proposition 4.5.3.

Lemma 4.4.3. *If ∇_1 and ∇_2 are two left-covariant torsionless connections on a bicovariant differential calculus $(\mathcal{E}, d)$ on $\mathcal{A}$, then $\nabla_1 - \nabla_2$ is an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$.*

Proof. If ∇_1 and ∇_2 are two torsionless connections, we have that $\wedge \circ \nabla_1 = -d = \wedge \circ \nabla_2$. Therefore,

$$\mathrm{Ran}(\nabla_1 - \nabla_2) \subseteq \mathrm{Ker}(\wedge) = \mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E}.$$

Moreover, by Lemma 4.4.2, if ω is an element of ${}_0\mathcal{E}$, then $(\nabla_1 - \nabla_2)(\omega)$ is in ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$, i.e, $(\nabla_1 - \nabla_2)(\omega)$ is invariant under $\Delta_{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$. Hence, by (4.2.5), $(\nabla_1 - \nabla_2)(\omega)$ is an element of ${}_0(\mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E}) = {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}$. □

4.4.1 A canonical bicovariant torsionless connection

In this subsection, we prove, by construction, the existence of a bicovariant torsionless connection on any bicovariant differential calculus which satisfies the condition that ${}_0\sigma$ is diagonalisable. Indeed, we will be using the map $Q = \wedge|_{\mathcal{F}} : \mathcal{F} \to \Omega^2(\mathcal{A})$ (Definition 4.2.9) which makes sense due to the splitting $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = (\mathcal{E} \otimes_{\mathcal{A}}^{\mathrm{sym}} \mathcal{E}) \oplus \mathcal{F}$ (Theorem 4.2.5) which in turn follows from the assumption of diagonalisability of the map ${}_0\sigma$. Let us recall that Q is a bimodule isomorphism from $\mathcal{F}$ to $\Omega^2(\mathcal{A})$.

Theorem 4.4.4. *Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus on $\mathcal{A}$ such that ${}_0\sigma$ is diagonalisable. Then $\mathcal{E}$ admits a bicovariant torsionless connection.*

Proof. The proof of existence of a torsionless connection ∇_0 follows exactly along the lines of Theorem 2.2.3 of Chapter 2. The only difference here is that we need to define ∇_0 in such a way that it remains bicovariant.

We define $\widetilde{\nabla}_0 : {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ by

$$\widetilde{\nabla}_0(\omega) = Q^{-1}(-d(\omega)).$$

Indeed, by Corollary 4.2.10 and (4.2.8), $\widetilde{\nabla}_0(\omega)$ is an element of ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ for all ω in ${}_0\mathcal{E}$. Let $\{\omega_i\}_i$ be a vector space basis of ${}_0\mathcal{E}$. By the right $\mathcal{A}$-totality of ${}_0\mathcal{E}$ in $\mathcal{E}$, we extend $\widetilde{\nabla}_0$ to a map $\nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ by the formula

$$\nabla_0(\sum_i \omega_i a_i) = \sum_i \widetilde{\nabla}_0(\omega_i) a_i + \sum_i \omega_i \otimes_{\mathcal{A}} da_i.$$

Since $\mathcal{E}$ is a free module with basis $\{\omega_i\}_i$, the above formula is well-defined. It follows that for all ω in ${}_0\mathcal{E}$ and for all a in $\mathcal{A}$,

$$\nabla_0(\omega a) = \widetilde{\nabla}_0(\omega) a + \omega \otimes_{\mathcal{A}} da.$$

Then, to verify that ∇_0 is a connection we compute the following for ω in ${}_0\mathcal{E}$ and a, b in $\mathcal{A}$.

$$\begin{aligned}\nabla_0(\omega ab) =& \widetilde{\nabla}_0(\omega)ab + \omega \otimes_{\mathcal{A}} d(ab) = \widetilde{\nabla}(\omega)ab + \omega \otimes_{\mathcal{A}} da.b + \omega \otimes_{\mathcal{A}} adb \\ =& (\widetilde{\nabla}(\omega)a + \omega \otimes_{\mathcal{A}} da)b + \omega a \otimes_{\mathcal{A}} db = \nabla_0(\omega a)b + \omega a \otimes_{\mathcal{A}} db\end{aligned}$$

Now we prove that ∇_0 is torsionless. Indeed, since by Definition 4.2.9, we have $\wedge \circ Q^{-1} = \mathrm{id}_{\Omega^2(\mathcal{A})}$, we can deduce that

$$\begin{aligned}\wedge \circ \nabla_0(\omega a) =& \wedge \circ (\widetilde{\nabla}_0(\omega)a + \omega \otimes_{\mathcal{A}} da) = \wedge \circ Q^{-1}(-d(\omega))a + \omega \wedge da \\ =& -d(\omega)a + \omega \wedge da = -d(\omega a).\end{aligned}$$

Before proceeding further, let us note that since ∇_0 coincides with $\widetilde{\nabla}_0$ on ${}_0\mathcal{E}$ and $\widetilde{\nabla}_0(\omega)$ belongs to ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ if ω is in ${}_0\mathcal{E}$, $\nabla_0(\omega)$ is in ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. We will use this fact in the rest of the proof where ω and a will stand for arbitrary elements of ${}_0\mathcal{E}$ and $\mathcal{A}$ respectively.

To show that ∇_0 is left-covariant, we observe that since $\nabla_0(\omega)$ is in ${}_0\mathcal{E}\otimes_{\mathbb{C}0}\mathcal{E}$, $\Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\nabla_0(\omega)) = 1\otimes_{\mathbb{C}}\nabla_0(\omega)$. Using this, we get

$$\begin{aligned}
&(\mathrm{id}\otimes_{\mathbb{C}}\nabla_0)(\Delta_{\mathcal{E}}(\omega a)) = (\mathrm{id}\otimes_{\mathbb{C}}\nabla_0)(\Delta_{\mathcal{E}}(\omega)\Delta(a))\\
=&(\mathrm{id}\otimes_{\mathbb{C}}\nabla_0)((1\otimes_{\mathbb{C}}\omega)(a_{(1)}\otimes_{\mathbb{C}}a_{(2)})) = a_{(1)}\otimes_{\mathbb{C}}\nabla_0(\omega a_{(2)})\\
=&a_{(1)}\otimes_{\mathbb{C}}(\nabla_0(\omega)a_{(2)} + \omega\otimes_{\mathcal{A}} da_{(2)}) = (1\otimes_{\mathbb{C}}\nabla_0(\omega))(a_{(1)}\otimes_{\mathbb{C}}a_{(2)}) + a_{(1)}\otimes_{\mathbb{C}}\omega\otimes_{\mathcal{A}} da_{(2)}\\
=&(1\otimes_{\mathbb{C}}\nabla_0(\omega))(a_{(1)}\otimes_{\mathbb{C}}a_{(2)}) + (da)_{(-1)}\otimes_{\mathbb{C}}\omega\otimes_{\mathcal{A}}(da)_{(0)}\ \big(\text{by part (i) of Lemma 1.3.16}\big)\\
=&\Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\nabla_0(\omega))\Delta(a) + \Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\omega\otimes_{\mathcal{A}} da) = \Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\nabla_0(\omega)a + \omega\otimes_{\mathcal{A}} da)\\
=&\Delta_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(\nabla_0(\omega a)).
\end{aligned}$$

Finally, we show that ∇_0 is also right-covariant. Let ω and a continue to denote elements of ${}_0\mathcal{E}$ and $\mathcal{A}$ respectively. Since $\mathcal{E}$ is a bicovariant bimodule, ${}_{\mathcal{E}}\Delta(\omega) = \omega_{(0)}\otimes_{\mathbb{C}}\omega_{(1)}$ belongs to ${}_0\mathcal{E}\otimes_{\mathbb{C}}\mathcal{A}$ by Lemma 1.2.12. Hence $\omega_{(0)}$ belongs to ${}_0\mathcal{E}$ and we are allowed to write

$$\nabla_0(\omega_{(0)}a_{(1)}) = Q^{-1}(-d(\omega_{(0)}))a_{(1)} + \omega_{(0)}\otimes_{\mathbb{C}}d(a_{(1)}).$$

Thus, we obtain

$$\begin{aligned}
&(\nabla_0\otimes_{\mathbb{C}}\mathrm{id})_{\mathcal{E}}\Delta(\omega a) = (\nabla_0\otimes_{\mathbb{C}}\mathrm{id})(\omega_{(0)}a_{(1)}\otimes_{\mathbb{C}}\omega_{(1)}a_{(2)})\\
=&\nabla_0(\omega_{(0)}a_{(1)})\otimes_{\mathbb{C}}\omega_{(1)}a_{(2)} = (Q^{-1}(-d(\omega_{(0)}))a_{(1)} + \omega_{(0)}\otimes_{\mathcal{A}} d(a_{(1)}))\otimes_{\mathbb{C}}\omega_{(1)}a_{(2)}\\
=&(Q^{-1}\otimes_{\mathbb{C}}\mathrm{id})\big(((-d)\otimes_{\mathbb{C}}\mathrm{id})(\omega_{(0)}\otimes_{\mathbb{C}}\omega_{(1)})\big)(a_{(1)}\otimes_{\mathbb{C}}a_{(2)}) + \omega_{(0)}\otimes_{\mathcal{A}} d(a_{(1)})\otimes_{\mathbb{C}}\omega_{(1)}a_{(2)}\\
=&(Q^{-1}\otimes_{\mathbb{C}}\mathrm{id})\big(((-d)\otimes_{\mathbb{C}}\mathrm{id})(\omega_{(0)}\otimes_{\mathbb{C}}\omega_{(1)})\big)(a_{(1)}\otimes_{\mathbb{C}}a_{(2)}) + \omega_{(0)}\otimes_{\mathcal{A}}(da)_{(0)}\otimes_{\mathbb{C}}\omega_{(1)}(da)_{(1)}\\
&\big(\text{by part (ii) of Lemma 1.3.16}\big)\\
=&(Q^{-1}\otimes_{\mathbb{C}}\mathrm{id})\big(((-d)\otimes_{\mathbb{C}}\mathrm{id})({}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega))\big)(\Delta(a)) + {}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega\otimes_{\mathcal{A}} da)\\
=&(Q^{-1}\otimes_{\mathbb{C}}\mathrm{id})\big({}_{\Omega^2(\mathcal{A})}\Delta(-d(\omega))\big)\Delta(a) + {}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega\otimes_{\mathcal{A}} da)\\
&\big(\text{since d is a bicovariant map from } \mathcal{E} \text{ to } \Omega^2(\mathcal{A}) \text{ by Proposition 1.3.20}\big)\\
=&{}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(Q^{-1}(-d(\omega)))\Delta(a) + {}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega\otimes_{\mathcal{A}} da)\\
&\big(\text{since Q is right covariant by Corollary 4.2.10}\big)\\
=&{}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\nabla_0(\omega))\Delta(a) + {}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega\otimes_{\mathcal{A}} da)\\
=&{}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\nabla_0(\omega)a + \omega\otimes_{\mathcal{A}} da)\\
=&{}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\nabla_0(\omega a)).
\end{aligned}$$

This finishes the proof. $\square$

4.4.2 Metric Compatibility of a bicovariant connection

In this subsection, we define the notion of metric-compatibility of a left-covariant connection with a left-invariant pseudo-Riemannian metric. We will need the map ${}_0(P_{\mathrm{sym}})$ introduced in Definition 4.2.6. Our definition coincides with that in the classical case (Proposition 4.4.8) and also with that in [51] for cocycle deformations of classical Lie groups. The latter statement is derived in Chapter 5.

Definition 4.4.5. *Let ∇ be a left-covariant connection on a bicovariant calculus $(\mathcal{E}, d)$ and g a left-invariant pseudo-Riemannian metric. Then we define*

$$\widetilde{\Pi^0_g}(\nabla) : {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} \to {}_0\mathcal{E} \text{ by the following formula :}$$

$$\widetilde{\Pi^0_g}(\nabla)(\omega_i \otimes_{\mathbb{C}} \omega_j) = 2(\mathrm{id} \otimes_{\mathbb{C}} g)(\sigma \otimes_{\mathbb{C}} \mathrm{id})(\nabla \otimes_{\mathbb{C}} \mathrm{id})_0(P_{\mathrm{sym}})(\omega_i \otimes_{\mathbb{C}} \omega_j). \tag{4.4.1}$$

Next, for all ω_1, ω_2 in ${}_0\mathcal{E}$ and a in $\mathcal{A}$, we define $\widetilde{\Pi_g}(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ by

$$\widetilde{\Pi_g}(\nabla) \circ \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}(\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} a) = \widetilde{\Pi^0_g}(\nabla)(\omega_1 \otimes_{\mathbb{C}} \omega_2) a + g(\omega_1 \otimes_{\mathcal{A}} \omega_2) da.$$

It is easy to see that $\widetilde{\Pi^0_g}(\nabla)$ indeed maps ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ inside ${}_0\mathcal{E}$. Indeed, let ω_1, ω_2 be elements of ${}_0\mathcal{E}$. Since ${}_0(P_{\mathrm{sym}})$ is a map from ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ to itself, ${}_0(P_{\mathrm{sym}})(\omega_1 \otimes_{\mathbb{C}} \omega_2)$ is in ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. Then, by Lemma 4.4.2, $(\nabla \otimes_{\mathbb{C}} \mathrm{id})({}_0(P_{\mathrm{sym}}))(\omega_1 \otimes_{\mathbb{C}} \omega_2)$ is in ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. Since σ is left-covariant and g is left-invariant, Proposition 4.1.17 and the second assertion of Lemma 4.3.4 imply that the element
$(\mathrm{id} \otimes_{\mathbb{C}} g)(\sigma \otimes_{\mathbb{C}} \mathrm{id})(\nabla \otimes_{\mathbb{C}} \mathrm{id})({}_0(P_{\mathrm{sym}}))(\omega_1 \otimes_{\mathbb{C}} \omega_2)$ belongs to ${}_0\mathcal{E}$.

Finally, by Proposition 4.1.7 and the notation adopted in Proposition 4.1.12, the map $\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ from ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} \otimes_{\mathbb{C}} \mathcal{A}$ to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is an isomorphism, hence $\widetilde{\Pi_g}(\nabla)$ is well-defined.

Remark 4.4.6. *If ∇ is left-covariant and g is left-invariant, the above argument shows that*

$$\widetilde{\Pi_g}(\nabla)({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \subseteq {}_0\mathcal{E}$$

and thus by Proposition 4.1.17, the map $\widetilde{\Pi_g}(\nabla)$ is left-covariant.

For the rest of the chapter, dg will denote the map

$$dg : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}, \; dg(e \otimes_{\mathcal{A}} f) = d(g(e \otimes_{\mathcal{A}} f)).$$

Now we define the notion of metric compatibility of a bicovariant connection.

Definition 4.4.7. *Suppose $(\mathcal{E}, d)$ is a left-covariant differential calculus over $\mathcal{A}$ and g is a left-invariant pseudo-Riemannian metric. We say that a left-covariant connection ∇ on $\mathcal{E}$ is compatible with g if, as maps from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{E}$,*

$$\widetilde{\Pi_g}(\nabla) = dg.$$

We now show that our formulation of metric-compatibility of a connection coincides with that in the classical case of commutative Hopf algebras.

Proposition 4.4.8. *The above definition of metric compatibility coincides with that in the classical case.*

Proof. Let G be a linear algebraic group, $\mathcal{A}$ be its (commutative) Hopf algebra of regular functions and g be a left-invariant pseudo-Riemannian metric on the classical space of forms. In this case, the canonical braiding map σ is equal to the flip map, i.e., for all e, f in $\Omega^1(\mathcal{A})$,

$$\sigma(e \otimes_{\mathcal{A}} f) = \mathrm{flip}(e \otimes_{\mathcal{A}} f) = f \otimes_{\mathcal{A}} e.$$

Since $g \circ \sigma = g$, we have $g(e \otimes_{\mathcal{A}} f) = g(f \otimes_{\mathcal{A}} e)$. Moreover, the map P_{sym} is equal to $\frac{1}{2}(1 + \sigma)$. Let us recall (Proposition 1.4.8) that a connection ∇ on $\Omega^1(\mathcal{A})$ is compatible with g if and only if

$$(g \otimes_{\mathcal{A}} \mathrm{id})[\mathrm{flip}_{23}(\nabla(e) \otimes_{\mathcal{A}} e') + e \otimes_{\mathcal{A}} \nabla(e')] = dg(e \otimes_{\mathcal{A}} e'),$$

for all e, e' in $\Omega^1(\mathcal{A})$. The left hand side of the above equation can be written as

$$g_{13}\big(\nabla(e) \otimes_{\mathcal{A}} e' + \nabla(e') \otimes_{\mathcal{A}} e\big),$$

where $g_{13} = (\mathrm{id} \otimes_{\mathcal{A}} g)(\mathrm{flip} \otimes_{\mathcal{A}} \mathrm{id})$.

Let $\{e_i\}_i$ be a basis of left-invariant one-forms of $\Omega^1(\mathcal{A})$. If e, e' belong to $\Omega^1(\mathcal{A})$, then there exist elements a_i, b_j in $\mathcal{A}$ such that $e = \sum_i e_i a_i$ and $e' = \sum_j e_j b_j$. If ∇ is metric compatible in the sense of Definition 4.4.7, i.e, $\widetilde{\Pi_g}(\nabla) - dg = 0$, then using the Leibniz properties of ∇ and d

and the equation $g(e_i \otimes_{\mathcal{A}} e_j) = g(e_j \otimes_{\mathcal{A}} e_i)$, we get

$$
\begin{aligned}
&g_{13}(\nabla(e) \otimes_{\mathcal{A}} e' + \nabla(e') \otimes_{\mathcal{A}} e)\\
=&g_{13}(\nabla(\sum_i e_i a_i) \otimes_{\mathcal{A}} \sum_j e_j b_j + \nabla(\sum_j e_j b_j) \otimes_{\mathcal{A}} \sum_i e_i a_i)\\
=&(\mathrm{id} \otimes_{\mathcal{A}} g)(\mathrm{flip} \otimes_{\mathcal{A}} \mathrm{id})(\nabla(\sum_i e_i) a_i \otimes_{\mathcal{A}} \sum_j e_j b_j + \nabla(\sum_j e_j) b_j \otimes_{\mathcal{A}} \sum_i e_i a_i)\\
+&(\mathrm{id} \otimes_{\mathcal{A}} g)(\mathrm{flip} \otimes_{\mathcal{A}} \mathrm{id})(\sum_i e_i \otimes_{\mathcal{A}} da_i \otimes_{\mathcal{A}} \sum_j e_j b_j + \sum_j e_j \otimes_{\mathcal{A}} db_j \otimes_{\mathcal{A}} \sum_i e_i a_i))\\
=&\sum_{ij} \big((\mathrm{id} \otimes_{\mathbb{C}} g)(\mathrm{flip} \otimes_{\mathbb{C}} \mathrm{id})(\nabla \otimes_{\mathbb{C}} \mathrm{id})(e_i \otimes_{\mathbb{C}} e_j + e_j \otimes_{\mathbb{C}} e_i)\big) a_i b_j\\
+&\sum_{ij} \big(da_i g(e_i \otimes_{\mathcal{A}} e_j) b_j + db_j g(e_j \otimes_{\mathcal{A}} e_i) a_i\big)\\
=&\sum_{ij} \big((\mathrm{id} \otimes_{\mathbb{C}} g)(\mathrm{flip} \otimes_{\mathbb{C}} \mathrm{id})(\nabla \otimes_{\mathbb{C}} \mathrm{id})((1 + \mathrm{flip})(e_i \otimes_{\mathbb{C}} e_j))\big) a_i b_j + \sum_{ij} \big(g(e_i \otimes_{\mathcal{A}} e_j) d(a_i b_j)\big)\\
=&\sum_{ij} \big(\widetilde{\Pi^0_g}(\nabla)(e_i \otimes_{\mathbb{C}} e_j) a_i b_j + g(e_i \otimes_{\mathcal{A}} e_j) d(a_i b_j)\big)\\
=&\widetilde{\Pi_g}(\nabla)(\sum_{ij} e_i \otimes_{\mathcal{A}} e_j a_i b_j) = dg(\sum_{ij} e_i \otimes_{\mathcal{A}} e_j a_i b_j) = dg(e \otimes_{\mathcal{A}} e').
\end{aligned}
$$

This argument is reversible and thus, our definition of metric compatibility coincides with that in the classical case. □

It is also true that our definition of metric compatibility coincides with that of [51] for cocycle deformations of classical Lie groups. We state this result at the end of this section (Proposition 4.4.13) but the proof is postponed till Chapter 5.

4.4.3 Covariance properties of the map $\widetilde{\Pi_g}(\nabla)$

Let us now derive some covariance properties of the maps $\widetilde{\Pi^0_g}(\nabla)$ and $\widetilde{\Pi_g}(\nabla) - dg$ which will be used in Section 4.5.

Lemma 4.4.9. *If ∇ is a bicovariant connection on $\mathcal{E}$ and g is a bi-invariant pseudo-Riemannian metric, then $\widetilde{\Pi^0_g}(\nabla)$ is a right-covariant map.*

Proof. The maps σ and ${}_0(P_{\mathrm{sym}})$ are bicovariant (Proposition 4.2.7). Therefore, if ∇ is also right-covariant, and g is bi-invariant (and hence by the first assertion of Lemma 4.3.4 also bicovariant), then $\widetilde{\Pi^0_g}(\nabla)$ is a composition of right-covariant maps and therefore, right-covariant. □

Proposition 4.4.10. *If the connection ∇ is left-covariant and the pseudo-Riemannian metric g is left-invariant, then the map $\widetilde{\Pi_g}(\nabla) - dg : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ is a left-covariant right $\mathcal{A}$-linear map. Moreover, if ∇ is bicovariant and the pseudo-Riemannian metric g is bi-invariant, then $\widetilde{\Pi_g}(\nabla) - dg$ is also a bicovariant map.*

Proof. We start by proving that $\widetilde{\Pi_g}(\nabla) - dg$ is a right $\mathcal{A}$-linear. Since $\{\omega \otimes_{\mathcal{A}} \omega' : \omega, \omega' \in {}_0\mathcal{E}\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$, it suffices to show that for all $\omega_1, \omega_2 \in {}_0\mathcal{E}$ and $a, b \in \mathcal{A}$, we have:

$$(\widetilde{\Pi_g}(\nabla) - dg)((\omega_1 \otimes_{\mathcal{A}} \omega_2 a)b) = \big((\widetilde{\Pi_g}(\nabla) - dg)(\omega_1 \otimes_{\mathcal{A}} \omega_2 a)\big)b.$$

This follows from the following computation:

$$\begin{aligned}
&(\widetilde{\Pi_g}(\nabla) - dg)((\omega_1 \otimes_{\mathcal{A}} \omega_2 a)b) \\
=&\widetilde{\Pi_g}(\nabla)(\omega_1 \otimes_{\mathbb{C}} \omega_2 ab + g(\omega_1 \otimes_{\mathcal{A}} \omega_2)d(ab) - dg(\omega_1 \otimes_{\mathcal{A}} \omega_2 ab) \\
=&\widetilde{\Pi_g}(\nabla)(\omega_1 \otimes_{\mathbb{C}} \omega_2)ab + g(\omega_1 \otimes_{\mathcal{A}} \omega_2)(da.b + adb) - dg(\omega_1 \otimes_{\mathcal{A}} \omega_2 a)b - g(\omega_1 \otimes_{\mathcal{A}} \omega_2 a)db \\
=&(\widetilde{\Pi_g}(\nabla)(\omega_1 \otimes_{\mathbb{C}} \omega_2)a + g(\omega_1 \otimes_{\mathcal{A}} \omega_2)d(a) - dg(\omega_1 \otimes_{\mathcal{A}} \omega_2 a))b \\
=&\big((\widetilde{\Pi_g}(\nabla) - dg)(\omega_1 \otimes_{\mathcal{A}} \omega_2 a)\big)b.
\end{aligned}$$

Now, we prove that $\widetilde{\Pi_g}(\nabla) - dg$ is a left-covariant map. Since g is left-invariant, for any ω_1, ω_2 in ${}_0\mathcal{E}$, $g(\omega_1 \otimes_{\mathcal{A}} \omega_2) \in \mathbb{C}$ by the second assertion of Lemma 4.3.4, and so $dg(\omega_1 \otimes_{\mathcal{A}} \omega_2) = 0$. Hence,

$$(\widetilde{\Pi_g}(\nabla) - dg)(\omega_1 \otimes_{\mathcal{A}} \omega_2) = \widetilde{\Pi_g^0}(\omega_1 \otimes_{\mathbb{C}} \omega_2),$$

which is in ${}_0\mathcal{E}$. Therefore, by Proposition 4.1.17 , the map $\widetilde{\Pi_g}(\nabla) - dg$ is a left-covariant map. Finally, if ∇ is bicovariant and g is bi-invariant, then by Lemma 4.4.9, $\widetilde{\Pi_g^0}(\nabla)$ is a right-covariant map. Moreover, g and d are bicovariant (first assertion of Lemma 4.3.4 and Proposition 1.3.15). Hence $\widetilde{\Pi_g}(\nabla) - dg$ is also a bicovariant map. $\square$

Corollary 4.4.11. *Suppose ∇ is a bicovariant connection and g is a bi-invariant pseudo-Riemannian metric on $(\mathcal{E}, d)$. Then the map $\widetilde{\Pi_g}(\nabla) - dg$ is a right-covariant $\mathbb{C}$-linear map from ${}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E}$ to ${}_0\mathcal{E}$.*

Proof. Since ${}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E} \subseteq {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ and $g({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \in \mathbb{C}.1$ (second assertion of Lemma 4.3.4), the map dg is equal to zero on ${}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E}$. Hence,

$$\widetilde{\Pi_g}(\nabla) - dg = \widetilde{\Pi_g}(\nabla) = \widetilde{\Pi^0_g}(\nabla) \text{ on } {}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E} \subseteq {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}.$$

However, as noted before, $\widetilde{\Pi_g}(\nabla)({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \subseteq {}_0\mathcal{E}$. The right-covariance follows from Proposition 4.4.10. $\square$

The following result is an immediate corollary of the proof of Proposition 4.4.10 and Definition 4.4.7.

Corollary 4.4.12. *A connection ∇ on a bicovariant calculus $(\mathcal{E}, d)$ is compatible with a bi-invariant pseudo-Riemannian metric g if and only if $\widetilde{\Pi^0_g}(\nabla) = 0$ as a map on ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$.*

Comparison with literature

Let us remark that in Lemma 3.4 of [51], Heckenberger and Schmüdgen prove an exact analogue of Corollary 4.4.12 for their formulation of metric compatibility.

We end this subsection by comparing our notion of metric-compatibility with that of Heckenberger and Schmüdgen ([51]). Before we state our result, let us recall that a left connection on $\mathcal{E}$ is a $\mathbb{C}$-linear map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ such that $\nabla(ae) = a\nabla(e) + da \otimes_{\mathcal{A}} e$. Similarly, a left $\mathcal{A}$-linear pseudo-Riemannian metric on $\mathcal{E}$ is a left $\mathcal{A}$-linear map $g : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{A}$ such that $g \circ \sigma = g$ satisfying the condition that if $g(e \otimes_{\mathcal{A}} f) = 0$ for all e in $\mathcal{E}$, then $f = 0$.

Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus and g a left $\mathcal{A}$-linear bi-invariant pseudo-Riemannian metric on $\mathcal{E}$. The authors of [51] call a left connection ∇ on $\mathcal{E}$ to be compatible with g if

$$(\mathrm{id} \otimes_{\mathbb{C}} g)(\nabla \otimes_{\mathbb{C}} \mathrm{id}) + (g \otimes_{\mathbb{C}} \mathrm{id})(\mathrm{id} \otimes_{\mathbb{C}} \sigma)(\mathrm{id} \otimes_{\mathbb{C}} \nabla) = 0 \text{ on } {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}.$$

Therefore, we need to define the analogue of our compatibility for a bicovariant left connection ∇ with respect to a left $\mathcal{A}$-linear bi-invariant pseudo-Riemannian metric g in order to compare our definition with that in [51]. To this end, we define a map

$$\widetilde{{}_L\Pi^0_g}(\nabla) := 2(g \otimes_{\mathbb{C}} \mathrm{id})(\mathrm{id} \otimes_{\mathbb{C}} \sigma)(\mathrm{id} \otimes_{\mathbb{C}} \nabla)_0(P_{\mathrm{sym}}) : {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E} \to {}_0\mathcal{E}.$$

Then as before, we define an extension $\widetilde{{}_L\Pi_g}(\nabla) : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ by

$$\widetilde{{}_L\Pi_g}(\nabla)\widetilde{v}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}(a\otimes_{\mathbb{C}}\omega_1\otimes_{\mathbb{C}}\omega_2) = a\widetilde{{}_L\Pi^0_g}(\nabla)(\omega_1\otimes_{\mathbb{C}}\omega_2) + (da)g(\omega_1\otimes_{\mathbb{C}}\omega_2),$$

where $\widetilde{v}^{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}} : \mathcal{A}\otimes_{\mathbb{C}}{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is the multiplication map which we know is an isomorphism from Proposition 4.1.7 and Corollary 4.1.8. We say that the bicovariant left connection ∇ is compatible with the left $\mathcal{A}$-linear bi-invariant pseudo-Riemannian metric g if

$$\widetilde{{}_L\Pi_g}(\nabla) = dg. \tag{4.4.2}$$

It is easy to check that this definition coincides with the definition of metric-compatibility in the classical case, and the proof goes along the lines of Proposition 4.4.8. Then a result analogous to Corollary 4.4.12 can be derived to deduce that

$$\widetilde{{}_L\Pi_g}(\nabla) = dg \text{ if and only if } \widetilde{{}_L\Pi^0_g}(\nabla) = 0. \tag{4.4.3}$$

The next result compares the above two definitions of metric-compatibility. However, since this result needs the definitions and some results on cocycle deformations, we have proved this at the end of Section 5.4.

Proposition 4.4.13. *Let $\mathcal{A}$ be the Hopf algebra of regular functions on a linear algebraic group, $(\mathcal{E}, d)$ be the classical bicovariant differential calculus on $\mathcal{A}$ and γ a normalised 2-cocycle on $\mathcal{A}$. Consider the bicovariant differential calculus $(\mathcal{E}_\gamma, d_\gamma)$ over the Hopf algebra $\mathcal{A}_\gamma$ (see Proposition 5.3.1) and let g' be a left $\mathcal{A}$-linear bi-invariant pseudo-Riemannian metric on $\mathcal{E}_\gamma$.*

A bicovariant left connection ∇' on $\mathcal{E}_\gamma$ is compatible with g' in the sense of (4.4.2) if and only if ∇ is compatible with g' in the sense of [51].

4.5 Existence and uniqueness of Levi-Civita connections

In this section, we will derive some sufficient conditions for the existence of Levi-Civita connections for bicovariant differential calculus on quantum groups. As before, unless otherwise mentioned, $(\mathcal{E}, d)$ will denote a bicovariant differential calculus on $\mathcal{A}$ such that the restricted braiding map ${}_0\sigma$ is diagonalisable, and g a bi-invariant pseudo-Riemannian metric on $\mathcal{E}$.

Definition 4.5.1. *Let $(\mathcal{E}, d)$ be a bicovariant differential calculus such that the map ${}_0\sigma$ is diagonalisable and g a pseudo-Riemannian bi-invariant metric on $\mathcal{E}$. A left-covariant connection ∇ on $\mathcal{E}$ is called a Levi-Civita connection for the triple $(\mathcal{E}, d, g)$ if it is torsionless and compatible with g.*

The strategy to derive our results are the same as in Chapter 2. However, since we are not working with a centered bimodule and the pseudo-Riemannian metric is only right $\mathcal{A}$-linear, the arguments become more delicate. Given a bicovariant differential calculus $(\mathcal{E}, d)$ and a bi-invariant pseudo-Riemannian metric g, we start by defining a map

$$\widetilde{\Phi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$$

and show (Proposition 4.5.3) that the isomorphism of $\widetilde{\Phi_g}$ guarantees the existence of a unique left-covariant Levi-Civita connection for the triple $(\mathcal{E}, d, g)$.

However, since our metric is bi-invariant, it is to be expected that our Levi-Civita connection should be bicovariant. This is the second main result of this section (Theorem 4.5.8) which requires the Hopf algebra $\mathcal{A}$ to be cosemisimple. We remark that the bicovariance of the Levi-Civita connection (with respect to a different metric-compatibility condition) for $SL_q(n), Sp_q(n)$ and $O_q(n)$ were derived in [51].

Finally, our third result is Theorem 4.5.9 where we prove that the map
$\widetilde{\Phi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$ is an isomorphism if and only if the map

$$({}_0(P_{\mathrm{sym}}))_{23} : ({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \otimes_{\mathbb{C}} {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$$

is an isomorphism. The proofs of Theorem 4.5.8 and Theorem 4.5.9 need some preparations which are made in Subsection 4.5.1.

The main steps involved in the proof are as follows:

Step 1: We prove that the isomorphism of

$$\widetilde{\Phi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$$

guarantees the existence of a unique left-covariant Levi-Civita connection.

Step 2: We prove that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \arrow[r, "\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}, {}_0\mathcal{E}}"] \arrow[d, "\widetilde{\Phi_g}"] & ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} ({}_0\mathcal{E})^* \arrow[r, "\mathrm{id}\otimes_{\mathbb{C}} V_g^{-1}"] & ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} {}_0\mathcal{E} \arrow[d, "({}_0(P_{\mathrm{sym}}))_{23}"] \\
\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, {}_0\mathcal{E}) & {}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})^* \arrow[l, "\zeta^{-1}_{{}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}"'] & {}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \arrow[l, "\mathrm{id}\otimes_{\mathbb{C}} V_{g(2)}"']
\end{tikzcd}$$

We note that by virute of Lemma 4.2.11 and Proposition 4.3.14, all the arrows in the diagram except possibly $({}_0(P_{\mathrm{sym}}))_{23} : ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$ have already been proved to be isomorphisms. Thus, the isomorphism of $({}_0(P_{\mathrm{sym}}))_{23}$ implies the isomorphism of $\widetilde{\Phi_g}$ so that by **Step 1**, we have the existence of a unique left-covariant Levi-Civita connection.

For **Step 2** and the right-covariance of the Levi-Civita connection, we need to introduce an auxiliary map $\widetilde{\Psi_g}$ and obtain certain isomorphisms. This is done in Subsection 4.5.1. In Subsection 4.5.2, we prove that that this connection is actually right-covariant if $\mathcal{A}$ is cosemisimple. Moreover, a metric-independent sufficient condition for the existence and uniqueness of Levi-Civita connections is derived in Subsection 4.5.3.

Definition 4.5.2. *The map* $\widetilde{\Phi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$ *is defined by the following formula:*

$$\widetilde{\Phi_g}(L) = 2(\mathrm{id} \otimes_{\mathbb{C}} g)\sigma_{12}(L \otimes_{\mathbb{C}} \mathrm{id})_0(P_{\mathrm{sym}}).$$

We start with the following proposition for which we will need a bicovariant torsionless connection whose existence was proved in Theorem 4.4.4.

Proposition 4.5.3. *Suppose* $(\mathcal{E}, d)$ *is a bicovariant differential calculus such that* ${}_0\sigma$ *is diagonalisable, and* g *is a bi-invariant pseudo-Riemannian metric. If the map* $\widetilde{\Phi_g}$ *is a vector space isomorphism from* $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$ *to* $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$, *then there exists a unique left-covariant connection on* $\mathcal{E}$ *which is torsionless and compatible with* g.

Proof. Recall the torsionless bicovariant connection ∇_0 constructed in Theorem 4.4.4. Then Corollary 4.4.11 allows us to view $dg - \widetilde{\Pi_g}(\nabla_0)$ as an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$. Since $\widetilde{\Phi_g}$ is an isomorphism, there exists a unique pre-image of the element $dg - \widetilde{\Pi_g}(\nabla_0)$ under the map $\widetilde{\Phi_g}$. Define the $\mathbb{C}$-linear map

$$\nabla_1 := \nabla_0 + \widetilde{\Phi_g}^{-1}(dg - \widetilde{\Pi_g}(\nabla_0)) : {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}.$$

Then, by Proposition 4.4.3, $\nabla_1 - \nabla_0$ is an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \subseteq \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})$. By the proof of Proposition 4.1.16, $\nabla_1 - \nabla_0$ extends to an element L in ${}^{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})$. Define a $\mathbb{C}$-linear map

$$\nabla = L + \nabla_0 : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}.$$

Since L and ∇_0 are both left-covariant maps, ∇ is a left-covariant map. Moreover, since ∇_0 is a connection and L is right $\mathcal{A}$-linear, it follows that ∇ is a also a connection, since

$$\begin{aligned} &\nabla(ea) = L(ea) + \nabla_0(ea) = L(e)a + \nabla_0(e)a + e \otimes_{\mathcal{A}} da \\ =&(L(e) + \nabla_0(e))a + e \otimes_{\mathcal{A}} da = \nabla(e)a + e \otimes_{\mathcal{A}} da. \end{aligned}$$

Now we prove that ∇ is torsionless. Since $(\nabla_1 - \nabla_0)$ is an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$, $L(\omega)$ is in ${}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ for all ω in ${}_0\mathcal{E}$. Since L is right $\mathcal{A}$-linear and the right $\mathcal{A}$-linear span of ${}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E} = \mathrm{Ran}({}_0(P_{\mathrm{sym}}))$ is equal to $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E} = \mathrm{Ran}(P_{\mathrm{sym}})$ (see (4.2.5)), $L(\omega)$ is in $\mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$ for all ρ in $\mathcal{E}$. Hence, $\wedge \circ L(\rho) = 0$ for all ρ in $\mathcal{E}$. Therefore, for all ρ in $\mathcal{E}$, we have

$$\wedge \circ \nabla(\rho) = \wedge \circ (L + \nabla_0)(\rho) = \wedge \circ \nabla_0(\rho) = -d(\rho).$$

Therefore, ∇ is torsionless.

Now we prove that ∇ is compatible with g. The fact that ∇ is torsionless means in particular that $(\nabla - \nabla_0)(\omega)$ is in $\mathrm{Ker}(\wedge) = \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E}$. Thus, $\nabla - \nabla_0$ is in $\mathrm{Hom}_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes^{\mathrm{sym}}_{\mathcal{A}} \mathcal{E})$ and so $\widetilde{\Phi}_g(\nabla - \nabla_0)$ is well-defined. From the definitions of $\widetilde{\Phi_g}$ and $\widetilde{\Pi_g}$, it is immediate that

$$\widetilde{\Pi_g}(\nabla) - \widetilde{\Pi_g}(\nabla_0) = \widetilde{\Phi_g}(\nabla - \nabla_0) \tag{4.5.1}$$

as maps on ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$.

By the definition of ∇,

$$\widetilde{\Phi}_g(\nabla - \nabla_0) = dg - \widetilde{\Pi}_g(\nabla_0) \text{ on } {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}. \tag{4.5.2}$$

Combining (4.5.1) and (4.5.2), we conclude that

$$\widetilde{\Pi}_g(\nabla) - \widetilde{\Pi}_g(\nabla_0) = dg - \widetilde{\Pi}_g(\nabla_0) \text{ on } {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}.$$

Since $\widetilde{\Pi}_g(\nabla) - dg$ is right $\mathcal{A}$-linear by Proposition 4.4.10 and $\{\omega_1 \otimes_{\mathcal{A}} \omega_2 : \omega_1, \omega_2 \in {}_0\mathcal{E}\}$ is right $\mathcal{A}$-total in $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$,

$$\widetilde{\Pi}_g(\nabla) - dg = 0 \text{ as maps on } \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}.$$

Hence, ∇ is compatible with g.

To show uniqueness, suppose ∇' is another torsionless left-covariant connection compatible with the metric g. Then, by Lemma 4.4.3, $\nabla - \nabla'$ is in $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$ and

$$\widetilde{\Phi_g}(\nabla - \nabla') = \widetilde{\Pi_g}(\nabla) - \widetilde{\Pi_g}(\nabla') = dg - dg = 0,$$

where we have used the fact that ∇ and ∇' are compatible with g. As $\widetilde{\Phi_g}$ is an isomorphism, $\nabla - \nabla' = 0$ as an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})$. Since $\nabla - \nabla'$ is a right $\mathcal{A}$-linear map, $\nabla = \nabla'$ on $\mathcal{E}$. □

Proposition 4.5.3 gives us a metric-dependent sufficient condition for the existence of a unique left-covariant Levi-Civita connection. Moreover, it also follows (by Theorem 4.5.8) that if $\mathcal{A}$ is cosemisimple and $(\mathcal{E}, d, g)$ satisfies the hypotheses of Proposition 4.5.3, then the left-covariant Levi-Civita connection is also bicovariant. However, we would like to have a metric independent sufficient condition. This is derived in Theorem 4.5.9. Before we prove either of these results, we will need some preparatory lemmas which are derived in the next subsection.

4.5.1 Some preparatory results

In order to derive the right-covariance of the Levi-Civita connection, we need to define an auxiliary map $\widetilde{\Psi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, {}_0\mathcal{E})$. In Proposition 4.5.6, we will prove that the map $\widetilde{\Psi_g}$ restricts to the map $\widetilde{\Phi_g}$. The goal of this subsection is to prove Proposition 4.5.7 which states that $\widetilde{\Psi_g}$ preserves right-covariance.

We start with an elementary lemma for which we recall that for finite dimensional vector spaces V, W, $\zeta_{V,W}$ will be the isomorphism from $W \otimes_{\mathbb{C}} V^*$ to $\mathrm{Hom}_{\mathbb{C}}(V, W)$ as introduced in Proposition 1.1.8. Moreover, $V_{g^{(2)}}$ will be the map defined in Definition 4.3.11.

Lemma 4.5.4. *For $\omega_1, \omega_2, \omega_3 \in {}_0\mathcal{E}$, we have that*

$$\begin{aligned}&\zeta_{{}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}}\big((\mathrm{id} \otimes_{\mathbb{C}} V_{g^{(2)}})(\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_3)\big) \circ {}_0(P_{\mathrm{sym}}) \\ =&\zeta_{{}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}}\big((\mathrm{id} \otimes_{\mathbb{C}} V_{g^{(2)}})(\mathrm{id} \otimes_{\mathbb{C}} {}_0(P_{\mathrm{sym}}))(\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_3)\big).\end{aligned} \tag{4.5.3}$$

Proof. Let ω_4, ω_5 be elements of ${}_0\mathcal{E}$. Then, by the definition of $\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}$,

$$
\begin{aligned}
&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\omega_1\otimes_{\mathbb{C}}\omega_2\otimes_{\mathbb{C}}\omega_3)\big)\circ{}_0(P_{\mathrm{sym}})(\omega_4\otimes_{\mathbb{C}}\omega_5)\\
=&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big(\omega_1\otimes_{\mathbb{C}}V_{g^{(2)}}(\omega_2\otimes_{\mathbb{C}}\omega_3)\big)\circ{}_0(P_{\mathrm{sym}})(\omega_4\otimes_{\mathbb{C}}\omega_5)\\
=&\omega_1V_{g^{(2)}}(\omega_2\otimes_{\mathbb{C}}\omega_3)({}_0(P_{\mathrm{sym}})(\omega_4\otimes_{\mathbb{C}}\omega_5))\\
=&\omega_1V_{g^{(2)}}(({}_0(P_{\mathrm{sym}})(\omega_2\otimes_{\mathbb{C}}\omega_3)))(\omega_4\otimes_{\mathbb{C}}\omega_5)\ (\text{by } 4.3.9)\\
=&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}{}_0(P_{\mathrm{sym}}))(\omega_1\otimes_{\mathbb{C}}\omega_2\otimes_{\mathbb{C}}\omega_3)\big)(\omega_4\otimes_{\mathbb{C}}\omega_5)
\end{aligned}
$$

This proves the lemma. □

Now we define the map $\widetilde{\Psi_g}$ and discuss its properties.

Definition 4.5.5. *We define a map* $\widetilde{\Psi_g}:\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})\to\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E})$ *by the following formula:*

$$\widetilde{\Psi_g}(L)=2(\mathrm{id}\otimes_{\mathbb{C}}g)\circ(L\otimes_{\mathbb{C}}\mathrm{id}).$$

Lemma 4.5.6. *If T is an element of* $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})$*, then we have that*

$$\widetilde{\Psi_g}(T)=2\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}(V_g)^{-1})(\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(T))).\tag{4.5.4}$$

Moreover, if T is an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}^{\mathrm{sym}}{}_0\mathcal{E})$*, then the following two equations hold:*

$$\widetilde{\Psi_g}(T)|_{{}_0\mathcal{E}\otimes_{\mathbb{C}}^{\mathrm{sym}}{}_0\mathcal{E}}=\widetilde{\Phi_g}(T),\tag{4.5.5}$$

$$\widetilde{\Phi_g}(L)=2\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}{}_0(P_{\mathrm{sym}}))(\mathrm{id}\otimes_{\mathbb{C}}(V_g)^{-1})(\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(L))\big).\tag{4.5.6}$$

Proof. Let $\{\omega_i\}_i$ be a vector space basis of ${}_0\mathcal{E}$. We will use the facts (Lemma 4.3.4 and Proposition 4.3.6) that the elements $g_{ij}=g(\omega_i\otimes_{\mathbb{C}}\omega_j)$ are scalars and moreover, there exist scalars g^{ij} such that

$$\sum_j g^{ij}g_{jk}=\delta_{ik}.1=\sum_j g_{ij}g^{jk}.\tag{4.5.7}$$

Suppose T is an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})$. Then there exist scalars T^m_{ij} such that

$$T(\omega_m)=\sum_{ij}\omega_i\otimes_{\mathbb{C}}\omega_jT^m_{ij}$$

for all m. Hence, by using the definition of $\zeta_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}$ and (4.5.7), we get

$$\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(T)=\sum_{ijkl}\omega_i\otimes_{\mathbb{C}}\omega_j\otimes_{\mathbb{C}}V_g(\omega_k)g^{lk}T^l_{ij}. \tag{4.5.8}$$

We claim that

$$\frac{1}{2}\widetilde{\Psi_g}(T)=\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}V_g^{-1})(\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(T))\big). \tag{4.5.9}$$

Indeed, for all m,n, we have

$$\begin{aligned}
&\frac{1}{2}\widetilde{\Psi_g}(T)(\omega_m\otimes_{\mathbb{C}}\omega_n)\\
=&(\mathrm{id}\otimes_{\mathbb{C}}g)(T\otimes_{\mathbb{C}}\mathrm{id})(\omega_m\otimes_{\mathbb{C}}\omega_n)\\
=&\sum_{ij}(\mathrm{id}\otimes_{\mathbb{C}}g)(\omega_i\otimes_{\mathbb{C}}\omega_j\otimes_{\mathbb{C}}\omega_nT^m_{ij})\\
=&\sum_{ij}\omega_ig(\omega_j\otimes_{\mathbb{C}}\omega_n)T^m_{ij}\\
=&\sum_{ijkl}\omega_ig(\omega_j\otimes_{\mathbb{C}}g^{lk}g(\omega_k\otimes_{\mathbb{C}}\omega_m)T^l_{ij}\omega_n)\\
=&\sum_{ijkl}\omega_ig^{(2)}((\omega_j\otimes_{\mathbb{C}}\omega_kg^{lk}T^l_{ij})\otimes_{\mathbb{C}}(\omega_m\otimes_{\mathbb{C}}\omega_n))\\
=&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big(\sum_{ijkl}\omega_i\otimes_{\mathbb{C}}V_{g^{(2)}}(\omega_j\otimes_{\mathbb{C}}\omega_kg^{lk}T^l_{ij})\big)(\omega_m\otimes_{\mathbb{C}}\omega_n)\\
=&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\sum_{ijkl}\omega_i\otimes_{\mathbb{C}}\omega_j\otimes_{\mathbb{C}}\omega_kg^{lk}T^l_{ij})\big)(\omega_m\otimes_{\mathbb{C}}\omega_n)\\
=&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}V_g^{-1})(\sum_{ijkl}\omega_i\otimes_{\mathbb{C}}\omega_j\otimes_{\mathbb{C}}V_g(\omega_k)g^{lk}T^l_{ij})\big)(\omega_m\otimes_{\mathbb{C}}\omega_n)\\
=&\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}}V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}V_g^{-1})(\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(T))\big)(\omega_m\otimes_{\mathbb{C}}\omega_n),
\end{aligned}$$

where, in the last step, we have used (4.5.8) and also the fact (Proposition 4.3.6) that V_g is a vector space isomorphism from ${}_0\mathcal{E}$ to $({}_0\mathcal{E})^*$. This proves (4.5.9).

Next, if T is an element of $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$, then $T(\omega_m)\in{}_0\mathcal{E}\otimes^{\mathrm{sym}}_{\mathcal{C}}{}_0\mathcal{E}\subseteq\mathcal{E}\otimes^{\mathrm{sym}}_{\mathcal{A}}\mathcal{E}$. Since $\sigma(X)=X$ for all X in $\mathcal{E}\otimes^{\mathrm{sym}}_{\mathcal{A}}\mathcal{E}=\mathrm{Ker}(\sigma-\mathrm{id})$, we get that

$$(\sigma T)(\omega_m)=\sigma(T(\omega_m))=T(\omega_m).$$

Hence,

$$\widetilde{\Phi_g}(T) = 2(\mathrm{id}\otimes_{\mathbb{C}} g)(\sigma\otimes_{\mathbb{C}}\mathrm{id})(T\otimes_{\mathbb{C}}\mathrm{id})({}_0(P_{\mathrm{sym}}))$$
$$=2(\mathrm{id}\otimes_{\mathbb{C}} g)(T\otimes_{\mathbb{C}}\mathrm{id})({}_0(P_{\mathrm{sym}})) = \widetilde{\Psi_g}(T)({}_0(P_{\mathrm{sym}})),$$

which proves (4.5.5). Finally, for proving (4.5.6), we use (4.5.9) and (4.5.5) to deduce that

$$\begin{aligned} & \widetilde{\Phi_g}(T) = \widetilde{\Psi_g}(T)({}_0(P_{\mathrm{sym}})) \\ = \;& 2\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}} V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}} V_g^{-1})(\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(T))\big)\circ {}_0(P_{\mathrm{sym}}) \\ = \;& 2\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}\big((\mathrm{id}\otimes_{\mathbb{C}} V_{g^{(2)}})(\mathrm{id}\otimes_{\mathbb{C}}{}_0(P_{\mathrm{sym}}))(\mathrm{id}\otimes_{\mathbb{C}} V_g^{-1})(\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}}(T))\big) \end{aligned}$$

and we have used (4.5.3) in the last step. This completes the proof of the lemma. □

For the rest of the subsection, we will be using the following notations:
The set of all right $\mathcal{A}$-linear left covariant maps from M to N will be denoted by the symbol ${}^{\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(M,N)$, the set of all right $\mathcal{A}$-linear right covariant maps from M to N will be denoted by $\mathrm{Hom}^{\mathcal{A}}_{\mathcal{A}}(M,N)$ and finally, the set of all right $\mathcal{A}$-linear bicovariant maps will be denoted by ${}^{\mathcal{A}}\mathrm{Hom}^{\mathcal{A}}_{\mathcal{A}}(M,N)$.

Proposition 4.5.7. *If T is an element of $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E})$, then $\widetilde{\Psi_g}(T)$ is an element of $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E})$. Moreover, $\widetilde{\Phi_g}$ restricts to map from $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$ to $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E},{}_0\mathcal{E})$.*

Proof. Let us first observe that ${}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ are indeed right $\mathcal{A}$-comodules under the coactions ${}_{\mathcal{E}}\Delta$ and ${}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta$. Indeed, by (1.2.4), there exist elements R_{ij} in $\mathcal{A}$ such that

$$ {}_{\mathcal{E}}\Delta(\omega_i) = \sum_j \omega_j\otimes_{\mathbb{C}} R_{ji} \text{ so that } {}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\omega_i\otimes_{\mathbb{C}}\omega_j) = \sum_{k,l}\omega_k\otimes_{\mathbb{C}}\omega_l\otimes_{\mathbb{C}} R_{ki}R_{lj}. \tag{4.5.10}$$

Now, let us recall that in the proof of Theorem 4.2.5, we have proved that $\mathcal{E}\otimes^{\mathrm{sym}}_{\mathcal{A}}\mathcal{E}$ is a bicovariant bimodule. Since ${}_0(\mathcal{E}\otimes^{\mathrm{sym}}_{\mathcal{A}}\mathcal{E}) = {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ by (4.2.5), we can again apply (1.2.4) to deduce that ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ is invariant under ${}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta$.

Now, we come to the proof of the result. Let T be an element of $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E})$. Then in the notations of Lemma 4.5.6, there exist scalars T^m_{ij} such that

$$T(\omega_m) = \sum_{ij} \omega_i \otimes_{\mathbb{C}} \omega_j T^m_{ij}.$$

Since T is right-covariant, applying Lemma 4.1.20 to the second equation of (4.5.10) yields

$$\sum_{ij,n} \omega_i \otimes_{\mathbb{C}} \omega_j \otimes_{\mathbb{C}} T^n_{ij} R_{nm} = \sum_{ij,kl} \omega_k \otimes_{\mathbb{C}} \omega_l \otimes_{\mathbb{C}} R_{ki} R_{lj} T^m_{ij}. \tag{4.5.11}$$

We note that $\zeta^{-1}_{{}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, {}_0\mathcal{E}}(T) = \sum_{ijkl} \omega_i \otimes_{\mathbb{C}} \omega_j \otimes_{\mathbb{C}} T^l_{ij} g^{lk} V_g(\omega_k)$.

Then, by (4.5.4) in Lemma 4.5.6,

$$\frac{1}{2} \zeta^{-1}_{{}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}}(\widetilde{\Psi_g}(T)) = \sum_{ijkl} \omega_i \otimes_{\mathbb{C}} T^l_{ij} g^{lk} V^{(2)}_g(\omega_j \otimes_{\mathbb{C}} \omega_k).$$

Hence,

$$\widetilde{\Psi_g}(T)(\omega_m \otimes_{\mathbb{C}} \omega_n) = 2 \sum_{ijkl} \omega_i T^l_{ij} g^{lk} g^{(2)}((\omega_j \otimes_{\mathbb{C}} \omega_k) \otimes_{\mathbb{C}} (\omega_m \otimes_{\mathbb{C}} \omega_n)). \tag{4.5.12}$$

Applying Lemma 4.1.20 to the map $\widetilde{\Psi_g}(T)$ and using (4.5.12), we can conclude that $\widetilde{\Psi_g}(T)$ is an element of $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, {}_0\mathcal{E})$ if and only if, for all m, n, the following equation holds:

$$\begin{aligned} &\sum_{ii'jkl} \omega_{i'} \otimes_{\mathbb{C}} R_{i'i} T^l_{ij} g^{lk} g^{(2)}((\omega_j \otimes_{\mathbb{C}} \omega_k) \otimes_{\mathbb{C}} (\omega_m \otimes_{\mathbb{C}} \omega_n)) \\ = &\sum_{ijkl,pq} \omega_i \otimes_{\mathbb{C}} T^l_{ij} g^{lk} g^{(2)}((\omega_j \otimes_{\mathbb{C}} \omega_k) \otimes_{\mathbb{C}} (\omega_p \otimes_{\mathbb{C}} \omega_q)) R_{pm} R_{qn}. \end{aligned} \tag{4.5.13}$$

Hence if we prove (4.5.13), we are done with the first part of the theorem.

Let us note that

$$\begin{aligned} &\sum_{ii'jkl} \omega_{i'} \otimes_{\mathbb{C}} R_{i'i} T^l_{ij} g^{lk} g^{(2)}((\omega_j \otimes_{\mathbb{C}} \omega_k) \otimes_{\mathbb{C}} (\omega_m \otimes_{\mathbb{C}} \omega_n)) \\ = &\sum_{ii'jkl} \omega_{i'} \otimes_{\mathbb{C}} R_{i'i} T^l_{ij} g^{lk} g(\omega_k \otimes_{\mathbb{C}} \omega_m) g(\omega_j \otimes_{\mathbb{C}} \omega_n) \ (\text{as } g(\omega_k \otimes_{\mathbb{C}} \omega_m) \in \mathbb{C}) \\ = &\sum_{ii'jklqs} \omega_{i'} \otimes_{\mathbb{C}} R_{i'i} T^l_{ij} g^{lk} g(\omega_k \otimes_{\mathbb{C}} \omega_m) g(\omega_s \otimes_{\mathbb{C}} \omega_q) R_{sj} R_{qn}, \end{aligned}$$

where, in the last step, we have used Proposition 4.3.10 by which we have

$$g(\omega_j \otimes_{\mathbb{C}} \omega_n) = \sum_{q,s} g(\omega_s \otimes_{\mathbb{C}} \omega_q) R_{sj} R_{qn}. \tag{4.5.14}$$

Let $L : \mathcal{A} \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{A})$ denote the left multiplication map. Since ${}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E})^* \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{A})$ is isomorphic to $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}) \otimes_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathcal{A})$, we can write

$$\begin{aligned}
&\sum_{ii'jkl} \omega_{i'} \otimes_{\mathbb{C}} R_{i'i} T^l_{ij} g^{lk} g^{(2)}((\omega_j \otimes_{\mathbb{C}} \omega_k) \otimes_{\mathbb{C}} (\omega_m \otimes_{\mathbb{C}} \omega_n)) \\
&= \sum_{ii'jklqs} [\omega_{i'} \otimes_{\mathbb{C}} V_g(\omega_s) \otimes_{\mathbb{C}} T^l_{ij} g^{lk} g(\omega_k \otimes_{\mathbb{C}} \omega_m) L_{(R_{i'i} R_{sj})}](\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&\quad (\text{ by } (4.5.14) \text{ and since } T^l_{ij},\ g^{lk},\ g_{lm} \text{ are scalars }) \\
&= \sum_{ii'jlqs} [(\mathrm{id} \otimes_{\mathbb{C}} V_g \otimes_{\mathbb{C}} L)(\omega_{i'} \otimes_{\mathbb{C}} \omega_s \otimes_{\mathbb{C}} T^l_{ij} \sum_k (g^{lk} g(\omega_k \otimes_{\mathbb{C}} \omega_m)) R_{i'i} R_{sj})](\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&= \sum_{ii'jlqs} [(\mathrm{id} \otimes_{\mathbb{C}} V_g \otimes_{\mathbb{C}} L)(\omega_{i'} \otimes_{\mathbb{C}} \omega_s \otimes_{\mathbb{C}} T^l_{ij} \delta_{lm} R_{i'i} R_{sj})](\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&= \sum_{ii'jqs} [(\mathrm{id} \otimes_{\mathbb{C}} V_g \otimes_{\mathbb{C}} L)(\omega_{i'} \otimes_{\mathbb{C}} \omega_s \otimes_{\mathbb{C}} R_{i'i} R_{sj} T^m_{ij})](\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&= \sum_{ijpq} [(\mathrm{id} \otimes_{\mathbb{C}} V_g \otimes_{\mathbb{C}} L)(\omega_i \otimes_{\mathbb{C}} \omega_j \otimes_{\mathbb{C}} T^p_{ij} R_{pm})](\omega_q \otimes_{\mathbb{C}} R_{qn}) \ \big(\text{by } (4.5.11)\big) \\
&= \sum_{ijpql} [(\mathrm{id} \otimes_{\mathbb{C}} V_g \otimes_{\mathbb{C}} L)(\omega_i \otimes_{\mathbb{C}} \omega_j \otimes_{\mathbb{C}} T^l_{ij} \delta_{lp} R_{pm})](\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&= \sum_{ijlpq} [(\mathrm{id} \otimes_{\mathbb{C}} V_g \otimes_{\mathbb{C}} L)(\omega_i \otimes_{\mathbb{C}} \omega_j \otimes_{\mathbb{C}} T^l_{ij} (\sum_k g^{lk} g(\omega_k \otimes_{\mathbb{C}} \omega_p)) R_{pm})](\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&= \sum_{ijklpq} (\omega_i \otimes_{\mathbb{C}} V_g(\omega_j) \otimes_{\mathbb{C}} T^l_{ij} g^{lk} g(\omega_k \otimes_{\mathbb{C}} \omega_p) L_{R_{pm}})(\omega_q \otimes_{\mathbb{C}} R_{qn}) \\
&= \sum_{ijklpq} \omega_i \otimes_{\mathbb{C}} T^l_{ij} g^{lk} g(\omega_j \otimes_{\mathbb{C}} \omega_q) g(\omega_k \otimes_{\mathbb{C}} \omega_p) R_{pm} R_{qn} \\
&= \sum_{ijkl,pq} \omega_i \otimes_{\mathbb{C}} T^l_{ij} g^{lk} g^{(2)}((\omega_j \otimes_{\mathbb{C}} \omega_k) \otimes_{\mathbb{C}} (\omega_p \otimes_{\mathbb{C}} \omega_q)) R_{pm} R_{qn}.
\end{aligned}$$

This proves (4.5.13) and therefore, $\widetilde{\Psi_g}(T)$ is right-covariant.

Now we prove the second assertion of the proposition. Let T be an element of $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$. Then the first assertion of the proposition implies that $\widetilde{\Psi_g}(T)$ belongs to $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}, {}_0\mathcal{E})$. However, by (4.5.5), $\widetilde{\Psi_g}(T)|_{{}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}} = \widetilde{\Phi_g}(T)$ and by the definition of $\widetilde{\Phi_g}$, we know that $\widetilde{\Phi_g}(T)$ belongs to $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$. Hence, we conclude that $\widetilde{\Phi_g}(T)$ belongs to $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$. This finishes the proof of the proposition. $\square$

4.5.2 Right-covariance of the unique left-covariant connection

In this subsection, we prove that the unique torsion-less left-covariant connection compatible with a bi-invariant pseudo-Riemannian metric, obtained under the hypothesis of Proposition 4.5.3, is actually a bicovariant connection if the Hopf algebra $\mathcal{A}$ is cosemisimple, i.e, if the category of finite dimensional comodules of $\mathcal{A}$ is semisimple. For right $\mathcal{A}$-comodules V and W, the symbol $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}(V,W)$ will continue to denote the set of all right-covariant complex linear maps from V to W.

If $\mathcal{A}$ is a cosemisimple Hopf algebra and V, W be finite dimensional comodules as above, then Proposition 1.1.11 implies that

$$\dim(\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}(V,W)) = \dim(\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}(W,V)).$$

Now, if $\mathcal{A}$ is a cosemisimple Hopf algebra and $(\mathcal{E}, d)$ be a differential calculus such that ${}_0\sigma$ is diagonalisable, then in the proof of Proposition 4.5.7, we have noted that ${}_0\mathcal{E}$ and ${}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ are right $\mathcal{A}$-comodules. Hence, we can conclude that

$$\dim(\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})) = \dim(\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})). \tag{4.5.15}$$

Then we have the following theorem.

Theorem 4.5.8. *Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus over a cosemisimple Hopf algebra $\mathcal{A}$ such that the map ${}_0\sigma$ is diagonalisable, and g is a bi-invariant pseudo-Riemannian metric. If the map $\widetilde{\Phi_g}$ is an isomorphism, then the unique left-covariant connection guaranteed by Proposition 4.5.3 is in fact a bicovariant connection.*

Proof. The proof follows from the claim that under the hypothesis of the theorem, the map $\widetilde{\Phi_g}$ is an isomorphism from $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$ onto $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$. Indeed, let us recall that in Proposition 4.5.3, under the assumption that the map $\widetilde{\Phi_g}$ is an isomorphism, we explicitly constructed a torsionless left-covariant connection ∇ compatible with g by the formula $\nabla := L + \nabla_0$.

Here ∇_0 is the torsionless bicovariant connection constructed in Theorem 4.4.4 and $L : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is the left-covariant right $\mathcal{A}$-linear extension (via Proposition 4.1.14) of the map

$$\widetilde{\Phi_g}^{-1}(dg - \widetilde{\Pi_g}(\nabla_0)) : {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}.$$

By Corollary 4.4.11, $dg - \widetilde{\Pi_g}(\nabla_0)$ is a right $\mathcal{A}$-covariant $\mathbb{C}$-linear map from ${}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}$ to ${}_0\mathcal{E}$. Hence, our claim implies that ${}_0L = \widetilde{\Phi_g}^{-1}(dg - \widetilde{\Pi_g}(\nabla_0))$ belongs to $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$.

Since L is left-covariant right $\mathcal{A}$-linear and ${}_0L = \widetilde{\Phi_g}^{-1}(dg - \widetilde{\Pi_g}(\nabla_0))$ is right-covariant, Proposition 4.1.19 implies that the extension L is a bicovariant right $\mathcal{A}$-linear map from $\mathcal{E}$ to $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$. Again by the right-covariance of ∇_0, $\nabla = L + \nabla_0$ is a right-covariant map as well.

So we are left with proving that the map $\widetilde{\Phi_g} : \mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \to \mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$ is an isomorphism. To this end, we observe that since $\widetilde{\Phi_g}$ is an isomorphism from $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$ to $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$, Proposition 4.5.7 implies that $\widetilde{\Phi_g}$ is a one-to-one map from $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$ into $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$. However, by (4.5.15),

$$\dim(\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})) = \dim(\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})).$$

Therefore, $\widetilde{\Phi_g}$ is a one-to-one and onto map from $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$ to $\mathrm{Hom}^{\mathcal{A}}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$. $\square$

4.5.3 Sufficient conditions for the isomorphism of $\widetilde{\Phi_g}$

In this subsection, we prove a metric-independent sufficient condition for the map $\widetilde{\Phi_g}$ to be an isomorphism. We will continue to use the notation $\zeta_{\mathcal{E},\mathcal{F}}$ introduced in Proposition 1.1.8.

Theorem 4.5.9. *Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus over a cosemisimple Hopf algebra $\mathcal{A}$ such that the map ${}_0\sigma$ is diagonalisable and g be a bi-invariant pseudo-Riemannian metric.*

The map $\widetilde{\Phi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}, {}_0\mathcal{E})$ is an isomorphism if and only if $({}_0(P_{\mathrm{sym}}))_{23} : ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}}^{\mathrm{sym}} {}_0\mathcal{E})$ is an isomorphism. In particular, Theorem 4.5.8 implies that under either of these assumptions, the triple $(\mathcal{E}, d, g)$ admits a unique bicovariant Levi-Civita connection.

Proof. Suppose $({}_0(P_{\mathrm{sym}}))_{23} : ({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E} \to {}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$ is an isomorphism. Since g is left-invariant, part (i) of Proposition 4.3.6 implies that $V_g^{-1}(({}_0\mathcal{E})^*) = {}_0\mathcal{E}$. By the first assertion of Lemma 4.2.11 and our hypothesis, we can conclude that the following map is an isomorphism:

$$({}_0(P_{\mathrm{sym}}))_{23}(\mathrm{id}\otimes_{\mathbb{C}}V_g^{-1})\zeta^{-1}_{{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E},{}_0\mathcal{E}} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \to {}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}).$$

Now, by Proposition 4.3.14, $V_{g^{(2)}}$ is an isomorphism from ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ to $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^*$. Finally, by the second assertion of Lemma 4.2.11, $\zeta_{{}_0\mathcal{E},{}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}}$ is an isomorphism from ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})^*$ to $\mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$. Therefore, by (4.5.6), is a composition of isomorphisms and hence an isomorphism itself.

Conversely, suppose $\widetilde{\Phi_g} : \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}, {}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}) \to \mathrm{Hom}_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}, {}_0\mathcal{E})$ is an isomorphism. If $({}_0(P_{\mathrm{sym}}))_{23}$ is not an isomorphism from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E}$ to ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$, then it is not one-to-one. Hence by (4.5.6), $\widetilde{\Phi_g}$ is not an isomorphism, which is a contradiction. □

Remark 4.5.10. *In Chapter 6, the isomorphism* $({}_0(P_{\mathrm{sym}}))_{23} : ({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E} \to {}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$ *for the Hopf algebra* $SU_q(2)$ *is verified by an explicit computation. We refer to Theorem 5.4.4 for a cocycle-twisted version of the above isomorphism.*

Our next proposition states that if $\sigma^2 = 1$, then the hypothesis of Theorem 4.5.9 is satisfied.

Proposition 4.5.11. *If* $\sigma^2 = 1$*, then the map* $({}_0(P_{\mathrm{sym}}))_{23}$ *is an isomorphism from* $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E}$ *to* ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$.

Proof. Since $\sigma^2 = 1$, ± 1 are the only eigenvalues of ${}_0\sigma$ in this case and so by (4.2.7), ${}_0(P_{\mathrm{sym}}) = \frac{1}{2}(1 + {}_0\sigma)$. Now, let X be an element of $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E}$ such that $({}_0(P_{\mathrm{sym}}))_{23}(X) = 0$. Then $(P_{\mathrm{sym}})_{(12)}(X) = X$ so that $({}_0\sigma)_{12}(X) = X$.

Moreover, $({}_0\sigma)_{23}(X) = (2({}_0(P_{\mathrm{sym}}))_{23} - 1)(X) = -X$. We further obtain that

$$({}_0\sigma)_{12}({}_0\sigma)_{23}({}_0\sigma)_{12}(X) = -X \quad \text{and} \quad ({}_0\sigma)_{23}({}_0\sigma)_{12}({}_0\sigma)_{23}(X) = X.$$

Since ${}_0\sigma$ is a braiding, this implies that $X = 0$. Hence $({}_0(P_{\mathrm{sym}}))_{23}$ is a one-to-one map from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E}$ to ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$ and therefore, by a dimension count, $({}_0(P_{\mathrm{sym}}))_{23}$ is also onto ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$. Hence $({}_0(P_{\mathrm{sym}}))_{23}$ is an isomorphism from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E}$ to ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$. □

Remark 4.5.12. *In Corollary 5.1.9, we show that the hypothesis of Proposition 4.5.11 holds for the space of one-forms for cocycle deformations of a linear algebraic group G whose category of finite dimensional representations is semisimple. Thus, for these examples, we indeed have a unique bicovariant Levi-Civita connection by Proposition 4.5.11 (see Proposition 5.4.5).*

Chapter 5

Levi-Civita connections on cocycle deformation of Hopf algebras

Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus over a Hopf algebra $\mathcal{A}$ such that ${}_0\sigma$ is diagonalisable in the sense of Chapter 4. In Theorem 4.5.9, we have proved a sufficient condition for the existence of a unique bicovariant Levi-Civita connection for every bi-invariant pseudo-Riemannian metric. This chapter discusses a class of examples of bicovariant differential calculi for which this sufficient condition is satisfied. Indeed, if $(\mathcal{E}, d)$ is a bicovariant differential calculus and γ is a 2-cocycle on $\mathcal{A}$ as in Definition 1.2.6, then Majid and Oeckl proved ([74]) that $(\mathcal{E}, d)$ deforms to a bicovariant differential calculus $(\mathcal{E}_\gamma, d_\gamma)$ on the deformed Hopf algebra $\mathcal{A}_\gamma$ (see Definition 1.2.7). The goal of this chapter is to show that if $(\mathcal{E}, d)$ satisfies the hypotheses of Theorem 4.5.9, then $(\mathcal{E}_\gamma, d_\gamma)$ also satisfies its hypotheses. Thus, we have a unique bicovariant Levi-Civita connection for every bi-invariant pseudo-Riemannian metric on $\mathcal{E}_\gamma$.

In Section 5.1, we recall the generalities on cocycle deformation of bicovariant bimodules from [74]. We have also borrowed some formulas from [8], where necessary. As a result, we observe that in the presence of a cocycle, the braiding map σ of $(\mathcal{E}, d)$ deforms to the braiding map of $(\mathcal{E}_\gamma, d_\gamma)$. In Section 5.2, we study cocycle deformations of pseudo-Riemannian metrics on bicovariant bimodules. The main result of this section is Theorem 5.2.5 where we prove that for a Hopf algebra $\mathcal{A}$, cocycle deformations of a bi-invariant pseudo-Riemannian metric g on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$ is a bi-invariant pseudo-Riemannian metric on the deformed bicovariant $\mathcal{A}_\gamma$-bimodule $\mathcal{E}_\gamma$. The contents of Section 5.1 and Section 5.2 are from [18].

In Section 5.3, we recall the cocycle deformation of bicovariant differential calculi from [74] and also discuss the deformation of the space of two-forms. Finally, in Section 5.4, we prove the main result of this chapter. We begin by discussing cocycle deformations of bicovariant connections on bicovariant differential calculi. Theorem 5.4.3 is the main result of this section which demonstrates that the sufficient condition of Theorem 4.5.9 holds for a cocycle deformed differential calculus provided it holds for the undeformed one. Theorem 5.4.5 discusses the concrete example of cocycle deformation of Hopf algebras of regular functions on a linear algebraic group and the existence and uniqueness of Levi-Civita connections therein. We end the section with the proof of a comparison of our notion of bicovariant Levi-Civita connections with that of [51] in the context of cocycle deformations. The contents of these two sections are from [17].

5.1 Cocycle deformation of bicovariant bimodules

If $(\mathcal{A}, \Delta)$ is a Hopf algebra and γ is a 2-cocycle as in Definition 1.2.6, then Definition 1.2.7 shows that $(\mathcal{A}, \Delta)$ can be twisted to a new Hopf algebra $(\mathcal{A}_\gamma, \Delta_\gamma)$, where $\mathcal{A}_\gamma$ is equal to $\mathcal{A}$ as a vector space, the coproduct Δ_γ is equal to Δ, and the algebra structure $*_\gamma$ on $\mathcal{A}_\gamma$ is defined by the following equation:

$$a *_\gamma b = \gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)}) a_{(2)} b_{(2)} \overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} b_{(3)}). \tag{5.1.1}$$

Here, $\overline{\gamma}$ is the convolution inverse to γ as in Definition 1.2.6.

In this section, we will discuss the cocycle deformation of bicovariant bimodules over Hopf algebras and the deformation of covariant bimodule maps. Throughout this section, we will make heavy use of the Sweedler notations as spelt out Subsection 1.2. In particular, using coassociativity of Δ, we will use the notation, $(\Delta \otimes_{\mathbb{C}} \mathrm{id})\Delta(a) = (\mathrm{id} \otimes_{\mathbb{C}} \Delta)\Delta(a) = a_{(1)} \otimes_{\mathbb{C}} a_{(2)} \otimes_{\mathbb{C}} a_{(3)}$. Also, when m denotes an element of a bicovariant bimodule M, we will use the notation

$$(\mathrm{id} \otimes_{\mathbb{C}} {}_M\Delta)\Delta_M(m) = (\Delta_M \otimes_{\mathbb{C}} \mathrm{id})_M\Delta(m) = m_{(-1)} \otimes_{\mathbb{C}} m_{(0)} \otimes_{\mathbb{C}} m_{(1)}. \tag{5.1.2}$$

Note that the index (0) in the above equation denote the comodule tensorand and non-zero indices indicate the coalgebra tensorand.

Then we have the following:

Proposition 5.1.1. *(Theorem 2.5 of [74]) Suppose M is a bicovariant $\mathcal{A}$-bimodule and γ is a 2-cocycle on $\mathcal{A}$. Then we have a bicovariant $\mathcal{A}_\gamma$-bimodule M_γ which is equal to M as a vector* *space but the left and right $\mathcal{A}_\gamma$-module structures are defined by the following formulas:*

$$a *_\gamma m = \gamma(a_{(1)} \otimes_{\mathbb{C}} m_{(-1)}) a_{(2)}.m_{(0)} \overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} m_{(1)}) \tag{5.1.3}$$

$$m *_\gamma a = \gamma(m_{(-1)} \otimes_{\mathbb{C}} a_{(1)}) m_{(0)}.a_{(2)} \overline{\gamma}(m_{(1)} \otimes_{\mathbb{C}} a_{(3)}), \tag{5.1.4}$$

for all elements m of M and for all elements a of $\mathcal{A}$. Here, $*_\gamma$ denotes the right and left $\mathcal{A}_\gamma$-module actions and $.$ denotes the right and left $\mathcal{A}$-module actions.

The $\mathcal{A}_\gamma$-bicovariant structures are given by

$$\Delta_{M_\gamma} := {}_M\Delta : M_\gamma \to \mathcal{A}_\gamma \otimes_{\mathbb{C}} M_\gamma \text{ and } {}_{M_\gamma}\Delta := {}_M\Delta : M_\gamma \to M_\gamma \otimes_{\mathbb{C}} \mathcal{A}_\gamma. \tag{5.1.5}$$

Now, we are in a position to state the following proposition regarding cocycle deformations of bicovariant maps between bicovariant bimodules.

Proposition 5.1.2. *(Theorem 2.5 of [74]) Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant $\mathcal{A}$-bimodules, $T : M \to N$ be a $\mathbb{C}$-linear bicovariant map and γ be a cocycle as above. Then there exists a map $T_\gamma : M_\gamma \to N_\gamma$ defined by $T_\gamma(m) = T(m)$ for all m in M. Thus, $T_\gamma = T$ as $\mathbb{C}$-linear maps. Moreover, we have the following:*

(i) the deformed map $T_\gamma : M_\gamma \to N_\gamma$ is an $\mathcal{A}_\gamma$ bicovariant map,

(ii) if T is a bicovariant right (respectively left) $\mathcal{A}$-linear map, then T_γ is a bicovariant right (respectively left) $\mathcal{A}_\gamma$-linear map,

(iii) if $(P, \Delta_P, {}_P\Delta)$ is another bicovariant $\mathcal{A}$-bimodule, and $S : N \to P$ is a bicovariant map, then $(S \circ T)_\gamma : M_\gamma \to P_\gamma$ is a bicovariant map and $S_\gamma \circ T_\gamma = (S \circ T)_\gamma$.

Remark 5.1.3. *From Proposition 5.1.2, it is clear that if M is a finite bicovariant bimodule (see Definition 4.1.3), then M_γ is also a finite bicovariant bimodule. We will only be dealing with deformations of finite bicovariant bimodules in this chapter.*

Recall that in Proposition 4.1.14, for a bicovariant right $\mathcal{A}$-linear map $T : M \to N$ between bicovariant $\mathcal{A}$-bimodules, we adopted the notation ${}_0T = T|_{{}_0M}$, where ${}_0M$ is the space of left-invariant elements of M. As a corollary to Proposition 5.1.2, we obtain:

Proposition 5.1.4. *Let $(M, \Delta_M, {}_M\Delta)$ and $(N, \Delta_N, {}_N\Delta)$ be bicovariant bimodules over a Hopf algebra $\mathcal{A}$, T be a bicovariant right $\mathcal{A}$-linear map from M to N and γ be a cocycle as above.*

Then

$$T_\gamma = \widetilde{u}^{N_\gamma} \circ ({}_0T \otimes_{\mathbb{C}} \mathrm{id}) \circ (\widetilde{u}^{M_\gamma})^{-1}, \tag{5.1.6}$$

where $\widetilde{u}^{M_\gamma} : {}_0(M_\gamma) \otimes_{\mathbb{C}} A_\gamma \to M_\gamma$ *and* $\widetilde{u}^{N_\gamma} : {}_0(M_\gamma) \otimes_{\mathbb{C}} A_\gamma \to N_\gamma$ *are the usual multiplication maps as in Proposition 4.1.7. In particular, the* $\mathbb{C}$*-linear map* ${}_0(T_\gamma)$ *from* ${}_0(M_\gamma) = {}_0M$ *to itself coincides with* ${}_0T$*. Moreover,* T_γ *is an invertible map if and only if* T *is invertible, and more generally,* λ *is an eigenvalue of* T_γ *if and only if it is an eigenvalue of* T*.*

Proof. Since T is a bicovariant right $\mathcal{A}$-linear map from M to N, by Proposition 5.1.2, T_γ is an $\mathcal{A}_\gamma$ bicovariant right linear map. Since ${}_0(M_\gamma) = {}_0M$ and ${}_0(N_\gamma) = {}_0N$ as vector spaces, and T_γ is a left-covariant map, hence for all m in ${}_0(M_\gamma)$, the element $T_\gamma(m)$ belongs to ${}_0(N_\gamma)$. Then we compute, for any m in ${}_0(M_\gamma)$ and any element a of $\mathcal{A}_\gamma$,

$$\begin{aligned}
&(\widetilde{u}^{N_\gamma})^{-1} \circ T_\gamma(m *_\gamma a) = (\widetilde{u}^{N_\gamma})^{-1}(T_\gamma(m) *_\gamma a) = T_\gamma(m) \otimes_{\mathbb{C}} a \text{ (by the definition of } \widetilde{u}^{N_\gamma}) \\
&= T(m) \otimes_{\mathbb{C}} a = ({}_0T)(m) \otimes_{\mathbb{C}} a = ({}_0T \otimes_{\mathbb{C}} \mathrm{id})(\widetilde{u}^{M_\gamma})^{-1}(m *_\gamma a),
\end{aligned}$$

as m belongs to ${}_0(M_\gamma)$. Thus we have that

$$(\widetilde{u}^{N_\gamma})^{-1} \circ T_\gamma = ({}_0T \otimes_{\mathbb{C}} \mathrm{id})(\widetilde{u}^{M_\gamma})^{-1}, \text{ i.e., } T_\gamma = \widetilde{u}^{N_\gamma} \circ ({}_0T \otimes_{\mathbb{C}} \mathrm{id}) \circ (\widetilde{u}^{M_\gamma})^{-1}.$$

Evaluating this equation on an element of ${}_0(M_\gamma) = {}_0M$ yields ${}_0(T_\gamma) = {}_0T$.

Finally, applying Proposition 4.1.16 to T_γ and using the fact that ${}_0(T_\gamma) = {}_0T$, we get that T_γ is invertible if and only if T is invertible. More generally, λ is an eigenvalue of T_γ if and only if it is an eigenvalue of T. □

The next result studies the monoidal property of cocycle deformations.

Proposition 5.1.5. *(Theorem 2.5 of [74]) Let* $(M, \Delta_M, {}_M\Delta)$ *and* $(N, \Delta_N, {}_N\Delta)$ *be bicovariant bimodules over a Hopf algebra* $\mathcal{A}$ *and* γ *be a cocycle as above. Then there exists a bicovariant* $\mathcal{A}_\gamma$*- bimodule isomorphism*

$$\xi : M_\gamma \otimes_{\mathcal{A}_\gamma} N_\gamma \to (M \otimes_{\mathcal{A}} N)_\gamma.$$

The isomorphism ξ *and its inverse are respectively given by*

$$\begin{aligned}
\xi(m \otimes_{\mathcal{A}_\gamma} n) &= \gamma(m_{(-1)} \otimes_{\mathbb{C}} n_{(-1)}) m_{(0)} \otimes_{\mathcal{A}} n_{(0)} \overline{\gamma}(m_{(1)} \otimes_{\mathbb{C}} n_{(1)}) \\
\xi^{-1}(m \otimes_{\mathcal{A}} n) &= \overline{\gamma}(m_{(-1)} \otimes_{\mathbb{C}} n_{(-1)}) m_{(0)} \otimes_{\mathcal{A}_\gamma} n_{(0)} \gamma(m_{(1)} \otimes_{\mathbb{C}} n_{(1)})
\end{aligned}$$

As an illustration, we make the following computation.

Lemma 5.1.6. *Suppose ω is in ${}_0\mathcal{E}$ and η is in $\mathcal{E}_0$. Then the following equation holds:*

$$\xi^{-1}(\gamma(\eta_{(-1)}\otimes_{\mathbb{C}}1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\overline{\gamma}(1\otimes_{\mathbb{C}}\omega_{(1)})) = \eta \otimes_{\mathcal{A}_\gamma} \omega.$$

Proof. Let us first clarify that we view $\gamma(\eta_{(-1)}\otimes_{\mathbb{C}}1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\overline{\gamma}(1\otimes_{\mathbb{C}}\omega_{(1)})$ as an element in $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. Then the equation holds because of the following computation:

$$\begin{aligned}
&\xi^{-1}(\gamma(\eta_{(-1)}\otimes_{\mathbb{C}}1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\overline{\gamma}(1\otimes_{\mathbb{C}}\omega_{(1)}))\\
=&\gamma(\eta_{(-1)}\otimes_{\mathbb{C}}1)\xi^{-1}(\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)})\overline{\gamma}(1\otimes_{\mathbb{C}}\omega_{(1)})\\
=&\gamma(\eta_{(-2)}\otimes_{\mathbb{C}}1)\overline{\gamma}(\eta_{(-1)}\otimes_{\mathbb{C}}1)\eta_{(0)} \otimes_{\mathcal{A}_\gamma} \omega_{(0)}\gamma(1\otimes_{\mathbb{C}}\omega_{(1)})\overline{\gamma}(1\otimes_{\mathbb{C}}\omega_{(2)})\\
&(\text{since } \omega \in {}_0\mathcal{E},\ \eta \in \mathcal{E}_0)\\
=&\epsilon(\eta_{(-2)})\epsilon(\eta_{(-1)})\eta_{(0)} \otimes_{\mathcal{A}_\gamma} \omega_{(0)}\epsilon(\omega_{(1)})\epsilon(\omega_{(2)}) \text{ (since } \overline{\gamma} \text{ and } \gamma \text{ are unital)}\\
=&\eta \otimes_{\mathcal{A}_\gamma} \omega.
\end{aligned}$$

□

Recall the braiding map $\sigma : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ for a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$, as in Proposition 1.3.17. We next study the deformation of σ. By Proposition 5.1.1, $\mathcal{E}_\gamma$ is a bicovariant $\mathcal{A}_\gamma$-bimodule. Then Proposition 1.3.17 guarantees the existence of a canonical braiding from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself. We show that this map is nothing but the deformation σ_γ of the map σ associated with the bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$. By the definition of σ_γ, it is a map from $(\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E})_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. However, by virtue of Proposition 5.1.5, the map ξ defines an isomorphism from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. By an abuse of notation, we will denote the map

$$\xi^{-1}\sigma_\gamma\xi : \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma \to \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$$

by the symbol σ_γ again.

Theorem 5.1.7. *(Theorem 2.5 of [74]) Let $\mathcal{E}$ be a bicovariant $\mathcal{A}$-bimodule and γ be a cocycle as above. Then the deformation σ_γ of σ is the unique bicovariant $\mathcal{A}_\gamma$-bimodule braiding map on $\mathcal{E}_\gamma$ given by Proposition 1.3.17.*

Proof. Since σ is a bicovariant $\mathcal{A}$-bimodule map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to itself, part (ii) of Proposition 5.1.2 implies that σ_γ is a bicovariant $\mathcal{A}_\gamma$-bimodule map from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma \cong \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself.

By Proposition 1.3.17, there exists a unique $\mathcal{A}_\gamma$-bimodule map σ' from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself such that $\sigma'(\omega \otimes_{\mathcal{A}_\gamma} \eta) = \eta \otimes_{\mathcal{A}_\gamma} \omega$ for all ω in ${}_0(\mathcal{E}_\gamma)$, η in $(\mathcal{E}_\gamma)_0$.

Since ${}_0(\mathcal{E}_\gamma) = {}_0\mathcal{E}$ and $(\mathcal{E}_\gamma)_0 = \mathcal{E}_0$, it is enough to prove that $\sigma_\gamma(\omega \otimes_{\mathcal{A}_\gamma} \eta) = \eta \otimes_{\mathcal{A}_\gamma} \omega$ for all ω in ${}_0\mathcal{E}$, η in $\mathcal{E}_0$.

We will need the concrete isomorphism between $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ and $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ defined in Proposition 5.1.5. Since ω is in ${}_0\mathcal{E}$ and η is in $\mathcal{E}_0$, this isomorphism maps the element $\omega \otimes_{\mathcal{A}_\gamma} \eta$ to $\gamma(1 \otimes_{\mathbb{C}} \eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}} \eta_{(0)}\overline{\gamma}(\omega_{(1)} \otimes_{\mathbb{C}} 1)$. Then, by the definition of σ_γ, we compute the following:

$$\begin{aligned}
&\sigma_\gamma(\omega \otimes_{\mathcal{A}_\gamma} \eta) = \sigma(\gamma(1 \otimes_{\mathbb{C}} \eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}} \eta_{(0)}\overline{\gamma}(\omega_{(1)} \otimes_{\mathbb{C}} 1))\\
=&\sigma(\epsilon(\eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}} \eta_{(0)}\epsilon(\omega_{(1)})) = \epsilon(\eta_{(-1)})\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\epsilon(\omega_{(1)})\\
=&\gamma(\eta_{(-1)} \otimes_{\mathbb{C}} 1)\eta_{(0)} \otimes_{\mathcal{A}} \omega_{(0)}\overline{\gamma}(1 \otimes_{\mathbb{C}} \omega_{(1)}) = \eta \otimes_{\mathcal{A}_\gamma} \omega,
\end{aligned}$$

where, in the last step we have used Lemma 5.1.6. □

Remark 5.1.8. *Proposition 5.1.1, Proposition 5.1.2, Proposition 5.1.5 and Theorem 5.1.7 together imply that the categories* ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ *and* ${}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}\mathcal{M}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}$ *are isomorphic as braided monoidal categories. This was the content of Theorem 2.5 of* [74].

However, in Theorem 5.1.7, we have emphasized in addition that the braiding on ${}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}\mathcal{M}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}$ *is precisely the Woronowicz braiding of Proposition 1.3.17.*

We end this section with some consequences of Theorem 5.1.7.

Corollary 5.1.9. *If the map* ${}_0(\sigma)$ *is diagonalisable, then the map* ${}_0(\sigma_\gamma)$ *is also diagonalisable.*

Proof. This is a consequence of Proposition 5.1.4, by which we have that the $\mathbb{C}$-linear maps ${}_0(\sigma_\gamma)$ and ${}_0\sigma$ coincide. □

Corollary 5.1.10. *If the unique bicovariant* $\mathcal{A}$*-bimodule braiding map* σ *for a bicovariant* $\mathcal{A}$*-bimodule* $\mathcal{E}$ *satisfies the equation* $\sigma^2 = 1$*, then the bicovariant* $\mathcal{A}_\gamma$*-bimodule braiding map* σ_γ *for the bicovariant* $\mathcal{A}_\gamma$*-bimodule* $\mathcal{E}_\gamma$ *also satisfies* $\sigma_\gamma^2 = 1$.

In particular, if $\mathcal{A}$ *is the commutative Hopf algebra of regular functions on a compact semisimple Lie group* G *and* $\mathcal{E}$ *is its canonical space of one-forms, then the braiding map* σ_γ *for* $\mathcal{E}_\gamma$ *satisfies* $\sigma_\gamma^2 = 1$.

Proof. By Theorem 5.1.7, σ_γ is the unique braiding map for the bicovariant $\mathcal{A}_\gamma$-bimodule $\mathcal{E}_\gamma$. Since, by our hypothesis, $\sigma^2 = 1$, the deformed map σ_γ also satisfies $\sigma_\gamma^2 = 1$ by part (iii) of Proposition 5.1.2.

Next, if $\mathcal{A}$ is a commutative Hopf algebra as in the statement of the corollary and $\mathcal{E}$ is its canonical space of one-forms, then we know that the braiding map σ is just the flip map, i.e. for all e, e' in $\mathcal{E}$,

$$\sigma(e \otimes_{\mathcal{A}} e') = e' \otimes_{\mathcal{A}} e,$$

and hence it satisfies $\sigma^2 = 1$. Therefore, for every cocycle deformation $\mathcal{E}_\gamma$ of $\mathcal{E}$, the corresponding braiding map satisfies $\sigma_\gamma^2 = 1$. □

5.2 Cocycle deformation of pseudo-Riemannian metrics

In this section, we will discuss the cocycle deformation of pseudo-Riemannian bi-invariant metrics on bicovariant bimodules. By Proposition 4.3.3, a pseudo-Riemannian bi-invariant metric g on a bicovariant bimodule $\mathcal{E}$ is a bicovariant map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{A}$. Hence, by Proposition 5.1.2, we have a right $\mathcal{A}_\gamma$-linear bicovariant map g_γ from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to $\mathcal{A}_\gamma$. We next show that this map g_γ is a pseudo-Riemannian bi-invariant metric on $\mathcal{E}_\gamma$ upto a suitable identification, by checking the conditions (i) and (ii) of Definition 4.3.1 for the map g_γ.

The proof of the equality $g_\gamma = g_\gamma \circ \sigma_\gamma$ is straightforward. However, checking condition (ii), i.e, verifying that the map V_{g_γ} is an isomorphism onto its image needs some work. The root of the problem is that we do not yet know whether $\mathcal{E}^* = V_g(\mathcal{E})$. Our strategy to verify condition (ii) is the following: we show that the right $\mathcal{A}$-module $V_g(\mathcal{E})$ is a bicovariant right $\mathcal{A}$-module (see Definition 4.1.1) in a natural way. Let us remark that since the map g (hence V_g) is not left $\mathcal{A}$-linear, $V_g(\mathcal{E})$ need not be a left $\mathcal{A}$-module. Since bicovariant right $\mathcal{A}$-modules and bicovariant maps can be deformed (Proposition 5.2.3), the map V_g deforms to a right $\mathcal{A}_\gamma$-linear isomorphism $(V_g)_\gamma$ from $\mathcal{E}_\gamma$ to $(V_g(\mathcal{E}))_\gamma$. Then in Theorem 5.2.5, we show that $(V_g)_\gamma$ coincides with the map V_{g_γ} and the latter is an isomorphism onto its image. This is the only section where we use the theory of bicovariant right modules (as opposed to bicovariant bimodules).

For the rest of the section, $\mathcal{E}$ will denote a bicovariant $\mathcal{A}$-bimodule. Moreover, $\{\omega_i\}_i$ will denote a basis of ${}_0\mathcal{E}$ and $\{\omega_i^*\}_i$ the dual basis, i.e, $\omega_i^*(\omega_j) = \delta_{ij}$. Let us recall that (1.2.4) implies the existence of elements R_{ij} in $\mathcal{A}$ such that

$$
{}_{\mathcal{E}}\Delta(\omega_i) = \sum_{ij} \omega_j \otimes_{\mathbb{C}} R_{ji}. \tag{5.2.1}
$$

We want to show that $V_g(\mathcal{E})$ is a bicovariant right $\mathcal{A}$-module in the sense of Definition 4.1.1. To this end, we recall that (Lemma 4.3.7) $V_g(\mathcal{E})$ is a free right $\mathcal{A}$-module with basis $\{\omega_i^*\}_i$. This allows us to make the following definition.

Definition 5.2.1. *Let $\{\omega_i\}_i$ and $\{\omega_i^*\}_i$ be as above and g a bi-invariant pseudo-Riemannian metric on $\mathcal{E}$. Then we can endow $V_g(\mathcal{E})$ with a left-coaction $\Delta_{V_g(\mathcal{E})} : V_g(\mathcal{E}) \to \mathcal{A} \otimes_{\mathbb{C}} V_g(\mathcal{E})$ and a right-coaction ${}_{V_g(\mathcal{E})}\Delta : V_g(\mathcal{E}) \to V_g(\mathcal{E}) \otimes_{\mathbb{C}} \mathcal{A}$, defined by the formulas*

$$
\Delta_{V_g(\mathcal{E})}(\sum_i \omega_i^* a_i) = \sum_i (1 \otimes_{\mathbb{C}} \omega_i^*)\Delta(a_i), \ \ {}_{V_g(\mathcal{E})}\Delta(\sum_i \omega_i^* a_i) = \sum_{ij} (\omega_j^* \otimes_{\mathbb{C}} S(R_{ij}))\Delta(a_i), \tag{5.2.2}
$$

where the elements R_{ij} are as in (5.2.1) and S is the antipode of the Hopf algebra $\mathcal{A}$.

Then we have the following result.

Proposition 5.2.2. *The triplet $(V_g(\mathcal{E}), \Delta_{V_g(\mathcal{E})}, {}_{V_g(\mathcal{E})}\Delta)$ is a bicovariant right $\mathcal{A}$-module. Moreover, the map $V_g : \mathcal{E} \to V_g(\mathcal{E})$ is bicovariant, i.e, we have*

$$
\Delta_{V_g(\mathcal{E})}(V_g(e)) = (\mathrm{id} \otimes_{\mathbb{C}} V_g)\Delta_{\mathcal{E}}(e), \ \ {}_{V_g(\mathcal{E})}\Delta(V_g(e)) = (V_g \otimes_{\mathbb{C}} \mathrm{id})_{\mathcal{E}}\Delta(e). \tag{5.2.3}
$$

Proof. The fact that $(V_g(\mathcal{E}), \Delta_{V_g(\mathcal{E})}, {}_{V_g(\mathcal{E})}\Delta)$ is a bicovariant right $\mathcal{A}$-module follows immediately from the definition of the maps $\Delta_{V_g(\mathcal{E})}$ and ${}_{V_g(\mathcal{E})}\Delta$. So we are left with proving (5.2.3). Let e be an element in $\mathcal{E}$. Then there exist elements a_i in $\mathcal{A}$ such that $e = \sum_i \omega_i a_i$. Hence, by (4.3.1), we obtain

$$
\begin{aligned}
\Delta_{V_g(\mathcal{E})}(V_g(e)) &= \Delta_{V_g(\mathcal{E})}(V_g(\sum_i \omega_i a_i)) = \Delta_{V_g(\mathcal{E})}(\sum_{ij} g_{ij}\omega_j^* a_i) = \sum_{ij} (1 \otimes_{\mathbb{C}} g_{ij}\omega_j^*)\Delta(a_i) \\
&= \sum_i ((\mathrm{id} \otimes_{\mathbb{C}} V_g)(1 \otimes_{\mathbb{C}} \omega_i))\Delta(a_i) = \sum_i (\mathrm{id} \otimes_{\mathbb{C}} V_g)(\Delta_{\mathcal{E}}(\omega_i))\Delta(a_i) \\
&= \sum_i (\mathrm{id} \otimes_{\mathbb{C}} V_g)\Delta_{\mathcal{E}}(\omega_i a_i) = (\mathrm{id} \otimes_{\mathbb{C}} V_g)\Delta_{\mathcal{E}}(e).
\end{aligned}
$$

This proves the first equation of (5.2.3).

For the second equation, we begin by making an observation. Since ${}_{\mathcal{E}}\Delta(\omega_i) = \sum_j \omega_j \otimes_{\mathbb{C}} R_{ji}$ ((5.2.1)), we have

$$\delta_{ij} = \epsilon(R_{ij}) = m(\mathrm{id} \otimes_{\mathbb{C}} S)\Delta(R_{ij}) = \sum_k R_{ik} S(R_{kj}).$$

Therefore, multiplying (4.3.5) by $S(R_{jm})$ on the right and summing over j, we obtain

$$\sum_j g_{ij} S(R_{jm}) = \sum_j g_{jm} R_{ji}. \tag{5.2.4}$$

Now by using (4.3.1), we compute

$$\begin{aligned}
{}_{V_g(\mathcal{E})}\Delta(V_g(e)) &= {}_{V_g(\mathcal{E})}\Delta(V_g(\sum_i \omega_i a_i)) = {}_{V_g(\mathcal{E})}\Delta(\sum_{ij} g_{ij}\omega_j^* a_i) = \sum_{ij} {}_{V_g(\mathcal{E})}\Delta(g_{ij}\omega_j^*)\Delta(a_i) \\
&= \sum_{ijk} (g_{ij}\omega_k^* \otimes_{\mathbb{C}} S(R_{jk}))\Delta(a_i) = (\sum_{ik} \omega_k^* \otimes_{\mathbb{C}} \sum_j g_{ij} S(R_{jk}))\Delta(a_i) \\
&= \sum_{ik} (\omega_k^* \otimes_{\mathbb{C}} \sum_j g_{jk} R_{ji})\Delta(a_i) \text{ (by (5.2.4))} \\
&= \sum_{ijk} (g_{jk}\omega_k^* \otimes_{\mathbb{C}} R_{ji})\Delta(a_i) = \sum_{ij} (V_g(\omega_j) \otimes_{\mathbb{C}} R_{ji})\Delta(a_i) \\
&= \sum_i ((V_g \otimes_{\mathbb{C}} \mathrm{id})(\sum_j \omega_j \otimes_{\mathbb{C}} R_{ji}))\Delta(a_i) = \sum_i (V_g \otimes_{\mathbb{C}} \mathrm{id})_{\mathcal{E}}\Delta(\omega_i)\Delta(a_i) \text{ (by (5.2.1))} \\
&= \sum_i (V_g \otimes_{\mathbb{C}} \mathrm{id})_{\mathcal{E}}\Delta(\omega_i a_i) = (V_g \otimes_{\mathbb{C}} \mathrm{id})_{\mathcal{E}}\Delta(e).
\end{aligned}$$

This finishes the proof. □

Now we recall that bicovariant right $\mathcal{A}$-modules (i.e., objects in the category ${}^{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$) can be deformed too.

Proposition 5.2.3. *(Theorem 5.7 of [84]) Let $(M, \Delta_M, {}_M\Delta)$ be a bicovariant right $\mathcal{A}$-module and γ be a 2-cocycle on $\mathcal{A}$. Then*

(i) M deforms to a bicovariant right $\mathcal{A}_\gamma$-module, denoted by M_γ,

(ii) if $(N, \Delta_N, {}_N\Delta)$ is another bicovariant right $\mathcal{A}$-module and $T : M \to N$ is a bicovariant right $\mathcal{A}$-linear map, then the deformation $T_\gamma : M_\gamma \to N_\gamma$ is a bicovariant right $\mathcal{A}_\gamma$-linear map,

(iii) T_γ, as in (ii), is an isomorphism if and only if T is an isomorphism.

Proof. We refer to Theorem 5.7 of [84] for proofs of (i) and (ii). Part (iii) follows by noting that since the map T is a bicovariant right $\mathcal{A}$-linear map, its inverse T^{-1} is also a bicovariant right $\mathcal{A}$-linear map. Thus, the deformation $(T^{-1})_\gamma$ of T^{-1} exists and is the inverse of the map T_γ. □

As an immediate corollary, we make the following observation.

Corollary 5.2.4. *Let g be a bi-invariant pseudo-Riemannian metric on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$. Then the following map is a well-defined isomorphism.*

$$(V_g)_\gamma : \mathcal{E}_\gamma \to (V_g(\mathcal{E}))_\gamma = (V_g)_\gamma(\mathcal{E}_\gamma).$$

Proof. Since both $\mathcal{E}$ and $V_g(\mathcal{E})$ are bicovariant right $\mathcal{A}$-modules, and V_g is a right $\mathcal{A}$-linear bicovariant map from $\mathcal{E}$ to $V_g(\mathcal{E})$ (Proposition 5.2.2), Proposition 5.2.3 guarantees the existence of the map $(V_g)_\gamma$ from $\mathcal{E}_\gamma$ to $(V_g(\mathcal{E}))_\gamma$. Since g is a pseudo-Riemannian metric, by (ii) of Definition 4.3.1, $V_g : \mathcal{E} \to V_g(\mathcal{E})$ is an isomorphism. Then, by (iii) of Proposition 5.2.3, $(V_g)_\gamma$ is also an isomorphism from $\mathcal{E}_\gamma$ to $(V_g(\mathcal{E}))_\gamma$. In particular, this implies that $(V_g(\mathcal{E}))_\gamma = (V_g)_\gamma(\mathcal{E}_\gamma)$. □

Now we are in a position to state and prove that there is an abundant supply of bi-invariant pseudo-Riemannian metrics on $\mathcal{E}_\gamma$. Since g is a map from $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ to $\mathcal{A}$, g_γ is a map from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ to $\mathcal{A}_\gamma$. But we have the isomorphism ξ from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ (Proposition 5.1.5). As in the case of the map σ_γ in Subsection 4.3, we will make an abuse of notation to denote the map $g_\gamma \xi^{-1}$ by the symbol g_γ.

Theorem 5.2.5. *If g is a bi-invariant pseudo-Riemannian metric on a bicovariant $\mathcal{A}$-bimodule $\mathcal{E}$ and γ is a 2-cocycle on $\mathcal{A}$, then g deforms to a right $\mathcal{A}_\gamma$-linear map g_γ from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to itself. Moreover, g_γ is a bi-invariant pseudo-Riemannian metric on $\mathcal{E}_\gamma$. Finally, any bi-invariant pseudo-Riemannian metric on $\mathcal{E}_\gamma$ is a deformation (in the above sense) of some bi-invariant pseudo-Riemannian metric on $\mathcal{E}$.*

Proof. Since g is a right $\mathcal{A}$-linear bicovariant map (Proposition 4.3.3), g indeed deforms to a right $\mathcal{A}_\gamma$-linear map g_γ from $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma \cong \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ (see Proposition 5.1.5) to $\mathcal{A}_\gamma$. The second assertion of Proposition 5.1.2 implies that g_γ is bicovariant. Then Proposition 4.3.3 implies that g_γ is bi-invariant. Since $g \circ \sigma = g$, part (iii) of Proposition 5.1.2 implies that

$$g_\gamma = (g \circ \sigma)_\gamma = g_\gamma \circ \sigma_\gamma.$$

This verifies condition (i) of Definition 4.3.1

Next, we prove that g_γ satisfies (ii) of Definition 4.3.1. Let ω be an element of ${}_0\mathcal{E} = {}_0(\mathcal{E}_\gamma)$ and η be an element of $\mathcal{E}_0 = (\mathcal{E}_\gamma)_0$. Then we have

$$\begin{aligned}
&(V_g)_\gamma(\omega)(\eta) = (V_g(\omega))_\gamma(\eta) = (V_g(\omega)(\eta)) \\
=&g(\omega \otimes_{\mathcal{A}} \eta) = g_\gamma(\overline{\gamma}(1 \otimes_{\mathbb{C}} \eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}_\gamma} \eta_{(0)}\gamma(\omega_{(1)} \otimes_{\mathbb{C}} 1)) \\
&(\text{ by the definition of } \xi^{-1} \text{ in } Proposition \ 5.1.5 \ ) \\
=&g_\gamma(\epsilon(\eta_{(-1)})\omega_{(0)} \otimes_{\mathcal{A}_\gamma} \eta_{(0)}\epsilon(\omega_{(1)})) = g_\gamma(\omega \otimes_{\mathcal{A}_\gamma} \eta) = V_{g_\gamma}(\omega)(\eta).
\end{aligned}$$

Then, by the right-$\mathcal{A}_\gamma$ linearity of $(V_g)_\gamma(\omega)$ and $V_{(g_\gamma)}(\omega)$, we get, for all a in $\mathcal{A}$,

$$V_{g_\gamma}(\omega)(\eta *_\gamma a) = V_{g_\gamma}(\omega)(\eta) *_\gamma a = (V_g)_\gamma(\omega)(\eta) *_\gamma a = (V_g)_\gamma(\omega)(\eta *_\gamma a).$$

Therefore, by the right $\mathcal{A}$-totality of $(\mathcal{E}_\gamma)_0 = \mathcal{E}_0$ in $\mathcal{E}_\gamma$, we conclude that the maps $(V_g)_\gamma$ and V_{g_γ} agree on ${}_0(\mathcal{E}_\gamma)$. But since ${}_0(\mathcal{E}_\gamma) = {}_0\mathcal{E}$ is right $\mathcal{A}_\gamma$-total in $\mathcal{E}_\gamma$ and both V_{g_γ} and $(V_g)_\gamma$ are right-$\mathcal{A}_\gamma$ linear, $(V_g)_\gamma = V_{g_\gamma}$ on the whole of $\mathcal{E}_\gamma$.

Next, since V_g is a right $\mathcal{A}$-linear isomorphism from $\mathcal{E}$ to $V_g(\mathcal{E})$, hence by Corollary 5.2.4, $(V_g)_\gamma$ is an isomorphism onto $(V_g(\mathcal{E}))_\gamma = (V_g)_\gamma(\mathcal{E}_\gamma) = V_{g_\gamma}(\mathcal{E}_\gamma)$. Therefore V_{g_γ} is an isomorphism from $\mathcal{E}_\gamma$ to $V_{g_\gamma}(\mathcal{E}_\gamma)$. Hence g_γ satisfies (ii) of Definition 4.3.1.

To show that every pseudo-Riemannian metric on $\mathcal{E}_\gamma$ is obtained as a deformation of a pseudo-Riemannian metric on $\mathcal{E}$, we view $\mathcal{E}$ as a cocycle deformation of $\mathcal{E}_\gamma$ under the cocycle $\overline{\gamma}$. Then given a pseudo-Riemannian metric g' on $\mathcal{E}_\gamma$, by the first part of this proof, $(g')_{\overline{\gamma}}$ is a bi-invariant pseudo-Riemannian metric on $\mathcal{E}$. Hence, $g' = ((g')_{\overline{\gamma}})_\gamma$ is indeed a deformation of the bi-invariant pseudo-Riemannian metric $(g')_{\overline{\gamma}}$ on $\mathcal{E}$. □

Remark 5.2.6. *We have actually used the fact that $\mathcal{E}$ is finite in order to prove Theorem 5.2.5. Indeed, since $\mathcal{E}$ is finite, we can use the results of Section 4.3 to derive Proposition 5.2.2 which is then used to prove Corollary 5.2.4. Finally, Corollary 5.2.4 is used to prove Theorem 5.2.5.*

Also note that the proof of Theorem 5.2.5 also implies that the maps $(V_g)_\gamma$ and V_{g_γ} are equal.

When g is a pseudo-Riemannian bicovariant bilinear metric on $\mathcal{E}$, then we have a much shorter proof of the fact that g_γ is a pseudo-Riemannian metric on $\mathcal{E}_\gamma$ which avoids the theory of bicovariant right $\mathcal{A}$-modules. We end this section with a brief discussion of the proof which is as follows:

We will work in the categories ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ and ${}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}\mathcal{M}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}$. Firstly, as g is bilinear, V_g is a morphism of the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$. and can be deformed to a bicovariant $\mathcal{A}_\gamma$-bilinear map $(V_g)_\gamma$ from $\mathcal{E}_\gamma$ to $(\mathcal{E}^*)_\gamma$. Similarly, g deforms to a $\mathcal{A}_\gamma$-bilinear map from $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ to $\mathcal{A}_\gamma$. Then as in the proof of Theorem 5.2.5, we can easily check that $(V_g)_\gamma = V_{g_\gamma}$.

On the other hand, from Proposition 4.3.9, we know that the left dual $\widetilde{\mathcal{E}}$ of $\mathcal{E}$ (in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$) is isomorphic to $\mathcal{E}^*$. Since g is bilinear, Proposition 4.3.9 implies that the morphism V_g (in the category ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$) is an isomorphism from $\mathcal{E}$ to $\mathcal{E}^*$.

Therefore, we have an isomorphism $(V_g)_\gamma$ is an isomorphism from $\mathcal{E}_\gamma$ to $(\mathcal{E}^*)_\gamma$. Since the functor from ${}^{\mathcal{A}}_{\mathcal{A}}\mathcal{M}^{\mathcal{A}}_{\mathcal{A}}$ to ${}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}\mathcal{M}^{\mathcal{A}_\gamma}_{\mathcal{A}_\gamma}$ sending M to M_γ is monoidal by Proposition 5.1.5, we can apply the second assertion of Proposition 1.1.10 to deduce that $(\mathcal{E}^*)_\gamma \cong (\mathcal{E}_\gamma)^*$. Thus $(V_g)_\gamma$ is an isomorphism from $\mathcal{E}_\gamma$ to $(\mathcal{E}_\gamma)^*$. As $(V_g)_\gamma = V_{g_\gamma}$, we deduce that V_{g_γ} is an isomorphism from $\mathcal{E}_\gamma$ to $(\mathcal{E}_\gamma)^*$. Since the equation $g_\gamma \circ \sigma_\gamma = g_\gamma$, this completes the proof.

5.3 Cocycle deformation of bicovariant differential calculi

In this section, we deform a first order bicovariant differential calculus $(\mathcal{E}_\gamma, d_\gamma)$ over $\mathcal{A}$ and see that $(\mathcal{E}_\gamma, d_\gamma)$ is a first order bicovariant differential calculus on $\mathcal{A}_\gamma$.

Since $d : \mathcal{A} \to \mathcal{E}$ is a bicovariant map between bicovariant bimodules, by Proposition 5.1.2, we have the map

$$d_\gamma := d : \mathcal{A}_\gamma \to \mathcal{E}_\gamma.$$

Proposition 5.3.1. *([74]) The tuple $(\mathcal{E}_\gamma, d_\gamma)$ is a first-order bicovariant differential calculus on $\mathcal{A}_\Omega$.*

Proof. Though the proof of this result is already available in Proposition 3.2 and Corollary 3.4 of [74], we provide the proof here in our notations for the sake of completeness. We start by proving that $d_\gamma : \mathcal{A}_\Omega \to \mathcal{E}_\gamma$ is a derivation. For a, b in $\mathcal{A}_\Omega$, we compute

$$\begin{aligned}
& d_\gamma(a *_\gamma b) \\
= \;& \gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)}) d(a_{(2)} b_{(2)}) \overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} b_{(3)}) \text{ (by 1.2.2)} \\
= \;& \gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)}) d(a_{(2)}) b_{(2)} \overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} b_{(3)}) + \gamma(a_{(1)} \otimes_{\mathbb{C}} b_{(1)}) a_{(2)} d(b_{(2)}) \overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} b_{(3)})
\end{aligned}$$

$$
\begin{aligned}
&= \gamma((da)_{(-1)} \otimes_{\mathbb{C}} b_{(1)})((da)_{(0)} b_{(2)} \overline{\gamma}((da)_{(1)} \otimes_{\mathbb{C}} b_{(3)})) \\
&\quad + \gamma(a_{(1)} \otimes_{\mathbb{C}} (db)_{(-1)}) a_{(2)} (db)_{(0)} \overline{\gamma}(a_{(3)} \otimes_{\mathbb{C}} (db)_{(1)}) \\
&\quad \text{(by part(iii) of Lemma 1.3.16)} \\
&= da *_{\gamma} b + a *_{\gamma} db,
\end{aligned}
$$

where in the last step we have used (5.1.3) and (5.1.4). This proves that d_γ is a derivation on $\mathcal{A}_\gamma$.

Next, we observe that since $\mathcal{E}_\gamma = \mathcal{E}, d_\gamma = d$, and $(\mathcal{E}, d)$ is a first order differential calculus on $\mathcal{A}$, $(\mathcal{E}_\gamma, d_\gamma)$ is a first order differential calculus on $\mathcal{A}_\gamma$ (see Definition 1.3.14). To prove that $(\mathcal{E}_\gamma, d_\gamma)$ is a left-covariant differential calculus, let $a_i, i = 1, \ldots, k$ be elements in $\mathcal{A}$ such that $\sum_k a_k *_\gamma db_k = 0$.

Now, since $d_\gamma = d$ and $\Delta_\gamma = \Delta$ as maps, we have

$$(d_\gamma \otimes_{\mathbb{C}} \mathrm{id}) \Delta_\gamma(a) = {}_{\mathcal{E}_\gamma}\Delta \circ d_\gamma \text{ and } (\mathrm{id} \otimes_{\mathbb{C}} d_\gamma) \Delta_\gamma = \Delta_{\mathcal{E}_\gamma} \circ d_\gamma.$$

Moreover, by Proposition 5.1.1, $\mathcal{E}_\gamma$ is a bicovariant $\mathcal{A}_\gamma$-bimodule. Thus, if $\sum_k a_k *_\gamma d(b_k) = 0$, we get

$$
\begin{aligned}
&\sum_k \Delta_\gamma(a_k) *_\gamma (\mathrm{id} \otimes_{\mathbb{C}} d_\gamma) \Delta_\gamma(b_k) = \sum_k \Delta_\gamma(a_k) *_\gamma \Delta_{\mathcal{E}_\gamma}(d_\gamma(b_k)) \\
= \;& \Delta_\gamma\Big(\sum_k a_k *_\gamma d(b_k)\Big) = 0.
\end{aligned}
$$

Therefore, $(\mathcal{E}_\gamma, d_\gamma)$ is a left-covariant first order differential calculus. Similarly, it can be proved that $(\mathcal{E}_\gamma, d_\gamma)$ is a right-covariant first order differential calculus. This completes the proof. $\square$

If $(\mathcal{E}, d)$ is a bicovariant differential calculus such that ${}_0\sigma$ is diagonalisable, then we have proved (Theorem 4.2.5) that $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} = \mathrm{Ker}(\wedge) \oplus \mathcal{F}$, where $\mathcal{F} = \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}({}_0\mathcal{F} \otimes_{\mathbb{C}} \mathcal{A})$. Here, ${}_0\mathcal{F}$ is the direct sum of eigenspaces of ${}_0\sigma$ corresponding to the eigenvalues which are not equal to 1 and $\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}$ is the isomorphism defined in (4.2.1). Moreover, we have a bicovariant $\mathcal{A}$-bilinear idempotent map P_{sym} on $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ with range $\mathrm{Ker}(\wedge)$ and kernel $\mathcal{F}$, defined by the equation (see Definition 4.2.6)

$$P_{\mathrm{sym}} = \widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}}({}_0(P_{\mathrm{sym}}) \otimes_{\mathbb{C}} \mathrm{id})(\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}})^{-1}.$$

Since $P_{\mathrm{sym}} : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$ is bicovariant, we have a deformed map $(P_{\mathrm{sym}})_\gamma : (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. With an abuse of notation, we will denote the map $\xi^{-1}(P_{\mathrm{sym}})_\gamma \xi : \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma \to \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ by the symbol $(P_{\mathrm{sym}})_\gamma$ again.

Now, let us consider the bicovariant differential calculus $(\mathcal{E}_\gamma, d_\gamma)$. By Proposition 5.3.1, we can apply Theorem 4.2.5 (to $(\mathcal{E}_\gamma, d_\gamma)$) to get a bicovariant $\mathcal{A}_\gamma$-bilinear idempotent $(P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$ on $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$. It is worthwhile to note that the map $(P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$ coincides with the cocycle deformation $(P_{\mathrm{sym}})_\gamma$ of the map P_{sym}. Indeed, since ${}_0(\sigma_\gamma) = {}_0\sigma$ on ${}_0(\mathcal{E}_\gamma) \otimes_{\mathbb{C}} {}_0(\mathcal{E}_\gamma) = {}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$, the kernel of $(P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$ is equal to $\widetilde{u}^{\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma}({}_0\mathcal{F} \otimes_{\mathbb{C}} \mathcal{A}_\gamma)$. However, using the isomorphism $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma \cong \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$, it is easy to check that

$$\begin{aligned}
&\widetilde{u}^{\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma}({}_0\mathcal{F} \otimes_{\mathbb{C}} \mathcal{A}_\gamma) = (\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}})_\gamma(({}_0\mathcal{F} \otimes_{\mathbb{C}} \mathcal{A})_\gamma) \\
=&((\widetilde{u}^{\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}})({}_0\mathcal{F} \otimes_{\mathbb{C}} \mathcal{A}))_\gamma = \mathcal{F}_\gamma = \mathrm{Ker}((P_{\mathrm{sym}})_\gamma).
\end{aligned}$$

On the other hand, by the definition of $(P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$,

$$\begin{aligned}
&\mathrm{Ran}((P_{\mathrm{sym}})_{\mathcal{E}_\gamma}) = \mathrm{Ker}(\sigma_\gamma - 1) = (\mathrm{Ker}(\sigma - 1))_\gamma \\
=&(\mathrm{Ran}(P_{\mathrm{sym}}))_\gamma = \mathrm{Ran}((P_{\mathrm{sym}})_\gamma)
\end{aligned}$$

Since $(P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$ and $(P_{\mathrm{sym}})_\gamma$ are both idempotents on $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ with the same kernel and the same range, we can conclude that $(P_{\mathrm{sym}})_\gamma = (P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$. We collect the observations made above in the following proposition.

Proposition 5.3.2. *Let $(\mathcal{E}, d)$ be a bicovariant differential calculus over $\mathcal{A}$ such that ${}_0\sigma$ is diagonalisable and γ be a 2-cocycle. Then the maps $(P_{\mathrm{sym}})_{\mathcal{E}_\gamma}$ and $(P_{\mathrm{sym}})_\gamma$ coincide. Moreover, we have*

$$\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma = \mathrm{Ker}(\wedge_\gamma) \oplus \mathcal{F}_\gamma = \mathrm{Ker}(\sigma_\gamma - 1) \oplus \mathcal{F}_\gamma.$$

Cocycle deformation of two-forms

In order to introduce the deformation of the space of two-forms, we need the deformation of the braiding map σ of the space of one-forms $\mathcal{E}$, which was discussed in Theorem 5.1.7. Utilising the map σ_γ we have the following result.

Proposition 5.3.3. *Let $\mathcal{E}$ be a bicovariant $\mathcal{A}$-bimodule and γ be a cocyle as above. Then the space of two-forms $\Omega^2(\mathcal{A}_\gamma)$ of the cocycle deformed algebra $\mathcal{A}_\gamma$ is the deformed bimodule*

$(\Omega^2(\mathcal{A}))_\gamma$. *Moreover, the deformation* d_γ *of the map* $d : \mathcal{E} \to \Omega^2(\mathcal{A})$ *is the bicovariant derivative map from the space of one-forms to the space of two-forms.*

Proof. By Theorem, 5.1.7, σ_γ is the canonical braiding map on $\mathcal{E}_\gamma$. Hence, the space of two-forms $\Omega^2(\mathcal{A}_\gamma) = (\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma)/\mathrm{Ker}(\sigma_\gamma - 1)$. Since $\sigma_\gamma = \sigma$ as vector space maps, $\mathrm{Ker}(\sigma_\gamma - 1) = \mathrm{Ker}(\sigma - 1)$ as vector spaces. Therefore, using the isomorphism $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma \cong (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$, we have that $\Omega^2(\mathcal{A}_\gamma) = ((\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})/\mathrm{Ker}(\sigma - 1))_\gamma$. Thus, we have the first part of the statement.
By Proposition 1.3.20, the map $d : \mathcal{E} \to \Omega^2(\mathcal{A})$ is a bicovariant map. Therefore, by Proposition 5.1.2, it deforms to the map

$$d_\gamma : \mathcal{E}_\gamma \to (\Omega^2(\mathcal{A}))_\gamma,$$

which satisfies the properties of Proposition 1.3.20. By the first part of this proof, $(\Omega^2(\mathcal{A}))_\gamma = \Omega^2(\mathcal{A}_\gamma)$. Hence, d_γ is the derivative map from the space of one-forms to the space of two-forms, and we are done with our proof. □

5.4 Existence and uniqueness of Levi-Civita connections

This section concerns the Levi-Civita connections on bicovariant differential calculus on cocycle deformations of Hopf algebras. We discuss the effect of cocycle deformations on the map P_{sym} as well as bicovariant connections. Finally, we prove the main theorem which states that if $(\mathcal{E}, d)$ is a bicovariant differential calculus such that ${}_0\sigma$ is diagonalisable and g is a pseudo-Riemannian bi-invariant metric on $\mathcal{E}$ such that $(\mathcal{E}, d, g)$ admits a bicovariant Levi-Civita connection ∇, then ∇ deforms to a bicovariant Levi-Civita connection for the deformed triple $(\mathcal{E}_\gamma, d_\gamma, g_\gamma)$.

We start by discussing bicovariant connections on $\mathcal{E}_\gamma$. Suppose that ∇ is a bicovariant connection on $\mathcal{E}$. Then Proposition 5.1.2 yields a $\mathbb{C}$-linear map ∇_γ from $\mathcal{E}_\gamma$ to $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. However, we would like to have the deformed map to take value in $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$. For this, we will need to use the isomorphism $\xi : \mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ introduced in Proposition 5.1.5.

The following lemma will be needed to prove that ∇_γ is actually a connection.

Lemma 5.4.1. *If* ∇ *is a bicovariant connection on a bicovariant differential calculus* $(\mathcal{E}, d)$ *and we write*

$${}_{\mathcal{E}\otimes_{\mathcal{A}}\mathcal{E}}\Delta(\nabla(e)) = (\nabla(e))_{(0)} \otimes_{\mathbb{C}} (\nabla(e))_{(1)},$$

then for all ω in ${}_0\mathcal{E}$ and a in $\mathcal{A}$, we have

$$\xi^{-1}((\nabla(\omega))_{(0)}a_{(1)}\overline{\gamma}((\nabla(\omega))_{(1)}\otimes_{\mathbb{C}}a_{(2)})) = \xi^{-1}(\nabla_\gamma(\omega)) *_\gamma a,$$

where $\nabla_\gamma : \mathcal{E}_\gamma \to (\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ is the deformation of the $\mathbb{C}$-linear bicovariant map $\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}$.

Proof. We will use the right $\mathcal{A}_\gamma$-module structure of $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$ and the bicovariance of the map ∇_γ (Proposition 5.1.2). In particular, this implies that if ω is in ${}_0\mathcal{E}$, then $\nabla_\gamma(\omega)$ is an element of ${}_0((\mathcal{E} \otimes \mathcal{E})_\gamma)$. Hence, we get:

$$\begin{aligned}\nabla_\gamma(\omega) * a &= (\nabla_\gamma(\omega))_{(0)}.a_{(1)}\overline{\gamma}((\nabla_\gamma(\omega))_{(1)}\otimes_{\mathbb{C}}a_{(2)})\\ &=(\nabla(\omega))_{(0)}.a_{(1)}\overline{\gamma}((\nabla(\omega))_{(1)}\otimes_{\mathbb{C}}a_{(2)}),\end{aligned}$$

where the equality is of elements in $(\mathcal{E} \otimes_{\mathcal{A}} \mathcal{E})_\gamma$. Then, using Lemma 5.1.6, by the right $\mathcal{A}_\gamma$-linearity of ξ, we have

$$\xi^{-1}((\nabla(\omega))_{(0)}a_{(1)}\overline{\gamma}((\nabla(\omega))_{(1)}\otimes_{\mathbb{C}}a_{(2)})) = \xi^{-1}(\nabla_\gamma(\omega)) *_\gamma a,$$

where the equality is of elements in $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$. This completes the proof of the lemma. □

By an abuse of notation, we will denote the map $\xi^{-1}\nabla_\gamma$ by the symbol ∇_γ again. Thus, ∇_γ takes value in $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ as desired. Then we have the following theorem.

Theorem 5.4.2. *Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus. Then a bicovariant connection ∇ deforms to a bicovariant connection ∇_γ on $\mathcal{E}_\gamma$. In fact, bicovariant connections on $\mathcal{E}$ and $\mathcal{E}_\gamma$ are in bijective correspondence.*

Proof. For ω in ${}_0\mathcal{E}$ and a in $\mathcal{A}$, we have

$$\begin{aligned}\nabla_\gamma(\omega *_\gamma a) &= \nabla_\gamma(\omega_{(0)}a_{(1)}\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}a_{(2)}))\\ &=\nabla_\gamma(\omega_{(0)}a_{(1)})\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}a_{(2)}) = \nabla(\omega_{(0)}a_{(1)})\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}a_{(2)})\\ &=(\nabla(\omega_{(0)})a_{(1)} + \omega_{(0)} \otimes_{\mathcal{A}} d(a_{(1)}))\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}a_{(2)})\\ &=\nabla(\omega_{(0)})a_{(1)}\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}a_{(2)}) + \omega_{(0)} \otimes_{\mathcal{A}} d(a_{(1)})\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}a_{(2)}).\end{aligned}$$

Now, by the right covariance of the maps ∇ and d (see (1.3.6)), the following equations hold:

$$\nabla(\omega_{(0)})\otimes_{\mathbb{C}}\omega_{(1)} = (\nabla(\omega))_{(0)}\otimes_{\mathbb{C}}(\nabla(\omega))_{(1)},\ d(a_{(1)})\otimes_{\mathbb{C}}a_{(2)} = (da)_{(0)}\otimes_{\mathbb{C}}(da)_{(1)},$$

and therefore, the above expression is equal to

$$\begin{aligned}&(\nabla(\omega))_{(0)}a_{(1)}\overline{\gamma}((\nabla(\omega))_{(1)}\otimes_{\mathbb{C}}a_{(2)})+\omega_{(0)}\otimes_{\mathcal{A}}(da)_{(0)}\overline{\gamma}(\omega_{(1)}\otimes_{\mathbb{C}}(da)_{(1)})\\ =&\nabla_\gamma(\omega)*_\gamma a+\omega\otimes_{\mathcal{A}_\gamma}d_\gamma a\end{aligned}$$

where we have used the two equations of Lemma 5.4.1. This proves that for all ω in ${}_0\mathcal{E}$ and a in $\mathcal{A}$,

$$\nabla_\gamma(\omega *_\gamma a)=\nabla_\gamma(\omega)*_\gamma a+\omega\otimes_{\mathcal{A}_\gamma}da. \tag{5.4.1}$$

Since ${}_0\mathcal{E}={}_0(\mathcal{E}_\gamma)$ is right $\mathcal{A}_\gamma$-total in $\mathcal{E}_\gamma$, we are left to prove that for all a,b in $\mathcal{A}$ and ω in ${}_0\mathcal{E}$,

$$\nabla_\gamma((\omega *_\gamma a)*_\gamma b)=\nabla_\gamma(\omega *_\gamma a)*_\gamma b+\omega *_\gamma a\otimes_{\mathcal{A}_\gamma}d_\gamma b.$$

But this follows easily from (5.4.1). Since the right and left comodule structure of the calculus and its deformation are the same, hence ∇_γ is also bicovariant.
To show that the bicovariant connections of $\mathcal{E}$ and $\mathcal{E}_\gamma$ are in a bijective correspondence, we consider the bicovariant calculus $(\mathcal{E},d)$ as a cocycle deformation of the calculus $(\mathcal{E}_\gamma,d_\gamma)$ under the cocycle $\overline{\gamma}$. If ∇' is a bicovariant connection on $(\mathcal{E}_\gamma,d_\gamma)$, then by the above argument, $(\nabla')_{\overline{\gamma}}$ is a bicovariant connection on $((\mathcal{E}_\gamma)_{\overline{\gamma}},(d_\gamma)_{\overline{\gamma}})=(\mathcal{E},d)$. Moreover, $\nabla'=((\nabla')_{\overline{\gamma}})_\gamma$ and hence is a cocycle deformation of a bicovariant connection on $(\mathcal{E},d)$ under the cocycle γ. □

Next we prove the main result of this section, namely that, if $(\mathcal{E},d)$ is a bicovariant differential calculus on $\mathcal{A}$ satisfying the conditions of Theorem 4.5.9 and g' is a pseudo-Riemannian bi-invariant metric on the deformed bimodule $\mathcal{E}_\gamma$, then there exists a unique left-invariant connection which is torsionless and compatible with g'. This is an analogue of Theorem 3.3.1 proved in Section 3 for Connes-Landi deformations of bimodules. We will continue to use the notations σ_γ, g_γ introduced in Theorem 5.3.1 and ∇_γ from Theorem 5.4.2. In particular, if g be a pseudo-Riemannian bi-invariant metric on $\mathcal{E}$, then g_γ is a pseudo-Riemannian bi-invariant metric on $\mathcal{E}_\gamma$ by Theorem 5.2.5.

Theorem 5.4.3. *Suppose $(\mathcal{E},d)$ is a bicovariant differential calculus on a Hopf algebra $\mathcal{A}$, σ be the corresponding braiding map and γ a 2-cocycle on $\mathcal{A}$. If ${}_0\sigma$ is diagonalisable and g is a pseudo-Riemannian bi-invariant metric on $\mathcal{E}$, then the following statements hold:*

(i) If ∇ is a bicovariant Levi-Civita connection for the triple $(\mathcal{E},d,g)$, then ∇ deforms to a bicovariant Levi-Civita connection ∇_γ for $(\mathcal{E}_\gamma,d_\gamma,g_\gamma)$.

(ii) In the set-up of (i), if we assume that ∇ is the unique Levi-Civita connection for $(\mathcal{E}, d, g)$, then ∇_γ is the unique bi-covariant Levi-Civita connection for $(\mathcal{E}_\gamma, d_\gamma, g_\gamma)$.

Proof. We start by proving that ∇_γ is torsionless and metric compatible. Since $\wedge$, ∇ and d are bicovariant, therefore the right $\mathcal{A}$-linear homomorphism $T_\nabla = \wedge \circ \nabla + d$ is also bicovariant. Therefore its cocycle deformation exists and

$$(T_\nabla)_\gamma = (\wedge \circ \nabla + d)_\gamma = \wedge_\gamma \circ \nabla_\gamma + d_\gamma = T_{\nabla_\gamma}.$$

Since ∇ is torsionless, we have that $T_{\nabla_\gamma} = 0$.

Now we prove that ∇_γ is compatible with the metric g_γ. The map $dg : \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E} \to \mathcal{E}$ is also a bicovariant map as Proposition 1.3.15 and Proposition 4.3.3 imply that d and g are bicovariant maps. Therefore, since ∇ is bicovariant and g is bi-invariant, Remark 4.4.6 and Proposition 4.4.10 imply that the map $\widetilde{\Pi_g}(\nabla) - dg$ is bicovariant. Therefore, the deformation of the map $\widetilde{\Pi_g}(\nabla) - dg$ exists and is equal to $\widetilde{\Pi_{g_\gamma}}(\nabla_\gamma) - d_\gamma g_\gamma$. Since $\widetilde{\Pi_g}(\nabla) - dg = 0$, therefore we have that $\widetilde{\Pi_{g_\gamma}}(\nabla_\gamma) - d_\gamma g_\gamma = 0$.
For the second part of the proof, assume that ∇' is a bicovariant Levi-Civita connection for the triple $(\mathcal{E}_\gamma, d_\gamma, g_\gamma)$. Viewing $(\mathcal{E}, d, g)$ as a cocycle deformation of $(\mathcal{E}_\gamma, d_\gamma, g_\gamma)$ under the cocycle $\overline{\gamma}$, by the first part of the proof, $(\nabla')_{\overline{\gamma}}$ is a bicovariant Levi-Civita connection on $(\mathcal{E}_\gamma, d_\gamma, g_\gamma)$. By our hypothesis, such a connection is unique. Hence $(\nabla')_{\overline{\gamma}} = \nabla$, and hence $\nabla' = \nabla_\gamma$. Thus $(\mathcal{E}_\gamma, d_\gamma, g_\gamma)$ admits a unique bicovariant Levi-Civita connection. $\square$

In Theorem 4.5.9, we proved that if the map $({}_0(P_{\mathrm{sym}}))_{23}$ is an isomorphism from $({}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} {}_0\mathcal{E}$ to ${}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes^{\mathrm{sym}}_{\mathbb{C}} {}_0\mathcal{E})$, then there exists a unique left-covariant Levi-Civita connection for $(\mathcal{E}, d, g)$. The next theorem shows that under the same assumption, $(\mathcal{E}_\gamma, d_\gamma)$ admits a unique left-covariant Levi-Civita connection for any bi-invariant pseudo-Riemannian metric.

Theorem 5.4.4. *Suppose $(\mathcal{E}, d)$ is a bicovariant differential calculus such that ${}_0\sigma$ is diagonalisable. If the map*

$$({}_0(P_{\mathrm{sym}}))_{23} : ({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}} {}_0\mathcal{E}) \otimes_{\mathbb{C}} {}_0\mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}} ({}_0\mathcal{E} \otimes_{\mathbb{C}}{}^{\mathrm{sym}} {}_0\mathcal{E})$$

is an isomorphism, then

(i) considering the map $(P_{\mathrm{sym}})_\gamma$ *as in Proposition 5.3.2, the following map is also an isomorphism:*

$$({}_0((P_{\mathrm{sym}})_\gamma))_{23} : ({}_0(\mathcal{E}_\gamma) \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0(\mathcal{E}_\gamma)) \otimes_{\mathbb{C}} {}_0(\mathcal{E}_\gamma) \to {}_0(\mathcal{E}_\gamma) \otimes_{\mathbb{C}} ({}_0(\mathcal{E}_\gamma) \otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0(\mathcal{E}_\gamma)),$$

(ii) for every bi-invariant pseudo-Riemannian metric g'*, the corresponding deformed calculus* $(\mathcal{E}_\gamma, d_\gamma)$ *admits a unique left-covariant connection which is torsionless and compatible with* g'*. Moreover, if* $\mathcal{A}$ *is cosemisimple, this connection is also right-covariant.*

Proof. The first part of the theorem follows by recalling that ${}_0(\mathcal{E}_\gamma) = {}_0\mathcal{E}$ and the fact that by Proposition 5.1.4, we have ${}_0(P_{\mathrm{sym}}) = {}_0((P_{\mathrm{sym}})_\gamma)$.

By Proposition 5.3.2, $(P_{\mathrm{sym}})_\gamma$ is the unique idempotent on $\mathcal{E}_\gamma \otimes_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ with range $\mathcal{E}_\gamma \otimes^{\mathrm{sym}}_{\mathcal{A}_\gamma} \mathcal{E}_\gamma$ and kernel $\mathcal{F}_\gamma$. The existence of a unique left-covariant Levi-Civita connection for $(\mathcal{E}_\gamma, d_\gamma, g')$ follows by combining the first part and Theorem 4.5.9.

If in addition, if $\mathcal{A}$ is cosemisimple, then $\mathcal{A}_\gamma$ is also cosemisimple and the right-covariance of the Levi-Civita connection follows from Theorem 4.5.9. $\square$

As a direct corollary to Theorem 5.4.4 and the existence-uniqueness theorem for Levi-Civita connection on a classical manifold, we have:

Proposition 5.4.5. *Let* $\mathcal{A}$ *be the Hopf algebra of regular functions on a linear algebraic group* G *whose category of finite dimensional representations is semisimple. Suppose* $(\mathcal{E}, d)$ *is the classical bicovariant differential calculus on* $\mathcal{A}$ *and* γ *a 2-cocycle on* $\mathcal{A}$*. If* g' *is a pseudo-Riemannian bi-invariant metric on the bicovariant differential calculus* $(\mathcal{E}_\gamma, d_\gamma)$ *over the Hopf algebra* $\mathcal{A}_\gamma$*, then there exists a unique bicovariant Levi-Civita connection for the triple* $(\mathcal{E}_\gamma, d_\gamma, g')$*.*

Proof. The map g' is a bi-invariant pseudo-Riemannian metric on $\mathcal{E}_\gamma$ and so by Theorem 5.2.5, there exists a bi-invariant pseudo-Riemannian metric g on $\mathcal{E}$ such that $g_\gamma = g'$. The Levi-Civita connection for the triple $(\mathcal{E}, d, g)$ is bicovariant. This is well-known and can also be seen using Proposition 4.5.11 and Theorem 4.5.9. Therefore, we can apply Theorem 5.4.3 to reach the desired conclusion. $\square$

We conclude this section by proving Proposition 4.4.13 stated in the previous chapter, which shows that our definition of metric-compatibility coincides with that in [51] in the case of cocycle deformation of the Hopf algebra of regular functions on a linear algebraic group. The proof will use the notations and discussions preceding the statement of Theorem 4.4.13.

Proof of Proposition 4.4.13: By our assumption, ∇' and g' are bicovariant. It can be easily checked that analogues of Theorem 5.4.2 for left connections and the third assertion of Theorem 5.2.5 for left $\mathcal{A}$-linear pseudo-Riemannian metrics hold. This implies that there exist a bicovariant left-connection ∇ on $\mathcal{E}$ and a left $\mathcal{A}$-linear bi-invariant pseudo-Riemannian metric g on $\mathcal{E}$ such that $\nabla' = \nabla_\gamma$ and $g' = g_\gamma$.

Now suppose that $\nabla' = \nabla_\gamma$ is such that (4.4.2) holds for the left $\mathcal{A}$-linear bi-invariant pseudo-Riemannian metric $g' = g_\gamma$. Then by (4.4.3), $\widetilde{{}_L\Pi^0_{g_\gamma}}(\nabla_\gamma) = 0$, i.e,

$$2(g_\gamma \otimes_{\mathbb{C}} \mathrm{id})(\mathrm{id} \otimes_{\mathbb{C}} \sigma_\gamma)(\mathrm{id} \otimes_{\mathbb{C}} \nabla_\gamma)_0((P_{\mathrm{sym}})_\gamma) = 0$$

as maps on ${}_0(\mathcal{E}_\gamma) \otimes_{\mathbb{C}0} (\mathcal{E}_\gamma) = {}_0\mathcal{E} \otimes_{\mathbb{C}0} \mathcal{E}$. Since the maps $g_\gamma, \sigma_\gamma, {}_0(P_{\mathrm{sym}})_\gamma$ coincide with $g, \sigma, {}_0(P_{\mathrm{sym}})$ respectively on ${}_0\mathcal{E} \otimes_{\mathbb{C}0} \mathcal{E}$, we can conclude that

$$2(g \otimes_{\mathbb{C}} \mathrm{id})(\mathrm{id} \otimes_{\mathbb{C}} \sigma)(\mathrm{id} \otimes_{\mathbb{C}} \nabla)_0(P_{\mathrm{sym}}) = 0$$

as maps on ${}_0\mathcal{E} \otimes_{\mathbb{C}0} \mathcal{E}$. But $\mathcal{E}$ is the classical space of forms on the group G and therefore, our definition of metric-compatibility coincides with that in [51]. Hence we have

$$(\mathrm{id} \otimes_{\mathbb{C}} g)(\nabla \otimes_{\mathbb{C}} \mathrm{id}) + (g \otimes_{\mathbb{C}} \mathrm{id})(\mathrm{id} \otimes_{\mathbb{C}} \sigma)(\mathrm{id} \otimes_{\mathbb{C}} \nabla) = 0$$

Applying the same argument as above, we deduce that

$$(\mathrm{id} \otimes_{\mathbb{C}} g_\gamma)(\nabla_\gamma \otimes_{\mathbb{C}} \mathrm{id}) + (g_\gamma \otimes_{\mathbb{C}} \mathrm{id})(\mathrm{id} \otimes_{\mathbb{C}} \sigma_\gamma)(\mathrm{id} \otimes_{\mathbb{C}} \nabla_\gamma) = 0,$$

i.e, $\nabla' = \nabla_\gamma$ is compatible with $g' = g_\gamma$ in the sense of [51].

The converse part follows similarly and this completes the proof. $\square$

Chapter 6

Levi-Civita connection on $SU_q(2)$

In this chapter, we will investigate the theory of Chapter 4, in particular Theorem 4.5.9, in the context of the $4D_{\pm}$ calculi of the Hopf algebra $SU_q(2)$ discussed in Example 1.2.9. The $4D_{\pm}$ calculi of $SU_q(2)$ were explicitly described in [93] and then [86], and we briefly recall the same in Section 6.1. In the same section, we verify that the diagonalisability condition of the map ${}_0\sigma$ (see (4.2.2)) is satisfied by the $4D_{\pm}$ calculi. Theorem 4.4.4 states that if the map ${}_0\sigma$ of a bicovariant differential calculus is diagonalisable, then it admits a canonical bicovariant torsionless connection. In Section 6.2, we provide an explicit construction of this torsionless connection for each of the $4D_{\pm}$ calculi. In Section 6.3, we will show that the metric-independent sufficiency condition of Theorem 4.5.9 is satisfied by both calculi, except for at most finitely many values of q, and hence we can conclude the existence of a unique bicovariant Levi-Civita connection, corresponding to each bi-invariant pseudo-Riemannian metric. Throughout the chapter, the symbol $\mathcal{A}$ will stand for the Hopf algebra $SU_q(2)$ and $\mathcal{E}$ for the bimodule of one-forms for the $4D_{\pm}$ calculi.

6.1 The $4D_{\pm}$ calculi on $SU_q(2)$ and the braiding map

Our main reference for the details of this section is [86].

Recall from Example 1.2.9 that for $q \in [-1,1]\backslash 0$, $SU_q(2)$ is the $*$-algebra generated by the two elements α, γ, and their adjoints, satisfying the following relations:

$$\alpha^*\alpha + \gamma^*\gamma = 1, \quad \alpha\alpha^* + q^2\gamma\gamma^* = 1,$$
$$\gamma^*\gamma = \gamma\gamma^*, \quad \alpha\gamma = q\gamma\alpha, \quad \alpha\gamma^* = q\gamma^*\alpha.$$

The comultiplication map Δ is given by

$$\Delta(\alpha) = \alpha \otimes_{\mathbb{C}} \alpha - q\gamma^* \otimes_{\mathbb{C}} \gamma, \quad \Delta(\gamma) = \gamma \otimes_{\mathbb{C}} \alpha + \alpha^* \otimes_{\mathbb{C}} \gamma.$$

In this chapter, we will denote this Hopf algebra by the symbol $\mathcal{A}$.

In [86], it is explicitly proven that there does not exists any three-dimensional bicovariant differential calculi and exactly two inequivalent four-dimensional calculi for $SU_q(2)$. We use the description of the two bicovariant calculi, $4\mathrm{D}_+$ and $4\mathrm{D}_-$, as given in [86]. We will rephrase some of the notations to fit our formalism.

For $q \in (-1,1)\backslash\{0\}$, the first order differential calculi $\mathcal{E}$ of each of the $4\mathrm{D}_+$ and $4\mathrm{D}_-$ calculi are bicovariant $\mathcal{A}$-bimodules such that the space ${}_0\mathcal{E}$ of one-forms invariant under the left coaction of $\mathcal{A}$ is a 4-dimensional vector space. We will denote a preferred basis of ${}_0\mathcal{E}$ by $\{\omega_i\}_{i=1,2,3,4}$. Here we have replaced the notation Ω_i in [86] with the symbol ω_i.

The following is the explicit description of the exterior derivative d on ${}_0\mathcal{E}$ for the preferred basis $\{\omega_i\}_{i=1}^4$ mentioned above.

Proposition 6.1.1. *(Equation (5.2) of [86]) Let $d : \mathcal{E} \to \Omega^2(\mathcal{A})$ be the exterior derivative of the $4D_\pm$ calculus.*

$$d(\omega_1) = \pm\sqrt{r}\omega_1 \wedge \omega_3, \qquad d(\omega_2) = \mp\frac{\sqrt{r}}{q^2}\omega_2 \wedge \omega_3,$$
$$d(\omega_3) = \pm\frac{\sqrt{r}}{q}\omega_1 \wedge \omega_2, \qquad d(\omega_4) = 0,$$

where the upper sign stand for $4D_+$ and the lower for $4D_-$, and $r = 1 + q^2$.

Next we will show that the map ${}_0\sigma$ for $SU_q(2)$ satisfies the diagonalisability condition by giving explicit bases for eigenspaces of ${}_0\sigma$. First we recall from [86], the explicit action of σ on elements $\omega_i \otimes_{\mathcal{A}} \omega_j$, $i, j = 1, 2, 3, 4$.

Lemma 6.1.2. *(Equation (4.1) of [86]) The action of σ on the preferred basis of the $4D_{\pm}$ calculi is given by*

$$\sigma(\omega_1 \otimes_{\mathcal{A}} \omega_1) = \omega_1 \otimes_{\mathcal{A}} \omega_1, \quad \sigma(\omega_2 \otimes_{\mathcal{A}} \omega_2) = \omega_2 \otimes_{\mathcal{A}} \omega_2, \quad \sigma(\omega_4 \otimes_{\mathcal{A}} \omega_4) = \omega_4 \otimes_{\mathcal{A}} \omega_4,$$
$$\sigma(\omega_1 \otimes_{\mathcal{A}} \omega_4) = \omega_4 \otimes_{\mathcal{A}} \omega_1, \quad \sigma(\omega_2 \otimes_{\mathcal{A}} \omega_4) = \omega_4 \otimes_{\mathcal{A}} \omega_2, \quad \sigma(\omega_3 \otimes_{\mathcal{A}} \omega_4) = \omega_4 \otimes_{\mathcal{A}} \omega_3,$$

$$\sigma(\omega_1 \otimes_{\mathcal{A}} \omega_2) = \omega_2 \otimes_{\mathcal{A}} \omega_1 + t\omega_3 \otimes_{\mathcal{A}} \omega_3 - \frac{q\sqrt{r}}{k}\omega_3 \otimes_{\mathcal{A}} \omega_4,$$
$$\sigma(\omega_2 \otimes_{\mathcal{A}} \omega_1) = \omega_1 \otimes_{\mathcal{A}} \omega_2 - t\omega_3 \otimes_{\mathcal{A}} \omega_3 + \frac{q\sqrt{r}}{k}\omega_3 \otimes_{\mathcal{A}} \omega_4,$$
$$\sigma(\omega_1 \otimes_{\mathcal{A}} \omega_3) = \frac{t}{q}\omega_1 \otimes_{\mathcal{A}} \omega_3 - \frac{\sqrt{r}}{k}\omega_1 \otimes_{\mathcal{A}} \omega_4 + \omega_3 \otimes_{\mathcal{A}} \omega_1,$$
$$\sigma(\omega_3 \otimes_{\mathcal{A}} \omega_1) = \omega_1 \otimes_{\mathcal{A}} \omega_3 + \frac{q^2\sqrt{r}}{k}\omega_1 \otimes_{\mathcal{A}} \omega_4 - qt\omega_3 \otimes_{\mathcal{A}} \omega_1,$$
$$\sigma(\omega_2 \otimes_{\mathcal{A}} \omega_3) = -qt\omega_2 \otimes_{\mathcal{A}} \omega_3 + \frac{q^2\sqrt{r}}{k}\omega_2 \otimes_{\mathcal{A}} \omega_4 + \omega_3 \otimes_{\mathcal{A}} \omega_2,$$
$$\sigma(\omega_3 \otimes_{\mathcal{A}} \omega_2) = \omega_2 \otimes_{\mathcal{A}} \omega_3 - \frac{\sqrt{r}}{k}\omega_2 \otimes_{\mathcal{A}} \omega_4 + \frac{t}{q}\omega_3 \otimes_{\mathcal{A}} \omega_2,$$
$$\sigma(\omega_3 \otimes_{\mathcal{A}} \omega_3) = t\omega_1 \otimes_{\mathcal{A}} \omega_2 - t\omega_2 \otimes_{\mathcal{A}} \omega_1 + (1-t^2)\omega_3 \otimes_{\mathcal{A}} \omega_3 + \frac{tq\sqrt{r}}{k}\omega_3 \otimes_{\mathcal{A}} \omega_4,$$
$$\sigma(\omega_4 \otimes_{\mathcal{A}} \omega_1) = \frac{t^2k}{q^2\sqrt{r}}\omega_1 \otimes_{\mathcal{A}} \omega_3 + (1+t^2)\omega_1 \otimes_{\mathcal{A}} \omega_4 - \frac{t^2k}{\sqrt{r}}\omega_3 \otimes_{\mathcal{A}} \omega_1,$$
$$\sigma(\omega_4 \otimes_{\mathcal{A}} \omega_2) = -\frac{t^2k}{\sqrt{r}}\omega_2 \otimes_{\mathcal{A}} \omega_3 + (1+t^2)\omega_2 \otimes_{\mathcal{A}} \omega_4 + \frac{t^2k}{q^2\sqrt{r}},$$
$$\sigma(\sigma_4 \otimes_{\mathcal{A}} \omega_3) = \frac{t^2k}{q\sqrt{r}}\omega_1 \otimes_{\mathcal{A}} \omega_2 - \frac{t^2k}{q\sqrt{r}}\omega_2 \otimes_{\mathcal{A}} \omega_1 - \frac{t^3k}{q\sqrt{r}}\omega_3 \otimes_{\mathcal{A}} \omega_3 + (1+t^2)\omega_3 \otimes_{\mathcal{A}} \omega_4,$$

where $r = 1+q^2$, $t = q - \frac{1}{q}$, $s = \frac{1+q^4}{q}$, $k = \frac{r+q^4}{r\pm s}$ for $4D_{\pm}$ respectively.

This lead us to the next result which states the minimal polynomial equation of the map ${}_0\sigma$ and its eigenspace decomposition. The minimal polynomial equation of the map σ appeared in Equation (6.13) of [20] and the eigenspace decomposition can be found in Chapter 8 of [11]. Hence, the following proposition merely collects these results in the notational formality required for this chapter.

Proposition 6.1.3. *For $SU_q(2)$, the map ${}_0\sigma : {}_0\mathcal{E} \otimes_{\mathbb{C}0} \mathcal{E} \to {}_0\mathcal{E} \otimes_{\mathbb{C}0} \mathcal{E}$ is diagonalisable and has the minimal polynomial equation*

$$({}_0\sigma - 1)({}_0\sigma + q^2)({}_0\sigma + q^{-2}) = 0.$$

Proof. The proof of this result is by explicit listing of eigenvectors of ${}_0\sigma$ for eigenvalues $1, -q^2, -q^{-2}$ and by a dimension argument. Throughout we make use of the canonical equivalence $\omega_i \otimes_{\mathcal{A}} \omega_j \mapsto \omega_i \otimes_{\mathbb{C}} \omega_j$ as stated in (i) of Theorem 4.1.11. Moreover, r, t, k will be as in Lemma 6.1.2.

By explicit computation (also derived in Equation (4.2) of [86]), we get that the following ten two-tensors are in the eigenspace of ${}_0\sigma$ corresponding to eigenvalue 1:

$$\omega_1 \otimes_{\mathbb{C}} \omega_1,\ \omega_2 \otimes_{\mathbb{C}} \omega_2,\ \omega_3 \otimes_{\mathbb{C}} \omega_3 + t\omega_1 \otimes_{\mathbb{C}} \omega_2,\ \omega_4 \otimes_{\mathbb{C}} \omega_4,$$
$$\omega_1 \otimes_{\mathbb{C}} \omega_2 + \omega_2 \otimes_{\mathbb{C}} \omega_1,\ \omega_2 \otimes_{\mathbb{C}} \omega_3 + q^2 \omega_3 \otimes_{\mathbb{C}} \omega_2,$$
$$q^2 \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_3 \otimes_{\mathbb{C}} \omega_1,\ \tfrac{t^2 k}{q^2 \sqrt{r}} \omega_2 \otimes_{\mathbb{C}} \omega_3 - \omega_2 \otimes_{\mathbb{C}} \omega_4 - \omega_4 \otimes_{\mathbb{C}} \omega_2,$$
$$\tfrac{t^2 k}{\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_1 \otimes_{\mathbb{C}} \omega_4 + \omega_4 \otimes_{\mathbb{C}} \omega_1,\ \tfrac{t^2 k}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_2 + \omega_3 \otimes_{\mathbb{C}} \omega_4 + \omega_4 \otimes_{\mathbb{C}} \omega_3.$$

Similarly, by explicit computation, the following three linearly independent two-tensors are in the eigenspace corresponding to the eigenvalue $-q^2$:

$$\tfrac{tqk}{\sqrt{r}} \omega_2 \otimes_{\mathbb{C}} \omega_3 - q^2 \omega_2 \otimes_{\mathbb{C}} \omega_4 - \tfrac{tk}{q\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_2 + \omega_4 \otimes_{\mathbb{C}} \omega_2,$$
$$-\tfrac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 - q^2 \omega_1 \otimes_{\mathbb{C}} \omega_4 + \tfrac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 + \omega_4 \otimes_{\mathbb{C}} \omega_1,$$
$$-\tfrac{tk}{\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_2 + \tfrac{tk}{\sqrt{r}} \omega_2 \otimes_{\mathbb{C}} \omega_1 + \tfrac{t^2 k}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_3 - q^2 \omega_3 \otimes_{\mathbb{C}} \omega_4 + \omega_4 \otimes_{\mathbb{C}} \omega_3.$$

Finally, the following three linearly independent two-tensors are in the eigenspace corresponding to the eigenvalue $-q^{-2}$:

$$\tfrac{tqk}{\sqrt{r}} \omega_2 \otimes_{\mathbb{C}} \omega_3 + \omega_2 \otimes_{\mathbb{C}} \omega_4 - \tfrac{tk}{q\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_2 - q^2 \omega_4 \otimes_{\mathbb{C}} \omega_2,$$
$$-\tfrac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_1 \otimes_{\mathbb{C}} \omega_4 + \tfrac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 - q^2 \omega_4 \otimes_{\mathbb{C}} \omega_1,$$
$$-\tfrac{tk}{\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_2 + \tfrac{tk}{\sqrt{r}} \omega_2 \otimes_{\mathbb{C}} \omega_1 + \tfrac{t^2 k}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_3 + \omega_3 \otimes_{\mathbb{C}} \omega_4 - q^2 \omega_4 \otimes_{\mathbb{C}} \omega_3.$$

We have thus accounted for sixteen linearly independent elements of ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. Since ${}_0\mathcal{E}$ has dimension 4, ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$ has dimension 16. Hence we have a basis, and in particular bases for the eigenspace decomposition, of ${}_0\mathcal{E} \otimes_{\mathbb{C}} {}_0\mathcal{E}$. Moreover, ${}_0\sigma$ satisfies the minimal polynomial

$$({}_0\sigma - 1)({}_0\sigma + q^2)({}_0\sigma + q^{-2}) = 0.$$

$\square$

6.2 A bicovariant torsionless connection

In this section, using the eigenspace decomposition of ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}$, we construct a bicovariant torsionless connection on the $4D_\pm$ calculus.

By Proposition 6.1.3, we have the eigenspace decomposition

$$ {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} = \mathrm{Ker}({}_0\sigma - \mathrm{id}) \oplus \mathrm{Ker}({}_0\sigma + q^2) \oplus \mathrm{Ker}({}_0\sigma + q^{-2}). \tag{6.2.1} $$

Since $\mathrm{Ker}(\wedge) = \mathrm{Ker}({}_0\sigma - \mathrm{id})$, we have that

$$ \mathrm{Ker}({}_0\sigma + q^2) \oplus \mathrm{Ker}({}_0\sigma + q^{-2}) \cong \Omega^2(\mathcal{A}), $$

with the isomorphism being given by $\wedge|_{\mathrm{Ker}({}_0\sigma+q^2)\oplus\mathrm{Ker}({}_0\sigma+q^{-2})}$. Let us denote $\mathrm{Ker}({}_0\sigma + q^2) \oplus \mathrm{Ker}({}_0\sigma + q^{-2})$ by ${}_0\mathcal{F}$ from now on. This is consistent with the notation adopted in Definition 4.2.2.

Before we state the main result of this section, let us make the following remark.

Remark 6.2.1. *Note that since any element ρ in the bicovariant bimodule $\mathcal{E}$ can be uniquely expressed as $\rho = \sum_i \omega_i a_i$ for some a_i in $\mathcal{A}$ (Proposition 4.1.7), a connection on $\mathcal{E}$ is determined by its action on the basis $\{\omega_i\}_i$.*

Theorem 6.2.2. *Let $\{\omega_i\}_i$ be the preferred basis for the $4D_\pm$ calculus on $SU_q(2)$. For $i = 1, 2, 3, 4$, we define*

$$ \nabla_0(\omega_i) = -(\wedge|_{{}_0\mathcal{F}})^{-1} \circ d(\omega_i) \in {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E}. $$

Then, ∇_0 extends to a bicovariant torsionless connection on $\mathcal{E}$. More explicitly,

$$
\begin{aligned}
\nabla_0(\omega_1) &= \mp \frac{qr}{tk(q^2+1)^2}\big(\frac{2tk}{q\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_3 + tq\omega_1\otimes_{\mathbb{C}}\omega_4 - \frac{2tqk}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_1 + tq\omega_4\otimes_{\mathbb{C}}\omega_1\big)\\
\nabla_0(\omega_2) &= \pm \frac{qr}{tk(q^2+1)^2}\big(\frac{2tqk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_3 - tq\omega_2\otimes_{\mathbb{C}}\omega_4 - \frac{2tk}{q\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_2 - tq\omega_4\otimes_{\mathbb{C}}\omega_2\big)\\
\nabla_0(\omega_3) &= \pm \frac{qr}{tk(q^2+1)^2}\big(\frac{2tk}{\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_2 - \frac{2tk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_1 - \frac{t^2k}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_3\\
&\quad + tq\omega_3\otimes_{\mathbb{C}}\omega_4 + tq\omega_4\otimes_{\mathbb{C}}\omega_3\big)\\
\nabla_0(\omega_4) &= 0,
\end{aligned}
$$

where the upper and lower signs stand for the $4D_+$ and $4D_-$ calculi respectively, and r, t, k are as in Lemma 6.1.2.

Proof. By the definition of ∇_0,

$$\wedge \circ \nabla_0(\omega_i) = - \wedge \circ (\wedge|_{0\mathcal{F}})^{-1} \circ d(\omega_i) = -d(\omega_i).$$

Therefore, for any element $\rho = \sum_i \omega_i a_i$ in $\mathcal{E}$,

$$\begin{aligned} \wedge \circ \nabla_0(\sum_i \omega_i a_i) &= \wedge \circ \sum_i (\nabla_0(\omega_i) a_i + \omega_i \otimes_{\mathcal{A}} a_i) \\ = & \quad -\sum_i \big(\wedge \circ (\wedge|_{0\mathcal{F}})^{-1} \circ d(\omega_i) a_i + \omega_i \wedge a_i \big) \\ = & \quad -\sum_i (d(\omega_i) a_i + \omega_i \wedge a_i) = -\sum_i d(\omega_i a_i). \end{aligned}$$

Hence ∇_0 is a torsionless connection. The construction of ∇_0 is the same as that in Theorem 4.4.4. Hence, by that theorem, our connection ∇_0 is bicovariant.

Now we derive ∇_0 explicitly on each ω_i using the formulas for $d(\omega_i)$ in Proposition 6.1.1. We have that $d(\omega_1) = \pm\sqrt{r}\omega_1 \wedge \omega_3$. The decomposition of $\omega_1 \otimes_{\mathbb{C}} \omega_3$ as a linear combination of the basis eigenvectors listed in Proposition 6.1.3 is given by

$$\begin{aligned} \omega_1 \otimes_{\mathbb{C}} \omega_3 = & \frac{2q^2}{(q^2+1)^2} \big(q^2 \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_3 \otimes_{\mathbb{C}} \omega_1 \big) \\ & - \frac{q^2\sqrt{r}}{k(q^2+1)^2} \big(\frac{t^2 k}{\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_1 \otimes_{\mathbb{C}} \omega_4 + \omega_4 \otimes_{\mathbb{C}} \omega_1 \big) \\ & - \frac{q\sqrt{r}}{tk(q^2+1)^2} \big(-\frac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 - q^2 \omega_1 \otimes_{\mathbb{C}} \omega_4 + \frac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 + \omega_4 \otimes_{\mathbb{C}} \omega_1 \big) \\ & - \frac{q\sqrt{r}}{tk(q^2+1)^2} \big(-\frac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_1 \otimes_{\mathbb{C}} \omega_4 + \frac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 - q^2 \omega_4 \otimes_{\mathbb{C}} \omega_1 \big). \end{aligned}$$

Since the first two terms in the above decomposition are elements of $\mathrm{Ker}({}_0\sigma - \mathrm{id}) = \mathrm{Ker}(\wedge)$, applying $\wedge$ on both sides, we have

$$\begin{aligned} \omega_1 \wedge \omega_3 = \wedge \Big(& - \frac{q\sqrt{r}}{tk(q^2+1)^2} \big(-\frac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 - q^2 \omega_1 \otimes_{\mathbb{C}} \omega_4 + \frac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 + \omega_4 \otimes_{\mathbb{C}} \omega_1 \big) \\ & - \frac{q\sqrt{r}}{tk(q^2+1)^2} \big(-\frac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_1 \otimes_{\mathbb{C}} \omega_4 + \frac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 - q^2 \omega_4 \otimes_{\mathbb{C}} \omega_1 \big) \Big), \end{aligned}$$

and since the last two terms in the decomposition are from ${}_0\mathcal{F}$,

$$\begin{aligned} (\wedge|_{0\mathcal{F}})^{-1}(\omega_1 \wedge \omega_3) = & - \frac{q\sqrt{r}}{tk(q^2+1)^2} \big(-\frac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 - q^2 \omega_1 \otimes_{\mathbb{C}} \omega_4 + \frac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 + \omega_4 \otimes_{\mathbb{C}} \omega_1 \big) \\ & - \frac{q\sqrt{r}}{tk(q^2+1)^2} \big(-\frac{tk}{q\sqrt{r}} \omega_1 \otimes_{\mathbb{C}} \omega_3 + \omega_1 \otimes_{\mathbb{C}} \omega_4 + \frac{tqk}{\sqrt{r}} \omega_3 \otimes_{\mathbb{C}} \omega_1 - q^2 \omega_4 \otimes_{\mathbb{C}} \omega_1 \big). \end{aligned}$$

Thus, by the construction of ∇_0, we have

$$\begin{aligned}\nabla_0(\omega_1) &= \mp\big(-\frac{qr}{tk(q^2+1)^2}\big(-\frac{tk}{q\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_3 - q^2\omega_1\otimes_{\mathbb{C}}\omega_4 + \frac{tqk}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_1 + \omega_4\otimes_{\mathbb{C}}\omega_1\big)\\ &\quad - \frac{qr}{tk(q^2+1)^2}\big(-\frac{tk}{q\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_3 + \omega_1\otimes_{\mathbb{C}}\omega_4 + \frac{tqk}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_1 - q^2\omega_4\otimes_{\mathbb{C}}\omega_1\big)\big)\\ &= \mp\frac{qr}{tk(q^2+1)^2}\big(\frac{2tk}{q\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_3 + tq\omega_1\otimes_{\mathbb{C}}\omega_4 - \frac{2tqk}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_1 + tq\omega_4\otimes_{\mathbb{C}}\omega_1\big)\end{aligned}$$

Proposition 6.1.1 also gives that $d(\omega_2) = \mp\frac{\sqrt{r}}{q^2}\omega_2\wedge\omega_3, d(\omega_3) = \pm\frac{\sqrt{r}}{q}\omega_1\wedge\omega_2$ and $d(\omega_4) = 0$. So, similarly, we have

$$\begin{aligned}\omega_2\otimes_{\mathbb{C}}\omega_3 =&\frac{2}{(q^2+1)^2}\big(\omega_2\otimes_{\mathbb{C}}\omega_3 + q^2\omega_3\otimes_{\mathbb{C}}\omega_2\big) - \frac{q^4\sqrt{r}}{k(q^2+1)^2}\big(\frac{t^2k}{q^2\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_3 - \omega_2\otimes_{\mathbb{C}}\omega_4 - \omega_4\otimes_{\mathbb{C}}\omega_2\big)\\ &+ \frac{q^3\sqrt{r}}{tk(q^2+1)^2}\big(\frac{tqk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_3 - q^2\omega_2\otimes_{\mathbb{C}}\omega_4 - \frac{tk}{q\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_2 + \omega_4\otimes_{\mathbb{C}}\omega_2\big)\\ &+ \frac{q^3\sqrt{r}}{tk(q^2+1)^2}\big(\frac{tqk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_3 + \omega_2\otimes_{\mathbb{C}}\omega_4 - \frac{tk}{q\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_2 - q^2\omega_4\otimes_{\mathbb{C}}\omega_2\big),\end{aligned}$$

and hence,

$$\nabla_0(\omega_2) = \pm\frac{qr}{tk(q^2+1)^2}\big(\frac{2tqk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_3 - tq\omega_2\otimes_{\mathbb{C}}\omega_4 - \frac{2tk}{q\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_2 - tq\omega_4\otimes_{\mathbb{C}}\omega_2\big).$$

Moreover,

$$\begin{aligned}\omega_1\otimes_{\mathbb{C}}\omega_2 =&\frac{2q^2}{(q^2+1)^2}\big(\omega_1\otimes_{\mathbb{C}}\omega_2 + \omega_2\otimes_{\mathbb{C}}\omega_1\big) + \frac{2tq^2}{(q^2+1)^2}\big(\omega_3\otimes_{\mathbb{C}}\omega_3 + t\omega_1\otimes_{\mathbb{C}}\omega_2\big)\\ &- \frac{q^3\sqrt{r}}{k(q^2+1)^2}\big(\frac{t^2k}{q\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_2 + \omega_3\otimes_{\mathbb{C}}\omega_4 + \omega_4\otimes_{\mathbb{C}}\omega_3\big)\\ &- \frac{q^2\sqrt{r}}{tk(q^2+1)^2}\big(-\frac{tk}{\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_2 + \frac{tk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_1 + \frac{t^2k}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_3 - q^2\omega_3\otimes_{\mathbb{C}}\omega_4 + \omega_4\otimes_{\mathbb{C}}\omega_3\big)\\ &- \frac{q^2\sqrt{r}}{tk(q^2+1)^2}\big(-\frac{tk}{\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_2 + \frac{tk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_1 + \frac{t^2k}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_3 + \omega_3\otimes_{\mathbb{C}}\omega_4 - q^2\omega_4\otimes_{\mathbb{C}}\omega_3\big),\end{aligned}$$

and hence,

$$\nabla_0(\omega_3) = \pm\frac{qr}{tk(q^2+1)^2}\big(\frac{2tk}{\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_2 - \frac{2tk}{\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_1 - \frac{t^2k}{\sqrt{r}}\omega_3\otimes_{\mathbb{C}}\omega_3 + tq\omega_3\otimes_{\mathbb{C}}\omega_4 + tq\omega_4\otimes_{\mathbb{C}}\omega_3\big)$$

Lastly, since $d(\omega_4) = 0$, $\nabla_0(\omega_4) = 0$

Thus, we are done with our proof. □

6.3 Existence of a unique bicovariant Levi-Civita connection

In this section, we prove that except for finitely many $q \in (-1,1)\backslash\{0\}$, the $4D_{\pm}$ calculi admit a unique bicovariant Levi-Civita connection for every bi-invariant pseudo-Riemannian metric (as defined in Definition 4.3.1) on $\mathcal{E}$. We achieve this by verifying the hypotheses of Theorem 4.5.9.

Recall that ((6.2.1)) for the $4D_{\pm}$ calculus, we had the decomposition

$$ {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} = \mathrm{Ker}({}_0\sigma - \mathrm{id}) \oplus {}_0\mathcal{F}, $$

where ${}_0\mathcal{F} := \mathrm{Ker}({}_0\sigma + q^2) \oplus \mathrm{Ker}({}_0\sigma + q^{-2})$.

Let us now denote $\mathrm{Ker}({}_0\sigma - \mathrm{id})$ by ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$. Moreover, as in Definition 4.2.2, we define the $\mathbb{C}$-linear map

$$ {}_0(P_{\mathrm{sym}}) : {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} \to {}_0\mathcal{E}\otimes_{\mathbb{C}}{}_0\mathcal{E} $$

to be the idempotent with range ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}$ and kernel ${}_0\mathcal{F}$. Since, ${}_0(P_{\mathrm{sym}})$ is the idempotent onto the eigenspace of ${}_0\sigma$ with eigenvalue one, and with kernel the eigenspaces with eigenvalues $-q^2$ and $-q^{-2}$, it is of the form (see (4.2.6))

$$ {}_0(P_{\mathrm{sym}}) = \frac{{}_0\sigma + q^2}{1+q^2} . \frac{{}_0\sigma + q^{-2}}{1+q^{-2}} . \tag{6.3.1} $$

.

Let us introduce the following notations:

$$ \begin{aligned} \nu_1 &= \omega_1\otimes_{\mathbb{C}}\omega_1, & \nu_2 =& \omega_2\otimes_{\mathbb{C}}\omega_2, \\ \nu_3 &= \omega_3\otimes_{\mathbb{C}}\omega_3 + t\omega_1\otimes_{\mathbb{C}}\omega_2, & \nu_4 =& \omega_4\otimes_{\mathbb{C}}\omega_4, \\ \nu_5 &= \omega_2\otimes_{\mathbb{C}}\omega_1 + \omega_1\otimes_{\mathbb{C}}\omega_2, & \nu_6 =& \omega_3\otimes_{\mathbb{C}}\omega_2 + \frac{1}{q^2}\omega_2\otimes_{\mathbb{C}}\omega_3, \\ \nu_7 &= \omega_3\otimes_{\mathbb{C}}\omega_1 + q^2\omega_1\otimes_{\mathbb{C}}\omega_3, & \nu_8 =& \omega_4\otimes_{\mathbb{C}}\omega_2 + \omega_2\otimes_{\mathbb{C}}\omega_4 - \frac{t^2k}{q^2\sqrt{r}}\omega_2\otimes_{\mathbb{C}}\omega_3, \\ \nu_9 &= \omega_4\otimes_{\mathbb{C}}\omega_1 + \omega_1\otimes_{\mathbb{C}}\omega_4 + \frac{t^2k}{\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_3, & \nu_{10} =& \omega_4\otimes_{\mathbb{C}}\omega_3 + \omega_3\otimes_{\mathbb{C}}\omega_4 + \frac{t^2k}{q\sqrt{r}}\omega_1\otimes_{\mathbb{C}}\omega_2. \end{aligned} \tag{6.3.2} $$

Then by the proof of Proposition 6.1.3, the set $\{\nu_i\}_{i=1}^{10}$ forms a basis of ${}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E} := \mathrm{Ker}({}_0\sigma - \mathrm{id})$.

Thus, an arbitary element of $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}0}\mathcal{E}$ is of the form

$$X = \sum_{ij} A_{ij}\nu_i\otimes_{\mathbb{C}}\omega_j,$$

where A_{ij} are some complex numbers.

Hence, if we show that $({}_0(P_{\mathrm{sym}}))_{23}(\sum_{ij} A_{ij}\nu_i\otimes_{\mathbb{C}}\omega_j) = 0$ implies that $A_{ij} = 0$ for all i, j, then $({}_0(P_{\mathrm{sym}}))_{23}$ is a one-one map from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}0}\mathcal{E}$ to ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$.

However, $\dim(({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}0}\mathcal{E}) = \dim({}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E}))$and so $({}_0(P_{\mathrm{sym}}))_{23}$ will be a vector space isomorphism from $({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}0}\mathcal{E}$ to ${}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$.

So let us suppose $\{A_{ij}\}_{ij}$ are complex numbers such that

$$({}_0(P_{\mathrm{sym}}))_{23}(\sum_{ij} A_{ij}\nu_i\otimes_{\mathbb{C}}\omega_j) = 0.$$

Then, by (6.3.1), we have

$$\big((q^2({}_0\sigma)_{23} + 1)(({}_0\sigma)_{23} + q^2)\big)(\sum_{ij} A_{ij}\nu_i\otimes_{\mathbb{C}}\omega_j) = 0. \tag{6.3.3}$$

We want to show that except for finitely many values of q, the above equation implies that all the A_{ij} are equal to 0. This involves a long computation, including a series of preparatory lemmas. We will be using the explicit form of ${}_0\sigma(\omega_i\otimes_{\mathbb{C}}\omega_j)$ as given in Lemma 6.1.2 as well as (6.3.2) to express the left hand side of (6.3.3) as a linear combination of basis elements $\omega_i\otimes_{\mathbb{C}}\omega_j\otimes_{\mathbb{C}}\omega_k$. Then we compare coefficients to derive relations among the A_{ij}. We do not provide the details of the computation. However, for the purposes of book-keeping, each equation is indexed by a triplet (i, j, k) meaning that it is obtained by collecting coefficients of the basis element $\omega_i\otimes_{\mathbb{C}}\omega_j\otimes_{\mathbb{C}}\omega_k$ in the expansion of the term

$$\big((q^2({}_0\sigma)_{23} + 1)(({}_0\sigma)_{23} + q^2)\big)(\sum_{mn} A_{mn}\nu_m\otimes_{\mathbb{C}}\omega_n).$$

Lemma 6.3.1. *We have the following equations:*

$$A_{11} = 0 \tag{1,1,1}$$

$$A_{12}(q^4+2) + (tA_{31} + A_{51} + A_{10,1}\frac{t^2k}{q\sqrt{r}})2q^2 + (A_{73}q^2 + A_{93}\frac{t^2k}{\sqrt{r}})2q(q^2-1) = 0 \tag{1,1,2}$$

$$\begin{aligned} &A_{13}(q^4+2q^2-1) + A_{14}(\frac{k}{\sqrt{r}}(q^2-2+q^{-2})) \\ &+(A_{71}q^2 + A_{91}\frac{t^2k}{\sqrt{r}})2q^2 + A_{91}(\frac{k}{\sqrt{r}}q^{-2}(q^2-1)) = 0 \end{aligned} \tag{1,1,3}$$

$$A_{13}(-q^2\frac{\sqrt{r}}{k}) + A_{14}(q^4+1) + (A_{71}q^2 + A_{91}\frac{t^2k}{\sqrt{r}})\frac{\sqrt{r}}{k}q^4 + A_{91}(q^4+1) = 0 \tag{1,1,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_1 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_1$, $\omega_1 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_2$, $\omega_1 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_3$ and $\omega_1 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. ☐

Lemma 6.3.2. *We have the following equations:*

$$\begin{aligned} &A_{12}(2q^2-1) + (tA_{31} + A_{51} + A_{10,1}\frac{t^2k}{q\sqrt{r}})(q^4+1) + (A_{73}q^2 + A_{93}\frac{t^2k}{\sqrt{r}})(-2q(q^2-1)) \\ &+A_{93}(-\frac{k}{q\sqrt{r}}(q^2-1)^2) + (A_{74}q^2 + A_{94})(-\frac{k}{q\sqrt{r}}(q^2-1)^2) = 0 \end{aligned} \tag{1,2,1}$$

$$tA_{32} + A_{52} + A_{10,2}\frac{t^2k}{q\sqrt{r}} = 0 \tag{1,2,2}$$

$$\begin{aligned} &(tA_{34} + A_{54} + A_{10,4}\frac{t^2k}{q\sqrt{r}})(-\frac{k}{\sqrt{r}}(q^2-1)^2) + (tA_{33} + A_{53} + A_{10,3}\frac{t^2k}{q\sqrt{r}})(-(q^2-1)^2) \\ &+(A_{72}q^2 + A_{92}\frac{t^2k}{\sqrt{r}})2q^2 + A_{92}(-\frac{k}{\sqrt{r}}(q^2-1)^2) = 0 \end{aligned} \tag{1,2,3}$$

$$\begin{aligned} &(tA_{33} + A_{53} + A_{10,3}\frac{t^2k}{q\sqrt{r}})\frac{q^4\sqrt{r}}{k} + (tA_{34} + A_{54} + A_{10,4}\frac{t^2k}{q\sqrt{r}})(q^4+1) \\ &+(A_{72}q^2 + A_{92}\frac{t^2k}{\sqrt{r}})(-q^2) + A_{92}(q^4+1) = 0 \end{aligned} \tag{1,2,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_1$, $\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_2$, $\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_3$ and $\omega_1 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. ☐

Lemma 6.3.3. *We have the following equations:*

$$\begin{aligned}
&A_{13}2q^2 + A_{14}\frac{k}{\sqrt{r}}(-q^2(q-q^{-1})^2)\\
&+(A_{71}q^2 + A_{91}\frac{t^2k}{\sqrt{r}})(-q^4+2q^2+1) + A_{91}\frac{k}{\sqrt{r}}(-(q^2-1)^2) = 0
\end{aligned} \qquad (1,3,1)$$

$$\begin{aligned}
&(tA_{33} + A_{53} + A_{10,3}\frac{t^2k}{q\sqrt{r}})2q^2 + (tA_{34} + A_{54} + A_{10,4}\frac{t^2k}{q\sqrt{r}})\frac{k}{\sqrt{r}}(q^2-2q+q^{-2})\\
&+(A_{72}q^2 + A_{92}\frac{t^2k}{\sqrt{r}})(q^4+2q^2-1) + A_{92}\frac{k}{\sqrt{r}}q^{-2}(q^2-1)^2 = 0
\end{aligned} \qquad (1,3,2)$$

$$\begin{aligned}
&(tA_{31} + A_{51} + A_{10,1}\frac{t^2k}{q\sqrt{r}})(-2q^3+2q) + A_{12}2q(q^2-1) + A_{93}(-\frac{k}{\sqrt{r}}q^{-2}(q^2-1)^3)\\
&+(A_{73}q^2 + A_{93}\frac{t^2k}{\sqrt{r}})(-q^4+6q^2-1) + (A_{74}q^2 + A_{94}\frac{t^2k}{\sqrt{r}})(-\frac{k}{\sqrt{r}}q^{-2}(q^2-1)^3) = 0
\end{aligned} \qquad (1,3,3)$$

$$\begin{aligned}
&(tA_{31} + A_{51} + A_{10,1}\frac{t^2k}{q\sqrt{r}})(-2q^3+2q) + (A_{73}q^2 + A_{93}\frac{t^2k}{\sqrt{r}})\frac{\sqrt{r}}{k}q^4\\
&+A_{93}(3(q^2-1)^2+2q^2) + (A_{74}q^2 + A_{94}\frac{t^2k}{\sqrt{r}})(q^4+1) = 0
\end{aligned} \qquad (1,3,4)$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_1 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_1$, $\omega_1 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_2$, $\omega_1 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_3$ and $\omega_1 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.4. *We have the following equations:*

$$A_{13}(-\frac{q^2\sqrt{r}}{k}) + A_{14}(q^4+1) + (A_{71}q^2 + A_{91}\frac{t^2k}{\sqrt{r}})\frac{q^4\sqrt{r}}{k} + A_{91}(q^4+1) = 0 \qquad (1,4,1)$$

$$\begin{aligned}
&(tA_{33} + A_{53} + A_{10,3}\frac{t^2k}{q\sqrt{r}})q^4\frac{\sqrt{r}}{k} + (tA_{34} + A_{54} + A_{10,4}\frac{t^2k}{q\sqrt{r}})(q^4+1)\\
&+(A_{72}q^2 + A_{92}\frac{t^2k}{\sqrt{r}})\frac{\sqrt{r}}{k}(-q^2) + A_{92}(q^4+1) = 0
\end{aligned} \qquad (1,4,2)$$

$$\begin{aligned}
&A_{12}(-\frac{r}{k}q^3) + (tA_{31} + A_{51} + A_{10,1}\frac{t^2k}{q\sqrt{r}})\frac{\sqrt{r}}{k}q^3 + A_{93}(q^4-1)\\
&+(A_{73}q^2 + A_{93}\frac{t^2k}{\sqrt{r}})\frac{\sqrt{r}}{k}q^2(q^2-1) + (A_{74}q^2 + A_{94}\frac{t^2k}{\sqrt{r}})(q^4+1) = 0
\end{aligned} \qquad (1,4,3)$$

$$A_{94} = 0 \qquad (1,4,4)$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_1 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_1$, $\omega_1 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_2$, $\omega_1 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_3$ and $\omega_1 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.5. *We have the following equations:*

$$A_{51} = 0 \tag{2,1,1}$$

$$\begin{aligned}&A_{52}(q^4+2) + A_{21}(2q^2) + (A_{63}q^{-2} + A_{83}\frac{t^2k}{q^2\sqrt{r}})2q(q^2-1)\\ &+A_{83}(\frac{k}{\sqrt{r}}q^{-1}(q^2-1)^2) + (A_{64}q^{-2} + A_{84}\frac{t^2k}{q^2\sqrt{r}})\frac{k}{\sqrt{r}}q(q^2-2+q^{-2}) = 0\end{aligned} \tag{2,1,2}$$

$$\begin{aligned}&A_{53}(q^4+2q^2-1) + A_{54}\frac{k}{\sqrt{r}}(q^2-2+q^{-2})\\ &+(A_{64}q^{-2} + A_{84}\frac{t^2k}{q^2\sqrt{r}})2q^2 + A_{81}\frac{k}{\sqrt{r}}q^{-2}(q^2-1) = 0\end{aligned} \tag{2,1,3}$$

$$A_{53}(-\frac{\sqrt{r}}{k}q^2) + A_{54}(q^4+1) + (A_{61}q^{-2} + A_{81}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}q^4 + A_{81}(q^4+1) = 0 \tag{2,1,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_2 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_1$, $\omega_2 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_2$, $\omega_2 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_3$ and $\omega_2 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.6. *We have the following equations:*

$$\begin{aligned}&A_{52}(2q^2-1) + A_{21}(q^4+1) + (A_{63}q^{-2} + A_{83}\frac{t^2k}{q^2\sqrt{r}})(-2q(q^2-1))\\ &+A_{83}(\frac{k}{q\sqrt{r}}(q^2-1)^2) + (A_{64}q^{-2} + A_{84}\frac{t^2k}{q^2\sqrt{r}})(-\frac{k}{\sqrt{r}}q(q^2-2+q^{-2})) = 0\end{aligned} \tag{2,2,1}$$

$$A_{22} = 0 \tag{2,2,2}$$

$$\begin{aligned}&A_{23}(-q^4+2q^2+1) + A_{24}(-\frac{k}{\sqrt{r}}(q^4-2q^2+1))\\ &+(A_{62}q^{-2} + A_{82}\frac{t^2k}{q^2\sqrt{r}})2q^2 + A_{82}(-\frac{k}{\sqrt{r}}(q^2-1)^2) = 0\end{aligned} \tag{2,2,3}$$

$$A_{23}\frac{q^4\sqrt{r}}{k} + A_{24}(q^4+1) + (A_{62}q^{-2} + A_{82}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}(-q^2) + A_{82}(q^4+1) = 0 \tag{2,2,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_2 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_1$, $\omega_2 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_2$, $\omega_2 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_3$ and $\omega_2 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.7. *We have the following equations:*

$$\begin{aligned}&A_{53}(2q^2)+A_{54}(q^4+1)+(A_{61}q^{-2}+A_{81}\frac{t^2k}{q^2\sqrt{r}})(-q^4+2q^2+1)\\ &+A_{81}\frac{k}{\sqrt{r}}(-(q^2-1)^2)=0\end{aligned}\tag{2,3,1}$$

$$\begin{aligned}&A_{23}2q^2+A_{24}\frac{k}{\sqrt{r}}(q^2-2+q^{-2})+(A_{62}q^{-2}+A_{82}\frac{t^2k}{q^2\sqrt{r}})(q^4+2q^2-1)\\ &+A_{82}\frac{k}{\sqrt{r}}q^{-2}(q^2-1)^2=0\end{aligned}\tag{2,3,2}$$

$$\begin{aligned}&A_{52}2q(q^2-1)+A_{21}(-2q^3+2q)+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})(-q^4+6q^2-1)\\ &+A_{83}\frac{k}{\sqrt{r}}(-q^{-2}(q^2-1)^3)+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})\frac{k}{\sqrt{r}}(-q(q-q^{-1})^3)=0\end{aligned}\tag{2,3,3}$$

$$\begin{aligned}&A_{21}\frac{\sqrt{r}}{k}q^3+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}q^4+A_{83}(3(q^2-1)^2+2q^2)\\ &+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})(q^4+1)=0\end{aligned}\tag{2,3,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_2\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_1$, $\omega_2\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_2$, $\omega_2\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_3$ and $\omega_2\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn}A_{mn}\nu_m\otimes_{\mathbb{C}}\omega_n)$. □

Lemma 6.3.8. *We have the following equations:*

$$A_{53}\frac{\sqrt{r}}{k}(-q^2)+A_{54}(q^4+1)+(A_{61}q^{-2}+A_{81}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}q^4+A_{81}(q^4+1)=0\tag{2,4,1}$$

$$A_{23}\frac{\sqrt{r}}{k}q^4+A_{24}(q^4+1)+(A_{62}q^{-2}+A_{82}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}(-q^2)+A_{82}(q^4+1)=0\tag{2,4,2}$$

$$\begin{aligned}&A_{52}\frac{\sqrt{r}}{k}(-q^3)+A_{21}\frac{\sqrt{r}}{k}q^3+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}q^2(q^2-1)\\ &+A_{83}(q^4-1)+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})(q^4+1)=0\end{aligned}\tag{2,4,3}$$

$$A_{84}=0\tag{2,4,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_2\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_1$, $\omega_2\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_2$, $\omega_2\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_3$ and $\omega_2\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn}A_{mn}\nu_m\otimes_{\mathbb{C}}\omega_n)$. □

Lemma 6.3.9. *We have the following equations:*

$$A_{71} = 0 \tag{3,1,1}$$

$$\begin{aligned}&A_{72}(q^4+2) + A_{61}2q^2 + A_{33}2q(q^2-1)\\ &+A_{10,3}\frac{k}{\sqrt{r}}q^{-1}(q^2-1)^2 + A_{34}\frac{k}{\sqrt{r}}q(q^2-2+q^{-2}) = 0\end{aligned} \tag{3,1,2}$$

$$A_{73}(q^4+2q^2-1) + A_{74}\frac{k}{\sqrt{r}}(q^2-2+q^{-2}) + A_{31}2q^2 + A_{10,1}\frac{k}{\sqrt{r}}q^{-2}(q^2-1) = 0 \tag{3,1,3}$$

$$A_{73}\frac{\sqrt{r}}{k}(-q^{-2}) + A_{74}(q^4+1) + A_{31}\frac{\sqrt{r}}{k}q^4 + A_{10,1}(q^4+1) = 0 \tag{3,1,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_3 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_1$, $\omega_3 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_2$, $\omega_3 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_3$ and $\omega_3 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.10. *We have the following equations:*

$$\begin{aligned}&A_{72}(2q^2-1) + A_{61}(q^4+1) + A_{33}2q(-(q^2-1))\\ &+A_{10,33}\frac{k}{\sqrt{r}}(-q^{-1}(q^2-1)^2) + A_{34}\frac{k}{\sqrt{r}}(-q(q^2-2+q^{-2})) = 0\end{aligned} \tag{3,2,1}$$

$$A_{62} = 0 \tag{3,2,2}$$

$$A_{63}(-q^4+2q^2+1) + A_{64}\frac{k}{\sqrt{r}}(-(q^4-2q^2+1)) + A_{32}2q^2 + A_{10,2}\frac{k}{\sqrt{r}}(-(q^2-1)^2) = 0 \tag{3,2,3}$$

$$A_{63}\frac{\sqrt{r}}{k}q^4 + A_{64}(q^4+1) + A_{32}\frac{\sqrt{r}}{k}(-q^2) + A_{10,2}(q^4+1) = 0 \tag{3,2,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_3 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_1$, $\omega_3 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_2$, $\omega_3 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_3$ and $\omega_3 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.11. *We have the following equations:*

$$A_{73}2q^2 + A_{74}\frac{k}{\sqrt{r}}(-(q^2-1)^2) + A_{31}(-q^4+2q^2+1) + A_{10,1}\frac{k}{\sqrt{r}}(-(q^2-1)^2) = 0 \tag{3,3,1}$$

$$A_{63}2q^2 + A_{64}\frac{k}{\sqrt{r}}(q^2-2+q^{-2}) + A_{32}(q^4+2q^2-1) + A_{10,2}\frac{k}{\sqrt{r}}q^{-2}(q^2-1)^2 = 0 \tag{3,3,2}$$

$$\begin{aligned}&A_{61}(-2q^3+2q) + A_{33}(-q^4+6q^2-1)\\ &+A_{10,3}\frac{k}{\sqrt{r}}(-q^{-2}(q^2-1)^3) + A_{34}\frac{k}{\sqrt{r}}(-q(q-q^{-1})^3) = 0\end{aligned} \tag{3,3,3}$$

$$A_{61}\frac{\sqrt{r}}{k}q^3 + A_{33}\frac{\sqrt{r}}{k}q^4 + A_{10,3}(3(q^2-1)^2+2q^2) + A_{34}(q^4+1) = 0 \tag{3,3,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_3 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_1$, $\omega_3 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_2$, $\omega_3 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_3$ and $\omega_3 \otimes_{\mathbb{C}} \omega_3 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.12. *We have the following equations:*

$$A_{73}\frac{\sqrt{r}}{k}(-q^2)+A_{74}(q^4+1)+A_{31}\frac{\sqrt{r}}{k}q^4+A_{10,1}(q^4+1)=0 \tag{3,4,1}$$

$$A_{63}\frac{\sqrt{r}}{k}q^4+A_{64}(q^4+1)+A_{32}\frac{\sqrt{r}}{k}(-q^2)+A_{10,2}(q^4+1)=0 \tag{3,4,2}$$

$$A_{72}\frac{\sqrt{r}}{k}(-q^3)+A_{61}\frac{\sqrt{r}}{k}q^3=0 \tag{3,4,3}$$

$$A_{10,4}=0 \tag{3,4,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_3 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_1$, $\omega_3 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_2$, $\omega_3 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_3$ and $\omega_3 \otimes_{\mathbb{C}} \omega_4 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.13. *We have the following equations:*

$$A_{91}=0 \tag{4,1,1}$$

$$\begin{aligned} &A_{92}(q^4+2)+A_{81}2q^2+A_{10,3}2q(q^2-1)\\ &+A_{43}\frac{k}{\sqrt{r}}q^{-1}(q^2-1)^2+A_{10,4}\frac{k}{\sqrt{r}}q(q^2-2+q^{-2})=0 \end{aligned} \tag{4,1,2}$$

$$A_{93}(q^4+2q^2-1)+A_{94}\frac{k}{\sqrt{r}}(q^2-2+q^{-2})+A_{10,1}2q^2+A_{41}\frac{k}{\sqrt{r}}q^{-2}(q^2-1)=0 \tag{4,1,3}$$

$$A_{93}\frac{\sqrt{r}}{k}(-q^2)+A_{94}(q^4+1)+A_{10,1}\frac{\sqrt{r}}{k}q^4+A_{41}(q^4+1)=0 \tag{4,1,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_4 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_1$, $\omega_4 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_2$, $\omega_4 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_3$ and $\omega_4 \otimes_{\mathbb{C}} \omega_1 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.14. *We have the following equations:*

$$\begin{aligned} &A_{92}(2q^2-1)+A_{81}(q^4+1)+A_{10,3}2q(q^2-1)+A_{43}\frac{k}{\sqrt{r}}(-q^{-1}(q^2-1)^2)\\ &+A_{10,4}\frac{k}{\sqrt{r}}q(q^2-2+q^{-2})=0 \end{aligned} \tag{4,2,1}$$

$$A_{82}=0 \tag{4,2,2}$$

$$A_{83}(-q^4+2q^2+1)+A_{84}\frac{k}{\sqrt{r}}(-q^4+2q^2-1)+A_{10,2}2q^2+A_{42}\frac{k}{\sqrt{r}}(q^2-1)^2=0 \tag{4,2,3}$$

$$A_{83}\frac{\sqrt{r}}{k}q^4+A_{84}(q^4+1)+A_{10,2}\frac{\sqrt{r}}{k}(-q^2)+A_{42}(q^4+1)=0 \tag{4,2,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_4 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_1$, $\omega_4 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_2$, $\omega_4 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_3$ and $\omega_4 \otimes_{\mathbb{C}} \omega_2 \otimes_{\mathbb{C}} \omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn} A_{mn}\nu_m \otimes_{\mathbb{C}} \omega_n)$. □

Lemma 6.3.15. *We have the following equations:*

$$\begin{aligned}&A_{93}(q^4+2q^2-1)+A_{94}\frac{k}{\sqrt{r}}(q^2-2+q^{-2})+A_{10,1}(-q^4+2q^2+1)\\&+A_{41}\frac{k}{\sqrt{r}}(-(q^2-1)^2)=0\end{aligned}\tag{4,3,1}$$

$$A_{83}2q^2+A_{84}\frac{k}{\sqrt{r}}(q^2-2+q^{-2})+A_{10,2}(q^4+2q^2-1)+A_{42}\frac{k}{\sqrt{r}}q^{-2}(q^2-1)^2=0\tag{4,3,2}$$

$$\begin{aligned}&A_{92}2q(q^2-1)+A_{81}(-2q^3+2q)+A_{10,3}(-q^4+6q^2-1)+A_{43}\frac{k}{\sqrt{r}}(-q^{-2}(q^2-1)^3)\\&+A_{10,4}\frac{k}{\sqrt{r}}(-q(q-q^{-1})^3)=0\end{aligned}\tag{4,3,3}$$

$$A_{81}\frac{\sqrt{r}}{k}q^3+A_{10,3}\frac{\sqrt{r}}{k}q^4+A_{43}(3(q^2-1)^2+2q^2)+A_{10,4}(q^4+1)=0\tag{4,3,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_4\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_1$, $\omega_4\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_2$, $\omega_4\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_3$ and $\omega_4\otimes_{\mathbb{C}}\omega_3\otimes_{\mathbb{C}}\omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn}A_{mn}\nu_m\otimes_{\mathbb{C}}\omega_n)$. □

Lemma 6.3.16. *We have the following equations:*

$$A_{93}\frac{\sqrt{r}}{k}(-q^2)+A_{94}(q^4+1)+A_{10,1}\frac{\sqrt{r}}{k}q^4+A_{41}(q^4+1)=0\tag{4,4,1}$$

$$A_{83}\frac{\sqrt{r}}{k}q^4+A_{83}(q^4+1)+A_{10,2}\frac{\sqrt{r}}{k}(-q^2)+A_{42}(q^4+1)=0\tag{4,4,2}$$

$$A_{92}\frac{\sqrt{r}}{k}(-q^3)+A_{81}\frac{\sqrt{r}}{k}q^3+A_{10,3}\frac{\sqrt{r}}{k}q^2(q^2-1)+A_{43}(q^4-1)+A_{10,4}(q^4+1)=0\tag{4,4,3}$$

$$A_{44}=0\tag{4,4,4}$$

Proof. The above equations are derived by comparing the coeffcients of $\omega_4\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_1$, $\omega_4\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_2$, $\omega_4\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_3$ and $\omega_4\otimes_{\mathbb{C}}\omega_4\otimes_{\mathbb{C}}\omega_4$ in $\big((q^2({}_0\sigma)_{23}+1)(({}_0\sigma)_{23}+q^2)\big)(\sum_{mn}A_{mn}\nu_m\otimes_{\mathbb{C}}\omega_n)$. □

Theorem 6.3.17. *For the $4D_\pm$ calculi, the map*

$$({}_0(P_{\mathrm{sym}}))_{23}:({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}0}\mathcal{E}\to{}_0\mathcal{E}\otimes_{\mathbb{C}}({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})$$

is an isomorphism except for, possibly, finitely many values of $q\in(-1,1)\backslash\{0\}$. Hence, for each bi-invariant pseudo-Riemannian metric g, there exists a unique bicovariant Levi-Civita connection for each calculus.

Proof. By the discussion preceding the above series of preparatory lemmas, we need to show that the system of equations given above admit only the trivial solution for A_{ij}, $i=1,\ldots,10$, $j=1,\ldots,4$. We then proceed to solve these equations for all A_{ij}. Note that the following variables are all identically zero in the above over-determined system:

A_{11} (by (1,1,1)), A_{94} (by (1,4,4)), A_{51} (by (2,1,1)), A_{22} (by (2,2,2)), A_{84} (by (2,4,4)), A_{71} (by (3,1,1)), A_{62} (by (3,2,2)), $A_{10,4}$ (by (3,4,4)), A_{91} (by (4,1,1)), A_{82} (by (4,2,2)) and A_{44} (by (4,4,4)).

This reduces the equations (1,3,1) and (1,4,1) to the following exact system of linear equations in the variables A_{13} and A_{14}, with the associated matrix having determinant $q^2(q^2+1)^2$:

$$A_{13}2q^2 + A_{14}\tfrac{k}{\sqrt{r}}(-q^2(q-q^{-1})^2) = 0$$
$$A_{13}(-\tfrac{q^2\sqrt{r}}{k}) + A_{14}(q^4+1) = 0$$

Hence the solution for the variables A_{13} and A_{14} is zero.

We repeat this process for the rest of the A_{ij}, identifying a subset of equations which has been reduced to an exact one due to the previously solved A_{ij}, and then concluding that the elements A_{ij} in the current set are also solved to be 0 except for at most finitely many value of $q \in (-1,1)\backslash\{0\}$.

(2,2,3) and (2,2,4) reduce to the following system of linear equations in A_{23} and A_{24} with determinant $(q^2+1)^2$:

$$A_{23}(-q^4+2q^2+1) + A_{24}(-\tfrac{k}{\sqrt{r}}(q^4-2q^2+1)) = 0$$
$$A_{23}\tfrac{q^4\sqrt{r}}{k} + A_{24}(q^4+1) = 0$$

(4,1,3), (4,1,4) and (4,3,1) reduce to the following system of linear equations in A_{41}, A_{93}, $A_{10,1}$ with determinant $2q^{10}-2q^4-2q^2+2$:

$$A_{93}(q^4+2q^2-1) + A_{10,1}2q^2 + A_{41}\tfrac{k}{\sqrt{r}}q^{-2}(q^2-1) = 0$$
$$A_{93}\tfrac{\sqrt{r}}{k}(-q^2) + A_{10,1}\tfrac{\sqrt{r}}{k}q^4 + A_{41}(q^4+1) = 0$$
$$A_{93}(q^4+2q^2-1) + A_{10,1}(-q^4+2q^2+1) + A_{41}\tfrac{k}{\sqrt{r}}(-(q^2-1)^2) = 0$$

(4,1,2), (4,2,1), (4,3,3) and (4,4,3) reduce to the following system of linear equations in A_{43}, A_{81}, A_{92}, $A_{10,3}$ with determinant $4q^{14}+10q^{12}-10q^{10}-8q^8+26q^4-26q^2+4$:

$$A_{92}(q^4+2)+A_{81}2q^2+A_{10,3}2q(q^2-1)+A_{43}\tfrac{k}{\sqrt{r}}q^{-1}(q^2-1)^2=0$$
$$A_{92}(2q^2-1)+A_{81}(q^4+1)+A_{10,3}2q(q^2-1)+A_{43}\tfrac{k}{\sqrt{r}}(-q^{-1}(q^2-1)^2)=0$$
$$A_{92}2q(q^2-1)+A_{81}(-2q^3+2q)+A_{10,3}(-q^4+6q^2-1)+A_{43}\tfrac{k}{\sqrt{r}}(-q^{-2}(q^2-1)^3)=0$$
$$A_{92}\tfrac{\sqrt{r}}{k}(-q^3)+A_{81}\tfrac{\sqrt{r}}{k}q^3+A_{10,3}\tfrac{\sqrt{r}}{k}q^2(q^2-1)+A_{43}(q^4-1)=0$$

(3,4,3), (3,1,2), (3,2,1) and (3,3,3) reduce to the following system of linear equations in A_{33}, A_{34}, A_{61}, A_{72} with determinant $-2q^2(q-1)^2(q+1)^2(q^2+1)^4$:

$$A_{72}\tfrac{\sqrt{r}}{k}(-q^3)+A_{61}\tfrac{\sqrt{r}}{k}q^3=0$$
$$A_{72}(q^4+2)+A_{61}2q^2+A_{33}(2q(q^2-1))+A_{34}\tfrac{k}{\sqrt{r}}q(q^2-2+q^{-2})=0$$
$$A_{72}(2q^2-1)+A_{61}(q^4+1)+A_{33}(-2q(q^2-1))+A_{34}\tfrac{k}{\sqrt{r}}(-q(q^2-2+q^{-2}))=0$$
$$A_{61}(-2q^3+2q)+A_{33}(-q^4+6q^2-1)+A_{34}\tfrac{k}{\sqrt{r}}(-q(q-q^{-1})^3)=0$$

(2,1,3) and (2,1,4) reduce to the following system of equations in A_{53} and A_{54} with determinant $q^4(q^2+1)^2$:

$$A_{53}(q^4+2q^2-1)+A_{54}\tfrac{k}{\sqrt{r}}(q^2-2+q^{-2})=0$$
$$A_{53}(-\tfrac{\sqrt{r}}{k}q^2)+A_{54}(q^4+1)=0$$

(1,1,2), (1,2,1), (1,3,3) and (1,3,4) reduce to a system of equations in A_{12}, A_{31}, A_{73}, A_{74} with determinant a non-zero polynomial in q:

$$A_{12}(q^4+2)+tA_{31}2q^2+A_{73}q^2 2q(q^2-1)=0$$
$$A_{12}(2q^2-1)+tA_{31}(q^4+1)+A_{73}q^2(-2q(q^2-1))$$
$$tA_{31}(-2q^3+2q)+A_{12}2q(q^2-1)+A_{73}q^2(-q^4+6q^2-1)+A_{74}q^2(-\tfrac{k}{\sqrt{r}}q^{-2}(q^2-1)^3)=0$$
$$tA_{31}(-2q^3+2q)+A_{73}q^2\tfrac{\sqrt{r}}{k}q^4+A_{74}q^2(q^4+1)=0$$

(2,1,2), (2,2,1), (2,3,3), (2,3,4) and (2,4,3) reduce to a system of equations in A_{21}, A_{52}, A_{63}, A_{64}, A_{83} with determinant a non-zero polynomial in q:

$$A_{52}(q^4+2)+A_{21}(2q^2)+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})2q(q^2-1)$$
$$+A_{83}(\frac{k}{\sqrt{r}}q^{-1}(q^2-1)^2)+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})\frac{k}{\sqrt{r}}q(q^2-2+q^{-2})=0$$
$$A_{52}(2q^2-1)+A_{21}(q^4+1)+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})(-2q(q^2-1))$$
$$+A_{83}(\frac{k}{q\sqrt{r}}(q^2-1)^2)+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})(-\frac{k}{\sqrt{r}}q(q^2-2+q^{-2}))=0$$
$$A_{52}2q(q^2-1)+A_{21}(-2q^3+2q)+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})(-q^4+6q^2-1)$$
$$+A_{83}\frac{k}{\sqrt{r}}(-q^{-2}(q^2-1)^3)+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})\frac{k}{\sqrt{r}}(-q(q-q^{-1})^3)=0$$
$$A_{21}\frac{\sqrt{r}}{k}q^3+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}q^4+A_{83}(3(q^2-1)^2+2q^2)$$
$$+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})(q^4+1)=0$$
$$A_{52}\frac{\sqrt{r}}{k}(-q^3)+A_{21}\frac{\sqrt{r}}{k}q^3+(A_{63}q^{-2}+A_{83}\frac{t^2k}{q^2\sqrt{r}})\frac{\sqrt{r}}{k}q^2(q^2-1)$$
$$+A_{83}(q^4-1)+(A_{64}q^{-2}+A_{84}\frac{t^2k}{q^2\sqrt{r}})(q^4+1)=0$$

(3,3,2) and (3,4,2) reduce to a system of equations in A_{32}, $A_{10,2}$ with determinant $q^4(q^2+1)^2$:

$$A_{32}(q^4+2q^2-1)+A_{10,2}\tfrac{k}{\sqrt{r}}q^{-2}(q^2-1)^2=0$$
$$A_{32}\tfrac{\sqrt{r}}{k}(-q^2)+A_{10,2}(q^4+1)=0$$

Finally, (4,2,3) reduces identically to $A_{42}=0$.

Hence we have shown that all A_{ij} are identically equal to zero except for atmost finitely many values of $q\in(-1.1)$. Therefore, $({}_0(P_{\mathrm{sym}}))_{23}|_{({}_0\mathcal{E}\otimes_{\mathbb{C}}{}^{\mathrm{sym}}{}_0\mathcal{E})\otimes_{\mathbb{C}}{}_0\mathcal{E}}$ is an isomorphism if q does not belong to this finite subset.

Since $SU_q(2)$ is a cosemisimple Hopf algebra, and we have shown that the map ${}_0\sigma$ is diagonalisable, by Theorem 4.5.9, for each bi-invariant pseudo-Riemannian metric g, each of the $4D_\pm$ calculi admits a unique bicovariant Levi-Civita connection for all but finitely many q. $\square$

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