Graphs with equal independence and matching number

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CERTIFICATE

This is to certify that the dissertation entitled **Graphs with equal independence and matching number** submitted by **Prajyot Subhash Patle** to Indian Statistical Institute, Kolkata, in partial fulfillment for the award of the degree of **Master of Technology in Computer Science** is a bonafide record of work carried out by him under my supervision and guidance. The dissertation has fulfilled all the requirements as per the regulations of this institute and, in my opinion, has reached the standard needed for submission.

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Abstract

Extremal graphs are graphs which sit at the extremes. In simpler words for a class of graphs which satisfy a certain property, extremal graphs are the ones which exhibit a minimum or maximum of that property. Here, we take a look at a property which is exhibited by any graph in general; $\delta\alpha \leq \Delta\mu$, where δ is the minimum degree of the graph, α is the size of the maximum independent set, Δ is the maximum degree, and μ is the size of the maximum matching of the graph. We first look at non-regular extremal graphs and regular extremal graphs (with degree 2 and 3) with respect to the above property as characterized by Mohr and Rautenbach. Later we try our hand at characterizing the regular extremal graphs using a general graph decomposition given jointly by Edmonds and Gallai. In doing so, we obtain a new proof for Mohr and Rautenbach's characterization of 3-regular extremal graphs and we believe our approach can be easily adapted to characterize k-regular extremal graphs for values of $k \geq 3$.

Contents

Chapter 1

Definitions

- Let $G(V, E)$ be an undirected graph where V is the set of vertices, and E is the set of edges.
- For $X \subseteq V$, let $G[X]$ be the induced subgraph formed by the vertex set X in G.
- Let δ and Δ be defined as the minimum and maximum degree of the graph respectively.
- Let α be defined as the size of a maximum independent set in the graph.
- Let μ be defined as the size of a maximum matching in the graph.
- An odd component is a connected component with an odd number of vertices.
- An even component is a connected component with an even number of vertices.
- For $X \subseteq V$, let $odd(X)$ be the number of odd components in the graph $G[V \setminus X]$.
- We call a graph $G(V, E)$ as hypomatchable if for every vertex $v \in V$, $G[V \setminus v]$ has a perfect matching.
- For $X \subseteq V$, let $\alpha(X)$ be the size of a maximum independent set in $G[X]$.
- For $X \subseteq V$, let $\mu(X)$ be the number of edges in a maximum matching of $G[X]$.

Chapter 2

Previous Work

2.1 Introduction

Caro, Davila and Pepper [1] showed the following theorem.

Theorem 1. *$\delta\alpha \leq \Delta\mu$ for any graph G, where δ is minimum degree, Δ is maximum degree, α is the independence number, and μ is the matching number.*

They asked for which graphs the above inequality becomes an equality. Mohr and Rautenbach [3] give a simplified proof of Theorem 1 and characterize all non-regular graphs that achieve equality. They further characterize all regular graphs of degree at most 3 that achieve equality.

Note that the independence number can be arbitrarily larger or smaller than the matching number in general. For example, the complete graph has $\alpha = 1$ and $\mu = \lfloor \frac{n}{2} \rfloor$, and on the other extreme, the star graph has $\alpha = n - 1$ and $\mu = 1$.

2.2 Non-regular Extremal Graphs

For $\delta < \Delta$, where both are positive integers, a bipartite graph is (δ, Δ)-regular if it has partite sets A and B, s.t. every vertex in A has degree δ and every vertex in B has degree Δ.

Theorem 2 (Mohr and Rautenbach)**.** *For non-regular graphs, the equality $\delta\alpha(G) = \Delta\mu(G)$ holds if and only if the graph is bipartite and (δ, Δ)-regular.*

2.3 Regular Extremal Graphs

Claim 2.1. *For every regular graph G, it follows from Theorem 1 that $\alpha(G) \leq \mu(G)$. So every component C of a regular graph G having $\alpha(G) = \mu(G)$ will have $\alpha(C) = \mu(C)$.*

Proof: Let there be some component C_1 such that $\alpha(C_1) < \mu(C_1)$. Since $\alpha(G) = \mu(G)$, there needs to be another component C_2 such that $\alpha(C_1) > \mu(C_1)$, which is a contradiction. □

Therefore we only need to characterize connected regular graphs G having $\alpha(G) = \mu(G)$.

2.3.1 2-regular extremal graphs

Since the connected 2-regular graphs are exactly the cycles, the equality $\alpha(G) = \mu(G)$ holds for all of them.

2.3.2 3-regular (cubic) extremal graphs

Before looking at cubic graphs which satisfy the above equality, we'll look at a substructure of these graphs, which Mohr and Rautenbach call a *bubble graph.*

Bubble graph

A graph G is a bubble with contact vertex z, and partition (I, R) if the vertex set of G can be partitioned into two sets I and R such that:

- Every vertex in $V(G) - z$ has degree 3 and z has degree 2.
- I is independent, and
- z lies in R and $G[R]$ contains exactly one edge.

Properties of a bubble graph G with partition (I, R) and contact vertex z:

- Not bipartite.
- $|R| = |I| + 1$; therefore, $|I| = (|V(G)| - 1)/2$
- $\alpha(G) = \alpha(G - z) = \mu(G) = \mu(G - z) = (|V(G)| - 1)/2$, i.e even if we omit z, it does not reduce the values of α and μ.
- If G is not 2-connected, then some induced subgraph G' of G is also a bubble with partition (I', R') such that $I' \subseteq I$ and $R' \subseteq R$.

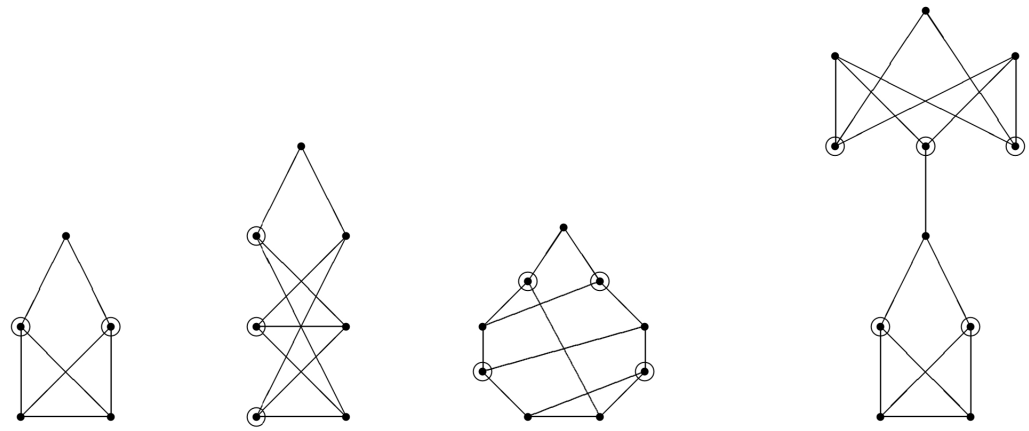

Figure 2.1: Examples of a *bubble* graph (picture taken from [3]). For each example, the top-most vertex is the contact vertex z, the encircled vertices belong to the independent set I and the remaining vertices belong to the vertex set R.

Special graph

This is the type of cubic graph which satisfies the equality. A graph is *special* if it is connected, cubic and its vertex set G can be partitioned into sets $V_0, V_1, \ldots, V_l$ such that:

- The graph $G[V_0]$ is a non-empty bipartite graph with partite sets I_0 and R_0 such that every vertex in R_0 has degree 3 in $G[V_0]$, and
- For every $i \in \{1, 2, \ldots, l\}$, the graph $G[V_i]$ is a 2-connected bubble.

For each bubble $G[V_i]$, the contact vertex z has degree 2 inside the bubble. Since the graph is connected and cubic, the contact vertex z must have an edge outside the bubble to which it belongs. If such an edge goes to the contact vertex of another bubble, the fact $G[V_0]$ is non-empty will make these 2 bubbles disconnected from the rest of the graph, which is contradiction. Therefore the edge from contact vertex z must go inside $G[V_0]$ and inside I_0 to be precise, since every vertex in R_0 has degree 3 in $G[V_0]$.

An example of a special graph can be seen in Figure 2.2

Theorem 3 (Mohr and Rautenbach)**.** *A connected cubic graph G satisfies $\alpha(G) = \mu(G)$ if and only if it is special.*

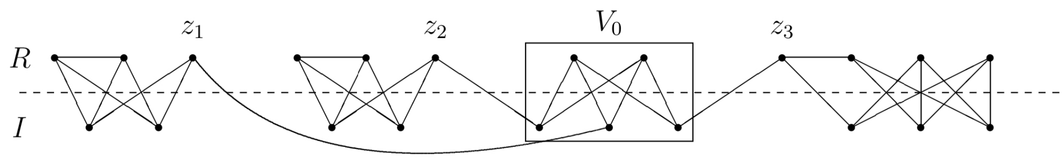

Figure 2.2: Example of a *special* graph (picture taken from [3]).

Chapter 3

Gallai-Edmonds Decomposition

The Gallai-Edmonds decomposition [4, 2] of a graph describes the structure of maximum matchings in the graph. In this decomposition, the vertex set of a graph $G(V, E)$ is partitioned or decomposed into three sets, A, B and C, where:

- A is the set of vertices such that $\forall v \in A$, there exists a maximum matching in G that does not cover v.
- B is defined to be the set of vertices in $V \setminus A$ such that for a vertex $v \in B$, there exists an edge $(u, v) \in E$ such that $u \in A$. In other words, B is the set of neighbours of the vertices in set A that lie outside A.
- $C = V \setminus (A \cup B)$, i.e the set of remaining vertices.

Observation 1. *If a graph G has a perfect matching, then A is empty.*

Such a decomposition exhibits the following properties:

- Each odd component in $G[V \setminus B]$ is hypomatchable.
- The vertices belonging to the odd components of $G[V \setminus B]$ are exactly the vertices of the set A.
- Each even component of $G[V \setminus B]$ has a perfect matching.
- The vertices belonging to the even components of $G[V \setminus B]$ are exactly the vertices of the set C.
- For every $X \subseteq B$, X has neighbours in greater than $|X|$ odd components of $G[V \setminus B]$. This implies that B has neighbours in greater than $|B|$ odd components of $G[V \setminus B]$.

Theorem 4. *There exists a matching in G which matches every vertex of the set B.*

Proof:
Since for every $X \subseteq B$, X has vertices in greater than $|X|$ number of odd components of $G[V \setminus B]$, according to Hall's theorem, there exists a matching covering every vertex of B, where each vertex of B is matched to a vertex inside a unique odd component. □

Theorem 5 (Tutte's theorem [4])**.** *For a graph $G(V, E)$, a perfect matching exists if and only if for all $X \subseteq V$, $odd(X) \leq |X|$.*

Corollary 1. *For a graph $G(V, E)$, if there exist $X \subseteq V$ such that $odd(X) > |X|$, a perfect matching for G does not exist. Moreover, for any matching M in G, at least $odd(X) - |X|$ vertices will remain unmatched.*

Tutte set. If for some $X \subseteq V$, we have $odd(X) > |X|$, we call X a *Tutte set* of the graph.

3.1 Construction of a maximum matching

Using properties of the Gallai-Edmonds decomposition, we can construct a matching M for a graph G where:

- Each vertex in an even component of $G[V \setminus B]$ is matched to another vertex in the same component; i.e. M restricted to an even component is a perfect matching of that even component (this is possible since every even component has a perfect matching).
- Each vertex in B is matched to a vertex in a unique odd component of $G[V \setminus B]$ (this is possible by Theorem 4).
- For each odd component Q such that there exist $v \in Q$ that is matched by M to a vertex of B, M restricted to $G[Q \setminus v]$ is a perfect matching (this is possible since the odd components are hypomatchable).
- For each odd component Q which does not contain a vertex matched to some vertex in set B, M restricted to $G[Q]$ is a matching of size $(|Q| - 1)/2$, i.e all but one vertex are matched (this is possible since the odd components are hypomatchable).

In this matching, all the even components have a perfect matching, thus all the vertices in C are matched. All vertices of B are also matched. All the vertices in $|B|$ odd components are matched. And all but one vertex in each of $odd(B) - |B|$ odd components are matched. Thus, except $odd(B) - |B|$ vertices, all other vertices are matched. Thus $|M| = (n - (odd(B) - |B|))/2$.

Lets call a matching with this structure a "Gallai-Edmonds Matching".

Claim 3.1. *A matching M is a maximum matching if and only if it is a Gallai-Edmonds matching.*

Proof:

1. Let M be a Gallai-Edmonds matching for a graph G. We know that $|M| = (n - (odd(B) - |B|))/2$. Since $odd(B) > |B|$, B is a Tutte set [2]. i.e. any matching M' in G will leave at least $odd(B) - |B|$ vertices as unmatched. This implies for any matching M' in G, $|M'| \leq (n - (odd(B) - |B|))/2$. Thus M is a maximum matching.

2. Let M be a maximum matching. As explained above $|M| \leq (n - (odd(B) - |B|))/2$ for a graph G. As a Gallai-Edmonds matching has size equal to $(n - (odd(B) - |B|))/2$ and M is a maximum matching, we have $|M| = (n - (odd(B) - |B|))/2$. Since B is a Tutte set, $odd(B) - |B|$ vertices in A are unmatched by M, and these vertices belong to distinct odd components of $G[V \setminus B]$. Since $|M| = (n - (odd(B) - |B|))/2$, every other vertex is matched by M. Then all the vertices in at least $|B|$ odd components of $G[V \setminus B]$ are matched by M. This means that a vertex in each of these components is matched to B. Thus each vertex in B is matched by M to a vertex in a unique odd component of $G[V \setminus B]$. Further, we can conclude that M restricted to an odd component of $G[V \setminus B]$ is a matching that matches every vertex except one. Since every vertex in B is matched to a vertex in A, every vertex in C must be matched to another vertex in C. This implies that M restricted to an even component of $G[V \setminus B]$ is a perfect matching. Thus M is a Gallai-Edmonds matching.

□

Note that every graph has canonical sets A, B, and C, i.e. these sets are defined according to the above rules as soon as a graph is defined. We'll hereafter look at a graph from the viewpoint of this decomposition.

3.2 Our Observations

Claim 3.2. *If a graph G contains a perfect matching, then $\alpha(G) \leq \mu(G)$.*

Proof: Let M be a perfect matching of G. Then $|M| = \mu(G)$. Every vertex of G is contained in some edge of M and any independent set of G can contain at most one vertex from each edge of M. Thus $\alpha(G) \leq |M| = \mu(G)$. □

Claim 3.3. *If G has a perfect matching, is k-regular and $\alpha = \mu$, then G is bipartite k-regular graph.*

Proof: Let M be a maximum matching in G of size $n/2$. Since every vertex of G is in some edge of M, every maximum independent set of size $n/2$ needs to contain exactly one vertex from each edge of M. Let us consider one such independent set I of size $n/2$. Since I is independent, all edges from the vertices of I go into the set $V \setminus I$. Thus a total of $kn/2$ edges go from I to $V \setminus I$. However $|V \setminus I|$ is also $n/2$, and thus the total degree of the vertices in $V \setminus I$, which is $kn/2$, is used up for absorbing the $kn/2$ edges coming from I. This implies that there does not exist an edge between two vertices of $V \setminus I$, which means that $V \setminus I$ is also an independent set. Therefore G is a bipartite k-regular graph. □

Note: Every bipartite k-regular graph has a perfect matching. Thus by Claim 3.3, the k-regular graphs containing a perfect matching and having $\alpha = \mu$ are exactly the bipartite k-regular graphs. Therefore, from this point on we'll assume that G is a graph without a perfect matching. Note that if G does not have a perfect matching, then $A \neq \emptyset$, and consequently the sets B and C are well-defined.

Claim 3.4. *For each even component P of $G[V \setminus B]$, $\alpha(P) \leq \mu(P)$.*

Proof: Since we know that every even component of $G[V \setminus B]$ has a perfect matching, by Claim 3.2, $\alpha(P) \leq \mu(P)$. □

Corollary 2. *Since the set C of a graph is made up of the vertices of the even components of $G[V \setminus B]$, it follows from Claim 3.4 that, the size of the maximum independent set of $G[C]$ is less than or equal to the size of its maximum matching, i.e. $\alpha(C) \leq \mu(C)$.*

Claim 3.5. *For each odd component Q of $G[V \setminus B]$ of size > 1, $\alpha(Q) \leq \mu(Q)$*

<u>Proof:</u> Let the odd component Q have n_q vertices, with independence number $\alpha(Q)$ and matching number $\mu(Q)$. Since every odd component of $G[V \setminus B]$ is hypomatchable, $\mu(Q) = (n_q - 1)/2$, i.e. the maximum matching of Q consists of $(n_q - 1)/2$ matching edges. Suppose for the sake of contradiction that $\alpha(Q) > \mu(Q) = (n_q - 1)/2$, i.e. $\alpha(Q) \geq (n_q + 1)/2$. Consider an independent set I of Q such that $|I| = \alpha(Q) \geq (n_q + 1)/2$. Since $|V(Q)| > 1$, there exists a vertex $u \in V(Q) \setminus I$. Since Q is hypomatchable, $G[V(Q) \setminus \{u\}]$ contains a perfect matching, say M. Since $|V(Q) \setminus \{u\}| = n_q - 1$, we have $|M| = (n_q - 1)/2$. But now I is an independent set of $G[V(Q) \setminus \{u\}]$ having size more than $|M|$, which contradicts Claim 3.2. □

Claim 3.6. *If G is k-regular then $\alpha(A \cup B) \leq \mu(A \cup B)$.*

<u>Proof:</u> Let M be any maximum matching of $G[A \cup B]$. Then by Claim 3.1, every vertex of B will be matched in M to a vertex in a unique odd component of $G[V \setminus B]$, and M will match every vertex except one in each odd component. Let $M' \subseteq M$ be the edges that have exactly one endpoint in B. We know that $|M'| = |B|$. Let $M'' = M \setminus M'$. The edges of M'' have both endpoints inside an odd component of $G[V \setminus B]$. Let I be any maximum independent set of $G[A \cup B]$. Let $\mathcal{D}$ denote the set of odd components $G[V \setminus B]$ of size more than 1. Let $X \in \mathcal{D}$. By Claim 3.1, $|M'' \cap E(X)| = \mu(X)$. By Claim 3.5, we know that $\alpha(X) \leq \mu(X)$, and therefore $|I \cap X| \leq |M'' \cap E(X)|$. Let S be the set of vertices that form singleton odd components of $G[V \setminus B]$. Let $|I \cap S| = t$. Since G is k-regular, the t vertices in $I \cap S$ have at least t neighbours in B. This means that $|I \cap B| \leq |B| - t$. So $|I \cap (S \cup B)| \leq |B| = |M'|$. Since in each odd component $X \in \mathcal{D}$, we have $|I \cap X| \leq |M'' \cap E(X)|$, we now have $\alpha(A \cup B) = |I| = \sum_{X \in \mathcal{D}} |I \cap X| + |I \cap (S \cup B)| \leq \sum_{X \in \mathcal{D}} |M'' \cap E(X)| + |M'| = |M| = \mu(A \cup B)$. □

Claim 3.7. *For a k-regular graph $G(V, E)$ with $\alpha = \mu$, the vertices that form the singleton odd components of $G[V \setminus B]$ are not part of any maximum independent set.*

<u>Proof:</u> Since G is k-regular and $\alpha = \mu$, Corollary 2 and Claim 3.6 imply that $\alpha(A \cup B) = \mu(A \cup B)$ and $\alpha(C) = \mu(C)$. Let I be a maximum independent set of G. Clearly, $|I \cap (A \cup B)| \leq \alpha(A \cup B) = \mu(A \cup B)$ and $|I \cap C| \leq \alpha(C) = \mu(C)$. Thus $\alpha = |I| = |I \cap (A \cup B)| + |I \cap C| \leq \alpha(A \cup B) + \alpha(C) = \mu(A \cup B) + \mu(C) = \mu$ (the last equality follows from Claim 3.1). Since $\alpha = \mu$, we then have $|I \cap (A \cup B)| + |I \cap C| = \alpha(A \cup B) + \alpha(C)$. As $|I \cap (A \cup B)| \leq \alpha(A \cup B)$ and $|I \cap C| \leq \alpha(C)$, this implies that $|I \cap (A \cup B)| = \alpha(A \cup B)$ and $|I \cap C| = \alpha(C)$. Let us denote $I \cap (A \cup B)$ by I'.

Let S be the set of vertices that form singleton odd components of $G[V \setminus B]$. Let M be a maximum matching of $A \cup B$. Then $|M| = \mu(A \cup B)$. Let $M' \subseteq M$ be the edges that have exactly one endpoint in B. We know by Claim 3.1 that $|M'| = |B|$. As

$\alpha(A \cup B) = \mu(A \cup B)$, we have $|I'| = |M|$. Let $T = I' \cap S$ and $|T| = t$. Since G is k-regular, T has at least t neighbours in B, which we denote by the set $N_B(T)$.

Suppose that $t \geq 1$. If $|N_B(T)| = t$, then since G is k-regular, $T \cup N_B(T)$ forms a connected component of G. Since G is connected, there are no other vertices in G, which implies that G is a k-regular bipartite graph on $2t$ vertices. Then G has a perfect matching, contradicting our assumption that G does not have a perfect matching. Thus we can conclude that $|N_B(T)| > t$. Since $T \subseteq I'$, we have $N_B(T) \cap I' = \emptyset$, which implies that $|I' \cap B| < |B| - t$. So $|I' \cap (S \cup B)| = |I' \cap S| + |I' \cap B| < t + |B| - t = |B| = |M'|$. Let $M'' = M \setminus M'$. Since $|I' \cap (S \cup B)| < |M'|$, we have $I' \setminus (S \cup B)| > |M''|$. This implies that there exists an odd component X of $G[V \setminus B]$, that is not a singleton component, in which $|I' \cap X| > |M'' \cap X|$. But for each such component X, by Claim 3.1, $|M'' \cap X| = \mu(X)$. Thus $\alpha(X) \geq |I' \cap X| > \mu(X)$, which contradicts Claim 3.5. Thus we can conclude that $t = 0$, or in other words, $T = \emptyset$. $\square$

Corollary 3. *For a k-regular graph $G(V, E)$ with $\alpha = \mu$, all the vertices in set B are part of the maximum independent set. This also implies that set B is an independent set.*

<u>Proof:</u> Consider any maximum independent set I of G and maximum matching M of G. We have $|I| = |M|$. Let $M' \subseteq M$ be the edges of M that are incident to vertices of B and let $M'' = M \setminus M'$. By Claim 3.1, $|M'| = |B|$. Since $|I \cap X| \leq |M'' \cap X|$ in every non-singleton odd component X of $G[V \setminus B]$ (by Claim 3.5), and $I \cap S = \emptyset$ (here, S is the set of vertices that form singleton odd components of $G[V \setminus B]$), we have $|I \cap B| \geq |M'| = |B|$, which implies that $|I \cap B| = |B|$. Thus $B \subseteq I$. $\square$

Claim 3.8. *For a graph $G(V, E)$ with maximum independent set I and maximum matching M, if $\alpha = \mu$, then*

- $|M \cap E(A \cup B)| = \mu(A \cup B) = \alpha(A \cup B) = |I \cap (A \cup B)|$
- $|M \cap E(C)| = \mu(C) = \alpha(C) = |I \cap C|$
- *For each even component X of $G[V \setminus B]$,*
  $|M \cap E(X)| = \mu(X) = \alpha(X) = |I \cap V(X)| = |V(X)|/2$
- *For each odd component P of $G[V \setminus B]$, where $|V(P)| > 1$,*
  $|M \cap E(P)| = \mu(P) = \alpha(P) = |I \cap V(P)| = |(V(P) - 1)/2|$

<u>Proof:</u> Let I be a maximum independent set in G, and M be a maximum matching in G. Since $\alpha = \mu$, then $|I| = |M|$. Let $M' \subseteq M$ be the edges in M that have both endpoints in $A \cup B$. Let $M'' = M \setminus M'$. Then by Claim 3.1, M'' contains the edges in M that have both endpoints in C. Also by Claim 3.1, it follows that M' is a maximum

matching in $A \cup B$ and M'' is maximum matching (in fact a perfect matching) in C. Thus $|M'| = \mu(A \cup B)$ and $|M''| = \mu(C)$. Let $I' = I \cap (A \cup B)$. Let $I'' = I \setminus I'$, i.e. $I'' = I \cap C$. By Claim 3.6, it implies that $|I'| \leq \alpha(A \cup B) \leq \mu(A \cup B) = |M'|$. By Corollary 2, it implies that $|I''| \leq \alpha(C) \leq \mu(C) = |M''|$. We know that $|I| = |M|$, which can also be written as $|I'| + |I''| = |M'| + |M''|$. Therefore $|I'| = |M'|$ and $|I''| = |M''|$. This implies that $|I'| = \alpha(A \cup B) = \mu(A \cup B)$ and $|I''| = \alpha(C) = \mu(C)$.

Let $\mathcal{D}$ be the set of even components in $G[V \setminus B]$. For each even component $X \in \mathcal{D}$, let $M_X \subseteq M''$ be the edges in M'' that have both endpoints in component X. Therefore $M = \bigcup_{X \in \mathcal{D}} M_X$. i.e M'' is partitioned into sets M_X of all even components. Since $|M''|$ is a perfect matching in C, this implies that for each $X \in \mathcal{D}$, M_X is a perfect matching in X, and $|M_X| = \mu(X)$. For each even component $X \in \mathcal{D}$, let $I_X = I'' \cap V(X)$. i.e. I'' is partitioned into sets I_X of all even components. Since for each $X \in D$, X is an even component and M_X is a perfect matching in it, we have $|M_X| = |V(X)|/2$. Claim 3.2 implies that for each $X \in \mathcal{D}$, $|I_X| \leq |M_X|$. We know that $|I''| = |M''|$, which can also be written as $\sum_{X \in \mathcal{D}} |I_X| = \sum_{X \in \mathcal{D}} |M_X|$. Therefore, for each $X \in \mathcal{D}, |I_X| = |M_X| = |V(X)|/2$, which implies that $|I_X| = \alpha(X)$ and $\alpha(X) = \mu(X)$.

Let $M_A \subseteq M'$ be the edges in M' that have both endpoints in A. Thus $M_A = M' \cap E(A)$. By Claim 3.1, M_A is a maximum matching in A. Thus $|M_A| = \mu(A)$. Let $I_A = I' \cap A$. Since $|I'| = |M'|$, and all vertices of B are included in I' (Corollary 3) and are matched by M' (Claim 3.1), we have that $|I_A| = |M_A|$. Let $\mathcal{T}$ be the set of odd components in $G[V\ B]$ of size > 1. For each odd component $P \in \mathcal{T}$, let $M_P \subseteq M_A$ be the edges in M_A that have both endpoints in component P. Therefore $M_A = \bigcup_{P \in \mathcal{T}} M_P$. i.e. M_A is partitioned into sets M_P of all components in $\mathcal{T}$. Since M_A is a maximum matching in A, this implies that for each $P \in \mathcal{T}$, M_P is a maximum matching in P, and $|M_P| = \mu(P)$. For each component $P \in \mathcal{T}$, let $I_P = I_A \cap V(P)$, i.e. I_A is partitioned into sets I_P of all components in $\mathcal{T}$. Since for each $P \in \mathcal{T}$, P is an odd component of size > 1 and M_P is a maximum matching in it, we have $|M_P| = |(V(P) - 1)/2|$. Claim 3.5 implies that for each $P \in \mathcal{T}$, $|I_P| \leq |M_P|$. We know that $|I_A| = |M_A|$, which can also be written as $\sum_{P \in \mathcal{T}} |I_P| = \sum_{P \in \mathcal{T}} |M_P|$. Therefore for each $P \in \mathcal{T}$, $|I_P| = |M_P| = |(V(P) - 1)/2|$, which implies that $|I_P| = \alpha(P)$ and $\alpha(P) = \mu(P)$. □

Claim 3.9. *For a k-regular graph $G(V, E)$ with $\alpha = \mu$, there does not exist any even component in $G[V \setminus B]$.*

<u>Proof:</u> Let I be a maximum independent set in G, and M be a maximum matching in G. Let $\mathcal{D}$ be the set of even components of $G[V \setminus B]$. Let $M' = M \cap E(A \cup B)$, $I' = I \cap (A \cup B)$, $M'' = M \setminus M'$, $I'' = I \setminus I'$, and for each even component X of $G[V \setminus B]$, let $M_X = M'' \cap E(X)$ and $I_X = I'' \cap V(X)$. Then by Claim 3.8:

- $|M'| = |M \cap E(A \cup B)| = \mu(A \cup B) = \alpha(A \cup B) = |I \cap (A \cup B)| = |I'|$

- $|M''| = |M \cap E(C)| = \mu(C) = \alpha(C) = |I \cap C| = |I''|$
- For each even component X of $G[V \setminus B]$,
  $|M_X| = |M'' \cap E(X)| = \mu(X) = \alpha(X) = |I'' \cap V(X)| = |I_X| = |V(X)|/2$

Since G is k-regular, with $\alpha = \mu$, by corollary 3, all the vertices in set B are part of the maximum independent set I. Thus for all $X \in \mathcal{D}$, there does not exist an edge between I_X and B. Since G is k-regular and $|I_X| = |V(X)|/2$, I_X has $k|V(X)|/2$ edges into $V(X) \setminus I_X$. Then the degrees of all the vertices in $V(X) \setminus I_X$ are used up to capture these edges, which implies that $V(X) \setminus I_X$ cannot have any edges into B. Thus X is a connected component in G, disconnected from B, which is a contradiction to our assumption that G is connected.

Therefore there does not exist any even component in $G[V \setminus B]$. □

Claim 3.10. *For a 3-regular graph $G(V, E)$ with $\alpha = \mu$, each odd component Q of $G[V \setminus B]$ of size > 1 is either a bipartite graph with 3 neighbours in B, or a bubble graph with 1 neighbour in B. Moreover, in the former case all the neighbours of B lie in one partite set of Q, and in the latter case, the contact vertex of the bubble is the only neighbour of B in Q.*

Proof: Since $\alpha = \mu$, then Corollary 2 and Claim 3.6 imply that $\alpha(A \cup B) = \mu(A \cup B)$ and $\alpha(C) = \mu(C)$.

Since $\alpha(A \cup B) = \mu(A \cup B)$, every maximum independent set contains B (Corollary 3) and every vertex in B is matched by any maximum matching (Claim 3.1), we have that $\alpha(A) = \mu(A)$. By Claim 3.5, this implies that for each odd component Q, $\alpha(Q) = \mu(Q)$. Let Q be an odd component with n_q vertices. Let I be a maximum independent set in G and let $I_Q = I \cap V(Q)$ By Claim 3.8, $|I_Q| = (n_q - 1)/2$. Corollary 3 implies that there does not exist an edge between the sets I_Q and B. Since G is 3-regular, there are exactly $3(n_q - 1)/2$ edges going from I_Q into $V(Q) \setminus I_Q$. Since $|V(Q) \setminus I_Q| = (n_q + 1)/2$, the total degree of vertices in $V(Q) \setminus I_Q$ is $3(n_q + 1)/2$ and $3(n_q - 1)/2$ edges from I_Q need to be absorbed by $V(Q) \setminus I_Q$, the set $V(Q) \setminus I_Q$ has a "free" degree of only 3. Since there is at least one edge from Q to B, and there are no edges from I_Q to B, there can only be two cases:

- Either $V(Q) \setminus I_Q$ has exactly 3 neighbours in B, and hence $V(Q) \setminus I_Q$ is an independent set, making Q a bipartite graph such that all neighbours of B in Q lie in one of the partite sets,
- or $V(Q) \setminus I_Q$ has exactly 1 neighbour in B, and there is one edge between two vertices of $V(Q) \setminus I_Q$. Hence Q is a bubble graph. The vertex of $V(Q) \setminus I_Q$ that has an edge to B is the contact vertex of the bubble and that vertex is the only neighbour of B in Q.

□

We are now ready to give our new proof for the theorem of Mohr and Rautenbach.

Theorem 6 (Mohr-Rautenbach)**.** *For a graph $G(V, E)$, if G is 3-regular with $\alpha = \mu$, then G is a special graph.*

Proof:
We only give a proof for the forward direction, since the proof for the backward direction is rather straightforward. If G has a perfect matching, then by Claim 3.3, G is a bipartite k-regular graph. Then we can set $V_0 = V(G)$ and there are no bubbles. Thus G is a special graph. So let us assume that G has no perfect matching. Then G has a Gallai-Edmonds decomposition in to the sets A, B, C as defined before. By Claim 3.9, we know that $C = \emptyset$. By Corollary 3, B is an independent set. By Claim 3.10, each odd component of $G[V \setminus B]$ that has size > 1 is either a bipartite graph such that all neighbours of B lie in one partite set or a bubble such that the only neighbour of B in it is its contact vertex. Let $\mathcal{P}$ denote the odd components in $G[V \setminus B]$ that are bipartite graphs and let $\mathcal{Q}$ denote the remaining odd components. Clearly, the singleton odd components of $G[V \setminus B]$ belong to $\mathcal{P}$. Define $V_0 = B \cup \bigcup_{P \in \mathcal{P}} V(P)$. Now it can be easily seen that $G[V_0]$ is a bipartite graph. Suppose $\mathcal{Q} = \{Q_1, Q_2, \ldots, Q_l\}$. We set $V_1 = V(Q_1), V_2 = V(Q_2), \ldots, V_l = V(Q_l)$. Then by Claim 3.10, for each $i \in \{1, 2, \ldots, l\}$, $G[V_i]$ is a bubble and the contact vertices of these bubbles are adjacent to vertices in one partite set of $G[V_0]$ (since the set B is contained in one partite set of $G[V_0]$). Note that it is possible that $G[V_i]$ for some $i \in \{1, 2, \ldots, l\}$ is a bubble, but is not a 2-connected bubble. Mohr and Rautenbach observe that if a bubble with partition (I, R) (where I is the independent set) is not 2-connected, then there exists $I' \subseteq I$ and $R' \subseteq R$ such that (I', R') is also a bubble. Then the subgraph of the bubble induced by the vertices in $(I \setminus I') \cup (R \setminus R')$ form a bipartite graph, and these vertices can be added into V_0 without affecting the fact that $G[V_0]$ is a bipartite graph (since $I \setminus I'$ will go into the same partite set as B in $G[V_0]$, the contact vertex of the bubble (I', R') will again have an edge into the partite set of $G[V_0]$ that contains B). Thus G is a special graph. □

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