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ON SOME CHARACTERISATIONS OF THE NORMAL LAW

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SUMMARY. Let $X_1, \ldots, X_n$ be independent variables and $Y_1, \ldots, Y_n$ be linear functions of $X_1, \ldots, X_n$. In this paper, the conditions under which the equations $E(Y_i | Y_{p+1}, \ldots, Y_{p+j}) = 0$, $i = 1, \ldots, p$, imply normality of $X_1, \ldots, X_n$ are examined. An important case considered is when $E(Y_1 | Y_2) = 0$ involving only two linear functions (with $p = 1, j = 0$), which provides a generalisation of the earlier results on the characterisation of the normal law by Darmois, Kagan, Linnik, Rao, Skitovich and others. The conditions imposed to ensure normality are of two types, one on the nature of the coefficients in the linear functions Y_i and another on the nature of the distributions of $X_1, \ldots, X_n$, such as identical distribution, existence of moments etc.

1. Introduction

The Darmois-Skitovich theorem asserts the normality of independent random variables $X_1, \ldots, X_n$ if there exist any two linear functions

$$\begin{array}{c} a_1X_1 + \ldots + a_nX_n \\ b_1X_1 + \ldots + b_nX_n \end{array} \qquad \ldots \quad (1.1)$$

with the condition $a_ib_i \neq 0$, $i = 1, \ldots, n$, which are independently distributed. In a recent paper, Kagan, Linnik and Rao (1965) showed that if $X_1, \ldots, X_n$ are $n(\geq 3)$ independent and identically distributed (i.i.d.) variables such that $E(X_i) = 0$ and the conditional expectation

$$E(\bar{X} | X_1 - \bar{X}, \ldots, X_n - \bar{X}) = 0 \qquad \ldots \quad (1.2)$$

where $n\bar{X} = (X_1 + \ldots + X_n)$, then X_i are normally distributed. The object of the paper is to provide a few other characterisations of the normal law, which are somewhat more general than the earlier theorems of Darmois-Skitovich and Kagan-Linnik-Rao, and which are of great interest in the theory of minimum variance unbiased estimation.

Instead of demanding stochastic independence of two linear functions as in the Darmois-Skitovich theorem, we impose the weaker condition of the conditional expectation of one given the other being zero

$$E(a_1X_1 + \ldots + a_nX_n | b_1X_1 + \ldots + b_nX_n) = 0 \qquad \ldots \quad (1.3)$$

and show that under some conditions, the X_i follow the normal law.

A surprising result is that if $X_1, \ldots, X_n$ are $n(\geq 3)$ i.i.d. variables such that $E(X_i) = 0$, $E(X_i^2) < \infty$ and

$$E(\bar{X} | X_i - \bar{X}) = 0 \quad \text{for any fixed } i, \qquad \ldots \quad (1.4)$$

then $X_1, ..., X_n$ are normally distributed. This result is also a generalisation of the Kagan-Linnik-Rao theorem.

It is also shown that if $X_1, ..., X_n$ are independent random variables (not necessarily i.i.d.) such that $E(X_i) = 0$, $i = 1, ..., n$ and there exist n linearly independent linear functions $\sum_1^n a_{ij}X_j$, $i = 1, ..., n$ such that

$$E(\Sigma a_{1i} X_i | \Sigma a_{2i} X_i, ..., \Sigma a_{ni} X_i) = 0 \quad \dots \quad (1.5)$$

then the X_i are normally distributed. This result provides another generalisation of the Kagan-Linnik-Rao theorem.

A further generalisation of this result is that if

$$E(\Sigma a_{ri} X_i | \Sigma a_{p+1,i} X_i, ..., \Sigma a_{ni} X_i) = 0, \quad r = 1, ..., p \quad \dots \quad (1.6)$$

then, under a simple condition on the coefficients, the X_i are normally distributed.

A more general problem is the examination of the condition

$$E(\Sigma a_{1i} X_i | \Sigma a_{2i} X_i, \Sigma a_{3i} X_i) = 0$$

involving the conditional expectation of one linear function given two linear functions of $n(> 3)$ independent variables. Again, under simple conditions on the coefficients, normality of the X_i follows.

Finally, some results obtained in an earlier paper of the author (Rao, 1969) on a characterisation of the normal law through an optimum property of least squares estimators are re-examined.

2. Solutions of some functional equations

Consider the functional equation

$$a_1 \psi(b_1 t) + a_2 \psi(b_2 t) = 0 \quad \dots \quad (2.1)$$

valid for all t in an interval $I = (-\delta, \delta)$, $\delta > 0$ where $a_1, a_2, b_1 \neq b_2$ are all fixed non-zero constants and ψ is a continuous function such that $\psi(0) = 0$. Let $\alpha = -(a_2/a_1)$ and $\beta = (b_2/b_1)$ with $|\beta| \leq 1$ without loss of generality. The solution of (2.1) depends on the nature of α, β and analytical properties of $\psi(t)$. Lemma 1 gives some conditions and the corresponding solutions. Note that the solutions given are valid only in the interval I, in which the functional equation (2.1) holds.

Lemma 1 : (i) *If* $|\alpha| < 1$, *or if* $|\alpha| = 1$ *and* $|\beta| < 1$, *then* $\psi(t) = 0$ *for all* t *in* I.

(ii)* *If* $\psi(t)$ *admits a derivative continuous at the origin (or, more generally, if* $\psi(t)$ *is of the form* $t\phi(t)$ *where* $\phi(t)$ *is continuous at the origin), then*

(a) $\psi(t) = ct$ *when* $|\alpha\beta| = 1$, $|\beta| < 1$, *and*

(b) $\psi(t) = 0$ *when* $|\alpha\beta| < 1$.

where, in (a), c *is a constant.*

* It is interesting to note that if $\psi(t)$ is of the form $|t|^\lambda \phi(t)$ where $\phi(t)$ is continuous at the origin, then

(a) $\psi(t) = c|t|^\lambda$ when $|\alpha||\beta|^\lambda = 1$, $|\beta| < 1$, and

(b) $\psi(t) = 0$ when $|\alpha||\beta|^\lambda < 1$.

To prove (i) let us observe that

$$\psi(t) = \alpha\psi(\beta t) = \alpha^2\psi(\beta^2 t) = \ldots = \alpha^n \psi(\beta^n t). \qquad \ldots \quad (2.2)$$

Hence
$$\psi(t) = \lim_{n\to\infty} \alpha^n \psi(\beta^n t)$$
$$= 0 \text{ if } |\alpha| < 1, \text{ or if } |\alpha| = 1 \text{ and } |\beta| \neq 1.$$

Note that, by our choice, $|\beta| \leqslant 1$.

To prove (ii), we substitute $t\phi(t)$ for $\psi(t)$ and obtain (for $t \neq 0$)

$$\phi(t) = \alpha\beta\phi(\beta t) = \alpha^2\beta^2\phi(\beta^2 t)$$
$$= \alpha^4\beta^4\phi(\beta^4 t) = \ldots = (\alpha\beta)^{2n}\phi(\beta^{2n} t). \qquad \ldots \quad (2.3)$$

Hence
$$\phi(t) = \lim_{n\to\infty} (\alpha\beta)^{2n}\phi(\beta^{2n} t)$$
$$= \phi(0) \text{ when } |\alpha\beta| = 1 \text{ and } \beta \neq 1; \text{ and}$$
$$= 0 \text{ when } |\alpha\beta| < 1.$$

Thus Lemma 1 is proved with $c = \phi(0)$.

Next, let us consider an equation of the form

$$a_1\psi(b_1 t) + \ldots + a_n\psi(b_n t) = 0 \qquad \ldots \quad (2.4)$$

valid for all t in an interval $I = (-\delta, \delta)$, $\delta > 0$, where $|b_n| > \max\{|b_1|, \ldots, |b_{n-1}|\}$. The equation (2.4) can then be written in the form

$$\psi(t) = -\{\alpha_1\psi(\beta_1 t) + \ldots + \alpha_{n-1}\psi(\beta_{n-1} t)\} \qquad \ldots \quad (2.5)$$

where
$$|\beta_i| < 1, \; i = 1, \ldots, n-1.$$

Lemma 2 : *If $\psi(t)$ admits a derivative which is continuous at the origin (or, more generally, $\psi(t)$ can be written in the form $t\phi(t)$ where $\phi(t)$ is continuous at the origin) $\Sigma\alpha_i\beta_i = -1$ and $\alpha_i\beta_i < 0$, $i = 1, \ldots, n-1$, then $\psi(t) = ct$.*

To prove the result, we substitute $t\phi(t)$ for $\psi(t)$ in (2.5) and for $t \neq 0$, divide both sides of the resulting relation by t, obtaining

$$\phi(t) = -\{\alpha_1\beta_1\phi(\beta_1 t) + \ldots + \alpha_{n-1}\beta_{n-1}\phi(\beta_{n-1} t)\}$$
$$= p_1\phi(\beta_1 t) + \ldots + p_{n-1}\phi(\beta_{n-1} t) \qquad \ldots \quad (2.6)$$

where all the p_i are positive and $\Sigma p_i = 1$, by assumption. From (2.6), we have

$$\phi(\beta_i t) = p_1\phi(\beta_1\beta_i t) + \ldots + p_{n-1}\phi(\beta_{n-1}\beta_i t)$$

so that
$$\phi(t) = \Sigma p_i\phi(\beta_i t) = \Sigma\Sigma p_i p_j\phi(\beta_i\beta_j t)$$
$$= \Sigma\Sigma q_{ij}\phi(\beta_i\beta_j t), \text{ where } \Sigma\Sigma q_{ij} = 1,$$

whence
$$\phi(t) - \phi(0) = \Sigma\Sigma q_{ij}[\phi(\beta_i\beta_j t) - \phi(0)]. \qquad \ldots \quad (2.7)$$

Thus proceeding, we obtain

$$\phi(t) - \phi(0) = \Sigma q_{i_1 \ldots i_m}[\phi(\beta_{i_1} \ldots \beta_{i_m} t) - \phi(0)] \qquad \ldots \quad (2.8)$$

where $\Sigma q_{i_1 \ldots i_m} = 1$, summation being over $1 \leqslant i_k \leqslant m$; $k = 1, 2, \ldots, n-1$. Now, for any fixed $t \neq 0$, and any given $\varepsilon < 0$, we choose m so large that $[\max|\beta_i|]^m < \eta/|t|$ where η is such that $|\phi(x) - \phi(0)| < \varepsilon$ for $|x| < \eta$. Then, the modulus of the right hand side of (2.8) is less than ε, so that, for any $\varepsilon > 0$, $|\phi(t) - \phi(0)| < \varepsilon$; in other words, $\phi(t) = \phi(0) = c$ (say). Then $\psi(t) = ct$. This proves Lemma 2.

Note that if we are using the hypothesis that $\psi'(t)$, the derivative of $\psi(t)$, exists and is continuous at the origin in proving Lemmas 1 and 2, the equations (2.3) and (2.6) will be written with ψ' in the place of ϕ. Then we obtain, in the same way, $\psi'(t) = \psi'(0) = c$, giving the solution $\psi(t) = ct$. In such a case the condition $|\alpha\beta| = 1$ of Lemma 1 and the condition $\Sigma\, \alpha_i\beta_i = -1$ of Lemma 2 can be replaced by the condition $\psi'(c) \neq 0$.

We quote some lemmas from previous papers of the author (Rao, 1966) and Zinger-Linnik (1955), Linnik (1964), which will be used in establishing some of the main results of the present paper.

Lemma 3 (Linnik, 1964 stated in an extended form in Rao, 1966) : *Consider the equation*

$$\psi_1(t_1+b_1t_2)+\psi_2(t_1+b_2t_2)+\ldots+\psi(t_1+b_rt_2) = A(t_1)+B(t_2)+Q(t_1, t_2)$$

in two real variables t_1, t_2, valid in $|t_1| < \delta$, $|t_2| < \delta$ where Q is a quadratic function. Let

(i) *$b_1, \ldots, b_r$ be all different, and*

(ii) *$\psi_1, \ldots, \psi_r$, A and B be continuous functions.*

Then $\psi_1, \ldots, \psi_r$, A and B are all polynomial functions of degree max $(2, r)$ at most in a neighbourhood of the origin.

Lemma 4 (Zinger-Linnik, 1955) : *Let $f_1, f_2, \ldots, f_n$ be characteristic functions and $\alpha_1, \ldots, \alpha_n$ positive constants. Let, in a neighbourhood $|t| < \delta$ of the origin, the relation*

$$\prod_{j=1}^{n} [f_j(t)]^{\alpha_j} = exp\left(i\mu t - \frac{1}{2}\sigma^2t^2\right)$$

hold. Then the f_j are all normal characteristic functions.

3. The main characterisation theorems on the normal law

Theorem 1 : *Let X_1, X_2 be two i.i.d. random variables such that $E(X_1) = 0$. Further, let there exist two linear functions $a_1X_1+a_2X_2$ and $b_1X_1+b_2X_2$, where all the coefficients are non-zero, such that*

$$E(a_1X_1+a_2X_2\,|\,b_1X_1+b_2X_2) = 0, \qquad \ldots \quad (3.1)$$

and $|b_2/b_1| \leq 1$ without loss of generality.

(i) *If $|a_2/a_1| < 1$, or if $|a_2/a_1| = 1$ and $|b_2/b_1| < 1$, then X_1, X_2 have degenerate distributions.*

(ii) *If $E(X_1^2) < \infty$, $a_1b_1+a_2b_2 = 0$* and $|b_2/b_1| < 1$, then X_1, X_2 are normally distributed.*

Proof : The condition (3.1) implies

$$E\left\{(a_1\, X_1+a_2\, X_2)e^{it(b_1X_1+b_2X_2)}\right\} = 0. \qquad \ldots \quad (3.2)$$

Let $f(t)$ denote the c.f. (characteristic function) of X_1 and $f'(t)$, the first derivative of $f(t)$, which exists since $E(X_1)$ exists. The equation (3.2) can be written as

$$a_1f'(b_1\, t)\, f(b_2\, t)+a_2f(b_1\, t)\, f'(b_2\, t) = 0 \qquad \ldots \quad (3.3)$$

*This condition can be replaced by the condition $E(X_1^2) \neq 0$, which follows by differentiating the equation (3.4) and putting $t = 0$. In such a case X_1, X_2 have a non-degenerate normal distribution.

valid for all t. Let us choose an interval $I = (-\delta, \delta)$ of t where $f(b_1 t)$ and $f(b_2 t)$ do not vanish. Dividing the equation (3.3) by the product $f(b_1 t)f(b_2 t)$ and denoting $\psi(t) = f'(t)/f(t)$, we obtain the functional equation

$$a_1\psi(b_1 t)+a_2\psi(b_2 t) = 0 \quad \ldots \quad (3.4)$$

valid in the interval $I = (-\delta, \delta)$.

Proof of (i) : The conditions on the coefficients are the same as those in (i) of Lemma 1. Hence $\psi(t) = 0$ in the interval $I = (-\delta, \delta)$ or $\log f(t) = c$ in I. By analytic continuation, $\log f(t) = c$ for all t, which proves (i) of Theorem 1.

Proof of (ii) : If $E(X^2) < \infty$, then $f''(t)$, the second derivative of $f(t)$ exists and is continuous everywhere, in which case the first derivative of $\psi(t)$ exists (in I) and is continuous at $t = 0$. Further the conditions on the coefficients satisfy those of (ii) (a) of Lemma 1. Hence $\psi(t) = ct$ in I or $\log f(t) = (c/2)t^2$ in $(-\delta, \delta)$. Hence by analytic continuation $\log f(t) = (c/2)t^2$ for all t, i.e., $f(t)$ is the c.f. of a normal distribution, which proves (ii).

Note 1 : Observe that Theorem 1 is not applicable when the condition (3.1) is of the form

$$E(X_1+X_2 \mid X_1-X_2) = 0 \quad \ldots \quad (3.5)$$

since the condition $|b_2/b_1| \neq 1$ is not satisfied. The condition (3.5) merely implies that the characteristic function of X_1 is symmetrical as shown by Kagan, Linnik and Rao (1965).

Note 2 : However, a condition such as

$$E(X_1+2X_2 \mid 2X_1-X_2) = 0 \quad \ldots \quad (3.6)$$

is sufficient to ensure the normality of X_1 and X_2 (if the variances are finite).

Note 3 : Suppose $E(|X_1|^s) < \infty$ for some integer s. Then the conditions

$$E(a_1X_1+a_2X_2 \mid b_1X_1+b_2X_2) = 0 \quad \ldots \quad (3.7)$$

$$\Sigma\, a_i b_i^r \neq 0, \; r = 2, \ldots, s$$

imply that X_1 and X_2 have a distribution which has the same moments upto order s as a normal distribution. The result is obtained by differentiating the relation (3.4) successively up to $(s-1)$ times and putting $t = 0$ in each equation.

Theorem 2* : *Let $X_1, \ldots, X_n$ be i.i.d. random variables such that $E(X_1) = 0$. Further let there exist two linear functions*

$$Y_1 = a_1X_1+\ldots+a_nX_n$$
$$Y_2 = b_1X_1+\ldots+b_nX_n \quad \ldots \quad (3.8)$$

such that $E(Y_1 \mid Y_2) = 0$, $|b_n| > \max(|b_1|, \ldots, |b_{n-1}|)$ and $a_n \neq 0$. (We need one of the coefficients in Y_2 to be numerically larger than the others and the corresponding coefficient in Y_1 to be non-zero).

If $E(X_1^2) < \infty$, $\Sigma a_i b_i = 0$ (or $E(X_1^2) \neq 0$) and $(a_i b_i / a_n b_n) < 0$ for $i = 1, \ldots, n-1$, then the X_i are normally distributed.

*While this paper was in press, A. M. Kagan communicated to me the result of Theorem 2, under different conditions on the random variables and the coefficients a_i and b_j.

Proof: The condition $E(Y_1 | Y_2) = 0$ gives the equation

$$a_1\psi(b_1 t) + \ldots + a_n\psi(b_n t) = 0. \qquad \ldots \quad (3.9)$$

Then the result of Theorem 2 follows by an application of Lemma 2.

Note: It follows from Theorem 2 that (if $n \geqslant 3$) the condition

$$E(\bar{X} | X_i - \bar{X}) = 0 \text{ for any given } i \qquad \ldots \quad (3.10)$$

is sufficient to ensure the normality of $X_1, \ldots, X_n$ (if the variance is finite). This result may be compared with the earlier result of Kagan, Linnik and Rao (1965) where the condition stated is

$$E(\bar{X} | X_1 - \bar{X}, \ldots, X_n - \bar{X}) = 0. \qquad \ldots \quad (3.11)$$

However, in the latter case we assume only the existence of the first moment, whereas in the case of the weaker condition (3.10), the existence of the second moment is assumed.

In Theorems 1 and 2, we have considered an equation involving the conditional expectation of one linear function given another. We shall now consider a condition of the type

$$E(Y_1 | Y_2, Y_3) = 0 \qquad \ldots \quad (3.12)$$

where Y_1, Y_2, Y_3 are linear functions. In such a case the normality of the basic variables follows under less severe conditions on the coefficients and the distributions of the random variables.

Let $X_1, \ldots, X_n$ be $n \geqslant 3$ independent random variables not necessarily identically distributed and consider three linear functions.

$$\begin{aligned} Y_1 &= a_1X_1 + \ldots + a_nX_n \\ Y_2 &= b_1X_1 + \ldots + b_nX_n \\ Y_3 &= c_1X_1 + \ldots + c_nX_n. \end{aligned} \qquad \ldots \quad (3.13)$$

Let $c_i/b_i = \beta_i$(when $c_i \neq 0$, $b_i \neq 0$), $c_i/a_i = \alpha_i$ (when $c_i \neq 0$, $a_i \neq 0$ and $b_i = 0$), $b_i/a_i = \delta_i$ (when $b_i \neq 0$, $a_i \neq 0$ and $c_i = 0$), and $a_i/b_i = \gamma_i$ (when $a_i \neq 0$, $b_i \neq 0$ and $c_i \neq 0$). Notice that in any given case the α's and δ's and other coefficients depend on the nature of the coefficients in (3.13). We prove the following theorem.

Theorem 3: *If*

(i) $a_i \neq 0$, $i = 1, \ldots, n$,

(ii) *for each* i, b_i *and* c_i *are not simultaneously zero,*

(iii) $(\beta_i) \neq (\beta_j)$ *for any* i, j *such that* b_i, b_j, c_i, c_j *are all different from zero, and*

(iv) *all* α_i *defined are of the same sign and all* δ_j *defined are of the same sign,*

then $E(Y_1 | Y_2, Y_3) = 0$ *implies that* $X_1, \ldots, X_n$ *are all normally distributed.*

Proof: Without loss of generality let $b_1 = \ldots = b_s = 0$, $c_{s+1}, \ldots, c_{s+k} = 0$ and $b_i \neq 0$, $c_i \neq 0$ for $i = s+k+1, \ldots, n$. Further let

$$Z_i = a_iX_i, \quad i = 1, \ldots, s+k; \; Z_i = b_iX_i, \quad i \geqslant s+k+1 \qquad \ldots \quad (3.14)$$

in which case the linear functions Y_1, Y_2, Y_3 can be written as

$$\begin{aligned} Y_1 &= Z_1 + \ldots + Z_{s+k} + \gamma_{s+k+1}Z_{s+k+1} + \ldots + \gamma_nZ_n \\ Y_2 &= \delta_{s+1}Z_{s+1} + \ldots + \delta_{s+k}Z_{s+k} + Z_{s+k+1} + \ldots + Z_n \\ Y_3 &= \alpha_1Z_1 + \ldots + \alpha_sZ_s + \beta_{s+k+1}Z_{s+k+1} + \ldots + \beta_nZ_n. \end{aligned} \qquad \ldots \quad (3.15)$$

The condition $$E(Y_1 \mid Y_2, Y_3) = 0$$

implies $$E\left(Y_1 e^{it_1 Y_2 + it_2 Y_3}\right) = 0. \qquad \ldots \quad (3.16)$$

Denoting $f_i'(t)/f_i(t)$ by $\psi_i(t)$ where $f_i(t)$ is the characteristic function of Z_i, the equation (3.16) reduces (in a domain $|t_1| < \delta$, $|t_2| < \delta$) to

$$\sum_{1}^{s} \psi_i(\alpha_i t_2) + \sum_{s+1}^{s+k} \psi_i(\delta_i t_1) + \sum_{s+k+1}^{n} \gamma_i \psi_i(t_1 + \beta_i t_2) = 0 \qquad \ldots \quad (3.17)$$

which can be written as

$$\sum_{s+k+1}^{n} \omega_i(t_1 + \beta_i t_2) = A(t_1) + B(t_2) \qquad \ldots \quad (3.18)$$

with obvious notation. We observe that $\beta_{s+k+1}, \ldots, \beta_n$ are all different by hypothesis and that all the functions in (3.18) are continuous and hence Lemma 3 applies giving that

$$\psi_i(t) = \gamma_i^{-1} \omega_i(t), \; i = s+k+1, \ldots, n$$

$$B(t) = -\sum_{1}^{s} \psi_i(\alpha_i t) \qquad \ldots \quad (3.19)$$

$$A(t) = -\sum_{s+1}^{s+k} \psi_i(\delta_i t)$$

are polynomials in t.

Now $\psi_{s+k+j}(t)$ is the derivative of the second characteristic of the random variable Z_{s+k+j} in which case the degree of $\psi_{s+k+j}(t)$ is at most unity. Thus Z_{s+k+j}, ..., Z_n are all normally distributed.

Further, the result that $\sum_{1}^{s} \psi_i(\alpha_i t)$ is polynomial in t implies that $\sum_{1}^{s} \alpha_i^{-1} \log f_i(\alpha_i t)$ is a ploynomial in t. Since α_i are all of the same sign, an application of Lemma 4 (Zinger-Linnik, 1955) gives that $f_i(\alpha_i t)$ is a polynomial of degree two at most or Z_i, $i = 1, \ldots, s$ are all normally distributed. Similarly Z_i, $i = s+1, \ldots, s+k$ are all normally distributed. Then $X_1, \ldots, X_n$ are all normal variables.

Note: In Theorem 3, the condition that β_i are different may appear somewhat restrictive. This can be relaxed in the following way.

Let $\beta_{i_1} = \beta_{i_2} = \ldots = \beta_{i_k} = \beta_i$ for a set of indices $i_1, i_2, \ldots, i_k$. Then the result of Theorem 3 is true if the corresponding $\gamma_{i_1}, \ldots, \gamma_{i_k}$ are of the same sign.

The result follows since the functional equation (3.17) gives us that

$$\sum_{j=1}^{k} \gamma_{i_j} \omega_{i_j}(t)$$

is a polynomial in t in an interval $(-\delta, \delta)$ or integrating with respect to t

$$\sum_{j=1}^{k} \gamma_{i_j} \log f_{i_j}(t)$$

is a polynomial in t in an interval $(-\delta, \delta)$ where $f_{i_j}(t)$ is the characteristic function of the random variable X_{i_j}. Then, if γ_{i_j} are all of the same sign, it follows from Lemma 4 that each $f(t)$ is at most a quadratic in t. Thus $X_{i_1}, \ldots, X_{i_k}$ are all normally distributed.

For example, if $X_1, \ldots, X_n$ are $n(\geqslant 3)$ independent and not necessarily identically distributed variables and $E(X_i) < \infty$, $i = 1, \ldots, n$, then the condition $E(\bar{X} \mid X_1 - \bar{X}, X_2 - \bar{X}) = 0$ implies that $X_1, \ldots, X_n$ are all normally distributed.

Theorem 4 : *Let $X_1, \ldots, X_n$ be $n \geqslant 3$ independent random variables not necessarily identically distributed. Let there exist n linear functions*

$$Y_i = a_{i1}X_1 + \ldots + a_{in}X_n, \quad i = 1, \ldots, n \qquad \ldots \quad (3.20)$$

such that the determinant $|(a_{ij})| \neq 0$ and $a_{11}, \ldots, a_{1n}$ are all different from zero. If $E(X_i) = 0$, $i = 1, \ldots, n$ and

$$E(Y_1 \mid Y_2, \ldots, Y_n) = 0 \qquad \ldots \quad (3.21)$$

then $X_1, \ldots, X_n$ are all normally distributed.

Proof : We can assume, without loss of generality, that $a_{11} = \ldots = a_{1n} = 1$. The condition (3.21) implies that

$$E\left(Y_1 e^{it_1 Y_2 + \cdots + it_{n-1} Y_n}\right) = 0$$

or, in a suitable domain $|t_i| < \delta$, $i = 1, 2, \ldots, n-1$,

$$\psi_1'(a_{21}t_1 + \ldots + a_{n1}t_{n-1}) + \ldots + \psi_n'(a_{2n}t_1 + \ldots + a_{nn}t_{n-1}) = 0. \qquad \ldots \quad (3.22)$$

Making the transformation

$$\begin{aligned} a_{21}t_1 + \ldots + a_{n1}t_{n-1} &= \tau_1 \\ \cdot \quad \cdots \quad \cdot \quad &\cdot \\ a_{2,n-1}t_1 + \ldots + a_{n,n-1}\, t_{n-1} &= \tau_{n-1} \end{aligned} \qquad \ldots \quad (3.23)$$

the equation (3.22) can be written as

$$\psi_1'(\tau_1) + \ldots + \psi_{n-1}'(\tau_{n-1}) + \psi'(c_1\tau_1 + \ldots + c_{n-1}\tau_{n-1}) = 0. \qquad \ldots \quad (3.24)$$

Substituting the value zero for all τ_i, except for τ_r and τ_s, we find ($E(X_i) = 0$ implies $\psi_i'(0) = 0$ for all i)

$$\psi_r'(\tau_r) + \psi_s'(\tau_s) = \psi_n'(c_r\tau_r + c_s\tau_s). \qquad \ldots \quad (3.25)$$

Hence, an application of Lemma 3 shows that $\psi_1, \ldots, \psi_n$ are all polynomials in a neighbourhood of the origin. Hence $X_1, \ldots, X_n$ are all normally distributed. Theorem 4 provides a direct generalisation of the Kagan-Linnik-Rao theorem.

Theorem 5 : *Let $X_1, \ldots, X_n$ be n i.i.d. random variables such that $E(X_1) = 0$ and $E(|X_1|^s) < \infty$, for some integer s. Let there exist two linear functions*

$$\begin{aligned} Y_1 &= a_1X_1 + \ldots + a_nX_n \\ Y_2 &= b_1X_1 + \ldots + b_nX_n \end{aligned} \qquad \ldots \quad (3.26)$$

such that $E(Y_1 \mid Y_2) = 0$. If $\Sigma a_i b_i^r \neq 0$, $r = 2, \ldots, s-1$, then the distribution of X_1 agrees with a normal distribution up to moments of order s.

Proof : We start with the equation

$$a_1\psi'(b_1 t) + \ldots + a_n\psi'(b_n t) = 0 \qquad \ldots \quad (3.27)$$

(valid in an interval around the origin), differentiate $r(\leqslant s-1)$ times and put $t = 0$, which gives the equations

$$(a_1 b_1^r + \ldots + a_n b_n^r)\, \kappa_{r+1} = 0 \qquad \ldots \quad (3.28)$$

where κ_{r+1} is the $(r+1)$-th cumulant of the distribution of X_1. Since $\Sigma a_i b_i^r \neq 0$ it follows from (3.28) that $\kappa_{r+1} = 0$, $r = 2, \ldots, s-1$. Hence the result of Theorem 5.

We shall now consider n linear functions of n independent non-degenerate variables $X_1, \ldots, X_n$,

$$Y_i = a_{i1}X_1 + \ldots + a_{in}X_n, \; i = 1, \ldots, n \qquad \ldots \quad (3.29)$$

satisfying the conditions

$$E(Y_i \mid Y_{p+1}, \ldots, Y_n) = 0, \quad i = 1, \ldots, p(>1). \qquad \ldots \quad (3.30)$$

If the determinant $|(a_{ij})| \neq 0$, then we can transform $Y_1, \ldots, Y_p$ to new variables $Z_1, \ldots, Z_p$ such that, by a suitable reordering of $X_1, \ldots, X_n$ if necessary and renaming the new variables, for convenience, again as $X_1, \ldots, X_n$,

$$\begin{aligned} Z_1 &= X_1 + w_{1p+1} X_{p+1} + \ldots + w_{1n} X_n \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ Z_p &= X_p + w_{pp+1} X_{p+1} + \ldots + w_{pn} X_n \end{aligned} \qquad \ldots \quad (3.31)$$

and also transform $Y_{p+1}, \ldots, Y_n$ to new variables $Z_{p+1}, \ldots, Z_n$ such that

$$\begin{aligned} Z_{p+1} &= u_{p+1,1} X_1 + \ldots + u_{p+1,p} X_p + X_{p+1} \\ &\ldots \quad \ldots \quad \ldots \quad \ldots \\ Z_n &= u_{n1} \quad X_1 + \ldots + u_{np} \quad X_p + X_n. \end{aligned} \qquad \ldots \quad (3.32)$$

It is easy to see that the condition (3.30) is equivalent to

$$E(Z_i \mid Z_{p+1}, \ldots, Z_n) = 0; \; i = 1, \ldots, p. \qquad \ldots \quad (3.33)$$

In the representation (3.31) we shall say that the variables X_{p+i} and X_{p+j} are connected if the coefficients of X_{p+i} and X_{p+j} are both non-zero in at least one equation (out of the p equations).

Theorem 6 : *If in the representation* (3.31), *the variable X_{p+i} is connected with at least one variable of the type X_{p+j}, for $i = 1, \ldots, n-p$ and in the representation* (3.32) *no column of the u_{ij} coefficients consists wholly of zeros, and $E(X_i) = 0$, $i = 1, \ldots, n$, then the condition*

$$E(Y_i \mid Y_{p+1}, \ldots, Y_n) = 0, \quad i = 1, \ldots, p(>1) \qquad \ldots \quad (3.34)$$

implies that $X_1, \ldots, X_n$ are normally distributed.

Proof: The condition (3.34) is equivalent to (3.33) and the condition

$$E(Z_i \mid Z_{p+1}, \ldots, Z_n) = 0 \qquad \ldots \quad (3.35)$$

gives (in a domain of the form $|t_j| < \delta$, $1 \leqslant j \leqslant n-p$)

$$\psi_i(u_{p+1,i} t_1 + \ldots + u_{ni} t_{n-p}) + w_{i,p+1} \psi_{p+1}(t_1) + \ldots + w_{in} \psi_n(t_{n-p}) = 0, \; i = 1, \ldots, p. \qquad \ldots \quad (3.36)$$

Let X_{p+r} and X_{p+s} be connected in the i-th equation of (3.31). Then substituting zero for all t_j except t_r and t_s in the corresponding relation of (3.36), we have (since $E(X_k) = 0$ implies $\psi_k(0) = 0$),

$$\psi_i(u_{p+r,i} t_r + u_{p+s,i} t_s) = -\{w_{i,p+r} \psi_{p+r}(t_r) + w_{i,p+s} \psi_{p+s}(t_s)\} \qquad \ldots \quad (3.37)$$

If $u_{p+r,i} = 0$, then putting $t_s = 0$, we find that $\psi_{p+r}(t_r) = 0$ which means that X_{p+r} is degenerate, contrary to assumption. Thus $u_{p+r,i}$ and $u_{p+s,i}$ are both different from zero. Now applying Lemma 3, X_i, X_{p+r} and X_{p+s} are normally distributed. Since by assumption the variable X_{p+i} is connected with another such variable for any given i, we have established that each variable of the type X_{p+i} is normally distributed. Since no column of the u_{ij} coefficients in (3.32) consists wholly of zeros, the argument of ψ_i in (3.36) is not zero for any i. Thus the i-th equation in (3.36) gives us an equation of the type (valid in an interval around the origin)

$$\psi_i(t) = -w_{i,p+r}\psi_{p+r}(t) \qquad \dots \quad (3.38)$$

for some r, where the coefficient $w_{i,p+r} \neq 0$, since X_i is not degenerate by assumption. Since X_{p+r} is a normal variable, so is X_i. Thus, $X_1, \dots, X_p$ are also normally distributed.

4. Characterisation of the normal law by properties of sample estimators

In this section we recall some earlier theorems of the author (Rao, 1959) characterising the normal distribution through the minimum variance property of least squares estimators in the Gauss-Markoff model.

Let $X_1, \dots, X_n$ be independent random variables such that

$$E(X_j) = a_{j1}\theta + \dots + a_{jm}\theta_m, \quad j = 1, \dots, n \qquad \dots \quad (4.1)$$

where $\theta_1, \dots, \theta_m$ are real unknown parameters and $V(X_j) = \sigma^2$ independent of j.

The least squares estimator of an estimable parametric function $q_1\theta_1 + \dots + q_m\theta_m$ is the linear function of $X_1, \dots, X_n$ which has the smallest variance in the class of all *linear unbiased estimators*. Thus the least squares estimator depends only on the matrix of coefficients (a_{ij}) in (4.1) and not on the exact distribution of $X_1, \dots, X_n$.

However, if $X_1, \dots, X_n$ have normal distributions with a common variance independent of $\theta_1, \dots, \theta_m$, then it is known (Rao, 1952) that the least squares estimator of $q_1\theta_1 + \dots + q_m\theta_m$ has minimum variance *in the class of all unbiased estimators.*

Now we raise the converse problem. Suppose that the least squares estimator of $q_1\theta_1 + \dots + q_m\theta_m$ is known to have minimum variance in the class of all unbiased estimators. Does it follow that $X_1, \dots, X_n$ are normally distributed ? The answer is in the affirmative under certain conditions.

We assume that $X_j - E(X_j)$, $j = 1, \dots, n$, have the same distribution $p_{\theta\phi}$ which may depend on $\theta = (\theta_1, \dots, \theta_m)$ and certain other unknown parameters ϕ. Of course, the mean of the distribution $p_{\theta\phi}$ is zero whatever θ and ϕ may be.

First, we consider the case where the rank of the matrix (a_{ij}) is unity, i.e., when there is only one independent estimable linear parametric function.

Theorem 7 : *Let the rank of* (a_{ij}) *be unity and let* $Y = b_1X_1 + \ldots + b_nX_n$ *be the least squares estimator of the essentially unique estimable linear parametric function. Let* $p_{\theta\phi}$ *admit moments up to order* $2s$ *for each* θ, ϕ, *and* $b_1, \ldots, b_n$ *be all different from zero without loss of generality (since we can consider only those variables with non-zero coefficients). If* $Y = b_1X_1 + \ldots + b_nX_n$ *has minimum variance in the class of all unbiased estimators which are polynomials of order* s *or less and the vector* $(b_1, \ldots, b_n)$ *is not proportional to a vector which has only* ± 1 *as its elements, then* X_i *agrees with a normal distribution up to moments of order* $(s+1)$.

Proof: In Theorem 1 of the earlier paper (Rao, 1959) it was stated that if moments of all orders exist and the least squares estimator has minimum variance in the class of all unbiased estimators, then X_i has normal distribution. The earlier theorem may appear to be a consequence of the present Theorem 7. But the proof is the same, and in fact, Theorem 7 is only a restatement of the earlier theorem.

We observe that when the rank of (a_{ij}) is unity, there exist $(n-1)$ linear functions

$$Z_j = c_{j1}X_1 + \ldots + c_{jn}X_n \qquad j = 1, \ldots, n-1 \qquad \ldots \quad (4.2)$$

such that $\quad E(Z_j) = 0, \ E(Z_j\, Y) = 0, \ E(Z_j\, Z_i) = 0, \ i \neq j.$

Since $E(Z_j\, Y) = 0$ and $V(Z_jY) < \infty$ if $s \geqslant 2$, it follows from a lemma proved in an earlier paper of the author (Rao, 1952) that if Y has minimum variance in the class of all unbiased estimators of the second degree, then

$$E[(Y)(Z_j\, Y)] = E(Z_j\, Y^2) = 0. \qquad \ldots \quad (4.3)$$

Similarly $\quad E(Z_j\, Y^r) = 0, \quad r = 1, \ldots, s; \ j = 1, \ldots, n-1. \qquad \ldots \quad (4.4)$

Now consider

$$\begin{aligned} \phi(t) &= E\{iZ_j\, e^{it(Y-E(Y))}\} \\ &= [c_{j1}\psi(b_1\, t) + \ldots + c_{jn}\psi(b_n\, t)]\, f(b_1\, t) \ldots f(b_n\, t) \end{aligned} \qquad \ldots \quad (4.5)$$

where $f(t)$ is the characteristic function of $p_{0\phi}$ and $\psi(t) = f'(t)/f(t)$. Since $p_{0\phi}$ has moments upto order $2s$, $\psi(t)$ is differentiable s times. Taking the r-th derivative of $\psi(t)$ for $r \leqslant s$, putting $t = 0$ and observing that $E(Z_j\, Y^r) = 0$, $r = 1, \ldots, s$, we obtain

$$(\Sigma c_{jk} b_k^r)\kappa_{r+1}(\theta, \phi) = 0, \qquad r \leqslant s \text{ and } j = 1, \ldots, n-1 \qquad \ldots \quad (4.6)$$

where $\kappa_{r+1}(\theta)$ is the $(r+1)$-th cumulant of $p_{\theta\phi}$. From (4.6) we deduce that $\Sigma c_{ji} b_i^r = 0$ and/or $\kappa_{r+1}(\theta, \phi) = 0$. Since $E(Z_j\, Y) = 0$, $\Sigma c_{jk} b_k = 0$, $j = 1, \ldots, n-1$. Then

$$[\Sigma c_{jk} b_k^r = 0, \ j = 1, \ldots, n-1] \Longrightarrow [b_j^r = \lambda_r\, b_j, \quad j = 1, \ldots, n] \qquad \ldots \quad (4.7)$$

which is false except when $r = 1$, or $b_j = 0$, or proportional to $\pm 1, j = 1, \ldots, n$. Consequently $\kappa_{r+1}\,(\theta, \phi) = 0$, $r = 2, \ldots, s$ and the proposition is proved.

Note 1 : If moments of all orders exist, then under the conditions of Theorem 7 on the coefficients $b_1, \ldots, b_n$ a necessary condition for the least squares estimator to be the minimum variance unbiased estimator is that the variables are normally distributed.

Note 2: The condition that $b_1, \ldots, b_n$ are not all proportional to ± 1 excludes the important case of $\bar{X} = (X_1 + \ldots + X_n)/n$ being the least squares estimator. In the earlier paper (Rao, 1959) a question was raised as to what further restrictions are needed to include this case. An answer is provided by Kagan (1966). Kagan assumes that $p_{0\phi}$, the distribution of $X_i - E(X_i)$ is independent of unknown parameters. It may be noted that in such a case admissibility and minimum variance of $\bar{X}$ in the class of polynomial estimators defined by Kagan are identical conditions.

In this paper we shall consider another formulation of the problem by demanding that the least squares estimator $\bar{X}$ of the unknown parameter θ has minimum mean square error in a suitably defined class of polynomial estimators. The theorem may be stated as follows.

Theorem 8: *Let $X_1, \ldots, X_n$ be i.i.d. (non-degenerate) variables with $E(X_j) = \theta$, an unknown parameter and let the distribution of $X_i - E(X_i)$ be $p_{0\phi}$ which may depend on unknown parameters θ and ϕ. Consider the class C_s of estimators*

$$\bar{X} + P_s(X_1, \ldots, X_n) \qquad \ldots \quad (4.8)$$

where P_s is a polynomial statistic of order less than or equal to s which is translation invariant, i.e.,

$$P_s(X_1 + c, \ldots, X_n + c) = P_s(X_1, \ldots, X_n) \qquad \ldots \quad (4.9)$$

for any constant c. If $\bar{X}$ has the minimum mean square error uniformly for each admissible value of (θ, ϕ) in the class of estimators C_s, and $E(X_1^{2s}) < \infty$ for each value of (θ, ϕ) then the moments up to order $(s+1)$ of the distribution of the X_i coincide with a normal distribution.

Proof: Consider in particular the statistic

$$\bar{X} + g(X_j - \bar{X})^r, \; r \leqslant s, \text{ fixed } j, \qquad \ldots \quad (4.10)$$

where g is a constant, which belongs to C_s, and the particular values $\theta = \theta_0, \phi = \phi_0$, of the parameters. Now for the special choice of the constant $g = g_0$,

$$-g_0 = \frac{E_{\theta_0, \phi_0}[(\bar{X} - \theta_0)(X_j - \bar{X})^r]}{E_{\theta_0, \phi_0}[(X_j - \bar{X})^{2r}]} \qquad \ldots \quad (4.11)$$

the estimator

$$\bar{X} + g_0(X_j - \bar{X})^r \qquad \ldots \quad (4.12)$$

has the mean square error at $\theta = \theta_0, \phi = \phi_0$

$$E_{\theta_0, \phi_0}[(\bar{X} - \theta_0)^2] - g_0^2 E_{\theta_0, \phi_0}[(X_j - \bar{X})^{2r}] \qquad \ldots \quad (4.13)$$

which is smaller than $E_{\theta_0, \phi_0}[(\bar{X} - \theta_0)^2]$ unless $g_0 = 0$, i.e.,

$$E_{\theta_0, \phi_0}[\bar{X} - \theta_0)(X_j - \bar{X})^r] = 0, \qquad r = 1, \ldots, s. \qquad \ldots \quad (4.14)$$

The equations (4.14) provide necessary conditions for the optimality of $\bar{X}$.

Now consider

$$\phi(t) = E_{\theta_0,\phi_0}\left[i(\bar{X}-\theta_0)\,e^{it(X_j-\bar{X})}\right]$$

$$=\frac{1}{n}\left[\psi\left(\frac{n-1}{n}t\right)+(n-1)\,\psi\left(-\frac{t}{n}\right)\right] f\left(\frac{n-1}{n}t\right) f\left(\frac{-t}{n}\right)^{n-1} \quad \ldots \quad (4.15)$$

where $f(t)$ is the c.f. of $p_{\theta_0\,\phi_0}$ and $\psi(t) = f'(t)/f(t)$. Taking the r-th derivative of $\phi(t)$, putting $t = 0$ and observing that $E[(X-\theta_0)(X-\bar{X})^r] = 0$, $r = 1, \ldots, s$ and $f'(0) = 0$ we obtain

$$\left[\left(\frac{n-1}{n}\right)^r+n-1\left(\frac{-1}{n}\right)^r\right]\kappa_{r+1}(\theta_0, \phi_0) = 0, \quad r \leqslant s \quad \ldots \quad (4.16)$$

where $\kappa_i(\theta_0, \phi_0)$ is the i-th cumulant of p_{θ_0,ϕ_0}. But

$$(n-1)^r+(-1)^r(n-1) \neq 0 \quad \ldots \quad (4.17)$$

unless $r=1$ or $n=2$ and r odd. So if $n>2$, it follows from (4.16) that $\kappa_{r+1}(\theta_0, \phi_0)=0$ for $r = 2, \ldots, s$, which proves the theorem.

Note 1 : In proving Theorem 8, we made use of only the subset of estimators

$$\bar{X}+g(X_j-\bar{X})^r, \quad r = 1, \ldots, s; \quad j \text{ fixed},$$

in the class C_s, so that we could have narrowed down the class of estimators in stating the theorem.

Note 2 : If $P_{\theta,\phi}$ is independent of the unknown parameters θ and ϕ, we could impose the *further* restriction of unbiasedness of estimators in defining the class C_s, as done by Kagan (1966).

Note 3 : If $P_{\theta,\phi}$ is independent of θ but depends on an unknown parameter ϕ, then we need only impose the condition that $\bar{X}$ has minimum mean square error uniformly in ϕ in the class C_s. This answers at least partially one of the questions raised by Kagan (1966), where the parent distribution is of the type $F[(X-\theta)/\phi]$, θ and ϕ being unknown parameters.

Note 4 : The other exceptional case where some of the coefficients are $+1$ and others are -1 can be handled in a similar way.

For further comments on the result of Theorem 8 the reader is referred to the earlier paper (Rao, 1959).

Now we consider the general case where the rank of (a_{ij}) is $r > 1$. Let $Y_1, \ldots, Y_r$ be the least squares estimators of r independent estimable linear parametric functions. There exist $n-r$ independent linear functions

$$Z_j = c_{j1}X_1+\ldots+c_{jn}X_n, \quad j = 1, \ldots, n-r \quad \ldots \quad (4.18)$$

whose expectations are zero identically (in θ), and for any fixed i

$$E(Z_jY_i) = 0, \quad j = 1, \ldots, n-r. \quad \ldots \quad (4.19)$$

Given a $n \times r$ matrix $A = (a_{ij})$ of rank r, we can by sweeping out the rows, express it in the form

$$\begin{pmatrix} I_r \\ \cdots \\ W \end{pmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ & & \cdots & \\ w_{r+1,1} & w_{r+1,2} & \cdots & w_{r+1,r} \\ & & \cdots & \\ w_{n1} & w_{n2} & & w_{nr} \end{bmatrix} \quad \ldots \quad (4.20)$$

by rearranging the rows of (a_{ij}) if necessary (which merely means writing the random variables $X_1, \ldots, X_n$ in a different order). Without loss of generality, we can consider $X_1, \ldots, X_n$ to be the order leading up to the swept out form (4.20) of the matrix (a_{ij}). We shall say the matrix (a_{ij}) is exceptional if, in the matrix W of the reduced form, (4.20) any row contains at most one non-zero element and the non-zero element, if any, is ± 1. We now state a theorem for the general case.

Theorem 9 : *Let $p_{0\phi}$ admit moments of order $2s$ and the matrix (a_{ij}) be not exceptional. If every least squares estimator has minimum variance in the class of all unbiased estimators which are polynomials of order $(s+1)$ or less, then X_i agrees with the normal distribution up to moments of order $(s+1)$.*

The proof is the same as that of Theorem 2 in the author's earlier paper (Rao, 1959) and it follows on the same lines as the proof of Theorem 7 of the present paper.

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