Chords in a Longest Cycle of a 3-Connected Graph

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CERTIFICATE

This is to certify that the dissertation entitled **Chords in a Longest Cycle of a 3-Connected Graph** submitted by **Deepak Chaudhary** to Indian Statistical Institute, Kolkata, in partial fulfillment for the award of the degree of **Master of Technology in Computer Science** is a bonafide record of work carried out by him under my supervision and guidance. The dissertation has fulfilled all the requirements as per the regulations of this institute and, in my opinion, has reached the standard needed for submission.

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Abstract

In 1976, Carsten Thomassen conjectured that no longest cycle in a 3-connected graph can be a chordless cycle. Although this conjecture was later proved for some special classes of graphs, the general case remains open. In this work, we study how Thomason's Lollipop Method was used by Thomassen to verify this conjecture for cubic graphs.

Contents

Chapter 1

Introduction

In 1976, Carsten Thomassen (see [7]) made the following conjecture:

Conjecture 1. *Every longest cycle in any 3-connected graph has a chord.*

In 1978, Andrew G. Thomason introduced "the lollipop method". Thomassen in 1996 used this method to prove Conjecture 1 for cubic graphs. After this many people tried their hands to prove this conjecture for different classes of graphs [3, 4, 5, 9]. Etienne Birmelé [1] in 2008 proved the conjecture for $K_{3,3}$-minor free graphs.

This work is a study of how Thomassen proved Conjecture 1 for cubic graphs. We have tried to explain the proof and the lollipop method required for it in a way we feel is easier to follow.

First, we give some basic definitions of graph theory.

1.1 Definitions

Let $G = (V, E)$ be an undirected simple graph with n vertices and m edges.

Two vertices $u, v \in V(G)$ are said to be *adjacent* if $(u, v) \in E(G)$.

The *neighbours* of a vertex $v \in V$ are all the vertices $u \in V$ such that $(u, v) \in E(G)$.

A *walk* in G is a sequence vertices $v_1, v_2, \ldots, v_t$ where $v_iv_{i+1} \in E(G)$ for each $i \in \{1, 2, \ldots, t-1\}$. The vertices $v_1, v_2, \ldots, v_t$ and the edges $v_1v_2, v_2v_3, \ldots, v_{t-1}v_t$ are said to occur in that order in the walk.

A *path* is a walk in which no vertex repeats.

A *trail* is a walk in which no edge repeats but vertices may repeat.

A *cycle* in a graph G is the nonempty trail $v_1, v_2, \ldots, v_t$ in which $v_1 = v_t$, but the vertices $v_1, v_2, \ldots, v_{t-1}$ are all pairwise distinct. Length of a cycle is defined as the number of vertices in that cycle, which is equal to the number of edges in that cycle.

A *chord* of a cycle in a graph is an edge between two vertices of the cycle that are not consecutive on the cycle.

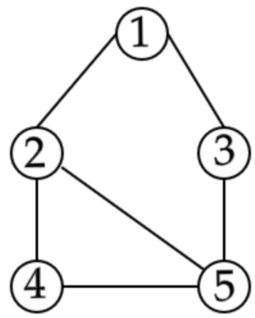

Figure 1.1: The edge (2,5) is a chord of the cycle $1, 2, 4, 5, 3, 1$

A *degree* of a vertex v is the number of edges incident on v. It is denoted as $d(v)$.

A *component* S of graph G is a maximal connected subgraph, i.e. there does not exists any other connected subgraph T of G such that S is a subgraph of T.

Connectivity of a graph G is defined as the minimum number of vertices that need to be removed to separate G into two or more components.

A *cut vertex* is a vertex whose removal will disconnect the graph G into 2 or more components.

A graph G with more than k vertices is said to be *k-connected* or *k-vertex-connected* if at least k vertices need to be removed to make the graph disconnected.

Theorem 1. *Every vertex in a k-connected graph has degree $\geq k$.*

Proof:
Let $G(V, E)$ be a k-vertex-connected graph. Let u be a vertex with degree $d < k$. If d adjacent vertices of u are removed, then it will disconnect the vertex u from the rest of the graph, which contradicts our assumption that G is k-connected. Hence, such a vertex can not exist. $\square$

Chapter 2

Thomason's Lollipop Method

In the 1940s, Smith (see [8]) proved that:

Theorem 2. *For any cubic graph, the number of Hamiltonian cycles containing an edge $e \in E(G)$ is even.*

Kotzig (see [6]) proved a similar result.

Theorem 3. *For any cubic and bipartite graph, the total number of Hamiltonian cycles is even.*

Thomason [6] generalizes these two theorems by introducing the Lollipop method. Thomason originally used this method for multigraphs. We will see in Chapter 3 how Carsten Thomassen proved Conjecture 1 for cubic graphs using this method. We present the lollipop method in the way it can be applied to simple graphs.

2.1 Lollipop Method

To understand the lollipop method we first need to understand some definitions.

2.1.1 Definitions

Let $G(V, E)$ be a simple undirected graph on n vertices.

A *Hamiltonian path* of G is a path in G that contains every vertex of G.

A *Hamiltonian cycle* of G is a cycle in G that contains all the vertices of G.

We fix a path $v_1, v_2, \ldots, v_m$ of G and call it the *stick* S.

We denote by $d(v)$ the degree of any vertex v in G.

For any vertex v, let $N(v)$ be the set of adjacent vertices of v. We denote by $\varepsilon(v)$ the number of edges between v and the vertices of the stick except the last vertex of the stick, i.e. $\varepsilon(v) = |N(v) \cap \{v_1, v_2, \ldots, v_{m-1}\}|$.

Then there are $d(v)-\varepsilon(v)$ edges between v and the vertices in $V(G)-\{v_1, v_2, v_3, \ldots, v_{m-1}\}$.

We say that a Hamiltonian path $u_1, u_2, \ldots, u_n$ *starts with the stick* S if for $1 \le i \le m$, $u_i = v_i$.

Let $h = v_1, v_2, \ldots, v_n$ be a Hamiltonian path of G starting with the stick S (notice that the first m vertices of h form the stick S). Now suppose that the last vertex v_n of h is adjacent to the vertex v_k where $m+1 \le k < n-1$. Notice that then $v_1, v_2, \ldots, v_k, v_n, v_{n-1}, v_{n-2}, \ldots, v_{k+1}$ is another Hamiltonian path, say h', of G. We then say that the Hamiltonian paths h and h' are *lollipop-related.* Notice that the relation "lollipop-related" is a symmetric relation.

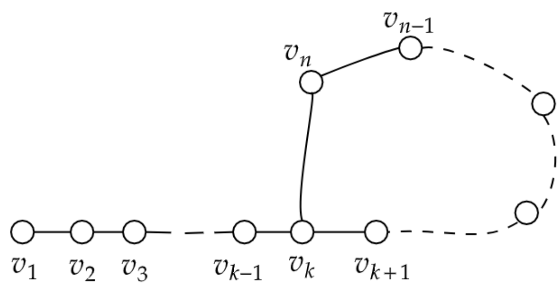

Figure 2.1: Two Hamiltonian paths $h = v_1, v_2, \ldots, v_n$ and $h' = v_1, v_2, \ldots, v_k, v_n, v_{n-1}, \ldots, v_{k+1}$ are lollipop-related.

A *lollipop graph of a graph* G *with respect to the stick* S, denoted as $\mathcal{L}(G,S)$, is a graph whose vertices represent the Hamiltonian paths of the graph G starting with the stick S. Two vertices in the lollipop graph are adjacent if the Hamiltonian paths that are represented by those vertices are lollipop-related.

Let $h = v_1, v_2, \ldots, v_n$ be a Hamiltonian path of G starting with the stick S. Let $F = \{v_n v_k \colon m \le k < n-1 \text{ and } v_n v_k \in E(G)\}$. Notice that we can associate every Hamiltonian path $h' = v_1, v_2, \ldots, v_k, v_n, v_{n-1}, \ldots, v_{k+1}$ that is lollipop-related to h with the edge $v_k v_n$ and vice versa. Thus the number of Hamiltonian paths that are lollipop-related to h is exactly $|F| = d(v) - \varepsilon(v) - 1$. This gives us the following observation.

Observation 1. *Let h be a Hamiltonian path of G starting with the stick S and ending at a vertex $w \in V(G)$. The degree of the vertex representing h in $\mathcal{L}(G,S)$ is $d(w) - \varepsilon(w) - 1$.*

2.2 Thomason's Theorem

Theorem 4 (Thomason 1978)**.** *The number of Hamiltonian paths of G starting with the stick S and ending in a vertex of the set $W = \{w \in V : d(w) - \varepsilon(w)$ is even$\}$ is even.*

Proof:

Notice that $W = \{w \in V : d(w) - \varepsilon(w) - 1 \text{ is odd}\}$. Let $\mathcal{H}$ be the collection of Hamiltonian paths of G starting with the stick S that end at a vertex of W. By Observation 1, $h \in \mathcal{H}$ if and only if h has odd degree in $\mathcal{L}(G, S)$. Since the number of odd degree vertices in any graph is even, $|\mathcal{H}|$ is also even. $\square$

Chapter 3

Thomassen's work on Cubic Graphs

In this chapter we will see how Thomassen [7] verified conjecture 1 for cubic graphs. But first we see a proof for the existence of a second Hamiltonian cycle C' given a Hamiltonian cycle C in a graph with some special properties using the Thomason's Lollipop method.

Lemma 1. *Let G be a simple graph and let $e = xy$ be an edge of G. Let T denote the vertices in $V(G) \setminus \{x, y\}$ defined as $T = \{v\colon$ there is a Hamiltonian path starting with the stick x, y and ending at $v\}$. If every vertex in T has odd degree in G, then the number of Hamiltonian cycles containing the edge xy is even.*

Proof:
We fix the stick S to be the path x, y. Then $\varepsilon(v)$ is the number of edges between vertex v and x. As the graph G is a simple graph, $\varepsilon(v)$ is either 0 or 1.

Let H be the set of Hamiltonian paths starting with the stick S.

Let H_v be the set of Hamiltonian paths starting with the stick S and ending at vertex v.

Let $v \in V(G) \setminus \{x, y\}$. Suppose that $\varepsilon(v) = 1$. Let $h \in H_v$. Suppose $h = x, y, v_1, v_2, \ldots, v_{n-3}, v$ (here $n = |V(G)|$). Then we define the path $h' := x, v, v_{n-3}, v_{n-2}, \ldots, v_2, v_1, y$. Note that h' is a Hamiltonian path starting at x and ending at y. For a vertex $v \in V(G) \setminus \{x, y\}$ such that $\varepsilon(v) = 1$, let $X_v = \{h' \colon h \in H_v\}$. Thus X_v is a set of Hamiltonian paths starting at x and ending at y. Then $|X_v| = |H_v|$. For a vertex $v \in V(G) \setminus \{x, y\}$ such that $\varepsilon(v) = 0$, we define $X_v = \emptyset$. Thus, $\mid X_v \mid = \varepsilon(v) \mid H_v \mid$.

Thus, $T = \{v \in V : H_v \neq \emptyset\}$
Let $T' = \{v \in T : \varepsilon(v) = 1\}$

Let $W = \{v \in V : d(v) - \epsilon(v) \text{ is even}\}$

Let $V' = V(G) \setminus \{x, y\}$. Consider any Hamiltonian path h starting from x and ending at y. Let $h = x, u_1, u_2, \ldots, u_{n-2}, y$. Then since $x, y, u_{n-2}, u_{n-3}, \ldots, u_1$ is a

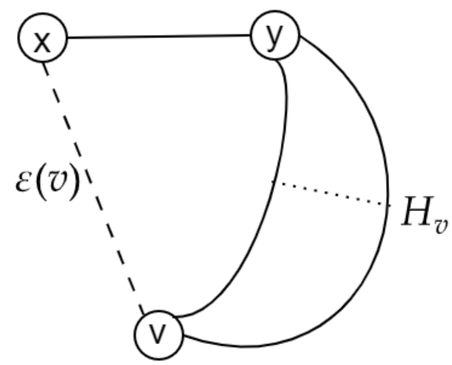

Figure 3.1: This illustrates an edge xy incident on vertex $x \in A$ and all the Hamiltonian paths starting from xy and ending at a vertex $v \in V(G) \setminus \{x, y\}$.

Hamiltonian path in H_{u_1}, we can conclude that $h \in X_{u_1}$. Thus the total number of Hamiltonian paths starting from x and ending at y
$= \sum_{v \in V'} \mid X_v \mid = \sum_{v \in V'} \varepsilon(v) \mid H_v \mid$
$= \sum_{v \in T} \varepsilon(v) \mid H_v \mid + \sum_{v \in (V' \setminus T)} \mid \varepsilon(v) \mid \mid H_v \mid$
$= \sum_{v \in T} \varepsilon(v) \mid H_v \mid \qquad \because \forall v \in (V' \setminus T), \mid H_v \mid = 0$
$= \sum_{v \in T'} \varepsilon(v) \mid H_v \mid + \sum_{v \in T \setminus T'} \mid \varepsilon(v) \mid \mid H_v \mid$
$= \sum_{v \in T'} \varepsilon(v) \mid H_v \mid \qquad \because \forall v \in (T \setminus T'), \varepsilon(v) = 0$
$= \sum_{v \in T'} \mid H_v \mid \qquad \because \forall v \in T', \varepsilon(v) = 1$
Recall that every vertex in set T has odd degree.

We can partition vertex set V of G into three sets. $V = (T - T') \cup (T') \cup (V - T)$

Notice here that $T' \subseteq W \qquad \because \forall v \in T' : \varepsilon(v) = 1$ and $d(v)$ is odd.

Also notice that $(T \setminus T') \cap W = \emptyset$, since $\forall v \in T \setminus T'$, we have that $d(v)$ is odd and $\varepsilon(v) = 0$.

By Theorem 4, the number of Hamiltonian paths starting with stick S (i.e. edge xy) and ending at a vertex in W is even. Consider a Hamiltonian path h starting with the stick S and ending at a vertex v in W. Clearly, $v \in T$ (since Hamiltonian paths starting with the stick S have to end at vertices in T, by definition of T). Also $v \notin T \setminus T'$ since $(T \setminus T') \cap W = \emptyset$. Thus $v \in T'$. Therefore, every Hamiltonian path starting with stick S and ending at a vertex in W must end at a vertex in T', which means that $W \subseteq T'$. Since $T' \subseteq W$, this implies that $W = T'$.

Therefore, $\sum_{v \in T'} \mid H_v$ is nothing but the number of Hamiltonian paths starting with the stick S and ending at vertices of W. Thus by Theorem 4, this number is even, which implies that total number of Hamiltonian paths starting from x and ending at y is even. Since this is same as the number of Hamiltonian cycles containing the edge xy, we now have the lemma. $\square$

Lemma 2 (Thomassen)**.** *Let G be a graph with a Hamiltonian cycle C. Suppose that for some set of vertices A, the subgraph $G - A$ has $\mid A \mid$ components each of which is a path whose end vertices are of odd degree in G. Then*

1. *For every Hamiltonian cycle C' of G, $C' - A = C - A$, and*

2. *Each edge of G incident to a vertex of A is included in an even number of Hamiltonian cycles of G.*

Proof:
Let G be a graph with a Hamiltonian cycle C and having a set $A \subseteq V(G)$ such that $G - A$ has $\mid A \mid$ components each of which is a path whose end vertices have odd degree in G. Let $a = \mid A \mid$ and $A = \{v_1, v_2, v_3, \ldots, v_a\}$. Let the paths which are the components of $G - A$ be $P_1, P_2, P_3, \ldots, P_a$.

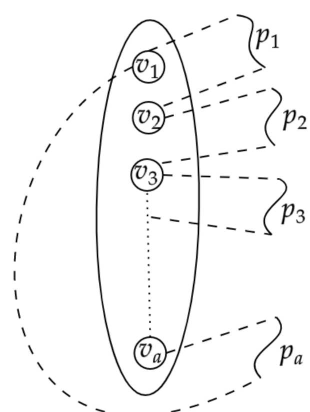

Figure 3.2: $A = \{v_1, v_2, \ldots, v_a\}$ and $P_1, P_2, \ldots, P_a$ are the components in $G - A$. The dotted line representS a Hamiltonian cycle in G.

1. To prove this, we only need to prove that for each $i \in \{1, 2, \ldots, a\}$, the vertices of P_i occur consecutively in the order in which they appear in P_i in any Hamiltonian cycle of G. Then it will follow that for any Hamiltonian cycle C' of G, the components of $C' - A$ will also be $P_1, P_2, \ldots, P_a$. We shall prove this as follows. Consider any Hamiltonian cycle C' of G. For any vertex x in C', we denote by x' the vertex that occurs next in C' after x (when traversing the Hamiltonian cycle C' in a fixed direction). For each $i \in \{1, 2, \ldots, a\}$, we

define a set $X_i \subseteq A$ as follows: $X_i = \{x \colon x \in A \text{ and } x' \in V(P_i)\}$. Clearly, the sets $X_1, X_2, \dots, X_a$ are all pairwise disjoint. For any $i \in \{1, 2, \dots, a\}$, the vertex in C' that occurs just before the first vertex in $V(P_i)$ while traversing the Hamiltonian cycle in the fixed direction starting from a vertex outside $V(P_i)$ belongs to A. Thus for each $i \in \{1, 2, \dots, a\}$, $X_i \neq \emptyset$. Then since $|A| = a$, it then follows that $|X_1| = |X_2| = \cdots = |X_a| = 1$. Now if the vertices of some path P_i did not occur consecutively in C', then $|X_i| \geq 2$, which is a contradiction to the previous observation. Thus we can conclude that for each $i \in \{1, 2, \dots, a\}$, the vertices of P_i occur consecutively in C'. Now since the subgraph of C' induced by $V(P_i)$ is a path, that subgraph must be isomorphic to the path P_i (since there are no edges in G between two non-consecutive vertices on P_i), which implies that the vertices of P_i occur in the same order in C' as they occur in C'.

A possible illustration of a Hamiltonian cycle C can be seen in the figure 3.2.

2. Let edge $e = xy$ where $x \in A$ and vertex v be any vertex of the graph G other than x, y as seen in figure 3.1. As before, it can be argued that every Hamiltonian path starting with a vertex in A will end at a vertex that is an endvertex of one of the paths in $G - A$. By our assumption, such vertices have odd degree in G. Thus we can now apply Lemma 1 to conclude that the number of Hamiltonian cycles containing the edge xy is even.

□

Let G be any graph. For any vertex $u \in V(G)$ and any subgraph F of G, define $E_u(F) = \{uv \in E(F) \colon v \in V(G)\}$.

Theorem 5 (Thomassen)**.** *Let G be a graph with a Hamiltonian cycle C. Let A be a vertex set in G such that*

(i) A is independent in C (i.e. A contains no two consecutive vertices of C), and

(ii) A is dominating in $G - E(C)$(i.e., every vertex of $G - A$ is joined to a vertex in A by some chord of C).

Then G has a Hamiltonian cycle C' distinct from C. Moreover, C' can be chosen such that

(iii) $C' - A = C - A$, and

(iv) there is a vertex v in A such that one of the edges of C' incident with v is in C and other is not in C.

Proof:
Let $X = \{x \in V(G) \colon x \text{ is a neighbour of a vertex in } A \text{ in } C\}$. For every vertex

$x \in X$, there exist at least one vertex in A that is connected to x through a chord of C (since A is a dominating set in $G - E(C)$). For each $x \in X$, choose one such chord e_x that connects it to a vertex of A. Define G' as the graph with vertex set $V(G') = V(G)$ and $E(G') = E(C) \cup \{e_x : x \in X\}$. Note that in G', every chord of C that is incident to a vertex in $V(G') \setminus A$ is also incident to a vertex in A. Thus $G' - A$ has exactly $|A|$ components, each of which is a path. Moreover, each of these paths have vertices from X as their endpoints. From the definition of G', it is clear that every vertex in X has degree 3 in G'. Thus G' and A' satisfy the conditions required to apply Lemma 2. Let e be any edge of C that is incident to a vertex of A. By Lemma 2, we know that the number of Hamiltonian cycles containing e is even, which implies that there is at least one Hamiltonian cycle C' of G' that is different from C. Moreover, we have $C - A = C' - A$. This proves (iii).

Let C' be a Hamiltonian cycle of G' that is different from C such that it contains the largest possible number of edges in C. We claim that C' satisfies (iv). Suppose not. Then for every vertex $v \in A$, $|E_v(C) \cap E_v(C')| \in \{0, 2\}$. Let $A' = \{v \in A : |E_v(C) \cap E_v(C')| = 0\}$. Let H be the graph with vertex set $V(H) = V(G')$ and $E(H) = E(C) \cup E(C')$. Let $X' = \{x \in V(G') : x \text{ is adjacent to a vertex of } A' \text{ in } C\}$. Clearly, $X' \subseteq X$. Consider a vertex $x \in X'$ that is a neighbour of a vertex in $a \in A'$ in C. If $|E_x(C') \cap E_x(C)| = 0$, then x will have degree 4 in G', which contradicts the fact that x has degree 3 in G'. If $|E_x(C') \cap E_x(C)| = 2$, then $xa \in E(C')$, which contradicts the fact that $a \in A'$. Thus we can conclude that $|E_x(C') \cap E_x(C)| = 1$. This implies that in H, the vertex x has degree 3. Any chord e of C in H is an edge of C'. Clearly, one endpoint v of e is a vertex in A since every chord of C in G' had this property. Then $|E_v(C) \cap E_v(C')| = 0$, which implies that $v \in A'$. Thus any chord of C in H has a vertex from A' as one of its endpoints. Therefore $H - A'$ has $|A'|$ components, each of which is a path whose both endpoints are in X'. Since every vertex in X' has degree 3 in H, we can now apply Lemma 2 to H and A'. Now consider any edge e of C' that is incident to a vertex of A'. By Lemma 2, the number of Hamiltonian cycles of H that contain e is even, which implies that there is a Hamiltonian cycle C'' different from C' that contains e. Note that C'' is also different from C since $e \notin E(C)$. Since by Lemma 2, we also have that $C' - A' = C'' - A'$, if at every vertex $a \in A'$, we have $E_a(C') = E_a(C'')$, then C' and C'' cannot be distinct. Thus there exists a vertex $z \in A'$ such that $E_z(C') \neq E_z(C'')$. Since the set of edges incident to z in H is exactly $E_z(C) \cup E_z(C')$, this means that $E_z(C'') \cap E_z(C) \neq \emptyset$. Let $f \in E_z(C'') \cap E_z(C)$. Clearly, $f \notin E(C')$. Now consider any edge $e \in E(C') \cap E(C)$. Clearly, e is not incident to a vertex of A', since for every vertex $v \in A'$, we have $|E_v(C) \cap E_v(C')| = 0$. Thus e is an edge of one of the connected components (which are paths) of $C' - A'$. Since $C'' - A = C' - A$, we have that $e \in C''$. Thus $E(C') \cap E(C) \subseteq E(C'') \cap E(C)$. Since $f \in E(C'') \cap E(C)$ and $f \notin E(C') \cap E(C)$, we now have a contradiction to the choice of C'.

$\square$

We give below a theorem of Fleischner and Steibitz [2] that will be useful for proving the next result.

Theorem 6 (Fleischner and Steibitz). *Let G be a graph whose edge set is the disjoint union of a Hamiltonian cycle and a collection of pairwise vertex-disjoint triangles. Then G is 3-colourable.*

Theorem 7 (Thomassen). *Any longest cycle in a 3-connected cubic graph has a chord.*

Proof:
Let G be a 3-connected cubic graph and let C be a longest cycle in it. Suppose for the sake of contradiction that C has no chord. Then let $H_1, H_2, \ldots, H_k$ be the connected components of $G - V(C)$. For each $i \in \{1, 2, \ldots, k\}$, let $N_C(H_i)$ denote the set of vertices in C that have neighbour in H_i. Since G is 3-connected, we have that $|N_C(H_i)| \geq 3$. Since G is cubic, each vertex of C belongs to $N_C(H_i)$ for at most one $i \in \{1, 2, \ldots, k\}$. Moreover, for each i, if two consecutive vertices of C belong to $N_C(H_i)$, then there is a cycle longer than C in G, which is a contradiction. For each i, choose distinct vertices $x_i, y_i, z_i \in N_C(H_i)$. Consider the graph F with vertex set $V(F) = V(C)$ and $E(F) = E(C) \cup \bigcup_{i\in\{1,2,\ldots,k\}} \{x_iy_i, y_iz_i, x_iz_i\}$. Clearly, F satisfies the requirements of Theorem 6, and therefore, F is 3-colourable. Let X denote one of the three colour classes of F. Clearly, for each $i \in \{1, 2, \ldots, k\}$, $|X \cap \{x_i, y_i, z_i\}| = 1$. Moreover, no two vertices in X are consecutive on C. We assume without loss of generality that $X = \{x_i \colon i \in \{1, 2, \ldots, k\}\}$. We now construct the graph G' from G by contracting the component H_i of $G - V(C)$ into x_i, for each $i \in \{1, 2, \ldots, k\}$. Then $V(G') = V(C)$ and for each $i \in \{1, 2, \ldots, k\}$, the vertex x_i has degree $|N_C(H_i)| - 1$ in G', the vertices y_i and z_i have degree 3 in G' and every other vertex has degree 2 in G'. Further, notice that every vertex in $V(G') \setminus X$ belongs to $N_C(H_i)$ for some $i \in \{1, 2, \ldots, k\}$, and hence is connected to $x_i \in X$ by a chord. Thus G', C, and X satisfy the requirements of Theorem 5. Thus there is a Hamiltonian cycle C' different from C in G' such that there exists a vertex $v \in X$ for which $|E_v(C) \cap E_v(C')| = 1$. It can be seen that C' can be extended into a cycle C'' of G as follows. Let $C' = v_1, v_2, \ldots, v_n, v_1$ where $V(G') = \{v_1, v_2, \ldots, v_n\}$. Let us assume without loss of generality that $x_1, x_2, \ldots, x_k$ appear in this order in C'. Let P_i denote the subpath of C' between the vertex after x_i and the vertex before x_{i+1} that does not contain any vertex of X (subscripts modulo k). Let p_i, q_i be the first and last vertices of the path P_i. Then C' is the cycle $x_1, p_1P_1q_1, x_2, p_2P_2q_2, x_3, \ldots, p_{k-1}P_{k-1}q_{k-1}, x_k, p_kP_kq_k, x_1$. If $|E_{x_i}(C) \cap E_{x_i}(C')| = 0$, then $\{q_{i-1}, p_i\} = \{y_i, z_i\}$, and we define Q_i to be the path in G between q_{i-1} and p_i whose internal vertices all lie in the component H_i of $G - V(C)$. On the other hand if $|E_{x_i}(C) \cap E_{x_i}(C')| = 2$, then q_{i-1}, p_i are the neighbours of x_i on C and we define Q_i to be simply the path q_{i-1}, x_i, p_i in G. Finally, if $|E_{x_i}(C) \cap E_{x_i}(C')| = 1$, then one of q_{i-1}, p_i belongs to $\{y_i, z_i\}$ and the other is a neighbour of x_i on C. If q_{i-1} is the neighbour of x_i on C, then we define Q_i to be the path q_{i-1}, x_iRp_i, where R is

the path in G between x_i and p_i whose internal vertices all lie in H_i. Otherwise, if p_i is the neighbour of x_i on C, then we define Q_i to be the path $q_{i-1}Rx_i, p_i$, where R is the path in G between q_{i-1} and x_i whose internal vertices all lie in H_i. Now consider the cycle C'' in G that is obtained from C' by replacing each subpath q_{i-1}, x_i, p_i with the path $q_{i-1}Q_ip_i$. Notice that if in G', there exist at least one $i \in \{1, 2, \ldots, k\}$, such that $|E_{x_i}(C) \cap E_{x_i}(C')| = 1$, then C'' is a longer cycle than C'. As observed earlier, the vertex $v \in X$ has the property that $|E_v(C) \cap E_v(C')| = 1$, and therefore C'' is a cycle in G that is longer than C' and hence also longer than C. This contradicts the fact that C is a cycle of maximum possible length in G. $\square$

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