TWO APPROXIMATIONS FOR THE DISTRIBUTION OF DOUBLE NON-CENTRAL BETA

By PIJUSH DASGUPTA

Indian Statistical Institute

SUMMARY. Two heuristic approximations—based on Laguerre series expansion and Jacobi series expansion—have been obtained in this paper for the distribution function of the random variable $T = \chi_2^2/(\chi_1^2+\chi_2^2)$ where χ_1^2 and χ_2^2 are independent chi-square variables. Accuracy of these approximations have been studied numerically.

1. Let $W = (W_1, W_2, \ldots, W_n)$ be an n-component random vector with expectation $E(W) = (\theta_1, \theta_2, \ldots, \theta_n)$, and let $Z_i = W_i - \theta_i$, $i = 1, 2, \ldots, n$. In the classical problem of analysis of variance with fixed effects, the following assumptions are made :

(1°) The random variables $Z_1, \ldots, Z_n$ are independent.

(2°) They have a common, but possibly unknown variance $V^*(Z_i) = \sigma^2$, $i = 1, 2, \ldots, n$.

(3°) They are distributed normally.

(4°) In the n-dimensional vector space V of all non-trivial linear compounds of W, there is a given subspace E of rank r, such that all linear compounds of W belonging to E have expected value zero.

Under assumptions (1°) through (4°), the problem is to test the null hypothesis H_0 that a given subspace $E_0 \supset E$ has property (4°). Let the rank of E_0 be $r+s$. Let Y and Y_0 denote respectively the projection of W on E and E_0. Then the variance-ratio statistic for H_0 is

$$F = \frac{r}{s} \cdot \frac{1-T}{T}$$

where

$$T = Y\, Y'/Y_0 Y_0'.$$

The non-null distribution of F when H_0 is not true, but assumptions (1°) through (4°) hold, has been studied extensively. We propose to investigate in a series of papers, the effect of relaxing the assumptions (1°)-(4°) on the sampling distribution of F, and examine the importance of these assumptions on the robustness of the F test in analysis of variance.

If only assumption (4°) is relaxed the distribution function of T is easily seen to be

$$P(x, r, \partial, \lambda) = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{e^{-(\lambda+\partial)}\partial^i \lambda^{k-i}}{(K-i)!\,i!} \times B\left(x, \frac{1}{2} r+i, \frac{1}{2} s+k-i\right) \qquad \ldots \quad (1)$$

where $\partial = \frac{1}{2} E(Y) E(Y)', \quad \lambda+\partial = \frac{1}{2} E(Y_0).E(Y_0)'$

and $$B(x, m, n) = \frac{\Gamma(m+n)}{\Gamma(m)\,\Gamma(n)} \int_0^x t^{m-1}(1-t)^{n-1}dt.$$

We shall call P the distribution function of a double non-central beta variable with degrees of freedom r, s and non-centrality parameters 2∂, 2λ.

If assumptions (1°)—(4°) hold, then $\partial = 0$, and P is the power function of the analysis of variance test. A non-zero value of ∂ indicates that assumption (4°) is not valid, and P the probability of rejection of H_0, gives us an idea of the robustness of the analysis of variance procedure. For example, whether the analysis of variance test would provide an unbiased means of detecting treatment effects in a randomised block experiment when blocks and treatments interact, can be judged by comparing P against the level of significance of the test.

In this paper, two heuristic approximations have been used for (1). No attempt was made to derive estimates of error of approximation in these cases, but extensive numerical computation shows that agreement with exact values is quite good.

2. It is easy to see that T can be written as $T = \frac{\chi_2^2}{\chi_1^2+\chi_2^2}$ where χ_1^2 and χ_2^2 are independely distributed as non-central χ^2 with degrees of freedom s and r and non-centrality parameters 2λ and 2∂.

Approximation I *for* (1) *using Laguerre polynomials.* Using a Laguerre series expansion for the density function of $X = \chi^2/2\rho$, where χ^2 is a non-central χ^2 variable with degrees of freedom n and non-centrality parameter λ^*, and $\rho = \frac{n+2\lambda^*}{n+\lambda^*}$, Roy and Mohammed (1964) have shown that the density of X can be approximated by

$$\phi(X) = p_m(x)[1+b_3^{(m)}+b_4^{(m)}]+p_{m+1}(x)[-3b_3^{(m)}-4b_4^{(m)}]$$
$$+p_{m+2}(x)[3b_3^{(m)}+6b_4^{(m)}]+p_{m+3}[-b_3^{(m)}-4b_4^{(m)}]+p_{m+4}[b_4^{(m)}] \quad \dots \quad (2)$$

where $m = (n+\lambda^*)^2/2(n+2\lambda^*)$, $p_m(x) = e^{-x}x^{m-1}/\Gamma(m)$,

$$b_3^{(m)} = \frac{\lambda^{*2}m}{3(n+2\lambda^*)^2},\ b_4^{(m)} = \frac{\lambda^{*2}m(n+4\lambda^*)}{4(n+2\lambda^*)^3}.$$

Hence the joint density function of $X_1 = \chi_1^2/2\rho_1$ and $X_2 = \chi_2^2/2\rho_2$, where χ_1^2 and χ_2^2 are independently distributed and

$$\rho_i = \frac{n_i+2\lambda_i^*}{n_i+\lambda_i^*},\ i = 1, 2,$$

is approximately given by

$$\phi(x_1, x_2) = \sum_{i=0}^{4}\sum_{j=0}^{4} A_i^{(m_1)} A_j^{(m_2)}\, p_{m_1+i}(x_1)\, p_{m_2+j}(x_2) \quad \dots \quad (3)$$

where $A_i^{(m)}$ is the coefficient of $p_{m+i}(x)$ in (2). Integrating (3) over region $\frac{x_2}{x_1+x_2} \leqslant z$, we get

$$\Pr\left\{\frac{x_2}{x_1+x_2} \leqslant z\right\} \simeq \sum_{i=0}^{4}\sum_{j=0}^{4} A_i^{(m_1)} A_j^{(m_2)} B_z(m_2+j,\ m_1+i).$$

Hence

$$P(x) = \Pr\{T \leqslant x\} = \Pr\left\{\frac{x_2}{x_1+x_2} \leqslant \frac{1}{1+\frac{\rho_2(1-x)}{\rho_1 x}} = Z\right\}$$

$$\simeq \sum_{i=0}^{4}\sum_{j=0}^{4} A_i^{(m_1)} A_j^{(m_2)} B_z(m_{2+j}, m_{1+i}) \qquad \dots \quad (4)$$

where $\lambda_1^* = 2\lambda,\ \lambda_2^* = 2\partial$ and $m_i = (n_i+\lambda_i^*)^2/2(n_i+2\lambda_i^*)$ $\quad i = 1, 2.$

It is to be noted that if one uses the approximation for non-central χ^2 as suggested by Patnaik (1949), an approximation for $P(x)$ is obtained as $P(X) \simeq B_z(m_2, m_1)$ where z, m_1 and m_2 are defined above. This approximation will be referred to as Patnaik's approximation.

Approximation-II *for* (1) *using Jacobi polynomials.* Let $f(T)$ be the density function of T. Following Roy (1965), the ratio $f(T)/\beta_T(a, b)$ where

$$\beta_y(a, b) = \frac{\Gamma(a+b)}{\Gamma(a)\ \Gamma(b)}\, y^{a-1}(1-y)^{b-1},$$

can be formally expanded in an infinite series involving polynomials as

$$\frac{f(T)}{\beta_T(a, b)} = \sum_{r=0}^{\infty} a_r J_r(T, a, b). \qquad \dots \quad (5)$$

The polynomials $J_r(T, a, b)$, known as Jacobi polynomials, are orthogonal with respect to the weight function $\beta_T(a, b)$ and are defined by

$$J_0(T, a, b) = 1$$

$$J_r(T, a, b) = \sum_{\nu=0}^{r} (-1)^\nu\, C_\nu(r\,|\,a, b)T^\nu \qquad \text{for } r = 1, 2, \dots$$

where

$$C_0(r\,|\,a, b) = a(a+1) \dots (a+r-1)/r!$$

$$C_\nu(r\,|\,a, b) = (r+a+b-1)(r+a+b) \dots (r+a+b+\nu-2)/\nu\,!$$

$$X(a+\nu)(a+\nu+1) \dots (a+r-1)/(r-\nu)!$$

$$\text{for } \nu = 1, 2, \dots, r-1$$

$$C_r(r\,|\,a, b) = (r+a+b-1)(r+a+b) \dots (2r+a+b-2)/r!$$

and $$\int_0^1 J_r\ J_s\ \beta\ dx = K_r(a, b)\partial_{rs}$$

where $$K_0(a, b) = 1$$

and $$K_r(a, b) = \frac{a(a+1)\ldots(a+r-1)\,b(b+1)\ldots(b+r-1)}{r!(2r+a+b-1)(a+b)(a+b+1)\ldots(a+b+r-2)} \quad \text{for} \quad r = 1, 2, \ldots$$

Multiplying both sides of (5) by $J_r\ (T, a, b)$ and integrating over T from 0 to 1, we get

$$a_r = \int_0^1 J_r f dT / K_r$$

because of the orthogonality property of the Jacobi polynomials. If we retain only the first five terms in (5) we get

$$f(T) \simeq \beta_T(a, b) \sum_{r=0}^{4} a_r J_r(T, a, b).$$

This in turn gives an approximation for $P(x)$

$$P(x) \simeq B_x(a, b) - \beta_x(a+1,\ b+1) \sum_{r=1}^{4} a_r^* J_{r-1}(x, a+1, b+1) \qquad \ldots \quad (6)$$

where

$$a_r^* = \frac{a_r \times ab}{r(a+b)(a+b+1)}.$$

Let us write m_g for the g-th moment of T about the origin. We now choose a and b to make $a_1 = a_2 = 0$. This is done by taking

$$a = \frac{m_1(m_1-m_2)}{m_2-m_1^2}, \qquad b = \frac{(m_2-m_1)(m_1-1)}{m_2-m_1^2}.$$

Write $C = a+b$ and get

$$a_3^* = \frac{C+5}{(b+1)(b+2)} \left\{ \frac{a}{3} - (a+2)m_1 + \frac{(C+2)(C+3)m_2}{a+1} - \frac{(C+2)(C+3)(C+4)}{3(a+1)(a+2)} m_3 \right\}$$

$$a_4^* = \frac{(C+7)(C+2)}{(b+1)(b+2)(b+3)} \left\{ \frac{a}{4} - (C+3)m_1 + \frac{3(C+3)(C+4)m_2}{2(a+1)} \right.$$

$$\left. - \frac{(C+3)(C+4)(C+5)}{(a+1)(a+2)} m_3 + \frac{(C+3)(C+4)(C+5)(C+6)}{4(a+1)(a+2)(a+3)} m_4 \right\}$$

Where

$$m_g = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \frac{\Gamma\left(\frac{r}{2}+i+g\right)\Gamma\left(\frac{r+s}{2}+k\right)}{\Gamma\left(\frac{r+s}{2}+k+g\right)\Gamma\left(\frac{r}{2}+i\right)} \frac{e^{-(\lambda+\delta)}\delta^i\lambda^{k-i}}{i!(k-i)!}.$$

If we ignore all the moments of order greater than two, then $P(x)$ can be approximated by $B_x(a, b)$ which is the first term of (6). The idea of approximating a distribution function by a beta series is not new. Rao (1951) has used such an approximation for the distribution of Wilks' criterion.

3. *Table.* The following table presents the exact value and different approximations for the function $P(x)$ at selected points. It appears from a study of this table that

(i) for most cases, Patnaik's approximation and the first term of approximation II are good enough for practical purposes ;

(ii) approximation I is better than Patnaik's approximation in most cases and

(iii) in majority of the cases approximation I seems to be better than approximation II. This is to be expected because in approximation I 25 terms are considered whereas in approximation II only five terms are used. Although approximation I uses 25 terms, it takes less time to compute than approximation II which requires the knowledge of first four exact moments of double non-central beta expressed in the form of double infinite series. If, however, for any given $(r, s, \partial, \lambda)$ tabulation is undertaken the total time taken by approximation II may be less as the moments need be calculated once.

TABLE. COMPARISON OF THE APPROXIMATIONS TO DOUBLE NON-CENTRAL BETA DISTRIBUTION

s	r	2λ	2∂	x	$P(x)$				
					Patnaik's approx.	approx.-I	first term of approx.-II	approx.II	exact
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
6.0	20.0	2.0	7.104	.6	.0520	.0531	.0553	.0530	.0528
6.0	20.0	2.0	7.104	.8	.4610	.4650	.4636	.4597	.4433
4.0	20.0	3.0	12.3731	.6	.0286	.0287	.0299	.0308	.0358
4.0	20.0	8.0	28.46	.8	.4152	.4002	.4375	.4439	.4782
10.0	20.0	24.0	2.8164	.4	.4925	.4925	.4781	.4864	.4948
10.0	20.0	24.0	2.8164	.6	.9602	.9573	.9618	.9606	.9571
10.0	20.0	8.2552	0.0	.4	.1555	.1565	.1577	.1574	.1563
10.0	20.0	8.2552	0.0	.6	.7115	.7124	.7031	.7072	.7123
10.0	20.0	8.2552	0.0	.8	.9880	.9877	.9900	.9924	.9870
10.0	20.0	12.0	3.9714	.4	.1509	.1521	.1545	.1532	.1519
10.0	20.0	12.0	3.9714	.6	.7287	.7275	.7108	.7241	.7284
10.0	20.0	12.0	3.9714	.8	.9924	.9913	.9950	.9930	.9906

REFERENCES

PATNAIK, P. B. (1949) : The non-central χ^2 and F-distribution and their applications. *Biometrika*, 26, 202–232.

RAO, C. R. (1951) : An asymptotic expansion of the distribution of Wilks' criterion. *Bull. Inst. Int.*, 33(2), 177–180.

ROY, J. (1965) : Power of the likelihood-ratio test used in analysis of dispersion. Paper presented at the International Symposium on Multivariate Analysis at Dayton.

ROY, J. and MOHAMMED, J. (1964) : An approximation to the non-central chi-square distribution. *Sankhyā*, Series A, 26, 81–84.

SZEGO, G. (1939) : Orthogonal polynomials. *Amer. Math. Soc. Colloquium Publication*, 23.

Paper received : May, 1967.

Revised : August, 1967.