Essays on sequencing problems with welfare bounds

A DISSERTATION PRESENTED

BY

SREOSHI BANERJEE

TO

THE ECONOMIC RESEARCH UNIT

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS

FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

IN THE SUBJECT OF

QUANTITATIVE ECONOMICS

INDIAN STATISTICAL INSTITUTE

KOLKATA, WEST BENGAL

INDIA

JULY 2021

Thesis advisor: Professor Manipushpak Mitra   Sreoshi Banerjee

Essays on sequencing problems with welfare bounds

Abstract

This is a comprehensive study of sequencing problems with welfare bounds. The sequencing framework comprises a finite set of agents and a single facility provider that processes their jobs sequentially. Each job is characterized by its per period waiting cost and processing time. The designer has to fix the order in which agents are served and the monetary compensations to be paid/received. The sequencing and queueing literature has studied the impact of imposing lower bounds on the utility function in various contexts. The most natural bound is the first come first serve protocol where there is a preexisting order in which agents arrive. From the cooperative game perspective, sequencing games with initial order was analyzed by Curiel et al. (1989) and, from the mechanism design perspective, the queueing problem was addressed by Chun et al. (2017) and by Gershkov & Schweinzer (2010). There are other fairness bounds that have been studied from the normative viewpoint. In queueing, the notion of identical costs bound (ICB), analogous to identical preferences lower bound,[1] has been analyzed by Maniquet (2003), Chun (2006b), Mitra (2007) and Chun & Yengin (2017). In the sequencing context, Mishra & Rangarajan (2007) and De (2013) study the expected cost bound where agents have identical urgency indices, implying that every possible ordering is equally likely. Chun & Yengin (2017) have introduced welfare lower bounds with the k-welfare lower bound guaranteeing each agent his utility at the kth queue position with zero transfer. In the queueing literature, Gershkov & Schweinzer (2010) honor an agent's existing service rights by defining individual rationality with respect to an existing mechanism (first come first serve and random arrival schedules). They have examined whether efficient reordering is possible when individuals are rational with respect to the status quo.

This thesis introduces a universal representation of all the previously studied welfare bounds in the literature. Such a generalized representation enriches the existing literature by allowing future studies to be more simplified and compact. We term this bound as the "generalized minimum welfare bound" (GMWB). It is type dependent and offers every agent a minimum guarantee on their utilities.

[1] Moulin (1990b), Moulin (1991b), Bevia (1996), Beviá (1998), Thomson (2003)

In other words, such an assurance puts an upper limit on the maximum disutility of waiting for a service and safeguards all agents against adverse circumstances.

The dissertation imposes the generalized minimum welfare bound property in the sequencing framework and studies its impact using three different approaches, viz.,the strategic approach, the egalitarian approach and the cooperative game approach. The strategic notion used in the first essay is that of *strategyproofness*. We characterize the entire class of mechanisms that satisfies outcome efficiency, strategyproofness and the GMWB property. The chapter provides relevant theoretical applications and also addresses issues of feasibility (or, budget balance). The second essay uses the classic Lorenz criterion that embodies the essence of egalitarianism in the distribution of the final outcome and can be used to make inequality comparisons. We find that the constrained egalitarian mechanism is the only Lorenz optimal mechanism in the class of feasible mechanisms satisfying the GMWB property. The final essay maps the sequencing problem to a characteristic form game using an optimistic and a pessimistic approach to define the worth of a coalition. Under both the approaches, the transfers are designed such that, every agent receives his share of Shapley value payoff as his final utility. We provide a necessary and sufficient condition for the allocation rule to satisfy GMWB. The paper also provides key insights on the existence of the core allocations in sequencing games.

Contents

Author List

The following authors contributed to Chapter 2: **Sreoshi Banerjee, Parikshit De, Manipushpak Mitra**.

The following authors contributed to Chapter 3: **Sreoshi Banerjee and Manipushpak Mitra**.
(Published in Economic Letters, August 2021)

The following authors contributed to Chapter 4: **Sreoshi Banerjee**.

To,

The girl who dared to dream and the tiny ray of hope that showed her the way.

Acknowledgments

I ardently believe that one page and a few hundred words will never be enough to justify the amount of gratitude I carry for every single person who has guided me and been by my side during this journey. Let me assure you, the sentences that follow this paragraph flow straight from the bottom of my heart and I will forever be indebted to all of you. After all, you have given life to this thesis.

- *Mani sir* : You have taught me the meaning of patience and persistence. From the very first day till this moment, you have been an unwavering support under every single circumstance. Be it be personal or professional, one quick conversation with you would fill me with immense positivity and strength. You have made me believe in myself and my work. You have steered me in the right direction and equipped me to be a quality researcher. You made this journey so easy and so comfortable that today, when I look back, I wonder how quickly the years have flown by. Thank you sir, for being the backbone of this thesis. You will continue to inspire me forever.

- *Ma and Baba* : I am not sure how to thank you because your love is and always will be unconditional. I am what I am because of you. The ray of hope that lives inside me, is you. The light that shows me the path, is you. Nothing can ever change between us. Despite our ups and downs, I am your little girl and will always be. You have shown me how to dream. You have taught me to be curious and hungry for knowledge. You have taught me to be sincere and hardworking. But, most importantly, you have taught me what being a good human being means. My parents have imparted in me their goodness and honesty. I promise to preserve this till my last breath.

  I would also like to thank my grandparents, especially *Dadan* for being my constant companion and my best friend. You are the only one who can make me smile even on the most difficult days.

- *Arunava sir, Debasis sir and Krishnendu sir* : You have always been there for me whenever I reached out for suggestions and guidance. Every discussion with you has been productive and fruitful. You have shown me immense warmth and helped me bring out the best version of myself and my work. I deeply express my gratitude to both you for being such a huge support during this journey.

- *Satya sir, Nityananda sir, Samarjit sir, Tarun sir, Anuj da, Soumyanetra mam and all the respected faculty members of the economic research unit* : Thank you for making me feel at home. I have enjoyed spending my hours working in the lab much more than anywhere else in the campus (including my own hostel room). Your presence has always made me feel protected and happy. Your smiling faces and welcoming gestures have meant a lot to me during all these years. For all I know, you became my local guardians so effortlessly. This department and your love shall always live within me.

- *Swarup da, Chunu da, Tara di and all my dearest staff members* : Thank you very much for being so affectionate and helpful. Each one of you became an important part of my day to day life. I remember opening the door to the office and being greeted with a warm smile every single time. You are like family to me.

- *Parikshit da, Arindam da, Tanmoy da, Chayanika di, Abhinandan da and all other beloved scholars* : We have made so many memories together. Right from our morning greetings to evening chai at the canteen to late night hostel addas; you all were with me throughout. Whenever the tables turned south, I could blindly count on you and the trust we developed over the years will only keep getting stronger.

- *Vaishali, Vikram, Sayantan, Mansi* : You guys are my pillars. You were there during all my highs and lows. You know me in and out and you have accepted me for who I am. I could sail through everything because I had your friendship with me. You guys are the soul of this thesis, my strongest supporters and my best critiques. I am the luckiest to have you in my life.

- I would specially like to thank *Saptarshi da and Sarvesh* for being such a good friend to me. I have absolutely enjoyed all our conversations and I have learnt so much from you.

- Finally, I bow my head to the institute for having faith in me and providing me with this opportunity. *Indian Statistical Institute*, I salute thee. I owe you my identity, my degree and my title. I hope to live up to your name.

1
Introduction

1.1 The problem of fair division

Fairness matters. It matters because every human being has an intuitive sense of the notion of fairness. A common man might not be able to tangibly define or measure the idea, but he understands whether or not he is being treated fairly by the society, the market and the government. The theories of distributive justice are concerned with the fair allocation of resources amongst diverse agents in an economy. Distribution of rewards or costs must happen in a reasonable fashion so that every member receives (or bears) a 'fair share'.

A fundamental concern of distributive justice involves the allocation of commonly owned resources among a group of individuals who have equal claim to the resource but diversified preferences. In such a fair division problem, the justness in the distribution of the final outcome can be measured using several fairness axioms that have been studied in the literature. Two such axioms are no-envy and egalitarian equivalence. No-envy emphasizes that no agent should prefer another agent's allocation in comparison to his own (see Tinbergen (1946) and Foley (1966)) and egalitarian equivalence requires an agent to be indifferent between his own bundle and a common reference bundle (see Pazner & Schmeidler (1978)). However, Yengin (2013a) pointed out that a mechanism satisfying these axioms might yield exceptionally small or large levels of welfare. A society would then prefer a mechanism which guarantees a minimum level of welfare to every individual over a mechanism where some agents end up with arbitrarily low welfare levels despite being envy free. Assuring every agent with a minimum guaranteed payoff serves as an indicator of how developed and cohesive an economy is. Knowledge about welfare levels in worst-case scenarios acts as a safety net and protects an agent against factors like, differences in preferences, for which no agent can be held individually responsible for.

The fairness literature has proposed and extensively studied welfare lower bounds that are unbiased, righteous and built on the notions of equity and fair-play. In the classic fair allocation problem, guaranteeing each agent a utility level that is atleast that of consuming an equal share of the resource, is one of the oldest axioms (see Steihaus (1948) and Dubins & Spanier (1961)). Due to an equal treatment of equals, each member receives the highest feasible utility outcome from an equal split of the resource. However, when the central planner needs to allocate an indivisible commodity along with monetary transfers to its agents, such a division is not well defined. Moulin (see Moulin (1990b) and Moulin (1991b)) suggested an alternate fairness axiom namely, the identical preferences lower bound.

In an economy where agents have varying preferences, we arbitrarily pick an agent and imagine a hypothetical situation where every other agent has preferences identical to him. We compute the common welfare level if equals are treated equally and allocations are Pareto efficient. Ideally, an agent should be concerned only about his own taste and should not have to bear the externalities caused by the heterogeneity of preferences in the actual economy. This bound promises each agent the benchmark welfare level (realized in the reference economy) such that no-one is worse off in the actual world than he/she was in the hypothetical setup. There are other welfare lower bounds that have been examined in the fairness literature that bind the maximum loss incurred by an agent. Some of them include the stand-alone lower bound (respecting an agent's autonomy; see Moulin (2003)), individual rationality (respecting the status quo) and the k-fairness criterion based on the Rawlsian maximin principle (see Porter et al. (2004) and Atlamaz & Yengin (2008)).

1.2 Sequencing games

I adopt the mechanism design approach to study 'sequencing' problems which belong to the class of finite decision problems. A position in a queue is an indivisible good and sequencing games deal with typical situations of allocating such goods amongst a finite set of agents. I provide a brief outline of the framework followed in this thesis.

- A sequencing problem consists of a finite set of agents and a single server/processing unit.
- Agents are in need of processing their respective jobs. These job and may differ across individuals in terms of their processing times.
- The service provider can process only one job at a time. Once a job starts processing, it cannot be interrupted.

- We allow for monetary transfers and preferences are quasi-linear over the positions in a queue and the transfers.
- Agents incur a per period cost of waiting and the total cost of completing a job depends upon his waiting time in the queue and the time taken to process his own job.
- The task of the planner is to decide the ordering of agents (allot each agent a position in the queue) to minimize the total cost of job completion in the economy and the transfer amounts to be made or received by each agent.

Sequencing problems have a well established literature beginning with Dolan (1978). Dolan studied sequencing games as incentive problems with the waiting costs as private information. He provided a mechanism which was incentive compatible but not budget balanced. Significant contributions to this field were later made by Suijs (1996), Mitra (2001), Mitra (2002), Hain & Mitra (2004) and several others. Our framework follows the structure adopted by Suijs (1996) where the perceived waiting cost of an agent is linear in time. A special class of sequencing games is referred to as queueing games where the job processing times are identical across agents. A sequencing rule is said to be outcome efficient if it minimizes the aggregate cost of job completion. Smith et al. (1956) provided a necessary and sufficient condition for a sequencing rule to be outcome efficient based on agents' urgency indices. An urgency index is the ratio of an agent's waiting cost to his processing time. Outcome efficiency is achieved when players in a queue are arranged in a decreasing order of their urgency indices.

A sequencing model with private information is implementable if and only if the mechanism is a Groves mechanism (Groves (1973)). Implementability means that for any agent, truth telling is the dominant strategy (no one is better off by misreporting their true type) and the sequencing rule is outcome efficient. Under relatively weak assumptions on the preference domain, Green & Laffont (1979), Holmström (1979) and Suijs (1996), have shown that the Groves mechanisms are the unique class that satisfies efficiency and dominant strategy incentive compatibility. However, Green & Laffont (1979) have also shown that the transfer payments of a Groves' scheme need not be budget balanced in an unrestricted domain. Further, Hurwicz & Walker (1990) showed that in a variety of exchange economies one cannot find mechanisms that satisfy both incentive compatibility and budget balancedness. Suijs (1996) deduced that linearity of costs over time is crucial for a sequencing problem to be first best implementable; meaning, that it is possible to design a mechanism where truth telling is dominant, the outcome is efficient and budget balancedness is maintained. Mitra (2002) analyzed a more general and natural class of cost functions to conclude that, for first best implementability, linearity of cost functions is not only sufficient but also necessary. A sequencing problem is first best implementable only if the cost function is linear.

1.3 Welfare bounds in sequencing games

The sequencing and queueing literature has studied the impact of imposing lower bounds on the utility function in various contexts. The most natural bound is the first come first serve protocol where there is a preexisting order in which agents arrive. From the cooperative game perspective, sequencing games with initial order was analyzed by Curiel et al. (1989) and, from the mechanism design perspective, the queueing problem was addressed by Chun et al. (2017) and by Gershkov & Schweinzer (2010). There are other fairness bounds that have been studied from the normative viewpoint. In queueing, the notion of identical costs bound (ICB), analogous to identical preferences lower bound [1], has been analyzed by Maniquet (2003), Chun (2006b), Mitra (2007) and Chun & Yengin (2017). In the sequencing context, Mishra & Rangarajan (2007) and De (2013) study the expected cost bound where agents have identical urgency indices, implying that every possible ordering is equally likely. Chun & Yengin (2017) have introduced welfare lower bounds with the k-welfare lower bound guar-

[1] Moulin (1990b), Moulin (1991b), Bevia (1996), Beviá (1998), Thomson (2003))

anteeing each agent his utility at the kth queue position with zero transfer. Starting from the last position, the center progressively reduces k (thus increasing the welfare levels) till there is a clash with certain budgetary requirements. In the queueing literature, Gershkov & Schweinzer (2010) honor an agent's existing service rights by defining individual rationality with respect to an existing mechanism (first come first serve and random arrival schedules). They have addressed the queueing problem of reordering an existing queue into its efficient order through trade.

This thesis introduces a universal representation of all the previously studied welfare bounds in the literature. Such a generalized representation enriches the existing literature by allowing future studies to be more simplified and compact. We term this bound as the "generalized minimum welfare bound" (GMWB). It is type dependent and offers every agent a minimum guarantee on their utilities. In other words, such an assurance puts an upper limit on the maximum disutility of waiting for a service and safeguards all agents against adverse circumstances. By virtue of the linear cost structure in our framework, the generalized minimum welfare bound can be decomposed and expressed as a product of two components - the per period waiting cost of an agent and the welfare parameter. The welfare parameter is simply a function of the vector of job processing times (which is common knowledge) and the form of this function varies with the specific bound under consideration. Thus, for a given sequencing problem, a mechanism is said to satisfy the generalized minimum welfare bound property if every agent receives a utility that is atleast as much as the guaranteed level.

The dissertation imposes the generalized minimum welfare bound property in the sequencing framework and studies its impact using three different approaches, viz.,the strategic approach, the egalitarian approach and the cooperative game approach. The strategic notion used in the first essay is that of *strategyproofness*. We characterize the entire class of mechanisms that satisfies outcome efficiency, strategyproofness and the GMWB property. The chapter provides relevant theoretical applications and also addresses issues of feasibility (or, budget balance). The second essay uses the classic Lorenz criterion that embodies the essence of egalitarianism in the distribution of the final outcome and can be used to make inequality comparisons. We find that the constrained egalitarian mechanism is the only Lorenz optimal mechanism in the class of feasible mechanisms satisfying the GMWB property. The final essay maps the sequencing problem to a characteristic form game using an optimistic and a pessimistic approach to define the worth of a coalition. Under both the approaches, the transfers are designed such that, every agent receives his share of Shapley value payoff as his final utility. In this scenario, the expected cost bound condition is shown to be necessary and sufficient for the generalized minimum welfare bound property to hold. The paper also provides key insights on the existence of core allocations associated with both the cooperative games.

1.3.1 A welfarist approach to sequencing problems with incentives

Literature

The notion strategyproofness incentivizes an agent to not misreport her true waiting cost irrespective of what she believes other agents to be doing.[2] Both sequencing and queueing problems have been extensively studied from the strategic view point in the last couple of years (see Chun et al. (2014a), Chun et al. (2019a), De & Mitra (2017), De & Mitra (2019), Mitra (2001), Mitra (2002) , Mitra & Mutuswami (2011) and Ramaekers & Kayi (2008)). From the normative viewpoint, notable contributions have been made by Chun (2006a), Chun (2006b), Chun (2004), Maniquet (2003), Mishra & Rangarajan (2007) and Moulin (2007).

There are several studies in the literature that have combined strategic and fairness properties to study the general class of allocation problems with heterogeneous indivisible goods and monetary transfers.[3] Within the scope of queueing problems, Kayı & Ramaekers (2010) study the no-envy

[2] The literature on strategyproofness is too vast. A comprehensive review of preferences can be found in Barberà (2011). Other examples where strategyproofness has been applied in different contexts include Chatterji & Sen (2011), Moreno & Moscoso (2013), Öztürk et al. (2014), Velez (2014), Westkamp (2013) and many more

[3] Atlamaz & Yengin (2008), Mukherjee (2014), Ohseto (2004), Ohseto (2006), Pápai (2003), Yengin (2012),

rule (see Foley (1966)) along with queue efficiency and strategyproofness. No-envy is a fairness criteria which states that no agent should prefer another agent's bundle to her own. Following this, Hashimoto & Saitoh (2012), study the relationship between equity and efficiency for queueing problems. They characterize the class of rules that satisfy strategyproofness, anonymity in welfare and budget balance. Chun et al. (2014b) impose the egalitarian equivalence condition (see Pazner & Schmeidler (1978)) to characterize a sub-family of VCG rules. An allocation rule is egalitarian equivalent if there is a reference bundle for every preference profile which makes an agent indifferent between her allocation bundle and the reference bundle. The normative distributive requirement of k-welfare lower bound has been studied by Chun & Yengin (2017) which requires that each agent should be guaranteed her utility at the k-th queue position with zero transfer. In the queueing context, they investigate the implication of such bounds along with queue efficiency and strategyproofness. For sequencing problems, De (2013) studies implementation of VCG mechanisms with egalitarian equivalence and expected costs bound.

Chapter Contribution

The "generalized minimum welfare bound" (GMWB) is a universal representation encompassing fairness bounds as well as naturally and artificially constructed bounds. We study the implications of this bound in the sequencing framework with incomplete information (Banerjee et al. (2020)). Our first result, identifies the "constrained welfare property" which is a condition that is both necessary and sufficient to obtain the class outcome efficient and strategyproof mechanisms that satisfy the generalized minimum welfare bound. Our second theorem characterizes the entire class of mechanisms that satisfies outcome efficiency, strategy proofness and GMWB. We term this as the class of "relative pivotal mechanisms". We also address the issue of finding those relative pivotal mechanisms that satisfy either feasibility or its stronger version, budget balance. Our paper proposes relevant theoretical ap-

Yengin (2013b), Yengin (2013a) and Yengin (2017).

plications namely; ex-ante initial order, identical costs bound and expected cost bound. The idea of identical costs bound is based on a reference economy where we pick an agent and all the other agents have identical waiting costs and processing times. The expected costs bound allows agents to arrive randomly such that every arrival order is equally likely. We also give insights on the issues of feasibility and/or budget balance.

1.3.2 Lorenz optimality in sequencing with welfare bounds

Literature

The second essay of this thesis applies the Lorenz criterion to identify the class of feasible mechanisms which satisfies the generalized minimum welfare bound and is Lorenz optimal. Under the egalitarian idea of distributive justice, members of a society are not only concerned with their share of final outcome, but also with the distribution of the final outcome across all its members. In such an economy, both the absolute and relative positions matter. The central planner values social welfare and equality in the distribution of outcomes whereas individuals are primarily driven by their self-seeking behavior. There is a notable literature that deals with the characterization of the Lorenz ordering as a plausible concept of inequality (see Atkinson et al. (1970), Dasgupta et al. (1973), Fields & Fei (1978),Sen et al. (1997)). Thomson (2012) develops three general approaches to obtain the Lorenz ranking of rules for the adjudication of claims. He develops a general criteria to enable Lorenz comparison across specific classes of rules. Dutta and Ray (Dutta & Ray (1989), Dutta & Ray (1991)) propose a constrained egalitarian solution concept for transferable utility games by combining commitment towards egalitarianism and promoting individual interest in a consistent fashion. The constrained egalitarian rule has also been proposed by Chun et al. (1998) to solve claim problems and attain certain objectives of equality. For queueing problems, Chun et al. (2019b) show that the constrained egalitarian mechanisms are Lorenz optimal amongst the class of mechanisms satisfying outcome efficiency, budget balance and the identical preferences lower bound (IPLB).

Chapter contribution

In the sequencing context, we explore the possibility of designing mechanisms which uphold the notion of justness and safeguard an agent's individual interest. Every agent is guaranteed a minimum level of utility by imposing the generalized minimum welfare bound (Banerjee & Mitra (2021)). This paper is an important generalization of Chun et al. (2019b). Our main result shows that the constrained egalitarian mechanism is Lorenz optimal in a broader class of feasible mechanisms satisfying the generalized minimum welfare bound. We also characterize the complete class of mechanisms that satisfies GMWB and feasibility (budget balance).

1.3.3 Existence of core in sequencing - an optimistic and a pessimistic approach

Literature

This paper adopts a cooperative approach to study sequencing problems Banerjee (2021). A popular approach to studying cost sharing problems (Moulin (2002)) involves associating an appropriate characterization form to the original problem and implementing solution concepts from the theory of cooperative games. The Shapley value is considered as an appropriate solution to fair division problems in general and has been shown to possess interesting fairness properties (Moulin (1992)). Maniquet (2003) has worked in the queueing framework and defines the worth of a coalition to be the minimum aggregate waiting cost of its members if they are to be served first in the queue. On the other hand, Chun (2006b) adopts a pessimistic approach towards evaluating the worth of a coalition by computing the minimum waiting cost of that coalition if they are served after the non-coalitional members. For the queueing problem, both Maniquet (2003) and Chun (2006b) axiomatically characterize their respective transfer rules (which generate an agent's corresponding Shapley value payoff) using classic fairness axioms.

Curiel et al. (1989) consider the class of sequencing games for which the initial order of the agents is known. They focus on sharing the savings in costs when switching from an initial ordering to an optimal ordering and show that the "mid-point" solution is in the core of an associated cooperative game. Mishra & Rangarajan (2007) extend the characterization of Maniquet for the one dimensional model to the general model of sequencing games. They provide a new set of axioms to characterize the Shapley value under efficient ordering. Moulin (2007) studies the strategic aspect like splitting and merging for a class of problems where agents have identical waiting costs but different job processing times. These are known as scheduling problems. He shows that the Shapley value solution is merge proof, but not splitproof.

Chapter contribution

The first half of the paper defines the characteristic form game for a given sequencing problem using the Maniquet's optimistic approach and Chun's pessimistic perspective. I compute the associated Shapley value payoffs in both the cases and the corresponding transfer amounts. The objective of this paper is to study the set of core allocations, if at all they exist. It can be observed that - using the optimistic approach, the core of the primal game is empty while the Shapley value belongs to its dual and using the pessimistic approach, the core of the dual game is empty while the Shapley value belongs to the core of the primal. The second half of the paper shows that an allocation rule which assigns utilities corresponding to an agent's Shapley payoff of the sequencing game, satisfies the generalized minimum welfare bound if and only if the expected costs bound holds. The expected costs bound guarantees each agent his expected cost when every possible ordering has an equal probability of arriving for the service.

2

A welfarist approach to sequencing problems with incentives

2.1 Introduction

This chapter adopts a holistic approach to analyze sequencing problems in a framework which focuses on prioritizing a customer's well-being under private information.

2.1.1 Purpose

We live in an instant world where time is precious and convenience is an essential prerequisite. The service sector is struggling under the burden of long waiting lines that hamper customer satisfaction and their long term loyalty. This paper adopts a holistic approach to analyze sequencing problems while prioritizing a customer's well-being. One might argue that a consumer can always exercise his option to walk away and not participate in the mechanism. However, if we look deeper into our lifestyles, there are multiple instances where waiting in a line to get our jobs processed is not optional but is either inevitable, voluntary or deemed as an absolute necessity. In such cases, the respective service platforms often make an attempt to smoothen out the disutility of waiting and render a fair treatment to all its customers. Our model offers the participating individuals a basic layer of protection against the agony of waiting in a queue to avail a service. This is done by guaranteeing each agent a minimum level of utility as and when his final welfare is realized. Such an assurance acts as a safety net for an agent and tends to improve a consumer's overall satisfaction even in the face of adverse circumstances. This welfarist approach can be justified through multiple real life examples of waiting-time guarantees on services as well as scenarios where offering such a guarantee could benefit both service providers and consumers at large.

Health care services and medical emergencies are an unavoidable part of our lives. In Sweden, long waiting lines for surgical procedures pose a threat to the quality of their health policy agenda. To reduce waiting lists, in 1992 the Swedish Government and the Federation of County Council agreed on an initiative to offer a maximum waiting-time guarantee. Patients awaiting medical procedures are guaranteed a waiting time no longer than 3 months from the physician's decision to treat/operate (seeHanning (1996)). Similarly, UK's national health service (NHS) provides emergency patients with a four hours target window within which 95 percent of the patients need to be discharged or transferred[1]. India faces a massive congestion of vehicles at the highway toll plazas. When an individual drives on the highway, waiting at a toll plaza to pay the toll tax is just as necessary as waiting at the airport check-in counter or the boarding gate before departure. The National Highway Authority of India (NHAI) ensures that the number of toll lanes/booths are such that, the service time per vehicle during peak hours is not more than 10 seconds. The NHAI rules also suggest an increase in the number of toll lanes if the waiting time of the users exceeds 3 minutes. Moreover, there are specific regions in the country where riders are exempted from paying the toll tax altogether if the total waiting time surpasses 3 minutes.

The COVID-19 pandemic has caused immense difficulties for customer care representatives at call centers[2]. With employees unable to work efficiently from home, callers are facing unprecedented waiting times to make essential inquiries and lodging complaints (broken gadgets, slow bandwidth, canceling airline tickets, etc). Hence, it is vital for government agencies/companies to ensure their existing clientele does not experience extreme discomfort and lose their patience. Moreover, prolonged queues at blood-donation clinics act as a major deterrent to voluntary donors [3]. Blood collection organizations aim to be donor-friendly in terms of their waiting time experience. Although the findings suggest a strong sense of commitment to donation, a waiting time guarantee is required to preserve donor satisfaction and avoid putting undue stress on voluntary donors.

2.1.2 Our framework

We work in a standard sequencing environment with a finite set of agents. In our model, each agent has a single job to process using a facility that can only serve one agent's requirement at a time. It is assumed that no job can be interrupted once it starts processing. A job is characterized by its processing time and an agent's waiting cost. The latter represents the disutility of waiting (per unit of

[1] https://www.nhsinform.scot/care-support-and-rights/health-rights/access/waiting-times
[2] https://www.washingtonpost.com/technology/2020/04/14/customer-service-coronavirus/
[3] See McKeever et al. (2006) and van Brummelen et al. (2018)

time). The processing time of all agents are publicly known while the waiting costs are private information. There is a well established literature in this direction.[4] We work in a private information set-up where agents have quasi-linear preferences and the mechanism designer allows for monetary incentives. Businesses often resort to monetary and non-monetary incentives to induce better queue management (express passes for peak hours at theme parks, off season discounts, airlines providing priority check-ins against a nominal fee, Amazon charging for faster deliveries and cashback offers for those willing to wait, etc). For sequencing problems, mechanism design under incomplete information was analyzed by Dolan (1978), Hain & Mitra (2004), Moulin (2007), Mitra (2002) and Suijs (1996). A special case of sequencing problems where the processing times of the agents are identical is called queueing problems. Queueing problems have also been analyzed extensively from both normative and strategic viewpoints.[5]

2.1.3 Contribution to the literature

The sequencing and queueing literature has studied the impact of imposing lower bounds on the utility function in various contexts. The most natural bound is the first come first serve protocol where there is a preexisting order in which agents arrive. From the cooperative game perspective, sequencing games with initial order was analyzed by Curiel et al. (1989) and, from the mechanism design perspective, the queueing problem was addressed by Chun et al. (2017) and by Gershkov & Schweinzer (2010). There are other fairness bounds that have been studied from the normative viewpoint. Identical cost bound (ICB)[6] requires that each agent receives at least the utility he could expect under the egalitarian solution if all agents were identical to him. For queueing problems, the notion of ICB was

[4] See De (2016), De (2013), De & Mitra (2017), De & Mitra (2019), Dolan (1978), Duives et al. (2015), Hain & Mitra (2004), Mitra (2002), Moulin (2007) and Suijs (1996).

[5] See Chun (2006a), Chun (2006b), Chun et al. (2014b), Chun et al. (2019a), Chun et al. (2017), Chun et al. (2019b), Hashimoto (2018), Kayı & Ramaekers (2010), Maniquet (2003), Mitra (2001), Mitra (2007), Mitra & Mutuswami (2011) and Mukherjee (2013).

[6] See Bevia (1996), Moulin (1991b), Moulin (1990a), Steihaus (1948) and Yengin (2013a).

analyzed by Maniquet (2003), Chun (2006b), Kayı & Ramaekers (2010) and Mitra (2007). Chun & Yengin (2017) have introduced *welfare lower bounds* with the *k-welfare lower bound* guaranteeing each agent his utility at the kth queue position with zero transfer. Starting from the last position, the center progressively reduces k (thus increasing the welfare levels) till there is a clash with certain budgetary requirements. In the queueing literature, Gershkov & Schweinzer (2010) honor an agent's existing service rights by defining individual rationality with respect to an existing mechanism (*first come first serve* and *random arrival schedules*). They have examined whether efficient reordering is possible when individuals are rational with respect to the status quo.

We introduce the "*generalized minimum welfare bound*", which is a compact and unified representation of all the existing bounds in the literature. Our welfarist approach gets enriched by this universal representation which encompasses the fairness bounds and any other naturally/artificially constructed bound. The generalized minimum welfare bound is type-dependent and guarantees an assured level of utility to every agent [7],[8]. By virtue of the linear cost structure, one can easily observe that such a bound can be decomposed and expressed as a product of two components- an agent's own waiting cost, θ_i (we do not consider interdependent waiting costs in this paper) and some function of the job processing time vector, $O_i(s)$. The component $O_i(s)$ is the *welfare parameter* which varies depending on the specific bound under consideration [9]. For instance, say mechanism μ_1 assures every agent his worst case utility, i.e., when he is placed in the last position. Let, mechanism μ_2 guarantee every agent the utility he would have obtained under the first come first serve protocol. The welfare parameter $O_i(s)$ under μ_1 is the sum of the processing times of all the agents while under μ_2, it is the sum of his own processing time and the processing time of all the agents preceding him in the initial

[7]This paper does not discuss the question of participation or tries to impose the individual rationality constraint (with respect to not getting the service) at any point.

[8]Every agent is entitled to getting his job processed. We work in a static framework where the facility starts operating only after the finite set of agents have arrived.

[9]The welfare parameter of any agent purely depends on the job processing time vector (s) and not on the waiting costs. Refer to how we define the job completion cost of an agent in the framework section below for further clarity.

order of arrival. The bound under μ_2 is stricter than μ_1 and guarantees a higher minimum welfare (unless of course the agent coincidentally occupies the last position in the initial order too!)

2.1.4 Results

Under private information, we study the implications of a generalized minimum welfare bound in a sequencing problem with monetary transfers. Our first result, identifies the "constrained welfare property" which is a condition that is both necessary and sufficient to obtain outcome efficient and strategyproof mechanisms that satisfy the generalized minimum welfare bound. Constrained welfare property requires that every agent's welfare parameter must be bounded below by his job completion time when he occupies the first position in the queue.

Given this property, our second theorem is a characterization result where we introduce the class of 'relative pivotal mechanisms' which is a strict subset of the set of all VCG mechanisms and satisfies the generalized minimum welfare bound. For any given vector of waiting costs, the main aspect of a relative pivotal mechanism is to construct a 'benchmark' waiting cost. This is based on an optimization exercise conducted using the welfare parameter of the agent and waiting costs of all other agents. Given the benchmark waiting costs of all agents, under the relative pivotal mechanism, the transfer of each agent has three parts. One part of the transfer depends on the difference between his welfare parameter and his job completion time with this benchmark waiting cost. The other part of the transfer involves calculating the externality caused by this agent with his waiting cost on all other agents relative to what would have happened if, ceteris paribus, this agent had the benchmark waiting cost. The third part of the transfer is any non-negative valued function that depends on the waiting cost of all other agents.

Moving forward, we address the issue of finding those relative pivotal mechanisms that satisfy either feasibility or its stronger version, budget balance.[10] We begin by identifying the "weighted net

[10] It is well-known that feasibility of a mechanism requires that the sum of transfers across all agents is non-

welfare" property which is a necessary condition to find mechanisms satisfying generalized minimum welfare bounds, outcome efficiency and feasibility. We show that when there are two agents, we can only get feasible (and not budget balanced) relative pivotal mechanisms if and only if each agent's welfare parameter equals the cost associated with getting served last. For more than two agents we show that if the welfare parameter of each agent is the cost associated with getting served last, then we can get budget balanced (hence, feasible) relative pivotal mechanism.

2.1.5 Applications

We apply our general results to sequencing problems with a natural ex-ante initial order (most commonly observed in our day to day lives). Our next application captures the essence of fairness by constructing an egalitarian bound that treats agents identically such that no agent suffers due to the heterogeneity of other's preferences. In our final application, we allow for random arrival of queues. In other words, every possible ordering of agents has an equal chance of arriving to avail a service.

For sequencing problems with initial order, there is a preexisting order on the agents. Any sequencing problems with a given initial order satisfies the constrained welfare property. Hence, for sequencing problems with initial order, achieving outcome efficiency and eliciting private information boils down to reordering the existing initial order to the outcome efficient order by using relative pivotal mechanisms. In this context we can show that there is no feasible (and hence no budget balanced) relative pivotal mechanism.[11]

Under Identical Costs Bound (ICB), every agent receives at least as much as his utility in the benchmark/reference economy. The reference economy for any agent i requires that all other agents have the same waiting cost and processing time as agent i. Since agents are identical in this sense, each of them has an equal right to the resource. As a consequence, agent i can occupy any position in the

positive and budget balance requires that the sum of transfers across all agents is zero.

[11] For the queueing problem this impossibility was shown by Chun et al. (2017) and our result generalizes it to the sequencing problems.

queue with an equal chance. To define the Expected Cost Bound (ECB) for sequencing problems, consider a reference economy where there are no transfers, agents arrive randomly and every arrival order is equally likely. ECB requires that the utility of each agent is no less than the expected cost of the agent associated with random arrival where each arrival order is equally probable. For queueing problems, the notions of ICB and ECB are equivalent. For all sequencing problems with ICB and ECB, both the constrained welfare property as well as the weighted net welfare property get satisfied. Given these two properties, we obtain the relative pivotal mechanisms with ICB and ECB. We also show that for both these bounds, when there are three agents, we can get feasible relative pivotal mechanisms only for queueing problems.

2.1.6 Implication in terms of queueing problems

For the queueing problems with generalized minimum welfare bounds that satisfy the constrained welfare property, one can give a more explicit form of the transfers associated with the relative pivotal mechanism. We characterize the set of all mechanisms satisfying outcome efficiency, strategyproofness and ICB (ECB) [12]. For more than two agents, we provide a sufficient restriction on the welfare parameter that guarantees the existence of budget balanced relative pivotal mechanisms. The sufficiency condition also becomes necessary when welfare parameters are equal across agents.

2.2 The framework

Consider a finite set of agents $N = \{1, 2, \ldots, n\}$ who want to process their jobs using a facility that can be used sequentially. The job processing time can be different for different agents. Specifically, for each agent $i \in N$, the job processing time is given by $s_i > 0$. Let $\theta_i S_i$ measure the cost of job completion for agent $i \in N$ where $S_i \in \mathbb{R}_{++}$ is the job completion time for this agent and $\theta_i \in$

[12] This is a generalization of the result by Chun & Yengin (2017) where we eliminate the gap between the necessary and sufficient conditions.

$\Theta := \mathbb{R}_{++}$ denotes his constant per-period waiting cost where $\mathbb{R}_{++}$ is the positive orthant of the real line $\mathbb{R}$. Due to the sequential nature of providing the service, the job completion time for agent i depends not only on his own processing time s_i, but also on the processing time of the agents who precede him in the order of service. By means of an order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N, one can describe the position of each agent in the order. Specifically, $\sigma_i = k$ indicates that agent i has the k-th position in the order. Let Σ be the set of $n!$ possible orders on N. We define $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ to be the predecessor set of i in the order σ. Similarly, $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ denotes the follower (or successor) set of i in the order σ. Given a vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ and an order $\sigma \in \Sigma$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. Note that, for any $i \in N$ we write, $\sum_{j \in P_i(\sigma)} s_j = 0$ if $P_i(\sigma) = \emptyset$. The agents have quasi-linear utility of the form $u_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i$ where σ is the order, $\tau_i \in \mathbb{R}$ is the transfer that he receives and the parameter of the model θ_i is the waiting cost. Given any processing time vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ define $A(s) = \sum_{j \in N} s_j$ and, with slight abuse of notation, we denote a *sequencing problem* by Ω and we denote the set of all sequencing problems with the set of agents N by $\mathcal{S}(N)$. A sequencing problem $\Omega \in \mathcal{S}(N)$ is called a *queueing problem* if $s = (s_1, \ldots, s_n)$ is such that $s_1 = \ldots = s_n$. We denotes the set of all queueing problems with the set of agents N by $\mathcal{Q}(N)$. Clearly, $Q(N) \subset \mathcal{S}(N)$ for any given N (such that N is a finite set and $n \geq 2$).

A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. For any $i \in N$, let θ_{-i}, denote the profile $(\theta_1 \ldots \theta_{i-1}, \theta_{i+1}, \ldots \theta_n) \in \Theta^{n-1}$ which is obtained from the profile θ by eliminating i's waiting cost. A mechanism $\mu = (\sigma, \tau)$ constitutes of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma$. Because the sequencing rule is a function (and not a correspondence) we will require a tie-breaking rule to reduce a correspondence to a function which, unless explicitly discussed, is assumed to be fixed. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile

$\theta \in \Theta^n$ with a tie situation between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any mechanism $\mu = (\sigma, \tau)$, if (θ'_i, θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $u_i(\mu_i(\theta'_i, \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta'_i, \theta_{-i})) + \tau_i(\theta'_i, \theta_{-i})$ where $\mu_i(\theta'_i, \theta_{-i}) := (\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}))$. Given any $\Omega \in \mathcal{S}(N)$, any $\theta \in \Theta^n$ and any order $\sigma \in \Sigma$, define the aggregate cost as $C(\sigma; \theta)$, that is, $C(\sigma; \theta) := \sum_{j \in N} \theta_j S_j(\sigma)$.

A sequencing rule is outcome efficient if it minimizes the aggregate job completion cost. A mechanism implements a sequencing rule in dominant strategies if the transfer is such that truthful reporting for any agent weakly dominates false reporting irrespective of what other agents declare. Implementation of outcome efficient sequencing rules in dominant strategies has been well studied in the literature on mechanism design under incomplete information. It is also well-known that, as long as preferences are 'smoothly connected' (see Holmström (1979)), outcome efficient rules can be implemented in dominant strategies if and only if the mechanism is a Vickrey-Clarke-Groves (VCG) mechanism (see Clarke Clarke (1971), Atlamaz & Yengin (2008) and Vickrey (1961)).

Definition 1. A sequencing rule σ^* is said to be *outcome efficient* if for any $\theta \in \Theta^n$, $\sigma^*(\theta) \in \operatorname{argmin}_{\sigma \in \Sigma} C(\sigma; \theta)$.

The ratio of the waiting cost and processing time of any agent i, that is, θ_i / s_i is known as the urgency index. From Smith et al. (1956) it follows that σ^* is outcome efficient if and only if the following holds: **(OE)** For any $\theta \in \Theta^n$, the selected order $\sigma^*(\theta)$ satisfies the following: For any $i, j \in N$, $\theta_i / s_i > \theta_j / s_j \Rightarrow \sigma^*_i(\theta) < \sigma^*_j(\theta)$. We say that a mechanism $\mu = (\sigma, \tau)$ satisfies outcome efficiency if $\sigma = \sigma^*$.

Suppose that a waiting cost of zero was admissible in the domain. Consider any outcome efficient order $\sigma^*(\theta)$ for $\theta \in \Theta^n$. We define the "induced" order $\sigma^*(0, \theta_{-i})$ as follows:

$$\sigma_j^*(0, \theta_{-i}) = \begin{cases} \sigma_j^*(\theta) - 1 & \text{if } j \in F_i(\sigma^*(\theta)), \\ \sigma_j^*(\theta) & \text{if } j \in P_i(\sigma^*(\theta)), \\ n & j = i \end{cases} \tag{2.1}$$

In words, given $\theta \in \Theta^n$ and given any $i \in N$, $\sigma^*(0, \theta_{-i})$ is the order formed by setting the waiting cost of agent i at zero and hence moving agent i to the last position (following the outcome efficiency condition of Smith et al. (1956) by admitting zero waiting cost of agent i) so that only the agents in the set behind $F_i(\sigma^*(\theta))$ move up by one position under the outcome efficient queue for the induced profile $(0, \theta_{-i})$.

Definition 2. For a sequencing rule σ, a mechanism $\mu = (\sigma, \tau)$ is *strategyproof* (dominant strategy incentive compatible) if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i, \theta_i' \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta); \theta_i) \geq u_i(\mu_i(\theta_i', \theta_{-i}); \theta_i). \tag{2.2}$$

For a given sequencing rule σ, strategyproofness of a mechanism $\mu = (\sigma, \tau)$ requires that the transfer rule τ is such that truthful reporting for any agent weakly dominates false reporting no matter what others' report.

Definition 3. A mechanism μ satisfies *feasibility* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) \leq 0$.

Definition 4. A mechanism μ satisfies *budget balance* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) = 0$.

2.2.1 Generalized minimum welfare bounds

Given any sequencing problem $\Omega \in \mathcal{S}(N)$, let $O_i(s)$ be the welfare parameter of agent i. Let $O(N; s) := (O_1(s), \ldots, O_n(s)) \in \mathbb{R}^n$ denote the welfare parameter vector. We represent a typical sequencing problem with generalized minimum welfare bounds by $\Gamma = (\Omega, O(N;s))$ where $\Omega \in \mathcal{S}(N)$ and the associated $O(N;s) \in \mathbb{R}^n$ is the welfare parameter vector.

Definition 5. For Γ, a mechanism $\mu = (\sigma, \tau)$ satisfies *generalized minimum welfare bounds* (GMWB) if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta_i, \theta_{-i}); \theta_i) \geq -\theta_i O_i(s). \tag{2.3}$$

2.3 GMWB, outcome efficiency and strategyproofness

Given any sequencing game with generalized minimum welfare bounds, $\Gamma = (\Omega, O(N;s))$, we first try to identify the restriction on $O(N;s)$ for which we can get a mechanism satisfying outcome efficiency, strategyproofness and GMWB. The property defined below puts a constraint on the welfare parameter, indicating that an agent will always need to incur atleast the cost of his own processing time. Thus, the GMWB is no less than the cost of serving that agent when he occupies the first position in the queue.

Definition 6. Any sequencing problem with generalized minimum welfare bounds $\Gamma = (\Omega, O(N;s))$ satisfies the *constrained welfare property* if $O(N;s) = (O_1(s), \ldots, O_n(s))$ is such that

$$O_i(s) \geq s_i \;\; \forall\, i \in N. \tag{2.4}$$

Let $\mathcal{G}(N)$ be the set of all Γ satisfying the constrained welfare property given by condition (2.4).

Theorem 1. The following statements are equivalent:

(SPC1) For a Γ we can find a mechanism that satisfies outcome efficiency, strategyproofness and GMWB.

(SPC2) Γ satisfies the constrained welfare property, that is, $\Gamma \in \mathcal{G}(N)$.

Given any $\Gamma \in \mathcal{G}(N)$ what is the set of all mechanisms that satisfy outcome efficiency, strategyproofness and GMWB? The next result answers this question. Before going to the result we introduce some notations and definitions. For any agent $i \in N$ and any given profile $\theta_{-i} \in \Theta^{n-1}$, define the function

$$T_i(x_i; \theta_{-i}) := \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i})) + \{S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)\}x_i, \text{ where } x_i \in \mathbb{R}_+. \quad (2.5)$$

Observe that if $O_i(s) > A(s) = \sum_{j \in N} s_j$, then $S_i(\sigma^*(x_i, \theta_{-i})) < O_i(s)$ for all $x_i \in \Theta$ and hence the function $T_i(x_i; \theta_{-i})$ has no maximum value $x_i \in \Theta$ though the function has a least upper bound if we set $x_i = 0$. Hence, if $O_i(s) > A(s)$, we have $T_i(x_i; \theta_{-i}) < T_i(0; \theta_{-i}) < \infty$ for all $x_i \in \Theta$.[13] One can also verify that even if $O_i(s) = A(s)$, we have $T_i(x_i; \theta_{-i}) \leq T_i(0; \theta_{-i}) < \infty$ for all $x_i \in \Theta$. However, if $O_i(s) < s_i$, then $S_i(\sigma^*(x_i, \theta_{-i})) > O_i(s)$ for all $x_i \in \Theta$ and the function $T_i(x_i; \theta_{-i})$ has neither a maximum nor a least upper bound. Hence, for the function $T_i(x_i; \theta_{-i})$ defined on $x_i \in \Theta$ to have a least upper bound, the constrained welfare property (of Definition 6) is necessary.

Definition 7. An outcome efficient mechanism $\mu^p = (\sigma^*, \tau^p)$ is called a *relative pivotal mechanism* if τ^p satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^p(\theta) = \{S_i(\sigma^*(\theta_i^*, \theta_{-i})) - O_i(s)\}\theta_i^* + RP_i(\theta) + h_i(\theta_{-i}), \quad (2.6)$$

where, given the function $T_i(x_i; \theta_{-i})$ (defined in (2.5)), $\theta_i^* \in \mathbb{R}_+$ is such that $T_i(\theta_i^*; \theta_{-i}) \geq T_i(x_i; \theta_{-i})$ for all $x_i \in \Theta$, $RP_i(\theta) := \sum\limits_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i$ and $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$.

Let $\mathcal{R}(N)$ denote the set of all relative pivotal mechanisms defined in Definition 7.

[13] Given (2.1), the order $\sigma^*(0; \theta_{-i})$ is well-defined and hence the function $T_i(x_i; \theta_{-i})$ is well-defined at $x_i = 0$.

Theorem 2. For any $\Gamma \in \mathcal{G}(N)$, an outcome efficient mechanism $\mu = (\sigma^*, \tau)$ satisfies strategyproofness and GMWB if and only if it is a relative pivotal mechanism, that is, $\mu \in \mathcal{R}(N)$.

We try and explain Definition 7 and Theorem 2. It is well-known from Holmström (1979) that for outcome efficiency and strategyproof it is necessary that the mechanism $\mu = (\sigma^*, \tau)$ be a VCG mechanism where the transfers satisfy the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$ where $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary. The relative pivotal mechanism given in Definition 7 is a VCG mechanism which is obtained for each agent $i \in N$ and each profile $\theta \in \Theta^n$ by substituting $g_i(\theta_{-i}) = T_i(\theta_i^*; \theta_{-i}) + h_i(\theta_{-i})$ where $T_i(\theta_i^*; \theta_{-i})$ (resulting from the optimization exercise in Definition 7) and the restriction $h_i(\theta_{-i}) \geq 0$ are necessary to satisfy the GMWB. After appropriate simplification of the VCG transfer $\tau_i(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$ by using $g_i(\theta_{-i}) = T_i(\theta_i^*; \theta_{-i}) + h_i(\theta_{-i})$ we get that for all $\theta \in \Theta^n$ and all $i \in N$,

$$\tau_i^p(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + T_i(\theta_i^*; \theta_{-i}) + h_i(\theta_{-i}). \tag{2.7}$$

Simplifying (2.7) we get a subset of VCG mechanisms which we call relative pivotal mechanisms (Definition 7). From the proof of Theorem 2 it is clear that given any relative pivotal mechanism $\mu^p = (\sigma^*, \tau^p) \in \mathcal{R}(N)$, for any $\theta \in \Theta^n$ and any $i \in N$, $u_i(\mu_i^p(\theta_i, \theta_{-i}); \theta_i) = -\theta_i O_i(s) + \{T_i(\theta_i^*; \theta_{-i}) - T_i(\theta_i; \theta_{-i}) + h_i(\theta_{-i})\} \geq -\theta_i O_i(s)$ since $T_i(\theta_i^*; \theta_{-i}) - T_i(\theta_i; \theta_{-i}) + h_i(\theta_{-i}) \geq 0$. Hence, GMWB is satisfied for all agents.

The sum $RP_i(\theta) = \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i$ in condition (2.6) captures the relative pivotal nature of this sub-class of VCG mechanisms. Given any profile $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ the 'benchmark' type θ_i^* of agent i is obtained from the optimization exercise in Definition 7 and if this θ_i^* is taken along with $\theta_{-i} \in \Theta^{n-1}$, then the resulting benchmark outcome efficient order is $\sigma^*(\theta_i^*, \theta_{-i})$. Given any $\theta_i \in \Theta$, this benchmark order $\sigma^*(\theta_i^*, \theta_{-i})$ may or may not be the same as the actual outcome efficient order $\sigma^*(\theta_i, \theta_{-i})$ though the relative order across the agents other than i remains unchanged.[14] Given $\sigma^*(\theta_i^*, \theta_{-i})$ and $\sigma^*(\theta_i, \theta_{-i})$, we can have the three mutually exclusive and exhaustive possibilities-(i) $P_i(\sigma^*(\theta_i, \theta_{-i})) \subset P_i(\sigma^*(\theta_i^*, \theta_{-i}))$, (ii) $P_i(\sigma^*(\theta_i, \theta_{-i})) = P_i(\sigma^*(\theta_i^*, \theta_{-i}))$, and, (iii) $P_i(\sigma^*(\theta_i^*, \theta_{-i})) \subset P_i(\sigma^*(\theta_i, \theta_{-i}))$.

(R1) If $P_i(\sigma^*(\theta_i, \theta_{-i})) \subset P_i(\sigma^*(\theta_i^*, \theta_{-i}))$ (so that $\theta_i^* \in [0, \theta_i)$), then relative to $\sigma^*(\theta_i^*, \theta_{-i})$, agent i has inflicted an incremental cost of $\theta_j s_i$ to each agent $j \in P_i(\sigma^*(\theta_i^*, \theta_{-i}) \setminus P_i(\sigma^*(\theta_i, \theta_{-i}))$ under the actual order $\sigma^*(\theta_i, \theta_{-i})$. Hence, for any $j \in P_i(\sigma^*(\theta_i^*, \theta_{-i}) \setminus P_i(\sigma^*(\theta_i, \theta_{-i}))$, we get $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta_i, \theta_{-i}))| = -1$. Therefore, using the sum in (2.6) it follows that agent i has to pay

$$RP_i(\theta) = \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta_i, \theta_{-i}))|)\theta_j s_i = -\sum_{j \in P_i(\sigma^*(\theta_i^*, \theta_{-i}) \setminus P_i(\sigma^*(\theta_i, \theta_{-i}))} \theta_j s_i.$$

When can we have $\theta_i^* = 0$? If for any agent $i \in N$ we have $O_i(s) \geq A(s)$, then for every $\theta_{-i} \in \Theta^{n-1}$, $T_i(x_i; \theta_{-i})$ is decreasing in $x_i \in \Theta$ implying that by setting $\theta_i^* = 0$ we get $T_i(0; \theta_{-i}) \geq T_i(x_i, \theta_{-i})$ for all $x_i \in \Theta$. In this case,

$$RP_i(\theta) = \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(0, \theta_{-i}))| - |P_j(\sigma^*(\theta_i, \theta_{-i}))|)\theta_j s_i = -\sum_{j \in F_i(\sigma^*(\theta_i^*, \theta_{-i})} \theta_j s_i.$$

(R2) If $P_i(\sigma^*(\theta_i, \theta_{-i})) = P_i(\sigma^*(\theta_i^*, \theta_{-i}))$, then $\sigma^*(\theta_i^*, \theta_{-i}) = \sigma^*(\theta_i, \theta_{-i})$ and agent i has neither inflicted any incremental cost to any other agent nor has agent i induced any incremental benefit for any other agent, that is, $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| = |P_j(\sigma^*(\theta_i, \theta_{-i}))|$

[14] Specifically, for any $\sigma^*(\theta_i^*, \theta_{-i})$ and $\sigma^*(\theta_i, \theta_{-i})$, the relative order across the agents other than i remains unchanged means that for any $j, k \in N \setminus \{i\}$ with $j \neq k$, $\sigma_j^*(\theta_i^*, \theta_{-i}) > \sigma_k^*(\theta_i^*, \theta_{-i})$ if and only if and $\sigma_j^*(\theta_i, \theta_{-i}) > \sigma_k^*(\theta_i, \theta_{-i})$.

for all $j \in N$. Hence, using the sum in (2.6), it follows that

$$RP_i(\theta) = \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i, \theta_{-i}))| - |P_j(\sigma^*(\theta_i^*, \theta_{-i}))|)\theta_j s_i = 0$$

.

(R3) If $P_i(\sigma^*(\theta_i^*, \theta_{-i})) \subset P_i(\sigma^*(\theta_i, \theta_{-i}))$ (so that $\theta_i^* > \theta_i$), then relative to the outcome efficient order $\sigma^*(\theta_i^*, \theta_{-i})$, agent i has given an incremental benefit of $\theta_j s_i$ to each $j \in P_i(\sigma^*(\theta_i, \theta_{-i})) \setminus P_i(\sigma^*(\theta_i^*, \theta_{-i})$ under the outcome efficient order $\sigma^*(\theta_i, \theta_{-i})$. Hence, for any $j \in P_i(\sigma^*(\theta_i, \theta_{-i})) \setminus P_i(\sigma^*(\theta_i^*, \theta_{-i})$, we have $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta_i, \theta_{-i}))| = 1$. Thus, from the sum in (2.6), it follows that agent i gets a reward of

$$RP_i(\theta) = \sum_{j \in N \setminus \{i\}} (|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta_i, \theta_{-i}))|)\theta_j s_i = \sum_{j \in P_i(\sigma^*(\theta_i, \theta_{-i})) \setminus P_i(\sigma^*(\theta_i^*, \theta_{-i})} \theta_j s_i.$$

Therefore, (R1), (R2) and (R3) explains how the sum $RP_i(\theta)$ in (2.6) for agent i with type θ_i, given θ_{-i} is calculated based on the difference in the cost of all other agents $N \setminus \{i\}$ that results from the actual profile specific outcome efficient order $\sigma^*(\theta_i, \theta_{-i})$ relative to the benchmark outcome efficient order $\sigma^*(\theta_i^*, \theta_{-i})$. What follows from the above discussion is that for all $\theta \in \Theta^n$ and each $i \in N$, either $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))| \in \{-1, 0\}$ for all $j \in N \setminus \{i\}$ or $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))| \in \{0, 1\}$ for all $j \in N \setminus \{i\}$. Equivalently, we cannot find a profile $\theta \in \Theta^n$ and an agent $i \in N$ such that $|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))| = -1$ for some agent $j \in N \setminus \{i\}$ and $|P_k(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_k(\sigma^*(\theta))| = 1$ for other agent $k \in N \setminus \{i, j\}$.

2.3.1 Feasibility and budget balance

Before going to our results on identifying relative pivotal mechanisms that ensures outcome efficiency, strategyproofness, GMWB and feasibility, we first drop the strategyproofness requirement and provide a necessary restriction for getting mechanisms that satisfy outcome efficiency, GMWB and feasibility.

Definition 8. A sequencing problem with generalized minimum welfare bounds $\Gamma = (\Omega, O(N,s)) \in \mathcal{G}(N)$ satisfies the property of *weighted net welfare* if

$$\mathcal{D}(O(N,s)) := \sum_{j \in N} s_j \left\{ O_j(s) - \left(\frac{s_j + A(s)}{2} \right) \right\} \geq 0. \tag{2.8}$$

For any sequencing problem with generalized minimum welfare bounds $\Gamma = (\Omega, O(N,s))$ with $O_i(s) = s_i$ for all $i \in N$, condition (2.8) fails to hold. For any sequencing problem with generalized minimum welfare bounds $\Gamma = (\Omega, O(N,s))$ with $O_i(s) \geq (s_i + A(s))/2$ for all $i \in N$, condition (2.8) is satisfied. Let $\overline{\mathcal{G}}(N) (\subset \mathcal{G}(N))$ denote the set of all sequencing problems with GMWB satisfying the constrained welfare property and the weighted net welfare.

Lemma 1. If for any $\Gamma = (\Omega, O(N,s)) \in \mathcal{G}(N)$, we can find a mechanism that satisfies outcome efficiency, GMWB and feasibility, then Γ must satisfy the weighted net welfare, that is, $\Gamma \in \overline{\mathcal{G}}(N)$.

Remark 1. For any sequencing problem with generalized minimum welfare bounds $\Gamma = (\Omega, O(N,s))$, a good way to explain condition (2.8) is in terms of mean $\mu(s)$, variance $V(s)$ and coefficient of variation $CoV(s) := \sqrt{V(s)}/\mu$ of the elements of the processing time vector $s = (s_1, \ldots, s_n)$. Specifically, an equivalent way of representing condition (2.8) is the following:

$$\sum_{j \in N} w_j(s) O_j(s) \geq \frac{\mu(s)}{2} \left[n + 1 + \{CoV(s)\}^2 \right], \tag{2.9}$$

where $w_i(s) := s_i/A(s)$ for all $i \in N$.[15]

(i) If we have the queueing problem, that is if $\Omega \in \mathcal{Q}(N)$ with $s_1 = \ldots = s_n = a > 0$, then $\mu(s) = a$, $CoV(s) = 0$ and $w_i(s) = 1/n$ for all $i \in N$. Condition (2.9) holds if and only if $\sum_{j \in N} O_j(s)/n \geq (n+1)a/2$. Moreover, if we also require that the generalized minimum welfare bound of all the agents are identical, that is $O_i(s) = B^*$ for all $i \in N$, then condition (2.9) requires $B^* \geq (n+1)a/2$.

(ii) It is well-known that $CoV(s) \leq \sqrt{n-1}$ for any positive integer n and any $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$. Therefore, a sufficient condition for (2.9) to hold for any sequencing problem with generalized minimum welfare bounds $\Gamma = (\Omega, O(N,s))$ is obtained by substituting $CoV(s) = \sqrt{n-1}$ in (2.9) that yields $\sum_{j \in N} w_j(s) O_j(s) \geq n\mu(s) = A(s)$.

Remark 2. Fix any N and any $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$. Let $\mathcal{O}(N,s)$ denote the set of welfare parameter vectors $O(N,s) = (O_1(s), \ldots, O_n(s))$ satisfying the constrained welfare property and the weighted net welfare. It is obvious that the set $\mathcal{O}(N,s)$ is non-empty and convex. It is non-empty since for $\bar{O}(N,s) = (\bar{O}_1(s), \ldots, \bar{O}_n(s))$ with $\bar{O}_i(s) = (s_i + A(s))/2$ for all $i \in N$, inequality (2.8) holds. For convexity of $\mathcal{O}(N,s)$, observe that if $O(N,s), O'(N,s) \in \mathcal{O}(N,s)$ so that $\mathcal{D}(O(N,s)) \geq 0$ and $\mathcal{D}(O'(N,s)) \geq 0$, then, given (2.8) it easily follows that for any $\lambda^* \in [0,1]$ we get $\mathcal{D}(\lambda^* O(N,s) + (1-\lambda^*)O'(N,s)) = \lambda^* \mathcal{D}(O(N,s)) + (1-\lambda^*)\mathcal{D}(O'(N,s)) \geq 0$ implying $\lambda^* O(N,s) + (1-\lambda^*)O'(N,s) \in \mathcal{O}(N,s)$. For any $i \in N$, define $E_i(s) := s_i + \left(\sum_{j \in N} s_j \sum_{k \in N \setminus \{j\}} s_k\right)/s_i$ and $O^i(N,s) := (E_i(s), s_{-i})$.[16] It is easy to verify that for any $i \in N$, $O^i(N,s) = (E_i(s), s_{-i}) \in \mathcal{O}(N,s)$ since $\mathcal{D}(O^i(N,s)) = 0$. Moreover, given (2.8) it is also obvious that for any $i \in N$ and any $\underline{O}(N,s) \in \mathcal{R}^n_{++}$ such that $O^i(N,s) \geq \underline{O}(N,s)$ and $\underline{O}(N,s) \neq O^i(N,s)$, we have $\underline{O}(N,s) \notin \mathcal{O}(N,s)$. Therefore, for any $i \in N$, $O^i(N,s)$ is a boundary point of the set

[15] To derive inequality (2.9) we have used the following equalities: $\sum_{j \in N} s_j^2 = nVar(s) + n\{\mu(s)\}^2 = n\{\mu(s)\}^2\{1 + Cov(s)\} = A(s)\mu(s)\{1 + Cov(s)\}$.

[16] Note that if $|N| = 2$, then $E_i = A(s)$ for any $i \in N$.

$\mathcal{O}(N,s)$. Further, for the same type of reasoning, $\bar{O}(N,s) = (\bar{O}_1(s),\ldots,\bar{O}_n(s)) \in \mathcal{O}(N,s)$ such that $\bar{O}_i(s) = (s_i + A(s))/2$ for all $i \in N$ is also a boundary point of $\mathcal{O}(N,s)$. However, one can verify that $\sum_{j\in N} w_j(s) O^j(N,s) = \bar{O}(N,s)$, that is, $\bar{O}(N,s)$ is a weighted sum of the elements of the set $\{\{O^i(N,s)\}_{i\in N}\}$ with weight $w_i(s) = s_i/A(s)$ for each $i \in N$. The set $\{\{O^i(N,s)\}_{i\in N}\}$ plays a key role in explaining the set $\mathcal{O}(N,s)$. For any $\lambda = (\lambda_1,\ldots,\lambda_n) \in [0,1]^n$ with $\sum_{j\in N}\lambda_j = 1$, consider the vector $\sum_{j\in N}\lambda_j O^j(N,s) = (\lambda_1 E_1(s) + (1-\lambda_1)s_1,\ldots,\lambda_n E_n(s) + (1-\lambda_n)s_n)$. One can verify that $\mathcal{O}(N,s)$ is a non-empty and convex set given by

$$\mathcal{O}(N,s) = \left\{O(N,s) \in \mathbb{R}^N_{++} \mid \exists\lambda \in [0,1]^n \text{ with } \sum_{j\in N}\lambda_j = 1, \text{ s.t. } O(N,s) \geq \sum_{j\in N}\lambda_j O^j(N,s)\right\}. \tag{2.10}$$

Therefore, the set $\mathcal{O}(N,s)$ is non-empty and convex with the added property that any element in this set weakly vector dominates some weighted sum of the elements of the set $\{\{O^i(N,s)\}_{i\in N}\}$.

Given Lemma 1, from now on we restrict our attention only to the set $\overline{\mathcal{G}}(N)$ of all sequencing problems with GMWB satisfying the constrained welfare property and the weighted net welfare, that is, we consider any $\Gamma = (\Omega, O(N,s))$ such that $O(N,s) \in \mathcal{O}(N,s)$ and the set $\mathcal{O}(N,s)$ is given by (2.10) of Remark 2.

Definition 9. An outcome efficient mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ is called a *minimal relative pivotal mechanism* if it is a relative pivotal mechanism with the property that for all $i \in N$ and all $\theta_{-i} \in \Theta^{n-1}$, $h_i(\theta_{-i}) = 0$, that is, for any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\hat{\tau}^p_i(\theta) = \{S_i(\sigma^*(\theta^*_i, \theta_{-i})) - O_i(s)\}\theta^*_i + RP_i(\theta), \tag{2.11}$$

where $\theta^*_i \in \mathbb{R}_+$ ensures $T_i(\theta^*_i; \theta_{-i}) \geq T_i(x_i; \theta_{-i})$ for all $x_i \in \Theta$ and we have
$RP_i(\theta) = \sum_{j\in N\setminus\{i\}} (|P_j(\sigma^*(\theta^*_i, \theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i$.

Observe that if a relative pivotal mechanism $\mu^p = (\sigma^*, \tau^p) \in \mathcal{R}(N)$ is feasible, then the minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ is also feasible since for any $\theta \in \Theta^n$ and any $i \in N$, $\tau_i^p(\theta) - \hat{\tau}_i^p(\theta) = h_i(\theta_{-i}) \geq 0$. Therefore, for any $\Gamma \in \mathcal{G}(N)$, if we want to check whether we can find a feasible relative pivotal mechanism or not, we simply need to check the prospect of feasibility with the minimal relative pivotal mechanism $\hat{\mu}^p$.

Proposition 1. For any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ such that $|N| = 2$ we have the following results:

(B2a) A feasible relative pivotal mechanism exists if and only if $O_1(s) \geq A(s)$ and $O_2(s) \geq A(s)$.

(B2b) There is no budget balanced relative pivotal mechanism.

Can we find budget balanced relative pivotal mechanisms for sequencing problems with GMWB satisfying the constrained welfare and the weighted net welfare when there are more than two agents?

Proposition 2. For any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ such that $|N| \geq 3$ and $O_i(s) \geq A(s)$ for all $i \in N$, we can find budget balanced relative pivotal mechanisms.

Remark 1 (ii) states that the *weighted average of the welfare parameters is no less than the aggregate processing time* (specifically, $\sum_{j \in N} w_j(s) O_j(s) \geq A(s)$) is a sufficient condition for weighted net welfare property. Proposition 1 shows that the *welfare parameter of each agent is no less than the aggregate processing time* is necessary and sufficient for feasibility relative pivotal mechanisms when there are two agents and Proposition 2 shows that the same condition is sufficient to get budget balanced relative pivotal mechanism when there are more than two agents. What can we say about obtaining feasible relative pivotal mechanism for any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ such that $|N| \geq 3$ and there exists at least one agent with $O_i(s) \in (s_i, A(s))$? It is difficult to answer this question in general as the transfers associated with any relative pivotal mechanism lacks closed form representation. However, the following example suggests that one would expect to get more restriction on the processing time of the agents (over and above what is required under the constrained welfare and weighted net welfare properties) to get feasible relative pivotal mechanisms.

Example 1. Consider any $\Gamma = (\Omega, O(N,s)) \in \mathcal{G}(N)$ such that $|N| = 3$ and $O_i(s) = s_i + \max_{j \neq i} s_j$ for all $i \in N$. Without loss of generality, assume that $s_1 \geq s_2 \geq s_3$. Observe that condition (2.8) holds since $\mathcal{D}(s) = s_1(s_2 - s_3)/2 + s_2(s_1 - s_3)/2 + s_3(s_1 - s_2)/2 \geq 0$. Hence, $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$. Consider the profile $\theta \in \Theta^3$ such that $\sigma^*_j(\theta) = n + 1 - j$ for all $j \in N$ and in particular $\theta_3/s_3 = a > \theta_2/s_2 = b > \theta_1/s_1 = c > 0$. Using the function $T_i(x_i; \theta_{-i})$ (in (2.5)), we can fix $\theta^*_1 = s_1 b, \theta^*_2 = s_2 c$ and $\theta^*_3 = s_3 c$. Then, using the transfers associated with the minimal relative pivotal mechanism (Definition 9), we get the following:

1. $\hat{\tau}_1(\theta) = s_1 s_3 b$,

2. $\hat{\tau}_2(\theta) = -c s_2(s_1 - s_3)$, and

3. $\hat{\tau}_3(\theta) = -c s_3(s_1 - s_2) - s_3 s_2 b$.

If $s_1 > s_2$ and $a > b > c + c[s_2(s_1 - s_3)/s_3(s_1 - s_2)]$, then $\sum_{j \in N} \hat{\tau}_j(\theta) = (b - c)s_3(s_1 - s_2) - c s_2(s_1 - s_3) > 0$ and feasibility gets violated. Hence, for feasibility to hold it is necessary that $s_1 = s_2 \geq s_3$ which is a restriction on the processing time vector $s = (s_1, s_2, s_3)$.

2.4 Applications

2.4.1 Sequencing with a given initial order

For a sequencing problem $\Omega \in \mathcal{S}(N)$ with initial order, there is a preexisting order in which the agents have arrived to use the facility and the job processing starts only after all agents have arrived to use the facility. This problem is the natural extension of the problem of reordering an existing queue (addressed by Chun et al. (2017) and by Gershkov & Schweinzer (2010)) to the sequencing problem. Suppose that initial order of arrival is $\sigma^0 \in \Sigma$. In this case, the welfare parameter vector is $O^{\sigma^0}(N,s) = (O_1^{\sigma^0}(s), \ldots, O_n^{\sigma^0}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{\sigma^0}(s) = s_i + \sum_{j \in P_i(\sigma^0)} s_j$ and hence for any profile $\theta \in \Theta^n$, $\sum_{j \in N} \theta_j O_j^{\sigma^0}(s) = C(\sigma^0, \theta)$. Let $\mathcal{I}(N) = \{(\Omega, O^{\sigma^0}(N,s)) \mid \Omega \in \mathcal{S}(N), \sigma^0 \in \Sigma\}$ denote the set of all sequencing problems with initial order. Every $(\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ satisfies the constrained welfare property since for each $i \in N$, $O_i^{\sigma^0}(s) = s_i + \sum_{j \in P_i(\sigma^0)} s_j \geq s_i$. Moreover importantly, every $(\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ satisfies the weighted net welfare since $\mathcal{D}(s) = \sum_{j \in N} s_j \{S_j(\sigma^0) - (s_j + A(s))/2\} = \sum_{j \in N}(s_j/2)\{(\sum_{k \in P_j(\sigma^0)} s_k - \sum_{k \in F_j(\sigma^0)} s_k\} = \sum_{j \in N} \sum_{k \in P_j(\sigma^0)} (s_j s_k/2) - \sum_{j \in N} \sum_{k \in F_j(\sigma^0)} (s_j s_k/2) = 0$ implying that condition (2.8) holds.[17] Hence, we get $\mathcal{I}(N) \subset \overline{\mathcal{G}}(N)$. One can check that the special feature of the relative pivotal mechanisms is that the function $T_i(x_i; \theta_{-i})$ (defined in (2.5)) has the following form:

$$T_i^I(x_i; \theta_{-i}) = \left[\sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j - \sum_{j \in P_i(\sigma^0)} s_j \right] x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i})).^{18} \tag{2.12}$$

2.4.2 Identical cost bounds

Identical cost bounds (ICB) requires that each agent $i \in N$ receives at least the utility he could expect if all agents were like him (both in terms of waiting cost as well as in terms of processing time) in a reference economy. This means that each agent $i \in N$ in his reference economy has an equal chance of facing each order from Σ. Thus, ICB requires that for any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i((n+1)s_i/2)$ where $\theta_i((n+1)s_i/2)$ represents the expected cost of agent i with waiting cost θ_i and processing time s_i when all agents have the same processing

[17] The reason for the last equality is the following: For any two agents $j, k \in N$, $\{k \in P_j(\sigma^0) \Leftrightarrow j \in P_k(\sigma^0)\}$ which implies that for any term of the form $s_j s_k/2$, there is exactly one term of the form $-s_j s_k/2$ that cancels it out.

[18] Note that for any $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ and any $x_i \in \mathbb{R}_+$, $\{S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)\}x_i = \left[\sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j + s_i - \sum_{j \in P_i(\sigma^0)} s_j - s_i \right] x_i = \left[\sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j - \sum_{j \in P_i(\sigma^0)} s_j \right] x_i$.

time s_i and agent i gets each of the positions 1 to n with probability $1/n$. For a sequencing problem $\Omega \in \mathcal{S}(N)$ with generalized minimum welfare bounds given by ICB, the welfare parameter vector is $O^s(N, s) = (O_1^{s_1}(s), \ldots, O_n^{s_n}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{s_i}(s) = (n+1)s_i/2$. Let $\mathcal{C}(N) = \{(\Omega, O^s(N, s)) \mid \Omega \in \mathcal{S}(N)\}$ denote the set of all sequencing problems with ICB and let Γ^s represent a typical sequencing problem with ICB in $\mathcal{C}(N)$. Since for any $(\Omega, O^s(N, s)) \in \mathcal{C}(N)$, $O_i^{s_i}(s) = (n+1)s_i/2 > s_i$ for every $i \in N$, the constrained welfare property is satisfied. Moreover, $\mathcal{D}(s) = \sum_{j \in N} s_j\{(n+1)s_j/2 - (s_j + A(s))/2\} = \sum_{j \in N} s_j\{\sum_{k \neq j}(s_j - s_k)\} = \sum_{j=1}^{n-1}\sum_{k>j}(s_j - s_k)^2 \geq 0$ and hence condition (2.8) also holds. Therefore, $\mathcal{C}(N) \subset \overline{\mathcal{G}}(N)$. One can easily verify that the special feature of the relative pivotal mechanisms in this context is that the function $T_i(x_i; \theta_{-i})$ (provided in (2.5)) has the following form:

$$T_i^C(x_i; \theta_{-i}) = \left[\sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j - \frac{(n-1)s_i}{2}\right] x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_i(\sigma^*(x_i, \theta_{-i})).^{19} \tag{2.13}$$

2.4.3 Expected cost bounds

The expected cost bounds (ECB) requires that the utility of each agent is no less than the expected cost of the agent associated with random arrival where each arrival order is equally likely. Formally, ECB requires the following property: For any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i \left(\sum_{\sigma \in \Sigma} \frac{S_i(\sigma)}{n!}\right)$. Define $\bar{S}_i := s_i + \sum_{j \in N \setminus \{i\}}(s_j/2)$ for each $i \in N$. It is quite easy to verify that for

[19] Observe that for any $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ and any $x_i \in \mathbb{R}_+$,

$$\{S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)\}x_i = \left[\sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j + s_i - \frac{(n+1)s_i}{2}\right] x_i = \left[\sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j - \frac{(n-1)s_i}{2}\right] x_i.$$

each agent $i \in N$, $\sum_{\sigma\in\Sigma} \frac{S_i(\sigma)}{n!} = \bar{S}_i$.[20] Therefore, an equivalent representation of the ECB requirement is that for any agent $i \in N$ and any profile $\theta \in \Theta^n$, $u_i(\sigma(\theta), \tau_i(\theta); \theta_i) \geq -\theta_i \bar{S}_i$.

For a sequencing problem $\Omega \in \mathcal{S}(N)$ with generalized minimum welfare bounds given the ECB conditions, the welfare parameter vector is $O^{\bar{S}}(N,s) = (O_1^{\bar{S}_1}(s), \ldots, O_n^{\bar{S}_n}(s)) \in \mathbb{R}^n_{++}$ where for each $i \in N$, $O_i^{\bar{S}_i}(s) = \bar{S}_i$. Let $\mathcal{E}(N) = \{(\Omega, O^{\bar{S}}(N,s)) \mid \Omega \in \mathcal{S}(N)\}$ denote the set of all sequencing problems with ECB and let $\Gamma^{\bar{S}}$ represent a typical sequencing problem with ECB in $\mathcal{E}(N)$. All sequencing problems with ECB as their generalized minimum welfare bounds satisfy the constrained welfare property. In particular, observe that for any $\Gamma^{\bar{S}} \in \mathcal{E}(N)$ and any $i \in N$, $O_i^{\bar{S}_i}(s) = \bar{S}_i = s_i + \sum_{j\in N\setminus\{i\}}(s_j/2) > s_i$ implying that the constrained welfare property given by condition (2.4) holds. Further, $\mathcal{D}(s) = \sum_{j\in N} s_j\{(s_j + A(s))/2 - (s_j + A(s))/2\} = 0$ and hence condition (2.8) also holds. Therefore, $\mathcal{E}(N) \subset \overline{\mathcal{G}}(N)$. One can verify that the special feature of the relative pivotal mechanisms in this context is that the function $T_i(x_i; \theta_{-i})$ (in condition (2.5)) has the following form:

$$T_i^E(x_i;\theta_{-i}) = \left[\sum_{k\in P_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2} - \sum_{k\in F_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2}\right] x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_i(\sigma^*(x_i,\theta_{-i})).^{21} \qquad (2.14)$$

Remark 3. Clearly, the bounds associated with ICB and ECB are different for any sequencing problem which is not a queueing problem, that is, for any $\Omega \in \mathcal{S}(N) \setminus \mathcal{Q}(N)$. However, for any queueing problem $\Omega \in \mathcal{Q}(N)$ with $s_1 = \ldots = s_n = a > 0$, $\bar{S}_i = (n+1)a/2$ for all $i \in N$ implying that the

[20] The equality $\sum_{\sigma\in\Sigma} \frac{S_i(\sigma)}{n!} = \bar{S}_i$ states that the average completion time of each agent i equals $\bar{S}_i$. The sum in $\bar{S}_i$ has two components-own processing time s_i and half of the total processing time of all other agents $j \neq i$. In any possible ordering $\sigma \in \Sigma$, an agent will always incur his own processing time and hence s_i enters $\bar{S}_i$ with probability one. Moreover, observe that any other agent $j \neq i$ precedes agent i in any ordering σ if and only if he does not precede agent i in the complement ordering σ^c. Therefore, when we consider all possible orderings to calculate agent i's average completion time, s_j for $j \neq i$ will occur in exactly half of the cases as a part of the completion time of agent i.

[21] Observe that for any $i \in N$, any $\theta_{-i} \in \Theta^{n-1}$ and any $x_i \in \mathbb{R}_+$, $\{S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)\}x_i = \left[\sum_{j\in P_i(\sigma^*(x_i,\theta_{-i}))} s_j + s_i - \sum_{j\in N\setminus\{i\}} \frac{s_j}{2} - s_i\right] x_i = \left[\sum_{k\in P_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2} - \sum_{k\in F_i(\sigma^*(x_i,\theta_{-i}))} \frac{s_k}{2}\right] x_i$.

notions of ICB and ECB are equivalent.

2.4.4 Feasibility and budget balance

Sequencing with given initial order

Using Proposition 1 it follows that if we consider any two agent sequencing problem with initial order $(\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$, then we cannot find a mechanism that satisfies outcome efficiency, strategyproofness, GMWB and feasibility since for any agent (i say) having first position in the initial order σ^0, $O_i(s) = s_i < A(s)$. The discussion to follow shows that this impossibility result holds in general for any sequencing problems with given initial order.

Remark 4. Consider any $\Gamma^0 = (\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ such that $|N| \geq 3$. We provide certain observations about the minimal relative mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ with the $T_i(x_i; \theta_{-i})$ function given by condition (2.12).

(IO1) Let $i \in N$ be that agent having first queueing position under that initial order σ^0, that is, $S_i(\sigma^0) = s_i$. Then, for any profile $\theta \in \Theta^n$, $\theta_i^* = s_i.\{\max\{\theta_j/s_j\}_{j\in N\setminus\{i\}}\}$ is a solution to the maximization of the function $T_i^I(x_i : \theta_{-i})$ and we select $\sigma^*(\theta_i^*, \theta_{-i})$ such that $P_i(\sigma^*(\theta_i^*, \theta_{-i})) = P_i(\sigma^0) = \emptyset$. Therefore, we have $\theta_i^*[S_i(\sigma^*(\theta_i^*, \theta_{-i})) - O_i(s)] = \theta_i^*[s_i - s_i] = 0$ and hence using (2.12) it follows that the transfer associated with the minimal relative pivotal mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ for agent $i \in N$ is

$$\hat{\tau}_i(\theta) = s_i \sum_{j \in P_i(\sigma^*(\theta))} \theta_j.$$

(IO2) Let $k \in N$ be that agent having last queueing position under that initial order σ^0, that is, $S_i(\sigma^0) = A(s) = \sum_{j\in N} s_j$. Then, using argument similar to the one used in (R1), it follows that for any $\theta \in \Theta^n$, $\theta_k^* = 0$ and $P_k(\sigma^*(0, \theta_{-k})) = P_i(\sigma^0) = N \setminus \{k\}$. Therefore, we have $\theta_i^*[S_i(\sigma^*(0_i,\theta_{-i})) - O_i(s)] = \theta_i^*[A(s) - A(s)] = 0$ and hence using (2.12) it follows that the transfer associated with the minimal relative pivotal mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ for agent $k \in N$ is

$$\hat{\tau}_k(\theta) = -s_k \sum_{j \in F_k(\sigma^*(\theta))} \theta_j.$$

Points (IO1) and (IO2) of Remark 4 show that given a sequencing problem with initial order σ^0, the explicit form of the minimal relative pivotal transfers of the agents having the first and last positions under the initial order σ^0 are easy to derive. However, it is difficult to get an explicit form of the minimal relative pivotal transfers for agents having other positions under the initial order σ^0. Despite this difficulty, using points (IO1) and (IO2) of Remark 4 and by appropriate construction of a profile we can prove the following impossibility result.

Proposition 3. For any $\Gamma^0 = (\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ with $|N| \geq 3$, there is no mechanism that satisfies outcome efficiency, strategyproofness, GMWB and feasibility.

ICB and ECB

Using Proposition 1 one can show that if we consider $(\Omega, O^c(N,s)) \in \mathcal{C}(N)$ with two agents $N = \{1,2\}$, then we cannot find a mechanism that satisfies outcome efficiency, strategyproofness, GMWB and feasibility since we require $3s_1/2 \geq A(s)$ and $3s_2/2 \geq A(s)$ to hold simultaneously which is impossible. Similarly, using Proposition 1 one can also show that if we consider $(\Omega, O^{\bar{S}}(N,s)) \in \mathcal{E}(N)$ with two agents $N = \{1,2\}$, then we cannot find a mechanism that satisfies outcome efficiency, strategyproofness, GMWB and feasibility since, for each $i,j \in \{1,2\}$ with $i \neq j$, we have $s_i + s_j/2 < A(s) = s_1 + s_2$. What happens when we have more that two agents?

Proposition 4. For any $(\Omega, O(N,s)) \in \mathcal{C}(N) \cup \mathcal{E}(N)$ such that $|N| = 3$, if we can find a feasible relative pivotal mechanism, then $\Omega \in \mathcal{Q}(N)$.

Proposition 4 states that when there are three agents, if we can find a mechanism satisfying outcome efficiency, strategyproofness, feasibility and, either ICB or ECB, then we must have a queueing problem. It is well-known from the existing literature on queueing problems that, when there are three or more agents we can find mechanisms that satisfy budget balance along with outcome efficiency, strategyproofness and ICB (or ECB).[22] Therefore, before concluding, we analyze queueing problems with GMWB in greater details.

2.5 Queueing problems

Throughout this section we assume without loss of generality that $s_1 = \ldots = s_n = 1$, and, given any queueing problem $\Omega \in \mathcal{Q}(N)$, we define the welfare parameter vector as $O(N) = (O_1, \ldots, O_n) \in \mathbb{R}^n$. Therefore, we represent any queueing problem with GMWB as $\Gamma^{\mathcal{Q}} = (\Omega, O(N))$. Any $\Gamma^{\mathcal{Q}} = (\Omega, O(N))$ satisfies the constrained welfare property if $O(N) = (O_1, \ldots, O_n)$ is such that $O_i \geq 1$ for all $i \in N$. One can easily verify that the special feature of the relative pivotal mechanisms in this context is that the function $T_i(x_i; \theta_{-i})$ (given by (2.5)) has the following form:

$$T_i^{Q}(x_i; \theta_{-i}) = [\sigma_i^*(x_i, \theta_{-i}) - O_i]\, x_i + \sum_{j \in N \setminus \{i\}} \sigma_j^*(x_i, \theta_{-i})\theta_j. \tag{2.15}$$

For any queueing problem $\Omega \in \mathcal{Q}(N)$, the welfare parameter vector associated with either ICB or ECB is $O^B(N) = (O_1^B, \ldots, O_n^B)$ where $O_i^B = \frac{n+1}{2}$ for all $i \in N$ (see Remark 3). Given (2.15) we get that the function $T_i^{Q}(x_i; \theta_{-i})$ has the following form:

$$T_i^{QB}(x_i; \theta_{-i}) = \left[\sigma_i^*(x_i, \theta_{-i}) - \frac{(n+1)}{2}\right] x_i + \sum_{j \in N \setminus \{i\}} \sigma_j^*(x_i, \theta_{-i})\theta_j. \tag{2.16}$$

[22] See Chun & Mitra (2014), Chun et al. (2019a) and Kayı & Ramaekers (2010) for a detailed discussions on symmetrically balanced VCG mechanisms.

The discussion to follow identifies the explicit forms of the relative pivotal mechanisms.

Definition 10. For σ^* and for any positive integer $K \leq |N|$, a mechanism $\mu^k = (\sigma^*, \tau^{(K)})$ is a *K-pivotal mechanism* if for any $\theta \in \Theta^n$ and any $i \in N$,

$$\tau_i^{(K)}(\theta) = \begin{cases} -\sum\limits_{j:\sigma_i^*(\theta) < \sigma_j^*(\theta) \leq K} \theta_j & \text{if } \sigma_i^*(\theta) < K, \\ 0 & \text{if } \sigma_i^*(\theta) = K, \\ \sum\limits_{j:K \leq \sigma_j^*(\theta) < \sigma_i^*(\theta)} \theta_j & \text{if } \sigma_i^*(\theta) > K. \end{cases} \tag{2.17}$$

See Mitra & Mutuswami (2011) who introduce and characterize the K-pivotal mechanisms for the queueing problems. Chun & Yengin (2017) also provide another characterization of these mechanism. We define a new set of mechanisms which are obtained by appropriately mixing different K-pivotal mechanisms.

Definition 11. For any queueing problem, a mechanism $\bar{\mu}^a = (\sigma^*, \bar{\tau}^a)$ is a *centered K-pivotal mechanism with non-negative intercepts* if for all $\theta \in \Theta^n$ and all $i \in N$,

$$\bar{\tau}_i^a(\theta) = H_i(\theta_{-i}) + \begin{cases} \tau_i^{\left(\frac{n+1}{2}\right)}(\theta) & \text{if } n \text{ is odd}, \\ \frac{1}{2}\tau_i^{\left(\frac{n}{2}\right)}(\theta) + \frac{1}{2}\tau_i^{\left(\frac{n}{2}+1\right)}(\theta) & \text{if } n \text{ is even}, \end{cases} \tag{2.18}$$

where for each $i \in N$, the function $H_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$.

Corollary 1. For any queueing problem $\Omega \in \mathcal{Q}(N)$, a mechanisms satisfies outcome efficiency, strategyproofness and ICB (ECB) if and only if it is a centered K-pivotal mechanism with non-negative intercepts.

Corollary 1 generalizes a result by Chun & Yengin (2017) on outcome efficient, strategyproofness and ICB (ECB) by eliminating the gap between their necessary and sufficient conditions.

Symmetrically balanced VCG mechanism

The symmetrically balanced VCG mechanism is defined for any queueing problem with three or more agents as follows.

Definition 12. Assume $|N| \geq 3$. The mechanism $\mu^S = (\sigma^*, \tau^S)$ is the *symmetrically balanced VCG mechanism* if for all profiles $\theta \in \Theta^n$ and all $i \in N$,

$$\tau_i^S(\theta) = \sum_{j \in P_i(\sigma^*(\theta))} \left(\frac{\sigma_j^*(\theta) - 1}{n-2} \right) \theta_j - \sum_{j \in F_i(\sigma^*(\theta))} \left(\frac{n - \sigma_j^*(\theta)}{n-2} \right) \theta_j. \tag{2.19}$$

From the existing literature on queueing problems it is well known that the symmetrically balanced VCG mechanism satisfies outcome efficiency, strategyproofness and ICB (ECB) when there are three or more agents (see Chun & Mitra (2014), Chun et al. (2019a) and Kayı & Ramaekers (2010)). Given Corollary 1 it means that the symmetrically balanced VCG mechanism is a centered K-pivotal mechanism with non-negative intercept when there are three or more agents. Given more than two agents, consider that centered K-pivotal mechanism with non-negative intercept for which the $H_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$ function for any $i \in N$ and any $\theta_{-i} \in \Theta^{|N| \setminus \{i\}}$ has the following form:

$$H_i(\theta_{-i}) = \begin{cases} \sum_{k=1}^{\frac{n}{2}-1} \left(\frac{k-1}{n-2}\right) \left\{\theta_{(k)}(\theta_{-i}) - \theta_{(n-k)}(\theta_{-i})\right\} & \text{if } n \text{ is even and } n \geq 4, \\ \sum_{k=1}^{\frac{n-1}{2}} \left(\frac{k-1}{n-2}\right) \left\{\theta_{(k)}(\theta_{-i}) - \theta_{(n-k)}(\theta_{-i})\right\} & \text{if } n \text{ is odd and } n \geq 3 \end{cases} \tag{2.20}$$

where for any $k \in \{1, \ldots, n-1\}$, $\theta_{(k)}(\theta_{-i})$ is the k-th ranked waiting cost from the profile $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$ so that $\theta_{(1)}(\theta_{-i}) \geq \ldots \geq \theta_{(n-1)}(\theta_{-i})$. One can verify that with the $H_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$ function given by (2.20), the resulting centered K-pivotal mechanism with non-negative intercept is the symmetrically balanced VCG mechanism.

2.5.1 Feasibility and budget balance

From Proposition 1 it follows if there are two agents, then for a queueing problem $\Omega \in \mathcal{Q}(\{1,2\})$ with the welfare parameter vector $O(\{1,2\}) = (O_1, O_2)$ we can find a mechanism satisfying outcome efficiency, strategyproofness, GMWB and feasibility if and only if $O_1 \geq 2$ and $O_2 \geq 2$.

From Lemma 1 it follows that for any queueing problem we can find mechanisms satisfying outcome efficiency, GMWB and feasiblity only if condition (2.8) holds. Condition (2.8) for any queueing problem reduces to the following inequality: $\sum_{j \in N} O_i/n \geq (n+1)/2$ (see Remark 1(i)). This inequality requires that the average of the welfare parameters of all the agents should be no less than $(n+1)/2$. The next result shows that if the wefare parameter of every agent is no less than $(n+1)/2$, then we can find mechanisms that satisfy outcome efficiency, strategyproofness, GMWB and budget balance.

Proposition 5. For any $\Gamma^{Q} = (\Omega, O(N))$ with $|N| \geq 3$ and $O_i \geq \frac{n+1}{2}$ for all $i \in N$, we can find mechanisms that satisfy outcome efficiency, strategyproofness, GMWB and budget balance.

To prove Proposition 5, we make use of the fact that for any queueing problem with three or more agents, the symmetrically balanced VCG mechanism satisfies outcome efficiency, strategyproofness, ICB (ECB) and, more importantly, budget balance (see Chun & Mitra (2014), Chun et al. (2019a) and Kayı & Ramaekers (2010)). Given Remark 1 (i), it also follows that if all agents have identical O_i's, that is, $O_i = B^*$ for all $i \in N$, then condition $O_i = B^* \geq \frac{n+1}{2}$ for all $i \in N$ is both necessary and sufficient for getting mechanisms that satisfy outcome efficiency, strategyproofness, GMWB and budget balance.

2.6 Conclusion

The "*generalized minimum welfare bound*" is imposed on an agent's utility function to offer him an assurance that his dissatisfaction level will not exceed a guaranteed amount. Such a comprehensive and an all-inclusive bound will make future studies more compact and convenient. We have already shown that GMWB is compatible with the standard desirable properties in the literature. An obvious extension would be applying GMWB to a dynamic sequencing framework. One can also explore the implications of this bound when the waiting costs of the agents are interdependent.

2.7 Appendix

Proof of Theorem 1: (SPC1) $\Rightarrow$ (SPC2). It is well-known that for an outcome efficient sequencing rule a mechanisms is strategyproof if and only if the associated transfer is a VCG transfer (see Holmström (1979)). The standard way of specifying the VCG transfers for any sequencing problem Ω is that for all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta)), \theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$, where for each $i \in N$ the function $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary.[23] If in addition we require generalized minimum welfare bounds to be met, then it is necessary that for any profile $\theta \in \Theta^N$ and any agent $i \in N$, $U_i(\sigma^*(\theta), \tau_i(\theta); \theta_i) = -C(\sigma^*(\theta); \theta) + g_i(\theta_{-i}) \geq -\theta_i O_i(s)$ implying that $g_i(\theta_{-i}) \geq C(\sigma^*(\theta); \theta) - \theta_i O_i(s)$. Since the function $g_i(\theta_{-i})$ is independent of agent i's waiting cost θ_i, we have the following:

$$g_i(\theta_{-i}) \geq \bar{g}_i(\theta_{-i}) := \sup_{x_i \in \Theta} \left[T_i(x_i; \theta_{-i})\right], \quad T_i(x_i; \theta_{-i}) := [C(\sigma^*(x_i, \theta_{-i}); x_i, \theta_{-i}) - x_i O_i(s)]. \quad (2.21)$$

Observe that $T_i(x_i; \theta_{-i}) = [S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)]x_i + \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(x_i, \theta_{-i}))$.

Consider any profile $\tilde{\theta} \in \Theta^n$ and any $i \in N$ such that $\tilde{\theta}_j / s_j = a > 0$ for all $j \in N \setminus \{i\}$.

[23] See Mitra (2002) and Suijs (1996).

Consider any $x_i', x_i'' \in \Theta$ such that $x_i'/s_i \geq a \geq x_i''/s_i$ and $x_i' > x_i''$. If $O_i(s) < s_i$, then we have

$$T_i(x_i';\tilde{\theta}_{-i}) - T_i(x_i'';\tilde{\theta}_{-i}) = (x_i' - x_i'')[s_i - O_i(s)] + \sum_{j\neq i} s_i s_j \left[\frac{\tilde{\theta}_j}{s_j} - \frac{x_i''}{s_i}\right] > 0. \tag{2.22}$$

Moreover, for any $x_i > s_i a$, $T_i(x_i;\tilde{\theta}_{-i}) = x_i[s_i - O_i(s)] + \sum_{j\in N\setminus\{i\}} \tilde{\theta}_j S_j(\sigma^*(x_i,\tilde{\theta}_{-i}))$ is increasing in x_i. Therefore, the x_i^* that maximizes $T_i(x_i;\tilde{\theta}_{-i})$ is then $x_i^* = \infty$ implying that we do not have a supremum. Therefore, for a supremum to exist it is necessary that $O_i(s) \geq s_i$.

$(\text{SPC2}) \Rightarrow (\text{SPC1})$. Consider any Γ that satisfies the constrained welfare property, that is, consider $\Gamma \in \mathcal{G}(N)$. For any profile $\theta \in \Theta^n$ and any $i \in N$, consider the type $x_i^* \in \Theta$ such that it is a supremum for the function $T_i(x_i, \theta_{-i})$.

Step 1: For any $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$, there exists $x_i^* \in \{\{s_i(\theta_k/s_k)\}_{k\in N\setminus\{i\}} \cup \{0\}\}$ such that $T_i(x_i^*;\theta_{-i}) \geq T_i(x_i;\theta_{-i})$ for all $x_i \in \Theta$.

Proof of Step 1: Consider any agent $i \in N$ and any $\theta_{-i} \in \Theta^{|N\setminus\{i\}|}$ and we define the vector $\tilde{R}(\theta_{-i}) = ((\tilde{R}_j(\theta_{-i}) = \theta_j/s_j))_{j\neq i})$ of agent specific waiting cost to processing time ratio of agents in $N\setminus\{i\}$ and $R(\theta_{-i}) = (R_1(\theta_{-i}) = \theta_{(1)}/s_{(1)}, \ldots, R_{n-1}(\theta_{-i}) = \theta_{(n-1)}/s_{(n-1)})$ be the permutation of $\tilde{R}(\theta_{-i})$ such that $R_1(\theta_{-i}) \geq \ldots \geq R_{n-1}(\theta_{-i})$. We divide the proof into two possibilities (a) $O_i(s) \in [s_i, A(s)]$ and (b) $O_i(s) > A(s)$.

Proof of Possibility (a): We first show that there exists $x_i^* \in [s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ that maximizes $T_i(x_i,\theta_{-i})$. Observe that for any $x_i \in \Theta$, the function $T_i(x_i;\theta_{-i}) = [S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)]x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i}))$. If $x_i > s_i R_1(\theta_{-i})$, then $S_i(\sigma^*(x_i,\theta_{-i})) = s_i$ and hence $T_i(x_i;\theta_{-i}) = [s_i - O_i(s)]x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i}))$ which is non-increasing in x_i since by interval property $s_i \leq O_i(s)$ implying that the coefficient of x_i in $T_i(x_i;\theta_{-i})$ is non-positive. Hence, (i) if a maxima exists then we can always find a waiting cost $x_i^* \leq s_i R_1(\theta_{-i})$ that achieves it. Similarly, if $y_i < s_i R_{n-1}(\theta_{-i})$, then $S_i(\sigma^*(y_i,\theta_{-i})) = A(s)$ and hence it follows that $T_i(y_i;\theta_{-i}) = [A(s) - O_i(s)]y_i + \sum_{j\in N\setminus\{i\}} \theta_j S_i(\sigma^*(y_i,\theta_{-i}))$ which is non-decreasing in y_i since by interval property $A(s) \geq O_i(s)$ implying that the coefficient of x_i in $T_i(x_i;\theta_{-i})$ is non-negative. Hence, (ii) if a maxima exists, then we can always find a waiting cost $x_i^* \geq s_i R_{n-1}(\theta_{-i})$ that achieves it.

The function $T_i(x_i;\theta_{-i})$ is continuous and concave in x_i on the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ and the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ is compact.[24]

Hence, the function $T_i(x_i;\theta_{-i})$ has a maxima in the interval $[s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$. Given $x_i^* \in [s_i R_{n-1}(\theta_{-i}), s_i R_1(\theta_{-i})]$ and given continuity of $T_i(x_i;\theta_{-i})$, for two agents the proof is complete since $x_i^* = s_i R_1(\theta_j) = s_i(\theta_j/s_j)$ and it follows that $T_i(\theta_i(\theta_j), \theta_j) = [s_i - O_i(s)]s_i(\theta_j/s_j) + \theta_j(s_i + s_j)$. Therefore, consider the more than two agents case. If there exists $k \in N\backslash\{i\}$ such that $x_i^* = s_i(\theta_{(k)}/s_{(k)})$ (so that $T_i(x_i^*;\theta_{-i}) = T_i(s_i(\theta_k/s_k);\theta_{-i}) \geq T_i(x_i;\theta_{-i})$ holds for all $x_i \in \Theta$), then the proof is complete. If not then suppose there exists $k \in \{1,\ldots,n-2\}$ such that $x_i^* \in (s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i}))$, that is,

$$T_i(x_i^*;\theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right] x_i^* + \sum_{j\in N\backslash\{i\}} \theta_j S_j(\sigma^*(x_i^*, \theta_{-i})).$$

If $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) > 0$, then for any $x_i \in (x_i^*, s_i R_k(\theta_{-i})]$, $\sigma^*(x_i, \theta_{-i}) = \sigma^*(x_i^*, \theta_{-i})$ and $T_i(x_i;\theta_{-i}) > T_i(x_i^*;\theta_{-i})$ since $T_i(x_i;\theta_{-i}) - T_i(x_i^*;\theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right](x_i - x_i^*) > 0$. Therefore we have a contradiction to our assumption that at x_i^* the function $T_i(x_i;\theta_{-i})$ is maximized. If $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) < 0$, then for any $x_i' \in [s_i R_k(\theta_{-i}), x_i^*)$, $\sigma^*(x_i', \theta_{-i}) = \sigma^*(x_i^*, \theta_{-i})$ and $T_i(x_i';\theta_{-i}) > T_i(x_i^*;\theta_{-i})$ since $T_i(x_i';\theta_{-i}) - T_i(x_i^*;\theta_{-i}) = \left[\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s)\right](x_i' - x_i^*) > 0$. Again we have a contradiction to our assumption that at x_i^* the function $T_i(x_i;\theta_{-i})$ is

[24]From the functional form of $T_i(x_i;\theta_{-i})$ and given outcome efficiency it is obvious that given any θ_{-i}, the function $T_i(x_i;\theta_{-i})$ is continuous in x_i on any open interval $(s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i}))$ for all $k \in \{1,\ldots,n-2\}$ and by using appropriate limit argument one can also show continuity at any point $s_i R_k(\theta_{-i})$ for $k \in \{1,\ldots,n-1\}$. For concavity note that for any $\theta_{-i} \in \Theta_{-i}$, for every $x_i \in (s_i R_{k+1}(\theta_i), s_i R_k(\theta_i))$ for all $k \in \{0,\ldots,n\}$, where $R_{n+1} = 0$ and $R_0 = \infty$, $T_i(x_i;\theta_{-i}) = [S_i(\sigma^*(x_i,\theta_i)) - O_i(s)]\, x_i + \sum_{j\in N\backslash\{i\}} \theta_j s_j(\sigma^*(x_i,\theta_i))$ is a straight line. Moreover, $S_i(\sigma^*(x_i,\theta_i))$ is non-increasing in $x_i \in \mathbb{R}_{++}$. Hence, the slope $S_i(\sigma^*(x_i,\theta_i)) - O_i(s)$ is also non-increasing for $x_i \in \mathbb{R}_{++}$. As a result the piece-wise linear continuous function $T_i(x_i;\theta_{-i})$ is concave for $x_i \in \mathbb{R}_{++}$.

maximized. Therefore, the only possibility left is $\sum_{r=1}^{k} s_{(r)} + s_i - O_i(s) = 0$. However, in that case $T_i(x_i^*; \theta_{-i}) = \sum_{j \in N \setminus \{i\}} \theta_j S_i(\sigma^*(x_i^*, \theta_{-i}))$ and for every $x_i \in [s_i R_{k+1}(\theta_{-i}), s_i R_k(\theta_{-i})]$ the function $T_i(x_i, \theta_{-i})$ attains its maximum value implying that $T_i(x_i^*; \theta_{-i}) = T_i(s_i R_{k+1}(\theta_{-i}); \theta_{-i}) = T_i(s_i R_k(\theta_{-i}); \theta_{-i})$ and Step 1 continues to be valid.

Proof of Possibility (b): If $O_i(s) > A(s)$, then for any $i \in N$ and any given $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, the function $T_i(x_i; \theta_{-i})$ on $\mathbb{R}_+$ is maximized if we set $x_i^* = 0$. Since the function $T_i(x_i; \theta_{-i})$ is only defined on the domain $\Theta^n = \mathbb{R}_+ \setminus \{0\}$, $x_i^* = 0$ acts as a supremum of the function $T_i(x_i; \theta_{-i})$ and that $T_i(0; \theta_{-i}) = \sum_{j \in N \setminus \{i\}} \theta_j S_j(\sigma^*(0, \theta_{-i})) > T_i(x_i; \theta_{-i})$ for all $x_i \in \Theta$.

Fix any $i \in N$. First, suppose that $O_i(s) \in [s_i, A(s)]$. Given the proof of Possibility (a) of Step 1 and given any $\theta_{-i} \in \Theta^{n-1}$, let us define $x_i^* := \theta_i^*$ so that $T_i(x_i^*; \theta_{-i}) = T_i(\theta_i^*; \theta_{-i})$ and there exists $k \in N \setminus \{i\}$ such that $\theta_i^* = s_i(\theta_k / s_k)$. Consider the VCG transfer having the following property: For all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i^*(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + \bar{g}_i(\theta_{-i})$ with $\bar{g}_i(\theta_{-i}) := T_i(\theta_i^*; \theta_{-i})$. Then for any given $\theta \in \Theta^n$ and any agent $i \in N$, we have $u_i(\mu_i^*(\theta), \theta_i) + \theta_i O_i(s) = -[S_i(\sigma^*(\theta) - O_i(s)]\theta_i + \bar{g}_i(\theta_{-i}) = T_i(\theta_i^*, \theta_{-i}) - T_i(\theta_i, \theta_{-i}) \geq 0$. The last inequality follows from the fact that $T_i(\theta_i, \theta_{-i}) \leq T_i(\theta_i^*, \theta_{-i})$ for all $\theta_i \in \Theta$. Hence, $u_i(\mu_i^*(\theta), \theta_i) \geq -\theta_i O_i(s)$ implying that this VCG transfer satisfies the GMWB for agent i. Next, suppose that $O_i(s) > A(s)$. Given the proof of Possibility (b) of Step 1 and given any $\theta_{-i} \in \Theta^{n-1}$, let us define $x_i^* := 0$ so that $T_i(x_i; \theta_{-i}) \leq T_i(0; \theta_{-i})$ for all $x_i \in \Theta$. Consider the VCG transfer having the following property: For all $\theta \in \Theta^n$ and for all $i \in N$, $\tau_i^*(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + \bar{g}_i(\theta_{-i})$ with $\bar{g}_i(\theta_{-i}) := T_i(0; \theta_{-i})$. Then for any given $\theta \in \Theta^n$ and any agent $i \in N$, we have $u_i(\mu_i^*(\theta), \theta_i) + \theta_i O_i(s) = -[S_i(\sigma^*(\theta) - O_i(s)]\theta_i + \bar{g}_i(\theta_{-i}) = T_i(0; \theta_{-i}) - T_i(\theta_i; \theta_{-i}) \geq 0$. Thus, using the constrained welfare property we have identified VCG transfers that satisfies GMWB. □

Proof of Theorem 2: For outcome efficiency and strategyproof it is necessary that the mechanism $\mu = (\sigma^*, \tau)$ must be VCG with transfers satisfying the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$, $\tau_i(\theta) = -C(\sigma^*(\theta); \theta) + \theta_i S_i(\sigma^*(\theta)) + g_i(\theta_{-i})$ where $g_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}$ is arbitrary. For the GMWB condition to hold, in addition, it is necessary that

(I) $g_i(\theta_{-i}) \geq \bar{g}_i(\theta_{-i}) = T_i(\theta_i^*;\theta_{-i}) \in \max_{x_i\in\Theta} T_i(x_i;\theta_{-i})$ and $T_i(x_i;\theta_{-i}) = [S_i(\sigma^*(x_i,\theta_{-i})) - O_i(s)]x_i + \sum_{j\in N\setminus\{i\}} \theta_j S_j(\sigma^*(x_i,\theta_{-i}))$ (see condition (2.21) in the proof of Theorem 1).

Hence, using (I) we can replace $g_i(\theta_{-i}) = h_i(\theta_{-i}) + T_i(\theta_i^*;\theta_{-i})$ where $h_i : \Theta^{|N\setminus\{i\}|} \to \mathbb{R}$ and $h_i(\theta_{-i}) \geq 0$. By substituting $g_i(\theta_{-i}) = h_i(\theta_{-i}) + T_i(\theta_i^*;\theta_{-i})$ in the transfer $\tau_i(\theta)$ and then simplifying it we get

$$\tau_i(\theta) = [S_i(\sigma^*(\theta_i^*,\theta_{-i})) - O_i(s)]\theta_i^* + \sum_{j\in N\setminus\{i\}} \theta_j \delta_{ji}(\theta) + h_i(\theta_{-i}), \tag{2.23}$$

where $\delta_{ji}(\theta) := \left(\sum_{k\in P_j(\sigma^*(\theta_i^*,\theta_{-i}))} s_k - \sum_{k\in P_j(\sigma^*(\theta))} s_k\right)$. Observe the following:

(a) If $P_i(\sigma^*(\theta)) = P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, then for any $j \in N\setminus\{i\}$ we have $P_j(\sigma^*(\theta)) = P_j(\sigma^*(\theta_i^*,\theta_{-i}))$, then it easily follows that $\delta_{ji}(\theta) = 0 = (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)s_i$.

(b) If $P_i(\sigma^*(\theta_i^*,\theta_{-i})) \subset P_i(\sigma^*(\theta))$, then for agent any $j \in P_i(\sigma^*(\theta)) \setminus P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, we have $P_j(\sigma^*(\theta_i^*,\theta_{-i})) \setminus P_j(\sigma^*(\theta)) = \{i\}$. Hence, $\delta_{ji}(\theta) = s_i = (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)s_i$.

(c) If $P_i(\sigma^*(\theta)) \subset P_i(\sigma^*(\theta_i^*,\theta_{-i}))$, then for any $j \in P_i(\sigma^*(\theta_i^*,\theta_{-i})) \setminus P_i(\sigma^*(\theta))$, it easily follows that $P_j(\sigma^*(\theta)) \setminus P_j(\sigma^*(\theta_i^*,\theta_{-i})) = \{i\}$. Therefore, we obtain $\delta_{ji}(\theta) = -s_i = (|P_j(\sigma^*(\theta_i^*,\theta_{-i}))| - |P_j(\sigma^*(\theta))|)s_i$.

By substituting the values of $\delta_{ji}(\theta)$ for possibilities (a), (b) and (c) in the sum $\sum_{j\in N\setminus\{i\}} \theta_j\delta_{ji}(\theta)$ of (2.23) we get the sum in (2.6).

From (I) condition (2.23) and the expansion of the sum $\sum_{j\in N\setminus\{i\}} \theta_j\delta_{ji}(\theta)$ summarized in (a), (b) and (c) we get $\tau = \tau^p$.

To prove the converse, observe that since any μ^p is a particular type of VCG transfers, μ^p is sufficient to ensure outcome efficency and strategyproofness. To complete the proof we need to check the sufficiency of GMWB with μ^p. Consider any relative pivotal mechanism μ^p. For any $\theta \in \Theta^n$ and any $i \in N$, we have $u_i(\sigma^*(\theta), \tau_i^p(\theta), \theta_i) + \theta_i O_i(s) = -\theta_i[S_i(\sigma^*(\theta)) - O_i(s)] + [S_i(\sigma^*(\theta_i^*, \theta_{-i})) - O_i(s)]\theta_i^* + \sum_{j \in N\setminus\{i\}}(|P_j(\sigma^*(\theta_i^*, \theta_{-i}))| - |P_j(\sigma^*(\theta))|)\theta_j s_i + h_i(\theta_{-i}) = T_i(\theta_i^*, \theta_{-i}) - T_i(\theta) + h_i(\theta_{-i}) \geq 0$. Therefore, $u_i(\mu_i^p(\theta), \theta_i) + \theta_i O_i(s) \geq 0$ implying $u_i(\mu_i^p(\theta), \theta_i) \geq -\theta_i O_i(s)$. Hence, any relative pivotal mechanism μ^p satisfies the relevant generalized minimum welfare bounds. □

Proof of Lemma 1: Suppose $\Gamma = (\Omega, O(N, s)) \in \mathcal{G}(N)$ is a problem for which we can find a mechanism that satisfies outcome efficiency, GMWB and feasibility and let $\mu = (\sigma^*, \tau)$ be such a mechanism. Then using GMWB it follows that for every $\theta \in \Theta^n$ and each $i \in N$, $u_i(\sigma^*(\theta), \tau(\theta); \theta_i) = -\theta_i S_i(\sigma^*(\theta)) + \tau_i(\theta) \geq -\theta_i O_i(s)$ implying that for all $i \in N$, $\tau_i(\theta) \geq \theta_i S_i(\sigma^*(\theta)) - \theta_i O_i(s)$. By summing the transfers over all agents and applying feasibility it follows that $C(\sigma^*(\theta); \theta) - \sum_{j \in N} \theta_j O_j(s) \leq 0$. Hence, for the mechanism $\mu = (\sigma^*, \tau)$ to satisfy outcome efficiency, GMWB and feasibility it is necessary that

$$\sum_{j \in N} \theta_j \left\{ O_j(s) - S_j(\sigma^*(\theta)) \right\} \geq 0, \;\; \forall \; \theta \in \Theta^n. \tag{2.24}$$

Consider a set of profiles, $\theta^t = (\theta_1^t, \ldots, \theta_n^t) \in \Theta^n$ defined for any positive integer t such that $\theta_j^t = s_j[1 - \{j/(2^t n)\}]$ for all $j \in N$. Observe that for any given t and any $l, m \in N$ such that $l < m$, $\theta_l^t / s_l > \theta_m^t / s_m$ so that for every positive integer t, we have the same outcome efficient order $\sigma^*(\theta^t) = (\sigma_1^0, \ldots, \sigma_n^0)$ with $\sigma_j^0 = j$ for all $j \in N$. Also observe that as $t \to \infty$, $\theta_j^t \to s_j > 0$. Given (2.24), the condition $\sum_{j \in N} \theta_j^t \left\{ O_j(s) - S_j(\sigma^0) \right\} \geq 0$ must hold for every positive integer t and hence it must also hold at the limiting value of t as well, that is, it must also hold when $\theta_j = s_j$ for all $j \in N$. Hence, it is also necessary that

$$\sum_{j \in N} s_j \left\{ O_j(s) - S_j(\sigma^0) \right\} \geq 0. \tag{2.25}$$

If we can show that the equality $\sum_{j\in N} s_j S_j(\sigma^0) = \sum_{j\in N} s_j\{s_j + A(s)\}/2$ holds, then one can easily verify that using this equality in (2.25) we get the result.[25] Hence, our final step is to show this equality. Observe that

$$\begin{aligned}
\sum_{j\in N} s_j S_j(\sigma^0) &= \sum_{j\in N} s_j \left(s_j + \sum_{k>j} s_k \right) = \sum_{j\in N} s_j^2 + \sum_{j\in N}\sum_{k>j} s_j s_k \\
&= \sum_{j\in N} s_j^2 + \sum_{j\in N} \left(\sum_{k\neq j} \frac{s_j s_k}{2} \right) = \sum_{j\in N} s_j \left(s_j + \sum_{k\neq j} \frac{s_k}{2} \right) \\
&= \sum_{j\in N} s_j \left(\frac{2s_j + \sum_{k\neq j} s_k}{2} \right) = \sum_{j\in N} s_j \left(\frac{s_j + A(s)}{2} \right).
\end{aligned} \tag{2.26}$$

Therefore, from (2.26) we get the required equality and the result follows. $\square$

Proof of Proposition 1: Consider any $\Gamma = (\Omega, O(N,s)) \in \mathcal{G}(N)$ with $N = \{1,2\}$ and, given constrained welfare property assume without loss of generality that $O_1(s) = s_1 + \lambda_1 s_2$ and $O_2(s) = s_2 + \lambda_2 s_1$ where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$. If $\theta = (\theta_1, \theta_2) \in \Theta^2$ is any profile such that $\theta_1/s_1 > \theta_2/s_2$, then, given $\theta_i^* = s_i\theta_j/s_j$ if $\lambda_i \in [0,1)$ and $\theta_i^* = 0$ if $\lambda_i \geq 1$ for any $i,j \in \{1,2\}$ such that $i \neq j$, from the definition of minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ it follows that

$$\hat{\tau}_1^p(\theta_1,\theta_2) = -\min\{\lambda_1, 1\}\theta_2 s_1 \text{ and } \hat{\tau}_2^p(\theta_1,\theta_2) = (1-\min\{\lambda_2,1\})\theta_1 s_2. \tag{2.27}$$

Therefore, from (2.27) it follows that

$$\hat{\tau}_1^p(\theta_1,\theta_2) + \hat{\tau}_2^p(\theta_1,\theta_2) = \left[(1-\min\{\lambda_2,1\})\theta_1 s_2 - \min\{\lambda_1,1\}\theta_2 s_1\right]. \tag{2.28}$$

[25] Specifically, if $\sum_{j\in N} s_j S_j(\sigma^0) = \sum_{j\in N} s_j\{s_j + A(s)\}/2$, then expanding the left hand side of (2.25) we get

$$\sum_{j\in N} s_j O_j(s) - \sum_{j\in N} s_j S_j(\sigma^0) = \sum_{j\in N} s_j O_j(s) - \sum_{j\in N} s_j \left(\frac{s_j + A(s)}{2}\right) = \sum_{j\in N} s_j \left\{ O_j(s) - \left(\frac{s_j + A(s)}{2}\right) \right\}.$$

Feasibility requires that $\hat{\tau}_1^p(\theta_1, \theta_2) + \hat{\tau}_2^p(\theta_1, \theta_2) \leq 0$ for all $\theta = (\theta_1, \theta_2) \in \Theta^2$ and for any θ_1 and any θ_2 such that $\theta_1/s_1 > \theta_2/s_2$, (I) $(1 - \min\{\lambda_2, 1\})\theta_1 s_2 \leq \min\{\lambda_1, 1\}\theta_2 s_1$. If $(1 - \min\{\lambda_2, 1\}) > 0$ (that is, if $\lambda_2 \in [0, 1)$), then given any $\theta_2 > 0$ and any $\lambda_1 \geq 0$, by taking any θ_1 sufficiently large such that $\theta_1 > \min\{\lambda_1, 1\}s_1\theta_2/(1 - \min\{\lambda_2, 1\})s_2$ and making it sufficiently large we have a violation of condition (I). Hence, $\lambda_2 \geq 1$. Similarly, if $\theta' = (\theta'_1, \theta'_2) \in \Theta^2$ is such that $\theta'_1/s_1 < \theta'_2/s_2$, then, given $\lambda_2 \geq 1$, from the definition of minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$ it follows that

$$\hat{\tau}_1^p(\theta'_1, \theta'_2) = (1 - \min\{\lambda_1, 1\})\theta'_2 s_1 \text{ and } \hat{\tau}_2^p(\theta'_1, \theta'_2) = -\theta'_1 s_2. \tag{2.29}$$

Feasibility requires that $\hat{\tau}_1^p(\theta'_1, \theta'_2) + \hat{\tau}_2^p(\theta'_1, \theta'_2) \leq 0$ for all $\theta' = (\theta'_1, \theta'_2) \in \Theta^2$ and hence given (2.29) for any θ'_1 and any θ'_2 such that $\theta'_1/s_1 < \theta'_2/s_2$, for feasibility it is necessary that (II) $(1 - \min\{\lambda_1, 1\})\theta'_2 s_1 \leq \theta'_1 s_2$. If $(1 - \min\{\lambda_1, 1\}) > 0$ (that is, $\lambda_1 \in [0, 1)$), then given any θ'_1, by taking $\theta'_2 > s_2\theta'_1/(1 - \min\{\lambda_1, 1\})s_1$ we have a violation of condition (II). Hence, we must also have $\lambda_1 \geq 1$. Therefore, for feasibility it is necessary that $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$, that is, $O_1(s) \geq A(s)$ and $O_2(s) \geq A(s)$.

Conversely, if $\lambda_1 \geq 1$ and $\lambda_2 \geq 1$, then, from the definition of minimal relative pivotal mechanism $\hat{\mu}^p = (\sigma^*, \hat{\tau}^p)$, it follows that for any $\theta \in \Theta^2$, any $i \in \{1, 2\}$ and any $j \in \{1, 2\}$ with $j \neq i$,

$$\hat{\tau}_i^p(\theta) = \begin{cases} -\theta_j s_i & \text{if } P_i(\sigma^*(\theta)) = \emptyset, \\ 0 & \text{if } P_i(\sigma^*(\theta_i(\theta_{-i}), \theta_{-i}) = \{j\}, \end{cases} \tag{2.30}$$

It is immediate from (2.30) that for all $\theta_1, \theta_2 \in \Theta$, then we get feasibility. Hence, we have the first part of the result.

The proof of the second part, that is, any relative pivotal mechanism given by (2.30) is not budget balanced, is a special case of Proposition 3 in De & Mitra (2019) where we need to replace linear sequencing rule by its special case of outcome efficient sequencing rule. □

Proof of Proposition 2: Consider any $\Gamma = (\Omega, O(N,s)) \in \overline{\mathcal{G}}(N)$ with the generalized minimum welfare bounds satisfying the following properties: $O_i(s) \geq A(s) = \sum_{j \in N} s_j$ for all $i \in N$. Observe that the constrained welfare property given by condition (2.4) holds for this example as well. For any $\theta \in \Theta^n$ and any $i \in N$, the function $T_i(x_i; \theta_{-i})$ (given by Definition 7)) has a supremum at $\theta_i^* = 0$ for all $i \in N$ implying that $P_i(\sigma^*(0,,\theta_{-i})) \cup \{i\} = n$ and hence $S_i(\sigma^*(0, \theta_{-i})) = A(s) \leq O_i(s)$. The reason is the following: For any $i \in N$ and any $x_i \in \Theta$ such that $P_i(\sigma^*(x_i, \theta_{-i})) \subset N \setminus \{i\}$ and $P_i(\sigma^*(x_i, \theta_{-i})) \neq N \setminus \{i\}$, the function $T_i(x_i; \theta_{-i}))$ is decreasing in x_i since $[S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)] = \sum_{j \in P_i(\sigma^*(x_i, \theta_{-i}))} s_j - \sum_{j \in N \setminus \{i\}} s_j = -\sum_{j \in F_i(\sigma^*(x_i, \theta_{-i}))} s_j$ is negative. Therefore, for any $i \in N, \theta_i^* = 0$ implying that agent i is always served last in the benchmark order $\sigma^*(0, \theta_{-i})$. Given $\theta_i^* = 0$, it is quite easy to verify that (I) $\theta_i^*[S_i(\sigma^*(x_i, \theta_{-i})) - O_i(s)] = 0$ and (II) $RP_i(\theta) = -\sum_{k \in F_i(\sigma^*(\theta))} \theta_k s_i$. Therefore, using (I) and (II) in Definition 7 we get that an outcome efficient mechanism $\mu^p = (\sigma^*, \tau^p)$ is a relative pivotal mechanism if τ^p satisfies the following property: For any profile $\theta \in \Theta^n$ and any agent $i \in N$,

$$\tau_i^p(\theta) = -\sum_{k \in F_i(\sigma^*(\theta))} \theta_k s_i + h_i(\theta_{-i}), \tag{2.31}$$

where $h_i : \Theta^{|N \setminus \{i\}|} \to \mathbb{R}_+$. Let $n \geq 3$ and for all $i \in N$ and all $\theta_{-i} \in \Theta^{|N \setminus \{i\}|}$, suppose we set $h_i(\theta_{-i}) = \sum_{j \in N \setminus \{i\}} \left\{ s_j \sum_{k \in F_j(\sigma^*(\theta_{-i}))} \theta_k \right\} / (n-2)$ in the transfer given by (2.31). One can then simplify the resulting transfers (2.31) and show that we get budget balance.[26] □

Proof of Proposition 3: Consider any $\Gamma^0 = (\Omega, O^{\sigma^0}(N,s)) \in \mathcal{I}(N)$ and, without loss of generality, assume σ^0 such that $\sigma_i^0 = i$ for all $i \in N$. Consider any $\theta \in \Theta^n$ such that $\theta_n/s_n > \theta_1/s_1 > \ldots > \theta_{n-1}/s_{n-1}$ so that $P_1(\sigma^*(\theta)) = \{n\}, P_j(\sigma^*(\theta)) = \{1, \ldots, j-1\} \cup \{n\}$ for all $j \in N \setminus \{1, n\}$ and $P_n(\sigma^*(\theta)) = \emptyset$. Consider the minimal relative pivotal mechanism $\hat{\mu} = (\sigma^*, \hat{\tau})$ (in Definition 9) with the $T_i(x_i; \theta_{-i})$ function given by (2.12). It is easy to verify the following:

[26] We do not provide a formal proof since it is a special case of the proof of Theorem 1 in De & Mitra (2019).

(i) Given $P_1(\sigma^0) = \emptyset$, from (IO1) of Remark 4 we have $\theta_1^* = s_1\theta_n/s_n$ and $P_1(\sigma^*(\theta_1^*, \theta_{-1})) = P_1(\sigma^0) = \emptyset$. Further, $P_n(\sigma^*(\theta_1^*, \theta_{-1})) \setminus P_n(\sigma^*(\theta)) = \{1\}$ and $P_j(\sigma^*(\theta_1^*, \theta_{-1})) = P_j(\sigma^*(\theta))$ for all $j \in N \setminus \{1, n\}$. Thus, $\hat{\tau}_1(\theta) = (|P_n(\sigma^*(\theta_1^*, \theta_{-1}))| - |P_n(\sigma^*(\theta))|)\theta_n s_1 = \theta_n s_1$.

(ii) Given $P_n(\sigma^0) = N \setminus \{n\}$, from condition (IO2) of Remark 4 we get $\theta_n^* = s_n\theta_{n-1}/s_{n-1}$ and $P_n(\sigma^*(\theta_n^*, \theta_{-n})) = P_n(\sigma^0) = N \setminus \{n\}$. Moreover, $P_j(\sigma^*(\theta)) \setminus P_j(\sigma^*(\theta_n^*, \theta_{-n})) = \{n\}$ for all $j \in N \setminus \{n\}$. Hence, the transfer of n is $\hat{\tau}_n(\theta) = \sum_{j\in N\setminus\{n\}}(|P_j(\sigma^*(\theta_n^*, \theta_{-n}))| - |P_j(\sigma^*(\theta))|)\theta_j s_n = -\sum_{j\in N\setminus\{n\}} \theta_j s_n$. Therefore, the transfer of agent n does not involve the waiting cost θ_n.

(iii) Finally, consider any $k \in N \setminus \{1, n\}$. Observe that if $x_k = s_k\theta_n/s_n$, then $T_k^I(x_k; \theta_{-k}))$ is decreasing in x_k since the coefficient of x_k, that is $[\sum_{j\in P_k(\sigma^*(x_k,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_j] = -\sum_{j=1}^{k-1} s_j < 0$. Hence, $\theta_k^* \neq s_k\theta_n/s_n$. Further, $(|P_n(\sigma^*(\theta_k^*, \theta_{-k}))| - |P_n(\sigma^*(\theta))|)\theta_n s_k = 0$ since $P_n(\sigma^*(\theta_k^*, \theta_{-k})) = P_n(\sigma^*(\theta)) = \emptyset$. Thus, the transfer of any agent $k \in N \setminus \{1, n\}$ does not involve the waiting cost θ_n of agent n and hence can be expressed in the following form: $\hat{\tau}_k(\theta) = \theta_k^*[\sum_{j\in P_k(\sigma^*(\theta_k^*,\theta_{-k}))} s_j - \sum_{j\in P_k(\sigma^0)} s_k] + \sum_{j\in N\setminus\{k,n\}}(|P_j(\sigma^*(\theta_k^*, \theta_{-k}))| - |P_j(\sigma^*(\theta))|)\theta_j s_k$.

From (i), (ii) and (iii) it follows that $\sum_{j\in N} \hat{\tau}_j(\theta) = \theta_n s_1 + \sum_{j\in N\setminus\{1\}} \hat{\tau}_j(\theta)$. From (i) and (iii) above it also follows that the sum $\sum_{j\in N\setminus\{1\}} \hat{\tau}_j(\theta)$ does not involve the waiting cost θ_n and hence by defining $\mathcal{T}(\sigma^*(\theta); \theta_{-n}) := \sum_{j\in N\setminus\{1\}} \hat{\tau}_j(\theta)$ we get

$$\sum_{j\in N} \hat{\tau}_j(\theta) = \theta_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}). \tag{2.32}$$

If $\sum_{j\in N} \hat{\tau}_j(\theta) > 0$, then we have a violation of feasibility and the proof is complete. Therefore, assume $\sum_{j\in N} \hat{\tau}_j(\theta) = \theta_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}) \leq 0$. Given that $\mathcal{T}(\sigma^*(\theta); \theta_{-n})$ is independent of θ_n, if we increase the waiting cost of agent n to any $y_n(> \theta_n)$ by keeping θ_{-n} fixed, then the outcome efficient order remains unchanged (that is, $\sigma^*(y_n, \theta_{-n}) = \sigma^*(\theta)$ for all $y_n > \theta_n$) and the transfers of all but agent 1 continues to remain unchanged due to above mentioned independence argument, that is, $\mathcal{T}(\sigma^*(y_n, \theta_{-n}); \theta_{-n}) = \mathcal{T}(\sigma^*(\theta); \theta_{-n})$ for all $y_n > \theta_n$. Hence, we have

$$\sum_{j \in N} \hat{\tau}_j(y_n, \theta_{-n}) = y_n s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}) \ \forall\, y_n > \theta_n. \tag{2.33}$$

Since the first term in the right hand side of condition (2.33) is increasing in y_n and the second term remains constant with a change in y_n, it follows that by making y_n sufficiently large (say some y_n^*) we get $\sum_{j \in N} \hat{\tau}_j(y_n^*, \theta_{-n}) = y_n^* s_1 + \mathcal{T}(\sigma^*(\theta); \theta_{-n}) > 0$ leading to a violation of feasibility. □

Proof of Proposition 4: Consider any $\Gamma^s = (\Omega, O^s(N, s)) \in \mathcal{C}(N)$ such that $|N| = 3$ and, without loss of generality, assume that $s_1 \geq s_2 \geq s_3$. Consider the profile $\theta \in \Theta^3$ such that $\sigma_j^*(\theta) = j$ for all $j \in N$ and in particular $\theta_1/s_1 = a > \theta_2/s_2 = b > \theta_3/s_3 = c > 0$ and assume that (i) $a > \max\{cs_1/s_2, bs_2/s_3\}$. Since $O_j^s(s) = (n+1)s_j/2 > s_i$ for all $j \in N$, using the function $T_j^C(x_j; \theta_{-j})$ given by (2.13), we can take $\theta_1^* = s_1 c$, $\theta_2^* = s_2 a$ and $\theta_3^* = s_3 a$. Then using the transfers associated with the minimal relative pivotal mechanism (Definition 9) with $T_j^C(x_j; \theta_{-j})$ given by (2.13) we get the following:

1. $\hat{\tau}_1(\theta) = -cs_1(s_1 - s_2) - bs_1 s_2$,
2. $\hat{\tau}_2(\theta) = as_2(s_1 - s_2)$, and
3. $\hat{\tau}_3(\theta) = as_3(s_1 - s_2) + bs_2 s_3$.

If $s_1 > s_3$, then $\sum_{j \in N} \hat{\tau}_j(\theta) = (s_1 - s_2)(as_2 - cs_1) + (s_1 - s_3)(as_3 - bs_2) = (s_1 - s_2)s_2[a - (cs_1/s_2)] + (s_1 - s_3)s_3[a - (bs_2/s_3)] > 0$ (due to (i)) and we have a contradiction to feasibility. Hence, for feasibility it is necessary that $s_1 \leq s_3$ implying $s_1 \geq s_2 \geq s_3 \geq s_1$. Hence, $s_1 = s_2 = s_3$.

Consider any $\Gamma^{\bar{S}} = (\Omega, O^{\bar{S}}(N, s)) \in \mathcal{E}(N)$ such that $|N| = 3$ and, without loss of generality, assume that $s_1 \geq s_2 \geq s_3$. Consider the profile $\theta \in \Theta^3$ such that $\sigma_j^*(\theta) = j$ for all $j \in N$ and in particular $\theta_1/s_1 = a > \theta_2/s_2 = b > \theta_3/s_3 = c > 0$. Since $O_j^{\bar{S}}(s) = (s_j + A(s))/2 > s_i$ for all $j \in N$, using the function $T_j^E(x_j; \theta_{-j})$ given by (2.14), we can take $\theta_1^* = s_1 b$, $\theta_2^* = s_2 a$ and $\theta_3^* = s_3 a$. Then using the transfers associated with the minimal relative pivotal mechanism (Definition 9) with $T_j^E(x_j; \theta_{-j})$ given by (2.14) we get the following:

1. $\hat{\tau}_1(\theta) = -s_1 b\left(\frac{s_2+s_3}{2}\right)$,
2. $\hat{\tau}_2(\theta) = s_2 a\left(\frac{s_1-s_3}{2}\right)$, and
3. $\hat{\tau}_3(\theta) = s_3 a\left(\frac{s_1-s_2}{2}\right) + s_2 s_3 b$.

If $s_1 > s_3$, then $\sum_{j\in N}\hat{\tau}_j(\theta) = \frac{(a-b)}{2}(s_2 s_1 + s_1 s_3 - 2 s_2 s_3) > \frac{(a-b)}{2}(s_2 s_3 + s_1 s_3 - 2 s_2 s_3) = \frac{(a-b)s_3(s_1-s_2)}{2} \geq 0$ and we have a contradiction to feasibility. Hence, for feasibility we need $s_1 \leq s_3$ implying $s_1 = s_2 = s_3$. □

Proof of Corollary 1: For any profile $\theta \in \Theta^n$ and $i \in N$, consider the type $\theta_i^* \in \Theta$ such that the function $T_i^{QB}(x_i, \theta_{-i})$ (defined in (2.16)) takes the maximum value, that is, $T_i^{QB}(\theta_i^*, \theta_{-i}) \geq T_i^{QB}(x_i, \theta_{-i})$ for all $x_i \in \Theta^n$. Let $\bar{r}(\theta_{-i}) = ((\bar{r}_j(\theta_{-i}) = \theta_j)_{j\neq i})$ be the vector of agent specific waiting cost in $N\backslash\{i\}$ and $r_i(\theta_{-i}) = (r_1(\theta_{-i}) = \theta_{(1)}, \ldots r_{n-1}(\theta_{-i}) = \theta_{(n-1)})$ be the permutation of $\bar{r}(\theta_{-i})$ such that $r_1(\theta_{-i}) \geq \ldots \geq r_{n-1}(\theta_{-i})$. We can verify that if n is odd, $\theta_i^* \in \{r_{\frac{n-1}{2}}(\theta_{-i}), r_{\frac{n+1}{2}}(\theta_{-i})\}$ and when n is even, $\theta_i^* = r_{\frac{n}{2}}(\theta_{-i})$. Using the resulting θ_i^* that maximizes the function $T_i^{QB}(x_i, \theta_{-i})$ (defined in (2.16)), we have the following forms of the relative pivotal mechanisms derived for the even and odd cases separately. If n is odd, then we get the transfer given by $\tau_i^{odd}(\theta) + h_i(\theta_{-i})$ where,

$$\tau_i^{odd}(\theta) = \begin{cases} -\sum\limits_{k\in F_i(\sigma^*(\theta))|1<\sigma_k^*(\theta)\leq\frac{n+1}{2}} \theta_k & \text{if } \sigma_i^*(\theta) < \frac{n+1}{2}, \\ 0 & \text{if } \sigma_i^*(\theta) = \frac{n+1}{2}, \\ \sum\limits_{k\in P_i(\sigma^*(\theta))|\frac{n+1}{2}\leq\sigma_k^*(\theta)<n} \theta_k & \text{if } \sigma_i^*(\theta) > \frac{n+1}{2}, \end{cases} \quad (2.34)$$

and if n is even, then we get the transfer given by $\tau_i^{even}(\theta) + h_i(\theta_{-i})$ where,

$$\tau_i^{even}(\theta) = \begin{cases} -\sum_{k \in F_i(\sigma^*(\theta)) | 1 < \sigma_k^*(\theta) \leq \frac{n}{2}} \theta_k - \frac{\theta_f}{2} & \text{if } \sigma_i^*(\theta) < \frac{n}{2}, \sigma_f^*(\theta) = \frac{n}{2} + 1 \text{ and } n > 2, \\ -\frac{\theta_f}{2} & \text{if } \sigma_i^*(\theta) = \frac{n}{2} \text{ and } \sigma_f^*(\theta) = \frac{n}{2} + 1, \\ \frac{\theta_p}{2} & \text{if } \sigma_i^*(\theta) = \frac{n}{2} + 1 \text{ and } \sigma_p^*(\theta) = \frac{n}{2}, \\ \sum_{k \in P_i(\sigma^*(\theta)) | \frac{n}{2} + 1 \leq \sigma_k^*(\theta) < n} \theta_k + \frac{\theta_p}{2} & \text{if } \sigma_i^*(\theta) > \frac{n}{2} + 1, \sigma_p^*(\theta) = \frac{n}{2} \text{ and } n > 2. \end{cases} \tag{2.35}$$

Observe that, $\tau_i^{odd}(\theta)$ is a K-pivotal mechanism with $K = \frac{n+1}{2}$ while $\tau_i^{even}(\theta)$ is the simple average of two K-pivotal mechanisms-one with $K = n/2$ and the other with $K = n/2 + 1$. We can then generally express,

$$\overline{\tau}_i^a(\theta) = H_i(\theta_{-i}) + \begin{cases} \tau_i^{(\frac{n+1}{2})}(\theta) + & \text{if } n \text{ is odd,} \\ \frac{1}{2}\tau_i^{(\frac{n}{2})}(\theta) + \frac{1}{2}\tau_i^{(\frac{n}{2}+1)}(\theta) & \text{if } n \text{ is even.} \end{cases} \tag{2.36}$$

□

Proof of Proposition 5: Given that for any queueing problem $\Omega \in \mathcal{Q}(N)$, the symmetrically balanced VCG mechanism satisfies outcome efficiency, strategyproofness, ICB (ECB) and budget balance, it follows that with $O_i = (n+1)/2$ for all $i \in N$ (which is the bound associated with ICB(ECB)), the result holds. In particular, for any $\theta \in \Theta^n$, the utility of an agent $i \in N$ associated with the symmetrically balanced VCG mechanism satisfies $U_i(\sigma^*(\theta), \tau_i^{sb}(\theta); \theta_i) \geq -\theta_i(n+1)/2$. Consider any queueing problem with generalized minimum welfare bounds satisfying the following property: For all $i \in N$, $O_i \geq (n+1)/2$ or equivalently, for each $i \in N$, there exists $\beta_i \geq 0$ such that $O_i = \left(\frac{n+1}{2}\right) + \beta_i$. With the symmetrically balanced VCG mechanism we have that for each $\theta \in \Theta^n$

and each $i \in N$,

$$U_i(\sigma^*(\theta), \tau_i^{sb}(\theta); \theta_i) \geq -\theta_i \left(\frac{n+1}{2}\right) \geq -\theta_i \left(\frac{n+1}{2}\right) - \beta_i, \text{ for any } \beta_i \geq 0.$$

Therefore, the symmetrically balanced VCG mechanism also ensures outcome efficiency, strategyproofness, budget balance and GMWB for any welfare parameter vector $O(N) = (O_1, \ldots, O_n)$ such that $O_i \geq (n+1)/2$ for all $i \in N$. $\square$

3

Lorenz optimality for sequencing problems with welfare bounds

3.1 Introduction

We work in a sequencing framework where the mechanism designer achieves fairness, through reduction of inequalities in the distribution of outcomes, while individuals are driven by their self-seeking behavior. This raises the following question: is it possible to design mechanisms that discretely protect an agent's self-interest while justly allocating the final outcome? Banerjee et al. (2020) have introduced the generalized minimum welfare bound (GMWB) which is a unified and comprehensive representation of several specific lower bounds that have been previously examined in the queueing literature (see Chun & Mitra (2014), Chun et al. (2017), Chun et al. (2019a), Chun & Yengin (2017), Gershkov & Schweinzer (2010), Kayı & Ramaekers (2010), Mitra & Mutuswami (2011)). Every agent is offered a protection in the form of a minimum guarantee on their utilities. In other words, the GMWB imposes a lower bound on each agent's utility function. However, despite our self-centric existence as independent entities, we all identify ourselves with the society we live in. The decision maker faces a moral commitment of being fair and enhancing the overall social well-being. We show that such a goal can be achieved within the sequencing framework. Our main theorem states that the constrained egalitarian mechanism is the only mechanism that is Lorenz optimal in the class of feasible mechanisms satisfying GMWB. A constrained egalitarian mechanism is a Pareto optimal mechanism that assigns to each agent either a common level of utility or the minimum guaranteed level under GMWB, whichever is higher.

There are two sides to distributive justice in the study of microeconomics: fairness in the distribution of rights and fairness in the outcome of the game. Procedural justice addresses the former while end-state justice addresses the latter. The collective welfare approach continues to remain the most influential application of economic analysis to distributive justice. The central idea of *welfarism* is that, it evaluates collective action on the basis of individual utility levels when comparing two outcomes. The collective utility function (CUF) aggregates individual utilities into a single utility index representing the social welfare (thus creating a social welfare ordering). The two primary examples are the classic utilitarian collective utility (sum of individual utilities) and the egalitarian collective utility (the minimal individual utility). When the aggregate sum of individual utilities is maximized, some individuals may have to compromise. For instance, if a majority of roommates want to watch only sports on the television, the minority who might wish to tune into some news channel, will never be granted so under the utilitarian rule. Such a CUF views individuals as mere entities producing social welfare where an agent can be sacrificed for the betterment of the collective utility. However, the egalitarian utility function embodies compensation and makes the worst off (most unhappy) individual as better off as possible (maximizes the minimum utility). For instance, equal time shares are allotted to both sports and new channels even though the minority is small enough. An ample amount of extra utility is not worth the tiny disutility suffered by the least well of agent(see Moulin (1991a)Moulin (2003)).

The tension between these two opposing ideas is age old and can be further highlighted through an example given by Mirrlees (see Moulin (1991a)). "There are two cars burning after an accident. The first car has four passengers and the second car has only one (all five an unconscious). By choosing to rescue the first car, we tend to maximize the expected number of survivors and behave like an utilitarian. On the contrary, an egalitarian rescuer, would toss a fair coin and choose which car to help. He would give everyone an equal chance of surviving."

The primary economic application of CUFs is to measure inequalities in the distribution of incomes. A social welfare ordering (SWO) is the ordinal counter part of CUFs. The Lorenz criterion is a classic in the discussion of stochastic dominance and is an equivalent formulation of the Pigou-Dalton principle (transferring some utility from one agent to the other so as to reduce the difference in their welfare should not reduce social welfare). The criterion clearly defines what it means for one utility profile to be better than another profile when the utilities are additively separable and the objective is to reduce inequalities in the allocation of payoffs. Observe that, the utilitarian approach is orthogonal to the concern for compensation and totally indifferent towards inequalities in distribution of the final outcome as long as the sum of utilities remains the same. On the other hand, an egalitarian strives to re-distribute welfare from the rich to the poor. The beauty of the configuration of this criterion is that, you can be certain that one distribution is better than the other, irrespective of whether you are fully utilitarian or fully egalitarian or anything in between. If the designer can find such a mechanism whose final profile of utilities Lorenz dominates all other feasible profiles, then it will unambiguously be the optimal choice for the designer.[1,2]

3.2 The framework and definitions

A finite set of agents $N = \{1, 2, \ldots, n\}$ want to process their jobs using a facility that can be used sequentially (one agent at a time). Each agent has a single job to process and the processing time differs across agents. An agent $i \in N$ is identified by his job processing time $s_i \in \mathbb{R}_{++}$ and his constant per period waiting cost $\theta_i \in \Theta := \mathbb{R}_{++}$ (where $\mathbb{R}_{++}$ is the positive orthant of the real line $\mathbb{R}$). An order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N describes the position of each agent where $\sigma_i = k$ indicates that agent i has the k-th position in the queue. Let Σ be the set of $n!$ possible orders on N. We denote $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ as the predecessor set and $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ as the successor set of i in the order σ. The processing time vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ is common knowledge. Given a vector $s \in \mathbb{R}^n_{++}$ and an order $\sigma \in \Sigma$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. Note that, for any $i \in N$ we write, $\sum_{j \in P_i(\sigma)} s_j = 0$ if $P_i(\sigma) = \emptyset$. The agents have quasi-linear utility of the form $u_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i$ where $\tau_i \in \mathbb{R}$ is the transfer that he receives (pays). A *sequencing problem* is denoted by Ω and the set of all sequencing problems with a finite set of agents N (with $n \geq 2$) is given by $\mathcal{S}(N)$. A sequencing problem $\Omega \in \mathcal{S}(N)$ is called a *queueing problem* if $s = (s_1, \ldots, s_n)$ is such that $s_1 = \ldots = s_n$.[3]

[1] The authors are grateful to Hervé Moulin for his contributory talks at the Blaise Pascal Chair Lecture: Ten Lessons on Microeconomic Fairness.

[2] There is a notable literature that deals with the characterization of the Lorenz ordering as a plausible concept of inequality (see Atkinson et al. (1970), Dasgupta et al. (1973), Fields & Fei (1978)). Dutta and Ray (Dutta & Ray (1989), Dutta & Ray (1991)) were the first to propose the constrained egalitarian allocations as a solution concept for transferable utility games. The constrained egalitarian rule has also been proposed by Chun et al. (1998) to solve claim problems and attain certain objectives of equality and by Chun et al. (2019b) for queueing problems.

[3] There is a well established literature in this direction. See De (De (2016), De (2013)), De and Mitra (De & Mitra (2017), De & Mitra (2019)), Dolan (1978), Duives et al. (2015), Hain & Mitra (2004), Mitra (2002), Moulin (2007) and Suijs (1996).

A special case of sequencing problems where the processing times of the agents are identical is called queueing problems.[4]

Consider any $\Omega \in \mathcal{S}(N)$. A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. A mechanism $\mu = (\sigma, \tau)$ constitutes of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \to \Sigma$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma$. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile $\theta \in \Theta^n$ with a tie situation between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \to \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, given any mechanism $\mu = (\sigma, \tau)$, if (θ'_i, θ_{-i}) is the announced profile when the true waiting cost of i is θ_i, then utility of i is $u_i(\mu_i(\theta'_i, \theta_{-i}); \theta_i) = -\theta_i S_i(\sigma(\theta'_i, \theta_{-i})) + \tau_i(\theta'_i, \theta_{-i})$ where $\mu_i(\theta'_i, \theta_{-i}) := (\sigma(\theta'_i, \theta_{-i}), \tau_i(\theta'_i, \theta_{-i}))$.

3.3 Individual interests and other properties

The generalized minimum welfare bound is an all-inclusive lower bound that guarantees an assured level of utility to every agent (see Banerjee et al. (2020)). We represent a typical sequencing problem with the generalized minimum welfare bound by $\Gamma = (\Omega, O(N; s))$ where $\Omega \in \mathcal{S}(N)$ and the associated $O(N; s) \in \mathbb{R}^n$ is the welfare parameter vector.

Definition 13. For Γ, a mechanism $\mu = (\sigma, \tau)$ satisfies *generalized minimum welfare bound* (or GMWB) if the transfer rule $\tau : \Theta^n \to \mathbb{R}^n$ is such that for any $i \in N$, any $\theta_i \in \Theta$ and any $\theta_{-i} \in \Theta^{n-1}$,

$$u_i(\mu_i(\theta_i, \theta_{-i}); \theta_i) \geq -\theta_i O_i(s). \tag{3.1}$$

[4]Queueing problems have also been analyzed extensively from both normative and strategic viewpoints. See Chun (2006a), Chun (2006b), Chun et al. (2014b), Chun et al. (2017), Chun et al. (2019b), Chun et al. (2019a), Hashimoto (2018), Kayı & Ramaekers (2010), Maniquet (2003), Mitra (2001), Mitra (2007), Mitra & Mutuswami (2011) and Mukherjee (2013).

where $O_i(s)$ is the *welfare parameter* of agent i.

Remark 5. The GMWB is a universal representation encompassing fairness bounds and naturally constructed bounds that have been earlier studied in the literature. Its most natural application can be observed in sequencing problems with an (ex-ante) initial order where the minimum utility guaranteed is based on the agents' natural order of arrival. Identical Costs Bound (ICB) is yet another specific application of GMWB which requires that each agent receives at least the utility he could expect under the egalitarian solution if all agents were identical to him. The benchmark economy, based on which the bound is constructed, consists of agents having identical waiting costs as well as processing times. GMWB can be applied to sequencing environments that allow random arrival of queues with equal probability. Thus, the Expected Costs Bound (ECB) assures each agent a utility no less than his expected cost when every arrival order is equally likely. For a more detailed discussion of GMWB see Banerjee et al. (2020).

Definition 14. A mechanism μ satisfies *feasibility* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) \leq 0$.

Definition 15. A mechanism μ satisfies *budget balance* if for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_i(\theta) = 0$.

A sequencing rule σ^* is *outcome efficient* if for any $\theta \in \Theta^n$, $\sigma^*(\theta) \in \text{argmin}_{\sigma \in \Sigma} C(\sigma; \theta)$. The ratio of the waiting cost and processing time of any agent i, that is, θ_i/s_i is known as the urgency index. A sequencing rule σ^* is outcome efficient if and only if for any $\theta \in \Theta^n$, the selected order $\sigma^*(\theta)$ satisfies the following: for any $i, j \in N$, $\theta_i/s_i > \theta_j/s_j \Rightarrow \sigma_i^*(\theta) < \sigma_j^*(\theta)$. This condition was first introduced by Smith et al. (1956). We say that a mechanism $\mu = (\sigma, \tau)$ satisfies outcome efficiency if $\sigma = \sigma^*$.

Definition 16. A mechanism $\mu = (\sigma, \tau)$ satisfies outcome efficiency if $\sigma = \sigma^*$.

Definition 17. A mechanism μ satisfies *Pareto optimality* if it satisfies outcome efficiency and budget balance.

For any $\theta \in \Theta^n$ and any given order $\sigma \in \Sigma$, let us denote the *aggregate cost* by $C(\sigma;\theta)$, that is, $C(\sigma;\theta) := \sum_{j\in N}\theta_i S_i(\sigma)$.

Definition 18. A sequencing rule $\sigma^b : \Theta^n \to \Sigma$ is said to be a *Bounded Aggregate Cost (BAC)* rule if,

$$C(\sigma(\theta);\theta) \leq \sum_{i\in N}\theta_i O_i(s),\ \forall\, \theta \in \Theta^n. \tag{3.2}$$

Lemma 2. The following statements are equivalent.

(a1) A sequencing rule $\sigma : \Theta^n \to \Sigma$ is such that we can associate a mechanism μ that is budget balanced and satisfies GMWB.

(a2) A sequencing rule $\sigma : \Theta^n \to \Sigma$ is such that we can associate a mechanism μ that is feasible and satisfies GMWB.

(a3) The sequencing rule is a BAC rule.

The proof of Lemma 2 is provided in the Appendix. Consider any sequencing rule $\sigma : \Theta^n \to \Sigma$. For any $\theta \in \Theta^n$ and any $i \in N$, define $\Delta_i(\sigma(\theta);\theta_i) := \theta_i O_i(s) - \theta_i S_i(\sigma(\theta))$ and define $\Delta(\sigma(\theta);\theta) := \sum_{j\in N}\Delta_j(\sigma(\theta);\theta_j) = \sum_{i\in N}\theta_i O_i(s) - C(\sigma(\theta);\theta)$. Note that, for any BAC rule σ^b, $\Delta(\sigma^b(\theta);\theta) \geq 0$ for all $\theta \in \Theta^n$. The BAC rule puts an upper bound on the aggregate job completion cost, such that, it does not exceed the sum total of guarantees to all the agents under GMWB. If for any problem $\Gamma = (\Omega, O(N;s))$ we can find a BAC rule, then the outcome efficient rule is also a BAC rule. Given the outcome efficient sequencing rule σ^*, for any profile $\theta \in \Theta^n$ the selected order $\sigma^*(\theta)$ minimizes the aggregate cost, that is $C(\sigma^*(\theta);\theta) \leq C(\sigma;\theta)$ for all $\sigma \in \Sigma$. Hence, if we know that a BAC rule σ^b exists for some $\Gamma = (\Omega, O(N;s))$, then the aggregate cost associated with outcome efficiency is clearly no more than the aggregate cost under the rule σ^b and hence also no more than the aggregate guaranteed under GMWB, that is, for any $\theta \in \Theta^n$, $C(\sigma^*(\theta);\theta) \leq C(\sigma^b(\theta);\theta) \leq \sum_{i\in N}\theta_i O_i(s)$ implying condition (3.2). For any $\Gamma = (\Omega, O(N,s))$ we can find a BAC rule (which implies getting a mechanism which satisfies outcome efficiency, GMWB and feasibility) only if $O(N, s)$ is such that (a) $\sum_{j \in N} s_j \left\{ O_j(s) - (s_j + A(s))/2 \right\} \geq 0$ (see Lemma 1 in Banerjee et al. (2020)). One can show that if $O_i(s) \geq (s_j + A(s))/2$ for all $i \in N$, then we can find a BAC rule. Given Lemma 2, for the rest of the paper we restrict our attention to those $\Gamma = (\Omega, O(N; s))$ for which BAC rules exists.

3.4 Lorenz optimality and the constrained egalitarian mechanism

The Lorenz criterion has been recognized as a fundamental tool to make inequality comparisons in the distribution of income (or wealth). We denote $U_i^{\mu}(\theta) \equiv u_i(\sigma_i(\theta), \tau_i(\theta); \theta_i)$ to denote agent i's utility from mechanism μ at a profile θ. The utility of all agents is given by $U^{\mu}(\theta) \equiv (U_i^{\mu}(\theta))_{i \in N}$. The permutation of $U^{\mu}(\theta)$ such that $\overline{U}_1^{\mu}(\theta) \leq \overline{U}_2^{\mu}(\theta) \leq \ldots \leq \overline{U}_n^{\mu}(\theta)$ is denoted by $\overline{U}^{\mu}(\theta)$.

Definition 19. Let Y be a set of mechanisms and let $\mu, \nu \in Y$. We say that μ *Lorenz dominates* ν if for all profiles θ and for every $k = 1, 2, \ldots, n$ we have, $\sum_{r=1}^{k} \overline{U}_r^{\mu}(\theta) \geq \sum_{r=1}^{k} \overline{U}_r^{\nu}(\theta)$. The mechanism μ is *Lorenz optimal* in Y if μ *Lorenz dominates* all ν in Y.

Remark 6. We have two utility profiles $\overline{U}^{\mu}(\theta)$ from mechanism μ and $\overline{U}^{\nu}(\theta)$ from mechanism ν. We begin by comparing the worst utility under mechanism μ with that of ν. Next, we compare the sum of utilities of the two worst individuals in the respective utility profiles and continue till we reach the last stage which compares the aggregate utility under both the mechanisms. In simpler terms, we begin with the max-min (egalitarian) comparison and end with the utilitarian comparison.

A constrained egalitarian mechanism is a Pareto optimal mechanism that assigns each agent a utility which is the maximum of either his guaranteed level under GMWB or a defined common quantity $\lambda(\theta)$.

Definition 20. A mechanism μ_c is said to be *constrained egalitarian* if it is Pareto optimal and for each $\theta \in \Theta^n$ there exists a unique $\lambda(\theta)$ such that for all $i \in N$, $U_i^{\mu_c}(\theta) = \max\{\lambda(\theta), -\theta_i O_i(s)\}$.

Before stating our main theorem, we provide a step by step algorithm to compute the constrained egalitarian mechanism μ_c.

Remark 7. Given μ_c is Pareto optimal, observe that $\sum_{j\in N} \max\{\lambda(\theta), -\theta_i O_i(s)\} = -C(\sigma^*(\theta);\theta)$. We have the following observations,

- If $U_i^{\mu_c}(\theta) = \lambda(\theta)$, then $\lambda(\theta) \geq -\theta_i O_i(s)$ and for any $j \in N$ such that $\theta_j O_j(s) \geq \theta_i O_i(s)$, $U_j^{\mu_c}(\theta) = \lambda(\theta)$.
- If $U_i^{\mu_c}(\theta) = -\theta_i O_i(s) > \lambda(\theta)$, then for all $j \in N$ such that $\theta_j O_j(s) \leq \theta_i O_i(s)$, $U_j^{\mu_c}(\theta) = -\theta_j O_j(s)$.

We can thus partition N as $N^1 \cup N^2$ where $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\}$ and $N^2 = \{i \in N | U_i^{\mu_c}(\theta) = -\theta_i O_i(s)\}$. Observe that for any $i \in N^1$ and any $j \in N^2$, $U_i^{\mu_c}(\theta) < U_j^{\mu_c}(\theta)$.

Remark 8. If a profile $\theta \in \Theta^n$ is such that $\theta \in I(\sigma^*)$ so that $\theta_1/s_1 = \ldots = \theta_n/s_n$ implies that the aggregate cost is a constant for all possible orders in Σ, then we have $U_j^{\mu_c}(\theta) = -\theta_j O_j(s)$ for all $j \in N$ and we have $N^1 = \emptyset$ and $N^2 = N$. Suppose there exists $i, j, k \in N$ such that $s_i \neq s_j \neq s_k \neq s_i$. Then one can also show that if a profile $\theta \in \Theta^n$ is such that $\theta_1 = \ldots = \theta_n$, then $N^1 = N$ and $N^2 = \emptyset$.

For any $\theta \in \Theta^n$, let $Q(\theta) = (Q_1(\theta) = -\theta_1 O_1(s), \ldots, Q_n(\theta) = -\theta_n O_n(s))$ be the vector of guarantees under GMWB for all the agents and $\tilde{Q}(\theta)$ be the permutation of $Q(\theta)$ such that $\tilde{Q}_1(\theta) \geq \ldots \geq \tilde{Q}_n(\theta)$ and, for every $k \in \{1, \ldots, n\}$. Let $\tilde{U}_k^{\mu}(\theta)$ be the utility associated with the agent with k-th ranked agent specific guarantee under GMWB $\tilde{Q}_k(\theta)$. The algorithm for calculating the profile contingent allocation associated with the constrained egalitarian mechanism μ_c is the following: Consider any $\theta \in \Theta^n$.

Step 1: If $-C(\sigma^*(\theta);\theta)/n \geq \tilde{Q}_1(\theta)$, then we stop and set $\lambda(\theta) = -C(\sigma^*(\theta);\theta)/n$ so that $\tilde{U}_r^{\mu_c}(\theta) = \lambda(\theta) = -C(\sigma^*(\theta);\theta)/n \geq \tilde{Q}_r(\theta)$ for all $r \in \{1, \ldots, n\}$. If $-C(\sigma^*(\theta);\theta)/n < \tilde{Q}_1(\theta)$, then set $\tilde{U}^{\mu_c}_1(\theta) = \tilde{Q}_1(\theta)$ and go to Step 2.

$\vdots$

Step k: Consider any $k > 1$. If $-C(\sigma^*(\theta);\theta) - \sum_{r=1}^{k-1} \tilde{Q}_j(\theta) \geq (n-k+1)\tilde{Q}_k(\theta)$, then stop and set $\lambda(\theta) = [-C(\sigma^*(\theta);\theta) - \sum_{r=1}^{k-1} \tilde{Q}_r(\theta)]/(n-k+1)$ so that (given the previous steps of the algorithm) $U^{\mu_c}_r(\theta) = \tilde{Q}_r(\theta)$ for all $r \in \{1,\ldots,k-1\}$ and $\tilde{U}^{\mu_c}_r(\theta) = \lambda(\theta) \geq \tilde{Q}_r(\theta)$ for all $r \in \{k,\ldots,n\}$. If $-C(\sigma^*(\theta);\theta) - \sum_{r=1}^{k-1} \tilde{Q}_j(\theta) < (n-k+1)\tilde{Q}_k(\theta)$, then set $\tilde{U}^{\mu_c}_k(\theta) = \tilde{Q}_k(\theta)$ so that (given the previous steps of the algorithm) we have $\tilde{U}^{\mu_c}_r(\theta) = \tilde{Q}_r(\theta)$ for all $r \in \{1,\ldots,k\}$, and then go to Step $(k+1)$.

Given that in each step of the algorithm the utility of at least one agent is determined and given $-C(\sigma^*(\theta);\theta) \geq -\sum_{r=1}^{n} \tilde{Q}_j(\theta)$ (since $\Delta(\sigma^*(\theta);\theta) \geq 0$ for any profile θ), we can have at most $n-1$ steps before the utilities of all the agents are determined. Observe that for any $\theta \in \Theta^n$, $\tilde{U}(\theta)$ is such that $\tilde{U}^{\mu_c}_1(\theta) \geq \ldots \geq \tilde{U}^{\mu_c}_n(\theta)$ and hence $\overline{U}^{\mu_c}(\theta)$ is such that $\overline{U}^{\mu_c}_1(\theta) = \tilde{U}^{\mu_c}_n(\theta) \leq \ldots \leq \overline{U}^{\mu_c}_n(\theta) = \tilde{U}^{\mu_c}_1(\theta)$.

Theorem 3. The mechanism μ is Lorenz optimal in the class of mechanisms satisfying feasibility and GMWB if and only if $\mu = \mu_c$.

Given Lemma 2, we try to identify the complete class of transfers that work.[5] For any sequencing rule σ, define $I(\sigma) := \{\theta \in \Theta \mid \Delta(\sigma(\theta);\theta) = 0\}$. In the previous section, we found that a BAC sequencing rule is both necessary and sufficient to find mechanisms that are feasible (budget balanced) and satisfy GMWB. In this section, we characterize this entire class of mechanisms.

[5] Lemma 2 and Theorem 3 are important generalizations of results from Chun et al. (2019b). They show that for queueing problems, constrained egalitarian mechanisms are Lorenz optimal amongst the class of mechanisms satisfying outcome efficiency, budget balance and the identical preferences lower bound (IPLB). In the sequencing context, we identify the class of mechanisms which are Lorenz optimal, feasible and satisfy GMWB. Note that, by construction, the generalized minimum welfare bound entails the existing fairness bounds as well as the natural bounds that are based on the initial queue.

Definition 21. For any sequencing rule σ, a mechanism μ is *F-acceptable* (*B-acceptable*) if the following conditions hold.

(f1) For any $\theta \in \Theta^n \setminus I(\sigma)$, there exists a vector $\beta(\theta) = (\beta_1(\theta), \ldots, \beta_n(\theta)) \in [0,1]^n$ such that $\sum_{j \in N} \beta_j(\theta) \leq 1 (= 1)$, and, for each $i \in N$, $\tau_i(\theta) = \beta_i(\theta)\Delta(\sigma(\theta); \theta) - \Delta_i(\sigma(\theta); \theta_i)$.

(f2) For any $\theta \in I(\sigma)$, $\tau_i(\theta) = -\Delta_i(\sigma(\theta); \theta_i)$ for any $i \in N$.

Lemma 3. For any BAC sequencing rule σ^b, a mechanism $\mu = (\sigma^b, \tau)$ satisfies GMWB and feasibility (budget balance) if and only if it is an F-*acceptable* (B-*acceptable*) mechanism.

For the proof of Lemma 3 see appendix. Let $\mathcal{W}$ denote the set of all Bounded Aggregate Cost (BAC) sequencing rules. We use $B(\sigma^b)$ to denote the set of all B-*acceptable* mechanisms associated with a given BAC sequencing rule $\sigma^b \in \mathcal{W}$ and we use $B(\mathcal{W}) = \{\{B(\sigma^b)\}_{\sigma^b \in \mathcal{W}}\}$ to denote the set of all possible B-*acceptable* mechanisms associated with all BAC sequencing rules. Similarly, we use $F(\sigma^b)$ to denote the set of all F-*acceptable* mechanisms associated with a given $\sigma^b \in \mathcal{W}$ and we use $F(\mathcal{W}) = \{\{F(\sigma^b)\}_{\sigma^b \in \mathcal{W}}\}$ to denote the set of all possible F-*acceptable* mechanisms associated with all possible BAC sequencing rules. Clearly, for each $\sigma^b \in \mathcal{W}$, $B(\sigma^b) \subset F(\sigma^b)$ and hence we have $B(\mathcal{W}) \subset F(\mathcal{W})$. We provide two more results and the proves of these two results are provided in the appendix.

Lemma 4. If a mechanism μ is Lorenz optimal in $F(\mathcal{W})$, then μ must be Pareto optimal, that is, $\mu \in B(\sigma^*)$.

Lemma 5. The constrained egalitarian mechanism μ_c is Lorenz optimal in set of all Pareto optimal mechanisms $B(\sigma^*)$.

Proof of Theorem 3: If a mechanism μ is in the class of mechanisms satisfying feasibility and GMWB, then by Lemma 2 the associated sequencing rule has to be a BAC rule and if the mechanism is also

Lorenz optimal, then by Lemma 4 the sequencing rule cannot be just any BAC rule but has to be the outcome efficient rule σ^*. Moreover, by Lemma 4 it also follows that any Lorenz optimal mechanism must also be budget balanced. Hence, if we have a Lorenz optimal mechanism satisfying feasibility and GMWB, then it has to be in the Pareto optimal set of mechanisms $B(\sigma^*)$. To complete the proof all we need to show is that the constrained egalitarian mechanism μ_c is Lorenz optimal in the set of all Pareto optimal mechanisms $B(\sigma^*)$. This is done in Lemma 5. □

3.5 Appendix

Proof of Lemma 2: If for a sequencing rule $\sigma : \Theta^n \to \Sigma$ we can associate a mechanism that satisfies GMWB and is budget balanced, then, given budget balance implies feasibility, for that same sequencing rule σ we have obtained a mechanism that satisfies GMWB and feasibility. Hence, (a1) implies (a2). To complete the proof we first show (a2) implies (a3) and then show (a3) implies (a1).

(a2)⇒(a3): Consider any sequencing rule σ for which we can associate a mechanism (σ, τ) that satisfies GMWB and feasibility. For any profile $\theta \in \Theta^n$ and any agent $i \in N$, the GMWB condition $u_i(\mu_i(\theta); \theta_i) \geq -\theta_i O_i(s)$ implies (A1) $\tau_i(\theta) \geq -\Delta_i(\sigma(\theta); \theta_i)$ for any $i \in N$. Fix any $i \in N$ and consider the sum $\sum_{j \neq i} \tau_j(\theta)$. Using (A1) for all $j \in N \setminus \{i\}$ and the feasibility condition we get $\tau_i(\theta) \leq -\sum_{j \neq i} \tau_j(\theta) \leq \sum_{j \neq i} \Delta_j(\sigma(\theta); \theta_j) = \Delta(\sigma(\theta); \theta) - \Delta_i(\sigma(\theta); \theta_i)$. Therefore, we also have (A2) $\tau_i(\theta) \leq \Delta(\sigma(\theta); \theta) - \Delta_i(\sigma(\theta); \theta_i)$ for any $i \in N$. Combining (A1) and (A2) we get

$$-\Delta_i(\sigma(\theta); \theta_i) \leq \tau_i(\theta) \leq \Delta(\sigma(\theta); \theta) - \Delta_i(\sigma(\theta); \theta_i), \ \forall \theta \in \Theta^n \text{ and } \forall i \in N. \tag{3.3}$$

Condition (3.3) implies that for all $\theta \in \Theta^n$

$$\Delta(\sigma(\theta); \theta) = \sum_{i \in N} \theta_i O_i(s) - C(\sigma(\theta); \theta) \geq 0. \tag{3.4}$$

Given the outcome efficient sequencing rule σ^*, for any profile $\theta \in \Theta^n$ the selected order $\sigma^*(\theta)$ minimizes the aggregate cost, that is $C(\sigma^*(\theta);\theta) \leq C(\sigma;\theta)$ for all $\sigma \in \Sigma$. Hence, if we know that for a rule σ^b condition (3.4) exists for some $\Gamma = (\Omega, O(N;s))$, then the aggregate cost associated with outcome efficiency is clearly no more than the aggregate cost under the rule σ^b and hence no more than the aggregate guaranteed under GMWB, that is, for any $\theta \in \Theta^n$, $C(\sigma^*(\theta);\theta) \leq C(\sigma^b(\theta);\theta) \leq \sum_{i \in N} \theta_i O_i(s)$ implying condition (3.4). Hence, a necessary condition for condition (3.4) to hold is that

$$\sum_{i \in N} \theta_i O_i(s) - C(\sigma^*(\theta);\theta) \geq 0,\ \forall \theta \in \Theta^n. \tag{3.5}$$

If $\Delta(\sigma(\theta);\theta)(= \sum_{i \in N} \theta_i O_i(s) - C(\sigma(\theta);\theta)) \geq 0$, then the BAC condition (3.2) holds and hence we have $\sigma = \sigma^*$.

(a3)⇒(a1): Consider any BAC rule σ^b and any associated mechanism $\mu^b = (\sigma^b, \tau^b)$ such that for any $\theta \in \Theta^n$, $\tau_i^b(\theta) = (1/n)\Delta(\sigma^b(\theta);\theta) - \Delta_i(\sigma^b(\theta);\theta_i)$ for each $i \in N$. Observe that for any $\theta \in \Theta^n$, $\sum_{j \in N} \tau_j^b(\theta) = \Delta(\sigma^b(\theta);\theta) - \sum_{j \in N} \Delta_j(\sigma^b(\theta);\theta) = \Delta(\sigma^b(\theta);\theta) - \Delta(\sigma^b(\theta);\theta) = 0$. Hence, the mechanism μ^b satisfies budget balancedness. Further, for any θ and any i, $u_i(\mu_i^b(\theta);\theta_i) = -\theta_i S_i(\sigma^b(\theta)) + \tau_i^b(\theta) = (1/n)\Delta(\sigma^b(\theta);\theta) - \theta_i O_i(s) \geq -\theta_i O_i(s)$ since BAC condition (3.2) implies $\Delta(\sigma^*(\theta);\theta) \geq 0$. Hence, the mechanism μ^b also satisfies GMWB. Thus, for σ^b, we have found μ^b that satisfies GMWB and is budget balanced. Since the selection of the BAC rule σ^b was arbitrary, the result follows. □

Proof of Lemma 3: Consider any BAC sequencing rule σ^b for which we can associate a mechanism $\mu = (\sigma^b, \tau)$ that satisfies GMWB and budget balancedness (feasibility). For any profile $\theta \in \Theta^n$ and any agent $i \in N$, the GMWB condition $u_i(\mu_i(\theta);\theta_i) \geq -\theta_i O_i(s)$ implies (B1) $\tau_i(\theta) \geq -\Delta_i(\sigma^b(\theta);\theta_i)$ for any $i \in N$. Fix any $i \in N$ and consider the sum $\sum_{j \neq i} \tau_j(\theta)$. Using (B1) for all $j \in N \setminus \{i\}$ and budget balance (feasibility) condition we get $\tau_i(\theta) = (\leq) - \sum_{j \neq i} \tau_j(\theta) \leq \sum_{j \neq i} \Delta_j(\sigma^b(\theta);\theta_j) = \Delta(\sigma^b(\theta);\theta) - \Delta_i(\sigma^b(\theta);\theta_i)$. Therefore, we also have (B2)

$\tau_i(\theta) \leq \Delta(\sigma^b(\theta);\theta) - \Delta_i(\sigma^b(\theta);\theta_i)$ for any $i \in N$. Combining (B1) and (B2) we get that for any Bounded Aggregate Cost (BAC) sequencing rule, the associated mechanism (σ^b, τ) satisfies GMWB and budget balance (feasibility) only if

$$-\Delta_i(\sigma^b(\theta);\theta_i) \leq \tau_i(\theta) \leq \Delta(\sigma^b(\theta);\theta) - \Delta_i(\sigma^b(\theta);\theta_i),\ \forall \theta \in \Theta^n \text{ and } \forall i \in N. \tag{3.6}$$

From condition (3.6) it follows that given any profile $\theta \in \Theta^n \setminus I(\sigma^b)$, for each $i \in N$ there exists $\beta_i(\theta) \in [0,1]$ such that $\tau_i(\theta) = \beta_i(\theta)\Delta(\sigma^b(\theta);\theta) - \Delta_i(\sigma^b(\theta);\theta_i)$. Therefore, using the definition $\Delta(\sigma^b(\theta);\theta) := \sum_{j\in N} \Delta_j(\sigma^b(\theta);\theta_j)$, it follows that $\sum_{j\in N} \tau_j(\theta) = \Delta(\sigma^b(\theta);\theta)\left(\sum_{j\in N} \beta_j(\theta)\right) - \Delta(\sigma^b(\theta);\theta)$. Hence, from budget balancedness (feasibility) we get $\sum_{j\in N} \beta_j(\theta) = 1$ ($\sum_{j\in N} \beta_j(\theta) \leq 1$). Moreover, from condition (3.6) it also follows that given any profile $\theta \in I(\sigma^b)$, for each $i \in N$, $\tau_i(\theta) = -\Delta_i(\sigma^b(\theta);\theta_i)$ (since $\Delta(\sigma^b(\theta);\theta) = 0$). This proves the necessity of B-*acceptable* (F-*acceptable*) mechanisms.

For the converse, consider any BAC sequencing rule σ^b and observe that any associated B-*acceptable* (F-*acceptable*) mechanism satisfies budget balancedness (feasibility). Moreover, for any B-*acceptable* (F-*acceptable*) mechanism μ, for any $\theta \in \Theta^n \setminus I(\sigma)$ and any $i \in N$, $u_i(\mu_i(\theta);\theta_i) = -\theta_i S_i(\sigma^b(\theta)) + \tau_i(\theta) = \beta_i(\theta)\Delta(\sigma^b(\theta);\theta) - \theta_i O_i(s) > -\theta_i O_i(s)$ (since for any profile $\theta \in \Theta^n \setminus I(\sigma)$, $\Delta(\sigma^b(\theta);\theta) > 0$). Moreover, for any $\theta \in I(\sigma^b)$ and any $i \in N$, $u_i(\mu_i(\theta);\theta_i) = -\theta_i S_i(\sigma^b(\theta)) + \tau_i(\theta) = -\theta_i O_i(s)$. Hence, we also have GMWB being satisfied for any B-*acceptable* (F-*acceptable*) mechanism μ. □

Proof of Lemma 4: Suppose $\mu = (\sigma^b, \tau)$ is Lorenz optimal in $F(\mathcal{W})$ and that it satisfies feasibility but is not budget balanced. Then there exists $\theta \in \Theta^n$ such that the associated allocation is $\mu(\theta) = (\mu_1(\theta) = (\sigma^b(\theta), \tau_1(\theta)), \ldots, \mu_n(\theta) = (\sigma^b(\theta), \tau_n(\theta)))$ and $\sum_{j\in N} \tau_j(\theta) < 0$. Consider a mechanism ν that retains the same rule σ^b and that for the same profile θ gives an allocation $\nu(\theta) = (\nu_1(\theta) = (\sigma^b(\theta), \tau'_1(\theta)), \ldots, \nu_n(\theta)) = (\sigma^b(\theta), \tau'_n(\theta)))$ where $\tau'_i(\theta) = \tau_i(\theta) + \varepsilon$ for all $i \in N$, where $\varepsilon = -\{\sum_{j\in N}\tau_j(\theta)\}/n > 0$. Observe that $\sum_{j\in N}\tau'_j(\theta) = \sum_{j\in N}\tau_j(\theta) + n\varepsilon = \sum_{j\in N}\tau_j(\theta) - \sum_{j\in N}\tau_j(\theta) = 0$. Therefore, this new set of transfers under ν for profile θ is budget balanced. More importantly, $u_i(\nu_i(\theta);\theta) - u_i(\mu_i(\theta);\theta) = \varepsilon > 0$ for all $i \in N$ implies that $u_i(\nu_i(\theta);\theta) > u_i(\mu_i(\theta);\theta) \geq -\theta_i O_i(s)$ for all $i \in N$. Thus, for the allocation $\nu(\theta)$ under the mechanism ν for the profile θ satisfies GMWB. Therefore, we have a violation of Lorenz optimality of μ since $\sum_{r=1}^{n}\overline{U}_r^{\nu}(\theta) = \sum_{i\in N}u_i(\nu_i(\theta);\theta) > \sum_{i\in N}u_i(\mu_i(\theta);\theta) = \sum_{r=1}^{n}\overline{U}_r^{\mu}(\theta)$. Hence, any Lorenz optimal mechanism must be budget balanced, that is, if $\mu \in F(\mathcal{W})$ is Lorenz optimal, then $\mu \in B(\mathcal{W})$. Therefore, to identify a Lorenz optimal mechanism from $F(\mathcal{W})$ we must restrict our search within the set $B(\mathcal{W})$.

Suppose, $\mu = (\sigma^b, \tau) \in B(\mathcal{W})$ is Lorenz optimal. Then it is necessary that for all profiles $\theta \in \Theta^n$, (I) $\sum_{r=1}^{n}\bar{U}_r^{\mu}(\theta) = \sum_{i\in N}\tau_i(\theta) - \sum_{i\in N}\theta_i S_i(\sigma^b(\theta)) \geq \sum_{i\in N}\hat{\tau}_i(\theta) - \sum_{i\in N}\theta_i S_i(\hat{\sigma}^b(\theta)) = \sum_{r=1}^{n}\bar{U}_r^{\nu}(\theta)$ where $\sigma^b, \hat{\sigma}^b \in \mathcal{W}$. Using budget balance, we know that $\sum_{i\in N}\tau_i(\theta) = \sum_{i\in N}\hat{\tau}_i(\theta) = 0$ and hence from condition (I) we get $\sum_{i\in N}\theta_i S_i(\sigma^b(\theta)) \leq \sum_{i\in N}\theta_i S_i(\hat{\sigma}^b(\theta))$ implying that the aggregate cost associated with the order $\sigma^b(\theta)$ is not higher than the aggregate cost associated with $\hat{\sigma}^b(\theta)$. Therefore, if $\mu = (\sigma^b, \tau) \in B(\mathcal{W})$ is Lorenz optimal, then for all $\hat{\sigma}^b \in \mathcal{W}$ such that $\hat{\sigma}^b \neq \sigma^b$ and all $\theta \in \Theta^N$, (II) $C(\sigma^b(\theta);\theta) \leq C(\hat{\sigma}^b(\theta);\theta)$. Condition (II) means that $\sigma^b = \sigma^*$, that is, the sequencing rule σ^b associated with the Lorenz optimal mechanism μ must be outcome efficient. □

Proof of Lemma 5: Given the constrained egalitarian mechanism μ_c, we first divide the set of all possible profiles into three mutually exclusive and exhaustive sets. These sets are the following:

1. Profiles $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} = \emptyset$.
2. Profiles $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} = N$.
3. Profiles $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} \neq \emptyset$ and we also have $N^2 = \{i \in N | U_i^{\mu_c}(\theta) = -\theta_i O_i(s)\} \neq \emptyset$.

Separately, for each of above three cases, we show that for any such profile θ, $\sum_{r=1}^{k}\overline{U}_r^{\mu_c}(\theta) \geq$

$\sum_{r=1}^{k} \overline{U}_r^{\mu}(\theta)$ for any $k \in \{1, \ldots, n\}$ and for any $\mu \in B(\sigma^*)$.

1. For any $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} = \emptyset$ and $N^2 = \{i \in N | U_i^{\mu_c}(\theta) = -\theta_i O_i(s)\} = N$, $\overline{U}_j^{\mu_c}(\theta) = -\theta_j O_j(s)$ for all $j \in N$. Further, for any B-*acceptable* mechanism $\mu \in B(\sigma^*)$, $\overline{U}_j^{\mu}(\theta) = \overline{U}_j^{\mu_c}(\theta) = -\theta_j O_j(s)$ for all $j \in N$. Hence, for any such profile θ, $\sum_{r=1}^{k} \overline{U}_r^{\mu_c}(\theta) = \sum_{r=1}^{k} \overline{U}_r^{\mu}(\theta)$ for any $k \in \{1, \ldots, n\}$ and for any $\mu \in B(\sigma^*)$.

2. For any $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} = N$ and $N^2 = \{i \in N | U_i^{\mu_c}(\theta) = -\theta_i O_i(s)\} = \emptyset$, it follows that $\overline{U}_j^{\mu_c}(\theta) = \lambda(\theta) = -C(\sigma^*(\theta); \theta)/n$ for all $j \in N$ (see Step 1 of the algorithm). Moreover, given $\sum_{j \in N} \overline{U}_j^{\mu_c}(\theta) = \sum_{j \in N} \overline{U}_j^{\mu}(\theta) = -C(\sigma^*(\theta); \theta)$ for any B-*acceptable* mechanism $\mu \in B(\sigma^*)$, it follows that for any mechanism $\mu \in B(\sigma^*) \setminus \{\mu_c\}$, for any $r = 1, \ldots, n$, $\bar{U}_r^{\mu}(\theta) = \lambda(\theta) + \varepsilon_r$ where $\varepsilon_1 \leq \ldots \leq \varepsilon_n$ and $\sum_{r=1}^{n} \varepsilon_r = 0$. Hence, $\sum_{r=1}^{k} \varepsilon_r \leq 0$ for all $k \in \{1, \ldots, n\}$. Therefore, $\sum_{r=1}^{k} \bar{U}_r^{\mu_c}(\theta) - \sum_{r=1}^{k} \bar{U}_r^{\mu}(\theta) = -\sum_{r=1}^{k} \varepsilon_r \geq 0$ for all $k \in \{1, \ldots, n\}$. Hence, for any such profile θ, we always have $\sum_{r=1}^{k} \overline{U}_r^{\mu_c}(\theta) \geq \sum_{r=1}^{k} \overline{U}_r^{\mu}(\theta)$ for any $k \in \{1, \ldots, n\}$ and for any $\mu \in B(\sigma^*)$.

3. To complete the proof we have to show that for any profile $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} \neq \emptyset$ and $N^2 = \{i \in N | U_i^{\mu_c}(\theta) = -\theta_i O_i(s)\} \neq \emptyset$, $\sum_{r=1}^{k} \overline{U}_r^{\mu_c}(\theta) \geq \sum_{r=1}^{k} \overline{U}_r^{\mu}(\theta)$ for any $k \in \{1, \ldots, n\}$ and for any $\mu \in B(\sigma^*)$. Suppose to the contrary that there exists a profile $\theta \in \Theta^n$ such that $N^1 = \{i \in N | U_i^{\mu_c}(\theta) > -\theta_i O_i(s)\} \neq \emptyset$ and $N^2 = \{i \in N | U_i^{\mu_c}(\theta) = -\theta_i O_i(s)\} \neq \emptyset$, there exists is a mechanism $\mu \in B(\sigma^*)$, and, there exists $k \in \{1, \ldots, n-1\}$, such that $\sum_{r=1}^{k} \bar{U}_r^{\mu}(\theta) > \sum_{r=1}^{k} \bar{U}_r^{\mu_c}(\theta)$. Without loss of generality, assume that $-\theta_1 O_1(s) \leq \cdots \leq -\theta_n O_n(s)$ so that there exists an $m \in N$ such that $U_i^{\mu_c}(\theta) = \lambda(\theta)$

for any $i \in \{1, \ldots, m\}$ and $U_i^{\mu_c}(\theta) = -\theta_i O_i(s)$ for any $i \in \{m+1, \ldots, n\}$.[6] If $k \geq m$, then there is $\ell > m$ such that $U_\ell^{\mu}(\theta) < U_\ell^{\mu_c}(\theta) = -\theta_\ell O_\ell(s)$. Hence, μ does not satisfy GMWB, a contradiction to $\mu \in B(\sigma^*)$. If $k < m$, then there is $\ell \in \{k+1, \ldots, m\}$ such that $U_\ell^{\mu}(\theta) < U_\ell^{\mu_c}(\theta) = \lambda(\theta)$. Since $(1/k)\sum_{i=1}^{k} \bar{U}_i^{\mu_c}(\theta) = \lambda(\theta)$, $(1/k)\sum_{i=1}^{k} \bar{U}_i^{\mu}(\theta) > \lambda(\theta)$ and $\bar{U}_\ell^{\mu}(\theta) < \lambda(\theta)$, we have a contradiction to the assumption that $k < \ell$.

□

[6]Note that for this profile $|N^1| = m$ and $|N^2| = n - m$.

4

Existence of core in sequencing problems-optimistic and pessimistic approach

4.1 Introduction

This paper adopts a cooperative approach to study sequencing problems. We work using the standard sequencing framework that has been consistently followed in this thesis. There is a finite set of agents who have to process their jobs at a service facility. Each agent has a single job to process and the service provider can only cater to one agent at a time. We assume that no job can be interrupted once it starts processing. A job is characterized by its processing time (which in our framework differs across agents) and an agent's per period waiting cost (representing the disutility of waiting in a queue). Both these parameters are observable and do not need to be evoked from the agents.[1] We assume that agents have quasi-linear preferences and monetary transfers are allowed. If agents are served in an efficient order, the immediate question that arises is, how to compensate the agents so that the cost burden gets divided across jobs in a fair and equitable manner.

The class of sequencing problems are a subclass of allocation problems with indivisible objects. This general class has been examined from the cooperative game point of view (see Abdulkadiroğlu & Sönmez (1998), Moulin (1992)) as well as from the fair allocation perspective (see Alkan et al. (1991), Crès & Moulin (1998), CreÁs & Moulin (1998); Tadenuma (1996), Tadenuma & Thomson (1991), Tadenuma & Thomson (1993), Tadenuma & Thomson (1995); Thomson (2003)). A popular approach to studying cost sharing problems (Moulin (2002)) involves associating an appropriate characterization form to the original problem and implementing solution concepts from the theory of cooperative games. The Shapley value is considered as an appropriate solution to fair division problems in general and has been shown to possess interesting fairness properties (Moulin (1992)).

Queueing problems are a special class of sequencing problems with identical job processing times. Maniquet (2003) studied queueing games and defines the worth of a coalition to be the (negative of the) minimum aggregate waiting cost of its members if they are to be served first in the queue. In other words, it is the least possible cost that the coalition incurs if no other agents (non-coalitional members) are present. He designed the monetary compensations (minimal transfer rule) so that every agent's utility corresponds to their respective Shapley values in the associated characteristic form game. Chun

[1] Sequencing problems with incentives have been analyzed extensively in the literature. A few notable contributions include Dolan (1978),Mendelson & Whang (1990), Suijs (1996), Mitra (2001), Mitra (2002), De (2016), Banerjee et al. (2020)

(2006b) adopted a pessimistic approach towards evaluating the worth of a coalition by computing the (negative of the) minimum waiting cost of that coalition if they are served after the non-coalitional members. He too showed that the utility of the agents under the maximal transfer rule corresponds to their respective Shapley values. Both Maniquet (2003) and Chun (2006b) characterize the Shapley value solution using classic fairness axioms.

Curiel et al. (1989) are the first to study cooperation in sequencing situations for which an initial order of agents exists before jobs start getting processed. Every sequencing problem is associated with a cooperative transferable utility (TU) game. The worth of a coalition is the maximum cost savings that can be obtained through admissible rearrangements. They introduce the equal gain splitting (EGS) rule and show that such an allocation rule assigns to each sequencing game a particular core allocation. Mishra & Rangarajan (2007) provide the Shapley value characterization for the two dimensional case when jobs are not identical (the more general class of sequencing games). Further, Moulin (2007) studies scheduling problems in which agent have linear waiting costs but arbitrary job lengths. The server can monitor the length of the job but not the identity of the user; thus leading merging, splitting or partially transferring jobs to offer cooperative strategic opportunities. It is shown that the Shapley value solution is merge proof, but not split proof.

4.1.1 Contribution

The paper is broadly divided into three sections. The first section defines the worth of a coalition using the optimistic approach of Maniquet (2003) and calculates the Shapley value solution of the associated cooperative form game. Maniquet provides an alternate way of defining the worth of a coalition in terms of its dual. For queueing games, he interprets this as the (negative of the) total waiting cost of its members as if they are the last to arrive. In other words, if they have to compensate the non-members for the additional cost imposed. Similarly, when the underlying rule is efficient, we define the dual for sequencing games as the aggregate job completion cost of the members in the grand coalition along with the incremental cost they impose on non-members. Our first theorem shows that the core of the primal game is empty while the Shapley value belongs to the core of the dual game, thus proving the non-emptiness of core in the dual game. The second section defines the worth of a coalition from the pessimistic viewpoint of Chun (2006b) and calculates the Shapley value of the corresponding cooperative game. Our second theorem states that the Shapley value belongs to the core of the primal game while the core of the associated dual game is empty. Under this approach, the worth of a coalition in the dual game can be interpreted as follows: it is the (negative of the) aggregate waiting cost incurred by the members as if they were the first to arrive, that is, if the non-members compensate the additional gain to the members of the coalition. More simply, the worth of a coalition is the (net) aggregate cost, which is the sum of its member's waiting cost in the grand coalition minus the incremental benefit received as compensation from the non-members. The last section connects the paper to the generalized minimum welfare bound introduced by Banerjee et al. (2020). For any given sequencing problem, we provide a necessary and sufficient condition on the constrained welfare parameter (defined in Banerjee et al. (2020)) for the allocation rule, assigning Shapley value payoffs of the corresponding sequencing game, to satisfy the generalized minimum welfare bound property. This result holds for both the optimistic approach by Maniquet and the pessimistic approach by Chun.

4.2 The model

A finite set of agents $N = \{1, 2, \ldots, n\}$ want to process their jobs. An agent $i \in N$ is identified by his job processing time $s_i \in \mathbb{R}_{++}$ and his constant per period waiting cost $\theta_i \in \Theta := \mathbb{R}_{++}$. An order $\sigma = (\sigma_1, \ldots, \sigma_n)$ on N describes the position of each agent where $\sigma_i = k$ indicates that agent i has the k-th position in the queue. Let Σ be the set of $n!$ possible orders on N. We denote $P_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j < \sigma_i\}$ as the predecessor set and $F_i(\sigma) = \{j \in N \setminus \{i\} \mid \sigma_j > \sigma_i\}$ as the successor set of i in the order σ. The processing time vector $s = (s_1, \ldots, s_n) \in \mathbb{R}^n_{++}$ is common knowledge. Given a vector $s \in \mathbb{R}^n_{++}$ and an order $\sigma \in \Sigma$, the cost of job completion for agent $i \in N$ is $\theta_i S_i(\sigma)$, where the job completion time is $S_i(\sigma) = \sum_{j \in P_i(\sigma)} s_j + s_i$. The agents have quasi-linear utility of the form $u_i(\sigma, \tau_i; \theta_i) = -\theta_i S_i(\sigma) + \tau_i$ where $\tau_i \in \mathbb{R}$ is the transfer that he receives (pays). A *sequencing problem* is denoted by $\Omega = (N, \theta, s)$ and the set of all sequencing problems with a finite set of agents N (with $n \geq 2$) is given by $\mathcal{S}(N)$.

Consider any $\Omega \in \mathcal{S}(N)$. A typical profile of waiting costs is denoted by $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$. An allocation $\mu = (\sigma, \tau)$ constitutes of a sequencing rule σ and a transfer rule τ. A *sequencing rule* is a function $\sigma : \Theta^n \rightarrow \Sigma$ that specifies for each profile $\theta \in \Theta^n$ a unique order $\sigma(\theta) = (\sigma_1(\theta), \ldots, \sigma_n(\theta)) \in \Sigma$. We use the following tie-breaking rule. We take the linear order $1 \succ 2 \succ \ldots \succ n$ on the set of agents N. For any sequencing rule σ and any profile $\theta \in \Theta^n$ with a tie situation between agents $i, j \in N$, we pick the order $\sigma(\theta)$ with $\sigma_i(\theta) < \sigma_j(\theta)$ if and only if $i \succ j$. A *transfer rule* is a function $\tau : \Theta^n \rightarrow \mathbb{R}^n$ that specifies for each profile $\theta \in \Theta^n$ a transfer vector $\tau(\theta) = (\tau_1(\theta), \ldots, \tau_n(\theta)) \in \mathbb{R}^n$. Specifically, for each $i \in N$, σ_i denotes agent i's position in the queue and τ_i is the monetary compensation (to be paid, when transfers are negative or received, when transfers are positive).

An allocation μ is *feasible* if and only if the sum of transfers is not positive, i.e., the set of feasible allocations $\mathcal{F}(\Omega)$ is defined by $\mu = (\sigma, \tau) \in \mathcal{F}(\Omega)$ if and only if $\forall i, j \in N$ we have, $\sum_{i \in N} \tau_i \leq 0$. For any $\theta \in \Theta^n$ and any given order $\sigma \in \Sigma$, we denote the *aggregate cost* by $C(\sigma; \theta)$, that is, $C(\sigma; \theta) := \sum_{j \in N} \theta_i S_i(\sigma)$. An allocation is *efficient* for $\Omega \in \mathcal{S}(N)$ whenever it minimizes the aggregate cost of job completion and no transfer is lost (sum of transfers in zero), i.e., for all $\mu = (\sigma, \tau) \in \mathcal{F}(\Omega)$, μ is efficient if and only if for all $\mu' = (\sigma', \tau') \in \mathcal{F}(\Omega)$: $C(\sigma; \theta) \leq C(\sigma', \theta)$ and $\sum_{i \in N} \tau_i = 0$. The efficient ordering of a sequencing problem is independent of the transfer. We use $\sigma^*(\Omega)$ to denote an efficient ordering of agents.

An allocation rule ψ associates to each problem $\Omega \in \mathcal{S}(N)$ a non empty set $\psi(\Omega)$ of allocations.

Definition 22. An allocation rule ψ satisfies *efficiency* if and only if for all $\Omega \in \mathcal{S}(\mathcal{N})$, $\mu = (\sigma, \tau) \in$

$\psi(\Omega)$, μ is efficient.

Definition 23. An allocation rule ψ satisfies *feasibility* if and only if for all $\Omega \in \mathcal{S}(\mathcal{N}), \mu = (\sigma, \tau) \in \psi(\Omega), \sum_{i \in N} \tau_i \leq 0$.

4.3 Sequencing games - an optimistic approach

We treat a sequencing problem $\Omega \in \mathcal{S}(N)$ as a cooperative game. We primarily focus on two ways of defining the worth of a coalition, depending on whether its members are being served first or last. This section is inspired by the approach introduced in Maniquet (2003). We also interpret its dual and compute the Shapley value payoffs. The worth of a coalition $S \subseteq N$ is denoted by $v^M(S)$. It is calculated by taking the sum of its members' job completion cost in an efficient ordering provided they have the power to be served first or are the first to arrive. Formally, for any $\Omega \in \mathcal{S}(N), S \subseteq N$,

$$v^M(S) = -\sum_{i \in S} \theta_i (s_i + \sum_{j \in P_i(\sigma^*(\theta_S))} s_j) = -\sum_{i \in S} \theta_i S_j(\sigma^*(\theta_S)) \tag{4.1}$$

where $\sigma^*(\theta_S)$ is an efficient ordering of the members of coalition S. Maniquet has provided an alternate way of defining the worth of a coalition. Consider the scenario where the members of the coalition S are the last to arrive. Then, the worth $w^M(S)$ of a coalition $S \subseteq N$ is the sum of its member's waiting cost in the grand coalition in addition to the incremental cost they impose on the non-members. It is as if the members of S are compensating the non-members for their additional waiting costs. For any $S \subseteq N$ we define,

$$w^M(S) = -\sum_{i \in S} \theta_i (s_i + \sum_{j \in P_i(\sigma^*(\theta_N))} s_j) - \sum_{i \in N \setminus S} \theta_i (\sum_{j \in P_i(\sigma^*(\theta_N))} s_j - \sum_{j \in P_i(\sigma^*(\theta_{N \setminus S}))} s_j) \tag{4.2}$$

where $\sigma^*(\theta_N)$ is an efficient ordering over the grand coalition and $\sigma^*(\theta_{N \setminus S})$ is an efficient ordering of members of $N \setminus S$.

Remark 9. Using equation (4.2), the worth $w^M(S)$ of a coalition $S \subseteq N$ can be expressed as $w^M(S) = -\sum_{i \in N} \theta_i (s_i + \sum_{j \in P_i(\sigma^*(\theta_N))} s_j) + \sum_{i \in N \setminus S} \theta_i (s_i + \sum_{j \in P_i(\sigma^*(\theta_{N \setminus S}))} s_j)$. It can be easily verified using equation (4.1) that $w^M(S) = v^M(N) - v^M(N \setminus S)$. Thus, the game w^M is the dual of the game v^M.

The contribution of an agent $i \in N$ to a coalition S in v^M ($i \notin S$) is given by;

$$v^M(S \cup \{i\}) - v^M(S) = -(s_i + \sum_{j \in P_i(\sigma^*(\theta_{S \cup \{i\}}))} s_j)\theta_i - s_i \sum_{j \in F_i(\sigma^*(\theta_{S \cup \{i\}}))} \theta_j$$

where $\sigma^*(\theta_{S \cup \{i\}})$ is an efficient ordering over the coalition $S \cup \{i\}$. The contribution is composed of the agent's individual cost of waiting and the cost he imposes on those agents who succeed him in the queue.

The Shapley value of each agent is his contribution to a coalition, when we consider all possible permutations of the formation of the grand coalition. In other words, it is the expected value of the contributions of the player over all possible orderings when each ordering is equally likely. For all $\Omega \in \mathcal{S}(N)$, $i \in N$, the payoff assigned to agent i is given by,

$$Sh_i(v^M) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(|N| - |S| - 1)!}{|N|!} [v^M(S \cup \{i\}) - v^M(S)]$$

Given the duality of v^M and w^M, the Shapley value payoffs are identical for these two games (see Kalai & Samet (1987)).

Lemma 6. Let σ^* be an efficient ordering on N. For any $i \in N$, the Shapley value of the game v^M is given by,

$$Sh_i(v^M) = -\theta_i s_i - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j / 2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i / 2 \tag{4.3}$$

Proof. A TU-game v can be expressed uniquely as a linear combination of unanimity games, $v = \sum_{T \subseteq N} \Delta_v u_T$, where the unanimity game u_T on N is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$

otherwise. For any $S \subseteq N$, the dividend $\Delta_v(S)$ is defined as follows: if $|S| = 1$, $\Delta_v(S) = v(S)$ and if $|S| > 1$, $\Delta_v(S) = v(S) - \sum_{\substack{T \subset S \\ T \neq S}} \Delta_V(T)$. We first prove the following claim,

Claim 1. The dividends $\delta_{v^M}(S)$ for any $S \subseteq N$ are given by,

$$\delta_{v^M}(S) = \begin{cases} -\theta_i s_i & \text{if } |S| = 1 \\ -min_{i,j \in S}\{\theta_i/s_i, \theta_j/s_j\} s_i s_j & \text{if } |S| = 2 \\ 0 & \text{if } |S| \geq 3 \end{cases} \tag{4.4}$$

Proof When $|S| = 1$, let $S = \{i\}$. We have $\delta_{v^M}(i) = v^M(i) = -\theta_i s_i$. If $|S| = 2$, let us assume $S = \{i, j\}$ such that $\theta_i/s_i \geq \theta_j/s_j$ without loss of generality. We then have $\delta_v^M(\{i,j\}) = v^M\{i,j\} - \delta_{v^M}(\{i\}) - \delta_{v^M}(\{j\}) = -\theta_j s_i = -min\{\theta_i/s_i, \theta_j/s_j\} s_i s_j$. If $|S| = 3$ and let $S = \{i, j, k\}$ such that $\theta_i/s_i \geq \theta_j/s_j \geq \theta_k/s_k$ without loss of generality. We define $\delta_v^M(\{i,j,k\}) = v^M\{i,j,k\} - \delta_{v^M}(\{i,j\}) - \delta_{v^M}(\{j,k\}) - \delta_{v^M}(\{i,k\}) - \delta_{v^M}(\{i\}) - \delta_{v^M}(\{j\}) - \delta_{v^M}(\{k\}) = -\theta_j s_i - \theta_k(s_i + s_j) + \theta_j s_i + \theta_k s_j + \theta_k s_i = 0$. By induction on the size of the coalition S, let us assume $\delta_{v^M}(S') = 0$ where $3 \leq |S'| \leq |S|$. Without loss of generality, let $S = \{1, 2, \ldots s\}$ such that $\theta_1/s_1 \geq \theta_2/s_2 \geq \ldots \geq \theta_s/s_s$. Using the induction hypothesis, $\delta_{v^M}(S) = v^M(S) - \sum_{T \subset S; |T|=2} \delta_{v^M}(T) - \sum_{T \subset S; |T|=1} \delta_{v^M}(T) = -\sum_{j \in S} \theta_j S_j(\sigma^*(\theta_S)) + \sum_{j \in S} \theta_j (\sum_{m \in P_j(\sigma^*(\theta_S))} s_m) + \sum_{j \in S} \theta_j s_j = 0$. This proves the claim.

The Shapley value of player $i \in N$ in the game v is given by $SV_i(v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{\Delta_v(S)}{|S|}$. By substituting Eq. (4.4) in this expression, we obtain

$$\begin{aligned} Sh_i(v^M) &= -\theta_i s_i - \frac{1}{2} \sum_{j \in N \setminus \{i\}} min\{\theta_i/s_i, \theta_j/s_j\} s_i s_j \\ &= -\theta_i s_i - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \end{aligned}.$$

This gives us the desired conclusion. □

Remark 10. For a sequencing problem $\Omega \in \mathcal{S}(N)$, let the allocation $\mu = (\sigma, \tau)$ give each agent his utility corresponding to the Shapley value of the game v^M (given by Lemma 6). With the sequencing rule σ^*, the transfer to an agent $i \in N$ is given by,

$$\tau_i = \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j / 2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i / 2$$

4.3.1 Existence of core

This section provides insights on the existence of the core in the above defined characteristic form games. We first provide a few preliminary definitions to understand the nature of allocations in the core and use the necessary and sufficient condition provided by Bondareva (1963) and Shapley (1967), for the core of a game to be non-empty. Let G^N denote the set of all characteristic form games with a finite set of players $N = \{1, 2, \ldots, n\}$.

Given a game $v \in G^N$, we define an *outcome* of the game (a payoff vector) as an n-coordinated vector $x = (x_1, x_2, \ldots, x_n)$. Note that, for a coalition $S \subseteq N$, $x(S)$ is the sum of individual payoffs assigned to the members in the coalition S. A pay-off vector x is *individually rational* if for every $i \in N$, we have $x_i \geq v(\{i\})$, where x_i is the payoff allotted to agent i. A pay-off vector x is *totally rational* if $x(N) = v(N)$. An *imputation* can then be defined as a pay-off vector which is both individually and totally rational. The *core* of v is the set of all those pay-off vectors that are imputations and satisfy $x(S) \geq v(S)$ for all non-empty coalitions $S \subset N$. The core of the game v is denoted by $C(v)$.

Definition 24. A collection $\Phi = \{T_1, T_2, \ldots T_k\} \subseteq 2^N$ of non-empty coalitions is *balanced* if for any $i \in N$, there exist positive numbers λ_{T_j}, $T_j \in \Phi$, such that $\sum_{\substack{T_j \in \Phi \\ i \ni T_j}} \lambda_{T_j} = 1$.

Theorem 4. The core of the game v^M is always empty, that is $C(v^M) = \varphi$. The Shapley value belongs to the core of its dual game, w^M.

Proof. We first show that the core of the cooperative game v^M is empty.

Let $\Phi = \{T_1, \ldots, T_k\}$ be a balanced family with corresponding balancing weights $\{\lambda_{T_j}\}_{T_j \in \Phi}$. Bondareva (1963) and Shapley (1967) have shown that the core of a game v is non-empty if and only if for all balanced collections $\Phi = \{T_1, \ldots, T_k\}$ and their corresponding balancing weights $\{\lambda_{T_j}\}_{T_j \in \Phi}$, the inequality $\sum_{T_j \in \Phi} \lambda_{T_j} v(T_j) \leq v(N)$ holds. For the game v^M, it follows that for any $T_j \subseteq N$, we have $v^M(T_j) = -\sum_{i \in T_j} \theta_i S_i(\sigma^*(\theta_{T_j}))$. The left hand side of the above inequality can be expressed as,

$$
\begin{aligned}
\sum_{T_j \in \Phi} \lambda_{T_j} v^M(T_j) &= -\sum_{T_j \in \Phi} \lambda_{T_j} \left[\sum_{i \in T_j} \theta_i S_i(\sigma^*(\theta_{T_j})) \right] \\
&= -\sum_{T_j \in \Phi} \lambda_{T_j} \left[\sum_{i \in T_j} \theta_i \left(s_i + \sum_{k \in P_i(\sigma^*(\theta_N)) \cap T_j} s_k \right) \right] \\
&= -\sum_{i \in N} \theta_i \left(\sum_{\substack{T_j \in \Phi \\ T_j \ni i}} \lambda_{T_j} \right) s_i - \sum_{i \in N} \theta_i \left[\sum_{k \in P_i(\sigma^*(\theta_N))} \left(\sum_{\substack{T_j \ni i \\ T_j \ni k}} \lambda_{T_j} \right) s_k \right] \\
&= -\sum_{i \in N} \theta_i \left(s_i + \sum_{k \in P_i(\sigma^*(\theta_N))} \left(\sum_{\substack{T_j \ni i \\ T_j \ni k}} \lambda_{T_j} \right) s_k \right) \\
&> \sum_{i \in N} \theta_i \left(s_i + \sum_{k \in P_i(\sigma^*(\theta_N))} s_k \right) \\
&= v(N)
\end{aligned}
$$

This proves the first statement.

To prove the second part, we first show that the allocation $(Sh_1(w^M), \ldots, Sh_n(w^M))$ is an imputation. Using equation (4.2) , we can write,

$$
\begin{aligned}
w^M(\{i\}) &= -\theta_i s_i - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_i s_j - \sum_{k\in N\setminus i} \theta_k \Big(\sum_{j\in P_k(\sigma^*(\theta_N))} s_j - \sum_{j\in P_k(\sigma^*(\theta_{N\setminus i}))} s_j \Big) \\
&= -\theta_i s_i - \sum_{j\in P_i(\sigma^*(\theta_N))} s_j\theta_i - \sum_{k\in F_i(\sigma^*(\theta_N))} \theta_k s_i \\
&< -\theta_i s_i - \sum_{j\in P_i(\sigma^*(\theta_N))} s_j\theta_i/2 - \sum_{k\in F_i(\sigma^*(\theta_N))} \theta_k s_i/2 \\
&= Sh_i(w^M).
\end{aligned}
\tag{4.5}
$$

Further,

$$
\begin{aligned}
\sum_{i\in N} Sh_i(w^M) &= -\sum_{i\in N} [\theta_i s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_j/2] \\
&= -\sum_{i\in N} [\theta_i s_i + \theta_i \sum_{j\in P_i(\sigma^*(\theta_N))} s_j] \\
&= w^M(N).
\end{aligned}
\tag{4.6}
$$

The next step is to prove that for all non-empty coalitions $S \subset N$, $\sum_{i\in S} Sh_i(w^M) \geq w^M(S)$. For any given coalition S we have the following,

$$
\begin{aligned}
w^M(S) &= -\sum_{i\in S} \theta_i \Bigg[s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\in S}} s_j + \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\notin S}} s_j \Bigg] - \sum_{i\in N\setminus S} \theta_i \Bigg[\sum_{j\in P_i(\sigma^*(\theta_N))} s_j - \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\notin S}} s_j \Bigg] \\
&= -\sum_{i\in S} \theta_i \Bigg[s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\in S}} s_j + \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\notin S}} s_j \Bigg] - \sum_{i\in N\setminus S} \theta_i \Bigg(\sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\in S}} s_j \Bigg) \\
&= -\sum_{i\in S} \theta_i \Bigg[s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\in S}} s_j + \sum_{\substack{j\in P_i(\sigma^*(\theta_N)) \\ j\notin S}} s_j \Bigg] - \sum_{i\in S} s_i \Bigg(\sum_{\substack{j\in F_i(\sigma^*(\theta_N)) \\ j\notin S}} \theta_j \Bigg)
\end{aligned}
$$

Given, the above expression of $w^M(S)$, we consider the following:

$$
\begin{aligned}
\sum_{i\in S} Sh_i(w^M) &= -\sum_{i\in S}\theta_i s_i - \sum_{i\in S}\theta_i\Bigg(\sum_{j\in P_i(\sigma^*(\theta_N))} s_j\Bigg)/2 - \sum_{i\in S} s_i\Bigg(\sum_{j\in F_i(\sigma^*(\theta_N))}\theta_j\Bigg)/2 \\
&= -\sum_{i\in S}\theta_i s_i - \sum_{i\in S}\theta_i\Bigg(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}} s_j\Bigg)/2 - \sum_{i\in S} s_i\Bigg(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\in S}}\theta_j\Bigg)/2 - \sum_{i\in S}\theta_i\Bigg(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}} s_j\Bigg)/2 \\
&\qquad - \sum_{i\in S} s_i\Bigg(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Bigg)/2 \\
&= -\sum_{i\in S}\theta_i s_i - \sum_{i\in S}\theta_i\Bigg(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}} s_j\Bigg) - \sum_{i\in S}\theta_i\Bigg(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}} s_j\Bigg)/2 - \sum_{i\in S} s_i\Bigg(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Bigg)/2 \\
&> -\sum_{i\in S}\theta_i\Bigg[s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}} s_j + \sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}} s_j\Bigg] - \sum_{i\in S} s_i\Bigg(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Bigg) \\
&= w^M(S).
\end{aligned}
$$

We have thus proved that for the game w^M, the Shapley value allocation vector belongs to the core of the game w^M. □

4.4 Sequencing games: a pessimistic approach

In this section, we map a sequencing problem $\Omega \in \mathcal{S}(N)$ to a characteristic form game in which we define the worth of a coalition using the perspective provided by Chun (2006b). The worth of a coalition $S \subseteq N$ is denoted by $v^C(S)$. Chun adopts a pessimistic approach by taking the sum of its members' job completion cost in an efficient ordering provided the members of S are served after the members of $N \setminus S$. Formally,

$$
v^C(S) = -\sum_{i\in S}\theta_i[S_i(\sigma^*(\theta_S)) + \sum_{k\in N\setminus S} s_k] = v^M(S) - \sum_{i\in S}\theta_i(\sum_{k\in N\setminus S} s_k) \qquad (4.7)
$$

where $\sigma^*(\theta_S)$ is an efficient ordering of the members in coalition S. We provide an alternate way of defining the worth of a coalition. Consider the scenario where the members of the coalition S are the first to arrive. Then, the worth $w^C(S)$ of a coalition $S \subseteq N$ is the sum of its members' waiting cost in the grand coalition minus the incremental gain of the non-members. The incremental gain is given by the difference in the job completion time when members of $N \backslash S$ are served in the grand coalition as opposed to being served after the coalition S. It is as if the non-members are compensating the amount of additional benefit, in terms of the reduction in their waiting costs, to the members of S. For any $S \subseteq N$ we define,

$$w^C(S) = -\sum_{i \in S} \theta_i(s_i + \sum_{j \in P_i(\sigma^*(\theta_N))} s_j) - \sum_{i \in N \backslash S} \theta_i(\sum_{j \in P_i(\sigma^*(\theta_N))} s_j - \sum_{j \in P_i(\sigma^*(\theta_{N \backslash S}))} s_j - \sum_{k \in S} s_k) \quad (4.8)$$

where $\sigma^*(\theta_N)$ is an efficient ordering over the grand coalition and $\sigma^*(\theta_{N\backslash S})$ is an efficient ordering of members in $N \backslash S$.

Remark 11. Using equation (4.8), the worth $w^C(S)$ of a coalition $S \subseteq N$ can be expressed as $w^C(S) = -\sum_{i \in N} \theta_i(s_i + \sum_{j \in P_i(\sigma^*(\theta_N))} s_j) + \sum_{i \in N \backslash S} \theta_i(s_i + \sum_{j \in P_i(\sigma^*(\theta_{N\backslash S}))} s_j + \sum_{k \in N \backslash (N \backslash S)} s_k)$. It can be easily verified using equation (4.7) that $w^C(S) = v^C(N) - v^C(N \backslash S)$. Thus, the game w^C is the dual of the game v^C.

Lemma 7. For any $i \in N$, the Shapley value of the game v^C is given by,

$$Sh_i(v^C) = -\theta_i(s_i + \sum_{j \neq i} s_j) + \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \quad (4.9)$$

Proof. A TU-game v can be expressed uniquely as a linear combination of unanimity games, $v = \sum_{T \subseteq N} \Delta_v u_T$, where the unanimity game u_T on N is given by $u_T(S) = 1$ if $T \subseteq S$, and $u_T(S) = 0$ otherwise. For any $S \subseteq N$, the dividend $\Delta_v(S)$ is defined as follows: if $|S| = 1$, $\Delta_v(S) = v(S)$ and if

$|S| > 1, \Delta_v(S) = v(S) - \sum\limits_{\substack{T \subset S \\ T \neq S}} \Delta_V(T)$. We first prove the following claim,

Claim 2. The dividends $\delta_{v^M}(S)$ for any $S \subseteq N$ are given by,

$$\delta_{v^C}(S) = \begin{cases} -\theta_i(s_i + \sum_{j \neq i} s_j) & \text{if } |S| = 1 \\ max_{i,j \in S}\{\theta_i/s_i, \theta_j/s_j\}s_i s_j & \text{if } |S| = 2 \\ 0 & \text{if } |S| \geq 3 \end{cases} \tag{4.10}$$

Proof When $|S| = 1$, let $S = \{i\}$. We have $\delta_{v^C}(i) = v^C(i) = -\theta_i(s_i + \sum_{j \neq i} s_j)$. If $|S| = 2$, let us assume $S = \{i,j\}$ such that $\theta_i/s_i \geq \theta_j/s_j$ without loss of generality. We then have $\delta_v^C(\{i,j\}) = v^C\{i,j\} - \delta_{v^C}(\{i\}) - \delta_{v^C}(\{j\}) = \theta_i s_j = max\{\theta_i/s_i, \theta_j/s_j\}s_i s_j$. If $|S| = 3$ and let $S = \{i,j,k\}$ such that $\theta_i/s_i \geq \theta_j/s_j \geq \theta_k/s_k$ without loss of generality. We define $\delta_v^C(\{i,j,k\}) = v^C\{i,j,k\} - \delta_{v^C}(\{i,j\}) - \delta_{v^C}(\{j,k\}) - \delta_{v^C}(\{i,k\}) - \delta_{v^C}(\{i\}) - \delta_{v^C}(\{j\}) - \delta_{v^C}(\{k\}) = -\theta_i(s_j + s_k) - \theta_j(s_i + s_k) - \theta_k(s_i + s_j) + \theta_i(s_j + s_k) + \theta_j(s_i + s_k) + \theta_k(s_i + s_j) = 0$. By induction on the size of the coalition S, let us assume $\delta_{v^M}(S^{'}) = 0$ where $3 \leq |S^{'}| \leq |S|$. Without loss of generality, let $S = \{1, 2, \ldots s\}$ be such that $\theta_1/s_1 \geq \theta_2/s_2 \geq \ldots \geq \theta_s/s_s$. By induction hypothesis, $\delta_{v^C}(S) = v^C(S) - \sum_{T \subset S; |T|=2} \delta_{v^C}(T) - \sum_{T \subset S; |T|=1} \delta_{v^C}(T) = -\sum_{i \in S} \theta_i(S_i(\sigma^*(\theta_S)) + \sum_{k \in N \setminus S} s_k) - \sum_{i \in S} \theta_i(\sum_{j \in F_i(\sigma^*(\theta_S))} s_j) + \sum_{i \in S} \theta_i(s_i + \sum_{j \neq i} s_j)$. Note that, the term $\sum_{j \neq i} s_j$ in the last expression can be written as, $\sum_{j \in S \setminus \{i\}} s_j + \sum_{j \in N \setminus S} s_j$. Further, the expression $\sum_{j \in S \setminus \{i\}} s_j$ can be expressed as $\sum_{j \in P_i(\sigma^*(\theta_S))} s_j + \sum_{j \in F_i(\sigma^*(\theta_S))} s_j$. We prove the claim by rewriting $\sum_{j \neq i} s_j$ in terms of these expressions. □

The Shapley value of player $i \in N$ in the game v is given by $SV_i(v) = \sum\limits_{\substack{S \subseteq N \\ i \in S}} \frac{\Delta_v(S)}{|S|}$. By substituting Eq. (4.10) in this expression, we obtain

$$
\begin{aligned}
Sh_i(v^C) &= -\theta_i(s_i + \sum_{j \neq i} s_j) + \frac{1}{2} \sum_{j \in N \setminus \{i\}} max\{\theta_i/s_i, \theta_j/s_j\} s_i s_j \\
&= -\theta_i \sum_{j \in N} s_j + \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_i s_j/2
\end{aligned}.
$$

This gives us the desired conclusion. □

Remark 12. For a sequencing problem $\Omega \in \mathcal{S}(N)$, let the allocation $\mu = (\sigma, \tau)$ give each agent his utility corresponding to the Shapley value of the game v^C (given by Lemma 7). With the sequencing rule σ^*, the transfer to an agent $i \in N$ is given by,

$$
\tau_i = \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_i s_j/2
$$

4.4.1 Existence of core

This section again explores the existence of core in the above defined characteristic form games. We observe that the Shapley value belongs to the core of the game v^C while the core of its dual w^C is empty.

Theorem 5. The Shapley value of the game v^C belongs to its core, that is, $Sh(v^C) \in C(v^C)$. The core of its dual w^C is empty.

Proof. We begin with the first statement.

To prove that $Sh(v^C) \in C(v^C)$, we first show that the allocation $(Sh_1(v^C), \ldots, Sh_n(v^C))$ is an imputation. Using equation (4.7) we can write,

$$
\begin{aligned}
v^C(\{i\}) &= -\theta_i(s_i + \sum_{j\in N\setminus\{i\}} s_j) \\
&= -\theta_i \sum_{j\in N} s_j \\
&< \sum_{j\in P_i(\sigma^*(\theta))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta))} \theta_i s_j/2 - \theta_i \sum_{j\in N} s_j \\
&= Sh_i(v^C)
\end{aligned}
$$

Further,

$$
\begin{aligned}
Sh_i(v^C) &= -\sum_{i\in N}[\theta_i \sum_{j\in N} s_j - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F(\sigma^*(\theta_N))} \theta_i s_j/2] \\
&= -\sum_{i\in N}[\theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j + \sum_{j\in F_i(\sigma^*(\theta_N))} s_j) - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F(\sigma^*(\theta_N))} \theta_i s_j/2] \\
&= -\sum_{i\in N}[\theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F(\sigma^*(\theta_N))} \theta_i s_j/2] \\
&= -\sum_{i\in N}[\theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2] \\
&= -\sum_{i\in N} \theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) - \sum_{i\in N}(\sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j)/2 + \sum_{i\in N}(\sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i)/2 \\
&= -\sum_{i\in N} \theta_i(s_i + \sum_{j\in P_i(\sigma^*(\theta_N))} s_j) \\
&= v^C(N).
\end{aligned}
$$

The next step is to prove that for all non-empty coalition $S \subset N$, we have $\sum_{i\in S} Sh_i(v^C) \geq v^C(S)$. For any given coalition S we have the following,

$$\sum_{i\in S} Sh_i(v^C) = -\sum_{i\in S}\theta_i\Big(\sum_{j\in N}s_j\Big) + \sum_{i\in S}s_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}\theta_j\Big)\Big/2 + \sum_{i\in S}\theta_i\Big(\sum_{j\in F_i(\sigma^*(\theta_N))}s_j\Big)\Big/2$$

$$= -\sum_{i\in S}\theta_i S_i(\sigma^*(\theta_N)) + \sum_{i\in S}s_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}\theta_j\Big)\Big/2 - \sum_{i\in S}\theta_i\Big(\sum_{j\in F_i(\sigma^*(\theta_N))}s_j\Big)\Big/2$$

$$= -\sum_{i\in S}\theta_i\Big(s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}s_j + \sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big) + \sum_{i\in S}s_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}\theta_j\Big)\Big/2$$

$$- \sum_{i\in S}\theta_i\Big(\sum_{j\in F_i(\sigma^*(\theta_N))}s_j\Big)\Big/2$$

$$= -\sum_{i\in S}\theta_i\Big(s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}s_j\Big) - \sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big) - \sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big/2$$

$$-\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in F_i(\sigma^*(\theta_N))\\ j\in S}}s_j\Big)\Big/2 + \sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}\theta_j\Big)\Big/2 + \sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big)\Big/2$$

$$= -\sum_{i\in S}\theta_i\Big(s_i + \sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in S}}s_j\Big) - \sum_{i\in S}\theta_i\Big(\sum_{j\in N\setminus S}s_j\Big)\Big/2$$

$$+ \Big[\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big) - \sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big]\Big/2$$

Claim 3. For any $i \in S$, the term $\Big[\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big) - \sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big]\Big/2 \geq 0$

Proof. Without loss of generality, let us assume that $\theta_1/s_1 \geq \ldots \geq \theta_n/s_n$. By the definition of outcome efficiency, for any $i \in N$ and for any $j \in P_i(\sigma*(\theta))$, where $\sigma^*(\theta)$ is an efficient ordering of agents in a non-increasing order of their urgency indices, we have $\theta_j/s_j \geq \theta_i/s_i$ implying $s_i\theta_j \geq s_j\theta_i$.

We can thus say, $\sum_{i\in S}\left[\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\left(s_i\theta_j-\theta_is_j\right)\right]\geq 0$ by the above argument.

Given the above claim, for any $S\subset N$ observe that,

$$\sum_{i\in S}Sh_i(v^C)-v^C(S)=\sum_{i\in S}\theta_i\Big(\sum_{j\in N\setminus S}s_j\Big)/2+\Big[\sum_{i\in S}s_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}\theta_j\Big)-\sum_{i\in S}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\notin S}}s_j\Big)\Big]/2$$
$$>0$$

This completes the proof.

We now prove the second part of the theorem. Let $\Phi=\{T_1,\ldots,T_k\}$ be a balanced family with corresponding balancing weights $\{\lambda_{T_j}\}_{T_j\in\Phi}$. Bondareva (1963) and Shapley (1967) have shown that the core of a game v is non-empty if and only if for all balanced collections $\Phi=\{T_1,\ldots,T_k\}$ and their corresponding balanced weights $\{\lambda_{T_j}\}_{T_j\in\Phi}$, the inequality $\sum_{T_j\in\Phi}\lambda_{T_j}v(T_j)\leq v(N)$ holds. For the game w^C, it follows from definition (4.8) that for any $T_j\subseteq N$, we have $w^C(T_j)=-\sum_{i\in S}\theta_i(s_i+\sum_{j\in P_i(\sigma^*(\theta_N))}s_j)-\sum_{i\in N\setminus S}\theta_i(\sum_{j\in P_i(\sigma^*(\theta_N))}s_j-\sum_{j\in P_i(\sigma^*(\theta_{N\setminus S}))}s_j-\sum_{k\in S}s_k)$. The left hand side of the above inequality can be expressed as,

$$\sum_{T_j\in\Phi}\lambda_{T_j}w^C(T_j)=-\sum_{T_j\in\Phi}\lambda_{T_j}\Big[\sum_{i\in T_j}\theta_i\Big(s_i+\sum_{j\in P_i(\sigma^*(\theta_N))}s_j\Big)$$
$$+\sum_{i\in N\setminus T_j}\theta_i\Big(\sum_{j\in P_i(\sigma^*(\theta_N))}s_j-\sum_{j\in P_i(\sigma^*(\theta_{N\setminus T_j}))}s_j-\sum_{k\in T_j}s_k\Big)\Big]$$

$$=-\sum_{i\in N}\theta_i\Big(\sum_{\substack{T_j\in\Phi\\ T_j\ni i}}\lambda_{T_j}\Big)(s_i+\sum_{j\in P_i(\sigma^*(\theta_N))}s_j)-\sum_{T_j\in\Phi}\lambda_{T_j}\Big[\sum_{i\in N\setminus T_j}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in T_j}}s_j-\sum_{k\in T_j}s_k\Big)\Big]$$
$$=-\sum_{i\in N}\theta_i(s_i+\sum_{j\in P_i(\sigma^*(\theta_N))}s_j)-\sum_{T_j\in\Phi}\lambda_{T_j}\Big[\sum_{i\in N\setminus T_j}\theta_i\Big(\sum_{\substack{j\in P_i(\sigma^*(\theta_N))\\ j\in T_j}}s_j-\sum_{k\in T_j}s_k\Big)\Big]$$

Note that, for any non-empty $T_j \subset N$, the expression $\sum_{T_j \in \Phi} \lambda_{T_j} w^C(T_j) - w^C(N) = -\sum_{T_j \in \Phi} \lambda_{T_j} \left[\sum_{i \in N \setminus T_j} \theta_i \left(\sum_{\substack{j \in P_i(\sigma^*(\theta_N)) \\ j \in T_j}} s_j - \sum_{k \in T_j} s_k \right) \right] > 0$ because $\sum_{\substack{j \in P_i(\sigma^*(\theta_N)) \\ j \in T_j}} s_j < \sum_{k \in T_j} s_k$. This proves that the core of the game w^C is empty. □

4.5 Shapley value and GMWB

Given any sequencing problem $\Omega \in \mathcal{S}(N)$, let $O_i(s)$ be the welfare parameter of agent i. Let $O(N;s) := (O_1(s), \ldots, O_n(s)) \in \mathbb{R}^n$ denote the welfare parameter vector. We represent a typical sequencing problem with generalized minimum welfare bounds by $\Gamma = (\Omega, O(N;s))$ where $\Omega \in \mathcal{S}(N)$ and the associated $O(N;s) \in \mathbb{R}^n$ is the welfare parameter vector. This section provides a necessary and sufficient condition on the welfare parameter $O_i(s)$, such that, an allocation rule assigning utilities corresponding to the Shapley value of the associated sequencing game satisfies the generalized minimum welfare bound.

Definition 25. For Γ, an allocation rule ψ satisfies the *generalized minimum welfare bound* if and only if for all $\mu = (\sigma, \tau) \in \psi(\Gamma), i \in N$:

$$u_i(\sigma, \tau_i; \theta_i) \geq -\theta_i O_i(s). \tag{4.11}$$

The generalized minimum welfare bound imposes a lower bound on each agent's utility function, in the form of a minimum guarantee. Banerjee et al. (2020) have introduced this bound which is a unified and comprehensive representation of several specific lower bounds that have been previously examined in the literature.

Proposition 6. For a given Γ, an allocation rule ψ which selects those allocations that assign utilities to agents corresponding to the Shapley value of the sequencing games $v^M(v^C)$, satisfies the generalized minimum welfare bound if and only if we have $O_i(s) \geq s_i + \sum_{j \neq i} s_j/2$.[2]

Proof. Part A. The first part of the proof considers the corresponding characteristic form game under the optimistic approach given by v^M. The transfers are designed so that the utility of each agent $i \in N$ is given by the Shapley value $Sh_i(v^M)$.

For a given sequencing problem Γ, the utility of player $i \in N$ (corresponding to the Shapley value of the game v^M) will satisfy the GMWB property if, $Sh_i(v^M) \geq -\theta_i O_i(s)$ implying $-\theta_i s_i - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \geq -\theta_i O_i(s)$. Or, $\theta_i(O_i(s) - s_i) - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \geq 0$. Let, $O_i(s) = s_i + \sum_{j \neq i} s_j/2 + \varepsilon_i$. Thus have, $\theta_i \varepsilon_i + \theta_i(\sum_{j \neq i} s_j/2) - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 \geq 0$. This implies, $\sum_{j \in F_i(\sigma^*(\theta_N))}(\theta_i s_j - \theta_j s_i)/2 + \theta_i \varepsilon_i \geq 0$. Or, $\sum_{j \in F_i(\sigma^*(\theta_N))}(u_i - u_j) + \theta_i \varepsilon_i \geq 0$. We must have $\varepsilon_i \geq 0$. This proves necessity.

For any $i \in N$, it is given that $O_i(s) \geq s_i + \sum_{j \neq i} s_j/2$. The utility of player i is given by his Shapley value $Sh_i(v^M)$. For any such player, consider the expression $Sh_i(v^M) + \theta_i O_i(s) = -\theta_i s_i - \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_i s_j/2 - \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \theta_i s_i + \theta_i \sum_{j \neq i} s_j/2$. Since $\sigma^*(\theta_N)$ is an efficient ordering of the members of the grand coalition (N), then for any agent $i \in N$, if an agent $j \in F_i(\sigma^*(\theta_N))$ we must have $\theta_i/s_i \geq \theta_j/s_j$. This means, $Sh_i(v^M) + \theta_i O_i(s) = \sum_{j \in F_i(\sigma^*(\theta_N))}(\theta_i s_j - \theta_j s_i) \geq 0$. For any $i \in N$, we have $Sh_i(v^M) \geq -\theta_i O_i(s)$. This proves sufficiency.

Part B. We define the associated cooperative game (v^C) using the pessimistic approach. The utility of each player $i \in N$ corresponds to the Shapley value of this game, $Sh_i(v^C)$.

For a given sequencing problem Γ, the utility of player $i \in N$ (corresponding to the Shapley value of the game v^C) will satisfy the GMWB property if, $Sh_i(v^C) \geq -\theta_i O_i(s)$ implying $-\theta_i(\sum_{j \in N} s_j) + \sum_{j \in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j \in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \geq -\theta_i O_i(s)$. Or, $\theta_i(O_i(s) - \sum_{j \in N} s_j) +$

[2] This bound coincides with the expected costs bound property studied in Banerjee et al. (2020) where the arrival of each possible ordering is equally like

$\sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 - \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \geq 0$. Let, $O_i(s) = s_i + \sum_{j\neq i} s_j/2 + \varepsilon_i$. Thus, $\theta_i\varepsilon_i - \theta_i(\sum_{j\neq i} s_j/2) + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 \geq 0$. This implies, $\sum_{j\in P_i(\sigma^*(\theta_N))}(\theta_j s_i - \theta_i s_j)/2 + \theta_i\varepsilon_i \geq 0$. Or, $\sum_{j\in P_i(\sigma^*(\theta_N))}(u_j - u_i) + \theta_i\varepsilon_i \geq 0$. We must have $\varepsilon_i \geq 0$. This shows the necessary part.

For any $i \in N$, it is given that $O_i(s) \geq s_i + \sum_{j\neq i} s_j/2$. The utility of player i is given by his Shapley value $Sh_i(v^C)$. For any such player, consider the expression $Sh_i(v^C) + \theta_i O_i(s) = -\theta_i(\sum_{j\in N} s_j) + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2 + \theta_i s_i + \theta_i \sum_{j\neq i} s_j/2 = -\theta_i \sum_{j\neq i} s_j/2 + \sum_{j\in P_i(\sigma^*(\theta_N))} \theta_j s_i/2 + \sum_{j\in F_i(\sigma^*(\theta_N))} \theta_i s_j/2$. Since $\sigma^*(\theta_N)$ is an efficient ordering of the members of the grand coalition (N), then for any agent $i \in N$, if an agent $j \in P_i(\sigma^*(\theta_N))$ we must have $\theta_j/s_j \geq \theta_i/s_i$. This means, $Sh_i(v^M) + \theta_i O_i(s) = \sum_{j\in P_i(\sigma^*(\theta_N))}(\theta_j s_i - \theta_i s_j) \geq 0$. For any $i \in N$, we have $Sh_i(v^C) \geq -\theta_i O_i(s)$. This proves the sufficiency part. $\square$

4.6 Conclusion

This paper maps sequencing problems to cooperative games and adopts an optimistic and a pessimistic approach to define the worth of a coalition. We study two solution concepts: the core, which deals with stability of feasible allocations and the Shapley value, which assigns the outcome in a fair and an impartial manner. We observe that, the Shapley value belongs to the core in two cases - 1) the worth of a coalition is the aggregate job completion cost of its members in the grand coalition and the compensation to the non-members for the additional cost imposed (as if the coalition arrives in the end) and, 2) the worth of a coalition is the minimum waiting cost incurred by its members only after the non-coalitional members get served (under the pessimistic assumption). On the other hand, the core of the associated cooperative game is empty in the following cases - 1) the worth of a coalition is the minimum waiting cost of its members when they have the power to be served first (under the optimistic assumption) and, 2) the worth of a coalition is aggregate waiting cost of its members in the grand coalition discounted by the additional benefit amount received from the non-coalitional members as compensation (as if the coalition has arrived in the beginning).

Under both the approaches, the transfers are designed, so that, the utility of each individual corresponds to the Shapley value payoff of the associated sequencing game. When we impose the generalized minimum welfare bound property, we observe that the expected costs bound condition (which guarantees each agent his expected cost when every arrival order is equally likely) is necessary and sufficient for the welfare bound to hold.

References

Abdulkadiroğlu, A. & Sönmez, T. (1998). Random serial dictatorship and the core from random endowments in house allocation problems. *Econometrica*, 66(3), 689–701.

Alkan, A., Demange, G., & Gale, D. (1991). Fair allocation of indivisible goods and criteria of justice. *Econometrica: Journal of the Econometric Society*, (pp. 1023–1039).

Atkinson, A. B. et al. (1970). On the measurement of inequality. *Journal of economic theory*, 2(3), 244–263.

Atlamaz, M. & Yengin, D. (2008). Fair groves mechanisms. *Social Choice and Welfare*, 31(4), 573–587.

Banerjee, S. (2021). Existence of core in sequencing problems- optimistic and pessimistic approach. Economic Research Unit, Indian Statistical Institute, Kolkata (unpublished).

Banerjee, S., De, P., & Mitra, M. (2020). A welfarist approach to sequencing problems with incentives.

Banerjee, S. & Mitra, M. (2021). Lorenz optimality for sequencing problems with welfare bounds. *Economics Letters*, 205, 109963.

Barberà, S. (2011). Strategyproof social choice. *Handbook of social choice and welfare*, 2, 731–831.

Bevia, C. (1996). Identical preferences lower bound solution and consistency in economies with indivisible goods. *Social Choice and Welfare*, 13(1), 113–126.

Beviá, C. (1998). Fair allocation in a general model with indivisible goods. *Review of Economic Design*, 3(3), 195–213.

Bondareva, O. N. (1963). Some applications of linear programming methods to the theory of cooperative games. *Problemy kibernetiki*, 10(119), 139.

Chatterji, S. & Sen, A. (2011). Tops-only domains. *Economic Theory*, 46(2), 255–282.

Chun, Y. (2004). *Consistency and monotonicity in sequencing problems*. Technical report, Working paper.

Chun, Y. (2006a). No-envy in queueing problems. *Economic Theory*, 29(1), 151–162.

Chun, Y. (2006b). A pessimistic approach to the queueing problem. *Mathematical Social Sciences*, 51(2), 171–181.

Chun, Y. & Mitra, M. (2014). Subgroup additivity in the queueing problem. *European Journal of Operational Research*, 238(1), 281–289.

Chun, Y., Mitra, M., & Mutuswami, S. (2014a). Characterizations of pivotal mechanisms in the queueing problem. *Mathematical Social Sciences*, 72, 62–66.

Chun, Y., Mitra, M., & Mutuswami, S. (2014b). Egalitarian equivalence and strategyproofness in the queueing problem. *Economic Theory*, 56(2), 425–442.

Chun, Y., Mitra, M., & Mutuswami, S. (2017). Reordering an existing queue. *Social Choice and Welfare*, 49(1), 65–87.

Chun, Y., Mitra, M., & Mutuswami, S. (2019a). A characterization of the symmetrically balanced vcg rule in the queueing problem. *Games and Economic Behavior*, 118, 486–490.

Chun, Y., Mitra, M., & Mutuswami, S. (2019b). Egalitarianism in the queueing problem. *Journal of Mathematical Economics*, 81, 48–56.

Chun, Y., Schummer, J., & Thomson, W. (1998). Constrained egalitarianism: a new solution for claims problems.

Chun, Y. & Yengin, D. (2017). Welfare lower bounds and strategy-proofness in the queueing problem. *Games and Economic Behavior*, 102, 462–476.

Clarke, E. H. (1971). Multipart pricing of public goods. *Public choice*, (pp. 17–33).

CreÁs, H. & Moulin, H. (1998). Sequencing with opting out: Improving upon random priority. *Working Paper*.

Crès, H. & Moulin, H. (1998). Free access to the commons: random priority versus average cost.

Curiel, I., Pederzoli, G., & Tijs, S. (1989). Sequencing games. *European Journal of Operational Research*, 40(3), 344–351.

Dasgupta, P., Sen, A., & Starrett, D. (1973). Notes on the measurement of inequality. *Journal of economic theory*, 6(2), 180–187.

De, P. (2013). Incentive and normative analysis on sequencing problem.

De, P. (2016). *Mechanism Design In Sequencing Problems*. PhD thesis, Indian Statistical Institute, Kolkata.

De, P. & Mitra, M. (2017). Incentives and justice for sequencing problems. *Economic Theory*, 64(2), 239–264.

De, P. & Mitra, M. (2019). Balanced implementability of sequencing rules. *Games and Economic Behavior*, 118, 342–353.

Dolan, R. J. (1978). Incentive mechanisms for priority queuing problems. *The Bell Journal of Economics*, (pp. 421–436).

Dubins, L. E. & Spanier, E. H. (1961). How to cut a cake fairly. *The American Mathematical Monthly*, 68(1P1), 1–17.

Duives, J., Heydenreich, B., Mishra, D., Müller, R., & Uetz, M. (2015). On optimal mechanism design for a sequencing problem. *Journal of scheduling*, 18(1), 45–59.

Dutta, B. & Ray, D. (1989). A concept of egalitarianism under participation constraints. *Econometrica: Journal of the Econometric Society*, (pp. 615–635).

Dutta, B. & Ray, D. (1991). Constrained egalitarian allocations. *Games and Economic Behavior*, 3(4), 403–422.

Fields, G. & Fei, J. C. H. (1978). On inequality comparisons. *Econometrica*, 46(2), 303–16.

Foley, D. K. (1966). *Resource allocation and the public sector*. Yale University.

Gershkov, A. & Schweinzer, P. (2010). When queueing is better than push and shove. *International Journal of Games Theory*, 39, 409–430.

Green, J. & Laffont, J.-J. (1979). *Incentives in public decision-making*. Elsevier North-Holland.

Groves, T. (1973). Incentives in teams. *Econometrica*, 41(4), 617–631.

Hain, R. & Mitra, M. (2004). Simple sequencing problems with interdependent costs. *Games and Economic Behavior*, 48(2), 271–291.

Hanning, M. (1996). Maximum waiting-time guarantee—an attempt to reduce waiting lists in sweden. *Health policy*, 36(1), 17–35.

Hashimoto, K. (2018). Strategy-proofness and identical preferences lower bound in allocation problem of indivisible objects. *Economic Theory*, 65(4), 1045–1078.

Hashimoto, K. & Saitoh, H. (2012). Strategy-proof and anonymous rule in queueing problems: a relationship between equity and efficiency. *Social Choice and Welfare*, 38(3), 473–480.

Holmström, B. (1979). Groves' scheme on restricted domains. *Econometrica: Journal of the Econometric Society*, (pp. 1137–1144).

Hurwicz, L. & Walker, M. (1990). On the generic nonoptimality of dominant-strategy allocation mechanisms: A general theorem that includes pure exchange economies. *Econometrica: Journal of the Econometric Society*, (pp. 683–704).

Kalai, E. & Samet, D. (1987). On weighted shapley values. *International journal of game theory*, 16(3), 205–222.

Kayı, Ç. & Ramaekers, E. (2010). Characterizations of pareto-efficient, fair, and strategy-proof allocation rules in queueing problems. *Games and Economic Behavior*, 68(1), 220–232.

Maniquet, F. (2003). A characterization of the shapley value in queueing problems. *Journal of Economic Theory*, 109(1), 90–103.

McKeever, T., Sweeney, M., & Staines, A. (2006). An investigation of the impact of prolonged waiting times on blood donors in ireland. *Vox sanguinis*, 90(2), 113–118.

Mendelson, H. & Whang, S. (1990). Optimal incentive-compatible priority pricing for the m/m/1 queue. *Operations research*, 38(5), 870–883.

Mishra, D. & Rangarajan, B. (2007). Cost sharing in a job scheduling problem. *Social Choice and Welfare*, 29(3), 369–382.

Mitra, M. (2001). Mechanism design in queueing problems. *Economic Theory*, 17(2), 277–305.

Mitra, M. (2002). Achieving the first best in sequencing problems. *Review of Economic Design*, 7(1), 75–91.

Mitra, M. (2007). Preferences lower bound in the queueing model. In *Econophysics of markets and business networks* (pp. 233–237). Springer.

Mitra, M. & Mutuswami, S. (2011). Group strategyproofness in queueing models. *Games and Economic Behavior*, 72(1), 242–254.

Moreno, D. & Moscoso, M. J. (2013). Strategy-proof allocation mechanisms for economies with public goods. *Economic Theory*, 52(1), 315–336.

Moulin, H. (1990a). Fair division under joint ownership: recent results and open problems. *Social Choice and Welfare*, 7(2), 149–170.

Moulin, H. (1990b). Uniform externalities: Two axioms for fair allocation. *Journal of Public Economics*, 43(3), 305–326.

Moulin, H. (1991a). *Axioms of cooperative decision making*. Number 15. Cambridge university press.

Moulin, H. (1991b). Welfare bounds in the fair division problem. *Journal of Economic Theory*, 54(2), 321–337.

Moulin, H. (1992). An application of the shapley value to fair division with money. *Econometrica: Journal of the Econometric Society*, (pp. 1331–1349).

Moulin, H. (2002). Axiomatic cost and surplus sharing. In K. J. Arrow, A. K. Sen, & K. Suzumura (Eds.), *Handbook of Social Choice and Welfare*, volume 1 of *Handbook of Social Choice and Welfare* chapter 6, (pp. 289–357). Elsevier.

Moulin, H. (2003). *Fair division and collective welfare*. MIT press.

Moulin, H. (2007). On scheduling fees to prevent merging, splitting, and transferring of jobs. *Mathematics of Operations Research*, 32(2), 266–283.

Mukherjee, C. (2013). Weak group strategy-proof and queue-efficient mechanisms for the queueing problem with multiple machines. *International Journal of Game Theory*, 42(1), 131–163.

Mukherjee, C. (2014). Fair and group strategy-proof good allocation with money. *Social Choice and Welfare*, 42(2), 289–311.

Ohseto, S. (2004). Implementing egalitarian-equivalent allocation of indivisible goods on restricted domains. *Economic Theory*, 23(3), 659–670.

Ohseto, S. (2006). Characterizations of strategy-proof and fair mechanisms for allocating indivisible goods. *Economic Theory*, 29(1), 111–121.

Öztürk, M., Peters, H., & Storcken, T. (2014). On the location of public bads: strategy-proofness under two-dimensional single-dipped preferences. *Economic Theory*, 56(1), 83–108.

Pápai, S. (2003). Groves sealed bid auctions of heterogeneous objects with fair prices. *Social choice and Welfare*, 20(3), 371–385.

Pazner, E. A. & Schmeidler, D. (1978). Egalitarian equivalent allocations: A new concept of economic equity. *The Quarterly Journal of Economics*, 92(4), 671–687.

Porter, R., Shoham, Y., & Tennenholtz, M. (2004). Fair imposition. *Journal of Economic Theory*, 118(2), 209–228.

Ramaekers, E. & Kayi, Ç. (2008). *An impossibility in sequencing problems*. Technical report.

Sen, A., Sen, M. A., Amartya, S., Foster, J. E., Foster, J. E., et al. (1997). *On economic inequality*. Oxford university press.

Shapley, L. S. (1967). On balanced sets and cores. *Naval Research Logistics Quarterly*, 14(4), 453–460.

Smith, W. E. et al. (1956). Various optimizers for single-stage production. *Naval Research Logistics Quarterly*, 3(1-2), 59–66.

Steihaus, H. (1948). The problem of fair division. *Econometrica*, 16, 101–104.

Suijs, J. (1996). On incentive compatibility and budget balancedness in public decision making. *Economic design*, 2(1), 193–209.

Tadenuma, K. (1996). Trade-off between equity and efficiency in a general economy with indivisible goods. *Social Choice and Welfare*, 13(4), 445–450.

Tadenuma, K. & Thomson, W. (1991). No-envy and consistency in economies with indivisible goods. *Econometrica: Journal of the Econometric Society*, (pp. 1755–1767).

Tadenuma, K. & Thomson, W. (1993). The fair allocation of an indivisible good when monetary compensations are possible. *Mathematical Social Sciences*, 25(2), 117–132.

Tadenuma, K. & Thomson, W. (1995). Refinements of the no-envy solution in economies with indivisible goods. *Theory and Decision*, 39(2), 189–206.

Thomson, W. (2003). On monotonicity in economies with indivisible goods. *Social Choice and Welfare*, 21(2), 195–205.

Thomson, W. (2012). Lorenz rankings of rules for the adjudication of conflicting claims. *Economic Theory*, 50(3), 547–569.

Tinbergen, J. (1946). Redelijke inkomensverdeling.

van Brummelen, S., van Dijk, N., van den Hurk, K., & de Kort, W. (2018). Waiting time-based staff capacity and shift planning at blood collection sites. *Health systems*, 7(2), 89–99.

Velez, R. A. (2014). Consistent strategy-proof assignment by hierarchical exchange. *Economic Theory*, 56(1), 125–156.

Vickrey, W. (1961). Counterspeculation, auctions, and competitive sealed tenders. *The Journal of finance*, 16(1), 8–37.

Westkamp, A. (2013). An analysis of the german university admissions system. *Economic Theory*, 53(3), 561–589.

Yengin, D. (2012). Egalitarian-equivalent groves mechanisms in the allocation of heterogenous objects. *Social Choice and Welfare*, 38(1), 137–160.

Yengin, D. (2013a). Identical preferences lower bound for allocation of heterogenous tasks and nimby problems. *Journal of Public Economic Theory*, 15(4), 580–601.

Yengin, D. (2013b). Population monotonic and strategy-proof mechanisms respecting welfare lower bounds. *Journal of Mathematical Economics*, 49(5), 389–397.

Yengin, D. (2017). No-envy and egalitarian-equivalence under multi-object-demand for heterogeneous objects. *Social Choice and Welfare*, 48(1), 81–108.