Local vs Global Incentive Compatibility in Mechanism Design

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Contents

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To Family and Friends.

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1
Introduction

This thesis consists of five chapters related to mechanism design theory. A brief introduction of the chapters are provided below.

1.1 Local Global Equivalence in Voting Models: A Characterization and Applications

This chapter considers a voting model where each voter's type is her preference. The type graph for a voter is a graph whose vertices are the possible types of the voter. Two vertices are connected by an edge in the graph if the associated types are "neighbours". A social choice function is locally strategy-proof if no type of a voter can gain by misrepresentation to a type that is a neighbour of her true type. A social choice function is strategy-proof if no type of a voter can gain by misrepresentation to an arbitrary type.
Local-Global equivalence (LGE) is satisfied if local strategy-proofness implies strategy-proofness. We identify a condition on the graph that characterizes LGE. Our notion of "localness" is perfectly general - we use this feature of our model to identify notions of localness according to which various models of multi-dimensional voting satisfy LGE. Finally, we show that LGE for deterministic social choice functions does not imply LGE for random social choice functions.

1.2 Local Global Equivalence in Voting Models Admitting Indifferences

This chapter considers the same voting framework as in the previous chapter, except that each agent's type is her weak preference, that is, preferences that can admit indifference. We provide a condition that is sufficient for LGE and another condition that is necessary. Moreover, the "gap" between the two conditions is small (in the sense that both conditions boil down to the single condition identified in Chapter 1 that characterizes LGE for the case of strict preferences). We use the sufficiency result to propose notions of localness according to which environments with the domain of single-plateaued preferences and the domain of all weak preferences, satisfies LGE.

1.3 Local Global Equivalence for Unanimous Social Choice Functions

In this chapter, we identify a condition on preference domains that ensures that every locally strategy-proof and unanimous random social choice function is also strategy-proof. Furthermore every unanimous, locally strategy-proof deterministic social choice function is also group strategy-proof. The condition identified is significantly weaker than the characterization condition for local-global equivalence without unanimity in Kumar et al. [33]. The condition is not necessary for equivalence with unanimous random/deterministic social choice functions. However, we show the weaker condition of connectedness remains necessary.

1.4 Pointwise Local Incentive Compatibility in Non-Convex Type-Spaces

In this chapter, we explore the equivalence of pointwise local incentive compatibility (PLIC) (Carroll [12]) and incentive compatibility (IC) in non-convex type-spaces. We provide a sufficient condition on a type-space called minimal richness for the said equivalence. Using this result, we show that PLIC and IC are equivalent on large class of non-convex type-spaces such as type-spaces perturbed by modularity and concave-modularity. The gross substitutes type-space and the generalized gross substitutes and complements type-space are important examples of type-spaces perturbed by modularity and concave-modularity, respectively. Finally, we provide a geometric property consisting of three conditions for the equivalence of PLIC and IC, and show that all the conditions are indispensable.

1.5 Local Incentive Compatibility in Ordinal Type-Spaces

This chapter explores the relation between different notions of local incentive compatibility (LIC) and incentive compatibility (IC) on ordinal type-spaces. In this context, we introduce the notion of ordinal local global equivalent (OLGE) and cardinal local global equivalent (CLGE) environments. First, we establish the equivalence between the two environments on strict ordinal type-spaces. Next, we consider ordinal type-spaces admitting indifference. We introduce the notion of almost everywhere IC and strong LIC, and provide a necessary and sufficient condition on ordinal type spaces for their equivalence. Finally, we provide results on how to (minimally) check the IC property of a given mechanism on any ordinal type-space and show that local types along with the boundary types form a minimal set of incentive constraints that imply full incentive compatibility.

2

Local Global Equivalence in Voting Models: A Characterization and Applications

2.1 Introduction

Mechanism design theory is concerned with models where agents have private information (called a type) which has to be elicited by the mechanism designer. The cornerstone of the theory is the collection of strategy-proofness constraints which ensure that agents do not have incentives to misreport their types (or manipulate). The standard assumption in the theory is that the proposed social choice function must be immune to *all* possible misreports of agents. There is, however considerable experimental evidence that agents do not always lie in an optimal payoff-maximizing way. For instance Fischbacher and Föllmi-Heusi [22] conduct an experiment where agents are paid money on the basis of a report of a privately observed roll of a die. In their results, only 20 percent of the subjects lie optimally, 39 percent are fully honest while the remaining lie "partially". Agents often choose to lie credibly by only misreporting to types that are "near" or "close to" their true types. We consider a model where an agent of a particular type can only misreport to an arbitrary set of pre-specified "local" types. Our main contribution is a complete answer to the following question: under what circumstances is immunity to misreporting via a "local" type (local strategy-proofness) equivalent to immunity to misreporting via an arbitrary type (strategy-proofness)?

The equivalence issue has important conceptual and practical implications.[1] If it is not satisfied, the mechanism designer can choose from a wider class of locally strategy-proof social choice functions. It may enable her, in principle, to avoid negative results such as the Gibbard-Satterthwaite Theorem (Gibbard [25], Satterthwaite [47]). On the other hand, suppose that the problem at hand satisfies equivalence. In order to verify that a social choice function is strategy-proof, it suffices to check that it is locally strategy-proof. The latter is a simpler task because it involves checking fewer constraints.

We consider a model where an agent's type is a strict preference ordering over a finite set of alternatives. There are no monetary tranfers. For convenience, we shall refer to this model as the voting model and to the agent as a voter, even though the model could apply to other settings such as matching. For our purpose, it will be sufficient to restrict attention to the case of a single voter.[2] The set of possible preferences is called a *domain*. An *environment* is an undirected graph whose vertices are preferences in the domain. The agent whose preference is specified by a particular vertex can only misreport to another preference (or vertex) if the two vertices are connected by an edge in the environment. The set of vertices connected by an edge to a vertex are its *neighbours*. A social choice function is locally strategy-proof if no type of the agent can gain by manipulating to a neighbour; it is strategy-proof if the agent cannot gain by manipulating to any vertex in the graph. An environment satisfies *local-global equivalence* or LGE if local strategy-proofness implies strategy-proofness.[3]

Section 4.2 of the paper contains some examples and observations that highlight the issues underlying LGE. It serves to motivate our main result in Section 4.4, Theorem 2.3.2 which is a characterization of environments that satisfy LGE. Section 2.4 contains discussion of the computational complexity of Property L and its relationship with earlier results in the literature. Section 2.5 applies Theorem 2.3.2 to multi-dimensional voting environments. Finally Section 2.6 uses Theorem 2.3.2 to construct an example of an environment where LGE holds but equivalence fails for random social choice functions.

The LGE property depends on the existence of certain types of paths in the environment. For every pair of preferences P and P' in the domain and alternative a, there must exist a path from P to P' satisfying a monotonicity property with respect to all alternatives that are ranked worse than a according to P. Specifically, the relative ranking of a and any alternative b ranked worse than a according to P, can change at most once along the path. We call this condition, Property L. According to Theorem 2.3.2, Property L is both necessary and sufficient for LGE.

One of the strengths of our approach is that our notion of neighbours in the definition of local strategy-proofness, is perfectly general. The earlier literature (discussed below) used the Kemeny distance

[1] They have also been discussed extensively in Carroll [12] and Sato [46].

[2] Our results can easily be interpreted in the multi-voter setting.

[3] The converse is of course, always true.

metric to define “localness”. Thus two preferences are neighbours if there is a single pair of consecutively ranked alternatives that are switched between the two preferences. Preferences that are neighbours in this sense will be referred to as being adjacent. A limitation of adjacency is that it excludes several multi-dimensional voting models that are of interest. In these models, an alternative is an m-tuple ($m > 1$) and preferences are typically assumed to satisfy some form of separability. Consequently, it is not always possible to switch a consecutively ranked pair of alternatives without affecting the ranking of other alternatives. We consider two such domains, separable domains and multi-dimensional single-peaked domains and propose natural notions of neighbours such that the resulting environments satisfy LGE.

The question of local-global equivalence also arises naturally in the context of random social choice functions. We follow the standard approach of comparing lotteries via stochastic dominance (see Gibbard [26]). Earlier results (again discussed below) suggest that environments that satisfy LGE for deterministic social choice functions also do so for random social choice functions. We use our characterization result for the deterministic case to show that this is not true generally. We construct an environment that satisfies Property L and therefore satisfies deterministic LGE. We also find a random social choice in the same environment that satisfies local strategy-proofness but violates strategy-proofness.

2.1.1 Related Literature

Two important papers on LGE in voting models are Carroll [12] and Sato [46]. Both papers use the adjacency version of localness. Carroll [12] considers random social choice functions and shows that specific preference domains, such as the set of all strict preferences, the set of all single-peaked preferences and particular subsets of single-crossing preferences satisfy LGE. Sato [46] provides a necessary condition and a stronger sufficiency condition for LGE in the context of deterministic social choice functions. Section 2.4.2 describes the relationship between Sato's results and ours in greater detail. As already mentioned, there are two significant ways in which our main result extends and refines the earlier analysis. The first is that our notion of neighbours is completely general and the second is that we have a complete characterization. Both aspects of our result permit a wider range of applications than was earlier possible.

Cho [18] provides sufficient conditions for LGE with random social choice functions. The notion of neighbours is once again, adjacency, but several notions of preference extensions to lotteries are considered. In particular, it shows that a stronger version of the sufficient condition proposed in Sato [46] (see Property U in Section 2.4.2) is sufficient for LGE if lotteries are compared via stochastic dominance. We show in Section 2.6 that the condition which is necessary and sufficient for LGE with deterministic social choice functions (using adjacency as the notion of localness), is *not* sufficient for LGE with random social choice functions.

There are several papers that investigate LGE in models where monetary transfers to agents are permitted and preferences are quasi-linear in the usual sense (see, for instance Carroll [12], Archer and Kleinberg [1] and Mishra et al. [39]). Although the basic question is the same, the flavour of the analysis and the results in the two models are very different from each other.

In a companion paper Kumar et al. [33], we consider a multi-voter model and address the following question: under what conditions on the environment is it the case that every locally strategy-proof social choice function that also satisfies the mild condition of unanimity,[4] is also strategy-proof? We show that a condition much weaker than Property L is sufficient for LGE in this sense for both deterministic and random social choice functions.

2.2 The Model

Let $A = \{a, b, \dots\}$ denote a finite set of alternatives with $|A| \geq 2$. Throughout the paper, we shall assume that there is a single voter. This assumption is without loss of generality as will soon be apparent.

A *preference* P is an antisymmetric, complete and transitive binary relation over A i.e. a *linear order*. Given $a, b \in A$, aPb is interpreted as "a is strictly preferred to b" according to P. Let $\mathcal{P}$ denote the set of all preferences - the set $\mathcal{P}$ will be referred to as the *universal domain*. We shall refer to an arbitrary set $\mathcal{D} \subseteq \mathcal{P}$ as a *domain*.

An *environment* is an (undirected) graph $G = \langle \mathcal{D}, \mathcal{E} \rangle$. The set of vertices of the graph is a domain $\mathcal{D}$. The set of edges is the set $\mathcal{E}$. If $P, P' \in \mathcal{D}$ and $(P, P') \in \mathcal{E}$, the two preferences are said to be *neighbours* or are *local*.

The notion of neighbours is perfectly general. One possible specification is the one used by Carroll [12] and Sato [46]. Fix a pair of preferences $P, P' \in \mathcal{D}$. Two alternatives a and b in A are *reversed* if aPb and $bP'a$, or bPa and $aP'b$. Let $P \triangle P' = \{\{a, b\} \subseteq A : a \text{ and } b \text{ are reversed in } P \text{ and } P'\}$ be the set of all reversed pairs of alternatives between P and P'. [5] Two preferences P and P' are called *adjacent* if $|P \triangle P'| = 1$.[6] An environment where neighbours are defined by adjacency will be referred to as an *adjacency environment*. Whenever the notion of neighbours is defined by adjacency, we shall denote the set of edges by $\mathcal{E}^{adj}$. An adjacency environment will typically be denoted by $G = \langle \mathcal{D}, \mathcal{E}^{adj} \rangle$. In Section 2.5, we shall provide an example of a non-adjacency environment.

Definition 2.2.1 *A Social Choice Function (SCF) is a map $f : \mathcal{D} \to A$.*

[4] A deterministic social choice function satisfies unanimity if it always picks an alternative in a profile where it is first-ranked by all voters. In the case of a random social choice function such an alternative is picked with probability one.

[5] We are guilty of abuse of notation here. Since a preference is an ordered pair, $P \triangle P'$ should include both ordered pairs, (a, b) and (b, a) if a and b are reversed in P and P'. In our notation, $P \triangle P'$ will include only the unordered pair $\{a, b\}$ in this case.

[6] An alternative and equivalent statement would be that the Kemeny distance between P and P' is exactly one.

Definition 2.2.2 *Consider an environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$. An SCF $f : \mathcal{D} \to A$ is locally manipulable at P if there exists $P' \in \mathcal{D}$ with $(P, P') \in \mathcal{E}$ such that $f(P')Pf(P)$. The SCF f is locally strategy-proof if it is not locally manipulable at any $P \in \mathcal{D}$.*

Consider a graph or an environment. An SCF labels each vertex of the graph with an alternative. It is locally strategy-proof if the voter with preference of a particular vertex cannot gain by misrepresenting her preference to one which is a neighbour of her true preference.

In contrast with local strategy-proofness, an SCF is *strategy-proof* if the voter cannot gain by an arbitrary misrepresentation.

Definition 2.2.3 *An SCF $f : \mathcal{D} \to A$ is manipulable at P if there exists $P' \in \mathcal{D}$ such that $f(P')Pf(P)$. The SCF f is strategy-proof if it is not manipulable at any $P \in \mathcal{D}$.*

A strategy-proof SCF is clearly locally strategy-proof. We investigate the structure of environment when the converse is true.

Definition 2.2.4 *The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies local-global equivalence (LGE) if every locally strategy-proof SCF $f : \mathcal{D} \to A$ is strategy-proof.*

The next subsection makes some important observations regarding LGE.

2.2.1 Preliminary observations

Our goal in this subsection is to illustrate the issues involved in LGE and to provide some intuition behind our result. We begin with some standard concepts from graph theory.

Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment. A *path* $\pi = (P^1, \ldots, P^t)$ is a sequence of *distinct* vertices in $\mathcal{D}$ satisfying the property that consecutive vertices are neighbours, i.e. $(P^k, P^{k+1}) \in \mathcal{E}$ for all $k = 1, \ldots t-1$.[7] Let $\Pi(P, P')$ denote the set of all paths from P to P' in G. For any path $\pi = (P^1, \ldots, P^s, P^{s+1}, \ldots, P^t)$, we let $\pi|_{[P^s, P^t]}$ denote the sub-path $(P^s, P^{s+1}, \ldots, P^t)$. We say G is *connected* if there exists a path between every pair of vertices in G i.e. $\Pi(P, P') \neq \emptyset$ for all $P, P' \in \mathcal{D}$.

The example below highlights the reasons why LGE may fail.

Example 2.2.5 Let $A = \{a, b, c, z, u, v, w\}$. Consider the adjacency environment $G = \langle \mathcal{D}, \mathcal{E}^{adj} \rangle$ where $\mathcal{D} = \{P^1, P^2, P^3, P^4, P^5\}$ (Table 2.2.1). It will be convenient to represent G by Figure 2.2.1.

[7] In other words, repetitions of vertices in a path are ruled out.

P^1	P^2	P^3	P^4	P^5
c	c	c	c	c
$[a]$	$[b]$	$[b]$	$[b]$	a
b	a	a	a	$[b]$
z	z	z	z	z
v	v	v	u	u
w	w	u	v	v
u	u	w	w	w

Table 2.2.1: Domain $\mathcal{D}$

$$P^1 \overset{\{a,b\}}{\rule{2cm}{0.4pt}} P^2 \overset{\{w,u\}}{\rule{2cm}{0.4pt}} P^3 \overset{\{v,u\}}{\rule{2cm}{0.4pt}} P^4 \overset{\{b,a\}}{\rule{2cm}{0.4pt}} P^5$$

Figure 2.2.1: The Environment $G = \langle \mathcal{D}, \mathcal{E}^{adj} \rangle$ [8]

The SCF $f : \mathcal{D} \to A$ picks a at P^1 and b at other preferences.[9] The SCF f is locally strategy-proof. However, it is not strategy-proof since the voter with preference P^5 can manipulate via P^1. $\square$

The cause of the failure of strategy-proofness while maintaining local strategy-proofness can be clearly identified from Example 2.2.5. Consider the path $\pi = (P^5, P^4, P^3, P^2, P^1)$. The outcome at P^5 is b. Since b "improves" at P^4 relative to P^5, local strategy-proofness implies that the outcome at P^4 must be b; otherwise the voter would manipulate locally to P^5. Local strategy-proofness also implies that the outcomes at P^3 and P^2 must be b. Note that b "declines" at P^1 with respect to a. There are two options at P^1 that are consistent with the requirement of local strategy-proofness (with respect to P^1). The outcome can remain b, or it can switch to a. In the former case, we maintain strategy-proofness since the outcome is b everywhere along the path π. However, if the outcome is a, a problem with strategy-proofness arises since a is preferred to b at P^5.

The failure of LGE in $G = \langle \mathcal{D}, \mathcal{E}^{adj} \rangle$ arises from an inherent asymmetry in the "monotonicity" requirement imposed by local strategy-proofness. If the outcome of an SCF at a preference improves[10] relative to a local preference, the same outcome continues to be chosen at the new neighbour preference. However, if the outcome at a preference falls relative to a local preference, the new outcome can either remain the same or switch to an alternative that has improved (relative to the original outcome) in the new preference. Combining the latter option together with an improvement in the same path, can lead to a failure of strategy-proofness without violating local strategy-proofness.

[8] Two vertices are connected by an edge in G if and only if the preferences represented by the vertices are adjacent. For instance, P^1 and P^2 are adjacent; in particular aP^1b and bP^2a. The edge between P^1 and P^2 is labelled $\{a, b\}$ in order to signify that the only "difference" between the two preferences is the ranking of a and b.

[9] This is indicated by the square brackets on the alternative chosen by f at each preference.

[10] We are intentionally informal in this description. These notions will be made precise in due course.

A key feature of the path π in Example 2.2.5 is that a and b switch relative ranking *more than once* in the path. Thus aP^5b, bP^4a and aP^1b. The preceding discussion makes it clear that such paths may be problematic for LGE.

Definition 2.2.6 *Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment and let $a, b \in A$. A path $\pi = (P^1, P^2, \ldots, P^t)$ satisfies no $\{a, b\}$-restoration if the relative ranking of a and b is reversed* [11] *at most once along π i.e. there do not exist integers q, r and s with $1 \leq q < r < s \leq t$ such that either (i) aP^qb, bP^ra and aP^sb or (ii) bP^qa, aP^rb and bP^sa.*[12]

Let $P, P' \in \mathcal{D}$ and $a, b \in A$ be such that aPb. We say that b *overtakes* a in path $\pi \in \Pi(P, P')$ if bP^la for some preference P^l in the path π. The notion of overtaking can be used to restate the definition of an $\{a, b\}$-restoration in an obvious way. For instance in case (i) of Definition 2.2.6, b overtakes a in the path $\pi^1 = (P^q, \ldots, P^r)$ and a overtakes b in the path $\pi^2 = (P^r, \ldots, P^s)$.

[11] Recall that a pair of alternatives a, b are reversed in the pair of preferences P and P' if they are ranked differently in P and P'.

[12] It is worth emphasizing that in our definition of "$\{a, b\}$-restoration", we are *not* referring to an ordered pair (a, b). Thus $\{a, b\}$-restoration and $\{b, a\}$-restoration are the same in our definition. We use expressions such as "the path has no $\{a, b\}$-restoration" and "the path has no restoration for the pair $\{a, b\}$" interchangeably.

P^0	P^6	P^7	P^8	P^9	P^{10}
c	a	a	a	a	a
a	c	c	c	c	c
b	b	z	z	z	b
z	z	b	b	b	z
v	u	u	v	v	v
u	v	v	u	w	w
w	w	w	w	u	u

Table 2.2.2: Preferences P^0 and $P^6, P^7, P^8, P^9, P^{10}$

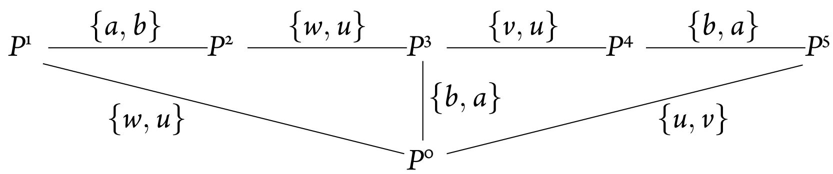

Figure 2.2.2: The Environment $\overline{G} = \left\langle \overline{\mathcal{D}}, \mathcal{E}^{adj} \right\rangle$

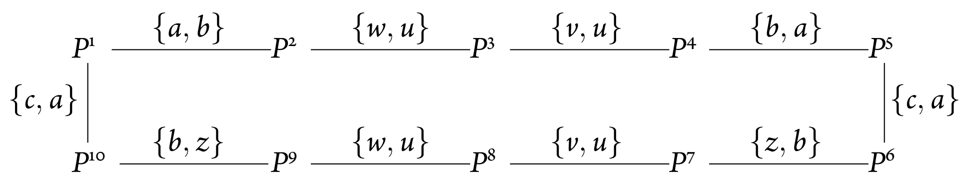

Figure 2.2.3: The Environment $G^* = \langle \mathcal{D}^*, \mathcal{E}^{adj} \rangle$

It will sometimes be useful to consider paths without restoration for a pair of alternatives. Let $P, P' \in \mathcal{D}$ and $a, b \in A$ be such that aPb. Let $\pi = (P^1, P^2, \ldots, P^t) \in \Pi(P, P')$ be a path without $\{a, b\}$-restoration. If $aP'b$, then aP^rb for all preferences P^r on the path π. Suppose $bP'a$ instead. Then there exists a unique preference P^r on π such that aP^sb for all $s = 1, \ldots r$ and bP^sa for all $s = r+1, \ldots, t$.

In order to further clarify the relationship between the LGE property and paths without restoration, we make two modifications of Example 2.2.5.

Example 2.2.7 As in Example 2.2.5, $A = \{a, b, c, z, u, v, w\}$. We consider six additional preferences $P^0, P^6, P^7, P^8, P^9, P^{10}$ as shown in Table 2.2.2. Let $\overline{\mathcal{D}}$ and $\mathcal{D}^*$ be the domains $\overline{\mathcal{D}} = \mathcal{D} \cup \{P^0\}$ and $\mathcal{D}^* = \mathcal{D} \cup \{P^6, P^7, P^8, P^9, P^{10}\}$. These domains are used to construct two adjacency environments $\overline{G} = \left\langle \overline{\mathcal{D}}, \mathcal{E}^{adj} \right\rangle$ and $G^* = \langle \mathcal{D}^*, \mathcal{E}^{adj} \rangle$. These environments are shown in Figures 2.2.2 and 2.2.3.

Consider $\overline{G}$ and a locally strategy-proof SCF $\bar{f} : \overline{\mathcal{D}} \to A$ such that $\bar{f}(P^5) = b$. Using the same arguments as in Example 2.2.5, along the path $\pi = (P^5, P^4, P^3, P^2, P^1)$, we can infer that local strategy-proofness implies $\bar{f}(P^k) = b$ for all $k = 5, 4, 3, 2$, and $\bar{f}(P^1)$ is either b or a. Due to the presence P^0, there is now another path $\bar{\pi} = (P^5, P^0, P^1)$ from P^5 to P^1. This path has no $\{a, b\}$-restoration. Furthermore, the path $\bar{\pi}$ has the following properties: (i) a and b are identically consecutively ranked, and

(ii) c always ranks above a, while z, u, v and w are all ranked below b. Clearly, b does not switch places with any other alternative along $\bar{\pi}$. As a result, local strategy-proofness forces the outcome of $\bar{f}$ to be b everywhere along $\bar{\pi}$ which rules out the manipulability of $\bar{f}$.

Now consider G^* and a locally strategy-proof SCF $f^* : \mathcal{D}^* \to A$ such that $f^*(P^5) = b$. Once again, local strategy-proofness along the path

$\pi = (P^5, P^4, P^3, P^2, P^1)$, implies that $f^*(P^k) = b$ for all $k = 5, 4, 3, 2$, and $f^*(P^1)$ is either b or a. Consider the path $\pi^* = (P^5, P^6, P^7, P^8, P^9, P^{10}, P^1)$. Observe that π^* has no restoration for a and any of the alternatives in the set $Z = \{b, z, u, v, w\}$ which are all ranked below a in P^5. Alternatives of Z switch places among themselves along π^* (see for example, the sub-path $(P^6, P^7, P^8, P^9, P^{10})$). Consequently, the local strategy-proofness of f^* does not preclude the outcomes for preferences along π^* from belonging to Z. Suppose $f^*(P^1) = a$. Since $f^*(P^5) = b$, local strategy-proofness implies that some alternative in Z must "jump above" a and then "jump below" a (in order to conform with P^1) along the path π^*.[13] However, this is explicitly ruled out by the observation that π^* has no restoration for a and any of the alternatives in Z. Therefore, it must be the case that $f^*(P^1) = b$. In fact, only one of two possibilities can arise: (i) $f^*(P^k) = b$ for all $k = 1, \ldots, 10$, or (ii) $f^*(P^k) = b$ for all $k = 1, 2, 3, 4, 5, 6, 10$ and $f^*(P^{k'}) = z$ for all $k' = 7, 8, 9$. In either case, f^* is strategy-proof.

We conclude with an important observation. The alternative c is always ranked above a along the path $\bar{\pi}$ in $\overline{G}$. However, the path π^* in G^* does not forbid restoration between a and alternatives better than a in the initial preference P^5. $\square$

We summarize the insights of Examples 2.2.5 and 2.2.7. There is "potential" for the failure of LGE whenever there is a path in an environment that has restoration for some pair of alternatives. However LGE can be restored by the existence of certain "other" paths in the environment. As the argument relating to π^* in G^* suggests, the existence of a path that satisfies no-restoration of an alternative with respect to all alternatives that are worse at a preference, is sufficient to ensure strategy-proofness and hence, LGE. In the next section, we show that this insight is general. In fact, this condition is also necessary though the argument establishing necessity, is more subtle.

2.3 The Main Result

The key condition for LGE is the *Lower Contour Set no-restoration* property which we define below.

For any $P \in \mathcal{D}$ and $a \in A$, the lower contour set of a at P is the set of alternatives strictly worse than a according to P, i.e. $L(a, P) = \{b \in A : aPb\}$.

[13] We can first easily rule out the possibility that c is chosen at some preference in the subpath $(P^6, P^7, P^8, P^9, P^{10})$. In that case, local strategy-proofness forces the outcome of f^* to be c everywhere in G^*.

Definition 2.3.1 *The environment G satisfies the Lower Contour Set no-restoration property (Property L) if, for all $P, P' \in \mathcal{D}$ and $a \in A$, there exists a path $\pi \in \Pi(P, P')$ such that for all $b \in L(a, P)$ the path π satisfies no $\{a, b\}$-restoration.*

Pick an arbitrary pair of preferences $P, P' \in \mathcal{D}$ and an alternative $a \in A$ which is not ranked last in P. Suppose $L(a, P) = \{b_1, \ldots, b_m\}$. If G satisfies Property L, there exists a path π from P to P' such that for *all* $b_i \in \{b_1, \ldots, b_m\}$ the path π has no $\{a, b_i\}$-restoration . More informally, if a lies above b_i in P' then it lies above b_i everywhere along the path π. On the other hand, if the ranking of a and b_i are reversed between P and P' there is a single reversal between a and b_i along the path π.

The environment G^* in Example 2.2.7 satisfies Property L. In G^*, there are exactly two paths between any pair of vertices, one "clockwise" path and the other, "counterclockwise". For instance, between P^1 and P^5, the paths $(P^1, P^2, P^3, P^4, P^5)$ and $(P^1, P^{10}, P^9, P^8, P^7, P^6, P^5)$ are the clockwise and counterclockwise paths respectively. These paths satisfy an important property. Fix an arbitrary pair of distinct preferences P and P'. If a path between P and P' possesses a restoration, say an $\{x, y\}$-restoration, and x is better than y in P, then the other path between P and P' must have no restoration for x and any alternative of $L(x, P)$. For example, consider P^1 and P^5. The clockwise path $(P^1, P^2, P^3, P^4, P^5)$ has $\{a, b\}$-restoration and aP^1b. The counterclockwise path $(P^1, P^{10}, P^9, P^8, P^7, P^6, P^5)$ has no $\{a, x\}$-restoration for all $x \in L(a, P^1)$. The counterclockwise path $(P^1, P^{10}, P^9, P^8, P^7, P^6, P^5)$ has both $\{c, a\}$-restoration and $\{b, z\}$-restoration, cP^1a and bP^1z. On the other hand, the clockwise path $(P^1, P^2, P^3, P^4, P^5)$ has no $\{c, x\}$-restoration for all $x \in L(c, P^1)$ and no $\{b, x\}$-restoration for all $x \in L(b, P^1)$. This property ensures that G^* satisfies Property L.

Theorem 2.3.2 *An environment satisfies LGE if and only if it satisfies Property L.*

Proof: **Sufficiency**: Suppose $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies Property L but fails LGE i.e. there exists a locally strategy-proof SCF $f : \mathcal{D} \to A$ that is not strategy-proof. Suppose f is manipulable at P. Define the alternative x^1 as follows: $x^1 = \max_P\{a \in A : f(\bar{P}) = a \text{ for some } \bar{P} \in \mathcal{D}\}$. In other words, x^1 is the highest-ranked alternative in the range of f according to P.[14] Let P' be such that $f(P') = x^1$. Since f is manipulable at P, we have $x^1 \neq f(P)$.

By Property L, there exists a path $\pi = (P^1, P^2, \ldots, P^t) \in \Pi(P, P')$ such that for all $z \in L(x^1, P)$ the path π has no $\{x^1, z\}$-restoration. Searching the path π backwards from P^t to P^1, let P^s be the first vertex such that $f(P^s) = x^2 \neq x^1$ i.e. $f(P^k) = x^1$ for all $s < k \leq t$. Note that P^s always exists since $f(P^t) \neq f(P^1)$. It follows from the definition of x^1 that $x^1P^1x^2$. Since $(P^s, P^{s+1}) \in \mathcal{E}$, local strategy-proofness implies $x^2P^sx^1$ and $x^1P^{s+1}x^2$. We therefore have an $\{x^1, x^2\}$-restoration on the path π, contradicting our hypothesis. Therefore, $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies LGE and completes the proof of the sufficiency part of Theorem 2.3.2.

[14] For later reference, $\max_P(B)$ refers to the P-maximal alternative in the set $B \subseteq A$.

Necessity: We define a class of SCFs that we will employ repeatedly in the proof.

Definition 2.3.3 *Fix an environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$. Let $a \in A$, $\hat{P} \in \mathcal{D}$ and let B be a non-empty set with $B \subseteq L(a, \hat{P})$. An SCF $f : \mathcal{D} \to A$ is monotonic with respect to $(a, B, \hat{P})$ if*

(i) $f(P) = a$ *if there is a path $\pi \in \Pi(\hat{P}, P)$ such that $B \subseteq L(a, \bar{P})$ for all $\bar{P} \in \pi$, and*

(ii) $f(P) = \max_P(B)$ *otherwise.*

Thus $f(P) = a$ if there exists a path from $\hat{P}$ to P such that no alternative $x \in B$ overtakes a along the path (note that $a\hat{P}x$). Clearly $f(\hat{P}) = a$. The next lemma shows that SCF f of Definition 2.3.3 is locally strategy-proof.

Lemma 2.3.1 *Suppose $f : \mathcal{D} \to A$ is monotonic with respect to $(a, B, \hat{P})$. Then f is locally strategy-proof.*

Proof: Pick an arbitrary pair $P, P' \in \mathcal{D}$ with $(P, P') \in \mathcal{E}$. We show either $f(P) = f(P')$, or $f(P)Pf(P')$ and $f(P')P'f(P)$ establishing local strategy-proofness.

Let $\mathcal{D}_a = \{\bar{P} \in \mathcal{D} : f(\bar{P}) = a\}$ denote the set of preferences which are associated to a at SCF f. There are four cases to consider.

Case 1: $P, P' \in \mathcal{D}_a$. Then $f(P) = f(P') = a$.

Case 2: $P, P' \notin \mathcal{D}_a$. Then $f(P) = \max_P(B)$ and $f(P') = \max_{P'}(B)$. Hence, either $f(P) = f(P')$ or $f(P)Pf(P')$ and $f(P')P'f(P)$ must hold.

Case 3: $P \in \mathcal{D}_a$ and $P' \notin \mathcal{D}_a$. Thus, $f(P) = a \neq b = \max_{P'}(B) = f(P')$. Since $P \in \mathcal{D}_a$, there exists a path $\pi = (P^1, \ldots, P^t) \in \Pi(\hat{P}, P)$ such that $B \subseteq L(a, P^k)$ for all $1 \leq k \leq t$ (recall Definition 2.3.3). Since $b \in B$, we have aPb. Next, suppose $aP'b$. Since $b = \max_{P'}(B)$, it follows that $B \subseteq L(a, P')$. Observe that P' must be distinct from the vertices in the path π; otherwise we would contradict the hypothesis that $P' \notin \mathcal{D}_a$. Since $(P, P') \in \mathcal{E}$, we now have a new path $\bar{\pi} = (P^1, \ldots, P^t, P') \in \Pi(\hat{P}, P')$ such that $B \subseteq L(a, \bar{P})$ for all $\bar{P} \in \bar{\pi}$. Consequently, Definition 2.3.3 implies $f(P') = a$. This contradicts our initial assumption that $f(P') = b$. Therefore, $bP'a$.

Case 4: $P \notin \mathcal{D}_a$ and $P' \in \mathcal{D}_a$. This case is symmetric to Case 3 above and is omitted.

This completes the proof of the lemma. ■

Lemma 2.3.1 and the LGE property implies that monotonic SCFs are also strategy-proof. This, in turn imposes certain no-restoration conditions on the environment. The rest of the proof essentially shows that Property L is the consequence of the strategy-proofness of monotonic SCFs.

Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment satisfying LGE. We show that G satisfies Property L. We begin with an observation.

Claim 2.3.1 *G is connected.*

Proof: Suppose the Claim is false. Then there exists a component G' of G such that $G' \neq \emptyset$ and G' is a strict subset of G.[15], i.e. there does not exist a path from any vertex in G' to any vertex not in G'. Denote the set of vertices in G' by $\mathcal{D}'$. Pick an arbitrary vertex P^* in $\mathcal{D}'$ and let $a, b \in A$ be such that aP^*b. Define the SCF f as follows: $f(P) = b$ for all vertices $P \in \mathcal{D}'$ and $f(P) = a$ for all $P \notin \mathcal{D}'$.

Clearly f is not strategy-proof because $f(P^*) = b$ while $f(P') = a$ for any $P' \notin \mathcal{D}'$. However f is locally strategy-proof because the outcome does not change if the voter misrepresents via a neighbouring preference. Thus LGE is violated. ■

Suppose G violates Property L i.e. there exist $P^0, P^1 \in \mathcal{D}$ and $a \in A$ such that every path of $\Pi(P^0, P^1)$ has an $\{a, x\}$-restoration for some $x \in L(a, P^0)$. In view of Claim 2.3.1, this statement cannot hold vacuously.

Let Γ be the set of alternatives in $L(a, P^0)$ that appear in some restoration with a on some path of $\Pi(P^0, P^1)$:

$$\Gamma = \{x \in L(a, P^0) : \text{there exists } \pi \in \Pi(P^0, P^1) \text{ with } \{a, x\}\text{-restoration}\}.$$

Then, the hypothesis for the contradiction can be restated as follows: each path of $\Pi(P^0, P^1)$ has an $\{a, x\}$-restoration for some $x \in \Gamma$.

For a specific path $\pi \in \Pi(P^0, P^1)$, let Γ_1^π denote the set of alternatives in $L(a, P^0)$ that appear in some restoration with a on the path π, i.e.

$$\Gamma_1^\pi = \{x \in L(a, P^0) : \pi \text{ has } \{a, x\}\text{-restoration}\}.$$

Let $\Gamma^1 \subseteq [\Gamma \cap L(a, P^1)]$ be the set of alternatives such that *every* path $\pi \in \Pi(P^0, P^1)$ has $\{a, x\}$-restoration for some $x \in \Gamma^1$. Note that either $\Gamma^1 \neq \emptyset$ or $\Gamma^1 = \emptyset$ holds, and every alternative in Γ^1 (if Γ^1 is non-empty) is ranked below a in both preferences P^0 and P^1. We show that each of the two possible cases $\Gamma^1 \neq \emptyset$ and $\Gamma^1 = \emptyset$ leads to a contradiction.

Case A: $\Gamma^1 \neq \emptyset$.

Let $f : \mathcal{D} \to A$ be the SCF which is monotonic with respect to (a, Γ^1, P^0). Note that f is well-defined since $\emptyset \neq \Gamma^1 \subseteq L(a, P^0)$. According to Lemma 2.3.1, f is locally strategy-proof. We show that f is not strategy-proof.

According to Definition 2.3.3, $f(P^0) = a$. Pick an arbitrary path $\pi \in \Pi(P^0, P^1)$. By definition, there exists $z \in \Gamma^1$ such that π has $\{a, z\}$-restoration, i.e. there exists $P^r \in \pi$ such that zP^ra. Hence

[15]We say that G' is a component of G if G' is a maximal connected subgraph of G.

$\Gamma^1 \nsubseteq L(a, P^r)$. Since π was chosen arbitrarily, there does not exist $\bar{\pi} \in \Pi(P^0, P^1)$ such that $\Gamma^1 \subseteq L(a, P^s)$ for all $P^s \in \bar{\pi}$. Consequently, Definition 2.3.3 implies $f(P^1) = \max_{P^1}(\Gamma^1) \equiv b$. Since $\Gamma^1 \subseteq L(a, P^1)$, we have $f(P^0) = aP^1b = f(P^1)$. Therefore, f is not strategy-proof and we have a contradiction to the assumption that G satisfies LGE.

This argument establishes that Case A cannot occur.

Case B: $\Gamma^1 = \emptyset$.

This case is more complicated than the earlier one. We begin with a series of claims.

Claim 2.3.2 *There exists a path $\pi \in \Pi(P^0, P^1)$ such that $\Gamma_1^\pi \cap L(a, P^1) = \emptyset$.*

Proof: Suppose Claim 2.3.2 is false. This implies that in each path of $\Pi(P^0, P^1)$, at least one alternative involved in a restoration with a is ranked below a in P^1, i.e. $\Gamma_1^\pi \cap L(a, P^1) \neq \emptyset$ for all $\pi \in \Pi(P^0, P^1)$. Let $\hat{\Gamma} = \bigcup_{\pi \in \Pi(P^0, P^1)} \left[\Gamma_1^\pi \cap L(a, P^1)\right]$. Then $\emptyset \neq \hat{\Gamma} \subseteq L(a, P^1)$ and Case A holds with $\Gamma^1 = \hat{\Gamma}$. ■

Following Claim 2.3.2, let $\pi^1 \in \Pi(P^0, P^1)$ be the path such that $\Gamma_1^{\pi^1} \cap L(a, P^1) = \emptyset$. Thus, xP^1a for all $x \in \Gamma_1^{\pi^1}$. Note that path π^1 has $\{a, x\}$-restoration only for all $x \in \Gamma_1^{\pi^1}$, and aP^0x for all $x \in \Gamma_1^{\pi^1}$. Searching the path π^1 from P^1 back to P^0, let $P^2 \in \pi^1 \backslash \{P^1\}$ be the the first vertex such that a overtakes some alternative of $\Gamma_1^{\pi^1}$. Note that preference P^2 always exists since xP^1a and aP^0x for all $x \in \Gamma_1^{\pi^1}$. Let Z be the (non-empty) subset of alternatives in $\Gamma_1^{\pi^1}$ that are overtaken by a in the reverse path from P^1 to P^2 i.e. $Z \subseteq \Gamma_1^{\pi^1}$ such that (i) aP^2z for all $z \in Z$, (ii) yP^2a for all $y \in \Gamma_1^{\pi^1} \backslash Z$ (if $Z \neq \Gamma_1^{\pi^1}$), and (iii) $x\bar{P}a$ for all $x \in \Gamma_1^{\pi^1}$ and all $\bar{P} \in \pi^1|_{[P^2, P^1]} \backslash \{P^2\}$. Thus, subpath $\pi^1|_{[P^2, P^1]}$ has no $\{a, x\}$-restoration for any $x \in \Gamma_1^{\pi^1}$, and hence, $P^2 \neq P^0$. Since π^1 has $\{a, x\}$-restoration only for all $x \in \Gamma_1^{\pi^1}$, path π^1 must have no $\{a, y\}$-restoration for any $y \in \Gamma \backslash \Gamma_1^{\pi^1}$ (if $\Gamma_1^{\pi^1} \neq \Gamma$). Therefore, subpath $\pi^1|_{[P^2, P^1]}$ has no $\{a, x\}$-restoration for any $x \in \Gamma$.

Claim 2.3.3 $\Gamma \cap L(a, P^1)$ *is a strict subset of* $\Gamma \cap L(a, P^2)$.

Proof: It follows from the definition of Z that if $\Gamma \cap L(a, P^1) \subseteq \Gamma \cap L(a, P^2)$, then $\Gamma \cap L(a, P^1)$ must be a strict subset of $\Gamma \cap L(a, P^2)$. Suppose it is not the case that $\Gamma \cap L(a, P^1) \subseteq \Gamma \cap L(a, P^2)$ i.e. there exists $x \in \Gamma \cap L(a, P^1)$ such that xP^2a. Then, we have aP^0x, xP^2a and aP^1x which imply the $\{a, x\}$-restoration on π^1 and $x \in \Gamma_1^{\pi^1} \cap L(a, P^1)$. This contradicts the hypothesis $\Gamma_1^{\pi^1} \cap L(a, P^1) = \emptyset$. ■

Claim 2.3.4 *For every $\hat{\pi} \in \Pi(P^0, P^2)$, there exists $x \in \Gamma$ such that $\hat{\pi}$ has $\{a, x\}$-restoration.*

Proof: Suppose there exists $\hat{\pi} \in \Pi(P^0, P^2)$ and $\hat{\pi}$ has no $\{a, x\}$-restoration for any $x \in \Gamma$. Clearly P^2 is a vertex common to both $\hat{\pi}$ and $\pi^1|_{[P^2, P^1]}$. Starting from P^1, proceed along the path which is the reverse of $\pi^1|_{[P^2, P^1]}$. Let $\tilde{P}$ be the first vertex in this reverse path which also belongs to $\hat{\pi}$. From our earlier remark, such a vertex must exist (it could be P^2). Now combine the sequences of vertices $\hat{\pi}|_{[P^0,\tilde{P}]}$ and $\pi^1|_{[\tilde{P},P^1]}$ to form the vertex sequence $\bar{\pi}$. By construction, $\bar{\pi}$ contains no repetition of vertices so that it is a path and $\bar{\pi} \in \Pi(P^0, P^1)$.

For convenience, let $\bar{\pi} = (\bar{P}^1, \ldots, \bar{P}^k, \ldots, \bar{P}^t)$ where $\bar{P}^k = \tilde{P}$, $\hat{\pi}|_{[P^0,\tilde{P}]} = (\bar{P}^1, \ldots, \bar{P}^k)$ and $\pi^1|_{[\tilde{P},P^1]} = (\bar{P}^k, \ldots, \bar{P}^t)$. Since $\bar{\pi} \in \Pi(P^0, P^1)$, the hypothesis for the contradiction of the necessity part of Theorem 2.3.2 implies $\Gamma_1^{\bar{\pi}} \neq \emptyset$. Therefore, there exists $b \in \Gamma$ such that $\bar{\pi}$ has $\{a, b\}$-restoration. Since neither $\hat{\pi}$ nor $\pi^1|_{[P^2,P^1]}$ have $\{a, b\}$-restoration and aP^0b, it must be the case that b overtakes a on the path $(\bar{P}^1, \ldots, \bar{P}^k)$ and then a overtakes b on the path $(\bar{P}^k, \ldots, \bar{P}^t)$. Thus we have i.e. $b\bar{P}^ka$ and $a\bar{P}^tb$. Now refer back to the path π^1. Since aP^0b, $b\tilde{P}a$ and aP^1b, path π^1 has $\{a, b\}$-restoration and hence, $b \in \Gamma_1^{\pi^1} \cap L(a, P^1)$. This contradicts the hypothesis $\Gamma_1^{\pi^1} \cap L(a, P^1) = \emptyset$. ■

We can now replace P^1 by P^2 in our earlier arguments and define Γ^2 in the same way as we defined Γ^1. Once again, there are two possibilities, $\Gamma^2 \neq \emptyset$ and $\Gamma^2 = \emptyset$. The former case leads to an immediate contradiction using the arguments in Case A. In the latter case, we can apply Claims 2.3.2, 2.3.3 and 2.3.4 to infer the existence of P^3 such that (i) $\Gamma \cap L(a, P^2)$ is a strict subset of $\Gamma \cap L(a, P^3)$, and (ii) every path $\pi \in \Pi(P^0, P^3)$ has $\{a, x\}$-restoration for some $x \in \Gamma$. Repeating the argument, it follows that the only way to avoid a contradiction via Case A is to find an infinite sequence of vertices $P^1, P^2, \ldots P^n, \ldots$ such that

$$[\Gamma \cap L(a, P^1)] \subset [\Gamma \cap L(a, P^2)] \subset \cdots \subset [\Gamma \cap L(a, P^n)] \cdots .$$ [16]

However this is impossible in view of the finiteness of G. Thus Case B cannot occur either and the proof is complete. ■

Property L can be simplified if an additional restriction is imposed on the domain.

For any preference P, $r_1(P)$ denotes the first-ranked alternative in P. A domain $\mathcal{D}$ satisfies *minimal richness* if for all $a \in A$, there exists $P \in \mathcal{D}$ such that $r_1(P) = a$.

[16] Each of the subset relations is strict.

Definition 2.3.4 *The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies Property L' if the following two conditions hold:*

1. *For all $P, P' \in \mathcal{D}$ with $r_1(P) = r_1(P') = a$, there exists a path $\pi = (P^1, \dots, P^t) \in \Pi(P, P')$ such that $r_1(P^k) = a$ for all $k = 1, \dots, t$.*

2. *For all $a \in A$ and $P' \in \mathcal{D}$ with $r_1(P') \neq a$, there exists $P \in \mathcal{D}$ with $r_1(P) = a$ and a path $\pi = (P^1, \dots, P^t) \in \Pi(P, P')$ such that for all $b \in A \backslash \{a\}$ the path π has no $\{a, b\}$-restoration.*

Property L' is easier to verify than Property L. In order to verify the latter, we have to find the existence of a suitable path for all pairs of preferences and all alternatives not ranked last in one of the preferences. For Part 1 of Property L', we only need to check for the existence of a path with a simple property for all pairs of preferences with the *same* first-ranked alternative. For Part 2 of Property L', we only need to verify the existence of appropriate paths for special pairs of preferences.

Proposition 2.3.1 *Properties L and L' are equivalent on all environments $G = \langle \mathcal{D}, \mathcal{E} \rangle$ where $\mathcal{D}$ is minimally rich.*

Proof: Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment where $\mathcal{D}$ is minimally rich. We first show that Property L implies Property L'.

Pick $P, P' \in \mathcal{D}$ such that $r_1(P) = r_1(P') = a$. Since G satisfies Property L, there exists a path π from P to P' such that for all $b \in L(a, P) = A \setminus \{a\}$ the path π has no $\{a, b\}$-restoration. Clearly, all preferences on this path must have a as the first-ranked alternative. In order to show Part 2 of Property L', consider $a \in A$ and $P' \in \mathcal{D}$ where $r_1(P') \neq a$. By minimal richness, we can find $P \in \mathcal{D}$ with $r_1(P) = a$. Property L implies the existence of a path π in $\Pi(P, P')$ such that for all $b \in L(a, P) = A \setminus \{a\}$ the path π has no $\{a, b\}$-restoration. This is precisely the path required to satisfy Part 2 of Property L'.

We now show that Property L' implies Property L. Pick $P, P' \in \mathcal{D}$ and $a \in A$. We have to show the existence of a path π in $\Pi(P, P')$ such that for all $b \in L(a, P)$ the path π has no $\{a, b\}$-restoration. There are four cases to consider.

Case 1: $r_1(P) = r_1(P') = a$. Part 1 of Property L' guarantees the existence of a path which satisfies the required condition.

Case 2: $r_1(P) = a$ and $r_1(P') \neq a$. According to Part 2 of Property L', there exist $P'' \in \mathcal{D}$ with $r_1(P'') = a$ and a path $\pi' \in \Pi(P'', P')$ such that π' has no $\{a, b\}$-restoration for any $b \neq a$. Let $\tilde{\pi} \in \Pi(P, P'')$ be the path whose existence is guaranteed by Part 1 of Property L'. Let $\tilde{P}$ be the first vertex in the path $\tilde{\pi}$ (proceeding from P towards P'') which lies on π'. Such a vertex must exist since P'' belongs to both $\tilde{\pi}$ and π'. Let π be the sequence of vertices obtained by concatenating the sub-paths $\tilde{\pi}|_{[P,\tilde{P}]}$ and $\pi'|_{[\tilde{P},P']}$. By construction, π does not contain any repetition of vertices. Therefore $\pi \in \Pi(P, P')$. Since there is no

$\{a, b\}$-restoration in π' for any $b \neq a$, there is no such restoration on its sub-path $\pi'|_{[\tilde{P},P']}$ either. Also a is first-ranked everywhere on the sub-path $\tilde{\pi}|_{[P,\tilde{P}]}$. Therefore π has no $\{a, b\}$-restoration for all $b \in A \setminus \{a\} = L(a, P)$.

Case 3: $r_1(P) \neq a$ and $r_1(P') = a$. According to Case 2, there exists a path $\pi' \in \Pi(P', P)$ that has no $\{a, b\}$-restoration for any $b \neq a$. Let π be the reverse of path π'. Then $\pi \in \Pi(P, P')$, and π has no $\{a, b\}$-restoration for all $b \in L(a, P)$.

Case 4: $r_1(P) \neq a$ and $r_1(P') \neq a$. By minimal richness, there exists $\bar{P} \in \mathcal{D}$ with $r_1(\bar{P}) = a$. Applying the argument in Case 3, there exists a path $\tilde{\pi} \in \Pi(P, \bar{P})$ with no $\{a, b\}$-restoration for any $b \in L(a, P)$. Applying Case 2, there exists a path $\hat{\pi} \in \Pi(\bar{P}, P')$ with no $\{a, b\}$-restoration for all $b \in A \setminus \{a\}$. Arguments similar to those in Case 2 can now be used to construct an appropriate path from P to P'. Let $\tilde{P}$ be the first vertex in the path $\tilde{\pi}$ (proceeding from P to $\bar{P}$) that also lies on $\hat{\pi}$. Let π be the sequence of vertices obtained by the concatenation of the sub-paths $\tilde{\pi}|_{[P,\tilde{P}]}$ and $\hat{\pi}|_{[\tilde{P},P']}$. Clearly $\pi \in \Pi(P, P')$. Since $\tilde{\pi}$ satisfies no $\{a, b\}$-restoration for all $b \in L(a, P)$ and $a = r_1(\bar{P})$, it follows that no alternative in $L(a, P)$ overtakes a in $\tilde{\pi}|_{[P,\tilde{P}]}$, i.e. $L(a, P) \subset L(a, \tilde{P})$. The sub-path $\hat{\pi}$ satisfies no $\{a, b\}$-restoration for all $b \neq a$; therefore the sub-path $\hat{\pi}|_{[\tilde{P},P']}$ satisfies no $\{a, b\}$-restoration for all $b \in L(a, P)$. We can summarize the argument thus far as follows. Pick an arbitrary $b \in L(a, P)$ and consider the path π. If $aP'b$, then b lies everywhere less preferred to a along π. If $bP'a$, then b is less preferred to a in π till $\tilde{P}$ and overtakes a once from $\tilde{P}$ to P'. In other words, π satisfies no $\{a, b\}$-restoration for all $b \in L(a, P)$. ■

In Section 2.5, we apply Property L' to various environments in order to show LGE.

2.4 Discussion

We comment on some aspects of our results.

2.4.1 Computational Complexity

The problem of determining whether an environment satisfies Property L, is not computationally hard. The *Depth First Search Algorithm* [17] for efficiently traversing graphs can be modified easily to construct an algorithm that decides whether an environment satisfies Property L. The worst case time complexity of the algorithm is $O\big(|A|^2|\mathcal{D}|(|\mathcal{D}| + |\mathcal{E}|)\big)$ which is polynomial in the parameters of the problem. The details of the argument can be found in Chatterjee [13].

[17] See Cormen et al. [19].

2.4.2 Relationship with Earlier Results

Carroll [12] proved that the the environments $\langle \mathcal{P}, \mathcal{E}^{adj} \rangle$ and $\langle \mathcal{D}^{SP}, \mathcal{E}^{adj} \rangle$ satisfy LGE.[18] Both these environments satisfy a stronger version of Property L which we refer to as Property U.

Definition 2.4.1 *The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies the universal pairwise no-restoration property (Property U) if for all $P, P' \in \mathcal{D}$, there exists a path in $\Pi(P, P')$ that satisfies no-restoration for all pairs $\{a, b\}$.*

Let $\pi \in \Pi(P, P')$ be the path that satisfies no-restoration for all pairs of alternatives as required by Property U. Then π also satisfies no $\{a, b\}$-restoration for any $a \in A$ and $b \in L(a, P)$. Clearly, Property L is satisfied. On the other hand, Property L does not imply Property U. In order to see this, consider the environment G^* in Example 2.2.7 which satisfies Property L. For the pair (P^1, P^5) the clockwise path has $\{a, b\}$-restoration while the counterclockwise path has $\{c, a\}$-restoration. Clearly, Property U is violated.

Sato [46] showed that Property P below is necessary for LGE in adjacency environments.

Definition 2.4.2 *The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies the pairwise no-restoration property (Property P) if for all $P, P' \in \mathcal{D}$, and $a, b \in A$, there exists a path in $\Pi(P, P')$ that satisfies no $\{a, b\}$-restoration.*

Example 3.2 in Sato [46] shows that Property P is not sufficient for LGE. The difficulty is that Property P does not specify the relationship between the no-restoration paths for *different* pairs of alternatives - the path satisfying no-restoration between P and P' for $\{a, b\}$ could be distinct from the no-restoration path between the same vertices for another pair $\{c, d\}$. Property L is clearly a strengthening of Property P.

Sato [46] also introduced a sufficient condition for LGE in adjacency environments (we refer to this condition as Property S for convenience) which is weaker than Property U.

Definition 2.4.3 *Let $G = \langle \mathcal{D}, \mathcal{E}^{adj} \rangle$ be an environment. Consider $P, P' \in \mathcal{D}$. A path $\pi = (P^1, P^2, \dots, P^t) \in \Pi(P, P')$ satisfies the antidote property with respect to the pair (P, P') if, for all pairs $a, b \in A$ such that π is with $\{a, b\}$-restoration and aP^1b, then for each $h \in \{1, \dots, t\}$ such that $bP^{h-1}a$ and $aP^h b$, there exists a path $\pi' \in \Pi(P, P^h)$ along which a does not overtake any alternative.*

The environment G satisfies Property S if, for every $P, P' \in \mathcal{D}$ there exists a path satisfying the antidote property with respect to (P, P').

Environment G^* in Example 2.2.7 violates Property S which establishes that Property S is stronger than Property L. Consider the pair (P^1, P^5). As noted earlier, the clockwise path from P^1 to P^5 has $\{a, b\}$-restoration since aP^1b, bP^4a and aP^5b. In order for it to satisfy the antidote property, a should not

[18] Recall that $\mathcal{P}$ is the set of all strict preferences. Also $\mathcal{D}^{SP}$ is the domain of single-peaked preferences. A formal definition of single-peaked preferences can be found in Section 2.5.

overtake any alternative in the counterclockwise path from P^1 to P^5. However a does overtake c on this path. Property L is nevertheless satisfied since there is no restoration with a and any of the alternatives ranked below a in P^1 along this path.

2.5 Multi-dimensional Voting: the separable domain and the multi-dimensional single-peaked domain

In this section, we apply our results to a well-known voting model. The set of alternatives has a Cartesian product structure, i.e. $A = \times_{j\in M} A_j$ where $M = \{1, 2, \ldots, m\}$ is a finite set of *components* with $m \geq 2$. For each $j \in M$, the component set A_j contains a finite number of elements with $|A_j| \geq 2$. For any $j \in M$, $A_{-j} = \times_{i\neq j} A_i$. An alternative $a \in A$ is an m-tuple $a \equiv (a_1, \ldots, a_m)$. We shall sometimes write a in the form (a_j, a_{-j}) where $a_j \in A_j$ and $a_{-j} \in A_{-j}$. A preference P is a linear order over A. A *marginal* preference over component j is a linear order over A_j.

A preference P is *separable* if, for all $a_j, b_j \in A_j$, $c_{-j}, d_{-j} \in A_{-j}$ and $j \in M$, $(a_j, c_{-j})P(b_j, c_{-j})$ implies $(a_j, d_{-j})P(b_j, d_{-j})$. Thus every separable preference P induces an m-tuple of marginal preferences $(P_1, \ldots, P_m)$.[19] Let $\mathcal{D}_S$ denote the set of *all* separable preferences. Note that for every component j and any marginal preference P_j over the component set A_j, there exists $P \in \mathcal{D}_S$ such that P induces the marginal preference P_j over A_j. There is a large literature on committee voting following Barberà et al. [5] which assumes separable preferences.

Another domain of preferences that we shall consider is that of multi-dimensional single-peaked preferences introduced by Barberà et al. [6]. (See also Le Breton and Sen [36]) This notion generalizes the well-known class of single-peaked preferences (see Moulin [41]). For this purpose, additional structure is introduced on each component set.

Let $\prec_j$ denote a linear order over A_j for each $j \in M$. A *grid* is an m-tuple $(\prec_1, \ldots, \prec_m)$.[20] Let P be a preference over A whose first-ranked alternative is x. Then P is *multi-dimensional single-peaked* with respect to the grid $(\prec_1, \ldots, \prec_m)$ if for all distinct $a, b \in A$, we have
$\left[x_j \preceq_j a_j \prec_j b_j \text{ or } b_j \prec_j a_j \preceq_j x_j \text{ for all } j \in M \text{ with } a_j \neq b_j\right] \Rightarrow [aPb]$.[21]

[19] The converse is not true however. Several preferences can induce the same tuple of marginal preferences. For instance, consider *additively separable* preferences. Preferences over each component j have a utility representation $u_j : A_j \to \Re$. Utility representations over A are obtained by summing utilities over components. By considering different affine transformations of u_j, one can obtain different preferences over A without changing marginal preferences. Details can be found in Le Breton and Sen [36].

[20] A grid can be interpreted as a product of lines. The notion of multi-dimensional single-peakedness can be generalized on a product of trees where our result still holds. For notational convenience, let $a_j \preceq_j b_j$ denote either $a_j \prec_j b_j$ or $a_j = b_j$.

[21] In the case where $m = 1$, multi-dimensional single-peakedness reduces to single-peakedness. The definition of multi-dimensional single-peakedness is silent regarding the comparison of some alternatives. For instance, suppose $m = 2$, $\prec$ is the $<$ ordering on real numbers and $A_1 = A_2 = \{0, 1\}$. Let $(0, 0)$ be the highest-ranked alternative in the multi-dimensional

P^1	P^2	P^3	P^4	P^5	P^6	P^7	P^8
$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(1,1)$	$(1,1)$
$(0,1)$	$(1,0)$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$
$(1,0)$	$(0,1)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,0)$	$(0,1)$
$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$

Table 2.5.1: Domains $\mathcal{D}_S$ and $\mathcal{D}_{MSP}$

The domain $\mathcal{D}_{MSP}$ contains preferences that are not separable (see Section 3 in Le Breton and Sen [36]). However $\mathcal{D}_S \cap \mathcal{D}_{MSP} \neq \emptyset$. In order to see this, pick an arbitrary m-tuple of marginal preferences $(P_1, \ldots, P_m)$ where each P_j, $j \in M$ is single-peaked with respect to $\prec_j$. Construct P as follows. For all distinct $c, d \in A$ with $c \neq d$, let j be the integer in M such that $c_j \neq d_j$ and $c_r = d_r$ for all $r < j$. Then cPd if and only if $c_j P_j d_j$. It is easy to verify that $P \in \mathcal{D}_S$. We also claim $P \in \mathcal{D}_{MSP}$. Suppose x is the first-ranked alternative in P. Pick distinct alternatives $a, b \in A$. Clearly, $a_j \neq b_j$ for some $j \in M$.
Assume further that $x_j \preceq_j a_j \prec_j b_j$ or $b_j \prec_j a_j \preceq_j x_j$ for all $j \in M$ with $a_j \neq b_j$. Let $k \in M$ be the lowest component such that $a_k \neq b_k$. By virtue of the single-peakedness of P_k, $x_k \preceq_k a_k \prec_k b_k$ or $b_k \prec_k a_k \preceq_k x_k$ implies $a_k P_k b_k$. Then, aPb follows directly from the construction of P.

We introduce a new notion of neighbours that applies to any domain which includes separable preferences. Let $P, P' \in \mathcal{D}_S$. We say that P and P' are *separably adjacent* (denoted by $(P, P') \in \mathcal{E}^{SA}$) if there exist $j \in M$ and $a_j, b_j \in A_j$ such that
$[\{x, y\} \in P \triangle P'] \Rightarrow [x_j = a_j, y_j = b_j$ and $x_k = y_k$ for all $k \neq j]$. Thus P and P' are separably adjacent if all pairs of alternatives that are reversed between P and P' differ in the values of exactly one component.[22] We emphasize that separable adjacency applies *only* to separable preferences.

Separable adjacency does not cover the standard adjacency case. We therefore consider a strengthened version of separable adjacency: P and P' are *adjacent-separably adjacent* (denoted by $(P, P') \in \mathcal{E}^{ASA}$)[23] if either $(P, P') \in \mathcal{E}^{adj}$ or $(P, P') \in \mathcal{E}^{SA}$ holds. Two separable preferences P and P' are neighbours in the ASA sense if one can be obtained from the other by a "minimal" change.

Example 2.5.1 Let $A = A_1 \times A_2$ with $A_1 = A_2 = \{0, 1\}$. In the special case $|A_j| = 2$ for all $j \in M$, we have $\mathcal{D}_S = \mathcal{D}_{MSP}$ implying that the environments $\langle \mathcal{D}_S, \mathcal{E}^{ASA} \rangle$ and $\langle \mathcal{D}_{MSP}, \mathcal{E}^{ASA} \rangle$ are the same. Table 2.5.1 lists the preferences in $\mathcal{D}_S$ and $\mathcal{D}_{MSP}$. Note that the domain satisfies minimal richness.

single-peaked preference $\bar{P}$. We must have $(0,0)\bar{P}(1,0)$, $(0,0)\bar{P}(0,1)$, $(0,0)\bar{P}(1,1)$, $(1,0)\bar{P}(1,1)$ and $(0,1)\bar{P}(1,1)$ by definition.

[22]Separably adjacency is based on a notion of Kemeny distance that applies to separable preferences. Two (separable) preferences are separably adjacent if they disagree on the relative ranking of two alternatives that differ in the values of exactly one component. Further analysis of separable adjacency can be found in Chatterji and Zeng [16].

[23]The acronym *ASA* stands for adjacent-separably adjacent.

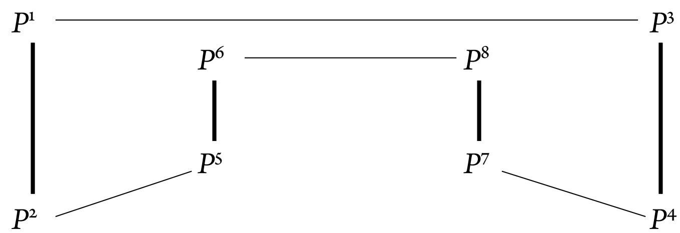

Figure 2.5.1: $\langle \mathcal{D}_S, \mathcal{E}^{ASA} \rangle$ and $\langle \mathcal{D}_{MSP}, \mathcal{E}^{ASA} \rangle$

This environment is shown in Figure 2.5.1. The thicker lines in the figure show the environment $\langle \mathcal{D}_S, \mathcal{E}^{adj} \rangle$, i.e. $\mathcal{E}^{adj} = \{(P^1, P^2), (P^3, P^4), (P^5, P^6), (P^7, P^8)\}$. The other edges in the figure belong to $\mathcal{E}^{SA}$. Note that $(P^1, P^2) \notin \mathcal{E}^{SA}$ since $P^1 \bigtriangleup P^2 = \big\{\{(0,1), (1,0)\}\big\}$. Also $P^1 \bigtriangleup P^3 = \big\{\{(0,0), (0,1)\}, \{(1,0), (1,1)\}\big\}$. Observe that the set of alternatives that are reversed between P^1 and P^3 can be obtained by switching the value of component 2 from 0 to 1 at different values of component 1. Clearly $(P^1, P^3) \in \mathcal{E}^{SA}$. On the other hand $(P^2, P^4) \notin \mathcal{E}^{SA}$ since $\{(0,0), (1,1)\} \in P^2 \bigtriangleup P^4$.

We will show later that the environment $\langle \mathcal{D}_{MSP}, \mathcal{E}^{ASA} \rangle$ satisfies Property L'. Clearly, Part 1 of Property L' is satisfied as indicated by the four thick edges in Figure 2.5.1. Now consider the preference P^1 and the alternative $(1,1)$ which is not first-ranked in P^1. We have $(1,1)$ first-ranked in preference P_8 and the path $(P^8, P^7, P^4, P^3, P^1)$ has no restoration for $(1,1)$ and any other alternative. Consequently, the requirement of Part 2 of Property L' is satisfied in this case. $\square$

Example 2.5.1 and Figure 2.5.1 also lead to the conclusion that the environments $\langle \mathcal{D}_S, \mathcal{E}^{SA} \rangle$, $\langle \mathcal{D}_{MSP}, \mathcal{E}^{SA} \rangle$, $\langle \mathcal{D}_S, \mathcal{E}^{adj} \rangle$ and $\langle \mathcal{D}_{MSP}, \mathcal{E}^{adj} \rangle$ fail LGE. The graphs in these environments are not connected which can be verified by inspection and by our earlier remarks.

According to the main result in the section, combining the adjacency and separable adjacency notions of neighbours with the separable and multi-dimensional single-peaked domains leads to LGE.

Proposition 2.5.1 *The environments $\langle \mathcal{D}_S, \mathcal{E}^{ASA} \rangle$ and $\langle \mathcal{D}_{MSP}, \mathcal{E}^{ASA} \rangle$ satisfy LGE.*

The proof of Proposition 2.5.1 can be found in the Appendix.

2.6 LGE and random social choice functions

In this section, we examine LGE in the context of random social choice functions. Our result is the following: an environment that satisfies LGE for deterministic social choice functions may not satisfy LGE for random social choice functions.

Let $\Delta(A)$ denote the set of probability distributions over A. An element $\lambda \in \Delta(A)$ will be referred to as a *lottery*. We let λ_a denote the probability with which $a \in A$ is selected by λ. Thus $0 \leq \lambda_a \leq 1$ and $\sum_{a \in A} \lambda_a = 1$.

A *Random Social Choice Function* (or RSCF) is a map $\phi : \mathcal{D} \to \Delta(A)$ that associates a lottery $\phi(P)$ with each $P \in \mathcal{D}$.

For every $P \in \mathcal{D}$, and $k = 1, 2, \ldots |A|$, let $r_k(P) \in A$ denote the k^{th} ranked alternative in P i.e. $r_k(P) = a$ implies $|\{b \in A : bPa\}| = k - 1$. The lottery λ *stochastically dominates* lottery λ' at $P \in \mathcal{D}$ (denoted by $\lambda P_{sd} \lambda'$) if $\sum_{k=1}^{t} \lambda_{r_k(P)} \geq \sum_{k=1}^{t} \lambda'_{r_k(P)}$ for all $t = 1, \ldots, |A|$.

Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment. A RSCF $\phi : \mathcal{D} \to \Delta(A)$ is *locally sd-strategy-proof* if $\phi(P) P_{sd} \phi(P')$ for all $(P, P') \in \mathcal{E}$. A RCSF $\phi : \mathcal{D} \to \Delta(A)$ is *sd-strategy-proof* if $\phi(P) P_{sd} \phi(P')$ for all $P, P' \in \mathcal{D}$.

The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies *random local-global equivalence or RLGE* if every locally sd-strategy-proof RSCF $\phi : \mathcal{D} \to \Delta(A)$ is also sd-strategy-proof.

In the case where a RSCF is deterministic, local sd-strategy-proofness and sd-strategy-proofness reduce to local strategy-proofness and strategy-proofness respectively. An immediate consequence of this observation is an environment that satisfies RLGE also satisfies LGE. The results of Carroll [12] and Cho [18] show that the converse is true for several special domains. The example below shows that LGE does not imply RLGE.

Example 2.6.1 Let $A = \{a, b, c, v, w, x, y, z\}$. The domain $\tilde{\mathcal{D}}$ is described in Table 2.6.1. The environment $\tilde{G} = \langle \tilde{\mathcal{D}}, \mathcal{E}^{adj} \rangle$ is shown in Figure 2.6.1.

By using arguments similar to those in Example 2.2.7, we can show that $\tilde{G}$ satisfies Property L. Therefore, Theorem 2.3.2 implies that $\tilde{G}$ satisfies LGE. We construct a RSCF which satisfies local sd-strategyproofness but not sd-strategy-proofness.

For any $d \in A$, we let e_d denote the degenerate lottery that picks d with probability one. Consider the RSCF $\phi : \tilde{\mathcal{D}} \to \Delta(A)$:

$$\phi(P^k) = \begin{cases} \frac{1}{2}e_a + \frac{1}{2}e_b & \text{if } k \in \{1, 10\}, \\ \frac{1}{2}e_a + \frac{1}{4}e_b + \frac{1}{4}e_c & \text{if } k \in \{2, 3, 4, 5\}, \\ \frac{1}{4}e_a + \frac{1}{2}e_b + \frac{1}{4}e_c & \text{if } k \in \{6, 7, 8, 9\}. \end{cases}$$

P^1	P^2	P^3	P^4	P^5	P^6	P^7	P^8	P^9	P^{10}
a	a	a	a	a	b	b	b	b	b
b	c	c	c	b	a	c	c	c	a
c	b	b	b	c	c	a	a	a	c
v	v	w	w	w	w	w	w	v	v
w	w	v	v	v	v	v	v	w	w
x	x	x	x	x	x	x	x	x	x
y	y	y	z	z	z	z	y	y	y
z	z	z	y	y	y	y	z	z	z

Table 2.6.1: Domain $\tilde{\mathcal{D}}$

$$\begin{tikzcd}
P^1 \arrow[r, dash, "\{b,c\}"] \arrow[d, dash, "\{a,b\}"'] & P^2 \arrow[r, dash, "\{v,w\}"] & P^3 \arrow[r, dash, "\{y,z\}"] & P^4 \arrow[r, dash, "\{c,b\}"] & P^5 \arrow[d, dash, "\{a,b\}"] \\
P^{10} \arrow[r, dash, "\{a,c\}"] & P^9 \arrow[r, dash, "\{v,w\}"] & P^8 \arrow[r, dash, "\{y,z\}"] & P^7 \arrow[r, dash, "\{c,a\}"] & P^6
\end{tikzcd}$$

Figure 2.6.1: $\tilde{G} = \langle \tilde{\mathcal{D}}, \mathcal{E}^{adj} \rangle$

In order to verify the local sd-strategy-proofness of ϕ it suffices to show that the voter cannot gain by manipulation in each of the following cases: (i) from P^1 to P^2 and vice versa, (ii) from P^5 to P^6 and vice versa and (iii) from P^9 to P^{10} and vice versa. This can be verified easily in each of the cases. Consider (i), for instance. Observe that c locally overtakes b from P^1 to P^2. Correspondingly, probability $\frac{1}{4}$ is transferred from b to c, (keeping other probabilities fixed) as we move from $\phi(P^1)$ to $\phi(P^2)$. Therefore, $\phi(P^2)P^2_{sd}\phi(P^1)$ and symmetrically, $\phi(P^1)P^1_{sd}\phi(P^2)$. The same argument can be made in cases (ii) and (iii).

However, it is not the case that $\phi(P^5)P^5_{sd}\phi(P^1)$ (in fact $\phi(P^1)P^5_{sd}\phi(P^5)$). Consequently ϕ is not sd-strategy-proof. $\square$

We make two observations about Example 2.6.1.

Observation 2.6.1 As mentioned earlier, Carroll [12] and Cho [18] have established the equivalence of local sd-strategy-proofness and sd-strategy-proofness in specific adjacency environments. These environments all satisfy Property U. The environment $\tilde{G}$ in Example 2.6.1 violates Property U since both the clockwise and counterclockwise paths between P^1 and P^5 have restorations.

Observation 2.6.2 The key feature of the example in Example 2.6.1 that makes the LGE and RLGE results differ is that some lotteries under ϕ have support $\{a, b, c\}$, e.g. $\phi(P^k)$, $k = 2, \ldots, 9$. However, no locally strategy-proof SCF can have a range that includes all three alternatives a, b and c. In order to see this, let $f : \tilde{\mathcal{D}} \to A$ be a locally strategy-proof SCF. Theorem 2.3.2 implies that f is strategy-proof. Suppose $\{a, b, c\} \subseteq Range(f) = \{d \in A : f(P) = d \text{ for some } P \in \tilde{\mathcal{D}}\}$. Thus, there exists a preference where f takes value a and another preference where f takes value b. Strategy-proofness immediately implies $f(P^k) = a$ for all $1 \leq k \leq 5$ and $f(P^l) = b$ for all $6 \leq l \leq 10$. Hence, we have a contradiction.

A characterization for RLGE appears to be significantly more difficult than that for LGE. In our companion paper Kumar et al. [33] we derive a weak sufficient condition for RLGE in multi-voter models where RSCFs satisfy the additional property of unanimity.

Appendix: Proof of Proposition 2.5.1

We begin by observing that both the separable domain $\mathcal{D}_S$ and the multi-dimensional single-peaked domain $\mathcal{D}_{MSP}$ satisfy the minimal richness property. Applying Theorem 2.3.2 and Proposition 2.3.1, it suffices to show that both domains satisfy Property L'. Furthermore both domains satisfy Part 1 of Property L' as is shown in Appendices E.2 and E.5 of Chatterji and Zeng [16]. Therefore, we only verify Part 2 of Property L'.[24]

We first investigate the separable domain $\mathcal{D}_S$. Next, we show Part 2 of Property L' on the intersection of the separable domain and the multi-dimensional single-peaked domain $\mathcal{D}_S \cap \mathcal{D}_{MSP}$, and then extend the result to the multi-dimensional single-peaked domain $\mathcal{D}_{MSP}$.

In the proofs, we shall occasionally employ a special type of separable preferences called lexicographic separable preferences. Let $(P_1, \ldots, P_m)$ be an m-tuple of marginal preferences and let P_0 be strict order over the set M. The preference P is *lexicographically separable* with respect to the $(m+1)$-tuple $(P_0, P_1, \ldots P_m)$ if, for all $a, b \in A$, $\left[a_j P_j b_j \text{ and } a_r = b_r \text{ for all } r \text{ such that } rP_0 j\right] \Rightarrow [aPb]$. In other words, a is ranked strictly better than b according to P if a_j is ranked higher than b_j according to the marginal preference P_j and $a_r = b_r$ for all components r that are ranked strictly higher than j according to the component preference P_0. We shall write a lexicographically separable preference P as $P \equiv (P_0, P_1, \ldots, P_m)$.

We first prove two preliminary lemmas.

Lemma 2.6.1 *Let distinct $P, P' \in \mathcal{D}_S$ induce the same marginal preferences. Then there exists a path from P to P' in $\langle \mathcal{D}_S, \mathcal{E}^{adj} \rangle$ such that there is no restoration for any pair of alternatives.*

Proof: This lemma follows from Fact 5 of Chatterji and Zeng [16]. ■

Lemma 2.6.2 *Fix marginal preferences $P_1, \ldots, P_m$. Let a be an alternative such that a_j is not the first-ranked element in P_j for some $j \in M$. For each component k, let $X_k = \{x_k \in A_k : x_k P_k a_k\} \cup \{a_k\}$. Let $X = X_1 \times \ldots \times X_m$. Pick component j, and let $b_j, c_j \in X_j$ or $b_j, c_j \in A_j \backslash X_j$ be consecutively ranked elements in P_j. Then there exists a separable ordering $\bar{P}(j)$ satisfying the following properties:*

[24] Part 1 of Property L' is the same as the interior+ property of Chatterji and Zeng [16]. Hence, we can directly apply their result for this part. However, Part 2 of Property L' is stronger than their exterior+ property so we have to show this independently.

1. $\bar{P}(j)$ *induces the marginal preferences* $P_1, \ldots, P_m$.

2. $[x\bar{P}(j)a] \Rightarrow$ *[for each* $k \in M$, *either* $x_k P_k a_k$ *or* $x_k = a_k$, *i.e.* $x \in X$].

3. (b_j, z_{-j}) *and* (c_j, z_{-j}) *are consecutively ranked in* $\bar{P}(j)$ *for all* $z_{-j} \in A_{-j}$.

Proof: We construct a partition of the set A. In order to do so, define the following sets: $A_{-j} = \times_{k \neq j} A_k$, $X_{-j} = \times_{k \neq j} X_k$, $Y_j = A_j \setminus X_j$, and $Y_{-j} = A_{-j} \setminus X_{-j}$. The sets X, $B = X_j \times Y_{-j}$, $C = Y_j \times X_{-j}$ and $D = Y_j \times Y_{-j}$ constitute a partition of the set A. The ordering $\bar{P}(j)$ is defined by the Conditions 1 and 2 below.

1. $X\bar{P}(j)B\bar{P}(j)C\bar{P}(j)D$ i.e. all alternatives in X are ranked above those in B which in turn are ranked above those in C, while all alternatives in D are ranked below those in C.

2. $\bar{P}(j)$ over X is lexicographically separable according to $(P_0(j), P_1, \ldots, P_m)$ where j is ranked last in the component preference $P_0(j)$ i.e. given $x, y \in X$, $[x_k P_k y_k$ and $x_r = y_r$ for all $rP_0(j)k] \Rightarrow [x\bar{P}(j)y]$. Similarly, $\bar{P}(j)$ is lexicographically separable over alternatives respectively in B, C and D with respect to $(P_0(j), P_1, \ldots, P_m)$.

Observe that a_k is the lowest ranked element in X_k according to P_k for all $k \in M$. Therefore, by the construction, a is the worst alternative in X according to $\bar{P}(j)$. As X is the highest-ranked block according to $\bar{P}(j)$, it follows that all alternatives x that are ranked higher than a according to $\bar{P}(j)$ must satisfy $x \in X$. This establishes Part 2 of Lemma 2.6.2.

To show that $\bar{P}(j)$ is a separable preference and satisfies Part 1 of Lemma 2.6.2, it suffices to show that for an arbitrary pair of alternatives that disagree in exactly one component, say $x = (x_k, z_{-k})$ and $y = (y_k, z_{-k})$, we have $[(x_k, z_{-k})\bar{P}(j)(y_k, z_{-k})]$
$\Rightarrow [x_k P_k y_k]$. If x and y both belong to one of the sets X, B, C or D, the result follows immediately. Henceforth, assume that x and y belong to two different sets of X, B, C and D.

Suppose $k = j$. We know either $z_{-j} \in X_{-j}$ or $z_{-j} \in Y_{-j}$. If $z_{-j} \in X_{-j}$, $(x_k, z_{-k})\bar{P}(j)(y_k, z_{-k})$ implies $x \in X$ and $y \in C$. Similarly, if
$z_{-j} \in Y_{-j}$, $(x_k, z_{-k})\bar{P}(j)(y_k, z_{-k})$ implies $x \in B$ and $y \in D$. Consequently, in both cases, $x_j \in X_j$ and $y_j \in Y_j$, and hence $x_j P_j y_j$.

Suppose $k \neq j$. Let z_{-jk} denote the vector z_{-k} with its element of component j deleted. Since $x\bar{P}(j)y$, and x and y agree on component j, we know either $x \in X$ and $y \in B$, or $x \in C$ and $y \in D$, both of which imply $(x_k, z_{-jk}) \in X_{-j}$ and $(y_k, z_{-jk}) \in Y_{-j}$. Since X_{-j} is a Cartesian product set, $(x_k, z_{-jk}) \in X_{-j}$ implies $x_k \in X_k$ and $z_{-jk} \in \times_{r \neq j,k} X_r$. Last, since $z_{-jk} \in \times_{r \neq j,k} X_r$, $(y_k, z_{-jk}) \notin X_{-j}$ implies $y_k \notin X_k$. Therefore, $x_k P_k y_k$.

Hence $\bar{P}(j)$ is a separable preference, and induces marginal preferences $P_1, \ldots, P_m$.

Part 3 of Lemma 2.6.2 is an immediate consequence of the fact that $\bar{P}(j)$ over alternatives of X and B respectively is lexicographically separable with respect to the component preference $P_0(j)$ where component j is ranked last. ■

We now show that the separable domain $\mathcal{D}_S$ satisfies Part 2 of Property L'.

Proof: Consider $P' \in \mathcal{D}_S$ and $a \in A$ such that a is not the first-ranked alternative in P'. Let $P'_1, \ldots, P'_m$ be the induced marginal preferences of P'. Without loss of generality, assume that $a_1, a_2, \ldots, a_r$, $r \leq m$, are not first-ranked in $P'_1, P'_2, \ldots, P'_r$ respectively, while $a_v = r_1(P'_v)$ for all $v = r+1, \ldots, m$. We will construct a sequence of preferences which are edges in $\langle \mathcal{D}_S, \mathcal{E}^{ASA} \rangle$ with the property that a keeps "rising" along the sequence. The sequence will terminate in a preference $P \in \mathcal{D}_S$ where a is first-ranked. Then, the reverse path from P to P' has no $\{a, b\}$-restoration for all $b \in A \backslash \{a\}$, as required by Part 2 of Property L'.

We start from P'_1. Let $\mathcal{P}_1$ denote the set of all marginal preferences over A_1. Pick a marginal ordering P_1 such that a_1 is first-ranked. By Proposition 4.1 of Sato [46], we have a path $\pi^1 = (P_1^1, \ldots, P_1^t)$ from P'_1 to P_1 in $\langle \mathcal{P}_1, \mathcal{E}^{adj} \rangle$ which has no restoration for any pair of elements of A_1.[25] Since $L(a_1, P'_1) \subset L(a_1, P_1)$, a_1 must keep rising along the path π^1 i.e. $L(a_1, P_1^k) \subseteq L(a_1, P_1^{k+1})$ for all $1 \leq k < t$. Therefore, for all $1 \leq k < t$, if a_1 is involved in the local switching elements across P_1^k and P_1^{k+1}, it is true that $x_1 P_1^k a_1$ and $a_1 P_1^{k+1} x_1$ for some $x_1 \in A_1$.

For each $k = 1, \ldots, t$, let $X_1^k = \{x_1 \in A_1 : x_1 P_i^k a_1\} \cup \{a_1\}$. For each $k = 1, \ldots, t-1$, consider (P_1^k, P_1^{k+1}) and let $P_1^k \triangle P_1^{k+1} = \big\{\{b_1^k, c_1^k\}\big\}$. Since $L(a_1, P_1^k) \subseteq L(a_1, P_1^{k+1})$, it must be the case that either $b_1^k, c_1^k \in X_1^k$ or $b_1^k, c_1^k \in A_1 \backslash X_1^k$. Next, for each $k = 1, \ldots, t$, by Lemma 3, let $\bar{P}^k(1) \in \mathcal{D}_S$ be such that (i) it induces the marginal preferences $P_1^k, P'_2, \ldots, P'_m$, (ii) if $x\bar{P}^k(1)a$, then for all $j \in M$, either $x_j = a_j$, or x_j is strictly better than a_j according to the j^{th} marginal ordering of $\bar{P}^k(1)$, and (iii) (b_1^k, z_{-1}) and (c_1^k, z_{-1}) are consecutively ranked in $\bar{P}^k(1)$ for all $z_{-1} \in A_{-1}$. Let $\hat{P}^k(1)$ be the ordering obtained by switching all alternatives of the type (b_1^k, z_{-1}) and (c_1^k, z_{-1}) for some $z_{-1} \in A_{-1}$. It is clear that $\hat{P}^k(1)$ is a separable preference with the same marginal preferences as $\bar{P}^k(1)$ for all components other than 1. For component 1, c_1^k is now ranked immediately above b_1^k, while the rankings of other elements are unchanged. Therefore, there are three properties of $\hat{P}^k(1)$ that are important: (i) $\big(\bar{P}^k(1), \hat{P}^k(1)\big) \in \mathcal{E}^{SA}$ and $\bar{P}^k(1) \triangle \hat{P}^k(1) = \big\{\{(b_1^k, z_{-1}), (c_1^k, z_{-1})\} : z_{-1} \in A_{-1}\big\}$, (ii) $L\big(a, \bar{P}^k(1)\big) \subseteq L\big(a, \hat{P}^k(1)\big)$ where the strict inclusion holds if and only if $a_1 = c_1^k$, and (iii) $\hat{P}^k(1)$ and $\bar{P}^{k+1}(1)$ have the same marginal preferences, and $L\big(a, \hat{P}^k(1)\big) \subseteq L\big(a, \bar{P}^{k+1}(1)\big)$ by Part 2 of Lemma 2.6.2 in the construction of $\bar{P}^{k+1}(1)$.

[25] For instance, we generate P_1 by moving a_1 directly to the top of P'_1 while keeping the rankings of other elements unchanged, and then construct the path from P'_1 to P_1 in $\langle \mathcal{P}_1, \mathcal{E}^{adj} \rangle$ by progressively moving a_1 to the top of P'_1.

Now, we have a sequence:

$$P' \to \bar{P}^1(1) \to \hat{P}^1(1) \to \bar{P}^2(1) \to \cdots \to \bar{P}^{t-1}(1) \to \hat{P}^{t-1}(1) \to \bar{P}^t(1).$$

Note that $\bar{P}^t(1)$ has marginal preference P_1 where a_1 is the first-ranked element. Since P' and $\bar{P}^1(1)$ have the same marginal preferences $P'_1, P'_2, \dots, P'_m$, we know that either $P = \bar{P}^1(1)$, or there exists a path $\bar{\pi}^0$ from P to $\bar{P}^1(1)$ in $\langle \mathcal{D}_S, \mathcal{E}^{adj} \rangle$ which has no restoration for any pair of alternatives (by Lemma 2.6.1). Similarly, for all $1 \leq k < t$, we know that either $\hat{P}^k(1) = \bar{P}^{k+1}(1)$, or there exists a path $\bar{\pi}^k$ from $\hat{P}^k(1)$ to $\bar{P}^{k+1}(1)$ in $\langle \mathcal{D}_S, \mathcal{E}^{adj} \rangle$ which has no restoration for any pair of alternatives. Since $\left(\bar{P}^k(1), \hat{P}^k(1)\right) \in \mathcal{E}^{SA}$ for all $k = 1, \dots, t-1$, we construct a concatenated path $\bar{\pi} = (\bar{\pi}^0, \bar{\pi}^1, \dots, \bar{\pi}^{t-1})$ from P' to $\bar{P}^t(1)$ in $\langle \mathcal{D}_S, \mathcal{E}^{ASA} \rangle$.[26] Recall that $L(a, P') \subseteq L(a, \bar{P}^1(1))$, $L(a, \bar{P}^k(1)) \subseteq L(a, \hat{P}^k(1))$ and $L(a, \hat{P}^k(1)) \subseteq L(a, \bar{P}^{k+1}(1))$ for all $k = 1, \dots, t-1$. Then, no restoration on subpaths $\bar{\pi}^0, \bar{\pi}^1, \dots, \bar{\pi}^{t-1}$ implies that a keeps rising along the path $\bar{\pi}$.

We can clearly repeat this procedure, progressively moving a_1 to the top in the marginal preference P_1, and then doing the same for a_2, through till a_r. The procedure generates a path in $\langle \mathcal{D}_S, \mathcal{E}^{ASA} \rangle$ culminating in a preference $P \in \mathcal{D}_S$ where a is first-ranked. Moreover if a overtakes some x at some preference on the path, it beats x at all preferences further along the path. It follows immediately that the reverse path from P to P' satisfies no $\{a, b\}$-restoration for all $b \in A \backslash \{a\}$. This establishes Part 2 of Property L', and hence proves Proposition 2.5.1 for the separable domain $\mathcal{D}_S$. ■

To show Part 2 of Property L' in the multi-dimensional single-peaked domain $\mathcal{D}_{MSP}$, we first consider the domain $\mathcal{D}_S \cap \mathcal{D}_{MSP}$. We make several observations. Firstly, $\mathcal{D}_S \cap \mathcal{D}_{MSP}$ satisfies Part 1 of Property L' by Appendix E.4 of Chatterji and Zeng [16]. Secondly, Lemma 2.6.1 remains valid in $\mathcal{D}_S \cap \mathcal{D}_{MSP}$ according to Fact 11 of Chatterji and Zeng [16]. Thirdly, Lemma 2.6.2 holds when we set the marginal preferences $P_1, \dots, P_m$ to be single-peaked with respect to $\prec_1, \dots, \prec_m$ respectively, and change preference $\bar{P}(j)$ to be both separable and multi-dimensional single-peaked. Finally, in the verification of Part 2 of Property L' in the separable domain, if we replace $\mathcal{D}_S$ with $\mathcal{D}_S \cap \mathcal{D}_{MSP}$, $\mathcal{P}_1$ with $\mathcal{S}_1$ which is the set of all single-peaked marginal preferences with respect to $\prec_1$, and the reference to Proposition 4.1 of Sato [46] with a reference to Proposition 4.2 of Sato [46], our earlier proof works for verifying Part 2 of Property L' in $\mathcal{D}_S \cap \mathcal{D}_{MSP}$. Therefore, $\mathcal{D}_S \cap \mathcal{D}_{MSP}$ satisfies Property L'.

To extend the result to the multi-dimensional single-peaked domain, we use the following lemma

[26] The concatenated path $\bar{\pi}$ has no repeated preference. Given two preferences $\hat{P}$ and $\tilde{P}$ in $\bar{\pi}$, we know $\hat{P} \in \bar{\pi}^k$ and $\tilde{P} \in \bar{\pi}^{k'}$ for some $0 \leq k, k' \leq t-1$. If $k = k'$, it is evident that $\hat{P} \neq \tilde{P}$ by the definition of the path $\bar{\pi}^k$. Next, assume $k < k'$. Note that $\hat{P}^{k'}(1)$ and $\bar{P}^{k'+1}(1)$ induce the same marginal preference $P_1^{k'+1}$ and the path $\bar{\pi}^{k'}$ connecting $\hat{P}^{k'}(1)$ and $\bar{P}^{k'+1}(1)$ has no restoration for any pair of alternatives. Then, $\tilde{P} \in \bar{\pi}^{k'}$ implies that $\tilde{P}$ induces the marginal preference $P_1^{k'+1}$. Symmetrically, $\hat{P}$ induces the marginal preference P_1^{k+1}, which is distinct from $P_1^{k'+1}$. Therefore, $\hat{P}$ and $\tilde{P}$ must be distinct.

which follows from Lemma 8 of Chatterji and Zeng [15].

Lemma 2.6.3 *Given distinct $P, P' \in \mathcal{D}_{MSP}$, let $r_1(P) = r_1(P')$. Then there exists a path from P to P' in $\langle \mathcal{D}_{MSP}, \mathcal{E}^{adj} \rangle$ such that there is no restoration for any pair of alternatives.*

We now show Part 2 of Property L' in the multi-dimensional single-peaked domain $\mathcal{D}_{MSP}$.

Proof: Consider $P' \in \mathcal{D}_{MSP}$ and $a \in A$ such that a is not the first-ranked alternative in P'. Let $r_1(P') = \bar{a}$. Fix $k \in M$. If $a_k = \bar{a}_k$, we pick an arbitrary single-peaked marginal preference P'_k that has a_k as the first-ranked element. If $a_k \neq \bar{a}_k$, we identify a particular single-peaked marginal preference P'_k which satisfies the following condition: $[x_k P'_k a_k] \Rightarrow [\bar{a}_k \preceq_k x_k \prec_k a_k$ or $a_k \prec_k x_k \preceq_k \bar{a}_k]$. The marginal preferences $P'_1, \ldots, P'_m$ are single-peaked by construction. Applying the counterpart of Lemma 2.6.2, we have $\bar{P}' \in \mathcal{D}_S \cap \mathcal{D}_{MSP}$ such that $\bar{P}'$ induces $P'_1, \ldots, P'_m$, and
$[x\bar{P}'a] \Rightarrow [\text{for all } k \in M, \text{ either } x_k = a_k \text{ or } x_k P'_k a_k]$. Note that $L(a, \bar{P}') \supseteq L(a, P')$. By Lemma 2.6.3, since $r_1(P') = r_1(\bar{P}')$, we have a path $\hat{\pi}$ from $\bar{P}'$ to P' in $\langle \mathcal{D}_{MSP}, \mathcal{E}^{adj} \rangle$ which has no restoration for any pair of alternatives. Moreover, since $\mathcal{D}_S \cap \mathcal{D}_{MSP}$ satisfies Property L', we have $P \in \mathcal{D}_S \cap \mathcal{D}_{MSP}$ that has a first-ranked, and a path $\bar{\pi}$ from P to $\bar{P}'$ in $\langle \mathcal{D}_S \cap \mathcal{D}_{MSP}, \mathcal{E}^{ASA} \rangle$ that has no $\{a, b\}$-restoration for all $b \neq a$.

Now, we have a concatenated path $\pi = (\bar{\pi}, \hat{\pi})$ from P to P' in $\langle \mathcal{D}_{MSP}, \mathcal{E}^{ASA} \rangle$.[27] We show that π has no $\{a, b\}$-restoration for all $b \neq a$. Fix an arbitrary $b \neq a$. If b overtakes a on path $\bar{\pi}$, then no $\{a, b\}$-restoration on $\bar{\pi}$ implies that b overtakes a on $\bar{\pi}$ exactly once, and $b\bar{P}'a$. Then, $L(a, \bar{P}') \supseteq L(a, P')$ implies $bP'a$, and no restoration on $\hat{\pi}$ from $\bar{P}'$ to P' implies $b\hat{P}a$ for all $\hat{P} \in \hat{\pi}$. Hence, the concatenated path π has no $\{a, b\}$-restoration. If b does not overtake a on path $\bar{\pi}$, then no $\{a, b\}$-restoration on $\bar{\pi}$ implies $a\bar{P}b$ for all $\bar{P} \in \bar{\pi}$, and hence $a\bar{P}'b$. Furthermore, no restoration on $\hat{\pi}$ implies that b can overtake a on $\hat{\pi}$ for at most once. Hence, the concatenated path π has no $\{a, b\}$-restoration. This establishes Part 2 of Property L', and hence proves Proposition 2.5.1 for the multi-dimensional single-peaked domain $\mathcal{D}_{MSP}$. ■

[27] By an argument similar to the earlier one, the concatenated path π has no repeated preference.

3

Local Global Equivalence in Voting Models Admitting Indifferences

3.1 Introduction

We consider a finite set of alternatives and a society of agents where each agent has a preference over alternatives.[1] The objective of the social planner is to construct social choice functions that aggregate the preferences of the agents in a way that reporting true preference is a dominant strategy for each agent. This property of a social choice function is called strategy-proofness. However, in reality, agents might be comfortable to misreport to preferences that are "local" to their true preference.[2] In such situations, it is sufficient for the social planner to consider social choice functions which ensure that the agents do not benefit from misreporting to a preference that is local to their true preference. Such social choice functions are called locally strategy-proof.

We consider a single agent model, which is without loss of generality in this setting. The set of admissible preferences is called a *domain*.[3] An *environment* is an undirected graph where the vertex set is

[1]The preferences can be weak, that is, it can admit indifference.

[2]Local preferences can be any arbitrary pre-specified set of preferences.

[3]Here, we allow the preferences to have indifferences.

the set of admissible preferences and two preference have an edge if and only if they are local. The agent having her true preference at a particular vertex can misreport to only those preferences (vertices) that have an edge with her true preference (vertex). A social choice function is called locally strategy-proof if for every possible true preference of the agent, she cannot be better off by misreporting to a local preference. A social choice function is strategy-proof if for every possible true preference of the agent, she cannot be better off by misreporting to any other preference. The main question we ask is the following: What are the environments where every locally strategy-proof social choice function is strategy-proof?

We investigate environments where preferences admit indifference. In this chapter, we extend Theorem 2.3.2 of Chapter 2 to preference domains with indifference. Unfortunately, a "clean" characterization result appears to be difficult to obtain in this case. We show that our earlier arguments can be modified to yield a condition that is sufficient for LGE and another condition that is necessary. Moreover the "gap" between the conditions is small.[4] We also provide some applications of our result where we use the sufficiency result to propose notions of neighbours according to which environments with the domain of single-plateaued preferences and the domain of all weak preferences, satisfies LGE.

3.2 Model

Let A be a finite set of alternatives with $|A| \geq 2$. Without loss of generality, we consider a single agent model as we did in the previous chapter. A weak preference denoted by R is a complete and transitive binary relation on A. The antisymmetric and symmetric parts of R are denoted by P and I respectively. Let $\mathcal{R}$ denote the set of all weak preferences on A. Recall that $\mathcal{P}$ is the set of all strict preferences on A. For every weak preference R and alternative a, $L(a, R) = \{x \in A : aPx\}$ is the *strict lower contour set* of a at R. Analogously, $\bar{L}(a, R) = \{x \in A : aRx\}$ is the *weak lower contour set* of a at R.

A domain $\mathcal{D}$ is a set of weak preferences. An environment G is a graph $G = \langle \mathcal{D}, \mathcal{E} \rangle$ where $\mathcal{D}$ and $\mathcal{E}$ are the set of vertices and edges in G respectively.

Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment. An SCF $f : \mathcal{D} \to A$ is *locally strategy-proof* if for all $(R, R') \in E$, we have $f(R)Rf(R')$. Furthermore, the SCF $f : \mathcal{D} \to A$ is *strategy-proof* if for all $R, R' \in \mathcal{D}$, we have $f(R)Rf(R')$. The environment G satisfies *local-global equivalence* (LGE) if every locally strategy-proof SCF is also strategy-proof.

These definitions are the natural counterparts of those in Chapter 2. The notion of a path with no restoration requires reformulation in this setting. Recall that a path $\pi \equiv (R^1, \ldots, R^t)$ in G is a sequence of preference in $\mathcal{D}$ such that $(R^s, R^{s+1}) \in \mathcal{E}$ for all $s = 1, \ldots, t-1$. The set of paths between R and R' where $R, R' \in \mathcal{D}$ is denoted by $\Pi(R, R')$.

[4] The "gap" is small in the sense that both conditions boil down to the single condition (Property L) obtained in the case of preference domains without indifference (as discussed in Chapter 2).

Fix $a, b \in A$. The path $\pi \equiv (R^1, \ldots, R^t)$ has (a, b)-*restoration* if there exist $1 \leq u < q < s \leq t$ such that one of the following three cases occurs:

(i) $aP^u b$, $bR^q a$ and $aR^s b$;

(ii) $aR^u b$, $bP^q a$ and $aR^s b$;

(iii) $aR^u b$, $bR^q a$ and $aP^s b$.

For an (a, b)-restoration, (weak) preferences over a and b reverse more than once along the path. However, the preference over the pair at one of the preferences, where reversal takes place, must be strict.

We introduce two variants of Property L below.

Definition 3.2.1 *The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies Properties WL if, for all $R, R' \in \mathcal{D}$ and $a \in A$, there exists a path π in $\Pi(R, R')$ such that for all $b \in L(a, R)$ the path π has no (a, b)-restoration.*

Definition 3.2.2 *The environment $G = \langle \mathcal{D}, \mathcal{E} \rangle$ satisfies Property SL if, for all $R, R' \in \mathcal{D}$ and $a \in A$, there exists a path π in $\Pi(R, R')$ such that for all $b \in \bar{L}(a, R) \setminus \{a\}$ the path π has no (a, b)-restoration.*

Pick R and R', distinct preferences in $\mathcal{D}$ and $a \in A$. Property *WL* guarantees the existence of a path π from R to R' such that for *any* x ranked *strictly lower* than a in R the path π has no (a, x)-restoration. On the other hand, the path whose existence is guaranteed under Property *SL* also satisfies no-restoration with respect to alternatives that are indifferent to a under R. Clearly, Property *SL* is a stronger property than Property *WL*. If there are no alternatives indifferent to a at R, then the path specified under Property *WL* satisfies the requirements of Property *SL*. Thus the two properties reduce to the Property L in the absence of indifference.

Suppose b is indifferent to a under R and the environment satisfies Property *SL*. Let $\pi \in \Pi(R, R')$ be the path specified by Property *SL*. Then the relative ranking of a and b along π must be one of the following (i) a and b are indifferent everywhere along the path (ii) a and b are indifferent to each other till some preference R^1; then a is strictly preferred to b everywhere from R^1 till R' (iii) a and b are indifferent to each other till some preference R^1; then b is strictly preferred to a everywhere from R^1 till R'

The example below highlights the two lower contour set properties.

Example 3.2.3 Let $A = \{a, b, x, y\}$. The domain $\hat{\mathcal{D}}$ consists of three preferences specified in Table 3.2.1. Note that the notation $\{a, b\}$ in a preference of Table 3.2.1 denotes that a and b are indifferent. The environment is $\hat{G} = \langle \hat{\mathcal{D}}, \mathcal{E} \rangle$ where $\mathcal{E} = \{(R^1, R^2), (R^2, R^3)\}$.

R^1	R^2	R^3
x	x	y
y	y	x
$\{a,b\}$	a	$\{a,b\}$
	b	

Table 3.2.1: The Domain $\hat{\mathcal{D}}$

It is clear from inspection that there is no restoration for any pair of alternatives where one is ranked strictly higher than the other. Consider the pair (R^1, R^3) and the alternative a. Note that $\Pi(R^1, R^3)$ contains the unique path (R^1, R^2, R^3) where there is (a,b)-restoration. Hence $\hat{G}$ satisfies Property *WL*, but fails Property *SL*. $\square$

3.3 The Main Result

We state the main result of this chapter.

Theorem 3.3.1 *Let $G = \langle \mathcal{D}, \mathcal{E} \rangle$ be an environment. If G satisfies LGE, it satisfies Property WL. If G satisfies Property SL, it satisfies LGE.*

Proof: The proof of the necessity part of Theorem 3.3.1 is essentially the same as that of its counterpart in Theorem 2.3.2 of Chapter 2. We therefore omit it and only provide the proof for the sufficiency part.

Suppose G satisfies Property *SL* but violates LGE. Therefore there exists $f : \mathcal{D} \to A$ such that f is locally strategy-proof but not strategy-proof. It follows that there exists $R^0, R' \in \mathcal{D}$ such that $x' = f(R')P^0 f(R^0) = x^0$. Let $x^1 \in \max_{R^0}\{x \in A : f(R) = x \text{ for some } R \in \mathcal{D}\}$ and $f(R^1) = x^1$. Clearly $x^1 P^0 x^0$ and $R^0 \neq R^1$. By Property *SL* property, there exists a path $\pi \in \Pi(R^0, R^1)$ such that for all $z \in \bar{L}(x^1, R^0) \setminus \{x^1\}$ the path π has no (x^1, z)-restoration.

Starting from R^1 and proceeding backwards along the path π, let R^2 be the first vertex such that $f(R^2) \equiv x^2 \neq x^1$. Let $\hat{R}^2$ denote the second vertex in the path from R^2 to R^1 along π. By construction, $f(\hat{R}^2) = x^1$.

Claim 3.3.1 *$x^1 I x^2$ for all vertices R on π between R^0 and $\hat{R}^2$.*

Proof: Since $(R^2, \hat{R}^2) \in \mathcal{E}$, local strategy-proofness implies $x^2 R^2 x^1$ and $x^1 \hat{R}^2 x^2$.

We first show $x^1 I^0 x^2$. By the definition of x^1, we know $x^1 R^0 x^2$. If $x^1 P^0 x^2$, then path π has (x^1, x^2)-restoration which contradicts our hypothesis regarding π. Therefore, $x^1 I^0 x^2$.

We next show $x^1 \hat{I}^2 x^2$. Suppose not, i.e. $x^1 \hat{P}^2 x^2$. Thus, $x^1 R^0 x^2$, $x^2 R^2 x^1$ and $x^1 \hat{P}^2 x^2$ so that we have an (x^1, x^2)-restoration on π. Hence, $x^1 \hat{I}^2 x^2$.

We further show $x^1 I^2 x^2$. Suppose not, i.e. $x^2 P^2 x^1$. Once again, $x^1 R^0 x^2$, $x^2 P^2 x^1$ and $x^1 \hat{R}^2 x^2$ imply the existence of an (x^1, x^2)-restoration on π - contradiction. Hence $x^1 I^2 x^2$.

Last, pick an arbitrary vertex R distinct from R^0, R^2 and $\hat{R}^2$ on the path π. We show $x^1 I x^2$. Note that R is before R^2 and $\hat{R}^2$ on path π. Since $x^1 R^0 x^2$ and $x^1 \hat{R}^2 x^2$, no (x^1, x^2)-restoration on π implies $x^1 R x^2$. Suppose $x^1 P x^2$. Then, we have $x^1 P x^2$, $x^2 R^2 x^1$ and $x^1 \hat{R}^2 x^2$ which implies (x^1, x^2)-restoration on π - contradiction. Therefore, $x^1 I x^2$ as required. ■

Starting from R^2 and proceeding backwards along the path π, let R^3 be the first vertex such that $f(R^3) \equiv x^3 \neq x^2$. Also, let $\hat{R}^3$ denote the second vertex in the path from R^3 to R^2 along π. Repeating this process, we can identify vertices $R^s, \hat{R}^s$ on π, $s = 2, \ldots, k$, and alternatives x^s, $s = 2, \ldots, k-1$ such that (i) $f(R^s) = x^s$, $s = 2, \ldots, k-1$, and $f(R^k) = x^0$ (ii) R^s is the first vertex in the path from R^{s-1} to R^0 such that $f(R^s) \neq f(R^{s-1})$, $s = 2, \ldots, k$, and (iii) $\hat{R}^s$ is the second vertex on the path from R^s to R^{s-1}, $s = 2, \ldots, k$.

By construction, $f(\hat{R}^s) = x^{s-1}$, $s = 2, \ldots, k$. For all $s = 2, \ldots, k-1$, since $(R^s, \hat{R}^s) \in \mathcal{E}$, local strategy-proofness of f implies $x^s R^s x^{s-1}$ and $x^{s-1} \hat{R}^s x^s$.

Claim 3.3.2 *For each $s = 2, \ldots, k-1$, $x^1 I x^s$ for all vertices R on π from R^0 to $\hat{R}^s$.*

Proof: We shall prove the claim by induction. Note that Claim 3.3.1 establishes the claim for the case $s = 2$. We impose the following induction hypothesis:

Pick an arbitrary s such that $2 < s \leq k-1$. For all $2 \leq s' < s$, we have $x^1 I x^{s'}$ for all R on π from R^0 to $\hat{R}^{s'}$.

We will show $x^1 I x^s$ for all R on π from R^0 to $\hat{R}^s$.

We have already noted that $x^s R^s x^{s-1}$ and $x^{s-1} \hat{R}^s x^s$. The induction hypothesis implies $x^1 I^s x^{s-1}$ and $x^1 \hat{I}^s x^{s-1}$. Hence $x^s R^s x^1$ and $x^1 \hat{R}^s x^s$. From the definition of x^1, it follows that $x^1 R^0 x^s$. Thus, $x^1 R^0 x^s$, $x^s R^s x^1$ and $x^1 \hat{R}^s x^s$. Then, no (x^1, x^s)-restoration on π implies $x^1 I^0 x^s$, $x^1 I^s x^s$ and $x^1 \hat{I}^s x^s$.

Finally pick an arbitrary R on the path π from R^0 to $\hat{R}^s$ distinct from R^0, R^s and $\hat{R}^s$. We show $x^1 I x^s$. Note that R is before R^s and $\hat{R}^s$ on path π. Since $x^1 I^0 x^s$ and $x^1 \hat{I}^s x^s$, no (x^1, x^s)-restoration on π implies $x^1 R x^s$. Suppose $x^1 P x^s$. Then, we have $x^1 P x^s$, $x^s R^s x^1$ and $x^1 \hat{R}^s x^1$ which is an (x^1, x^k)-restoration on π - contradiction. Hence, $x^1 I x^s$, as required. ■

Since $(R^k, \hat{R}^k) \in \mathcal{E}$ and $f(R^k) = x^0$, local strategy-proofness implies $x^0 R^k x^{k-1}$ and $x^{k-1} \hat{R}^k x^0$. Claim 3.3.2 implies $x^1 I^k x^{k-1}$ and $x^1 \hat{I}^k x^{k-1}$ so that $x^0 R^k x^1$ and $x^1 \hat{R}^k x^0$. Then, by the assumption $x^1 P^0 x^0$, we have an (x^1, x^0)-restoration on π. Hence $x^1 P^0 x^0$ cannot hold and f is strategy-proof. This completes the proof of sufficiency. ■

Although Property *SL* is sufficient for LGE, it is not necessary. Consider the environment $\hat{G}$ in Example 3.2.3. We know that it violates Property *SL* - however it is an LGE environment. Observe that

R	P^a	$\bar{P}^a$
x	x	y
y	y	x
$\{c, d\}$	c	d
$\{a, b\}$	d	c
e	a	a
	b	b
	e	e

Table 3.4.1: a-straightenings of R

(R^1, R^3) is the only pair of preferences that are not neighbours. In order for an SCF to satisfy local strategy-proofness but fail strategy-proofness, there must be a manipulation from R^1 to R^3 or vice versa. The only alternatives that are candidates for the outcomes at R^1 and R^3 for a manipulation to occur, are a and b. This is so because (a, b) is the only pair for which a restoration occurs between R^1 and R^3. However a and b are indifferent to each other in both preferences. Therefore a manipulation cannot occur.

3.4 Further Applications

In this section, we provide some applications of Theorem 3.3.1. Our main result in this section shows how an LGE environment consisting of strict preferences can be embedded in an environment consisting of weak preferences so that the larger environment also satisfies the LGE property.

Let $\mathcal{D} \subseteq \mathcal{R}$ be a domain and $\mathcal{D}_P = \mathcal{D} \cap \mathcal{P}$. We assume $\mathcal{D}_I = \mathcal{D} \setminus \mathcal{D}_P \neq \emptyset$, i.e. $\mathcal{D}$ contains some preference with indifferences.

Let $R \in \mathcal{D}_I$ and $a \in A$. We say that the strict ordering $P^a \in \mathcal{D}_P$ is an "a-straightening" of R if, for all $b \in A$ we have (i) $bPa \Leftrightarrow bP^a a$, and (ii) $aRb \Rightarrow aP^a b$. Thus P^a is a strict ordering in the domain $\mathcal{D}$ with the property that alternatives ranked strictly above a and below a remain ranked above a and below a respectively. In addition, the indifference class to which a belongs in R is "broken" into a strict order where a is preferred to all other alternatives in the indifference class containing a. In Table 3.4.1 both P^a and $\bar{P}^a$ are a-straightenings of R.

Let $G^* = \langle \mathcal{D}, \mathcal{E}^* \rangle$ be the environment where $\mathcal{E}^*$ is defined as follows: for all $R \in \mathcal{D}_I$ and $a \in A$, there exists an a-straightening of R, P^a such that $(R, P^a) \in \mathcal{E}^*$.

Before stating it formally, we provide an informal description of the main result of this subsection. Consider an environment consisting of a domain with only strict preferences and an arbitrary set of edges. A new environment is created by adding a set of weak preferences to the domain and a set of new edges satisfying the following property: for every new weak preference R added and for every alternative a, there is an edge connecting R with a strict preference belonging to the original domain which is an a-straightening of R. The original environment is thus a sub-environment of the new environment. According to our result, the new environment satisfies LGE whenever the original environment satisfies LGE.

Proposition 3.4.1 *Let $G_P = \langle \mathcal{D}_P, \mathcal{E} \rangle$ be an LGE environment. The environment $G = \langle \mathcal{D}, \mathcal{E} \cup \mathcal{E}^* \rangle$ is also an LGE environment.*

Proof: In view of Theorem 3.3.1, it suffices to show that G satisfies Property SL, i.e. for all $R_1, R_2 \in \mathcal{D}$ and $a \in A$, there exists a path in $\mathcal{E} \cup \mathcal{E}^*$ from R_1 to R_2 having no (a, b)-restoration for all $b \in \bar{L}(a, R_1) \setminus \{a\}$. Let P_1^a and P_2^a be the a-straightenings of R_1 and R_2 respectively such that $(R_1, P_1^a) \in \mathcal{E}^*$ and $(R_2, P_2^a) \in \mathcal{E}^*$. If R_i, for $i = 1, 2$, is a strict preference, then let $R_i = P_i^a$. Since G_P is an LGE environment, it follows from Theorem 2.3.2 (in Chapter 2) that G_P satisfies Property L, i.e. there exists a path $\pi' \equiv (P_1^a, \ldots, P_2^a)$ in G_p such that for all $z \in L(a, P_1^a)$ the path π' has no (a, z)-restoration. By construction, $(R_1, P_1^a) \in \mathcal{E}^*$ and $(R_2, P_2^a) \in \mathcal{E}^*$. Therefore $\pi \equiv (R_1, \pi', R_2)$ is a path in G. We will show that π has no (a, b)-restoration for all $b \in \bar{L}(a, R^0) \setminus \{a\}$.

Let b be an arbitrary alternative in $\bar{L}(a, R^1) \setminus \{a\}$. By the definition of a-straightening, we must have $aP_1^a b$. Therefore π' has no (a, b)-restoration. There are two possibilities to consider here.

The first is that b does not overtake a on the path π'. Thus $aP_2^a b$. Again by the definition of a-straightening, $aR^2 b$. Therefore along the path π, we have $aR^1 b$, a strictly preferred to b along the path π' and $aR^2 b$. Clearly there is no (a, b)-restoration on π.

The second case is that b overtakes a on the path π'. Since there is no (a, b)-restoration on the path π', we have $bP_2^a a$. By the definition of a-straightening, we must have $bP^2 a$. So along the path π, we have $aR^1 b$, b overtaking a exactly once along the path π' and remaining strictly preferred to a at R^2. Once again there is no (a, b)-restoration along the path π.

This completes the proof. ■

Proposition 3.4.1 can be interpreted as specifying a notion of localness that guarantees LGE in a domain with weak preferences. The result can be applied quite generally with one caveat: for every weak ordering R in the domain and alternative a, there must exist a strict preference in the original domain that is an a-straightening of R. We provide a couple of examples where the result can be applied and one where it cannot.

Let $\prec$ be a strict ordering of the elements of A. Let $\underline{a}, \overline{a} \in A$ with $\underline{a} \preceq \overline{a}$. An *interval* denoted by $[\underline{a}, \overline{a}]$ is a subset of A such that $\underline{a}, \overline{a} \in [\underline{a}, \overline{a}]$ and $b \in [\underline{a}, \overline{a}]$ whenever $\underline{a} \prec b \prec \overline{a}$. A weak ordering R is *single-plateaued* if there exists an interval $[\underline{a}, \overline{a}]$ such that the following properties hold:

(i) $bIcPz$ for all $b, c \in [\underline{a}, \overline{a}]$ and for all $z \notin [\underline{a}, \overline{a}]$,

(ii) bPc whenever $c \prec b \prec \underline{a}$, and

(iii) bPc whenever $\overline{a} \prec b \prec c$.

We let $\mathcal{D}_{SPL}$ denote the set of all single-plateaued preferences over A (keeping the ordering $\prec$ fixed).

Single-plateaued preferences have been extensively studied in the literature (See Berga [10] and Barberà [4]). Alternatives in the interval $[\underline{a}, \overline{a}]$ are the peaks or the maximal elements in A according to R. On either side of the interval, preferences decline strictly as alternatives move further away from the relevant end-point of the interval. No assumptions are made regarding the ranking of alternatives on different sides of the interval $[\underline{a}, \overline{a}]$. In particular, we do not preclude the possibility that there exist $b, c \in A$ such that $b \prec \underline{a}$ and $\overline{a} \prec c$ and bIc.

A single-peaked preference is a special case of a single-plateaued preference where the interval of peaks consists of a single peak and the preference is strict. Let $\mathcal{D}_{SP}$ denote the set of such preferences. Clearly $\mathcal{D}_{SP} \subset \mathcal{D}_{SPL}$.

Let $R \in \mathcal{D}_{SPL} \setminus \mathcal{D}_{SP}$ and consider an arbitrary $a \in A$. Suppose a belongs to a non-empty indifference class, i.e. there exists another alternative b such that aIb. Suppose a belongs to the interval of peaks $[\underline{a}, \overline{a}]$. Assume for notational convenience that $[\underline{a}, \overline{a}]$ is the set $\{a_1, a_2, \ldots, a_{k-1}, a_k, a_{k+1} \ldots, a_T\}$ where $a_1 \prec a_2 \prec \cdots \prec a_{k-1} \prec a_k \prec a_{k+1} \prec \cdots \prec a_T$, $\underline{a} = a_1$, $a_k = a$ and $\overline{a} = a_T$. Let P^a be a strict ordering where alternatives in the set $[\underline{a}, \overline{a}]$ are ranked above all other alternatives in the following way: $a_k P^a a_{k-1} P^a \cdots P^a a_1 P^a a_{k+1} P^a \cdots P^a a_T$. Alternatives ranked below those in $[\underline{a}, \overline{a}]$ in R are ranked in the same way in P^a. If there are other indifference classes in R with more than one alternative, they must be of the form $\{c, d\}$ with $c \prec \underline{a} \preceq \overline{a} \prec d$. Then c and d are ranked consecutively in P^a with $cP^a d$. It is easy to verify that $P^a \in \mathcal{D}_{SP}$ - therefore it is an a-straightening of R.

There are two other cases to consider regarding the ranking of a in R. The first is that a does not belong to $[\underline{a}, \overline{a}]$ but to another indifference class containing one other alternative, say b. The procedure described in the previous paragraph with some minor modifications can be used to construct an a-straightening of R. The modifications are as follows: (i) an arbitrary alternative $a_k \in [\underline{a}, \overline{a}]$ is chosen to be the first-ranked alternative in P^a and (ii) $aP^a b$ where aIb. The second case is that a belongs to a singleton indifference class in R. The procedure used in the earlier case with only the first modification again yields P^a which is an a-straightening of R.

The arguments in the previous paragraphs establish that for each $R \in \mathcal{D}_{SPL} \setminus \mathcal{D}_{SP}$ and $a \in A$, there exists $P^a \in \mathcal{D}_{SP}$ which is an a-straightening of R. Let $\mathcal{E}^*$ be the set of edges in the environment $\langle \mathcal{D}_{SPL}, \mathcal{E}^* \rangle$ where $(R, P^a) \in \mathcal{E}^*$ for each $R \in \mathcal{D}_{SPL} \setminus \mathcal{D}_{SP}$ and $a \in A$.

Proposition 3.4.2 *The environment $\langle \mathcal{D}_{SPL}, \mathcal{E}^{adj} \cup \mathcal{E}^* \rangle$ satisfies LGE.*

Proof: We know from Carroll [12] and Sato [46] that the environment $G_{SP} = \langle \mathcal{D}_{SP}, \mathcal{E}^{adj} \rangle$ satisfies LGE. The result follows by an immediate application of Proposition 3.4.1. ■

There is a variant of the domain of single-plateaued preferences where Proposition 3.4.1 cannot be applied. Suppose that part (i) of the definition of single plateaued preferences is retained but (ii) and (iii) are modified as follows: (ii') bRc whenever $c \prec b \prec \underline{a}$ and (iii') bRc whenever $\overline{a} \prec b \prec c$. Indifference is now permitted on the same side of the interval of peaks. Let $\mathcal{D}'_{SPL}$ denote the set of all such preferences.

Consider an example where $A = \{a_1, a_2, a_3, a_4\}$ where $a_1 \prec a_2 \prec a_3 \prec a_4$. Let R be the preference $a_4 I a_3 P a_2 I a_1$ so that $R \in \mathcal{D}'_{SPL}$. Let P^{a_1} be a a_1-straightening of R. We must have $a_4 P^{a_1} a_1$, $a_3 P^{a_1} a_1$ and $a_1 P^{a_1} a_2$. But then P^{a_1} is not single-peaked with respect to $\prec$. Hence there does not exist a a_1-straightening of R and Proposition 3.4.1 cannot be applied.

Our second application of Proposition 3.4.1 concerns $\mathcal{R}$, the domain of all weak preferences over A. Recall that $\mathcal{P}$ is the domain of all strict preferences over A. Let $R \in \mathcal{R} \setminus \mathcal{P}$ and let a be an arbitrary alternative. Let P^a be a strict ordering such that (i) $bPa \Leftrightarrow bP^a a$, and (ii) $aRb \Rightarrow aP^a b$. By construction P^a is an a-straightening of R. Let $\mathcal{E}^*$ be the set of edges in the environment $\langle \mathcal{R}, \mathcal{E}^* \rangle$ where $(R, P^a) \in \mathcal{E}^*$ for each $R \in \mathcal{R} \setminus \mathcal{P}$ and $a \in A$.

Proposition 3.4.3 *The environment $\langle \mathcal{R}, \mathcal{E}^{adj} \cup \mathcal{E}^* \rangle$ satisfies LGE.*

Proof: We know from Carroll [12] and Sato [46] that the environment $G_P = \langle \mathcal{P}, \mathcal{E}^{adj} \rangle$ satisfies LGE. The result follows by an immediate application of Proposition 3.4.1. ■

Sato [45] provides another notion of localness for $\mathcal{R}$ that ensures LGE. Let $|A| = m$. For any $k \in \{1, \ldots, m(m-1)\}$, let $\mathcal{E}^k$ be the set of edges in the environment $\langle \mathcal{R}, \mathcal{E}^k \rangle$ where $(R, R') \in \mathcal{E}^k$ if the Kemeny distance between R and R' does not exceed k. According to Theorem 3.1 in Sato [45], $\langle \mathcal{R}, \mathcal{E}^k \rangle$ is LGE if and only if $k \geq m - 1$.

Proposition 3.4.3 is independent of Sato's result. Comparing the set of edges in the two environments, the set of edges in ours is sparser though their nature is different. In our environment, vertices in $\mathcal{P}$ are connected by an edge if only if they are adjacent - in terms of the notion of Kemeny distance used in Sato [45], these vertices are connected by an edge if only if their Kemeny distance is 2. On the other hand, each vertex in $\mathcal{R} \setminus \mathcal{P}$ in our environment is only required to be connected to a vertex in $\mathcal{P}$ by an edge. This involves connecting a preference where all alternatives are indifferent to one where preferences are strict. The Kemeny distance between these preferences is $\frac{m(m-1)}{2}$ which is strictly greater than $m - 1$ if $m > 2$. In the Sato environment where $k = m - 1$, these preferences would not be connected by an edge. However, many vertices in $\mathcal{P}$ would be connected by an edge in his environment which are not connected by an edge in ours.

4

Local Global Equivalence for Unanimous Social Choice Functions

4.1 Introduction

The theory of mechanism design investigates the objectives that can be achieved by a group of agents (or a planner) when these objectives depend on information held privately by the agents. Agents must be induced to reveal their private information truthfully: in more formal terms, the Random/Deterministic Social Choice Function (RSCF/DSCF) representing the objectives of the planner must be *incentive compatible* or *strategy-proof*. A RSCF/DSCF is strategy-proof if no agent can gain by misrepresenting her preferences irrespective of the preference announcements of the other agents. In particular, in the random setting, we use the stochastic dominance notion for strategy-proofness. In many contexts, it is plausible to assume that an agent can only misrepresent to a "local" preference. The class of locally strategy-proof RSCFs should, in principle be larger than the class of strategy-proof RSCFs. However, Carroll [12] and Sato [46] demonstrate that for many important preference domains and a natural notion of localness (adjacency), the classes of locally strategy-proof and strategy-proof RSCFs/DSCFs coincide. We shall refer to this property as local-global equivalence. This property has important theoretical and practical implications which are discussed in both papers.

Kumar et al. [32] formulate the local-global equivalence problem more generally, in the context of an "environment". An environment is a graph where the nodes represent admissible preferences and the edges, the notion of localness. They characterize environments that satisfy local-global equivalence. The necessary and sufficient condition for local-global equivalence requires the existence of certain kinds of paths in the graph. An important aspect of the paper is that it considers a single-agent model. Our goal in this paper is to show that in a multi-agent problem, a much weaker condition is sufficient, when the set of RSCFs under consideration satisfy the familiar and mild efficiency property of *unanimity*. We note that imposing unanimity in a single-agent model renders it trivial — it is an interesting requirement only in a multi-agent problem.[1]

We consider a model with a finite number of alternatives. A preference *domain* is a collection of strict orderings of the alternatives. A pair of preferences is local if there is a single pair of alternatives whose ranking is reversed between the two preferences.[2] We consider RSCFs that satisfy unanimity, i.e. those that respect consensus amongst agents. A domain satisfies equivalence if every unanimous locally strategy-proof RSCF is also strategy-proof.

In this setting, we show that a condition first identified in Sato [46] (which we refer to as Property P) has very important implications. This condition was shown to be necessary (but not sufficient) in the single-agent problem by Sato [46]. Property P is a weak condition, which specifies for every pair of alternatives, the existence of a path where preferences over this pair are not reversed more than once.[3] In contrast, the necessary and sufficient condition in Kumar et al. [32] (which they call Property L) requires the existence of a path that satisfies no-restoration with respect to *all* alternatives in an appropriate lower contour set.[4]

We prove two main results using Property P. We show that it is sufficient for equivalence. In contrast, Kumar et al. [32] show that the stronger Property L is not sufficient for local-global equivalence for RSCFs in the single-agent model.[5] Furthermore, a stronger result in the deterministic setting is true: every unanimous, locally strategy-proof DSCF on a domain satisfying Property P is also group strategy-proof. Our result is independent of the results in the existing literature on domains where strategy-proofness and group strategy-proofness are equivalent (see Section 4.4.1). Our overall

[1] Formally, the models in Kumar et al. [32], Carroll [12] and Sato [46] are also multi-agent models. Since they do not impose unanimity, the multi-agent model is indistinguishable from its single-agent counterpart. For this reason, we choose to refer to the models in these papers as single-agent models.

[2] This is the "adjacency" notion of localness used in Carroll [12] and Sato [46].

[3] Further discussion of domains satisfying Property P can be found in Section 4.3.

[4] Sato [46] and Carroll [12] also provide stronger sufficient conditions for equivalence in the single-agent model.

[5] Cho [18] also considers the local-global equivalence issue for RSCFs in the single-agent model. The paper provides sufficient conditions for a variety of lottery comparisons.

conclusion is that imposing the requirement of unanimity leads to a considerable weakening of the conditions required for equivalence in both random and deterministic settings.

As mentioned earlier, Property P is a weak condition. It is satisfied by several familiar domains such as the universal domain and the single-peaked domain. However it is not a necessary condition for a domain to satisfy equivalence of local strategy-proofness and strategy-proofness for unanimous RSCFs/DSCFs. In Section 4.4.1, we construct an example demonstrating this fact. We also show that the weaker condition of connectedness remains necessary for equivalence.

In recent work, Hong and Kim [28] independently derive a condition slightly weaker than our Property P and show that it is sufficient for equivalence.[6] They focus on ordinal Bayesian incentive compatible DSCFs and dictatorial domains. In contrast, we study RSCFs and extend our result for DSCFs to cover group strategy-proofness. We discuss their condition further in Section 4.4.1 where we also show that it, like Property P, is not necessary for equivalence.

The paper is organized as follows. Section 4.2 describes the model. Section 4.3 introduces and discusses Property P, which is the key condition for our results. The main results are in Section 4.4 while Section 4.5 discusses issues regarding necessity.

4.2 The Model

Let $A = \{a, b, \dots\}$ denote a finite set of alternatives with $|A| \geq 3$. Let $N = \{1, 2, \dots, n\}$ denote a finite set of voters with $n \geq 2$. A preference P_i of voter i is an antisymmetric, complete and transitive binary relation over A, i.e. a *linear order*. Given $a, b \in A$, aP_ib is interpreted as "a is strictly preferred to b" according to P_i. Let $r_k(P_i)$, $k = 1, \dots, |A|$ denote the k^{th} ranked alternative in preference P_i, i.e. $[r_k(P_i) = a] \Leftrightarrow \left[\left|\{x \in A : xP_ia\}\right| = k - 1\right]$. Let $\mathcal{P}$ denote the set of all preferences - the set $\mathcal{P}$ will be referred to as the *universal domain*. We shall refer to an arbitrary set $\mathcal{D} \subseteq \mathcal{P}$ as a *domain*.[7] A preference profile is an n-tuple $P = (P_1, P_2, \dots, P_n)$.

Fix a pair of preferences $P_i, P_i' \in \mathcal{D}$. Two alternatives a and b are *reversed* between P_i and P_i' if aP_ib and $bP_i'a$, or bP_ia and $aP_i'b$ hold. Accordingly, two preferences P_i and P_i' are *adjacent/local*, denoted by $P_i \sim P_i'$, if there exists exactly one pair of alternatives that are reversed between P_i and P_i'; formally, there exists $1 \leq k < |A|$ such that $r_k(P_i) = r_{k+1}(P_i')$, $r_k(P_i') = r_{k+1}(P_i)$ and $r_l(P_i) = r_l(P_i')$ for all $l \notin \{k, k+1\}$. A *path* $\pi \equiv (P_i^1, \dots, P_i^t)$ is a sequence of non-repeated preferences in $\mathcal{D}$ satisfying the property that consecutive preferences are adjacent, i.e. $P_i^k \sim P_i^{k+1}$ for all $k = 1, \dots t-1$. The set of all paths from P_i to P_i' where $P_i, P_i' \in \mathcal{D}$ is denoted by $\Pi(P_i, P_i')$. The domain $\mathcal{D}$ is *connected* if there exists a path between every

[6]The two conditions are equivalent if the domain satisfies the following richness property: for every alternative a, there exists a preference in the domain whose first-ranked alternative is a.

[7]We assume that all voters have the same preference domain $\mathcal{D}$.

pair $P_i, P'_i \in \mathcal{D}$.

Our model is identical to the models in Sato [46] and Carroll [12]. It is a special case of the model in Kumar et al. [32] where the notion of localness is completely general. On the other hand, we consider a many-agent setting while Kumar et al. [32] only consider the single-agent problem.

Let $\Delta(A)$ denote the set of probability distributions over A. An element $\lambda \in \Delta(A)$ will be referred to as a *lottery*. We let λ_a denote the probability with which $a \in A$ is selected by λ. Thus $0 \leq \lambda_a \leq 1$ and $\sum_{a \in A} \lambda_a = 1$. Given a preference P_i, the lottery λ *stochastically dominates* lottery λ' according to P_i (denoted by $\lambda P_i^{sd} \lambda'$) if $\sum_{k=1}^{t} \lambda_{r_k(P_i)} \geq \sum_{k=1}^{t} \lambda'_{r_k(P_i)}$ for all $1 \leq t \leq |A|$.

Observation 4.2.1 Fix P_i and $\lambda, \lambda' \in \Delta(A)$ such that $\lambda P_i^{sd} \lambda'$. Pick $a, b \in A$ such that aP_ib. Let $\hat{\lambda} \in \Delta(A)$ be such that (i) $\hat{\lambda}_b > \lambda'_b$, (ii) $\hat{\lambda}_a < \lambda'_a$ and (iii) $\hat{\lambda}_c = \lambda'_c$ for all $c \notin \{a, b\}$. Then $\lambda P_i^{sd} \hat{\lambda}$. The lottery $\hat{\lambda}$ is obtained by transferring probability weight from an alternative a to a less preferred one b, in λ' while keeping all other probabilities unchanged. It is easy to verify that $\lambda' P_i^{sd} \hat{\lambda}$ from which $\lambda P_i^{sd} \hat{\lambda}$ follows immediately.

Definition 4.2.1 *A Random Social Choice Function (RSCF) is a map $\phi : \mathcal{D}^n \to \Delta(A)$.*

Given $a \in A$, let $\phi_a(P)$ denote the probability with which a is selected at the profile P. A Deterministic Social Choice Function (DSCF) $f : \mathcal{D}^n \to \Delta(A)$ is a particular RSCF such that for each $P \in \mathbb{D}^n$, $f_a(P) = 1$ for some $a \in A$. Henceforth, for ease of presentation, we write a DSCF as $f : \mathbb{D}^n \to A$, where an alternative is selected at each preference profile.

We require all RSCFs under consideration to satisfy the property of *unanimity*. This is a weak form of efficiency where the RSCF selects a commonly first-ranked alternative with probability 1 whenever it exists.

Definition 4.2.2 *A RSCF $\phi : \mathcal{D}^n \to \Delta(A)$ is unanimous if for all $P \in \mathcal{D}^n$,*

$$[r_1(P_i) = a \text{ for all } i \in N] \Rightarrow [\phi_a(P) = 1].$$

Correspondingly, a DSCF $f : \mathcal{D}^n \to A$ is *unanimous* if for all $P \in \mathcal{D}^n$, we have $[r_1(P_i) = a \text{ for all } i \in N] \Rightarrow [f(P) = a]$. In order to avoid trivial considerations, we assume throughout that $\mathcal{D}$ contains at least two preferences with distinct peaks.

A RSCF is locally strategy-proof if a voter cannot gain by a misrepresentation to an adjacent preference (in other words, according to the sincere preference, the social lottery induced by any misrepresentation to an adjacent preference is always stochastically dominated by the lottery delivered by truthtelling). On the other hand, a RSCF is strategy-proof if a voter cannot gain by an arbitrary misrepresentation.

Definition 4.2.3 *A RSCF $\phi : \mathcal{D}^n \to A$ is locally manipulable by an agent $i \in N$ at profile $P = (P_i, P_{-i})$ if there exists $P'_i \in \mathcal{D}$ with $P_i \sim P'_i$ such that $\phi(P_i, P_{-i}) P_i^{sd} \phi(P'_i, P_{-i})$ does not hold,*
i.e. $\sum_{k=1}^{t} \phi_{r_k(P_i)}(P_i, P_{-i}) < \sum_{k=1}^{t} \phi_{r_k(P_i)}(P'_i, P_{-i})$ for some $1 \leq t < |A|$. The RSCF ϕ is locally strategy-proof if it is not locally manipulable by any agent at any profile.

Definition 4.2.4 *A RSCF $\phi : \mathcal{D}^n \to A$ is manipulable by an agent $i \in N$ at profile $P = (P_i, P_{-i})$ if there exists $P'_i \in \mathcal{D}$ such that $\phi(P_i, P_{-i}) P_i^{sd} \phi(P'_i, P_{-i})$ does not hold,*
i.e. $\sum_{k=1}^{t} \phi_{r_k(P_i)}(P_i, P_{-i}) < \sum_{k=1}^{t} \phi_{r_k(P_i)}(P'_i, P_{-i})$ for some $1 \leq t < |A|$. The RSCF ϕ is strategy-proof if it is not manipulable by any agent at any profile.

A strategy-proof RSCF is clearly locally strategy-proof. We investigate the structure of domains where the converse is true for *all* unanimous RSCFs.

Definition 4.2.5 *The domain $\mathcal{D}$ satisfies local-global equivalence for unanimous RSCFs (uLGE) if every unanimous and locally strategy-proof RSCF $\phi : \mathcal{D}^n \to \Delta(A)$, $n \geq 2$, is strategy-proof.*

We can correspondingly define local-global equivalence for DSCFs. A DSCF $f : \mathcal{D}^n \to A$ is *locally strategy-proof* (respectively, *strategy-proof*) if for all $i \in N$, $P_i, P'_i \in \mathcal{D}$ with $P_i \sim P'_i$ (respectively, $P_i, P'_i \in \mathcal{D}$) and $P_{-i} \in \mathcal{D}^{n-1}$, either $f(P_i, P_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P_{-i}) P_i f(P'_i, P_{-i})$ holds. The domain $\mathcal{D}$ satisfies local-global equivalence for unanimous DSCFs if every unanimous and locally strategy-proof DSCF $f : \mathcal{D}^n \to A$, $n \geq 2$, is strategy-proof.

In the next section, we provide a sufficient condition for uLGE.

4.3 A Sufficient Condition

In this section, we introduce Property P that is central to our results. Let $\mathcal{D}$ be a domain and $a, b \in A$ be a pair of alternatives. A path $\pi = (P_i^1, \ldots, P_i^t)$ satisfies *no $\{a, b\}$-restoration* if the relative ranking of a and b is reversed *at most* once along π, i.e. there does not exist integers q, r and s with $1 \leq q < r < s \leq t$ such that either (i) $a P_i^q b$, $b P_i^r a$ and $a P_i^s b$, or (ii) $b P_i^q a$, $a P_i^r b$ and $b P_i^s a$.[8]

Sato [46] introduces the *pairwise no-restoration property*. This property requires that for every pair of distinct preferences and a pair of alternatives, there exists a path between the preferences that satisfies no-restoration with respect to the pair of alternatives.

Definition 4.3.1 *The domain $\mathcal{D}$ satisfies the pairwise no-restoration property (Property P) if for all distinct $P_i, P'_i \in \mathcal{D}$ and distinct $a, b \in A$, there exists a path $\pi = (P_i^1, \ldots, P_i^t) \in \Pi(P_i, P'_i)$ with no $\{a, b\}$-restoration.*

[8]It is worth emphasizing that in our definition of "$\{a, b\}$-restoration", we are *not* referring to an ordered pair $\{a, b\}$. Thus $\{a, b\}$-restoration and $\{b, a\}$-restoration are the same in our definition.

Property P is satisfied by the universal domain and the domain of single-peaked preferences. Conversely, Chatterji et al. [17] show that any domain satisfying Property P and some additional regularity conditions must either be a sub-domain of the domain of single-peaked preferences or a *hybrid* domain which is a "perturbation" of the single-peaked domain. Alternatives are again ordered as in the single-peaked domain. Alternatives are partitioned into three segments, left, middle and right. A hybrid domain consists of all preferences satisfying the following property: preferences in the left and right segments are single-peaked while being unrestricted in the middle segment. Hybrid domains cover the universal domain and the single-peaked domain as special cases, the former in the case where the middle segment is the entire set of alternatives and the latter where the middle segment is the null set.

Sato [46] shows that Property P is necessary but not sufficient for the equivalence of local strategy-proofness and strategy-proofness for DSCFs (henceforth called LGE) in a single-agent model (or equivalently without imposing unanimity). Kumar et al. [32] formulate the *lower contour set no-restoration property* (Property L) that is necessary and sufficient condition for LGE in a more general model. Property L is satisfied if for all $P_i, P_i' \in \mathcal{D}$ and $a \in A$, there exists a path $\pi = (P_i^1, \ldots, P_i^t) \in \Pi(P_i, P_i')$ such that for all $b \in L(a, P_i) = \{z \in A : aP_iz\}$ the path π has no $\{a, b\}$-restoration.

Property P is a weaker than Property L. This is illustrated in the example below which is adapted from Example 3.2 in Sato [46].

Example 4.3.2 Let $A = \{x, y, z, u, v, w\}$. The domain $\mathcal{D}$ is specified in Table 4.3.1 below. Figure 4.3.1 (below) shows all paths induced by the adjacent preferences in $\mathcal{D}$.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}
x	x	y	y	y	x	x	z	z	z
z	y	x	x	x	y	z	x	x	x
y	z	z	z	z	z	y	y	y	y
v	v	v	v	u	u	u	u	v	v
w	w	w	u	v	v	v	v	u	w
u	u	u	w	w	w	w	w	w	u

Table 4.3.1: The Domain $\mathcal{D}$

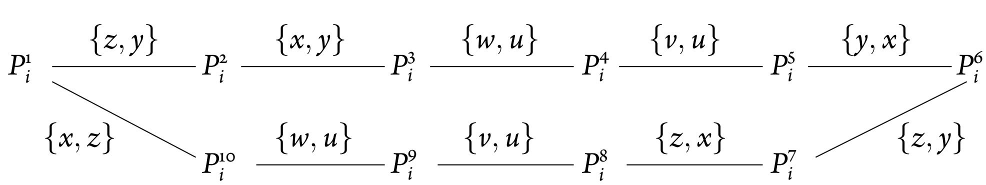

Figure 4.3.1: Paths induced by the adjacent preferences in $\mathcal{D}$

Figure 4.3.1 highlights an important property of $\mathcal{D}$ — there are exactly two paths between any pair of preferences. For example, between P_i^1 and P_i^7, there is a path $(P_i^1, P_i^2, P_i^3, P_i^4, P_i^5, P_i^6, P_i^7)$ and another path $(P_i^1, P_i^{10}, P_i^9, P_i^8, P_i^7)$. We shall refer to the former as the "clockwise" path and the latter as the "counter clockwise" path between P_i^1 and P_i^7. We shall in fact, refer to the clockwise and counter clockwise paths between any pair of preferences in $\mathcal{D}$. It can be verified that for any pair of distinct preferences and alternatives, either the clockwise path or the counter clockwise path is a path without restoration for the alternatives. Therefore, $\mathcal{D}$ satisfies Property P. However, it fails Property L, e.g. $z, y \in L(x, P_i^1)$, and the clockwise path from P_i^1 to P_i^7 has an $\{x, y\}$-restoration while the counter clockwise path from P_i^1 to P_i^7 has an $\{x, z\}$-restoration. We know there that LGE fails for $\mathcal{D}$. For instance, let $N = \{1, 2\}$ and consider the following DSCF:

$$f(P_1, P_2) = \begin{cases} z & P_1 = P_i^1, \\ y & P_1 = P_i^2, \text{ and} \\ r_1(P_1) & \text{otherwise.} \end{cases}$$

It is easy to verify that f is locally strategy-proof but fails strategy-proofness, e.g. $f(P_1^6, P_2) = x$, $f(P_1^1, P_2) = z$ and $xP_1^1 z$.[9] It also violates unanimity, e.g. $f(P_1^1, P_2^2) = z \neq x$. Our result implies that *every* locally strategy-proof RSCF that fails to be strategy-proof on this domain must violate unanimity. Furthermore, every DSCF satisfying unanimity and local strategy-proofness is group strategy-proof. □

4.4 Main Results

Kumar et al. [32] show that Property L does not guarantee that locally strategy-proof RSCFs are also strategy-proof. In this section, we show that this equivalence holds for unanimous RSCFs defined over domains satisfying the weaker Property P.

Theorem 4.4.1 *If a domain satisfies Property P, it satisfies uLGE.*

Proof: Pick a domain $\mathcal{D}$ that satisfies Property P. Consider an arbitrary locally strategy-proof RSCF $\phi : \mathcal{D}^n \to \Delta(A)$ that satisfies unanimity. We will show that ϕ is strategy-proof. We begin with an observation.

Observation 4.4.1 Consider $P_i, \bar{P}_i \in \mathcal{D}$ such that $P_i \sim \bar{P}_i$; in particular xP_iy and $y\bar{P}_ix$. If $\phi(P_i, P_{-i}) \neq \phi(\bar{P}_i, P_{-i})$ for some $P_{-i} \in \mathcal{D}^{n-1}$, then it must be the case that (i) $\phi_y(\bar{P}_i, P_{-i}) > \phi_y(P_i, P_{-i})$,

[9] Here, (P_1^6, P_2) is a preference profile where agent 1's preference is P_i^6 and agent 2's preference is P_2 which is arbitrary. Similarly, (P_1^1, P_2) is a profile where agent 1's preference is P_i^1 and agent 2's preference is P_2.

(ii) $\phi_x(\bar{P}_i, P_{-i}) < \phi_x(P_i, P_{-i})$ and (iii) $\phi_z(\bar{P}_i, P_{-i}) = \phi_z(P_i, P_{-i})$ for all $z \notin \{x, y\}$. These properties are well-known in the literature. Gibbard [26] refers to Parts (i) and (ii) as the property of being *non-perverse* and Part (iii) as the property of being *localized*.

Lemma 4.4.1 *Let $P_i, \bar{P}_i \in \mathcal{D}$ be such that $P_i \sim \bar{P}_i$ and $r_1(P_i) = r_1(\bar{P}_i)$. Then $\phi(P_i, P_{-i}) = \phi(\bar{P}_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}^{n-1}$.*

Proof: Assume w.l.o.g. that i is agent 1. Let $P_1, \bar{P}_1 \in \mathcal{D}$ be such that $P_1 \sim \bar{P}_1$ and $r_1(P_1) = r_1(\bar{P}_1) = a$. Let x, y be the alternatives that are reversed between P_1 and $\bar{P}_1$ with xP_1y and $y\bar{P}_1x$.

Let $P^k \equiv (P_1, P_2, \dots, P_k, P_1, \dots, P_1)$, i.e. P^k is the profile where agents 1 and $k+1, \dots, n$ have the preference P_1 while agents $2, \dots, k$ have preferences specified in the profile P_{-1}. Here $k \in \{1, \dots, n\}$ where $P^1 = (P_1, P_1, \dots, P_1)$ and $P^n = (P_1, P_2, \dots, P_n)$.

Let $\bar{P}^k \equiv (\bar{P}_1, P_2, \dots, P_k, P_1, \dots, P_1)$, i.e. $\bar{P}^k$ is the profile where agent 1 has the preference $\bar{P}_1$, agents $k+1, \dots, n$ have the preference P_1 and agents $2, \dots, k$ have preferences specified in the profile P_{-1}. Again $k \in \{1, \dots, n\}$ where $\bar{P}^1 = (\bar{P}_1, P_1, \dots, P_1)$ and $\bar{P}^n = (\bar{P}_1, P_2, \dots, P_n)$.

We will prove $\phi(P^n) = \phi(\bar{P}^n)$ by induction on k. Observe that $\phi_a(P^1) = \phi_a(\bar{P}^1) = 1$ since ϕ satisfies unanimity. Assume that $\phi(P^{k-1}) = \phi(\bar{P}^{k-1})$ for $k - 1 < n$. We will show that $\phi(P^k) = \phi(\bar{P}^k)$.

We assume w.l.o.g. xP_ky. Since $\mathcal{D}$ satisfies Property P, there exists a path $(P_k^1, \dots, P_k^T) \in \Pi(P_1, P_k)$ such that xP_k^ry for all $r \in \{1, \dots, T\}$.

Let $P^{k,r} \equiv (P_1, P_2, \dots, P_{k-1}, P_k^r, P_1, \dots, P_1)$ and $\bar{P}^{k,r} \equiv (\bar{P}_1, P_2, \dots, P_{k-1}, P_k^r, P_1, \dots, P_1)$. The induction hypothesis implies that $\phi(P^{k,1}) = \phi(\bar{P}^{k,1})$. Suppose $\phi(P^{k,T}) \neq \phi(\bar{P}^{k,T})$. Let t be the smallest integer in the set $\{1, \dots, T\}$ such that $\phi(P^{k,t}) \equiv \lambda \neq \bar{\lambda} \equiv \phi(\bar{P}^{k,t})$. Clearly, $t > 1$. Observe that the profiles $P^{k,t}$ and $\bar{P}^{k,t}$ differ only in the preferences of agent 1 with P_1 in the former profile and $\bar{P}_1$ in the latter. Thus, local strategy-proofness implies $\lambda P_1^{sd} \lambda'$ and then Observation 4.4.1 implies $\bar{\lambda}_y - \lambda_y > 0$, $\bar{\lambda}_x - \lambda_x < 0$ and $\bar{\lambda}_z = \lambda_z$ for all $z \notin \{x, y\}$. By the induction hypothesis, let $\phi(P^{k,t-1}) = \phi(\bar{P}^{k,t-1}) \equiv \lambda'$. Observe that the profiles $P^{k,t-1}$ and $P^{k,t}$ (respectively, profiles $\bar{P}^{k,t-1}$ and $\bar{P}^{k,t}$) differ only in the preferences of agent k being P_k^{t-1} in the former profile and P_k^t in the latter. Since $P_k^{t-1} \sim P_k^t$, local strategy-proofness implies that both λ and $\bar{\lambda}$ stochastically dominate λ' according to P_k^t, and λ' stochastically dominates both λ and $\bar{\lambda}$ according to P_k^{t-1}, and moreover there must be exactly one pair of alternatives which are reversed between P_k^{t-1} and P_k^t. This pair cannot be $\{x, y\}$ because xP_k^ry for all P_k^r belonging to the path π. Suppose this pair is $\{a, x\}$ with $a \neq y$: in this case, by Part (iii) of Observation 4.4.1, $\lambda'_y = \lambda_y$ and $\lambda'_y = \bar{\lambda}_y$ contradicting our hypothesis that $\bar{\lambda}_y - \lambda_y > 0$. If the pair is $\{a, y\}$ with $a \neq x$, we contradict our assumption $\bar{\lambda}_x - \lambda_x < 0$. Similarly if the pair is $\{a, b\}$ with $a \neq x$ and $y \neq b$, we contradict both $\bar{\lambda}_y - \lambda_y > 0$ and $\bar{\lambda}_x - \lambda_x < 0$. This completes the proof. ■

Lemma 4.4.2 *Let $P \equiv (P_i, P_{-i}) \in \mathcal{D}^n$ be a profile and $a \in A$ be an alternative. Let $\bar{P}_i \in \mathcal{D}$ and suppose there exists a path $\pi = (P_i^1, \ldots, P_i^T) \in \Pi(P_i, \bar{P}_i)$ such that $a \neq r_1(P_i^k)$ for all $k \in \{1, \ldots, T\}$. Then $\phi_a(P_i, P_{-i}) = \phi_a(\bar{P}_i, P_{-i})$.*

Proof: Suppose the Lemma is false. Let $t \geq 2$ be the smallest integer in the set $\{1, \ldots, T\}$ such that $\phi_a(P_i^{t-1}, P_{-i}) \neq \phi_a(P_i^t, P_{-i})$. Consider the preferences P_i^{t-1} and P_i^t. If $r_1(P_i^{t-1}) = r_1(P_i^t)$, we have an immediate contradiction to Lemma 4.4.1. The remaining possibility is $r_1(P_i^{t-1}) \neq r_1(P_i^t)$. Here, there must be a reversal of the first and second ranked alternatives in P_i^{t-1} to obtain P_i^t. By assumption, a cannot be first or second ranked in either P_i^{t-1} or P_i^t; otherwise a would be ranked first in either P_i^{t-1} or P_i^t. Then, Part (iii) of Observation 4.4.1 implies $\phi_a(P_i^{t-1}, P_{-i}) = \phi_a(P_i^t, P_{-i})$ contradicting our initial assumption. ■

We can now complete the proof of the result. Let $P = (P_i, P_{-i})$ be a profile and $\bar{P}_i \in \mathcal{D}$. We will show $\phi(P_i, P_{-i}) P_i^{sd} \phi(\bar{P}_i, P_{-i})$. Pick an arbitrary path $\pi = (P_i^1, \ldots, P_i^t, \ldots, P_i^T) \in \Pi(P_i, \bar{P}_i)$. We will prove the result by induction on t.

The conclusion for the initial step ($t = 2$) follows from local strategy-proofness. Assume that $\phi(P_i, P_{-i}) P_i^{sd} \phi(P_i^{t-1}, P_{-i})$ for some $t > 2$. We will show $\phi(P_i, P_{-i}) P_i^{sd} \phi(P_i^t, P_{-i})$. If $\phi(P_i^{t-1}, P_{-i}) = \phi(P_i^t, P_{-i})$, then the result follows immediately. Assume therefore $\phi(P_i^{t-1}, P_{-i}) \neq \phi(P_i^t, P_{-i})$. Immediately, since $P_i^{t-1} \sim P_i^t$, by Lemma 4.4.1, it must be the case that $r_1(P_i^{t-1}) \equiv a \neq b \equiv r_1(P_i^t)$. Thus, we know that the only reversal between P_i^{t-1} and P_i^t is of a and b, and hence by Observation 4.4.1, $\phi_b(P_i^t, P_{-i}) > \phi_b(P_i^{t-1}, P_{-i})$, $\phi_a(P_i^t, P_{-i}) < \phi_a(P_i^{t-1}, P_{-i})$ and $\phi_c(P_i^t, P_{-i}) = \phi_c(P_i^{t-1}, P_{-i})$ for all $c \notin \{a, b\}$. Consequently, if aP_ib, the conclusion follows from Observation 4.2.1. For the remainder of the argument, we assume bP_ia.

Let b be the q^{th}-ranked alternative in P_i, i.e. $b = r_q(P_i)$ where $1 \leq q < |A|$. Pick an arbitrary integer K between 1 and $|A|$. We will show $\sum_{s=1}^K \phi_{r_s(P_i)}(P_i, P_{-i}) \geq \sum_{s=1}^K \phi_{r_s(P_i)}(P_i^t, P_{-i})$ thereby establishing $\phi(P_i, P_{-i}) P_i^{sd} \phi(P_i^t, P_{-i})$. We consider two cases.

Suppose $1 \leq K < q$. Then the alternatives ranked above the K^{th}-ranked alternative in P_i do not involve either a or b. By virtue of Part (iii) of Observation 4.4.1, the total probability on these alternatives is unchanged between $\phi(P_i^{t-1}, P_{-i})$ and $\phi(P_i^t, P_{-i})$. In conjunction with the induction hypothesis, we have $\sum_{s=1}^K \phi_{r_s(P_i)}(P_i, P_{-i}) \geq \sum_{s=1}^K \phi_{r_s(P_i)}(P_i^t, P_{-i})$ as required.

Suppose $q \leq K \leq |A|$. Pick an arbitrary $c \in A$ such that bP_ic. Since $b = r_1(P_i^t)$, we must have bP_i^tc. Property P implies the existence of a path $\bar{\pi} \in \Pi(P_i, P_i^t)$ such that $b\bar{P}_i^r c$ for all $\bar{P}_i^r$ along the path $\bar{\pi}$. Hence, $r_1(\bar{P}_i^r) \neq c$ for all $\bar{P}_i^r$ along $\bar{\pi}$. Applying Lemma 4.4.2, we can conclude $\phi_c(P_i, P_{-i}) = \phi_c(P_i^t, P_{-i})$. Consequently the total probability of alternatives ranked strictly below the K^{th}-ranked alternative in P_i is the same in $\phi(P_i, P_{-i})$ and $\phi(P_i^t, P_{-i})$. Equivalently, the total probability of alternatives ranked above the K^{th}-ranked alternative in P_i is the same in $\phi(P_i, P_{-i})$ and $\phi(P_i^t, P_{-i})$, i.e. $\sum_{s=1}^K \phi_{r_s(P_i)}(P_i, P_{-i}) \geq \sum_{s=1}^K \phi_{r_s(P_i)}(P_i^t, P_{-i})$. This completes the proof. ■

Theorem 4.4.1 leads immediately to the following corollary.

Corollary 4.4.1 *If a domain satisfies Property P, it satisfies local-global equivalence for unanimous DSCFs.*

The arguments in the proof of Theorem 4.4.1 can be used to show that any locally strategy-proof and unanimous RSCF defined on a domain satisfying Property P also satisfies the important property of *tops-onlyness.*[10]

Definition 4.4.2 *A RSCF $\phi : \mathcal{D}^n \rightarrow \Delta(A)$ satisfies the tops-only property if for all $P, P' \in \mathcal{D}^n$, we have $[r_1(P_i) = r_1(P'_i)$ for all $i \in N] \Rightarrow [\phi(P) = \phi(P')]$.*

Suppose a RSCF satisfies the tops-only property. Then its value at any profile depends only on the peaks of the agent preferences in the profile.

Corollary 4.4.2 *If the domain $\mathcal{D}$ satisfies Property P, every unanimous and locally strategy-proof RSCF $\phi : \mathcal{D}^n \rightarrow \Delta(A)$ satisfies the tops-only property.*

Proof: Fix a unanimous and locally strategy-proof RSCF $\phi : \mathcal{D}^n \rightarrow \Delta(A)$. To verify the tops-only property, it suffices to show that for all $i \in N$, $P_i, P'_i \in \mathcal{D}$ and $P_{-i} \in \mathcal{D}^{n-1}$, $[r_1(P_i) = r_1(P'_i)] \Rightarrow [\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})]$.

Pick $i \in N$, $P_i, P'_i \in \mathcal{D}$ and $P_{-i} \in \mathcal{D}^{n-1}$ such that $r_1(P_i) = r_1(P'_i) \equiv x$. If $P_i \sim P'_i$, Lemma 4.4.1 immediately implies $\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})$. Suppose it is not the case that $P_i \sim P'_i$. To show $\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})$, it suffices to show $\phi_a(P_i, P_{-i}) = \phi_a(P'_i, P_{-i})$ for all $a \in A\backslash\{x\}$. Pick $a \in A\backslash\{x\}$. Since $r_1(P_i) = r_1(P'_i) = x \neq a$, Property P implies the existence of a path $\pi \in \Pi(P_i, P'_i)$ such that $xP^r_i a$ for all P^r_i along the path π. Thus, $r_1(P^r_i) \neq a$ for all P^r_i along the path π. Then, Lemma 4.4.2 implies $\phi_a(P_i, P_{-i}) = \phi_a(P'_i, P_{-i})$. This also implies $\phi_x(P_i, P_{-i}) = \phi_x(P'_i, P_{-i})$ so that $\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})$, as required. ■

Corollary 4.4.2 generalizes Theorem 1 of Chatterji and Zeng [15] on domains satisfying Property P. Their strategy-proofness is weakened to local strategy-proofness, their Interior property becomes redundant and the requirement of their Exterior property is met by Property P. For instance, the domain in Example 4.3.2 violates the Interior property but satisfies Property P.

[10] See Chatterji and Sen [14] and Chatterji and Zeng [15] for a discussion of this property.

4.4.1 Group Strategy-proofness

Our goal in this subsection is to show that when turning to the deterministic setting, any locally strategy-proof, unanimous DSCF defined on a domain satisfying Property P also satisfies the stronger property of group strategy-proofness, i.e. no coalition of agents can strictly improve by a joint misrepresentation of their preferences.[11] We denote a coalition by $S \subseteq N$ where S is non-empty. A preference profile for the coalition S is denoted by P_S and a preference profile $P \in \mathcal{D}^n$ is written as (P_S, P_{-S}).

Definition 4.4.3 *A DSCF $f : \mathcal{D}^n \to A$ is group manipulable by a coalition $S \subseteq N$ at profile $P = (P_S, P_{-S})$ if there exists $P'_S \in \mathcal{D}^{|S|}$ such that $f(P'_S, P_{-S})P_i f(P_S, P_{-S})$ for all $i \in S$. The DSCF is group strategy-proof if it is not group manipulable by any coalition at any profile.*

Our main result in this section is the following.

Theorem 4.4.4 *If the domain $\mathcal{D}$ satisfies Property P, every unanimous and locally strategy-proof DSCF is group strategy-proof.*

Proof: Let $\overline{A} = \{a \in A : r_1(P) = a \text{ for some } P \in \mathcal{D}\}$ be the set of alternatives that are first-ranked for some preferences in $\mathcal{D}$. Recall that $\mathcal{D}$ is assumed to contain at least two preferences with distinct peaks. Hence, $|\overline{A}| \geq 2$. Fix a unanimous and locally strategy-proof DSCF $f : \mathcal{D}^n \to A$. The range of f is defined as $R(f) = \{a \in A : f(P) = a \text{ for some } P \in \mathcal{D}^n\}$. Unanimity implies $\overline{A} \subseteq R(f)$. Lemmas 4.4.1, 4.4.2 and Corollary 4.4.1 hold for f, i.e. f is strategy-proof.

Lemma 4.4.3 $R(f) = \overline{A}$.

Proof: Suppose not, i.e. there exists $P = (P_1, P_2, \ldots, P_n) \in \mathcal{D}^n$ such that $f(P) = a \notin \overline{A}$. Let $r_1(P_1) = x$. Thus $x \in \overline{A}$. Let $P' = (P'_1, P'_2, \ldots, P'_n) \in \mathcal{D}^n$ be a preference profile such that $P'_i = P_1$ for all $i \in N$. For each $i \in \{2, \ldots, n\}$, we pick an arbitrary path $\pi_i \in \Pi(P_i, P'_i)$.[12] Since $a \notin \overline{A}$, there does not exist any preference P^r_i in the path π_i with $r_1(P^r_i) = a$. We can move from P to P' by changing P_i to P'_i for each i ranging from $i = 2$ to $i = n$. According to paths $\pi_2, \ldots, \pi_n$, by repeatedly applying Lemma 4.4.2, we have $f(P') = f(P) = a \neq x$ which contradicts unanimity. Therefore $R(f) = \overline{A}$. ■

In order to prove the theorem, we will prove the following equivalent reformulation of group strategy-proofness: for all $S \subseteq N$, $P_S, P'_S \in \mathcal{D}^{|S|}$ and $P_{-S} \in \mathcal{D}^{|N \setminus S|}$, either $f(P_S, P_{-S}) = f(P'_S, P_{-S})$ or $f(P_S, P_{-S}) P_i f(P'_S, P_{-S})$ for some $i \in S$.

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[11] In the random setting, the notion of group strategy-proofness is too demanding. For instance, Corollary 1 of Morimoto [40] implies that "most" unanimous and strategy-proof RSCFs defined on the domain of single-peaked preferences, which of course satisfies Property P, are group manipulable.

[12] If $P_i = P'_i$, π_i is the null path that begins and terminates at P_i.

We will prove this by induction on the cardinality of S. The case where $|S| = 1$ reduces to strategy-proofness which is implied by Corollary 4.4.1. Assume that the statement above holds for all $S \subseteq N$ such that $|S| \leq t - 1 < n$. We will show that the statement holds for all $S \subseteq N$ where $|S| = t$.

Suppose not, i.e. there exists $S \subseteq N$ (with $|S| = t$) such that $f(P'_S, P_{-S}) = b$, $f(P_S, P_{-S}) = a$ and bP_ia for all $i \in S$. Since $b \in R(f)$, Lemma 4.4.3 implies that there exists $P_i^* \in \mathcal{D}$ such that $r_1(P_i^*) = b$. Furthermore, since $f(P'_S, P_{-S}) = b$, strategy-proofness implies $f(P_S^*, P_{-S}) = b$ where every voter of S has the preference P_i^*.

Since $f(P_S, P_{-S}) = a \neq b = f(P_S^*, P_{-S})$, we have a voter $j \in S$ such that $P_j \neq P_i^*$. By Property P, we have a path $\pi = (P_j^1, \ldots, P_j^v) \in \Pi(P_j, P_i^*)$ with no $\{a, b\}$-restoration. Since bP_ja and bP_i^*a, no $\{a, b\}$-restoration on π implies bP_j^ka for all $k = 1, \ldots, v$. Hence, $r_1(P_j^k) \neq a$ for all $k = 1, \ldots, v$. Since $f(P_j, P_{S\setminus\{j\}}, P_{-S}) = a$, Lemma 4.4.2 implies $f(P_i^*, P_{S\setminus\{j\}}, P_{-S}) = a$.[13] Since $f(P_i^*, P_{S\setminus\{j\}}, P_{-S}) = a$ and $f(P_i^*, P^*_{S\setminus\{j\}}, P_{-S}) = b$, coalition $S \setminus \{j\}$ can group manipulate at profile $(P_i^*, P_{S\setminus\{j\}}, P_{-S})$, which contradicts the induction hypothesis. This completes the proof. ∎

There are some papers that investigate preference domains on which equivalence of strategy-proofness and group strategy-proofness holds. Barberà et al. [7] consider a more general setting than ours in the following respects: (i) the alternative set is either finite or infinite, (ii) preferences can admit indifference, (iii) preference domains can vary across different voters, and (iv) unanimity is not exogenously imposed on DSCFs. On the other hand, our result has a weaker premise — local strategy-proofness rather than strategy-proofness. In addition our Property P is far simpler (especially in the computational sense) than their *sequential inclusion condition*.[14] The latter is a condition imposed on preference profiles while Property P is a condition imposed only on preferences in a domain. Our result is not implied by theirs - for example, the domain of single-peaked preferences on a tree introduced by Demange [20] is covered by our condition but not by theirs.

Property P is also independent of the sufficient condition identified in Le Breton and Zaporozhets [37] for the equivalence of strategy-proofness and group strategy-proofness. For instance, consider a domain $\mathcal{D}$ consisting of the three preferences $P_i^1 = (a\,b\,c\,d)$, $P_i^2 = (a\,b\,d\,c)$ and $P_i^3 = (b\,a\,d\,c)$.[15] This domain satisfies Property P but violates the richness condition of Le Breton and Zaporozhets [37] — though bP_i^1c and cP_i^1d, there exists no preference $P_i \in \mathcal{D}$ such that $r_1(P_i) = b$ and cP_id.

[13] Here agent j has the preference P_i^* in the profile $(P_i^*, P_{S\setminus\{j\}}, P_{-S})$.

[14] According to Section 4.1 of Kumar et al. [32], verifying whether Property L, which as mentioned is significantly stronger than Property P, is satisfied is not computationally hard.

[15] For notational convenience, we specify preferences here horizontally. For instance, $P_i^1 = (a\,b\,c\,d)$ represents that a is top-ranked, b is second-ranked, c is third-ranked, and d is bottom-ranked.

4.5 Necessity

We have already shown that Property P guarantees uLGE and ensures that in the deterministic setting, local strategy-proofness implies group strategy-proofness. However, it is not a necessary condition for uLGE as Example 4.5.1 shows.

Example 4.5.1 Let $A = \{a, b, c, d, x, y\}$. The domain $\mathcal{D}$ is specified in Table 4.5.1 and Figure 4.5.1 (below) illustrates the path induced by the adjacent preferences in $\mathcal{D}$. Note that there is a single path between P_i^1 and P_i^6 which has a $\{b, c\}$-restoration. It follows that $\mathcal{D}$ violates Property P.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6
a	a	a	a	a	b
b	c	c	c	b	a
c	b	b	b	c	c
d	d	d	y	y	y
x	x	y	d	d	d
y	y	x	x	x	x

Table 4.5.1: The Domain $\mathcal{D}$

$$P_i^1 \ \overset{\{b,c\}}{\overline{\hspace{2cm}}}\ P_i^2 \ \overset{\{x,y\}}{\overline{\hspace{2cm}}}\ P_i^3 \ \overset{\{d,y\}}{\overline{\hspace{2cm}}}\ P_i^4 \ \overset{\{c,b\}}{\overline{\hspace{2cm}}}\ P_i^5 \ \overset{\{a,b\}}{\overline{\hspace{2cm}}}\ P_i^6$$

Figure 4.5.1: The path induced by the adjacent preferences in $\mathcal{D}$

Let $\phi : \mathcal{D}^n \to \Delta(A)$ be an arbitrary unanimous and locally strategy-proof RSCF. Observe that the first-ranked alternatives in each of the preferences in $\mathcal{D}$ is either a or b. In order for ϕ to satisfy unanimity, it must be the case that a and b exhaust the whole probability at each preference profile, i.e. $\phi_a(P) + \phi_b(P) = 1$ for all $P \in \mathcal{D}^n$.[16] Consequently, at profiles (P_i, P_{-i}) and (P'_i, P_{-i}) such that $P_i \sim P'_i$, zP_iz', $z'P'_iz$ and $\{z, z'\} \neq \{a, b\}$, we have $\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})$. Therefore, the $\{b, c\}$-restoration alluded to earlier, is irrelevant. Finally, since the path of Figure 4.5.1 has no $\{a, b\}$-restoration, it is easy to show that $\mathcal{D}$ satisfies uLGE following the proof of Theorem 4.4.1. $\square$

[16] Let $\mathcal{D}^1 = \{P_i^1, P_i^2, P_i^3, P_i^4, P_i^5\}$ and $\mathcal{D}^2 = \{P_i^6\}$. Given a profile $P \in \mathcal{D}^n$, if $P_i \in \mathcal{D}^1$ for all $i \in N$, unanimity implies $\phi_a(P) = 1$. Symmetrically, $\phi_b(P) = 1$ if $P_i \in \mathcal{D}^2$ for all $i \in N$. Suppose $\phi_z(P) > 0$ for some $z \in A \backslash \{a, b\}$ and some $P \in \mathcal{D}^n$. It must be the case that $P_i \in \mathcal{D}^1$ and $P_j \in \mathcal{D}^2$ for some $i, j \in N$. Assume for notational convenience that $P_i \in \mathcal{D}^1$ for all $i = 1, \ldots, s$ and $P_j = P_i^6$ for all $j = s+1, \ldots, n$, where $1 \leq s < n$. Let $P'_{s+1} = \cdots = P'_n = P_i^5$ and $P^\ell = (P_1, \ldots, P_s, P'_{s+1}, \ldots, P'_\ell, P_{\ell+1}, \ldots, P_n)$ for all $\ell = s+1, \ldots, n$. Thus, $P'_j \sim P_j$ for all $j = s+1, \ldots, n$, and unanimity implies $\phi_a(P^n) = 1$ and hence $\phi_z(P^n) = 0$. Consequently, Observation 4.4.1 implies $0 = \phi_z(P^n) = \cdots = \phi_z(P^{s+1}) = \phi_z(P) > 0$. Contradiction.

Hong and Kim [28] restrict attention to DSCFs, focus on ordinal Bayesian incentive compatibility, and establish uLGE for domains satisfying a property called *Sparsely Connected Domain without Restoration* (or SCD). This property requires the existence of paths without restoration for all pairs of alternatives such that at least one of the two alternatives is first-ranked in some preference in the domain. This condition is slightly weaker than Property P since the no-restoration requirement is imposed only on a subset of all pairs of alternatives. However, SCD is not necessary for uLGE either. For instance, the domain in Example 4.5.1 violates SCD because the path of Figure 4.5.1 has a $\{b, c\}$-restoration and b is first-ranked in P_i^6.

A characterization of domains that satisfy uLGE remains an open problem. However, we are able to show that uLGE implies connectedness of a domain.

Proposition 4.5.1 *If a domain satisfies uLGE, it is connected.*

Proof: Pick a domain $\mathcal{D}$ that satisfies uLGE. Suppose that domain $\mathcal{D}$ is not connected. We can then partition $\mathcal{D}$ into two non-empty subsets $\mathcal{D}^1$ and $\mathcal{D}^2$ such that there does not exist any $P_i \in \mathcal{D}^1$ and $P_i' \in \mathcal{D}^2$ with $P_i \sim P_i'$.

There are several cases to consider. In each one, we find a set of agents and construct a unanimous, locally strategy-proof and manipulable DSCF. We begin with an observation that we will use frequently.

Observation 4.5.1 We consider a particular class of DSCFs in this setting. We say that a DSCF f is *local* if for all $i \in N$, $P_{-i} \in \mathcal{D}^{n-1}$, $j \in \{1, 2\}$ and $P_i, P_i' \in \mathcal{D}^j$,

$$[f(P_i, P_{-i}) \neq f(P_i', P_{-i})] \Rightarrow [f(P_i, P_{-i}) = r_1(P_i) \text{ and } f(P_i', P_{-i}) = r_1(P_i')].$$

Suppose that agent i's true preference is $P_i \in \mathcal{D}^j$ for some $j \in \{1, 2\}$. A local misrepresentation of P_i is some preference P_i' that also belongs to $\mathcal{D}^j$. Thus local DSCFs are locally strategy-proof.

Case 1: There exist $\bar{P}_i \in \mathcal{D}^1$ and $\hat{P}_i \in \mathcal{D}^2$ such that $r_1(\bar{P}_i) = r_1(\hat{P}_i)$.

Let $N = \{1, 2\}$. Consider the following DSCF:

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } P_1, P_2 \in \mathcal{D}^1 \text{ or } P_1, P_2 \in \mathcal{D}^2, \text{ and} \\ r_1(P_2) & \text{otherwise.} \end{cases}$$

The outcome at each preference profile is the first-ranked alternative of some voter's preference; it is evident that f is unanimous. It is easy to verify that f is local.[17] Then, Observation 4.5.1 implies local

[17] For agent 1, pick $P_1, P_1' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_2 \in \mathcal{D}$. If $f(P_1, P_2) \neq f(P_1', P_2)$, we can deduce that $P_2 \in \mathcal{D}^j$. Hence $f(P_1, P_2) = r_1(P_1)$ and $f(P_1', P_2) = r_1(P_1')$. For agent 2, fix $P_2, P_2' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_1 \in \mathcal{D}$. If $f(P_1, P_2) \neq f(P_1, P_2')$, we immediately deduce that $P_1 \notin \mathcal{D}^j$. Hence $f(P_1, P_2) = r_1(P_2)$ and $f(P_1, P_2') = r_1(P_2')$. Therefore, f is local.

strategy-proofness. However, f is not strategy-proof. Suppose $r_1(\bar{P}_i) = r_1(\hat{P}_i) = x$. Recall that $\mathcal{D}$ is assumed to contain at least two preferences with distinct peaks. Therefore, there exists $P_2 \in \mathcal{D}$ such that $r_1(P_2) = y \neq x$. Suppose $P_2 \in \mathcal{D}^2$. Then $f(\bar{P}_i, P_2) = y$ and $f(\hat{P}_i, P_2) = x$.[18] Agent 1 will then manipulate at $(\bar{P}_i, P_2)$ via $\hat{P}_i$. If $P_2 \in \mathcal{D}^1$, we have $f(\bar{P}_i, P_2) = x$ and $f(\hat{P}_i, P_2) = y$. Then, agent 1 will manipulate at $(\hat{P}_i, P_2)$ via $\bar{P}_i$. This contradicts the hypothesis that $\mathcal{D}$ satisfies uLGE.

Case 1 implies that all preferences with the same first-ranked alternative must belong to the same subset of $\mathcal{D}$, i.e. $[P'_i \in \mathcal{D}^j$ and $r_1(P''_i) = r_1(P'_i)] \Rightarrow [P''_i \in \mathcal{D}^j]$, for $j = 1, 2$. Let $\tau(\mathcal{D}^j) = \{a \in A : r_1(P_i) = a$ for some $P_i \in \mathcal{D}^j\}$, for $j = 1, 2$. We consider two cases, labelled Case 2 and 3. In each case, we show the existence of a unanimous, locally strategy-proof and manipulable DSCF.

Case 2: $|\tau(\mathcal{D}^j)| > 1$ for some $j \in \{1, 2\}$.

Assume w.l.o.g. that $|\tau(\mathcal{D}^2)| > 1$. Let $x, y \in \tau(\mathcal{D}^2)$ and $P^*_i \in \mathcal{D}^1$. Assume w.l.o.g. that $xP^*_i y$. Let $N = \{1, 2\}$. Consider the following DSCF:

$$f(P_1, P_2) = \begin{cases} r_1(P_1) & \text{if } P_1, P_2 \in \mathcal{D}^1 \text{ or } P_1, P_2 \in \mathcal{D}^2, \text{ and} \\ y & \text{otherwise.} \end{cases}$$

Let (P_1, P_2) be a profile such that $r_1(P_1) = r_1(P_2)$. By virtue of our assumption, it must be the case that $P_1, P_2 \in \mathcal{D}^j$, for some $j \in \{1, 2\}$. Since f picks an agent's first-ranked alternative in such a profile, it is clear that f satisfies unanimity. Again f is local.[19] So Observation 4.5.1 implies that f is locally strategy-proof. Finally, we show that f is not strategy-proof. Since $x \in \tau(\mathcal{D}^2)$, there exists $P_1 \in \mathcal{D}^2$ with $r_1(P_1) = x$. By construction, $f(P_1, P^*_i) = y$ and $f(P_1, P_1) = x$. Since $xP^*_i y$, agent 2 manipulates at (P_1, P^*_i) via P_1. Therefore Case 2 cannot occur.

Case 3: $|\tau(\mathcal{D}^1)| = |\tau(\mathcal{D}^2)| = 1$.

Let $\tau(\mathcal{D}^1) = \{x\}$ and $\tau(\mathcal{D}^2) = \{y\}$. Recall that $|A| \geq 3$. Accordingly, we consider two subcases: (A) there exists $P^*_i \in \mathcal{D}$ such that $r_{|A|}(P^*_i) = z \notin \{x, y\}$, and (B) $r_{|A|}(P_i) \in \{x, y\}$ for all $P_i \in \mathcal{D}$.

Case 3A: Assume w.l.o.g. that $r_1(P^*_i) = x$, i.e. $P^*_i \in \mathcal{D}^1$. By assumption, $yP^*_i z$. Let $N = \{1, 2\}$ and consider

[18]Here $(\bar{P}_i, P_2)$ is the profile where agent 1's preference is $\bar{P}_i$ and agent 2's preference is P_2. Similarly $(\hat{P}_i, P_2)$ is the profile where agent 1's preference is $\hat{P}_i$ and agent 2's preference is P_2.

[19]Arguing as we did in Footnote 17, by picking $P_1, P'_1 \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_2 \in \mathcal{D}$, we can infer $[f(P_1, P_2) \neq f(P'_1, P_2)] \Rightarrow [f(P_1, P_2) = r_1(P_1)$ and $f(P'_1, P_2) = r_1(P'_1)]$. Next, fixing $P_2, P'_2 \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_1 \in \mathcal{D}$, we always have $f(P_1, P_2) = f(P_1, P'_2)$ by the construction of f. Therefore, f is local.

the following DSCF:

$$f(P_1, P_2) = \begin{cases} x & \text{if } P_1, P_2 \in \mathcal{D}^1, \\ y & \text{if } P_1, P_2 \in \mathcal{D}^2, \text{ and} \\ z & \text{otherwise.} \end{cases}$$

It is easy to verify that f satisfies unanimity. Local strategy-proofness follows again from Observation 4.5.1 as f is local.[20] Again f is not strategy-proof. Pick $P_2 \in \mathcal{D}^2$. By construction $f(P_i^*, P_2) = z$ and $f(P_2, P_2) = y$. Since yP_i^*z, agent 1 manipulates at (P_i^*, P_2) via P_2.

Case 3B: Since $|A| \geq 3$, there must exist $z \in A \backslash \{x, y\}$ and $\hat{P}_i \in \mathcal{D}$ such that $z\hat{P}_i y$ or $z\hat{P}_i x$ holds. We assume w.l.o.g. that $z\hat{P}_i y$. Thus $\hat{P}_i \in \mathcal{D}^1$. Let $N = \{1, 2, 3\}$ and consider the following DSCF.

$$f(P_1, P_2, P_3) = \begin{cases} x & \text{if } P_1, P_2, P_3 \in \mathcal{D}^1, \\ y & \text{if } P_1, P_2, P_3 \in \mathcal{D}^2, \\ y & \text{if } P_i \in \mathcal{D}^2 \text{ for some } i \in \{1, 2, 3\} \text{ and } P_j \in \mathcal{D}^1 \text{ for all } j \neq i, \text{ and} \\ z & \text{if } P_i \in \mathcal{D}^1 \text{ for some } i \in \{1, 2, 3\} \text{ and } P_j \in \mathcal{D}^2 \text{ for all } j \neq i. \end{cases}$$

In order to show unanimity, we need to only consider profiles where all agents have preferences belonging to the same $\mathcal{D}^j$. In each of these cases, f picks the commonly first-ranked alternative. Also f is local,[21] and we can deduce that f is locally strategy-proof from Observation 4.5.1. Finally we show that f is not strategy-proof. Consider the profile $(\hat{P}_i, \hat{P}_i, P_3)$ where voters 1 and 2 report the preference $\hat{P}_i$, and voter 3 reports a preference $P_3 \in \mathcal{D}^2$. By construction, $f(\hat{P}_i, \hat{P}_i, P_3) = y$. Consider another profile $(\hat{P}_i, P_3, P_3)$ where voter 1 reports the preference $\hat{P}_i$, and voters 2 and 3 reports the preference P_3. By construction, $f(\hat{P}_i, P_3, P_3) = z$. Consequently agent 2 will manipulate at $(\hat{P}_i, \hat{P}_i, P_3)$ via P_3 since $z\hat{P}_i y$.

This concludes the proof of Proposition 4.5.1. ■

[20] Fixing $P_1, P_1' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_2 \in \mathcal{D}$, we always have $f(P_1, P_2) = f(P_1', P_2)$ by the construction of f. Symmetrically, fixing $P_2, P_2' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_1 \in \mathcal{D}$, we also have $f(P_1, P_2) = f(P_1, P_2')$ by the construction of f. Therefore, f is local vacuously.

[21] Fixing arbitrary $i \in N$, $P_i, P_i' \in \mathcal{D}^j$ for some $j \in \{1, 2\}$ and $P_{-i} \in \mathcal{D}^{n-1}$, it is easy to show that $f(P_i, P_{-i}) = f(P_i', P_{-i})$ by the construction of f. Therefore, f is local vacuously.

5

Pointwise Local Incentive Compatibility in Non-Convex Type-Spaces

5.1 Introduction

We consider standard mechanism design problem where a set of agents have valuations for each alternative in a finite set of alternatives. Based on these valuations, the planner has to select an alternative to be shared by all the agents and some payment for each agent. Such a decision scheme is called a mechanism. Agents evaluate their net utilities by means of quasilinear utility functions. A mechanism is incentive compatible (IC) if no agent can increase his/her net utility by misreporting his/her type.

An important problem in mechanism design is to characterize all IC mechanisms for a given type-space. Except for the case when the type-space is $\mathbb{R}^{|A|}$, where A is the set of alternatives, this turns out to be a hard problem. As an intermediate step, researchers have got interested in exploring if the requirement of IC can be reduced considerably.[1] Pointwise local IC (PLIC) (Carroll [12]) turns out to be a way. A mechanism is PLIC if no agent can increase his/her net utility by misreporting to a type that is "close" to his/her sincere type.

[1]For the importance of identifying a minimal set of incentive constraints that will imply full incentive compatibility - see discussions in Chapter 7 of Fudenberg and Tirole [23], Armstrong [2] and Chapter 6 in Vohra [50].

The notion of close types varies from person to person and also depends on the behavioral aspect of the agents. Often agents might not be willing to go for "large" lies (possibly due to the presence of a monitoring technology that detects large lies and punishes them or due to social stigma, fear of loss of reputation, etc.) and choose to lie credibly by only deviating to small neighborhoods of their true types. Moreover, on type-spaces where the equivalence of PLIC and IC holds, poitwise local incentive compatibility is a significantly simpler way to check whether a mechanism is IC or not.[2]

PLIC ensures that a mechanism is IC on the types that are sufficiently close (with respect to Euclidean distance) to each other. More formally, PLIC requires that for every type t there is a neighborhood of t such that the mechanism is IC on both (t, s) and (s, t) for every type s in that neighborhood.[3] Carroll [12] showed that PLIC is equivalent to IC on any convex type-space. To the best of our knowledge, nothing is known about the said equivalence on other type-spaces, despite the fact that there are several important non-convex type-spaces such as the gross substitute one in combinatorial auction.[4]

The crucial fact about convex type-space is that the line joining any two types lie in the type-space. A natural step to get out of the convex type-space would be to consider a type-space that is polygonally connected: between every two types there is a (finite) sequence of lines in the type-space that join them. However, polygonal connectedness alone cannot guarantee the equivalence of PLIC and IC (See Examples 5.5.2, 5.5.3 and 5.5.4). We strengthen it by introducing a condition called minimal richness and show that it is sufficient for the equivalence of PLIC and IC. As applications of our result, we show that PLIC and IC are equivalent on large class of non-convex type-spaces such as type-spaces perturbed by modularity and concave-modularity. Further, we show that the gross substitutes type-space and the generalized gross substitutes and complements type-space are important examples of type-spaces perturbed by modularity and concave-modularity, respectively.

The Gross substitutes type-space has been extensively studied in the literature in various contexts such as matching, mechanism design, equilibrium and algorithms (see Ausubel and Milgrom [3], Gul and Stacchetti [27], Paes Leme [43]). The gross-substitutability condition was first introduced by Kelso and Crawford [30] in the context of two sided matching markets of workers and firms. They showed that gross-substitutability is a sufficient condition for the existence of Walrasian equilibria. Later, Shioura and Yang [48] generalized the gross-substitutability condition to generalized gross substitutes and complements condition where they allow multiple objects of the same kind and also allow for some complementarities across objects.

[2] See Carroll [12] for a detailed explanation on the importance of pointwise local incentive compatibility (PLIC).

[3] A mechanism is IC on a pair of types (t, s) if an agent with sincere type t cannot manipulate (that is, cannot increase his/her net utility) by reporting the type as s.

[4] It is worth mentioning that characterizing all type-spaces where PLIC and IC are equivalent is a long standing open problem and is considered as a hard problem as well.

Recently Kushnir and Lokutsievskiy [34] proved that every monotone allocation function defined on the gross-substitutes type space and the generalized gross substitutes and complements type-space is also cyclically monotone.[5] Our paper complements their paper by establishing the equivalence of PLIC and IC and thereby making the problem of designing mechanisms quite tractable on these domains (see Section 5.6 for a detailed discussion on the connection between our paper and that of Kushnir and Lokutsievskiy [34]).

Next, we provide a geometric condition on a type-space for the equivalence of PLIC and IC. We identify three conditions and show that together these conditions ensure the equivalence of PLIC and IC. Further, we show that these three conditions are indispensable, that is, if we drop any of the conditions, then the equivalence of PLIC and IC is no longer guaranteed.

5.2 Preliminaries

We consider a one-agent model in this paper. This is without loss of generality for our analysis.[6]

Let A be a finite set of alternatives with $|A| = n$. For any given subset X of $\mathbb{R}^n$, by $\partial(X)$ we denote the boundary of X. A type t is a mapping from A to $\mathbb{R}$ that represents the valuation of each alternative in A. We view a type as an element of $\mathbb{R}^n$ (with an arbitrary but fixed indexation of the alternatives). By relative valuation of an alternative a with respect to another alternative b at a type t, we mean the number $t(a) - t(b)$. For two types t and t', we denote the line joining them by $[t, t']$.[7] A subset T of $\mathbb{R}^n$ is called a type-space. A polygonal path from t to t' in T is a finite collection of types $(t = t^1, \dots, t' = t^k)$ such that $[t^l, t^{l+1}]$ lies in T for all $l \in \{1, \dots, k-1\}$. A type-space T is polygonally connected if for every $t, t' \in T$, there exists a polygonal path from t to t' in T. An allocation rule is a map $f : T \to A$ and a payment rule is a map $p : T \to \mathbb{R}$. A (direct) mechanism μ is a pair consisting of an allocation rule f and a payment rule p.

Definition 5.2.1 *A mechanism (f, p) is incentive compatible (IC) on a pair of types (t, s) if*

$$t(f(t)) - p(t) \geq t(f(s)) - p(s).$$

It is IC on a type-space T if it is IC on every pair of types $(t, s) \in T \times T$.

The notion of pointwise local incentive compatibility (PLIC) is introduced in Carroll [12]. A

[5] An allocation function f on a type-space T is monotone (or, 2-cycle monotone) if for all $t, t' \in T, t(f(t)) - t(f(t')) + t'(f(t')) - t'(f(t)) \geq 0$, and it is cyclically monotone if for any integer r and any points $t^0, t^1, \dots, t^r = t^0$ in T, $\sum_{k=0}^{r-1} t^k(f(t^k)) - t^k(f(t^{k+1})) \geq 0$.

[6] All the results of this paper can be generalized to the case of more than one agent in a systematic manner (see Carroll [12], Mishra et al. [39], etc.).

[7] More formally, $[t, t'] = \{\alpha t + (1-\alpha)t' \mid \alpha \in [0, 1]\}$.

mechanism is PLIC on a type-space T if for every $t \in T$, there exists an $\varepsilon > 0$ such that it is IC on (t, s) and (s, t) for every $s \in T$ with $||t - s|| < \varepsilon$.[8]

5.3 Result on minimally rich type-spaces

We introduce the notion of minimally rich type-spaces in this section and show that PLIC and IC are equivalent on such type-spaces. As an application in Section 5.4, we consider type-spaces that arise in the context of combinatorial auctions and show that any type-space that is closed under scaling and closed under modular/concave-modular perturbations is also minimally rich. Gross substitutes (GS) and the generalized gross substitutes and complements (GGSC) type-spaces are important examples of such type-spaces.

A type-space is minimally rich if for any two types t and t' in it and for each alternative a, there is a type s satisfying the following two properties: (i) the lines joining s to both t and t' lie in the type-space, and (ii) for every alternative z, if the relative valuation of a with respect to z (weakly) increases from s to t', then it will also (weakly) increase from from t to s. Notice that minimally rich type-spaces are polygonally connected.

Definition 5.3.1 *A type-space T is* ***minimally rich*** *if for all distinct $t, t' \in T$ and all $a \in A$, there exists $s \in T$ such that*

(i) $[s, t]$ *and* $[s, t']$ *lie in T, and*

(ii) $s(a) - s(z) \geq t(a) - t(z)$ *for all $z \in A$ such that* $t'(a) - t'(z) \geq s(a) - s(z)$.

We explain the implication of minimal richness with some figures for the case where there are two-dimensions (that is, two objects). Let $A = \{a, b\}$. Suppose the valuation of a is represented on the horizonal axis and the valuation of b is represented on the vertical axis. Consider two types t and t'. Without loss of generality, assume $t(a) - t(b) \leq t'(a) - t'(b)$. See Figure 5.3.1 for such two types t and t'. Suppose that the line $[t, t']$ does not lie in the type-space (this is not shown in the figure). To satisfy the minimal richness condition for t and t', we need to find two types s and $\bar{s}$ (not necessarily distinct) for a and b, respectively, such that the lines $[s, t]$, $[s, t']$, $[\bar{s}, t]$ and $[\bar{s}, t']$ lie in the type-space, s lies strictly below the slope 1 line passing through t, and $\bar{s}$ lies strictly above the slope 1 line passing through t'. In Figure 5.3.1, the shaded portions in red and blue are the feasible regions for s, and the shaded portions in grey and blue are the feasible regions for $\bar{s}$. It is worth noting that there are so many choices for s and $\bar{s}$, which in turn corroborates that the minimal richness condition is not much demanding. In Figure 5.3.2, we have

[8] We denote the Euclidean norm of a vector $t \in \mathbb{R}^n$ by $||t||$.

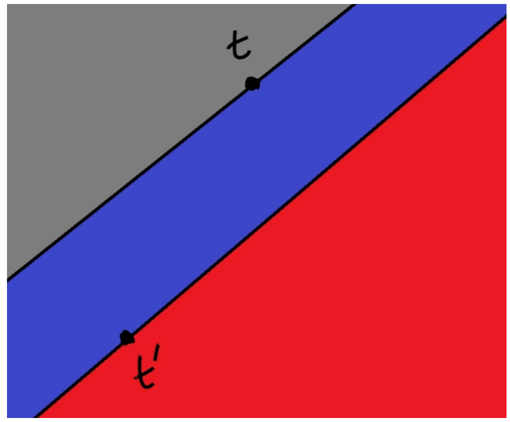

Figure 5.3.1

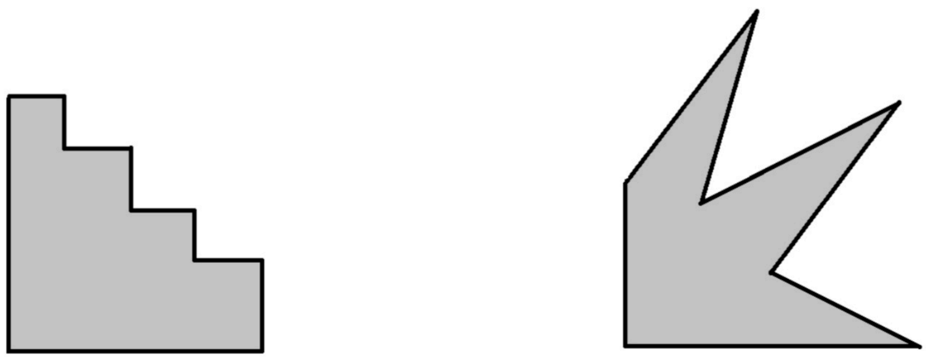

Figure 5.3.2

provided two minimally rich type spaces (marked by the shaded region), one can verify in above discussed way that they are indeed minimally rich.

Theorem 5.3.2 *A mechanism on a minimally rich type-space is IC if and only if it is PLIC.*

The proof of this Theorem is relegated to Appendix 5.7.2.

5.4 Application to combinatorial auction model

Combinatorial auctions are mechanisms where agents are asked to report valuations for combinations of objects, often referred to as "bundles" or "packages", instead of individual objects. Thus, agents are allowed to express their preferences more fully which often leads to greater auction revenues and improved economic efficiency. In what follows, we present two important classes of type-spaces that arise in the context of combinatorial auction model.

5.4.1 Type-spaces perturbed by modularity

Let $E = \{1, \ldots, k\}$ be the set of objects. The set of alternatives is $A = 2^E$, that is, the set of all possible subsets of E. For ease of presentation, let us denote the cardinality of A by n, that is, $n = 2^k$. Thus, a type is an element of $\mathbb{R}^n$.

We say that a type-space $T \subseteq \mathbb{R}^n$ is closed under scaling if for any $t \in T$ and any scalar $\lambda \geq 0$, we have $\lambda \cdot t \in T$.[9] Given $t \in T$ and a vector $m \in \mathbb{R}^k$, we define a type $t_m \in \mathbb{R}^n$ where $t_m(S) = t(S) + \sum_{i \in S} m(i)$ for every $S \subseteq E$. We say that T is closed under modular perturbations if for any $t \in T$ and any vector $m \in \mathbb{R}^k$, we have $t_m \in T$. A type t is **modular** if $t(S) = \sum_{i \in S} t(i)$ for all $S \subseteq E$.

Proposition 5.4.1 *Let $T \subseteq \mathbb{R}^n$ be a type-space that is closed under scaling and closed under modular perturbations. Then, T is minimally rich.*

The proof of this proposition is relegated to Appendix 5.7.3.

An important example of a type-space that is closed under scaling and closed under modular perturbations is the gross substitutes type-space.[10]

The gross substitutes type-space is well-studied in the literature in the context of matching, auction, etc., (see, e.g., Murota [42] and Paes Leme [43] for extensive surveys). This notion was introduced by Kelso and Crawford [30] as a sufficient condition for the existence of Walrasian equilibrium. In what follows, we define demand correspondence and gross substitutes type-space. These definitions are based on the notion of a price vector.

A price p (vector) for individual objects in E is an element of $\mathbb{R}^k$. The price of a bundle S is $p(S) = \sum_{i \in S} p(i)$. The demand correspondence for a price $p \in \mathbb{R}^k$ and a type t is defined as

$$D(t, p) = \arg\max_{S \subseteq E} \{t(S) - p(S)\}.$$

[9] For any type $t = (t_1, \ldots, t_n) \in T$ and any scalar $\lambda \geq 0$, $\lambda \cdot t = (\lambda t_1, \ldots, \lambda t_n)$.

[10] The fact that gross substitutes type-space is closed under scaling and closed under modular perturbations is well known in the literature.

In other words, the demand correspondence for p and t contains those bundles whose net valuation (valuation minus price) according to p and t is the maximum.

A type t satisfies the gross-substitutability condition if, roughly speaking, its demand correspondence satisfies a (partial) independence property with respect to (increasing) price. This is in the sense that if we increase the price of some objects (while keeping that unchanged for the others), then, in some sense, the "demand" of the objects whose prices are not changed will not be affected. More formally, if we go from one price vector to a higher price vector (that is, if we weakly increase the price of each object), then for each demanded bundle S at the former price there will be a demanded bundle S' at the higher price that contains all objects in S whose prices are not changed.

Definition 5.4.1 (Kelso and Crawford [30]) *A type t satisfies the gross-substitutability condition if for all $p, p' \in \mathbb{R}^k$ with $p' \geq p$, we have $S \in D(t,p)$ implies there exists $S' \in D(t,p')$ with $\{i \in S \mid p(i) = p'(i)\} \subseteq S'$.*

Reijnierse et al. [44] and Fujishige and Yang [24] present a characterization of gross-substitutability condition purely in terms of inequalities involving the agent's valuations. For instance, if $|E| = 2$ (say $E = \{i, j\}$), then any type t satisfies the gross-substitutability condition if and only if $t(\{i,j\}) + t(\emptyset) \leq t(\{i\}) + t(\{j\})$.

A type-space is gross substitutes if it contains all types satisfying the gross-substitutability condition. It is well-known that the gross substitutes type-space is not convex (see Example 3 in Kushnir and Lokutsievskiy [34]).

The following corollary is obtained from Theorem 5.3.2, Proposition 5.4.1 and the fact that the gross substitutes type-space is closed under scaling and closed under modular perturbations.

Corollary 5.4.1 *A mechanism on the gross substitutes type-space is IC if and only if it is PLIC.*

5.4.2 Type-spaces perturbed by concave-modularity

As before, let the set of objects be $E = \{1, \ldots, k\}$. The number of units available for object j is a_j. The set of alternatives A is the set of all feasible object bundles which is defined as $A = \{(z_1, \ldots, z_k) \mid z_i \in \mathbb{Z} \text{ and } 0 \leq z_i \leq a_i \text{ for all } i \in E\}$.[11] Let $n = |A|$. Thus, a type is an element of $\mathbb{R}^n$.

As defined in the previous subsection, we say that a type-space $T \subseteq \mathbb{R}^n$ is closed under scaling if for any $t \in T$ and any scalar $\lambda \geq 0$, we have $\lambda \cdot t \in T$. Given concave functions $g_i : \{0, 1, \ldots, a_i\} \to \mathbb{R}$ for each $1 \leq i \leq k$, we define $\tilde{g} = (g_1, \ldots, g_k)$. For any $t \in T$, we define a type $t_{\tilde{g}} \in \mathbb{R}^n$ where

[11] We denote by $\mathbb{Z}$ the set of all integers.

$t_{\tilde{g}}(z) = t(z) + \sum_{i=1}^{k} g_i(z_i)$ for all $z \in A$. We say that T is closed under concave-modular perturbations if for any $t \in T$ and concave functions $g_i : \{0, 1, \ldots, a_i\} \to \mathbb{R}$ for each $1 \le i \le k$, we have $t_{\tilde{g}} \in T$. A type $m : A \to \mathbb{R}$ *modular-concave* if there exists a concave function $g_i : \{0, 1, \ldots, a_i\} \to \mathbb{R}$ for each $1 \le i \le k$ such that $m(z) = \sum_{i=1}^{k} g_i(z_i)$ for all $z \in A$.

Proposition 5.4.2 *Let $T \subseteq \mathbb{R}^n$ be a type-space that is closed under scaling and closed under concave-modular perturbations. Then, T is minimally rich.*

The proof of this proposition is relegated to Appendix 5.7.4.

An important example of a type-space that is closed under scaling and closed under concave-modular perturbations is the generalized gross substitutes and complements type-space (for details see the proof of Theorem 3 in Kushnir and Lokutsievskiy [34]).

We introduce the notion of generalized gross substitutes and complements type-space. Shioura and Yang [48] introduces the notion of generalized gross substitutes and complements (GGSC) type-space. A set $C \subseteq \mathbb{Z}^k$ is a integer convex set if it contains all integer vectors in its convex hull.[12]

The objects are partitioned into two classes E_1 and E_2, that is, $E = E_1 \cup E_2$ with $E_1 \cap E_2 = \emptyset$. The objects are substitutes within classes and complements across the classes. For instance, in the problem of allocation of spectrum licenses, radio spectrum licenses are substitutes within each region, but complements across regions.[13] We denote the total number of units in a class $E_r \in \{E_1, E_2\}$ in a bundle $z \in A$ by $z(E_r)$, that is, $z(E_r) = \sum_{l \in E_r} z_l$.

We now extend the notion of demand correspondence defined in Subsection 5.4.1. Note that the price of a bundle $z \in A$ is $z \cdot p$, where $p \in \mathbb{R}^k$ is the price vector of individual objects. Therefore, for a price $p \in \mathbb{R}^k$ and a type t, we define demand correspondence as

$$D(p, t) = \underset{z \in A}{\arg\max}\{t(z) - p \cdot z\}.$$

For $r \in \{1, 2\}$ and $i \in E_r$, a bundle z' is an improvement of a bundle z with respect to E_r except for i if $z'_l \ge z_l$ for all $l \in E_r \setminus \{i\}$, and $z'_l \le z_l$ for all $l \in E_r^c$. Let us denote by $\chi_i \in \mathbb{R}^k$ the vector whose i-th component is 1 and other components are 0. Let $\chi_0 = (0, \ldots, 0) \in \mathbb{R}^k$ be the null vector.

Definition 5.4.2 *A type t satisfies the generalized gross substitutes and complements condition if for each price $p \in \mathbb{R}^k$,*

(i) $D(p, t)$ is an integer convex set, and

[12] Shioura and Yang [48] use the term discrete convex set instead of integer convex set.
[13] This example is taken from Kushnir and Lokutsievskiy [34].

(ii) for each $z \in D(p,t)$, each $r = 1, 2$, each $i \in E_r$, and each $\delta > 0$, there exists an improvement z' of z with respect to E_r except for i such that $z' \in D(p + \delta\chi_i, t)$ and

$$z(E_r) - z(E_r^c) \geq z'(E_r) - z'(E_r^c).$$

Similar to the characterization of gross-substitutability condition purely in terms of inequalities involving agent's valuations provided in Reijnierse et al. [44] and Fujishige and Yang [24], Shioura and Yang [48] (Theorem 3.3) proves that any type t satisfies the generalized gross substitutes and complements condition if and only if it is GM-concave.

Let $U = \operatorname{diag}(1, \ldots, 1, -1, \ldots, -1)$ be a diagonal $k \times k$ matrix that contains 1 in the first $|E_1|$ diagonal entries and -1 in the remaining $|E_2|$ diagonal entries. For $z = (z_1, \ldots, z_k) \in \mathbb{Z}^k$, define $\operatorname{supp}(z) = \{i \in E \mid z_i > 0\}$.

A type $t : A \to \mathbb{R}$ is called GM-concave if for all $z, z' \in A$ and all $i \in \operatorname{supp}\big(U(z - z')\big)$, there exists $j \in \operatorname{supp}\big(U(z' - z)\big) \cup \{0\}$ such that

$$t(z) + t(z') \leq t\big(z - U(\chi_i - \chi_j)\big) + t\big(z' + U(\chi_i - \chi_j)\big). \tag{5.1}$$

A type-space is generalized gross substitutes and complements if it contains all GM-concave types. It can be verified that the gross substitutes type-space is a special case of the generalized gross substitutes and complements type-space.

We obtain the following corollary from Theorem 5.3.2, Proposition 5.4.2 and the fact that the generalized gross substitutes and complements type-space is closed under scaling and closed under concave-modular perturbations.

Corollary 5.4.2 *A mechanism on the generalized gross substitutes and complements type-space is IC if and only if it is PLIC.*

5.5 A simple geometric structure of non-convex type-spaces for the equivalence of PLIC and IC

In this section we present a simple geometric structure of non-convex type-spaces that guarantees the equivalence of pointwise local incentive compatibilty (PLIC) and incentive compatibilty (IC). For ease of presentation, we assume in this section that $A = \{1, \ldots, n\}$, that is, the alternatives are indexed by the numbers $1, \ldots, n$. For a type $t \in \mathbb{R}^n$, we denote by $D(t) = \{s \in \mathbb{R}^n \mid$ there exists $c \in \mathbb{R}$ such that $s(i) = t(i) + c$ for all $i \in \{1, \ldots, n\}\}$ the set of points that

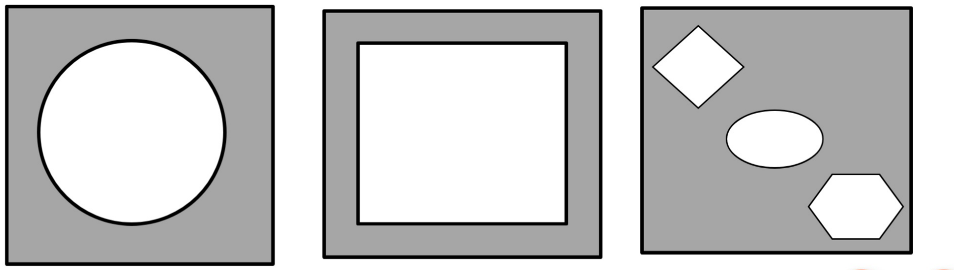

Figure 5.5.1

have the same relative difference between the alternatives as in t. Let $C = \prod_{i=1}^{n}[a_i, b_i]$, where $a_i, b_i \in \mathbb{R}$ with $a_i < b_i$, be a set in $\mathbb{R}^n$, and let $\partial(C)$ denote the boundary of C. Suppose that $T \subseteq C$ is such that T is open in C, $\partial(C) \subseteq T$, and for each $t \in T$, there is $s \in \partial(C) \cap D(t)$ such that the line $[t, s]$ lies in T. In Figure 5.5.1, we provide some examples of such a set T (marked by the shaded region) in two dimensions to illustrate its structure, and in Figures 5.5.2, 5.5.3 and 5.5.4, we provide examples of a set (marked by the shaded region) that does not satisfy the above mentioned property.[14] The type-spaces marked by shaded portion in Figures 5.5.1, 5.5.3 and 5.5.4 does not include the boundary of the inner shape(s) whereas the type-space marked by shaded portion in Figure 5.5.2 includes the boundary of the inner shape.

Theorem 5.5.1 *Let $T \subseteq C$ is such that*

(i) T is open in C,

(ii) $\partial(C) \subseteq T$, and

(iii) for each $t \in T$, there is $s \in \partial(C) \cap D(t)$ such that the line $[t, s]$ lies in T.

A mechanism on T is IC if and only if it is PLIC.

The proof of Theorem 5.5.1 is relegated to Appendix 5.7.5.

The equivalence between PLIC and IC in Theorem 5.5.1 is guaranteed by the existence of a certain kind of polygonal path between every two types lying in the type-space. The specified polygonal path satisfies some monotonic condition over the relative valuation between alternatives.[15]

As we have demonstrated by Figure 5.5.1, the main importance of Theorem 5.5.1 is that its conditions are geometrically easy to check. Additionally, Theorem 5.5.1 provides a geometric insight on the kind of subsets of $\mathbb{R}^n$ (say X) for which PLIC and IC are equivalent on the complement, that is, $\mathbb{R}^n \setminus X$. Note that the geometric property identified in the above theorem is very different from the minimal richness condition (Definition 5.3.1). For example, type-spaces identified in Figure 5.5.1 satisfy the geometric

[14] For more details, see Examples 5.5.2, 5.5.3 and 5.5.4.

[15] For details, see the proof of Theorem 5.5.1.

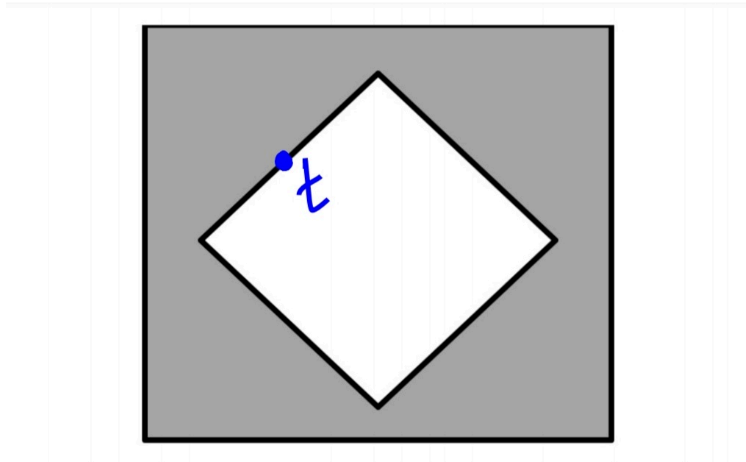

Figure 5.5.2

property proposed in Theorem 5.5.1 but do not satisfy the minimal richness condition. Also, type-spaces identified in Figure 5.3.2 satisfy the minimal richness condition but do not satisfy the geometric property described in Theorem 5.5.1.

It is worth mentioning that conditions of Theorem 5.5.1 are indispensable, that is, if we drop any condition of Theorem 5.5.1, then the equivalence of PLIC and IC is no longer guaranteed. We provide examples below to support this statement. For simplicity, let us assume $A = \{a, b\}$. The valuation of the alternative a is represented on the horizontal axis and the valuation of the alternative b is represented on the vertical axis.

Example 5.5.2 *Suppose we drop Condition (i) of Theorem 5.5.1. Consider the type-space T (marked by shaded portion) in Figure 5.5.2 where T includes the boundary of the inner square. Notice that this figure satisfies Conditions (ii) and (iii) of Theorem 5.5.1. The inner square has sides of slopes 1 and −1. Note that T includes the boundary of the inner square, and hence it is not open in C. Suppose that the side of the inner square containing the point t marked in Figure 5.5.2 lie on the line having slope 1 and passing through origin. Define a mechanism* $\mu = (f, p)$ *such that* $f(t) = a$*,* $f(t') = b$ *for every* $t' \in T \setminus \{t\}$*, and* $p(\bar{t}) = 0$ *for every* $\bar{t} \in T$*. Consider a neighborhood of t such that it does not intersect with any type* $\bar{t}$ *with* $\bar{t}(a) - \bar{t}(b) > 0$*, and similarly for any* $\bar{t}$ *with* $\bar{t}(a) - \bar{t}(b) > 0$*, consider its neighborhood such that it does not intersect t. It can be verified that* μ *is PLIC with such neighbourhoods. However,* μ *violates IC on every pair* (t', t) *with* $t'(a) - t'(b) > 0$.

Example 5.5.3 *Suppose we drop Condition (ii) of Theorem 5.5.1. Consider the type-space T (marked by shaded portion) in Figure 5.5.3 where T does not include the boundary of the cut out portion from the square.*

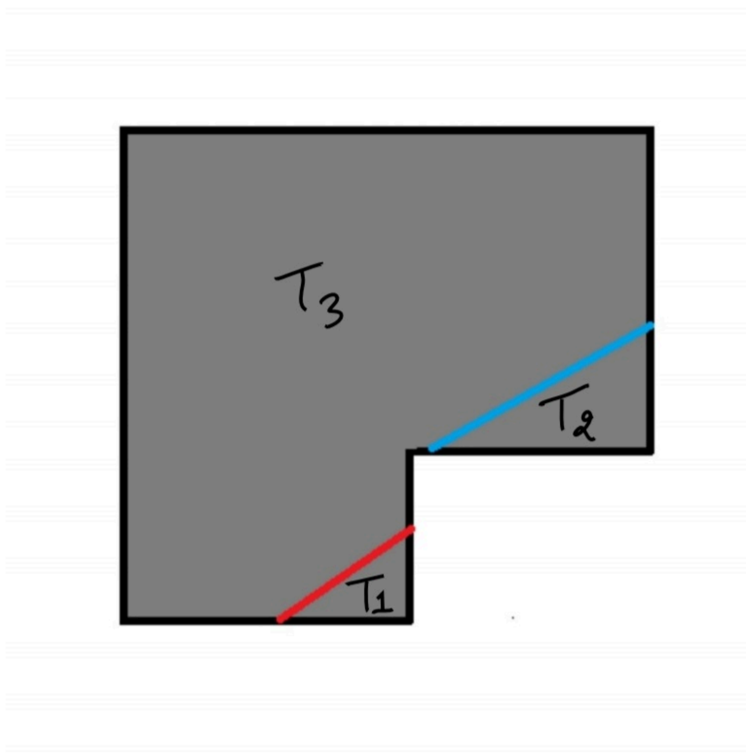

Figure 5.5.3

Clearly, T does not contain the boundary of the square. Notice that this figure satisfies Conditions (i) and (iii) of Theorem 5.5.1. The red line has slope 1 and the vertical intercept is 0, i.e., slope 1 line passing through the origin. The blue line also has slope 1 but the vertical intercept is 1. T_1, T_2 and T_3 forms a partition of T as depicted in Figure 5.5.3. Define a mechanism $\mu = (f, p)$ such that $f(t) = a$ for every $t \in T_1 \cup T_2$, $f(t') = b$ for every $t' \in T_3$, $p(t) = 2$ for every $t \in T_1 \cup T_3$, and $p(t') = 1$ for every $t' \in T_2$. For any given type $t \in T_1$, consider a neighborhood of t such that it does not intersect with any type belonging to T_2, and similarly for any type $\bar{t} \in T_2$, consider its neighborhood such that it does not intersect with any type belonging to T_1. It can be verified that μ is PLIC with such neighbourhoods. However, μ violates IC on every pair (t, t') with $t \in T_1$ and $t' \in T_2$.

Example 5.5.4 *Suppose we drop Condition (iii) of Theorem 5.5.1. Consider the type-space T (marked by shaded portion) in Figure 5.5.4. It does not include the boundary of inner shape. Notice that this figure satisfies Conditions (i) and (ii) of Theorem 5.5.1. The red line has slope 1 and the vertical intercept is 0, i.e., slope 1 line passing through the origin. The blue line also has slope 1 but the vertical intercept is 1. The subsets T_1, T_2 and T_3 form a partition of T as depicted in Figure 5.5.4. Notice that every type in T_1 violates Condition (iii) of Theorem*

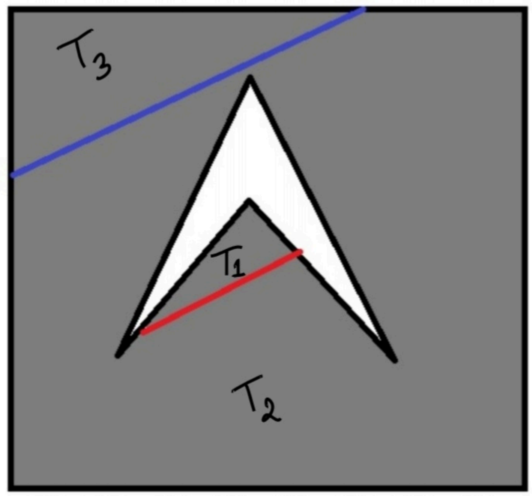

Figure 5.5.4

5.5.1. Define the mechanism $\mu = (f, p)$ such that $f(t) = b$ for every $t \in T_1 \cup T_3$, $f(t') = a$ for every $t' \in T_2$, $p(t) = 2$ for every $t \in T_1 \cup T_2$, and $p(t') = 3$ for every $t' \in T_3$. For any given type $t \in T_1$, consider a neighborhood of t such that it does not intersect with any type belonging to T_3, and similarly for any type $\bar{t} \in T_3$, consider its neighborhood such that it does not intersect with any type belonging to T_1. It can be verified that μ is PLIC with such neighbourhoods. However, μ violates IC on every pair (t, t') with $t \in T_3$ and $t' \in T_1$.

5.6 Discussion: Comparison of our results with Kushnir and Lokutsievskiy [34]

Kushnir and Lokutsievskiy [34] studies a research question that is conceptually different from ours. They study type-spaces where monotonicity implies cyclical monotonicity for allocation rules. Let us first formally introduce the main result of Kushnir and Lokutsievskiy [34].

An allocation rule f on a type-space T is monotone (or, 2-cycle monotone) if for all $t, t' \in T$, we have $t(f(t)) - t(f(t')) + t'(f(t')) - t'(f(t)) \geq 0$, and it is cyclically monotone if for any integer r and any points $t^0, t^1, \ldots, t^r = t^0$ in T, we have $\sum_{k=0}^{r-1} t^k(f(t^k)) - t^k(f(t^{k+1})) \geq 0$.

Definition 5.6.1 *A type-space T is simply connected if* *(i) T is path-connected, i.e., any two types in T can be connected by a path lying entirely in T, and*

(ii) any loop in T can be continuously contracted to a point.

A type-space being simply connected ensures that the type-space do not have certain kind of "holes".[16] A particular class of simply connected type-spaces are *star-shaped* type-spaces. Formally, a type-space T is star-shaped if there exists a type $m \in T$ such that $[m, t] \subseteq T$ for all $t \in T$. We call such types m as a **center** of the star-shaped type-space T.

For any allocation rule $f : T \to A$ and any ordered pair $a, b \in A$, we define $l_{ab} = \inf_{t \in T : f(t) = a} t(a) - t(b)$. For each $a \in A$, we define $T^f_a = \{t \in T \mid t(a) - t(b) \geq l_{ab} \text{ for all } b \in A\}$.

Definition 5.6.2 *An allocation rule $f : T \to A$ satisfies the local-to-global condition if for every $a, b \in f(T)$ with $T^f_a \cap T^f_b = \emptyset$, there exists a finite sequence of alternatives $(a = a_0, \ldots, a_r = b)$ such that $T^f_{a_k} \cap T^f_{a_{k+1}} \neq \emptyset$, $k = 0, \ldots, r-1$, and $l_{ab} \geq \sum_{k=0}^{r-1} l_{a_k a_{k+1}}$.*

The main result of Kushnir and Lokutsievskiy [34] is as follows.

Theorem 5.6.3 *(Kushnir and Lokutsievskiy [34]) Let $T \subseteq \mathbb{R}^{|A|}$ be a type-space and $f : T \to A$ be an allocation rule. Suppose that*

(i) T is simply connected,

(ii) For each $a \in A$, T^f_a is either path-connected or empty, and

(iii) f satisfies the local-to-global condition.

Then if f is monotone (or, 2-cycle monotone), it is also cyclically monotone.

It is important to note that Condition (ii) and Condition (iii) of Theorem 5.6.3 are defined for a given allocation rule, and consequently, Theorem 5.6.3 does not present direct conditions on a type-space so that monotonicity will imply cyclical monotonicity. Using Theorem 5.6.3 Kushnir and Lokutsievskiy [34] proved that every monotone allocation rule defined on the gross substitutes type-space and the generalized gross substitutes and complements type-space is also cyclically monotone. It is understood from the proof that a direct condition on type-spaces for the equivalence of monotonicity and cyclical monotonicity can be derived from Theorem 5.6.3. We state below these conditions as a corollary of Theorem 5.6.3.

[16] See footnote 8 in Kushnir and Lokutsievskiy [34] for a formal definition of simply connectedness.

Corollary 5.6.1 *Let T be a star-shaped type-space and $M \subseteq T$ be the collection of all centers of T. Suppose that for every $a \in A$ and every $t \in T$ there exists $m \in M$ such that $m(a) - m(b) \geq t(a) - t(b)$ for all $b \in A \setminus \{a\}$. Then every monotone (or, 2-cycle monotone) allocation rule on T is cyclically monotone.*

Now we are ready to compare the results of our paper with Theorem 5.6.3 in Kushnir and Lokutsievskiy [34] (along with Corollary 5.6.1). The main assumptions on type-spaces in Theorem 5.6.3 and Corollary 5.6.1 are simply connected and star-shaped, respectively. Our main assumption on type-spaces is minimal richness (Definition 5.3.1). The notion of minimal richness is flexible enough to accomodate type-spaces that are not simply connected (and hence not star-shaped). For example, let $A = \{a, b\}$ and consider the type-space $\mathbb{R}^2 \setminus \{(0, 0)\}$ where the valuation of a is represented on the horizontal axis and the valuation of b is represented on the vertical axis.[17] Note that $\mathbb{R}^2 \setminus \{(0, 0)\}$ is not simply connected because any loop around origin cannot be contracted continuously to a point. However, $\mathbb{R}^2 \setminus \{(0, 0)\}$ is minimally rich.[18] This can be verified using similar arguments provided after the definition of minimal richness (Definition 5.3.1) in Section 5.3. Moreover, minimal richness is a weaker condition than the conditions identified in Corollary 5.6.1, that is, every type-space satisfying conditions of Corollary 5.6.1 is minimally rich but the converse is not true. For instance, the examples of minimally rich type-spaces illustrated in Figure 5.3.2 are star-shaped but violates the other condition stated in Corollary 5.6.1, i.e. there exists a type t belonging to the type-space such that $m(a) - m(b) < t(a) - t(b)$ for every type m that belongs to the center of the type-space. Similarly, the type-spaces illustrated in Figure 5.5.1 satisfy our geometric property provided in Theorem 5.5.1 but are not simply connected.

Summarizing, although both Kushnir and Lokutsievskiy [34] and we have used the same property of the gross substitutes type-space (and the generalized gross substitutes and complements type-space) to prove our respective results, there are some differences between the properties required in general for these results. This is natural as the implications of our main result and that of Kushnir and Lokutsievskiy [34] are quite different.

5.7 Appendix

5.7.1 A useful lemma

In this subsection, we present a lemma that we will use in deriving the rest of the results of the paper. The lemma provides a sufficient condition for a mechanism to be IC on a pair of types based on its IC

[17] The same arguments work if we consider the type-space where finitely many points are deleted from $\mathbb{R}^2$, but for simplicity we just delete the origin.

[18] For more than two alternatives, we can construct such examples that are not simply connected but satisfies the minimal richness condition.

property over a sequence of types. As we show in Sections 5.3 and 5.5, this simple lemma is quite powerful in deducing a wide range of results.

Lemma 5.7.1 *A mechanism $\mu = (f, p)$ on a type-space T is IC on a pair of types (t, t') if there is a finite sequence of types $(t = t^1, \ldots, t^k = t')$ in T such that for all $l < k$,*
(i) μ is IC on (t^l, t^{l+1}), and
(ii) $t^1(f(t^{l+1})) - t^1(f(t^l)) \leq t^l(f(t^{l+1})) - t^l(f(t^l))$.

The proof of this lemma is quite straightforward; we provide it here for the sake of completeness.[19]
Proof: Consider a mechanism $\mu = (f, p)$ on a type-space T. Let (t, t') be a pair of types in T for which there is a finite sequence of types $(t = t^1, \ldots, t^k = t')$ in T such that for all $l < k$, (i) μ is IC on (t^l, t^{l+1}), and (ii) $t^1(f(t^{l+1})) - t^1(f(t^l)) \leq t^l(f(t^{l+1})) - t^l(f(t^l))$. We show that μ is IC on the pair (t, t').

We prove this by induction. By the assumption, μ is IC on (t^1, t^2). Suppose μ is IC on (t^1, t^l) for some $l < k$. This yields

$$t^1(f(t^1)) - p(t^1) \geq t^1(f(t^l)) - p(t^l). \tag{5.2}$$

Since μ is IC on (t^l, t^{l+1}), we have

$$t^l(f(t^l)) - p(t^l) \geq t^l(f(t^{l+1})) - p(t^{l+1}). \tag{5.3}$$

Adding $t^1(f(t^l))$ to both sides of (5.3) and doing some rearrangement, we obtain

$$t^1(f(t^l)) - p(t^l) \geq t^1(f(t^l)) + t^l(f(t^{l+1})) - t^l(f(t^l)) - p(t^{l+1}). \tag{5.4}$$

Combining (5.4) and Part (ii) of the condition in the lemma, we have
$t^1(f(t^l)) - p(t^l) \geq t^1(f(t^{l+1})) - p(t^{l+1})$. This, together with (5.2) gives
$t^1(f(t^1)) - p(t^1) \geq t^1(f(t^{l+1})) - p(t^{l+1})$, which implies μ is IC on (t^1, t^{l+1}). This completes the proof. ■

5.7.2 Proof of Theorem 5.3.2

Proof: The "only if" part of the theorem follows from the definition. We proceed to prove the "if" part of the theorem. Let (f, p) be an PLIC mechanism on a minimally rich type-space $T \subseteq \mathbb{R}^n$. We will show that for any $t, t' \in T$, (f, p) is IC on (t, t').

Fix any $t, t' \in T$. If $[t, t'] \subseteq T$, then by Carroll [12] it follows that (f, p) is IC on (t, t').[20] Suppose $[t, t'] \nsubseteq T$. Let $f(t') = a$. Since T satisfies minimal richness, there exists $s \in T$ satisfying conditions (i)

[19] We prove this using a familiar idea in the mechanism design literature related to supermodularity/revealed preference argument (for example see Lemma 2 in **?**], Theorem 1 in **?**]).
[20] Carroll [12] shows that any PLIC mechanism on a type-space T is IC on (t, t') if $[t, t'] \subseteq T$.

and (ii) of Definition 5.3.1. Assume $f(s) = b$. By condition (i), the line $[s, t']$ lies in T. Therefore, by Carroll [12], (f, p) is IC on both (t', s) and (s, t'). This implies $t'(a) - t'(b) \geq s(a) - s(b)$.

Therefore, by condition (ii) we have that,

$$s(a) - s(b) \geq t(a) - t(b). \tag{5.5}$$

By condition (i), both lines $[t, s]$ and $[s, t']$ lie in T. Therefore, by Carroll [12], (f, p) is IC on both (t, s) and (s, t'). This, together with the facts that $f(t') = a$, $f(s) = b$ and $s(a) - s(b) \geq t(a) - t(b)$, Lemma 6.7.3 implies that (f, p) is IC on (t, t'). This completes the proof of the theorem. ■

5.7.3 Proof of Proposition 5.4.1

Proof: Let T denote a type-space that is closed under scaling and closed under modular perturbations. Let us denote the zero vector by $\mathbf{0}$. Suppose M is the set of all modular types. First we show that $M \subseteq T$. Since T is closed under scaling, $\mathbf{0} \in T$. Also, since T is closed under modular perturbations, $\mathbf{0}_m \in T$ for every $m \in \mathbb{R}^k$. By definition, every modular type can be written as $\mathbf{0}_m$ for some $m \in \mathbb{R}^k$. Hence, $M \subseteq T$.

Next we show that $[t, m]$ lies in T for all $t \in T$ and $m \in M$. Take any $t \in T$ and $m \in M$. Pick any $0 < \lambda < 1$. Since $m \in M$, we have $(1 - \lambda) \cdot m \in M$, and since T is closed under scaling, we have $\lambda \cdot t \in T$. These, together with the fact that T is closed under modular perturbations, imply that $\lambda \cdot t + (1 - \lambda) \cdot m \in T$. Since $0 < \lambda < 1$ is arbitrary, this implies that the line $[t, m] \subseteq T$.

Now we show that T satisfies minimal richness. Fix any $t, t' \in T$ and $H \subseteq E$. To prove minimal richness we need to show that there exists a modular type $m \in T$ satisfying $m(H) - m(F) \geq t(H) - t(F)$ for all $F \subseteq E$ such that $t'(H) - t'(F) \geq m(H) - m(F)$. We prove something even stronger: there exists a modular type $m \in T$ such that $m(H) - m(F) \geq t(H) - t(F)$ for all $F \subseteq E$.

Since A is finite, there exists $c > 0$ such that $c \geq t(H) - t(F)$ for all $F \subseteq E$. Define a modular type $m \in M$ such that $m(\emptyset) = 0$, $m(i) = c$ for $i \in H$, and $m(i) = -c$ for $i \in E \setminus H$. Clearly $m(H) - m(F) \geq c$ for all $F \subseteq E$ with $F \neq H$. Therefore, $m(H) - m(F) \geq t(H) - t(F)$ for all $F \subseteq E$. Since $m \in M$, both $[t, m]$ and $[m, t']$ lie in T, and hence T is minimally rich. ■

5.7.4 Proof of Proposition 5.4.2

Proof: Let T denote a type-space that is closed under scaling and closed under concave-modular perturbations. Let M denote the set of all modular-concave types. Similar to the proof of Proposition 5.4.1, it follows that $M \subseteq T$ and $[t, m]$ lies in T for all $m \in M$ and $t \in T$.

We show that T satisfies minimal richness. Fix any $t, t' \in T$ and $z \in A$. To prove minimal richness we need to show that there exists a modular-concave type $m \in T$ satisfying $m(z) - m(z') \geq t(z) - t(z')$ for all $z' \in A$ such that $t'(z) - t'(z') \geq m(z) - m(z')$. As we did in the case of proving Proposition 5.4.1, we prove something stronger: there exists a modular-concave type $m \in T$ such that $m(z) - m(z') \geq t(z) - t(z')$ for all $z' \in A$.

Since A is finite, there exists $c > 0$ such that $c \geq t(z) - t(z')$ for all $z' \in A$. For each $i = 1, \ldots, k$, define the concave function g_i such that $g_i(0) = 0$ and $g_i(j) = -c|z_i - j|$ for all $j = 1, \ldots, a_i$. Consider the modular-concave type m defined by $m(\bar{z}) = \sum_{i=1}^{k} g_i(\bar{z}_i)$ for all $\bar{z} \in A$. We have $m(z) - m(z') = \sum_{i=1}^{k}(g_i(z_i) - g_i(z'_i)) = c\sum_{i=1}^{k} |z_i - z'_i| \geq c$ for all $z' \in A \setminus \{z\}$. Therefore, $m(z) - m(z') \geq t(z) - t(z')$ for all $z' \in A$. Since $m \in M$, both $[t, m]$ and $[m, t']$ lie in T, and hence T is minimally rich. ■

5.7.5 Proof of Theorem 5.5.1

Proof: First we prove a claim that will be used in the proof of the Theorem 5.5.1. We use the following terminologies in the proof. A polygonal path from t to t' in T is a finite collection of types $(t = t^1, \ldots, t' = t^k)$ such that $[t^l, t^{l+1}]$ lies in T for all $l \in \{1, \ldots, k-1\}$. An alternative i weakly (or strictly) improves from a type t to another type t' if $t(i) - t(j) \leq t'(i) - t'(j)$ for all $j \in A \setminus \{i\}$ (or, $t(i) - t(j) < t'(i) - t'(j)$ for all $j \in A \setminus \{i\}$). An alternative i weakly (or strictly) improves along a polygonal path $(t^1, \ldots, t^k)$ if i weakly (or strictly) improves from the type t^l to t^{l+1} for all $l \in \{1, \ldots, k-1\}$. For any alternative $i \in A$, let $\alpha_i = \inf\{t(i) \mid t \in C \setminus T\}$ and $\beta_i = \sup\{t(i) \mid t \in C \setminus T\}$. Since $\partial(C) \subseteq T$, $a_i < \alpha_i \leq \beta_i < b_i$. Let $U(i) = \{t \in T \mid t(i) \in (\beta_i, b_i]\}$ and $L(i) = \{t \in T \mid t(i) \in [a_i, \alpha_i)\}$. For any $i \in A$, we will often refer to $U(i)$ and $L(i)$ as hollow faces of the cuboid.

Claim 5.7.1 *For every $s, s' \in T$ and $i \in A$, there exist (not necessarily distinct) $t^1, t^2, t^3 \in T$ such that*

(i) $t^1 \in D(s)$,

(ii) *i weakly improves from t^1 to t^2,*

(iii) $[s, t^1], [t^1, t^2]$, *and* $[t^2, t^3]$ *lie in T, and*

(iv) *there exists a polygonal path from s' to t^3 along which i strictly improves.*

Proof: Fix $s, s' \in T$ and $i \in A$. We need to find $t^1, t^2, t^3 \in T$ satisfying conditions (i), (ii), (iii), and (iv). Let $L = \{t \in C \mid t(i) = b_i,\ t(j) = a_j \text{ for some } j \in A \setminus \{i\}\}$. Notice that $L \subseteq T$. We distinguish the following two cases:

Case (i): Suppose $s' \in L$. Take any $s \in T$. Set $t^3 = s'$. Hence condition (iv) is vacuously satisfied. By the assumption on T, there is $t^1 \in \partial(C) \cap D(s)$ such that the line $[s, t^1]$ lies in T.[21] Since $t^1 \in D(s)$, condition (i) is satisfied. Now we have to find $t^2 \in T$ such that the lines $[t^1, t^2]$, $[t^2, s']$ lie in T and i weakly improves from t^1 to t^2. Since $t^1 \in \partial(C)$, there exists $j \in A$ such that $t^1(j) \in \{a_j, b_j\}$. We further distinguish the following two subcases:

Case (i.a): Suppose $j \neq i$. Define $t^2 \in T$ such that $t^2(i) = b_i$ and $t^2(l) = t^1(l)$ for all $l \in A \setminus \{i\}$.[22] Note that for any type t lying on the line $[t^1, t^2]$, $t(j) = t^1(j) \in \{a_j, b_j\}$. Therefore, the line $[t^1, t^2]$ lies in $\partial(C)$. Since $\partial(C) \subseteq T$, it follows that the line $[t^1, t^2]$ lies in T. Since $s'(i) = b_i = t^2(i)$, it follows by using a similar logic that the line $[t^2, s']$ lies in T. These, together with the fact that the line $[s, t^1]$ lies in T, implies that condition (iii) is satisfied. By the construction of t^2, $t^2(i) - t^2(l) \geq t^1(i) - t^1(l)$ for all $l \in A \setminus \{i\}$. Hence condition (ii) is also satisfied, and thereby the proof for this subcase is complete.

Case (i.b): Suppose $j = i$. Then $t^1(i) \in \{a_i, b_i\}$. Since $s' \in L$, there exists $k \in A \setminus \{i\}$ such that $s'(k) = a_k$. Define $t^2 \in T$ such that $t^2(i) = t^1(i)$, $t^2(k) = s'(k) = a_k$ and $t^2(l) = t^1(l)$ for all $l \in A \setminus \{i, k\}$.[23] Since $t^2(i) = t^1(i)$ and $t^2(k) = s'(k) = a_k$, by using a similar logic as in Case (i.a), it follows that the lines $[t^1, t^2]$ and $[t^2, s']$ lie in T. This, together with the fact that the line $[s, t^1]$ lies in T, implies that condition (iii) is satisfied. By the construction of t^2, $t^2(i) - t^2(l) \geq t^1(i) - t^1(l)$ for all $l \in A \setminus \{i\}$. Hence condition (ii) is also satisfied, and thereby the proof for the subcase is completed.

Case (ii): Suppose $s' \in T \setminus L$. Take any $s \in T$. Define $t^3 \in T$ such that $t^3(i) = b_i$ and $t^3(l) = a_l$ for all $l \in A \setminus \{i\}$. Since by construction $t^3 \in L$, by Case (i) there exist $t^1, t^2 \in T$ such that conditions (i), (ii) and (iii) are satisfied. Therefore, we only need to show that condition (iv) is also satisfied, i.e., there exists a polygonal path from s' to t^3 along which i strictly improves. By the assumption on T, there exists $\bar{t} \in \partial(C) \cap D(s')$ such that the line $[s', \bar{t}]$ lies in T. Since $\bar{t} \in \partial(C)$, there exists $j \in A$ such that $\bar{t}$ belongs to $L(j) \cup U(j)$. Since $[s', \bar{t}]$ lies in T and $\bar{t} \in \partial(C)$, there exists a type $\tilde{t} \notin \partial(C)$ on the line $[s', \bar{t}]$ such that $\tilde{t}$ belongs to the same hollow face as $\bar{t}$. Since $\tilde{t}$ lies on the line $[s', \bar{t}]$, $\tilde{t} \in D(s')$. Define $\widehat{T} = \{t \in T \mid \tilde{t}(i) < t(i),\ t(l) = \tilde{t}(l) \text{ for all } l \in A \setminus \{i\} \text{ and } t \text{ belongs to the same hollow face as } \bar{t}\}$. Since T is open in C, there exists $\hat{t} \in \widehat{T} \setminus \partial(C)$ such that the line $[s', \hat{t}]$ lies in T. Note that $\hat{t}$ belongs to $L(j) \cup U(j)$ and $s'(i) - s'(l) < \hat{t}(i) - \hat{t}(l)$ for all $l \in A \setminus \{i\}$. We further distinguish the following two subcases:

Case (ii.a): Suppose $j \neq i$. Define $t^4 \in T$ such that $t^4(i) = b_i$ and $t^4(l) = \hat{t}(l)$ for all $l \in A \setminus \{i\}$. Since $\hat{t} \notin \partial(C)$, we have $a_l < \hat{t}(l) < b_l$ for every $l \in A$. Hence $\hat{t}(i) - \hat{t}(l) < t^4(i) - t^4(l)$ for all $l \in A \setminus \{i\}$ and $t^4 \notin L$. Note that for any type t lying on the line $[\hat{t}, t^4]$, $t \in L(j) \cup U(j)$. Therefore, the line $[\hat{t}, t^4]$ lies in

[21] If s itself is a point in $\partial(C)$, then we can set $s = t^1$.

[22] In $\mathbb{R}^3$, we can view t^2 as the foot of the perpendicular from t^1 to the face of the cuboid having b_i as the valuation of the alternative i for every type.

[23] In $\mathbb{R}^3$, we can view t^2 as the foot of the perpendicular from t^1 to the face of the cuboid having a_k as the valuation of the alternative k for every type.

$L(j) \cup U(j)$. Since $L(j) \cup U(j) \subseteq T$, it follows that the line $[\hat{t}, t^4]$ lies in T. Since $t^3(i) = b_i = t^4(i)$, it follows by using a similar logic that the line $[t^4, t^3]$ lies in T. Since $t^4 \notin L$, we have $t^4(i) - t^4(l) < t^3(i) - t^3(l)$ for all $l \in A \setminus \{i\}$. Therefore, $(s', \hat{t}, t^4, t^3)$ is a polygonal path from s' to t^3 along which i strictly improves. Hence, condition (iv) is also satisfied, and thereby the proof for this subcase is complete.

Case (ii.b): Suppose $j = i$. Define $t^4 \in T \cap (L(i) \cup U(i))$ such that $t^4(i) = \hat{t}(i) + \varepsilon < b_i,\ t^4(k) = a_k$ and $t^4(l) = \hat{t}(l)$ for some $\varepsilon > 0,\ k \in A \setminus \{i\}$ and all $l \in A \setminus \{i, k\}$. Note that such a type t^4 can always be found since T is open in C and $t^4 \notin L$. Since $t^3(k) = a_k = t^4(k)$ and $t^4 \notin L$, by using a similar logic as in Case (ii.a), it follows that $(s', \hat{t}, t^4, t^3)$ is a polygonal path from s' to t^3 along which i strictly improves. Hence, condition (iv) is satisfied, and thereby the proof for this subcase is complete.

Since Cases (i) and (ii) are exhaustive, this completes the proof of the claim. ■

Having proved Claim 5.7.1, now we proceed towards the proof of Theorem 5.5.1. Consider a PLIC mechanism $\mu = (f, p)$ on T and consider two arbitrary types s and s' in T. We show that μ is IC on (s, s'). Let $f(s') = i$. By Claim 5.7.1, there exist $t^1, t^2, t^3 \in T$ such that (i), (ii), (iii), and (iv) are satisfied. Suppose $(s' = s^1, \ldots, s^k = t^3)$ be a polygonally connected path from s' to t^3 satisfying (iv). By Carroll [12], it follows that every PLIC mechanism is IC on a line. Hence, μ is IC on both (s^l, s^{l+1}) and (s^{l+1}, s^l) for all $l \in \{1, \ldots, k-1\}$. This, together with the facts that $f(s') = i$ and i strictly improves along $(s' = s^1, \ldots, s^k = t^3)$, implies that $f(s^l) = i$ for every $l \in \{1, \ldots, k\}$. Now consider the polygonally connected path $(s, t^1, t^2, t^3 = s^k, \ldots, s^1 = s')$ from s to s'. Since this path is polygonally connected, μ is IC on every pair of consecutive types in $(s, t^1, t^2, t^3 = s^k, \ldots, s^1 = s')$, thereby satisfying condition (i) of Lemma 6.7.3. We show that $(s, t^1, t^2, t^3 = s^k, \ldots, s^1 = s')$ satisfies condition (ii) of Lemma 6.7.3. By condition (i), (ii), and (iii) of Claim 5.7.1 and the fact that $f(s^l) = i$ for every $l \in \{1, \ldots, k\}$, it follows that condition (ii) of Lemma 6.7.3 is also satisfied. Therefore, by Lemma 6.7.3, we obtain that μ is IC on (s, s'). This completes the proof of the Theorem. ■

6

Local Incentive Compatibility in Ordinal Type-Spaces

6.1 Introduction

We consider standard mechanism design problems when agents have quasi linear utility function. There is a finite set of alternatives and a finite set of agents. Agents' types are their valuations for the alternatives. A mechanism is incentive compatible (IC) if it is not possible for any agent to increase his/her (net) utility by misreporting his/her sincere type in any way. It is locally IC (LIC) if it is not possible for an agent to increase his/her (net) utility by misreporting to a type that lies in a small "neighborhood" of his/her sincere type. In other words, LIC is a weakening of IC where IC is required to be satisfied for deviations within a small neighborhood.

Characterizing all IC mechanisms on a given type-space is an important problem in mechanism design. However, despite its importance, the structure of IC mechanisms is known only for the case when the type-space is unrestricted (that is, $\mathbb{R}^{|A|}$, where A is the set of alternatives) (see Lavi et al. [35] for details) and finding this structure on other type-spaces seem to be a hard problem. As an intermediate step, researchers have got interested in exploring if the requirement of IC can be reduced considerably.[1] Local IC (LIC) turns out to be a way.

[1] For the importance of identifying a minimal set of incentive constraints that imply full incentive compatibility - see discussions in Chapter 7 of Fudenberg and Tirole [23], Armstrong [2] and Chapter 6 in Vohra [50].

The notion of LIC depends on the notion of neighborhood one intends to consider. Carroll [12] considered neighborhoods with respect to Euclidean distance. We refer to this notion as pointwise LIC (PLIC). He showed that if the type-space is convex, then PLIC is equivalent to IC. To the best of our knowledge, nothing is known about the equivalence of LIC and IC on other type-spaces.

An ordinal domain is a collection of ordinal preferences. In contrast to cardinal environments (that is, for type-spaces), the relation between LIC and IC is well-explored for ordinal domains (see Kumar et al. [32] for details). A type represents an ordinal preference if for any two alternatives a and b, a is preferred to b implies the valuation of a will be higher than b. A type-space is called ordinal if it is induced by an ordinal domain, that is, it contains all types representing some preference in an ordinal domain.[2] An ordinal domain/type-space is strict if it does not admit indifference.

The mechanism design literature generally considers geometric restrictions such as connectedness and convexity on type-spaces. While these are simplifying technical assumptions, they exclude ordinal restrictions such as single-peaked or single-crossing or single-dipped preferences that arise in several economic problems. For instance, in a problem where the location of a public good on a street needs to be decided, subsidies can be given to the people who reside far away from the chosen location. Similarly, in determining the budget for infrastructure, industrial development, etc., subsidies can be given to poor people (or whoever derives relatively lesser externalities from a decision). Barzel [8], Stiglitz [49], and Bearse et al. [9] consider the problem of setting the level of tax rates to provide public funding in the education sector, and Ireland [29] and Epple and Romano [21] consider the same problem in the health insurance market.[3] Our analysis enables one to analyze these problems as a mechanism design problem with transfers. Mishra et al. [39] explains how single-peakedness arises in a private good scheduling problem. Some other papers that deal with mechanism design in ordinal type-spaces are Mishra et al. [38], Carbajal and Müller [11], Mishra et al. [39], etc.

An ordinal domain satisfies ordinal local global equivalence (OLGE) if every locally incentive compatible social choice function on that domain is incentive compatible. The notion of OLGE is defined in Kumar et al. [32], where it is shown that a strict ordinal domain is OLGE if and only if it satisfies "Property L". Almost all well-known domains such as single-peaked, single-crossing, single-dipped, etc., satisfies OLGE. An ordinal type-space satisfies cardinal local global equivalence (CLGE) if every locally incentive compatible mechanism on that type-space is incentive compatible. We characterize strict CLGE type-spaces by showing that a strict ordinal domain is OLGE if and only if the corresponding strict type-space is CLGE. It is worth mentioning that our result applies to type-spaces that are not necessarily convex, not even connected. The relaxation of connectedness or convexity is not a trivial extension. For

[2]All the results of this paper also hold if we consider types that are bounded below or bounded above (for instance, non-negative types).

[3]Individuals' preferences are considered to be single-peaked in such scenarios.

instance, the equivalence of PLIC and IC does not hold on non-connected type-spaces, consequently we introduce the notion of uniform LIC (ULIC) and establish the equivalence of ULIC and IC on such type-spaces.

Indifference occurs naturally in preferences, therefore we explore the equivalence of LIC and IC on ordinal type-spaces admitting indifferences. We introduce the notion of almost everywhere IC. A mechanism is almost everywhere IC if it is IC outside a set of (Lebesgue) measure zero (thus, such a mechanism is IC except for some rare (measure zero) situations). We suitably define the notion of LIC to take care of indifference. We call it strong LIC and provide a necessary and sufficient condition on an ordinal type-space for the equivalence of strong LIC and almost everywhere IC. The closure of single-peaked or single-crossing type-spaces, single-plateaued type-spaces, etc., are non-convex type-spaces that satisfy the necessary and sufficient condition. As a corollary, we establish the equivalence of PLIC and almost everywhere IC on these type-spaces. To see the novelty of our analysis, note that the equivalence of PLIC and IC does not hold on such type-spaces (see Example 1 in Mishra et al. [39]), and that is why it is important to see the extent to which IC can be ensured by PLIC on such non-convex type-spaces. What our result says is that the said equivalence actually holds but only in an almost everywhere sense. Mishra et al. [39] consider the same problem for a particular type of mechanisms, called payments-only mechanisms and show that PLIC and IC are equivalent for such mechanisms. Our result complements their result by showing that one can drop payment-onlyness by requiring almost everywhere IC instead of IC.

Finally, we consider the problem of checking whether a given mechanism is IC or not on an arbitrary ordinal type-space. We show that to ensure IC of a mechanism, apart from checking the local types, one needs to check only the "boundary types". Thus, local types and boundary types form a minimal set of incentive constraints that imply full incentive compatibility. Since the boundary types have Lebesgue measure zero, this result reduces the complexity of checking whether a mechanism is IC or not in a considerable manner.

A salient feature of our result is that we deduce them for arbitrary notion of localness. To see the importance of this framework, note that the notion of localness is very subjective and may vary from person to person. Not only that, the standard notion of adjacent localness does not apply to multi-dimensional ordinal domains (See Kumar et al. [32] for details). Our general framework enables one to apply our results to any such scenario.

6.2 An example to illustrate our results

Let there be a finite set of alternatives A. For simplicity, assume $A = \{a, b, c\}$. We denote, for instance, by abc an ordinal preference where a is preferred to b, and b is preferred to c. Consider the following set of preferences: $\mathcal{D} = \{abc, bac, bca, cba\}$.[4] A social choice function (SCF) f on $\mathcal{D}$ is a mapping from $\mathcal{D}$ to A.[5] An SCF f is strategy-proof on a pair of preference (P, P') in $\mathcal{D}$ if, when an agent's true preference is P, it is not beneficial for him/her to misreport it as P', that is, if $f(P')$ is either equal to $f(P)$ or worse than $f(P)$ according to the preference P. An SCF f is strategy-proof on $\mathcal{D}$ if it is strategy-proof on every pair of preferences in $\mathcal{D}$.

Question 1. How can we check if a given SCF f on $\mathcal{D}$ is strategy-proof or not?

Answer. Clearly, checking the definition of strategy-proofness for every pair of preferences is time consuming (exponential in time). Kumar et al. [32] provides a simpler way (that works for arbitrary set of preferences) to resolve the problem. Let us call two preferences local if they differ minimally, that is, exactly one pair of alternatives change their relative ranking between the two preferences. For instance, abc and bac are local, bac and bca are local, etc. We call an SCF local strategy-proof if it is strategy-proof on every pair of local preferences.

Construct a graph, say G, with vertices as the elements of $\mathcal{D}$ where there is an edge between two preferences if and only if they are local. Kumar et al. [32] shows that if the graph G satisfies a property called Property L, then every local strategy-proof SCF on $\mathcal{D}$ will be strategy-proof.[6] It is known that the graph G satisfies Property L and hence to check whether f is strategy-proof or not one needs to check if f is strategy-proof on every pair of local preferences. This provides a significantly simpler way to check if an SCF is strategy-proof or not. See Sato [46] and Kumar et al. [32] for more details on the importance of strategy-proofness over pairs of local preferences. □

What we do in this paper is to consider the cardinal version of the problem. For a preference, say abc, we denote by T_{abc} all utility functions (valuations) that represent the ordinal preference abc, that is, utility of a is higher than utility of b and utility of b is higher than utility of c. Consider the type-space $T = T_{abc} \cup T_{bac} \cup T_{bca} \cup T_{cba}$. We will refer the elements of T as types. A mechanism on T is a pair (f, p) where $f : T \to A$ is an SCF and $p : T \to \mathbb{R}$ is a payment function. A mechanism $\mu = (f, p)$ is incentive compatible (IC) on a pair of types (t, t') in $T \times T$ if the net utility (after deducting the payment) at the type t cannot be strictly increased by misreporting the type as t', that is, if $t(f(t)) - p(t) \geq t(f(t')) - p(t')$. A mechanism is IC on T if it is IC on every pair of types in $T \times T$.

Question 2. How can we check if a given mechanism μ on T is IC or not?

[4] Such a set of preferences is called single-peaked with respect to the prior order $\prec := a \prec b \prec c$ on A.

[5] We consider one agent model in this paper which is without loss of generality for the problem we deal here.

[6] See Definition 6.4.2 for the description of Property L.

Answer. We answer this question by using the approach of local incentive compatibility (LIC). Let us call two types t and t' local if they represent the same preference or two preferences that are local in $\mathcal{D}$ (as we have defined in answering Question 1). We say that a mechanism μ on T is LIC if it is IC on every pair of local types in T, that is, μ is IC on every pair of types (t, t') where $t, t' \in T_P \cup T_{P'}$ for some local preferences P and P' in $\mathcal{D}$. We prove in this paper that if a mechanism on T is LIC, then it will be IC on T.

The novelty of our result is that it explores how the connection between local incentive compatibility and incentive compatibility extends from ordinal domains to cardinal ones. For a slightly more formal description of the result, let $\widehat{\mathcal{D}}$ be an arbitrary collection of preferences and let $\widehat{T}$ be the set of all types representing preferences in $\widehat{\mathcal{D}}$. We prove that the following two statements are equivalent:

(a) Every local strategy-proof SCF (as defined in answering Question 1) on $\widehat{\mathcal{D}}$ is strategy-proof.

(b) Every LIC mechanism (as defined in answering Question 2) on $\widehat{T}$ is IC.

Summarizing, not only we provide a significantly simpler way to check if a mechanism is IC or not, we establish the connection between ordinal domains and cardinal domains in the context of the implication of local strategy-proofness/incentive compatibility. □

Note that the preferences we have considered so far are strict and consequently the types represented by them are strict as well. In real life, preferences can be weak (that is, can have indifference), and hence the types. Since T consists of all (strict) types representing the set of strict preferences $\mathcal{D}$, a type representing a weak preference compatible with $\mathcal{D}$ will lie in the closure of T, which we denote by $cl(T)$.[7]

Question 3: How can we check if a given mechanism μ on $cl(T)$ is IC or not?

Answer. Once again we resort to the approach of local incentive compatibility (LIC). Since we need to take care of the indifference, we mildly strengthen LIC by introducing strong LIC. The strengthening is natural: we say that a mechanism on $cl(T)$ is strong LIC if it is IC on every pair (t, t') where $t, t' \in cl(T_P \cup T_{P'})$ for some local preferences P and P' in $\mathcal{D}$. Verbally speaking, for a strong LIC mechanism μ on $cl(T)$, in addition to μ being IC on every pair of local types (as defined in answering Question 2), μ is also IC on pairs of types admitting indifference that are arbitrarily close to some type belonging to the set of types representing the two local preferences in $\mathcal{D}$. Recall that the notion of LIC only applies to types that do not admit indifference, hence this mild strengthening of LIC is necessary and arises naturally.

Incidentally, strong LIC is not strong enough to imply IC. Mishra et al. [39] gives an example of a strong LIC mechanism on $cl(T)$ that fails to be IC (see Example 1 in Mishra et al. [39]). Therefore, we proceed to find the extent to which strong LIC ensures IC on $cl(T)$. Our finding is quite assuring: strong

[7] A weak preference is compatible with a strict preference if the former can be obtained by making some consecutively ranked alternatives indifferent in the latter.

LIC implies IC "almost everywhere". More precisely, we show that if μ is strong LIC then it is IC on every pair (t, t') where $t \in cl(T)$ and $t' \in T$. Note that this implies if μ is strong LIC then it may fail to be IC only on pairs (t, t') where t' admits indifference, that is, $t' \in cl(T) \setminus T$. Since the (Lebesgue) measure of $cl(T) \setminus T$ is zero, it justifies that μ is indeed almost everywhere IC.

Next, we further push the implication of strong LIC by showing that if μ is strong LIC, then it can only violate IC on a pair (t, t') where t', in addition to admitting indifference, belongs to the boundary of $cl(T)$ (see Remark 6.5.4 for details).

Finally, we provide a quite general result regarding the equivalence between strong LIC and almost everywhere IC. We characterize all ordinal type-spaces admitting indifference where the said equivalence holds. □

Remark 6.2.1 *It is worth mentioning that in this paper, we deal with arbitrary notions of localness that are formulated by means of arbitrary graphs. It is for simplicity we have considered the particular notion of localness (this is called adjacent localness in the literature) in this section.*

6.3 Model

We consider a one-agent model in this paper. This is without loss of generality for our analysis.[8]

Let A be a finite set of alternatives with $|A| = n$. For any given subset X of $\mathbb{R}^n$, by $cl(X)$ we denote the closure of X. A type t is a mapping from A to $\mathbb{R}$ that represents the valuation of each alternative in A. We view a type as an element of $\mathbb{R}^n$ (with an arbitrary but fixed indexation of the alternatives). A type t is strict if $t(a) \neq t(b)$ for all $a, b \in A$, otherwise it is a weak type.[9] By relative valuation of an alternative a with respect to another alternative b at a type t, we mean the number $t(a) - t(b)$. For two types t and t', we denote the line joining them by $[t, t']$.[10] A subset T of $\mathbb{R}^n$ is called a type-space. An allocation rule is a map $f : T \to A$ and a payment rule is a map $p : T \to \mathbb{R}$. A (direct) mechanism μ is a pair consisting of an allocation rule f and a payment rule p.

Definition 6.3.1 *A mechanism (f, p) is incentive compatible (IC) on a pair of types (t, s) if*

$$t(f(t)) - p(t) \geq t(f(s)) - p(s).$$

It is IC on a type-space T if it is IC on every pair of types $(t, s) \in T \times T$.

[8] All the results of this paper can be generalized to the case of more than one agent in a systematic manner (see Carroll [12], Mishra et al. [39], etc.).

[9] Note that strict types are not special cases of weak types.

[10] More formally, $[t, t'] = \{(1 - \alpha)t + \alpha t' \mid \alpha \in [0, 1]\}$.

The relation between LIC and IC is well-studied for social choice functions on ordinal domains (see Kumar et al. [32] for details); in this paper we extend this study for cardinal environments. In line with Kumar et al. [32], we consider a general notion of localness represented by means of a graph on an ordinal domain.

A preference on A is a weak linear order, that is, a complete and transitive binary relation on A. If it is additionally antisymmetric, it is called a strict preference, otherwise it is called a weak preference.[11] For a weak preference R, we denote its strict part by P and the indifference part by I. We denote the set of all preferences on A by $\mathcal{P}(A)$ and the set of all strict preferences on A by $\widehat{\mathcal{P}}(A)$. An ordinal domain $\mathcal{D}$ is a subset of $\mathcal{P}(A)$ and a strict ordinal domain $\widehat{\mathcal{D}}$ is a subset of $\widehat{\mathcal{P}}(A)$.

We deal with type-spaces that have some additional structure. For a type t and a preference R, we say that t represents R (or R represents t) if for all $a, b \in A$, aRb if and only if $t(a) \geq t(b)$. We denote the preference that a type t represents by $prfn(t)$, and the set of types that a preference R represents by $type(R)$.[12] Similarly, for an ordinal domain $\mathcal{D}$, we denote the set of all types that represent some preference in the domain by $type(\mathcal{D})$, that is, $type(\mathcal{D}) = \{t \in \mathbb{R}^{|A|} \mid prfn(t) \in \mathcal{D}\}$, and for a type-space T, the set of all preferences that are represented by some type in T by $prfn(T)$. A type-space T is called strict if $t(a) \neq t(b)$ for all $t \in T$ and all $a, b \in A$. We say that T is an ordinal type-space if $T = type(\mathcal{D})$ for some $\mathcal{D} \subseteq \mathcal{P}(A)$.

Let $\mathcal{D}$ be a domain. An ordinal environment is a pair $(\mathcal{D}, G)$, where $\mathcal{D}$ is an ordinal domain and $G = \langle \mathcal{D}, E \rangle$ is an (undirected) graph on $\mathcal{D}$. Two preferences in $\mathcal{D}$ are called G-local if they form an edge in G. A path $(P^1, \ldots, P^k)$ from P^1 to P^k is G-local if every two consecutive preferences in it are G-local. An ordinal environment $(\mathcal{D}, G)$ is called strict if $\mathcal{D}$ is a collection of strict preferences.

We introduce the notion of cardinal environment in a natural way. A cardinal environment is a pair (T, G), where T is a type-space and G is an undirected graph on $prfn(T)$. Two types t and t' in T are said to be G-local if $prfn(t) = prfn(t')$ or $prfn(t)$ and $prfn(t')$ are G-local. A cardinal environment (T, G) is called strict if T is a strict type-space.

A mechanism μ on a cardinal environment (T, G) is LIC if it is IC on every pair of G-local types, that is, μ is IC on $type(\{R, R'\}) \cap T$ for all G-local preferences R and R' in $prfn(T)$.

A social choice function (SCF) on an ordinal domain $\mathcal{D}$ is a mapping $g : \mathcal{D} \to A$. It is IC on a pair of preferences (R, R') if $g(R)Rg(R')$. An SCF $g : \mathcal{D} \to A$ is LIC on an ordinal environment $(\mathcal{D}, G)$ if it is IC on every pair of G-local preferences, and it is IC on $\mathcal{D}$ if it is IC on every pair of preferences in $\mathcal{D}$.

An ordinal environment $(\mathcal{D}, G)$ is called ordinal local global equivalent (OLGE) if every LIC SCF on

[11] Note that strict preferences are not special cases of weak preferences

[12] All the results of this paper also hold if we weaken the assumption on $type(R)$ to be the set of types that are bounded below (for instance, non-negative types) or bounded above. More formally, there exists a real number L such that $type(R) = \{t \mid t \text{ represents } R \text{ and } t(x) \geq L \text{ for every } x \in A\}$ or $type(R) = \{t \mid t \text{ represents } R \text{ and } t(x) \leq L \text{ for every } x \in A\}$.

$(\mathcal{D}, G)$ is IC on $\mathcal{D}$. Similarly, a cardinal environment (T, G) is called cardinal local global equivalent (CLGE) if every LIC mechanism on (T, G) is IC on T.

6.4 A characterization of strict CLGE type-spaces

Kumar et al. [32] provide a necessary and sufficient condition for a strict ordinal environment to be OLGE. We generalize their result for strict cardinal environments. We begin with defining some notions that are provided in Kumar et al. [32].[13]

Definition 6.4.1 *A pair of alternatives $\{a, b\}$ has a restoration in a G-local path $(P^1, \dots, P^k)$ if there exist $1 \leq r < s < t \leq k$ such that either $[aP^r b, bP^s a$, and $aP^t b]$ or $[bP^r a, aP^s b$, and $bP^t a]$.*

Definition 6.4.2 *Given a strict ordinal environment $(\widehat{\mathcal{D}}, G)$, for $P, P' \in \widehat{\mathcal{D}}$ and $a \in A$, we say that a G-local path π from P to P' satisfies the Lower Contour Set no-restoration property (Property L) with respect to a if for all $b \in L(a, P)$ where $L(a, P) = \{z \in A \mid aPz\}$ the path π has no $\{a, b\}$-restoration.*

The strict ordinal environment $(\widehat{\mathcal{D}}, G)$ satisfies Property L if for all distinct $P, P' \in \widehat{\mathcal{D}}$ and all $a \in A$, there exists a G-local path from P to P' satisfying Property L with respect to a.

The following theorem in Kumar et al. [32] provides a necessary and sufficient condition for a strict ordinal environment to be OLGE.

Theorem 6.4.3 *Kumar et al. [32] A strict ordinal environment $(\widehat{\mathcal{D}}, G)$ is OLGE if and only if it satisfies Property L.*

Our next theorem generalizes Theorem 6.4.3 for strict cardinal environments.

Theorem 6.4.4 *A strict ordinal environment $(\widehat{\mathcal{D}}, G)$ is OLGE if and only if the strict cardinal environment $\left(type(\widehat{\mathcal{D}}), G\right)$ is CLGE.*

The proof of this theorem is relegated to Appendix 6.7.1. It follows from Theorem 6.4.3 and Theorem 6.4.4 that a strict cardinal environment $\left(type(\widehat{\mathcal{D}}), G\right)$ is CLGE if and only if $(\widehat{\mathcal{D}}, G)$ satisfies Property L.

It is shown in Kumar et al. [32] that well-known multi-dimensional ordinal domains such as the separable domain and the multi-dimensional single-peaked domain are OLGE. It follows from Theorem 6.4.4 that the cardinal environments of these domains are CLGE.

[13] See Kumar et al. [32] for verbal (and detailed) explanations of these notions.

6.4.1 The case of adjacent localness

The main objective of this paper is to characterize ordinal type-spaces so that a restricted version of IC, called local IC (LIC), becomes equivalent to IC. Although we have presented results for a general notion of localness by means of graphs, we specifically deal with two particular kinds of localness that are practically important. A mechanism is point-wise LIC (PLIC) on a type-space T if for every $t \in T$, there exists an $\varepsilon > 0$ such that it is IC on (t, s) and (s, t) for every $s \in T$ with $||t - s|| < \varepsilon$.[14,15] For a given $\varepsilon > 0$, a mechanism on a type-space T is called ε-LIC if it is IC on every pair of types $(t, s) \in T \times T$ having (Euclidean) distance less than ε, that is, for all $t, s \in T$, $||t - s|| < \varepsilon$ implies the mechanism is IC on (t, s). A mechanism is called uniformly LIC (ULIC) if it is ε-locally IC for some $\varepsilon > 0$.

Fact 6.4.1 *(Carroll [12]) If a type-space T is convex, then every PLIC mechanism on T is IC on T.*

We explain the practical difference between PLIC and ULIC. According to PLIC, one has the freedom to choose different ε for different types, whereas in ULIC one has to chose the same ε for all types. If the infimum value of the ε's chosen for different types in case of PLIC is positive, then that value can be taken as the choice of the ε in ULIC, and consequently, the two notions will become equivalent. On other hand, if the said infimum is zero, then ULIC becomes slightly stronger than PLIC. This slight strengthening of PLIC widens its applicability. To see this, consider the situation where there are just two alternatives, say a and b, and the type-space is $T = \{t \in \mathbb{R}^2 \mid t(a) \neq t(b)\}$. Thus, T is disconnected and can be written as a union of two disjoint open spaces $T^1 = \{t \in \mathbb{R}^2 \mid t(a) < t(b)\}$ and $T^2 = \{t \in \mathbb{R}^2 \mid t(a) > t(b)\}$. In such situations, one can define neighborhoods of the points in T^1 such that none of them intersects T^2, and those of the points in T^2 such that none of them intersects T^1. Thus, PLIC with such neighborhoods does not put any constraint on a pair of types (s, t) where $s \in T^1$ and $t \in T^2$, and consequently, cannot imply IC on T. However, ULIC imposes IC on certain pairs of types (sufficiently close ones) that are on the other sides of the boundary of T^1 and T^2, and thereby retains the "possibility" of implying IC on T. In fact, as we show in this paper, ULIC indeed implies IC on T.

It is worth mentioning that ULIC is as useful as PLIC for practical purposes. In reality, if one wants to check (by means of a program/device) whether some mechanism is LIC or not, he/she can only check it for some given neighborhood of each type, not for a sequence of neighborhoods whose size converges to zero.

In this subsection, we deal with a specific case where the notion of localness for the ordinal environment is "adjacency" and that for the cardinal environment is a weaker version of ULIC which we call adjusted LIC.

[14] We denote the Euclidean norm of a vector $t \in \mathbb{R}^n$ by $||t||$.

[15] The notion of PLIC is introduced in Carroll [12].

For some $1 \leq k \leq n$, we denote the k-th ranked alternative of a strict preference P by $P(k)$. Two strict preferences P and P' are said to be adjacent local if they differ by the ranking of two consecutively ranked alternatives, that is, there is $1 \leq k < n$ such that $P(k) = P'(k+1)$, $P(k+1) = P'(k)$, and $P(l) = P'(l)$ for all $l \notin \{k, k+1\}$. We write G^{ad} when localness is defined by adjacency, that is, there is an edge between P and P' in G^{ad} if and only if P and P' are adjacent.

A mechanism μ is said to be adjusted LIC (ALIC) on a strict type-space $\hat{T}$ if (i) for every type t in $\hat{T}$, there is a neighborhood around t such that μ is IC on both (t, s) and (s, t) for all types s in that neighborhood, and (ii) for every type $\bar{t}$ that lies on the boundary of $\hat{T}$ (that is, in $cl(\hat{T}) \setminus \hat{T}$), there is a neighborhood of $\bar{t}$ such that μ is IC on every pair of strict types in that neighborhood.

Part (i) of the definition of ALIC is the same as PLIC. As we have explained in the beginning of this subsection, PLIC (with suitably chosen arbitrarily small neighborhoods) is unable to "spread" IC between two components of a disconnected type-space. Part (ii) of the definition of ALIC ensures the said spread in a natural way: it requires IC for types that are arbitrarily close but on the opposite sides of the boundary of the strict type-space. It does this by considering an arbitrary neighborhood of a type that lies on the boundary of $\hat{T}$ (and hence not in $\hat{T}$) and requiring IC for all pairs of types of $\hat{T}$ in this neighborhood.

Definition 6.4.5 *A mechanism μ on a strict type-space $\hat{T}$ is said to adjusted locally IC (ALIC) if it is PLIC and for every $\bar{t} \in cl(\hat{T}) \setminus \hat{T}$, there exists an open neighborhood $N(\bar{t}) \subseteq cl(\hat{T})$ of $\bar{t}$ such that for all $t', t'' \in N(\bar{t}) \cap \hat{T}$, μ is IC on (t', t'').*

The following corollary is obtained from Theorem 6.4.4.

Corollary 6.4.1 *If a strict ordinal environment $(\widehat{\mathcal{D}}, G^{ad})$ is OLGE, then a mechanism on type$(\widehat{\mathcal{D}})$ is IC if and only if it is ALIC.*

The proof of this corollary is relegated to Appendix 6.7.2.

Since ULIC implies ALIC by definition, it follows from Corollary 6.4.1 that if a strict ordinal environment satisfies OLGE with respect to adjacency localness, then ULIC and IC are equivalent on its cardinal version.

Corollary 6.4.2 *If a strict ordinal environment $(\widehat{\mathcal{D}}, G^{ad})$ is OLGE, then a mechanism on type$(\widehat{\mathcal{D}})$ is IC if and only if it is ULIC.*

A large class of strict ordinal environments of practical importance, such as single-peaked, single-dipped, single-crossing, etc., are OLGE with respect to the adjacency localness.[16] Corollary 6.4.1 implies that ALIC (and hence, ULIC) and IC are equivalent on their corresponding type-spaces. It should be noted that PLIC and IC are not equivalent on these type-spaces as the type-spaces are not connected.

[16]For the definition of these domains see Mishra et al. [39] and Carroll [12].

6.5 Ordinal domains admitting indifference

In this section, we consider ordinal environments admitting indifference where local structure is given by means of a graph over the strict preferences. Since in such an environment, LIC does *not* impose any restriction on weak preferences, it cannot ensure IC on the whole domain as well. So, we impose additional requirements on weak preferences in order to ensure IC.

For an ordinal domain $\mathcal{D}$, we denote its maximal strict ordinal subset by $strict(\mathcal{D})$, that is, $strict(\mathcal{D}) = \{P \in \mathcal{D} \mid P \text{ is a strict preference}\}$.

In order to generalize Theorem 6.4.4 for ordinal domains allowing indifferences, we introduce the notion of weak-compatibility. For a weak preference R, we say a strict preference $\hat{P}$ is compatible with R if aPb implies $a\hat{P}b$ for all $a, b \in A$. For instance, if $R = [ab]c[de]f$, then the following preferences are compatible with R: *abcdef*, *abcedf*, *bacdef*, and *bacedf*.[17] Weak compatibility says that for every weak preference R in $\mathcal{D}$, there exists a strict preference in $\mathcal{D}$ that is compatible with R.

Let $\mathcal{D}$ be an ordinal domain. Let G be an (undirected) graph on $strict(\mathcal{D})$. A mechanism μ is **strong LIC** on the cardinal environment $(type(\mathcal{D}), G)$ if it is LIC, and additionally IC on every pair of types $(\bar{t}, \hat{t})$ such that there is $P \in strict(\mathcal{D})$ so that $\hat{t}$ is a strict type in $type(P)$ and $\bar{t}$ is a weak type in $cl(type(P)) \cap type(\mathcal{D})$.

Now, we introduce the notion of almost everywhere IC. We use the following notation to ease the presentation. For a type-space T, we denote its maximal strict subset by $strict(T)$, that is, $strict(T) = \{t \in T \mid t(a) \neq t(b) \text{ for all distinct } a, b \in A\}$. A mechanism on a type-space T is **almost everywhere IC**, if it is IC on every pair of types in $T \times strict(T)$. Thus, an almost everywhere IC mechanism might fail to become IC on a pair of types $(t, \bar{t})$ only if $\bar{t}$ is a weak type that lies in T. Since the (Lebesgue) measure of the set $T \setminus strict(T)$ is zero, the measure (in the product space) of the set of pairs on which an almost everywhere IC mechanism may fail to be IC is also zero, which justifies the name. Note that almost everywhere IC implies strong LIC by definition.

As we have mentioned in Subsection 6.5.1, the equivalence between strong LIC and IC does not hold on ordinal type-spaces admitting indifference. Our next theorem establishes the extent to which IC is implied by strong LIC. It turns out that strong LIC implies IC "almost everywhere".

Theorem 6.5.1 *Let $\mathcal{D}$ be an ordinal domain. Then, the following two statements are equivalent.*

(i) Every strong LIC mechanism on the environment $(type(\mathcal{D}), G)$ is almost everywhere IC.

(ii) The domain $\mathcal{D}$ satisfies weak-compatibility and the environment $(strict(\mathcal{D}), G)$ is OLGE.

[17] By $[ab]c$, we denote a weak preference where a and b are indifferent, and are preferred to c. Similarly by abc, we denote a strict preference where a is preferred b, and b is preferred to c.

The proof of this theorem is relegated to Appendix 6.7.3.

6.5.1 Closure of type-spaces of strict ordinal domains

Let $\widehat{\mathcal{D}}$ be a strict ordinal domain and let $cl(type(\widehat{\mathcal{D}}))$ be the closure of $type(\widehat{\mathcal{D}})$. Since $cl(type(\widehat{\mathcal{D}}))$ is closed, Part (ii) of Definition 6.4.5 is vacuously true, and consequently, the notion of ALIC boils down to that of PLIC. However, Corollary 6.4.1 does not hold anymore, that is, PLIC does not imply IC on $cl(type(\widehat{\mathcal{D}}))$ (see Example 1 in Mishra et al. [39] for details). It is worth mentioning that PLIC implies strong LIC in adjacency environments.[18] Therefore the equivalence of strong LIC and IC cannot hold in such environments. The following corollary, which is obtained from Theorem 6.5.1, says that a version of Corollary 6.4.1 holds if we weaken IC by almost everywhere IC.

Corollary 6.5.1 *If a strict ordinal environment $(\widehat{\mathcal{D}}, G^{ad})$ is OLGE, then every PLIC mechanism on $cl(type(\widehat{\mathcal{D}}))$ is almost everywhere IC.*

The proof of this corollary is relegated to Appendix 6.7.4.

Corollary 6.5.1 applies to closure of single-peaked or single-crossing type-spaces or single-plateaued type-spaces.[19] Corollary 6.5.1 also applies to closure of single-peaked domain on a tree (Demange [20]).

6.5.2 LIC vs. IC for a given mechanism

Having a characterization of ordinal type-spaces such that the equivalence of strong LIC and almost everywhere IC holds, the next natural step is to look at the extent to which we can push the almost everywhere IC property on such type-spaces. As discussed earlier, an almost everywhere IC mechanism might fail to be IC only on the pairs $(t, \bar{t})$ where $\bar{t}$ is a weak type in the type-space.[20] We show that if $\bar{t}$ is a weak type lying in the *interior* of the type-space, then such mechanisms are bound to be IC on pairs $(t, \bar{t})$. Hence, we establish the fact that an almost everywhere IC mechanism might fail to be IC only on the pairs $(t, \bar{t})$ where $\bar{t}$ is a weak type lying on the boundary of the type-space, thereby modifying our previous result. We further identify the possible outcomes of a given mechanism at such a weak type $\bar{t}$ (that is, we identify possible values of $f(\bar{t})$) such that the mechanism is IC on pairs $(t, \bar{t})$.

For a set $T \subseteq \mathbb{R}^n$, by T^o we denote the interior of the set T, that is, $T^o = \{t \in T \mid \text{ there exists } \varepsilon > 0 \text{ such that } s \in T \text{ for every } s \text{ with } d(t, s) < \varepsilon\}$. By $\widehat{\partial} T$ we denote the points in T that lie on the boundary of T, that is, $\widehat{\partial} T = T \setminus T^o$.

[18] For a formal proof, see the proof of Corollary 6.5.1.

[19] For the definition of single-plateau domain see Berga [10].

[20] We use t here to denote a generic element of the type-space.

Theorem 6.5.2 *Let a strict ordinal environment $(\widehat{\mathcal{D}}, G^{ad})$ be OLGE and let $T = cl(type(\widehat{\mathcal{D}}))$. Suppose μ is an arbitrary PLIC mechanism on T. Then,*

(i) μ is IC on $T \times T^o$, and

(ii) μ is IC on $T \times \{\bar{t}\}$ for all $\bar{t} \in \widehat{\partial}T$ such that there exists $P \in \widehat{\mathcal{D}}$ with $\bar{t} \in cl(type(P))$ and $f(\bar{t})Pz$ for every z with $\bar{t}(f(\bar{t})) = \bar{t}(z)$.

The proof of this theorem is relegated to Appendix 6.7.5.

Remark 6.5.3 *For simplicity we present Theorem 6.5.2 for adjacent localness and PLIC mechanisms but it can be suitably formulated for arbitrary notion of localness and strong LIC mechanisms.*

Remark 6.5.4 *Example 1 in Mishra et al. [39] presents a single-peaked type-space $cl(T)$ (as described in Section 6.2) where they construct a PLIC mechanism that fails to be IC. It follows from part (i) of Theorem 6.5.2 in our paper that a PLIC mechanism on such a type-space T can violate IC only on types lying in $T \times \widehat{\partial}T$. For such a violation on any pair of types (t, t'), it must be the case that t' lie in either $type(c[ba])$ or $type(a[bc])$.[21] It further follows from part(ii) of Theorem 6.5.2 that the outcome at type t' must be either a if $t' \in type(c[ba])$ or c if $t' \in type(a[bc])$. Thus, the counter example (Example 1 in Mishra et al. [39]) was the only way (upto symmetry) to construct a PLIC mechanism that violates IC.*

6.6 Discussion: a stronger version of Theorem 6.4.4

The statement of Theorem 6.4.4 requires *all* types to be present for each ordinal preference. Since requiring all types representing an ordinal preference might be restrictive for practical applications, we extract out the types, the presence of which is sufficient for the proof of Theorem 6.4.4. The objective is to emphasize that Theorem 6.4.4 holds for much weaker environments rather than just strict ordinal type-spaces.

We introduce a property called $\hat{L}$ for a cardinal environment and show that any cardinal environment satisfying this property is CLGE. Property $\hat{L}$ is a suitable adaptation of Property L for cardinal environments. Thus, instead of providing a sufficient condition on an ordinal domain to ensure CLGE on its corresponding type-space, we provide a sufficient condition on a strict type-space directly.

We say an alternative a overtakes another alternative b from a strict preference P to another strict preference P' if bPa and $aP'b$. Recall that Property L says that between any two strict preferences P and P' and any alternative a, there exists a local path π from P to P' such that for all $b \in L(a, P)$ the path π has no $\{a, b\}$-restoration. Below, we introduce the cardinal version of Property L, which we call Property $\hat{L}$.

[21]Recall that by $c[ba]$, we denote a weak preference where c is preferred to both a and b, and a and b are indifferent.

Definition 6.6.1 *A strict cardinal environment $(\hat{T}, G)$ satisfies Property $\hat{L}$ if for all $t, t' \in \hat{T}$ and all $a \in A$, there exists a G-local path $(P^1, \ldots, P^k)$ in $prfn(\hat{T})$ with $P^1 \in prfn(t)$ and $P^k \in prfn(t')$ satisfying Property L with respect to a such that for all $l < k$ and all $t^{l+1} \in type(P^{l+1}) \cap \hat{T}$, there exists $t^l \in type(P^l) \cap \hat{T}$ such that*

(i) $t^l(a) - t^l(x) > t^{l+1}(a) - t^{l+1}(x)$ for all x that a does not overtake from P^l to P^{l+1}, and

(ii) $t^l(a) - t^l(y) \geq t(a) - t(y)$ for all y that a overtakes from P^l to P^{l+1}.

Theorem 6.6.2 *A strict cardinal environment $(\hat{T}, G)$ is CLGE if $\hat{T}$ satisfies Property $\hat{L}$.*

The proof of this theorem follows by using similar arguments as in the proof of Theorem 6.4.4.

6.7 Appendix

Before we begin the proofs, we note some facts below regarding the IC property. Facts 6.7.1 and 6.7.2 follow from the definition of IC property.

Fact 6.7.1 *Suppose that a mechanism (f, p) is IC on both pairs of types (t, s) and (s, t). Suppose further that $f(t) = f(s)$. Then, $p(t) = p(s)$.*

Fact 6.7.2 *Suppose that a mechanism (f, p) is IC on both pairs of types (t, s) and (s, t). Suppose further that the relative valuation of $f(t)$ with respect to some alternative a is increased from t to s, that is, $s(f(t)) - s(a) > t(f(t)) - t(a)$. Then, $f(s) \neq a$.*

Our next fact provides a sufficient condition for a mechanism to be IC on a pair of types based on its IC property over a sequence of types. This fact was first used in Kumar and Roy [31]; see Appendix A.1 of the paper for a formal proof.

Fact 6.7.3 *A mechanism $\mu = (f, p)$ on a type-space T is IC on a pair of types (t, t') if there is a finite sequence of types $(t = t^1, \ldots, t^k = t')$ in T such that for all $l < k$,*
(i) μ is IC on (t^l, t^{l+1}), and
(ii) $t^1(f(t^{l+1})) - t^1(f(t^l)) \leq t^l(f(t^{l+1})) - t^l(f(t^l))$.

6.7.1 Proof of Theorem 6.4.4

Proof: **If part:** The proof of the if part is rather straightforward; we provide it for the sake of completeness. Let $\big(type(\widehat{\mathcal{D}}), G\big)$ be a CLGE strict cardinal environment. We show that $(\widehat{\mathcal{D}}, G)$ is OLGE. Suppose not. Then there exists an SCF $\phi : \widehat{\mathcal{D}} \to A$ that is LIC on $(\widehat{\mathcal{D}}, G)$ but fails to be IC on $\widehat{\mathcal{D}}$. Therefore, there exists

$P, P' \in \widehat{\mathcal{D}}$ and $x, y \in A$ such that $\phi(P') = x, y = \phi(P)$, and xPy. We complete the proof of the if part by constructing a mechanism (f, p) that is LIC on $\big(type(\widehat{\mathcal{D}}), G\big)$ but fails to be IC on $type(\widehat{\mathcal{D}})$, which will lead to a contradiction to the fact that $\big(type(\widehat{\mathcal{D}}), G\big)$ is CLGE.

Define $f(s) = \phi(prfn(s))$ and $p(s) = 0$ for all $s \in type(\widehat{\mathcal{D}})$. The fact that (f, p) is LIC on $\big(type(\widehat{\mathcal{D}}), G\big)$ follows straightforwardly from the definition of G-local types and the fact that ϕ is LIC on $(\widehat{\mathcal{D}}, G)$. Fix any $t \in type(P)$ and $t' \in type(P')$. By the definition of f, $f(t) = y$ and $f(t') = x$. This, together with the facts that xPy and $p(t) = 0 = p(t')$, implies that $t(f(t)) - p(t) < t(f(t')) - p(t')$, and hence, it follows that (f, p) is not IC on (t, t'), a contradiction. This completes the proof of the if part of the theorem.

Only if part: Let $(\widehat{\mathcal{D}}, G)$ be an OLGE strict ordinal environment. We show that the environment $\big(type(\widehat{\mathcal{D}}), G\big)$ is CLGE. Let (f, p) be an LIC mechanism on $\big(type(\widehat{\mathcal{D}}), G\big)$. We need to show that (f, p) is IC on $type(\widehat{\mathcal{D}})$.

Consider arbitrary $t, t' \in type(\widehat{\mathcal{D}})$. By the definition of $type(\widehat{\mathcal{D}})$, there are $P, P' \in \widehat{\mathcal{D}}$ such that $t \in type(P)$ and $t' \in type(P')$. Without loss of generality, let us assume that the alternatives in A are indexed as $a_1, \ldots, a_n$ such that $a_1 P a_2 P \ldots P a_n$. Suppose $f(t') = a_j$ for some $j \in \{1, \ldots, n\}$. We proceed to show that (f, p) is IC on (t, t').

If $P = P'$, then t and t' are G-local types, and hence the proof follows by the assumption of the theorem that (f, p) is IC on every pair of G-local types. So, assume $P \neq P'$. Since $(\widehat{\mathcal{D}}, G)$ is OLGE, by Theorem 6.4.3, it satisfies Property L. This, together with the fact that $f(t') = a_j$, implies that there exists a G-local path $\pi = (P^1, \ldots, P^k)$ from P to P' satisfying Property L with respect to a_j. Since π has no (a_j, x)-restoration for all $x \in \{a_{j+1}, \ldots, a_n\}$, it follows that
$L(a_j, P^{l+1}) \setminus \{a_1, \ldots, a_{j-1}\} \subseteq L(a_j, P^l) \setminus \{a_1, \ldots, a_{j-1}\}$ for all $l \in \{1, \ldots, k-1\}$.

Let $l_1 \geq 1$ be the minimum number with the property that for each $l \in \{l_1 + 1, \ldots, k\}$ there exist $t^l \in type(P^l)$ such that $f(t^l) = a_j$. Note that such an l_1 will always exist as $f(t') = a_j$ and $t' \in type(P^k)$.

Claim 1. There exists $\tilde{t}^1 \in type(P^{l_1+1})$ such that $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$.

Proof of Claim 1. By the definition of l_1, for each $l \in \{l_1 + 1, \ldots, k\}$ there exist $t^l \in type(P^l)$ such that $f(t^l) = a_j$. Now we show that $p(t^l) = p(t')$ for each $l \in \{l_1 + 1, \ldots, k\}$. First we show that $p(t^{k-1}) = p(t')$. Since P^k and P^{k-1} are G-local preferences and (f, p) is LIC on $\big(type(\widehat{\mathcal{D}}), G\big)$, (f, p) is IC on $type(\{P^k, P^{k-1}\})$, and hence, is IC on (t^{k-1}, t') and (t', t^{k-1}). Since $f(t^{k-1}) = f(t')$, the fact that $p(t^{k-1}) = p(t')$ now follows from Fact 6.7.1. By using this argument repeatedly, it follows that for all $l \in \{l_1 + 1, \ldots, k\}$, $p(t^l) = p(t')$. Set $t^{l_1+1} = \tilde{t}^1$. This completes the proof of the claim. $\square$

Also note that since $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$, by the definition of incentive compatibility it follows that (f, p) is IC on $(\tilde{t}^1, t')$.

If $l_1 = 1$, then by Claim 1, there exists $\tilde{t}^1$ in $type(P^2)$ such that $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$. Since $t \in type(P)$, $\tilde{t}^1 \in type(P^2)$, and (f, p) is IC on $type(\{P = P^1, P^2\})$, it follows that

$t(f(t)) - p(t) \geq t(f(\tilde{t}^1)) - p(\tilde{t}^1)$. Since $f(\tilde{t}^1) = f(t')$ and $p(\tilde{t}^1) = p(t')$, this implies $t(f(t)) - p(t) \geq t(f(t')) - p(t')$, which shows that (f, p) is IC on (t, t') thereby completing the proof of the Theorem.

Suppose $l_1 > 1$. Then by Claim 1 there exists $\tilde{t}^1 \in type(P^{l_1+1})$ such that $f(\tilde{t}^1) = f(t') = a_j$ and $p(\tilde{t}^1) = p(t')$. We proceed to Step 1.

Step 1. In this step, we show that if $l_1 > 1$, then there exists $\tilde{t}^2 \in type(P^{l_1})$ such that $f(\tilde{t}^2) \in \{a_1, \ldots, a_{j-1}\}$ and $\tilde{t}^2(f(\tilde{t}^1)) - \tilde{t}^2(f(\tilde{t}^2)) \geq t(f(\tilde{t}^1)) - t(f(\tilde{t}^2))$.

Since $l_1 > 1$, by the definition of l_1, we must have $f(s) \neq a_j$ for all $s \in type(P^{l_1})$.[22]

Claim 2. $L(a_j, P^{l_1+1}) \nsubseteq L(a_j, P^{l_1})$.

Proof of Claim 2. Assume for contradiction that $L(a_j, P^{l_1+1}) \subseteq L(a_j, P^{l_1})$. Consider a type $t^{l_1} \in type(P^{l_1})$ such that $t^{l_1}(a_j) - t^{l_1}(x) > \tilde{t}^1(a_j) - \tilde{t}^1(x)$ for all $x \in A \setminus \{a_j\}$. Such a type can be found since $L(a_j, P^{l_1+1}) \subseteq L(a_j, P^{l_1})$. Since P^{l_1} and P^{l_1+1} are G-local preferences, (f, p) is IC on $type(P^{l_1}, P^{l_1+1})$, and hence, is IC on $(t^{l_1}, \tilde{t}^1)$ and $(\tilde{t}^1, t^{l_1})$. This, together with the facts that $t^{l_1}(a_j) - t^{l_1}(x) > \tilde{t}^1(a_j) - \tilde{t}^1(x)$ for all $x \in A \setminus \{a_j\}$ and $f(\tilde{t}^1) = a_j$, implies that $f(t^{l_1}) = a_j$. This leads to a contradiction to the fact that $f(s) \neq a_j$ for all $s \in type(P^{l_1})$. This completes the proof of the claim. $\square$

Since $L(a_j, P^{l_1+1}) \nsubseteq L(a_j, P^{l_1})$, it must be that $a_l P^{l_1} a_j$ and $a_j P^{l_1+1} a_l$ for some $l \in \{1, \ldots, n\}$. Let $B_1 = \{a_l \mid a_l P^{l_1} a_j \text{ and } a_j P^{l_1+1} a_l\}$. Note that since $L(a_j, P^{l_1+1}) \setminus \{a_1, \ldots, a_{j-1}\} \subseteq L(a_j, P^{l_1}) \setminus \{a_1, \ldots, a_{j-1}\}$, we must have $B_1 \subseteq \{a_1, \ldots, a_{j-1}\}$. Choose $\tilde{t}^2 \in type(P^{l_1})$ such that

(i) $\tilde{t}^2(a_j) - \tilde{t}^2(x) > \tilde{t}^1(a_j) - \tilde{t}^1(x)$ for all $x \in A \setminus B_1$, and

(ii) $\tilde{t}^2(a_j) - \tilde{t}^2(y) \geq t(a_j) - t(y)$ for all $y \in B_1$.

We explain how such a choice of $\tilde{t}^2$ is possible. Note that (i) implies that the relative valuation of a_j with respect to each alternative in $A \setminus B_1$ is increased from $\tilde{t}^1$ to $\tilde{t}^2$. This can be assured by the fact that there is no $z \in A \setminus B_1$ such that $z P^{l_1} a_j$ and $a_j P^{l_1+1} z$. Similarly, (ii) can be assured by means of the fact that the relative ordering of a_j with any alternative in B_1 is the same in both P and P^{l_1}.

Since (f, p) is IC on $type(\{P^{l_1}, P^{l_1+1}\})$ and $f(s) \neq a_j$ for all $s \in type(P^{l_1})$, (i) implies that $f(\tilde{t}^2) \in B_1$. This, together with (ii) and the fact that $a_j = f(\tilde{t}^1)$, implies that $\tilde{t}^2(f(\tilde{t}^1)) - \tilde{t}^2(f(\tilde{t}^2)) \geq t(f(\tilde{t}^1)) - t(f(\tilde{t}^2))$. This completes Step 1. $\square$

Note that since $\tilde{t}^2 \in type(P^{l_1})$, $\tilde{t}^1 \in type(P^{l_1+1})$ and P^{l_1} and P^{l_1+1} are G-local preferences, (f, p) is IC on $(\tilde{t}^2, \tilde{t}^1)$.

We now complete the proof of the Theorem by using Step 1 recursively. Let $f(\tilde{t}^2) = b_2$, where $b_2 \in \{a_1, \ldots, a_{j-1}\}$. Since $(\widehat{\mathcal{D}}, G)$ is OLGE, by Theorem 6.4.3, it satisfies Property L. This, together with

[22] Otherwise $l_1 - 1$ would satisfy the requirement of the definition of l_1 contradicting the fact that l_1 is the minimum number satisfying this requirement.

the fact that $f(\tilde{t}^2) = b_2$, implies there exists a G-local path $(\hat{P}^1, \ldots, \hat{P}^r)$ from P to P^{l_1} satisfying Property L with respect to b_2.

Let $l_2 \geq 1$ be the minimum number with the property that for each $l \in \{l_2 + 1, \ldots, r\}$ there exist $t^l \in type(\hat{P}^l)$ such that $f(t^l) = b_2$. Note that such an l_2 will always exist as $f(\tilde{t}^2) = b_2$ and $\tilde{t}^2 \in type(\hat{P}^r)$.

Using similar logic as in Claim 1, it follows that there exists $\tilde{t}^3 \in type(\hat{P}^{l_2+1})$ such that $f(\tilde{t}^3) = f(\tilde{t}^2) = b_2$ and $p(\tilde{t}^3) = p(\tilde{t}^2)$. This, together with the definition of incentive compatibility implies that (f, p) is IC on $(\tilde{t}^3, \tilde{t}^2)$.

If $l_2 = 1$, we have $\tilde{t}^3 \in type(\hat{P}^2)$ such that $f(\tilde{t}^3) = f(\tilde{t}^2) = b_2$ and $p(\tilde{t}^3) = p(\tilde{t}^2)$. Since $t \in type(P)$, $\tilde{t}^3 \in type(\hat{P}^2)$, and (f, p) is IC on $type(\{P = \hat{P}^1, \hat{P}^2\})$, it follows that $t(f(t)) - p(t) \geq t(f(\tilde{t}^3)) - p(\tilde{t}^3)$. Since $f(\tilde{t}^3) = f(\tilde{t}^2)$ and $p(\tilde{t}^3) = p(\tilde{t}^2)$, this implies $t(f(t)) - p(t) \geq t(f(\tilde{t}^2)) - p(\tilde{t}^2)$, which shows that (f, p) is IC on $(t, \tilde{t}^2)$. Hence we have a finite sequence of types $(t, \tilde{t}^2, \tilde{t}^1, t')$ such that (f, p) is IC on $(t, \tilde{t}^2)$, $(\tilde{t}^2, \tilde{t}^1)$ and $(\tilde{t}^1, t')$. Therefore the sequence of types $(t, \tilde{t}^2, \tilde{t}^1, t')$ satisfies condition (i) of Fact 6.7.3. Further note that since $f(\tilde{t}^1) = f(t')$, the fact that $\tilde{t}^1(f(t')) - \tilde{t}^1(f(\tilde{t}^1)) \geq t(f(t')) - t(f(\tilde{t}^1))$ is trivially satisfied (both sides being 0). This, together with step 1 implies that we have

(i) $\tilde{t}^2(f(\tilde{t}^1)) - \tilde{t}^2(f(\tilde{t}^2)) \geq t(f(\tilde{t}^1)) - t(f(\tilde{t}^2))$, and

(ii) $\tilde{t}^1(f(t')) - \tilde{t}^1(f(\tilde{t}^1)) \geq t(f(t')) - t)f(\tilde{t}^1)$.

Hence the sequence of types $(t, \tilde{t}^2, \tilde{t}^1, t')$ satisfies condition (ii) of Fact 6.7.3. Therefore, by Fact 6.7.3, it follows that (f, p) is IC on (t, t') thereby completing the proof of the Theorem.

Suppose $l_2 > 1$. Then, by using similar logic as in Step 1, there exists $\tilde{t}^4 \in type(\hat{P}^{l_2})$ such that $f(\tilde{t}^4) \in U(b_2, P)$ where $U(b_2, P) = \{z \in A \mid zPb_2\}$ and $\tilde{t}^4(f(\tilde{t}^3) - \tilde{t}^4(f(\tilde{t}^4)) \geq t(f(\tilde{t}^3)) - t(f(\tilde{t}^4))$.

Continuing in this manner, either we end up showing (f, p) is IC on (t, t') or we can construct a finite sequence $(\tilde{t}^{2u}, \tilde{t}^{2u-1}, \ldots, \tilde{t}^4, \tilde{t}^3, \tilde{t}^2, \tilde{t}^1)$ such that

(i) t^{2j} and t^{2j-1} are G-local types for all $j \in \{1, \ldots, u\}$,

(ii) $f(\tilde{t}^{2j}) = b_{2j}$ for all $j \in \{1, \ldots, u\}$,

(iii) $b_{2(j+1)}Pb_{2j}$ for all $j \in \{1, \ldots, u-1\}$,

(iv) $f(\tilde{t}^{2j+1}) = f(\tilde{t}^{2j})$ and $p(\tilde{t}^{2j+1}) = p(\tilde{t}^{2j})$ for all $j \in \{1, \ldots, u-1\}$, and

(v) $\tilde{t}^{2j}(f(\tilde{t}^{2j-1})) - \tilde{t}^{2j}(f(\tilde{t}^{2j})) \geq t(f(\tilde{t}^{2j-1})) - t(f(\tilde{t}^{2j}))$ for all $j \in \{1, \ldots, u\}$.

Since $b_{2(j+1)}Pb_{2j}$ for all $j \in \{1, \ldots, u-1\}$ and the process has not terminated, it must be that $f(\tilde{t}^{2u}) = a_1 = r_1(P)$. Let $\tilde{t}^{2u} \in type(P^{l_u})$ for some $P^{l_u} \in \widehat{\mathcal{D}}$. Since $(\widehat{\mathcal{D}}, G)$ is OLGE and $f(\tilde{t}^{2u}) = a_1$, by Theorem 6.4.3, there must exist a G-local path $\pi = (\bar{P}^1, \ldots, \bar{P}^w)$ from P to P^{l_u} satisfying Property L with respect to a_1. This means $L(a_1, \bar{P}^{l+1}) \subseteq L(a_1, \bar{P}^l)$ for all $l \in \{1, \ldots, w-1\}$.

Claim 3. (f, p) is IC on $(t, \tilde{t}^{2u})$.

Proof of Claim 3. Using similar logic as in the proof of Claim 2 (by using w in place of $l_1 + 1$, $w - 1$ in place of l_1 and a_1 in place of a_j), it follow that there exists $t^{w-1} \in t(\bar{P}^{w-1})$ such that $f(t^{w-1}) = a_1 = f(\tilde{t}^{2u})$.

The fact that $p(t^{w-1}) = p(\tilde{t}^{2u})$ now follows from Fact 6.7.1.

By using this fact repeatedly, it follows that for all $l \in \{2, \ldots, w-2\}$, there exists $t^l \in type(\bar{P}^l)$ such that $f(t^l) = a_1 = f(\tilde{t}^{2u})$ and $p(t^l) = p(\tilde{t}^{2u})$, which in particular means that there exists $t^2 \in type(\bar{P}^2)$ such that $f(t^2) = f(\tilde{t}^{2u}) = a_1$ and $p(t^2) = p(\tilde{t}^{2u})$. Since $t \in type(P)$, $t^2 \in type(\bar{P}^2)$, and (f, p) is IC on $type(\{P = \bar{P}^1, \bar{P}^2\})$, it follows that $t(f(t)) - p(t) \geq t(f(t^2)) - p(t^2)$. Since $f(t^2) = f(\tilde{t}^{2u}) = a_1$ and $p(t^2) = p(\tilde{t}^{2u})$, this implies $t(f(t)) - p(t) \geq t(f(\tilde{t}^{2u})) - p(\tilde{t}^{2u})$, which shows that (f, p) is IC on $(t, \tilde{t}^{2u})$ thus proving the claim. □

Now we show that (f, p) is IC on (t, t'). Consider the sequence of types $(t, \tilde{t}^{2u}, \tilde{t}^{2u-1}, \ldots, \tilde{t}^4, \tilde{t}^3, \tilde{t}^2, \tilde{t}^1, t')$. By construction, (f, p) is IC on each pair of consecutive types, and hence, the sequence of types $(t, \tilde{t}^{2u}, \tilde{t}^{u-1}, \ldots, \tilde{t}^4, \tilde{t}^3, \tilde{t}^2, \tilde{t}^1, t')$ satisfies condition (i) of Fact 6.7.3. Moreover, since $f(\tilde{t}^{2j+1}) = f(\tilde{t}^{2j})$ for all $j \in \{1, \ldots, u-1\}$, it follows that $\tilde{t}^{2j+1}(f(\tilde{t}^{2j})) - \tilde{t}^{2j+1}(f(\tilde{t}^{2j+1})) = 0 \geq 0 = t(f(\tilde{t}^{2j})) - t(f(\tilde{t}^{2j+1}))$ for all $j \in \{1, \ldots, u-1\}$. Similarly, since $f(\tilde{t}^1) = f(t')$, it follows that $\tilde{t}^1(f(t')) - \tilde{t}^1(f(\tilde{t}^1)) = 0 \geq 0 = t(f(t')) - t(f(\tilde{t}^1))$. These, together with the fact that $\tilde{t}^{2j}(f(\tilde{t}^{2j-1})) - \tilde{t}^{2j}(f(\tilde{t}^{2j})) \geq t(f(\tilde{t}^{2j-1})) - t(f(\tilde{t}^{2j}))$ for all $j \in \{1, \ldots, u\}$, imply that the sequence of types $(t, \tilde{t}^{2u}, \tilde{t}^{2u-1}, \ldots, \tilde{t}^4, \tilde{t}^3, \tilde{t}^2, \tilde{t}^1, t')$ satisfies condition (ii) of Fact 6.7.3. Hence, by Fact 6.7.3, it follows that (f, p) is IC on (t, t'), which completes the proof of the only if part of Theorem.

■

6.7.2 Proof of Corollary 6.4.1

Proof: IC implies ALIC by definition, we show the converse. Consider an ALIC mechanism $\mu = (f, p)$ on $type(\widehat{\mathcal{D}})$. We show that μ is IC. It follows from Theorem 6.4.4 that the cardinal environment $(type(\widehat{\mathcal{D}}), G^{ad})$ is CLGE. Therefore, to show that μ is IC, it is sufficient to show that it is IC on any pair of G^{ad}-local types. Consider two G^{ad}-local types t and t', and the line $[t, t']$. Let P and P' (not necessarily distinct), respectively, be the adjacent preferences that t and t' represent. Since P and P' are adjacent, there will be at most one point in the line $[t, t']$ that does not lie in $type(\widehat{\mathcal{D}})$. Such a point (or type) will lie on the boundary of $type(P)$ and $type(P')$ and will represent some weak preference and hence outside the domain $\widehat{\mathcal{D}}$. Let t^* be that point (if it exists). By means of ALIC, we can choose $\bar{t}$ and $\hat{t}$ in $type(\widehat{\mathcal{D}}) \cap [t, t']$ such that μ is IC on $(\bar{t}, \hat{t})$ and $(\hat{t}, \bar{t})$, and the lines $[t, \bar{t}]$ and $[\hat{t}, t']$ lie in $type(\widehat{\mathcal{D}})$. These, together with Fact 6.4.1 and the fact that implications of ALIC and PLIC are the same in the interior of a type-space, implies that μ is IC on any two types of the sequence $(t, \bar{t}, \hat{t}, t')$. Let us rename the sequence $(t, \bar{t}, \hat{t}, t')$ as $(t = t^1, \bar{t} = t^2, \hat{t} = t^3, t' = t^4)$. Note that the sequence (t^1, t^2, t^3, t^4) satisfies Condition (i) of Fact 6.7.3 because of the fact that μ is IC on any two consecutive types of the sequence. Let $f(t^i) = a^i$ for each $i \in \{1, 2, 3, 4\}$.

Claim. (t^1, t^2, t^3, t^4) satisfies Condition (ii) of Fact 6.7.3.

Proof of the claim. Since μ is IC on any two consecutive types of the sequence (t^1, t^2, t^3, t^4),

$$t^i(a^i) - t^i(a^{i+1}) \geq t^{i+1}(a^i) - t^{i+1}(a^{i+1}) \text{ for every } i \in \{1, 2, 3\}. \tag{6.1}$$

We need to show that $t^i(a^{i+1}) - t^i(a^i) \geq t^1(a^{i+1}) - t^1(a^i)$ for all $i \in \{1, 2, 3\}$ which would then establish that (t^1, t^2, t^3, t^4) satisfies Condition (ii) of Fact 6.7.3. Since $(t = t^1, \ldots, t^4 = t')$ is a finite sequence of types in $[t, t']$, there exists $0 = \beta_1 < \beta_2 < \beta_3 < \beta_4 = 1$ such that $t^i = (1 - \beta_i)t^1 + \beta_i t^4$ for all $i \in \{1, 2, 3\}$. Fix any $i \in \{1, 2, 3\}$. By (6.1), we have

$$t^i(a^i) - t^i(a^{i+1}) \geq t^{i+1}(a^i) - t^{i+1}(a^{i+1}). \tag{6.2}$$

Substituting $t^i = (1 - \beta_i)t^1 + \beta_i t^4$ and $t^{i+1} = (1 - \beta_{i+1})t^1 + \beta_{i+1} t^4$ in (6.2), we get

$$\begin{aligned} & t^1(a^i) - t^1(a^{i+1}) + \beta_i[(t^4(a^i) - t^4(a^{i+1})) - (t^1(a^i) - t^1(a^{i+1}))] \\ \geq\ & t^1(a^i) - t^1(a^{i+1}) + \beta_{i+1}[(t^4(a^i) - t^4(a^{i+1})) - (t^1(a^i) - t^1(a^{i+1}))]. \end{aligned} \tag{6.3}$$

Since $\beta_i < \beta_{i+1}$, from (6.3) we conclude that

$$(t^4(a^i) - t^4(a^{i+1})) - (t^1(a^i) - t^1(a^{i+1})) \leq 0. \tag{6.4}$$

Substituting $t^i = (1 - \beta_i)t^1 + \beta_i t^4$ in $t^i(a^{i+1}) - t^i(a^i)$, we get

$$t^i(a^{i+1}) - t^i(a^i) = (1 - \beta_i)(t^1(a^{i+1}) - t^1(a^i)) + \beta_i(t^4(a^{i+1}) - t^4(a^i)). \tag{6.5}$$

Since $\beta_i \geq 0$, (6.4) and (6.5) together imply $t^1(a^{i+1}) - t^1(a^i) \leq t^i(a^{i+1}) - t^i(a^i)$. This implies that (t^1, t^2, t^3, t^4) satisfies Condition (ii) of Fact 6.7.3 which completes the proof of the claim. □

Therefore, by applying Fact 6.7.3 to the sequence (t^1, t^2, t^3, t^4), we obtain that μ is IC on (t, t'). This completes the proof of the corollary. ■

6.7.3 Proof of Theorem 6.5.1

Proof: [**Proof of (i) implies (ii)**] Suppose (i) holds but (ii) does not hold. Since (ii) does not hold, either the domain $\mathcal{D}$ does not satisfy weak-compatibility or the environment $(strict(\mathcal{D}), G)$ is not OLGE. We distinguish these two cases.

Case A. Suppose that the domain $\mathcal{D}$ does not satisfy weak-compatibility.

Since $\mathcal{D}$ does not satisfy weak-compatibility, there exists a weak preference $R \in \mathcal{D}$ for which there is no strict preference in $strict(\mathcal{D})$ that is compatible with R. First, note that R cannot be indifferent over all the alternatives in A, that is, it is not possible that aIb for all $a, b \in A$. This is because, if R is so, then any strict preference in $strict(\mathcal{D})$ (recall that $strict(\mathcal{D}) \neq \emptyset$ by our assumption) is compatible with R. So, let us assume that aPb for some $a, b \in A$.

Consider the mechanism $\mu = (f, p)$ such that $f(t) = b$ for all $t \in type(R)$ and $f(t) = a$ for all other types, and $p(t) = 0$ for all $t \in type(\mathcal{D})$. We claim that this mechanism is strong LIC but not almost everywhere IC.

Since both f and p are constant (equal to a and 0, respectively) over all strict types in $type(\mathcal{D})$, the mechanism μ is LIC. To see that μ is strong LIC, consider any pair of types $(\bar{t}, \hat{t})$ where $\hat{t}$ is a strict type in $type(P^*)$ and $\bar{t}$ is a weak type in $cl(type(P^*)) \cap type(\mathcal{D})$ for some $P^* \in strict(\mathcal{D})$. Since there is no strict preference in $strict(\mathcal{D})$ that is compatible with R, we have $\bar{t} \notin type(R)$.

By the construction of f, this means $f(\bar{t}) = f(\hat{t}) = a$. This, together with the fact that the payment function is constant everywhere (equal to 0), implies that μ is IC on $(\bar{t}, \hat{t})$. Therefore, μ is strong LIC. Finally, we show that μ is not almost everywhere IC. Consider any type $\bar{\bar{t}}$ in $type(R)$ and any strict type $\hat{\hat{t}}$ in $type(\mathcal{D})$. By the definition of f, we have $f(\bar{\bar{t}}) = b$ and $f(\hat{\hat{t}}) = a$. Since aPb, we have $\bar{\bar{t}}(a) > \bar{\bar{t}}(b)$. This, together with the fact that $p(\bar{\bar{t}}) = p(\hat{\hat{t}}) = 0$, implies $\bar{\bar{t}}(f(\bar{\bar{t}})) - p(\bar{\bar{t}}) < \bar{\bar{t}}(f(\hat{\hat{t}})) - p(\hat{\hat{t}})$, and hence, μ is not IC on the pair $(\bar{\bar{t}}, \hat{\hat{t}})$. Since $\hat{\hat{t}}$ is a strict type in $type(\mathcal{D})$, this means μ is not almost everywhere IC, which is a contradiction to (i). This completes the proof for Case A.

Case B. Suppose that the environment $(strict(\mathcal{D}), G)$ is not OLGE.

Since the environment $(strict(\mathcal{D}), G)$ is not OLGE, there is an SCF g on $strict(\mathcal{D})$ that is LIC on $(strict(\mathcal{D}), G)$ but not IC. Let P and P' be two preferences in $strict(\mathcal{D})$ on which g fails to be IC, that is, $g(P')Pg(P)$.

In what follows, we construct a mechanism $\mu = (f, p)$ that is strong LIC on $(type(\mathcal{D}), G)$ but not IC, and thereby arrive at a contradiction to (i). Consider a strict type $\hat{t} \in type(\mathcal{D})$. Define $f(\hat{t}) = g(prfn(\hat{t}))$. This is well-defined as there is a unique $prfn(\hat{t})$ in $\mathcal{D}$ for such strict types $\hat{t}$. Next, consider a weak type $\bar{t}$ and consider the (strict) preferences in $\mathcal{D}$ that is compatible with the weak preference that represent $\bar{t}$, that is, the preferences $\mathcal{P}(\bar{t}) = \{\hat{P} \in strict(\mathcal{D}) \mid \hat{P} \text{ is compatible with } prfn(\bar{t})\}$. Let $P^* \in \mathcal{P}(\bar{t})$ be such that $\bar{t}(g(P^*)) \geq \bar{t}(g(\hat{P}))$ for all $\hat{P} \in \mathcal{P}(\bar{t})$. Define $f(\bar{t}) = g(P^*)$. Take the payment function p to be identically zero, that is, $p(t) = 0$ for all $t \in type(\mathcal{D})$.

Since g is not IC on (P, P'), by the definition, μ is not IC on any pair of types (t, t') such that $t \in type(P)$ and $t' \in type(P')$. Therefore, μ is not almost everywhere IC. We claim that the mechanism μ is strong LIC. The fact that μ is LIC follows from the fact that g is LIC. To see that μ is IC on every pair of types $(\bar{t}, \hat{t})$ where $\hat{t}$ is a strict type in $type(\tilde{P})$ and $\bar{t}$ is a weak type in $cl(type(\tilde{P})) \cap type(\mathcal{D})$ for some (strict) preference $\tilde{P} \in \mathcal{D}$, consider such a pair of types $(\bar{t}, \hat{t})$. We need to show $\bar{t}(f(\bar{t})) \geq \bar{t}(f(\hat{t}))$. Since $\hat{t}$ is a strict type, by the definition of f, $f(\hat{t}) = g(\tilde{P})$. Moreover, since $\bar{t}$ is a weak type and $\tilde{P}$ is compatible with the weak preference that represent $\bar{t}$, by the definition of f, we have $\bar{t}(f(\bar{t})) \geq \bar{t}(g(\tilde{P}))$. Combining these observations, we obtain $\bar{t}(f(\bar{t})) \geq \bar{t}(f(\hat{t}))$. This shows that the mechanism μ is strong LIC, completing the proof by contradicting (i).

Proof of (ii) implies (i): Consider an OLGE environment $(strict(\mathcal{D}), G)$ such that $\mathcal{D}$ satisfies weak-compatibility. We show that every strong LIC mechanism on $(type(\mathcal{D}), G)$ is almost everywhere IC. Consider a strong LIC mechanism $\mu = (f, p)$. To show that it is almost everywhere IC, we need to show that it is IC on every pair of types $(t, \hat{t})$ where $\hat{t}$ is a strict type in $type(\mathcal{D})$. Fix any pair of types $(t, \hat{t})$ such that $\hat{t}$ is a strict type in $type(\mathcal{D})$. We distinguish two cases based on the structure of t.

Case 1. Suppose t is a strict type.

Since $(strict(\mathcal{D}), G)$ OLGE, by Theorem 6.4.4, the environment $(type(strict(\mathcal{D})), G)$ is CLGE. This means every LIC mechanism on $(type(strict(\mathcal{D})), G)$ is IC. Since strong LIC implies LIC, it follows that μ is IC on $type(strict(\mathcal{D}))$, and in particular, IC on $(t, \hat{t})$. This completes the proof for Case 1.

Case 2. Suppose t is a weak type.

For notational convenience, let us denote t by $\bar{t}$. If $\bar{t}$ is such that $\bar{t}(x) = \bar{t}(y)$ for every $x, y \in A$, then $\bar{t} \in cl(type(prfn(\hat{t}))) \cap type(\mathcal{D})$. Since μ is strong LIC, μ is IC on $(\bar{t}, \hat{t})$, which completes the proof of the theorem. Now assume that $\bar{t}(x) \neq \bar{t}(y)$ for some $x, y \in A$.

Let P^* be a strict preference in $\mathcal{D}$ that is compatible with the weak preference representing $\bar{t}$. Such a preference exists since $\mathcal{D}$ satisfies the weak-compatibility property. Let B be the set of alternatives that have the same valuation as $f(\hat{t})$ in $\bar{t}$, and are preferred to $f(\hat{t})$ in P^*, that is, $B = \{b \in A \mid \bar{t}(b) = \bar{t}(f(\hat{t})) \text{ and } bP^*f(\hat{t})\}$. Notice that since $\bar{t}(x) \neq \bar{t}(y)$ for some $x, y \in A$, $A \setminus (B \cup f(\hat{t})) \neq \emptyset$. Let $\widetilde{T}$ be the set of types $\tilde{t}$ representing the preference P^* such that the relative valuation of $f(\hat{t})$ with respect to any alternative in $A \setminus (B \cup f(\hat{t}))$ strictly increases from $\bar{t}$ to $\tilde{t}$, that is, $\widetilde{T} = \{\tilde{t} \in type(P^*) \mid \tilde{t}(f(\hat{t})) - \tilde{t}(z) > \bar{t}(f(\hat{t})) - \bar{t}(z) \text{ for all } z \in A \setminus (B \cup f(\hat{t}))\}$. Since P^* is a strict preference that is compatible with the weak preference representing $\bar{t}$ and $A \setminus (B \cup f(\hat{t})) \neq \emptyset$, we have $\widetilde{T} \neq \emptyset$. We distinguish two further subcases.

Case 2.1. Suppose that there exists $\tilde{t} \in \widetilde{T}$ such that $f(\tilde{t}) \notin B$.

Since $f(\tilde{t}) \notin B$, by the definition $\widetilde{T}$, we have

$$\tilde{t}(f(\hat{t})) - \tilde{t}(f(\tilde{t})) \geq \bar{t}(f(\hat{t})) - \bar{t}(f(\tilde{t})), \tag{6.6}$$

where the equality holds only when $f(\hat{t}) = f(\tilde{t})$. Consider the sequence $(\bar{t}, \tilde{t}, \hat{t})$. We apply Fact 6.7.3 to this sequence. Since $\tilde{t}$ is a strict type in $type(P^*)$ and $\bar{t}$ is a weak type in $cl(type(P^*)) \cap type(\mathcal{D})$ and μ is strong

LIC, μ is IC on $(\bar{t}, \tilde{t})$. Moreover, since both $\tilde{t}$ and $\hat{t}$ are strict types, by Case 1, it follows that μ is IC on $(\tilde{t}, \hat{t})$. Thus, μ is IC on both the pairs $(\bar{t}, \tilde{t})$ and $(\tilde{t}, \hat{t})$, and thereby satisfies the Condition (i) of Fact 6.7.3. Furthermore, Condition (ii) of Fact 6.7.3 follows from (6.6). Therefore, the sequence $(\bar{t}, \tilde{t}, \hat{t})$ satisfies the conditions of Fact 6.7.3 and hence μ is IC on $(\bar{t}, \hat{t})$. This completes the proof for Case 2.1.

Case 2.2. Suppose that Case 2.1 does not hold, that is, for all $\tilde{t} \in \widetilde{T}$, we have $f(\tilde{t}) \in B$.

Let $\widetilde{B}$ be the set of outcomes of f on $\widetilde{T}$, that is, $\widetilde{B} = \{f(\tilde{t}) \mid \tilde{t} \in \widetilde{T}\}$. Let $\tilde{b}$ be the worst alternative in $\widetilde{B}$ according to P^*, that is, $bP^*\tilde{b}$ for all $b \in \widetilde{B} \setminus \{\tilde{b}\}$. Let $t_{\tilde{b}} \in \widetilde{T}$ be a type such that $f(t_{\tilde{b}}) = \tilde{b}$. Let $T_{\tilde{b}}$ be the set of strict types in $type(\mathcal{D})$ such that the relative valuation of $\tilde{b}$ with repect to any other alternative in $\widetilde{B}$ is greater than that in $t_{\tilde{b}}$, that is,
$T_{\tilde{b}} = \{\tilde{\tilde{t}} \in type(\mathcal{D}) \mid \tilde{\tilde{t}} \text{ is a strict type and } \tilde{\tilde{t}}(\tilde{b}) - \tilde{\tilde{t}}(z) > t_{\tilde{b}}(\tilde{b}) - t_{\tilde{b}}(z) \text{ for all } z \in \widetilde{B} \setminus \{\tilde{b}\}\}$. Note that since μ is strong LIC and the types in $T_{\tilde{b}}$ are strict, by Case 1, μ is IC on any pair of types in $T_{\tilde{b}}$. By the construction of the type-space $T_{\tilde{b}}$ and Fact 6.7.2, this implies that the outcome of f at any type in $T_{\tilde{b}}$ cannot be in the set $\widetilde{B} \setminus \{\tilde{b}\}$.

Consider the types in $\widetilde{T}_{\tilde{b}} = T_{\tilde{b}} \cap \widetilde{T}$. Since there is no restriction on the types in $\widetilde{T}$ about the relative valuation of $\tilde{b}$ with respect to any other alternative in $\widetilde{B} \setminus \{\tilde{b}\}$, we have $\widetilde{T}_{\tilde{b}} \neq \emptyset$. Moreover, since both $\widetilde{T}$ and $T_{\tilde{b}}$ put no restriction on the relative valuation of $\tilde{b}$ with respect to $f(\hat{t})$ (except that the said relative valuation is positive), the difference of the valuation of $\tilde{b}$ and $f(\hat{t})$ can be arbitrarily small in the types in $\widetilde{T}_{\tilde{b}}$, that is, $\inf_{\tilde{\tilde{t}} \in \widetilde{T}_{\tilde{b}}} \tilde{\tilde{t}}(\tilde{b}) - \tilde{\tilde{t}}(f(\hat{t})) = 0$. By the definition of $T_{\tilde{b}}$, the outcome of f at any type in $T_{\tilde{b}}$ cannot be in the set $\widetilde{B} \setminus \{\tilde{b}\}$. Moreover, by the assumption of Case 2.2, the outcome of f at any type in $\widetilde{T}$ has to be in the set $\widetilde{B}$, it follows that the outcome of any type in $\widetilde{T}_{\tilde{b}}$ is $\tilde{b}$.

Consider any type $\tilde{t}_{\tilde{b}}$ in $\widetilde{T}_{\tilde{b}}$. Since μ is strong LIC and both $\tilde{t}_{\tilde{b}}$ and $\hat{t}$ are strict types, by Case 1, μ must be IC on $(\tilde{t}_{\tilde{b}}, \hat{t})$. Therefore,

$$p(\tilde{t}_{\tilde{b}}) - p(\hat{t}) \leq \tilde{t}_{\tilde{b}}(\tilde{b}) - \tilde{t}_{\tilde{b}}(f(\hat{t})). \tag{6.7}$$

Since $f(\tilde{t}_{\tilde{b}}) = \tilde{b}$ for all types $\tilde{t}_{\tilde{b}} \in \widetilde{T}_{\tilde{b}}$ and μ is IC on $\widetilde{T}_{\tilde{b}}$, by Fact 6.7.1 we have $p(\tilde{t}_{\tilde{b}})$ must be the same for all types in $\widetilde{T}_{\tilde{b}}$. Let $p(\tilde{t}_{\tilde{b}}) = c$ for all $\tilde{t}_{\tilde{b}} \in \widetilde{T}_{\tilde{b}}$ and for some $c \in \mathbb{R}$. Taking infimum on both sides of (6.7) and doing some rearrangement, we obtain

$$-p(\hat{t}) \leq -c \tag{6.8}$$

Since $\bar{t}(f(\hat{t})) = \bar{t}(\tilde{b})$, adding $\bar{t}(f(\hat{t}))$ to the left side of (6.8) and $\bar{t}(\tilde{b})$ to the right side of (6.8), we get

$$\bar{t}(f(\hat{t})) - p(\hat{t}) \leq \bar{t}(\tilde{b}) - c.$$

Fix any $\tilde{t}_{\tilde{b}} \in \widetilde{T}_{\tilde{b}}$. Since $f(\tilde{t}_{\tilde{b}}) = \tilde{b}$, this implies

$$\bar{t}(f(\hat{t})) - p(\hat{t}) \leq \bar{t}(f(\tilde{t}_{\tilde{b}})) - c \tag{6.9}$$

Now, since μ is strong LIC and $\bar{t}$ is a weak type in $cl(type(P^*)) \cap type(\mathcal{D})$ and $\tilde{t}_{\tilde{b}}$ is a strict type in $type(P^*)$, μ is IC on $(\bar{t}, \tilde{t}_{\tilde{b}})$. This implies

$$\bar{t}(f(\bar{t})) - p(\bar{t}) \geq \bar{t}(f(\tilde{t}_{\tilde{b}})) - c \tag{6.10}$$

By (6.9), this yields

$$\bar{t}(f(\bar{t})) - p(\bar{t}) \geq \bar{t}(f(\hat{t})) - p(\hat{t}), \tag{6.11}$$

which concludes that μ is IC on $(\bar{t}, \hat{t})$. ■

6.7.4 Proof of Corollary 6.5.1

Proof: Let $(\widehat{\mathcal{D}}, G^{ad})$ be a strict ordinal OLGE environment and let μ be a PLIC mechanism on $cl(type(\widehat{\mathcal{D}}))$. We show that μ is almost everywhere IC. Let $\mathcal{D}$ be the set of all preferences representing the types in $cl(type(\widehat{\mathcal{D}}))$, that is, $\mathcal{D} = prfn(cl(type(\widehat{\mathcal{D}})))$. By definition, $\mathcal{D}$ satisfies weak compatibility. Since $(\widehat{\mathcal{D}}, G^{ad})$ is OLGE, by Theorem 6.5.1, this implies that every strong LIC mechanism on $(cl(type(\widehat{\mathcal{D}})), G^{ad})$ is almost everywhere IC. So, to show that μ is almost everywhere IC on $cl(type(\widehat{\mathcal{D}}))$, it is sufficient to show that μ is strong LIC on $(cl(type(\widehat{\mathcal{D}})), G^{ad})$.

Consider any pair of G^{ad}-local preferences (P, P') in $\widehat{\mathcal{D}}$. Since P and P' are adjacent local, $cl(type(\{P, P'\}))$ is convex (see Fact 1 in Mishra et al. [39] for details). Because, μ is PLIC, it follows from Fact 6.4.1 that it is IC on $cl(type(\{P, P'\}))$. In particular, it is IC on (i) any pair of strict types (t, t') in $type(\{P, P'\})$, and (ii) every pair of types $(\bar{t}, \hat{t})$ in $cl(type(P))$ where $\bar{t}$ is a weak type and $\hat{t}$ is a strict type. Since P and P' are arbitrary G^{ad}-local types, it follows that μ is strong LIC. ■

6.7.5 Proof of Theorem 6.5.2

Proof: Let $(\widehat{\mathcal{D}}, G^{ad})$ be a strict ordinal OLGE environment and let $T = cl(type(\widehat{\mathcal{D}}))$. Consider a PLIC mechanism μ on T. By Corollary 6.5.1, μ is almost everywhere IC on T. We first prove a claim.

Claim 6.7.1 *Let a weak type $\bar{t} \in T \setminus strict(T)$ be such that there exists a strict type $\hat{t} \in strict(T)$ with the property that*

(i) $\hat{t}(f(\bar{t})) - \hat{t}(x) > \bar{t}(f(\bar{t})) - \bar{t}(x)$ *for all* $x \in A \setminus \{f(\bar{t})\}$, *and*

(ii) μ *is IC on* $(\hat{t}, \bar{t})$.

Then, μ *is IC on* $T \times \{\bar{t}\}$.

Proof of the claim: Since μ is almost everywhere IC on T, it is IC on the pair $(\bar{t}, \hat{t})$. Moreover, by Condition (ii) of the claim, μ is IC on the pair $(\hat{t}, \bar{t})$. Thus, μ is IC on both $(\bar{t}, \hat{t})$ and $(\hat{t}, \bar{t})$. By Condition (i) of the claim, the relative valuation of $f(\bar{t})$ with respect to any other alternative is increased from $\bar{t}$ to $\hat{t}$. Therefore, by Fact 6.7.2, we have $f(\bar{t}) = f(\hat{t})$. This, together with the fact that μ is IC on both pairs $(\bar{t}, \hat{t})$ and $(\hat{t}, \bar{t})$, implies by Fact 6.7.1 that $p(\bar{t}) = p(\hat{t})$.

Now, since μ is almost everywhere IC, it is IC on $T \times \{\hat{t}\}$. Because $f(\bar{t}) = f(\hat{t})$ and $p(\bar{t}) = p(\hat{t})$, it follows that μ is IC on $T \times \{\bar{t}\}$. This completes the proof of the claim. □

We are now ready to prove the theorem.

Proof of (i): We show that μ is IC on $T \times T^o$. Since μ is almost everywhere IC on T, by definition μ is IC on $T \times \mathit{strict}(T)$. Note that T^o might contain weak types. So, we need to show that μ is IC on $T \times (T^o \setminus \mathit{strict}(T))$. Consider any $\bar{t} \in T^o \setminus \mathit{strict}(T)$. By the definition of T^o, there exists $\varepsilon_1 > 0$ such that $\{s \in \mathbb{R}^n \mid d(\bar{t}, s) < \varepsilon_1\} \subset T$. Also, by the definition of a PLIC mechanism, there exists $\varepsilon_2 > 0$ such that μ is IC on $(\bar{t}, s)$ and $(s, \bar{t})$ for every $s \in T$ with $d(\bar{t}, s) < \varepsilon_2$. Consider a type $\hat{t}$ in $\mathit{strict}(T)$ with $d(\bar{t}, \hat{t}) < \mathit{min}\{\varepsilon_1, \varepsilon_2\}$ such that $\hat{t}(f(\bar{t})) - \hat{t}(x) > \bar{t}(f(\bar{t})) - \bar{t}(x)$ for all $x \in A \setminus \{f(\bar{t})\}$. Such a type can be constructed from $\bar{t}$ by lowering the valuation of each alternative other than $f(\bar{t})$ by an arbitrarily small amount. This, together with the facts that $\bar{t} \in T \setminus \mathit{strict}(T)$, μ is almost everywhere IC on T, and μ is IC on $(\hat{t}, \bar{t})$, implies by Claim 6.7.1 that μ is IC on $T \times \{\bar{t}\}$. Since $\bar{t} \in T^o \setminus \mathit{strict}(T)$ is arbitrary, it follows that μ is IC on $T \times (T^o \setminus \mathit{strict}(T))$. This completes the proof of Part (i) of the theorem.

Proof of (ii): Let $\bar{t} \in \widehat{\partial} T$ such that there exists $P \in \widehat{\mathcal{D}}$ with $\bar{t} \in \mathit{cl}(\mathit{type}(P))$ and $f(\bar{t})Pz$ for every z with $\bar{t}(f(\bar{t})) = \bar{t}(z)$. Since μ is PLIC and $\mathit{cl}(\mathit{type}(P))$ is convex, by Fact 6.4.1, μ is IC on $\mathit{cl}(\mathit{type}(P))$. Since $f(\bar{t})Pz$ for every z with $\bar{t}(f(\bar{t})) = \bar{t}(z)$, starting from the type $\bar{t}$, we can construct a type $\hat{t} \in \mathit{type}(P)$ by suitably lowering the valuation of each alternative other than $f(\bar{t})$ such that $\hat{t}(f(\bar{t})) - \hat{t}(x) > \bar{t}(f(\bar{t})) - \bar{t}(x)$ for all $x \in A \setminus \{f(\bar{t})\}$. This, together with the facts that μ is almost everywhere IC on T and μ is IC on $(\hat{t}, \bar{t})$, implies by Claim 6.7.1 that μ is IC on $T \times \{\bar{t}\}$. This completes the proof of Part (ii) of the theorem. ■

References

[1] Aaron Archer and Robert Kleinberg. Truthful germs are contagious: a local-to-global characterization of truthfulness. *Games and Economic Behavior*, 86:340–366, 2014.

[2] Mark Armstrong. Optimal multi-object auctions. *The Review of Economic Studies*, 67(3):455–481, 2000.

[3] Lawrence M Ausubel and Paul R Milgrom. Ascending auctions with package bidding. *Advances in Theoretical Economics*, 1(1), 2002.

[4] Salvador Barberà. Indifferences and domain restrictions. *Analyse & Kritik*, 29(2):146–162, 2007.

[5] Salvador Barberà, Hugo Sonnenschein, and Lin Zhou. Voting by committees. *Econometrica*, 59(3): 595–609, 1991.

[6] Salvador Barberà, Faruk Gul, and Ennio Stacchetti. Generalized median voter schemes and committees. *Journal of Economic Theory*, 61(2):262–289, 1993.

[7] Salvador Barberà, Dolors Berga, and Bernardo Moreno. Individual versus group strategy-proofness: when do they coincide? *Journal of Economic Theory*, 145(5):1648–1674, 2010.

[8] Yoram Barzel. Private schools and public school finance. *Journal of Political Economy*, 81(1): 174–186, 1973.

[9] Peter Bearse, Gerhard Glomm, and Eckhard Janeba. Composition of government budget, non-single peakedness, and majority voting. *Journal of Public Economic Theory*, 3(4):471–481, 2001.

[10] Dolors Berga. Strategy-proofness and single-plateaued preferences. *Mathematical Social Sciences*, 35 (2):105–120, 1998.

[11] Juan Carlos Carbajal and Rudolf Müller. Implementability under monotonic transformations in differences. *Journal of Economic Theory*, 160:114–131, 2015.

[12] Gabriel Carroll. When are local incentive constraints sufficient? *Econometrica*, 80(2):661–686, 2012.

[13] Siddharth Chatterjee. Computational complexity of local-global equivalence. Available at https://drive.google.com/file/d/1XdHYBqxzc3CEXt2ahJP1giMSUoKQFOPp/view, 2020.

[14] Shurojit Chatterji and Arunava Sen. Tops-only domains. *Economic Theory*, 46(2):255–282, 2011.

[15] Shurojit Chatterji and Huaxia Zeng. On random social choice functions with the tops-only property. *Games and Economic Behavior*, 109:413–435, 2018.

[16] Shurojit Chatterji and Huaxia Zeng. Random mechanism design on multidimensional domains. *Journal of Economic Theory*, 182:25–105, 2019.

[17] Shurojit Chatterji, Souvik Roy, Sadhukhan Soumyarup, Arunava Sen, and Huaxia Zeng. Probabilistic fixed ballot rules and hybrid domains. *Journal of Mathematical Economics*, 100:102656, 2022.

[18] Wonki Jo Cho. Incentive properties for ordinal mechanisms. *Games and Economic Behavior*, 95: 168–177, 2016.

[19] T.H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein. *Introduction to Algorithms (2nd. Edition)*. MIT Press, Boston, 2001.

[20] Gabrielle Demange. Single-peaked orders on a tree. *Mathematical Social Sciences*, 3(4):389–396, 1982.

[21] Dennis Epple and Richard E Romano. Public provision of private goods. *Journal of political Economy*, 104(1):57–84, 1996.

[22] Urs Fischbacher and Franziska Föllmi-Heusi. Lies in disguise - an experimental study on cheating. *Journal of the European Economic Association*, pages 525–547, 2013.

[23] Drew Fudenberg and Jean Tirole. Game theory. *MIT Press, Cambridge*, 1991.

[24] Satoru Fujishige and Zaifu Yang. A note on kelso and crawford's gross substitutes condition. *Mathematics of Operations Research*, 28(3):463–469, 2003.

[25] Allan Gibbard. Manipulation of voting schemes: a general result. *Econometrica*, pages 587–601, 1973.

[26] Allan Gibbard. Manipulation of schemes that mix voting with chance. *Econometrica*, 45:665–681, 1977.

[27] Faruk Gul and Ennio Stacchetti. Walrasian equilibrium with gross substitutes. *Journal of Economic theory*, 87(1):95–124, 1999.

[28] Miho Hong and Semin Kim. Unanimity and local incentive compatability in sparsely connected domains. https://www.dropbox.com/s/2zsq1sobefsd9mi/ULIC_Full_200925.pdf?dl=0, 2020. unpublished manuscript.

[29] Norman J Ireland. The mix of social and private provision of goods and services. *Journal of Public Economics*, 43(2):201–219, 1990.

[30] Alexander S Kelso and Vincent P Crawford. Job matching, coalition formation, and gross substitutes. *Econometrica: Journal of the Econometric Society*, pages 1483–1504, 1982.

[31] Ujjwal Kumar and Souvik Roy. Local incentive compatability in non-convex type-spaces. https://mpra.ub.uni-muenchen.de/110872/1/MPRA_paper_110872.pdf, 2021. unpublished manuscript.

[32] Ujjwal Kumar, Souvik Roy, Arunava Sen, Sonal Yadav, and Huaxia Zeng. Local global equivalence in voting models: A characterization and applications. *Theoretical Economics*, 16(4):1195–1220, 2021.

[33] Ujjwal Kumar, Souvik Roy, Arunava Sen, Sonal Yadav, and Huaxia Zeng. Local global equivalence for unanimous social choice functions. *Games and Economic Behavior*, 130:299–308, 2021.

[34] Alexey I Kushnir and Lev V Lokutsievskiy. When is a monotone function cyclically monotone? *Theoretical Economics*, 16(3):853–879, 2021.

[35] Ron Lavi, Ahuva Mualem, and Noam Nisan. Two simplified proofs for roberts' theorem. *Social Choice and Welfare*, 32(3):407–423, 2009.

[36] Michael Le Breton and Arunava Sen. Separable preferences, strategyproofness and decomposability. *Econometrica*, 67:605–628, 1999.

[37] Michel Le Breton and Vera Zaporozhets. On the equivalence of coalitional and individual strategy-proofness properties. *Social Choice and Welfare*, 33(2):287–309, 2009.

[38] Debasis Mishra, Anup Pramanik, and Souvik Roy. Multidimensional mechanism design in single peaked type spaces. *Journal of Economic Theory*, 153:103–116, 2014.

[39] Debasis Mishra, Anup Pramanik, and Souvik Roy. Local incentive compatibility with transfers. *Games and Economic Behavior*, 100:149–165, 2016.

[40] Shuhei Morimoto. Group strategy-proof probabilistic voting with single-peaked preferences. https://www.biz.tmu.ac.jp/wp-content/uploads/sites/9/2020/05/RP-22_Morimoto.pdf, 2020. unpublished manuscript.

[41] Hervé Moulin. On strategy-proofness and single peakedness. *Public Choice*, 35(4):437–455, 1980.

[42] Kazuo Murota. Discrete convex analysis: A tool for economics and game theory. *Journal of Mechanism and Institution Design*, 1(1):151–273, 2016.

[43] Renato Paes Leme. Gross substitutability: An algorithmic survey. *Games and Economic Behavior*, 106:294–316, 2017.

[44] Hans Reijnierse, Anita van Gellekom, and Jos AM Potters. Verifying gross substitutability. *Economic Theory*, 20(4):767–776, 2002.

[45] Shin Sato. Strategy-proofness and the reluctance to make large lies: the case of weak orders. *Social Choice and Welfare*, 40(2):479–494, 2013.

[46] Shin Sato. A sufficient condition for the equivalence of strategy-proofness and nonmanipulability by preferences adjacent to the sincere one. *Journal of Economic Theory*, 148(1):259–278, 2013.

[47] Mark Allen Satterthwaite. Strategy-proofness and arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10(2):187–217, 1975.

[48] Akiyoshi Shioura and Zaifu Yang. Equilibrium, auction, and generalized gross substitutes and complements. *Journal of the Operations Research Society of Japan*, 58(4):410–435, 2015.

[49] JE Stiglitz. The demand for education in public and private school systems. *Journal of Public Economics*, 3(4):349–385, 1974.

[50] Rakesh V Vohra. *Mechanism design: a linear programming approach*, volume 47. Cambridge University Press, 2011.

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List of Publication(s)/Submitted Article(s)

- **Published Papers**
  (1) "Local Global Equivalence in Voting Models: A Characterization and Applications" *Theoretical Economics*, 2021, 16, 1195-1220.
  (2) "Local Global Equivalence for Unanimous Social Choice Functions" *Games and Economic Behavior*, 2021, 130, 299-308.
- **Completed Papers**
  (1) "Local Incentive Compatibility in Ordinal Type-Spaces".
  (2) "Local Global Equivalence in Voting Models Admitting Indifferences".
  (3) "Pointwise Local Incentive Compatibility in Non-Convex Type-Spaces".