DISTRIBUTION OF MOST SIGNIFICANT DIGIT IN CERTAIN FUNCTIONS WHOSE ARGUMENTS ARE RANDOM VARIABLES

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SUMMARY. It is empirically well established that in large collections of numbers the proportions of entries with the most significant digit A is $\log_{10}(A+1)/A$. The property of the most significant digit has been studied in the present paper. It has been proved that when random numbers or their reciprocals are raised to higher and higher powers, they have log distribution of most significant digit in the limit. The property is also demonstrated in the limit by the products of random numbers as the number of terms in the product becomes higher and higher. The property is not, however, demonstrated by higher roots of the random numbers or their reciprocals in the limit. In fact there is a concentration at some particular digit. It has been shown that if X has log distribution of the most significant digit, so does $1/X$ and CX, C being any constant, under stronger conditions.

In statistical tables more entries start with a smaller most significant digit (m.s.d.). The smaller is the digit, the more is the proportion of entries starting with it and the proportion of entries having m.s.d. A is approximately $\log_{10}(A+1)/A$. It means that one expects to see entries having one as the m.s.d. a little less than seven times more frequently than entries having nine as the m.s.d. This 'abnormal law' viz. the logarithmic law governing the distribution of m.s.d. has been found to be surprisingly accurate on empirical verification with large volume of data. Benford (1938) studied such a distribution wherever large volume of data is present. Attempts have been made to explore whether the nature of the entries presented in the table affects the distribution of m.s.d. [Furry and Hurtwitz (1945); Goudsmit and Furry (1944)]. Pinkham (1961) justifies the logarithmic law because only such a distribution of m.s.d. is invariant under a change of scale of the entries. Recently, Flehinger (1966) has studied the properties of the set of integers and has found a regular limiting process which leads to a probability measure on the set of integers with initial digit $\leqslant A$ that agrees with the logarithmic law.

In the present paper a direct approach is made to study the distribution of m.s.d. on certain common arithmetic operations like raising to powers or extracting roots of random numbers. The distribution of m.s.d. was also sutdied on the product of a number of random numbers. Some of the simulation results as obtained with five digit tested random numbers are given in Tables 1 and 2.

It is shown that the frequencies had a tendency to come closer to the expected frequencies under the logarithmic law as the power or the number of terms in the product was increasing. A theoretical justification was sought. The following sections describes the results.

TABLE 1. DISTRIBUTION OF SIGNIFICANT DIGITS IN POWERS OF RANDOM NUMBERS

digit	power							
	1	2	3	4	5	6	7	8
1	6,765	11,528	13,610	14,786	15,310	15,636	15,901	16,300
2	6,500	8,746	9,545	9,741	9,807	10,042	10,362	10,357
3	6,618	7,613	7,536	7,486	7,583	7,502	7,502	7,382
4	6,655	6,479	6,255	6,281	6,164	6,036	6,034	5,975
5	6,683	5,880	5,576	5,263	5,275	5,250	5,094	5,011
6	6,607	5,404	4,862	4,714	4,648	4,561	4,349	4,380
7	6,631	4,946	4,625	4,207	4,014	4,017	3,990	4,003
8	6,665	4,755	4,174	3,956	3,687	3,589	3,459	3,449
9	6,687	4,580	3,817	3,566	3,422	3,268	3,219	3,083
total	60,000	60,000	60,000	60,000	60,000	60,000	60,000	60,000

TABLE 2. DISTRIBUTION OF SIGNIFICANT DIGITS IN PRODUCTS OF 5 AND 20 RANDOM NUMBERS

digit	no. of terms in products		expected frequency under logarithmic law
	5	20	
1	15,442	17,134	18,062
2	9,278	10,915	10,565
3	7,266	7,366	7,496
4	6,025	5,807	5,815
5	5,321	4,854	4,751
6	4,588	3,082	4,017
7	4,103	3,822	3,480
8	4,027	3,207	3,069
9	3,800	2,733	2,745
total	60,000	60,000	60,000

Let X be a continuous random variable having a uniform distribution in the interval (0, 1) i.e. the probability density function (p.d.f.) of X is given by

$$f(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $M(X)$ be the most significant digit of X. It is clear that $M(X)$ has a discrete distribution taking values 1, 2, ..., 9 with probabilities 1/9 when X is uniformly distributed on (0, 1).

Theorem 1 : *If X is a random variable having a uniform distribution in the interval (0, 1) and $Y = X^n$ where n is an integer, then*

(i) $P_A(n)$ = *probability that the m.s.d. of Y is A*

$$= P_n\{M(Y) = A\}$$

$$\begin{cases} [(A+1)^{1/m} - A^{1/m}]/(10^{1/m}-1) & \text{for } n > 0 \\ \left[\dfrac{(A+1)^{1/m} - A^{1/m}}{(10^{1/m}-1)}\right]\left[\dfrac{10}{A(A+1)}\right]^{1/m} & \text{if } n < 0 \end{cases}$$

$$A = 1, 2, \ldots, 9 \text{ and } m = |n|.$$

(ii) *$P_A(n)$ is a monotonic function of m for large m,*

(iii) $\lim_{m\to\infty} P_A(n) = \log_{10}(A+1)/A.$

Proof: *Case* I, $n > 0$: The p.d.f. of Y is given by

$$f(y) = \begin{cases} \dfrac{1}{n} y^{1/n-1} & 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

So $$P\{M(y) = A\} = P_A(n) = \sum_{r=1}^{\infty} \int_{A/10^r}^{(A+1)/10^r} \frac{1}{n} y^{1/n-1}\, dy, \qquad A = 1, 2, \ldots, 9,$$

as Y can have the significant digit A if Y lies in any of the intervals $A/10^r$, $(A+1)/10^r$, $r = 1, 2, \ldots$.

Thus,

$$P_A(n) = \sum_{r=1}^{\infty} [y^{1/n}]_{A/10^r}^{(A+1)/10^r} = \sum_{r=1}^{\infty} [1/10^{r/n}][(A+1)^{1/n} - A^{1/n}]$$

$$= [(A+1)^{1/n} - A^{1/n}]/[10^{1/n}-1] = [(A+1)^{1/m} - A^{1/m}]/[10^{1/m}-1] \quad \ldots \quad (1)$$

since $m = |n| = n$ here.

The result (1) can also be derived as follows :

$$P_A(n) = \sum_{r=1}^{\infty} P[(A/10^r) \leqslant X^n < (A+1)/10^r] = \sum_{r=1}^{\infty} P[(A/10^r)^{1/n} \leqslant X < [(A+1)/10^r]^{1/n}]$$

$$= [(A+1)^{1/n} - A^{1/n}] \sum_{r=1}^{\infty} 1/10^{1/n} = [(A+1)^{1/n} - A^{1/n}]/[10^{1/n}-1].$$

The result (ii) will be established by proving the following result :

Result 1 : For large m, $P_A(n)$ is monotonically increasing for $A = 1$ or 2 and decreasing for $A = 3, 4, \ldots, 9$.

*Proof :**

$$P_A(n) = [(A+1)^{1/m} - A^{1/m}]/[10^{1/m} - 1] = [e^{\lambda x} - e^{\mu x}]/[e^{\nu x} - 1] = f(x), \text{ say,}$$

where $\quad x = 1/m,\ \lambda = \log_e(A+1),\ \ \mu = \log_e A$ and $\nu = \log_e 10$.

$$\frac{df(x)}{dx} = f'(x) = \frac{(e^{\nu x}-1)(\lambda e^{\lambda x} - \mu e^{\mu x}) - (e^{\lambda x} - e^{\mu x})\nu e^{\nu x}}{(e^{\nu x}-1)^2}.$$

For sufficiently small x the numerator of $f'(x)$ can be approximated as

$$\left(\nu x + \frac{\nu^2 x^2}{2}\right)(\lambda + \lambda^2 x - \mu - \mu^2 x) - \nu(1+\nu x)\left[(\lambda-\mu)x + \frac{\lambda^2-\mu^2}{2}x^2\right] + O(x^3)$$

$$= \nu x(\lambda-\mu) - \nu x^2(\lambda^2-\mu^2) + \frac{\nu^2 x^2}{2}(\lambda-\mu) - \nu(\lambda-\mu)x - \nu^2 x^2(\lambda-\mu) - \nu x^2\left(\frac{\lambda^2-\mu^2}{2}\right) + O(x^3)$$

$$= \frac{\nu(\lambda-\mu)}{2}(\lambda+\mu-\nu)x^2 + O(x^3).$$

Now, $\nu > 0$ and $(\lambda-\mu) > 0$ and hence $f'(x)$ will be positive or negative according as $\lambda+\mu-\nu$ is positive or negative. It is easily seen that

$$\lambda+\mu-\nu = \log_e \frac{A(A+1)}{10}$$

and so $\qquad \lambda+\mu-\nu < 0 \quad \text{for} \quad A = 1, 2$

and $\qquad \lambda+\mu-\nu > 0 \quad \text{for} \quad A = 3, 4, \ldots, 9.$

This proves the result.

To establish (iii) we apply Hospital's rule giving

$$\lim_{m\to\infty} P_A(n) = \lim_{m\to\infty} \frac{(A+1)^{1/m}\log_e(A+1) - A^{1/m}\log_e A}{10^{1/m}\log_e 10}$$

$$= \log_e\left(\frac{A+1}{A}\right)\Big/\log_e 10 = \log_{10}\frac{A+1}{A}.$$

* We are indebted to Dr. B. Ramachandran for suggesting this proof.

Case II, $n < 0$: The p.d.f. of Y is given by

$$f(y) = \begin{cases} \frac{1}{m} y^{-(m+1)/m} & 1 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Y can have the m.s.d. A if it lies in the interval $[A(10^r), (A+1)10^r]$, $r = 0, 1, 2, \ldots$

Thus

$$P_A(n) = \sum_{r=0}^{\infty} \int_{A.10^r}^{(A+1)10^r} \frac{1}{m} y^{-(m+1)/m} \, dy = \sum_{r=0}^{\infty} \left[-y^{-1/m}\right]_{A.10^r}^{(A+1)10^r}$$

$$= \left[\frac{1}{A^{1/m}} - \frac{1}{(A+1)^{1/m}}\right] \frac{10^{1/m}}{10^{1/m}-1} = \frac{(A+1)^{1/m} - A^{1/m}}{10^{1/m}-1} \cdot \left[\frac{10}{A(A+1)}\right]^{1/m}. \quad \ldots \quad (2)$$

The monotonicity of $P_A(n)$ in this case can be established as in the previous case, the result being for large m, $P_A(n)$ is monotonically decreasing for $A = 1, 2$ and monotonically increasing for $A = 3, 4, \ldots, 9$.

To prove (iii) for this case it is sufficient to prove that $\lim_{m \to \infty} \left[\frac{10}{A(A+1)}\right]^{1/m}$ is unity, which is true.

The probabilities $P_A(n)$ have been tabulated for certain values of n in the appendix. For $m > 6$ the probabilities $P_A(n)$ can be seen, from the appendix, to be monotonic.

Result 2 : Let $Y = X^{1/n}$ where X is having a uniform distribution in the interval (0, 1). Then

(i) $\phi_A(n)$ = probability that m.s.d. of Y is A

$$= \begin{cases} \frac{(A+1)^m - A^m}{10^m - 1} & \text{for } n > 0 \\ \frac{(A+1)^m - A^m}{10^m - 1} \cdot \left[\frac{10}{A(A+1)}\right]^m & \text{for } n < 0 \end{cases}$$

$$A = 1, 2, \ldots, 9 \quad \text{and} \quad m = |n|.$$

(iia) $$\lim_{m \to \infty} \phi_A(n) = \begin{cases} 1 & \text{for } A = 9 \\ 0 & \text{for } A = 1, 2, \ldots, 8, \quad n > 0. \end{cases}$$

(iib) $$\lim_{m \to \infty} \phi_A(n) = \begin{cases} 1 & \text{for } A = 1 \\ 0 & \text{for } A = 2, 3, \ldots, 9, \; n < 0. \end{cases}$$

Proof : Case I, $n > 0$: The p.d.f. of Y is given by

$$f(y) = \begin{cases} ny^{n-1} & 0 < y < 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\phi_A(n) = \sum_{r=1}^{\infty} \int_{A/10^r}^{(A+1)/10^r} ny^{n-1}dy = \sum_{r=1}^{\infty} \; [(A+1)^n/10^r - A^n/10^r]$$

$$= \frac{(A+1)^n - A^n}{10^n - 1} = \frac{(A+1)^m - A^m}{10^m - 1}$$

$$\lim_{m \to \infty} \frac{(A+1)^m - A^m}{10^m - 1} = \lim_{m \to \infty} \frac{\left(\frac{A+1}{10}\right)^m - \left(\frac{A}{10}\right)^m}{1 - \left(\frac{1}{10}\right)^m}$$

$$= \begin{cases} 0 & \text{if} \quad A = 1, 2, \dots 8 \\ 1 & \text{if} \quad A = 9. \end{cases}$$

Case II, $n < 0$: The p.d.f. of Y is given by

$$f(y) = \begin{cases} m \; y^{-m-1} & \text{if} \quad 1 < y < \infty \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $$\phi_A(n) = \sum_{r=0}^{\infty} \int_{A(10^r)}^{(A+1)10^r} my^{-m-1}dy$$

$$= [A^{-m} - (A+1)^{-m}] \sum_{r=0}^{\infty} \frac{1}{10^{rm}} = \frac{(A+1)^m - A^m}{10^m - 1} \; \frac{10^m}{A^m(A+1)^m}.$$

In this case,

$$\lim_{n \to \infty} \phi_A(n) = \lim_{m \to \infty} \left[\left\{ \left(\frac{1}{A}\right)^m - \left(\frac{1}{A+1}\right)^m \right\} \cdot \; \frac{1}{1 - \left(\frac{1}{10}\right)^m} \right]$$

$$= \begin{cases} 1 & \text{if} \quad A = 1 \\ 0 & \text{if} \quad A = 2, 3, \dots, 9. \end{cases}$$

Theorem 2 : *Let $X_1, X_2, \ldots, X_n$ be independent random variables, each X_i having a uniform distribution over the interval* (0, 1). *Then*

(i) *the p.d.f. of $Y = X_1 X_2 \ldots X_n$ is given by*

$$f(y) = \begin{cases} \dfrac{(-1)^{n-1}}{(n-1)!} (\log_e y)^{n-1} & 0 < y < 1 \\ 0 & \textit{otherwise.} \end{cases}$$

(ii) $\pi_A(n) =$ *probability that the m.s.d. of Y is A*

$$= \textit{coefficient of } t^{n-1} \text{ in } \frac{(A+1)^{1-t} - A^{1-t}}{10^{1-t}-1} \cdot \frac{1}{1-t}$$

(iii) $$\lim_{n \to \infty} \pi_A(n) = \log_{10} \frac{A+1}{A} .$$

Proof : (i) We derive first the p.d.f. of $X_1.X_2$. Let $Y_1 = X_1.X_2$ and $Y_2 = X_2$. The Jacobian of transformation is

$$|J| = \left| \frac{\partial(x_1, x_2)}{\partial(y_1, y_2)} \right| = \frac{1}{X_2} = \frac{1}{Y_2} .$$

So the joint density function of Y_1 and Y_2 is

$$f(y_1, y_2) = \begin{cases} \dfrac{1}{y_2} & 0 < y_1 < 1 \text{ and } y_1 \leqslant y_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the marginal density function of Y_1 is

$$f(y_1) = \int_{y_1}^{1} \frac{1}{y_2} \, dy_2 = -\log_e y_1, \quad 0 < y_1 < 1.$$

We shall prove (i) by induction. We have established (i) when $n = 2$.

Let the result be true for $X_1, X_2, \ldots, X_r$. Let X_{r+1} be another independent random variable. Let $Y = X_1 . X_2 \ldots X_r$, $Y_1 = Y. X_{r+1}$ and $Y_2 = X_{r+1}$.

$$f(y) = \begin{cases} \dfrac{(-1)^{r-1}}{(r-1)!} (\log_e y)^{r-1} & 0 \leqslant y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

The joint density function of Y_1 and Y_2 is given by

$$f(y_1, y_2) = \begin{cases} \dfrac{(-1)^{r-1}}{(r-1)!} \left(\log_e \dfrac{y_1}{y_2} \right)^{r-1} \dfrac{1}{y_2} & 0<y_1<1 \text{ and } y_1 \leqslant y_2 < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Thus, the marginal density function of Y_1 is

$$f(y_1) = \frac{(-1)^{r-1}}{(r-1)!} \int_{y_1}^{1} \left[(\log_e y_1 - \log_e y_2)^{r-1} \frac{1}{y_2} \right] dy_2$$

$$= \frac{(-1)^{r-1}}{(r-1)!} \Big[(\log_e y_1)^{r-1}(\log_e y_2) - \binom{r-1}{1} (\log_e y_1)^{r-2} \frac{(\log_e y_2)^2}{2}$$

$$+ \ldots + (-1)^{r-1} \frac{1}{r} (\log_e y_2)^r \Big]_{y_1}^{1}$$

$$= \frac{(-1)^{r-1}}{(r-1)!} (\log_e y_1)^r \left[1 - \frac{1}{2}\binom{r-1}{1} + \frac{1}{3}\binom{r-1}{2} + \ldots + (-1)^{r-1} \frac{1}{r} \right] (-1)$$

$$= \frac{(-1)^{r-1}}{(r-1)!} (\log_e y_1)^r \frac{1}{r} (-1)$$

$$= \frac{(-1)^r}{r!} (\log_e y_1)^r.$$

This completes the proof of result (i).

(ii) The probability that the m.s.d. of $Y = X_1, \ldots, X_n$ is A is given by

$$\pi_A(n) = \frac{(-1)^{n-1}}{(n-1)!} \sum_{r=1}^{\infty} \int_{A/10^r}^{(A+1)/10^r} (\log_e y)^{n-1} dy$$

$$= \frac{(-1)^{n-1}}{(n-1)!} \sum_{r=1}^{\infty} \left[y \left\{ (\log_e y)^{n-1} - \binom{n-1}{1} (\log_e y)^{n-2} + \ldots + (-1)^{n-1}(n-1)! \right\} \right]_{A/10^r}^{(A+1)/10^r}$$

$$= \sum_{r=1}^{\infty} \left[y \left\{ 1 - \frac{1}{1!} \log y + \frac{1}{2!} (\log y)^2 - \ldots + \frac{(-1)^{n-1}}{(n-1)!} (\log y)^{n-1} \right\} \right]_{A/10^r}^{(A+1)/10^r}$$

$$= \sum_{r=1}^{\infty} \left[y \times \text{coeff. of } t^{n-1} \text{ in } \frac{e^{-t \log_e y}}{1-t} \right]_{A/10^r}^{(A+1)/10^r}$$

$$= \sum_{r=1}^{\infty} \left[\text{coeff. of } t^{n-1} \text{ in } \frac{e^{\log_e y(1-t)}}{1-t} \right]_{A/10^r}^{(A+1)/10^r}$$

$$= \sum_{r=1}^{\infty} \left[\text{coeff. of } t^{n-1} \text{ in} \frac{1}{1-t}. \quad \{(A+1)^{1-t} - A^{1-t}\}. \frac{1}{10^{r(1-t)}} \right]$$

$$= \text{coeff. of } t^{n-1} \text{ in} \left[\left\{ \frac{(A+1)^{1-t} - A^{1-t}}{1-t} \right\} \sum_{r=1}^{\infty} \frac{1}{10^{r(1-t)}} \right]$$

$$= \text{coeff. of } t^{n-1} \text{ in } \frac{(A+1)^{1-t} - A^{1-t}}{10^{1-t} - 1} \cdot \frac{1}{1-t} \quad \text{where } |t| < 1.$$

(iii) Let $$q(t) = \frac{(A+1)^{1-t}-A^{1-t}}{10^{1-t}-1} \qquad t \neq 1$$

$$= a_0+a_1 t+a_2 t^2+\ldots+a_n t^n+\ldots .$$

Then $$\pi_A(n) = \text{coeff. of } t^{n-1} \text{ in } \frac{q(t)}{1-t}$$

$$= a_0+a_1+\ldots+a_{n-1}.$$

Hence $$\lim_{n\to\infty} \pi_A(n) = a_0+a_1+\ldots+a_n+\ldots$$

$$= \lim_{t\to 1} q(t).$$

By continuity at $t = 1$, so that $q(1) = \log_{10}(A+1)/A$, $q(t)$ is analytic in the disc $|t| < |1+(2\pi i/\nu)|$ where $\nu = \log_e 10$, so that the power series expansion $\sum_{n=0}^{\infty} a_n t^n$ is valid in the disc. In particular,

$$q(1) = \Sigma a_n.$$

Thus $$\lim_{n\to\infty} \pi_n(A) = \log_{10} \frac{A+1}{A} .$$

The results which are derived above are valid with respect to any base β though the particular case of the base being ten has been discussed.

Definition 1 : The most significant digit of a positive valued random variable X is said to follow a log distribution if

$$\text{prob}\,\{\text{m.s.d. of } X \leqslant A\} = P\{M(x) \leqslant A\} = \log_{10}(A+1) \text{ for } A = 1, 2, \ldots, 9.$$

For any given value x of X, $M(x)$ i.e. the m.s.d. of x is uniquely defined viz. $x = 10^r M(x) + q(x)$, $r = 0, \pm 1, \pm 2, \ldots, q(x) < 10^r M(x)$ and $M(x) = 1, 2, \ldots, 9$.

Let $M'(x)$ be defined as a function of X such that for any given $X = x$, x can be represented uniquely as $x = 10^r.M'(x)$, $r = 0, \pm 1, \pm 2, \ldots, 1 \leqslant M'(x) < 10$.

Definition 2 : The most significant digit of a positive valued random variable X is said to follow log distribution strongly, if

$$\text{prob}\,\{M'(x) \leqslant y\} = \log_{10} y, \quad 1 \leqslant y < 10. \qquad \ldots \quad (3)$$

It can be seen that the limiting distribution of the most significant digit of powers and products of random numbers is strongly logarithmic.

Theorem 3 : *If the distribution of the most significant digit of a random variable X is strongly logarithmic so are the distribution of the m.s.d. of $1/X$ and CX, $C > 0$.*

Proof : Since the m.s.d. of X is strongly logarithmic

$$\text{prob}\,\{M'(x) \leqslant y\} = \log_{10} y = \sum_{-\infty}^{\infty} \text{prob}\,\{10^r \leqslant X < y.10^r\}, \quad 1 \leqslant y < 10. \quad \ldots \quad (4)$$

The most significant digit of $1/X$ will be less than y if $\frac{10^r}{y} \leqslant X < 10^r$ for some r.

Hence
$$P\left\{M\left(\frac{1}{X}\right) \leqslant y\right\} = \sum_{-\infty}^{\infty} \text{prob}\left\{\frac{10^r}{y} \leqslant X < 10^r\right\}$$

$$= 1-\log_{10}\left(\frac{10}{y}\right) = \log_{10} y.$$

Similarly

$$\text{prob}\,\{M'(C\,X) \leqslant y\} = \sum_{-\infty}^{\infty} \text{prob}\,\{10^r \leqslant CX < y\,10^r\}$$

$$= \sum_{-\infty}^{\infty} \text{prob}\left\{\frac{10^r}{C} \leqslant X < \frac{Y}{C}\,10^r\right\}. \quad \ldots \quad (5)$$

Let $1 < C \leqslant 10$, then $1 \leqslant \frac{10}{C} = x' < 10$ and $1 \leqslant yx' < 100$. Hence from (5),

$$\text{prob}\,\{M'(CX) \leqslant y\} = \sum_{-\infty}^{\infty} P\{10^{r-1}\,x' \leqslant X < yx'.10^{r-1}\} \quad \ldots \quad (6)$$

Case I : Let $1 \leqslant yx' < 10$, so that

$$P\{M'(CX) \leqslant y\} = \sum_{-\infty}^{\infty} P\{10^{r-1}\,x' \leqslant X < 10^{r-1}\,yx'\}$$
$$= 1-\log_{10} x'-1+\log_{10} yx'$$
$$= \log_{10} y.$$

Case II : *Let* $10 \leqslant yx' < 100$, so that

$$\text{prob}\,\{M'(Cx) \leqslant y\} = \sum_{-\infty}^{\infty} P\left\{10^{r-1}\,x' \leqslant X < 10^r\left(\frac{yx'}{10}\right)\right\}$$

$$= \sum_{-\infty}^{\infty}\left[P\{10^{r-1}\,x' \leqslant X < 10^r\}+P\left\{10^r \leqslant X < \left(\frac{yx'}{10}\right).10^r\right\}\right]$$

$$= 1-\log_{10} x'+\log_{10} \frac{yx'}{10}$$

$$= \log_{10} y.$$

Therefore the prob $\{M(CX) \leqslant y\}$ is given by $\log_{10} y$ for any $1 \leqslant C < 10$, and hence for any $C > 0$, since any $C' > 0$ can be expressed as $C' = 10^m C$ where $1 \leqslant C < 10$, m being any integer.

Appendix

TABLE A.1. PROBABILITY OF m.s.d. OF x^n FOR POSITIVE INTEGER n

n	m.s.d.								
	1	2	3	4	5	6	7	8	9
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111	0.1111
2	0.1016	0.1470	0.1230	0.1002	0.0087	0.0098	0.0845	0.0703	0.0750
3	0.2252	0.1579	0.1257	0.1062	0.0926	0.0830	0.0754	0.0604	0.0644
4	0.2431	0.1630	0.1260	0.1042	0.0896	0.0790	0.0709	0.0640	0.0594
5	0.2542	0.1659	0.1261	0.1030	0.0870	0.0766	0.0683	0.0618	0.0565
6	0.2618	0.1678	0.1261	0.1021	0.0862	0.0750	0.0665	0.0509	0.0546
7	0.2672	0.1690	0.1260	0.1014	0.0853	0.0738	0.0653	0.0586	0.0533
8	0.2714	0.1700	0.1259	0.1009	0.0845	0.0730	0.0644	0.0577	0.0523
9	0.2746	0.1707	0.1259	0.1004	0.0839	0.0723	0.0636	0.0569	0.0516
10	0.2772	0.1713	0.1258	0.1001	0.0835	0.0718	0.0631	0.0563	0.0510
15	0.2850	0.1730	0.1256	0.0991	0.0820	0.0702	0.0614	0.0546	0.0492
20	0.2890	0.1738	0.1254	0.0986	0.0813	0.0694	0.0605	0.0537	0.0489
25	0.2914	0.1742	0.1254	0.0982	0.0809	0.0689	0.0600	0.0532	0.0478
30	0.2930	0.1746	0.1253	0.0980	0.0806	0.0686	0.0597	0.0529	0.0475
40	0.2950	0.1749	0.1252	0.0977	0.0803	0.0681	0.0592	0.0524	0.0470
50	0.2962	0.1752	0.1252	0.0976	0.0800	0.0679	0.0590	0.0522	0.0468
60	0.2970	0.1753	0.1251	0.0975	0.0799	0.0677	0.0588	0.0520	0.0466
70	0.2976	0.1754	0.1251	0.0974	0.0798	0.0676	0.0587	0.0519	0.0465
80	0.2980	0.1755	0.1251	0.0973	0.0797	0.0675	0.0586	0.0518	0.0464
90	0.2983	0.1756	0.1251	0.0973	0.0797	0.0675	0.0585	0.0517	0.0463
100	0.2986	0.1756	0.1250	0.0972	0.0796	0.0674	0.0585	0.0517	0.0463
150	0.2994	0.1758	0.1250	0.0971	0.0795	0.0673	0.0583	0.0515	0.0461
200	0.2998	0.1759	0.1250	0.0971	0.0794	0.0672	0.0582	0.0514	0.0460
250	0.3001	0.1759	0.1250	0.0970	0.0794	0.0671	0.0582	0.0514	0.0460
300	0.3002	0.1759	0.1250	0.0970	0.0793	0.0671	0.0582	0.0513	0.0459
400	0.3004	0.1760	0.1250	0.0970	0.0793	0.0671	0.0581	0.0513	0.0459
500	0.3005	0.1760	0.1250	0.0970	0.0793	0.0670	0.0581	0.0513	0.0459
600	0.3006	0.1760	0.1250	0.0970	0.0793	0.0670	0.0581	0.0512	0.0458
700	0.3007	0.1760	0.1250	0.0970	0.0792	0.0670	0.0581	0.0512	0.0458
800	0.3007	0.1760	0.1250	0.0970	0.0792	0.0670	0.0581	0.0512	0.0458
900	0.3008	0.1760	0.1250	0.0969	0.0792	0.0670	0.0580	0.0512	0.0458
1000	0.3008	0.1760	0.1250	0.0969	0.0792	0.0670	0.0580	0.0512	0.0458
∞	0.3010	0.1761	0.1249	0.0969	0.0791	0.0669	0.0580	0.0512	0.0458

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TABLE A.2. PROBABILITY OF m.s.d. OF x^n FOR NEGATIVE INTEGER n.

n	m.s.d. 1	2	3	4	5	6	7	8	9
(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)
1	0.5556	0.1852	0.0926	0.0556	0.0370	0.0265	0.0198	0.0154	0.0123
2	0.4283	0.1898	0.1131	0.0771	0.0570	0.0443	0.0357	0.0296	0.0250
3	0.3850	0.1873	0.1183	0.0843	0.0644	0.0514	0.0425	0.0359	0.0310
4	0.3635	0.1852	0.1205	0.0877	0.0681	0.0552	0.0461	0.0394	0.0343
5	0.3508	0.1837	0.1216	0.0896	0.0703	0.0575	0.0484	0.0416	0.0364
6	0.3423	0.1827	0.1223	0.0909	0.0718	0.0590	0.0499	0.0431	0.0379
7	0.3363	0.1818	0.1228	0.0918	0.0729	0.0602	0.0510	0.0442	0.0389
8	0.3318	0.1812	0.1231	0.0925	0.0737	0.0610	0.0519	0.0451	0.0397
9	0.3284	0.1807	0.1233	0.0930	0.0743	0.0616	0.0526	0.0457	0.0404
10	0.3256	0.1803	0.1235	0.0934	0.0748	0.0622	0.0531	0.0462	0.0409
15	0.3173	0.1789	0.1241	0.0946	0.0763	0.0638	0.0547	0.0478	0.0425
20	0.3132	0.1783	0.1243	0.0952	0.0770	0.0646	0.0555	0.0487	0.0433
25	0.3108	0.1778	0.1244	0.0955	0.0774	0.0650	0.0560	0.0492	0.0438
30	0.3091	0.1776	0.1245	0.0958	0.0777	0.0653	0.0563	0.0495	0.0441
40	0.3071	0.1772	0.1246	0.0961	0.0781	0.0657	0.0567	0.0499	0.0445
50	0.3059	0.1770	0.1247	0.0962	0.0783	0.0660	0.0570	0.0501	0.0448
60	0.3051	0.1768	0.1247	0.0963	0.0785	0.0661	0.0572	0.0503	0.0449
70	0.3045	0.1767	0.1248	0.0964	0.0786	0.0663	0.0573	0.0504	0.0450
80	0.3041	0.1766	0.1248	0.0965	0.0786	0.0663	0.0574	0.0505	0.0451
90	0.3037	0.1766	0.1248	0.0965	0.0787	0.0664	0.0574	0.0506	0.0452
100	0.3035	0.1765	0.1248	0.0966	0.0787	0.0665	0.0575	0.0506	0.0453
150	0.3026	0.1764	0.1249	0.0967	0.0789	0.0666	0.0577	0.0508	0.0454
200	0.3022	0.1763	0.1249	0.0967	0.0790	0.0667	0.0577	0.0509	0.0455
250	0.3020	0.1763	0.1249	0.0968	0.0790	0.0668	0.0578	0.0510	0.0456
300	0.3018	0.1762	0.1249	0.0968	0.0790	0.0668	0.0578	0.0510	0.0456
400	0.3016	0.1762	0.1249	0.0968	0.0791	0.0668	0.0579	0.0510	0.0456
500	0.3015	0.1762	0.1249	0.0968	0.0791	0.0669	0.0579	0.0511	0.0457
600	0.3014	0.1762	0.1249	0.0969	0.0791	0.0669	0.0579	0.0511	0.0457
700	0.3014	0.1762	0.1249	0.0969	0.0791	0.0669	0.0579	0.0511	0.0457
800	0.3013	0.1761	0.1249	0.0969	0.0791	0.0669	0.0579	0.0511	0.0457
900	0.3013	0.1761	0.1249	0.0969	0.0791	0.0669	0.0579	0.0511	0.0457
1000	0.3013	0.1761	0.1249	0.0969	0.0791	0.0669	0.0579	0.0511	0.0457
∞	0.3010	0.1761	0.1249	0.0969	0.0791	0.0669	0.0580	0.0512	0.4058

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