Improved lower and upper bounds on the span of distance labeling for some infinite regular grids

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by

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DECLARATION

Thesis Title	Improved lower and upper bounds on the span of distance labeling for some infinite regular grids
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I declare that the results presented in this thesis entitled *Improved lower and upper bounds on the span of distance labeling for some infinite regular grids* are the result of my own research work. I also declare that I have not presented or submitted the thesis for evaluation of any other degree.

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ABSTRACT

The channel assignment problem, popularly known as CAP, is one of the elementary and much studied topic in the field of wireless communication. The basic purpose for studying CAP is to find out solutions such that wireless communication becomes interference free with using spectrum as less as possible during the communication. Often the CAP is modeled as an $L(k_1, \ldots, k_\ell)$-vertex (edge) labeling problem of a graph, where $k_1, \ldots, k_\ell$ are non-negative integers. In $L(k_1, \ldots, k_\ell)$-vertex (edge) labeling problem, labels are assigned to the vertices (edges) of a graph in such a way that the absolute difference between the labels assigned to any pair of vertices (edges) located at distance i, $1 \leq i \leq \ell$, is k_i. One of the objective of $L(k_1, \ldots, k_\ell)$-vertex (edge) labeling of a graph G is to find a labeling of the vertices (edges) such that the span for the corresponding labeling is minimum among all $L(k_1, \ldots, k_\ell)$-vertex (edge) labelings of G, where span denotes the difference between maximum and minimum labels used for a labeling. Regular grid graphs are common choices for modeling CAP because of their natural resemblance to cellular network for regular geometric pattern. Consequently, various studies of $L(k_1, k_2, \ldots, k_\ell)$-vertex (edge) labeling have been done for infinite regular grids such as infinite hexagonal (T_3), square (T_4), triangular (T_6) and infinite 8-regular grid (T_8) grids. In this thesis, we first derive the exact values of the span of $L(1,2)$-edge labeling problem for T_3 and T_4. Then we improve the lower bound on the span of $L(1,2)$-edge labeling problem for T_6. Next by improving the lower bound, we derive the exact value of the span of $L(1,2)$-edge labeling of T_8. Next we attempt to derive theoretically the lower bound on the span of $L(k_1, k_2)$-vertex labeling problem for T_6 for $k_1 \leq k_2$. For this problem, the previous results were obtained partially through computer simulations. We find that our theoretically obtained results exactly coincide with the known results for the sub interval $0 \leq \frac{k_1}{k_2} \leq \frac{1}{3}$ but provide loose bound for the other sub interval $\frac{1}{3} \leq \frac{k_1}{k_2} \leq 1$. Next we derive improved lower bound on the span of $L(2,1)$-edge labeling problem for T_6. Next we study the $L(\underbrace{1, 1, \ldots, 1}_{\ell})$-vertex labeling problem for T_3. The exact value of the span of $L(\underbrace{1, 1, \ldots, 1}_{\ell})$-vertex labeling problem for T_3 has not been determined yet for any even $\ell \geq 8$, rather the value of

the span was conjectured. We prove this conjecture for $\ell \geq 8$. In all the cases we analyze the structural properties of the underlined graphs and based on which the results are obtained theoretically.

LIST OF PUBLICATIONS FROM THE CONTENT OF THE THESIS

In conferences/Journal (Published/Accepted/Submitted)

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4. [1] Susobhan Bandopadhyay, Sasthi C. Ghosh and **Subhasis Koley**. $L(2,1)$-edge labeling of infinite triangular grid. In: Proceedings of the *22nd Italian Conference on Theoretical Computer Science (ICTCS 2021), Bologna, Italy, September 13-15, 2021. CEUR Workshop Proceedings, vol. 3072, pp. 179-192, 2021*.

5. **Subhasis Koley** and Sasthi C. Ghosh. Deriving bounds on $L(k_1, k_2)$ labelling of triangular lattice by exploring underlined graph structures. In: Proceedings of the *7th International Conference on Mathematics and Computing (ICMC 2021), Shibpur, India, March 02-05, 2021. Springer, vol. 1412, pp. 591-601, 2021*.

6. [1] Susobhan Bandopadhyay, Sasthi C. Ghosh and **Subhasis Koley**. Improved Bounds on the span of L(1,2)-edge labeling of some infinite regular grids. In: Proceedings of the *18th Cologne-Twente Workshop on Graphs and Combinatorial Optimization (CTW 2020), September 14-16, 2020* (Graphs and Combinatorial Optimization: from Theory to Applications, AIRO Springer Series, Vol. 5, pp. 53-65, 2020).

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NOTATIONS

Notation	Explanation
CAP	Channel Assignment Problem
$d(u,v)$	distance between two vertices u and v of a graph
$d'(e_1,e_2)$	distance between two edges e_1 and e_2 of a graph
$L(G)$	Line graph of $G(V,E)$
T_3	Infinite hexagonal grid
T_4	Infinite square grid
T_6	Infinite triangular grid
T_8	Infinite octagonal grid
$\lambda_{k_1,k_2,\ldots,k_\ell}(G)$	Span for $L(k_1,k_2,\ldots,k_\ell)$-vertex labeling of the graph G
$\lambda'_{k_1,k_2,\ldots,k_\ell}(G)$	Span for $L(k_1,k_2,\ldots,k_\ell)$-edge labeling of the graph G
$\sigma_{k_1,k_2,\ldots,k_\ell}(G)$	Span for circular $L(k_1,k_2,\ldots,k_\ell)$-vertex labeling of the graph G
$\sigma'_{k_1,k_2,\ldots,k_\ell}(G)$	Span for circular $L(k_1,k_2,\ldots,k_\ell)$-edge labeling of the graph G

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Chapter 1

Introduction

Channel assignment problem (CAP) is one of the fundamental problems in wireless communication networks. In wireless communication networks, frequency bands are assigned to the transmitters for data communication. On the receiver's side, the receiver captures the signal intended for it and processes it. Therefore, if more than one proximity transmitters use same frequency band or overlapping frequency bands concurrently for communication, then interference will definitely occur and the communication will become noisy. So, frequency bands assigned to proximity transmitters must be distinct and must have predefined gap between them if the transmitters wants to communicate at a same time. It is well known that number of transmitters and receivers engaged in communication are far more than total number of available frequency bands for a fairly big communication network as frequency resources are limited. So for CAP, it is a challenging problem to assign frequency bands to transmitters using available frequency bands such that the communication becomes interference free. In other word, it can be said that CAP is an optimization problem where the objective is to find least number of frequency bands needed for interference free communication. This is also one of the rudimentary problems in wireless communication.

Various methods have been employed in wireless communication networks such that number of frequency bands needed for communication can be made as less as possible for interference free communication. One possible method is to assign non overlapping frequency bands at proximity transmitters which are engaged in communication at the same time and to reuse frequency bands at transmitters which are at far distance apart. This is particularly possible because effect of interference fades with distance. In practice, the effect of interference is taken into consideration up to some n ($n > 1$) hop distances. As generally interference diminishes with distance, gap between any pair of frequency bands assigned to the transmitters at one hop distance is kept at maximum, the corresponding gaps are reduced with increase of hop distances and the gap is kept at minimum when the transmitters are at n distance apart. However, in ad hoc network and device to device (D2D) communication, users, located at near distance, can communicate among them without the help of any base station. If two devices can sense each other (distance 1 apart in this context), proper parameter settings of multiple input multiple output (MIMO) antenna can be done so that the required frequency gap between those transmitting devices may be made shorten. If two transmitting devices can not sense each other (distance 2 apart in this context), then due to the hidden terminal problem, the required frequency gap between them may be higher than that of the required gap when the devices can sense each other. In both the cases, one of the the objective of `CAP` can be finding out the least number of frequency bands such that communication can be made without interference.

The optimization problem stated above has been encountered in several ways. Some of them use neural network based approach, some use genetic algorithm where some of them use search method [1, 2, 3]. There also have some other methods also. Among them, in one approach, `CAP` has been formulated as a vertex coloring problem where the vertices of the graph can be thought as transmitters and colors assigned to the vertices can be viewed as frequency bands assigned to the corresponding transmitters. Here an edge between two vertices represent that the corresponding transmitters are interfering to each other. Whereas, in another approach, `CAP` has been formulated as an edge coloring problem where each edge represents the communication link between a pair of users and colors assigned to the edges can be viewed as frequency bands assigned to the corresponding links. Therefore determining the minimum color needed to color the vertices/edges is equivalent to determining minimum frequency spectrum required for interference free communication. Hale [4] first formulated the `CAP` as a classical graph coloring problem. But it is not able to model the scenario when effect of interference persists beyond one hop distance. So to capture the effect of interference for more than one hop distance, $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling problem and $L(k_1, k_2, \ldots, k_\ell)$-edge labeling problem was introduced [5], [6], [7]. Now, it is very natural to model the cellular network as infinite regular grids for the regular pattern of the cellular network. So, investigating $L(k_1, k_2, \ldots, k_\ell)$ labeling for different infinite grids is also an interesting problem. In this thesis we explore and derive several results for $L(k_1, k_2, \ldots, k_\ell)$ labeling for different types of infinite grids. Below we formally define different versions of $L(k_1, k_2, \ldots, k_\ell)$ labeling problems and describe different types of infinite regular grids.

$L(k_1, k_2, \ldots, k_\ell)$-vertex labeling

For $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling problem, the absolute difference between colors assigned to two vertices $u, v \in V$ at distance i ($1 \leq i \leq l$) is at least k_i. Here distance between any two vertices u and v is the minimum number of edges in E that connect u and v and it is denoted as $d(u, v)$. For any coloring function f of the vertices of G that admits $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling for the graph G, the span is defined as $\max_{u \in V} f(u) - \min_{v \in V} f(v)$. Here $\lambda_{k_1, k_2, \ldots, k_\ell}(G)$ is the minimum span over all such f. The objective of $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling problem is to find $\lambda_{k_1, k_2, \ldots, k_\ell}(G)$. Therefore finding minimum span for $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling of a graph $G(V, E)$ is analogically same as determining the least frequency bandwidth for interference free communication if the transmitters are represented by the vertices of G, colors of the vertices represent frequency bands assigned to the transmitters, distance between any two vertices in G represents the hop distance between them.

$L(k_1, k_2, \ldots, k_\ell)$-edge labeling

Like $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling problem, $L(k_1, k_2, \ldots, k_\ell)$-edge labeling problem was also introduced in [7]. For $L(k_1, k_2, \ldots, k_\ell)$-edge labeling problem, the absolute difference between colors assigned to two edges $e_1, e_2 \in E$ at distance i ($1 \leq i \leq l$) is at least k_i. Here distance between any two edges e_1 and e_2 is represented as $d'(e_1, e_2)$ and if $d'(e_1, e_2) = k$ then least number of edges in E that connect e_1 and e_2 is $k - 1$, where $k \geq 1$. For any coloring function f' of the edges of G that admits $L(k_1, k_2, \ldots, k_\ell)$-edge labeling for the graph G, the span is defined as $\max_{e_1 \in E} f'(e_1) - \min_{e_2 \in E} f'(e_2)$. Here $\lambda'_{k_1, k_2, \ldots, k_\ell}(G)$ is the least span over all such f'. The objective of $L(k_1, k_2, \ldots, k_\ell)$-edge labeling problem is to find $\lambda'_{k_1, k_2, \ldots, k_\ell}(G)$. Here also, finding minimum span for $L(k_1, k_2, \ldots, k_\ell)$-edge labeling of a graph $G(V, E)$ is analogically same as determining least frequency bandwidth for interference free communication if the communication links formed between the users are represented by the edges of G, colors of the edges represent frequency bands assigned to the links, distance between any two edges in G represents the hop distance between them.

Circular $L(k_1, k_2, \ldots, k_\ell)$ labeling

One interesting variant of $L(k_1, k_2, \ldots, k_\ell)$ labeling problem is circular $L(k_1, k_2, \ldots, k_\ell)$ labeling problem [8]. Both $L(k_1, k_2, \ldots, k_\ell)$-vertex and edge labeling problems have their corresponding circular labeling versions. As in this thesis we deal with edge version of the circular labeling, we define it formally. For $\ell + 1$ given integers $n, k_1, k_2, \ldots, k_\ell$, an n-circular-$L(k_1, k_2, \ldots, k_\ell)$-edge labeling f' of the edges of a graph G assigns integers from the set of $\{0, 1, \ldots, n-1\}$ to the edges of G in a manner that $|f'(e_1) - f'(e_2)|_n \geq k_i$ if $d'(e_1, e_2) = i$ $(1 \leq i \leq l)$. It is noted that $|x|_n = \min\{x, n-x\}$. Here $d'(e_1, e_2)$ is the distance between the two edges e_1 and e_2. Here the circular span is equal to n. Note that $\sigma'_{k_1,k_2,\ldots,k_\ell}(G)$ is the least n among all circular $L(k_1, k_2, \ldots, k_\ell)$-edge labeling of G. The objective of n-circular $L(k_1, k_2, \ldots, k_\ell)$-edge labeling is to find $\sigma'_{k_1,k_2,\ldots,k_\ell}(G)$.

Infinite regular grids

Regular grid graphs are the graphs which are formed by tilling a two dimensional plane with regular two dimensional geometric patterns. An infinite regular hexagonal (T_3), square (T_4) and triangular (T_6) grids are formed by tilling a two dimensional plane regularly with regular hexagons, squares and equilateral triangles respectively. An infinite 8-regular grid is a regular grid where degree of each vertex is 8. In Figure 1.1, Figure 1.3, Figure 1.4 and Figure 1.5, T_3, T_4, T_6 and T_8 have been shown respectively. In Figure 1.2, brick structure representation of T_3 has been shown. Note that the degrees of T_3, T_4, T_6 and T_8 are `three`, `four`, `six` and `eight` respectively.

In next section, the motivation behind our study is stated.

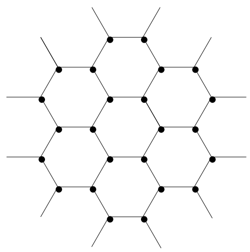


Figure 1.1: A part of infinite Hexagonal grid T_3

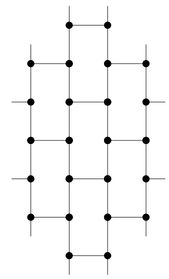


Figure 1.2: A part of brick structure presentation of T_3

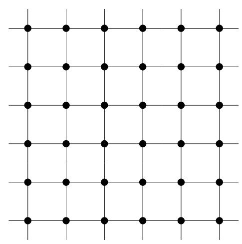


Figure 1.3: A part of infinite Square grid T_4

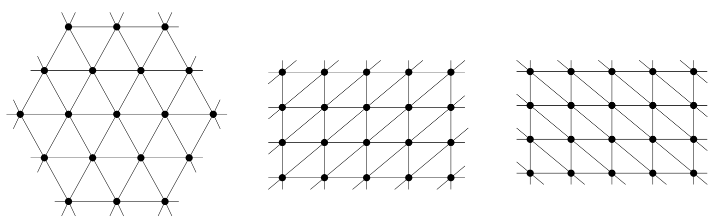


Figure 1.4: Three representations of infinite Triangular grid T_6

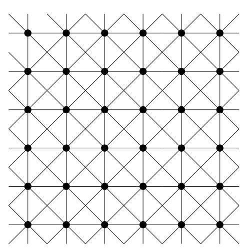


Figure 1.5: A part of infinite 8-regular grid grid T_8

1.1 Motivation

Several studies have been made for both vertex and edge version of $L(k_1, k_2, \ldots, k_\ell)$ labeling problem for T_3, T_4, T_6 and T_8 [9, 10, 11, 12, 13, 14, 15, 16, 17, 18]. But in some of the studies, rather finding an exact value of the minimum span, lower and upper bounds were given, in some of the studies the value of minimum span was conjectured and in some of the studies the values for minimum span were obtained with the help of computer simulation. In this thesis, we study some of these problems stated above and give improved results.

Specially, in this thesis, first we derive the exact values of $\lambda'_{1,2}(T_\Delta)$ for $\Delta = 3$ and $\Delta = 4$. Previously upper and lower bounds were given for these. Then we improve the lower bound of $\lambda'_{1,2}(T_\Delta)$ for $\Delta = 6$. Next by improving the existing lower bound of $\lambda'_{1,2}(T_8)$ we determine the exact value of $\lambda'_{1,2}(T_8)$. After that, we determine the lower bounds of $\lambda_{k_1,k_2}(T_6)$ theoretically when $k_1 \leq k_2$ for some sub intervals by examining the underlined graph structures, where the known results were partially based on computer simulations. Next we prove that $\lambda'_{2,1}(T_6) = 16$ which was posed as a conjecture. After that we study circular $L(2,1)$-edge labeling problem for T_6 and derive a labeling function to show the upper bound of $\sigma'_{2,1}(T_6)$. Note that no labeling function was known for circular $L(2,1)$-edge labeling of T_6. Next we study $L(\underbrace{1,1,\ldots,1}_{\ell})$-vertex labeling problem for T_3. The exact values of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ are known for all odd ℓ and even $\ell < 8$ but the corresponding values were conjectured for all even $\ell \geq 8$. Here we settle the conjecture for even $\ell \geq 8$.

1.2 Literature survey

In this thesis we deal with $L(1,2)$-edge labeling for T_3, T_4, T_6, T_8, $L(k_1,k_2)$-vertex labeling of T_6 when $k_1 \leq k_2$, $L(2,1)$-edge labeling and circular $L(2,1)$-edge labeling for T_6 and $L(\underbrace{1,1,\ldots,1}_{\ell})$-vertex labeling for T_3 when $\ell = 8$. First of all in the following paragraph we will present a general literature survey of distance labeling and then in subsequent paragraphs we will focus on the literature on distance labeling for infinite regular grid graphs.

Hale [4] dealt with different frequency assignment problems and model it as a vertex coloring problem. Later the concept of $L(k_1,k_2)$-vertex labeling was introduced to incorporate the effect of interference at two hop distance into the formulation [6]. Griggs and yeh [5] posed some fundamental issues regarding $L(2,1)$-vertex labeling problem. For a graph G, they proved that determining $\lambda_{2,1)}(G)$ is an NP-complete problem. Further different NP-hardness results for $L(2,1)$-vertex labeling and $L(k_1,k_2)$-vertex labeling for $k_1 > k_2 \geq 1$ are studied in [19, 20, 21, 22]. A detailed survey for related NP-hard result can be found in [9]. In [5], Griggs and Yeh showed that for a graph G with maximum degree Δ, $\lambda_{2,1}(G) \leq \Delta^2 + 2\Delta$. They also proposed a conjecture that given a graph G with maximum degree $\Delta \geq 2$, $\lambda_{2,1}(G) \leq \Delta^2$. Later Chang and Kuo [23], Král'and Skrekovski [24] and Goncalves [25] improved the upper bound obtained in [5]. Later the conjecture proposed in [5] was settled for sufficiently big value of Δ in [26] [27]. Extensive study has been done to determine bounds of the minimum spans for $L(0,1)$, $L(1,1)$, $L(2,1)$ as well as $L(k_1, k_2)$-vertex labeling for both of $k_1 \geq k_2$ and $k_1 < k_2$ for paths, cycles cliques, wheels, product of some known graph classes, planar and outer planar graph, chordal graph, regular graph, bipartite graph, kneser graph, intersection graph and graphs with given maximum degree and those can be found in [2, 28, 9, 5, 29, 30, 31, 32, 33]. Later Calamoneri [34] and Soumen [35] studied $L(2,1)$ labeling and $L(p,1)$ labeling problem for oriented planar graphs and oriented graphs respectively. Deng et al. [36] studied the $L(d,1)$-vertex labeling problem for generalised Petersen graphs. Regarding infinite regular grids, several results for $L(k_1, k_2)$-vertex labeling were presented in [37, 38, 39, 11, 40, 13, 12, 41, 42, 43]. Studies have been made for $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling for different values of $k_1, k_2, \ldots, k_\ell$ when $\ell > 2$ and the results can be found in [10, 44, 45, 46, 47, 48, 49, 50]. Several studies have also been made for $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling in [51, 52, 53, 54, 55] for different types of graphs for particular values of $k_1, k_2, \ldots, k_\ell$ when $\ell = 3$. Duan et al. [56] determined the value of $\lambda_{3,2,1}(T_3)$ and $\lambda_{3,2,1}(T_4)$. Later Atta and Mahapatra [57] studied the problem for T_4 in more general way and obtained the value of $\lambda_{D,2,1}(T_4)$ for all integer $D \geq 4$. Duan et al. [58] gave bounds for $\lambda_{3,2,1}(T_6)$ and Calamoneri [59] further improved the bound of $\lambda_{3,2,1}(T_6)$ and conjectured the exact value of $\lambda_{3,2,1}(T_6)$. Shao and Vesel [60] proved the conjecture posed in [59]. Later Das et al. [61] investigated the problem in a more general way and gave exact value of $\lambda_{4,2,1}(T_6)$ and lower bounds of $\lambda_{d,2,1}(T_6)$ for $d \geq 5$. Soumen et al. extended the problem to $L(k, k-1, \ldots, 1)$-vertex labeling problem of T_6 and gave upper bound $\lambda_{k,k-1,\ldots,1}(T_6)$ in [62]. For T_8, Calamoneri [59] studied $L(2,1,1)$ and $L(3,2,1)$-vertex labeling of T_8. Like distance vertex labeling, $L(k_1, k_2, \ldots, k_\ell)$-edge labeling of various types of graphs has also been studied. But the focus for $L(k_1, k_2, \ldots, k_\ell)$-edge labeling is limited for $\ell = 2$. Authors in [7] studied $L(k_1, k_2)$-edge labeling problem for complete graphs, trees, cubes and joins for $k_1 = 1, 2$ and $k_2 = 1$. For $L(k_1, k_2)$-edge labeling of T_3, T_4, T_6 and

T_8 when $k_1 = 1,2$ and $k_2 = 1,2$ have been studied in [14, 15, 16, 17]. Now we will present the literature suited for our interest in next paragraphs.

In this thesis, first we study the $L(1,2)$-edge labeling problem for T_Δ when $\Delta = 3,4,6$. In [7], Georges and Mauro first studied the $L(k_1,k_2)$-edge labeling problem for *trees, n-cubes, complete graphs and joins* when $k_1 = 1,2$ and $k_2 = 1$. Later Chen and Lin [14] investigated the $L(k_1,k_2)$-edge labeling problem on $K_{1,3}$ free graphs. They also provided the upper bound of $\lambda'_{k_1,k_2}(G)$ with respect to maximum degree of the line graph of G. Further Dan and Lin [16] evaluated the upper and lower bounds for $\lambda'_{1,2}(T_3)$, $\lambda'_{1,2}(T_4)$ and $\lambda'_{1,2}(T_6)$ but there are gaps in between the lower and upper bounds. In this thesis, we improve $\lambda'_{1,2}(T_\Delta)$ for each of $\Delta = 3,4,6$.

Next we study the problem of $L(1,2)$-edge labeling problem of infinite 8-regular grid T_8. In [17], it was shown that $25 \leq \lambda'_{1,2}(T_8) \leq 28$. In this thesis, we determine that $\lambda'_{1,2}(T_8) \geq 28$. As it was shown $\lambda'_{1,2}(T_8) \leq 28$ in [17], we conclude that $\lambda'_{1,2}(T_8) = 28$.

Next we study the lower bound of $\lambda_{k_1,k_2}(T_6)$ when $k_1 \leq k_2$. For a graph G, it can be shown from the scaling lemma in [13], $\lambda_{k_1,k_2}(G) = k_2 * \lambda_{\frac{k_1}{k_2},1}(G) = k_2 * \lambda_{h,1}(G)$, where $h = \dfrac{k_1}{k_2}$. Determining $\lambda_{k_1,k_2}(G)$ is equivalent to evaluating $\lambda_{h,1}(G)$ and multiplying it with k_2. So, we actually discuss $L(h,1)$-vertex labeling for T_6 when $0 \leq h \leq 1$ here. Efforts have been made to determine bounds of $\lambda_{h,1}(T_\Delta)$ when $0 \leq h \leq 1$ for $\Delta = 3,4,6$. Griggs and Jin [13] determined the values of $\lambda_{h,1}(T_3)$ and $\lambda_{h,1}(T_4)$ when $0 \leq h \leq 1$ but gave lower and upper bounds for $\lambda_{h,1}(T_6)$ for different intervals for $0 \leq h \leq 1$. But the proposed method in [13] for determining $\lambda_{h,1}(T_6)$ was partly computer assisted. Later, Daniel Král and Petr Skoda [12] gave exact values of $\lambda_{h,1}(T_6)$ for different finer sub intervals for $0 \leq h \leq 1$. In this case too the proof techniques depend on partially computer simulation. We study the $L(h,1)$-vertex labeling problem for T_6 when $0 \leq h \leq 1$. In our approach, we theoretically determine the lower bounds of $\lambda_{h,1}(T_6)$ for two sub intervals when $0 \leq h \leq \dfrac{1}{2}$ and $h \geq \dfrac{1}{2}$. We find that our theoretically obtained results exactly coincide with the known results for the sub interval $0 \leq h \leq \dfrac{1}{3}$ but provide loose bound for the other sub interval $h \geq \dfrac{1}{3}$.

Next we study $L(2,1)$-edge labeling problem and circular $L(2,1)$-edge labeling problem for T_6. Lin and Wu [15] studied $L(k_1,k_2)$-edge labeling and circular $L(k_1,k_2)$-edge labeling for T_3, T_4 and T_6 for $k_1 = 1,2$ and $k_2 = 1$. They obtained the exact values of $\lambda'_{1,1}(T_3)$, $\lambda'_{2,1}(T_3)$, $\sigma'_{2,1}(T_3)$, $\lambda'_{1,1}(T_6)$ and gave upper and lower bounds for $\lambda'_{2,1}(T_4)$, $\sigma'_{2,1}(T_4)$ and $\sigma'_{2,1}(T_6)$. It was conjecture in [15] that $\lambda'_{2,1}(T_6) = 16$. Later Calamoneri [17] improved some of the bounds for $\lambda'_{k_1,k_2}(T_\Delta)$ for $\Delta = 3,4,6$ when $k_1 = 1,2$ and $k_2 = 1$, but the conjecture remains unsettled. In our work, we prove the conjecture.

Next we investigate the $L(k_1,k_2,\ldots,k_\ell)$-vertex labeling problem for T_3 for all even $\ell \geq 8$ and $k_1 = k_2 = \cdots, k_\ell = 1$. Research work of ℓ distance vertex labeling for T_3 have been studied by several authors [63, 64, 65, 18, 66, 67, 68]. Jacko and Jendrol' [18] studied the $L(k_1,k_2,\ldots,k_\ell)$-vertex labeling problem for T_3 when $k_1 = k_2 = \cdots = k_\ell = 1$ and determined the value of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ for all odd ℓ and even $\ell < 8$. However, they conjectured the value of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ for all even $\ell \geq 8$. In our work, we determine the exact values of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ for all even $\ell \geq 8$ and the obtained values coincide with the conjecture values posed in [18]. Thus we settle the conjecture.

1.3 Scope of the thesis

We study the $L(1,2)$-edge labeling problem for T_Δ when $\Delta = 3,4,6$ in chapter 2. For $\lambda'_{1,2}(T_\Delta)$, upper and lower bounds were given in [14, 15, 16, 17]. In this chapter, we establish improved bounds for each of T_3, T_4, T_6. The existing best results for T_3, T_4 were $7 \leq \lambda'_{1,2}(T_3) \leq 8$ [16] and $10 \leq \lambda'_{1,2}(T_4) \leq 11$ [16] respectively. Here, by improving the lower bounds, we determine that $\lambda'_{1,2}(T_3) = 7$ and $\lambda'_{1,2}(T_4) = 11$. Given a graph $G(V,E)$, its *line graph* $L(G)(V',E')$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ have an edge if and only if their corresponding edges share a common vertex in G. To obtain the bounds for T_3 and T_4, we investigate vertex labeling of $L(T_3)$, $L(T_4)$ and derive the values of $\lambda_{1,2}(L(T_3))$ and $\lambda_{1,2}(L(T_4))$ because it is known that for any graph $G(V,E)$, $\lambda'_{1,2}(G) = \lambda_{1,2}(L(G))$. To prove the results, we identify some sub graphs of $L(T_3)$ and $L(T_4)$ and we analyze some structural properties of those sub graphs and thus we obtain the improved results. For T_6, the existing best results were $16 \leq \lambda'_{1,2}(T_6) \leq 20$ [17] respectively. Here too, by improving the lower bounds, we determine that $18 \leq \lambda'_{1,2}(T_6) \leq 20$. However, for T_6, we do not consider the line graphs of T_6 for high degrees of $L(T_6)$. Here bounds are obtained based on identifying some sub graphs of T_6 and and analysing some structural properties of those sub graphs.

In next chapter (chapter 3), we study $L(1,2)$- edge labeling problem for infinite 8-regular grid T_8. This problem was studied in [17] and it was shown that $25 \leq \lambda'_{1,2}(T_8) \leq 28$. In this chapter, we prove that $\lambda'_{1,2}(T_8) \geq 28$. As in [17], $\lambda'_{1,2}(T_8) \leq 28$, so $\lambda'_{1,2}(T_8) = 28$. To prove the lower bound of $\lambda'_{1,2}(T_8)$, we use the structural properties of T_8. More specifically, we first identify a sub graph G' of T_8 where no two edges can have the same color. After that we consider all edges where a pair of consecutive colors $(c, c \pm 1)$ can be used in G'. Based on this, we identify the sub graphs in T_8 where c and $c \pm 1$ can not be used. Then using the structural properties of those sub graphs we conclude how many additional colors other the colors used in G' must be required to color T_8 by using the pigeon hole principle and accordingly derive the span.

In chapter 4, we study the $L(k_1, k_2)$-vertex labeling problem for T_6 when $k_1 \leq k_2$. For a graph G, it follows from the scaling lemma in [13] that $\lambda_{k_1,k_2}(G) = k_2 * \lambda_{h,1}(G)$, where $h = \dfrac{k_1}{k_2}$. In [13], bounds of $\lambda_{h,1}(T_6)$ were obtained for $0 \leq h \leq 1$. But some of the bounds were not tight and moreover, the bounds were obtained partially based on computer simulation. More specifically, the bounds [13] are obtained by considering all possible $L(h,1)$ labeling of three induced sub graphs of T_6 having 7, 19 and 37 nodes using computer simulation. Later, by improving these bounds, exact values of $\lambda_{h,1}(T_6)$ were obtained by Král and Skoda [12] for different sub intervals of $0 \leq h \leq 1$. But here also, the bounds are obtained through brute force computer simulations on the induced sub-graphs of T_6 having 81, 100, 169 and 225 nodes. Here, we attempt to determine the minimum span of $L(h,1)$-vertex labeling for T_6 by examining the underlined graph structures with a theoretical proof for the bounds. Our result exactly coincides with the results previously obtained when $0 \leq h \leq \dfrac{1}{3}$. When $\dfrac{1}{3} \leq h \leq 1$, however, results obtained in [13, 12] are finer than ours. In our work, we introduce the notion of color class to represent all the colors within a specific interval. We also identify a subset of vertices in T_6 where colors from the same color class can not be used. Later we show that how many color classes are required to color the vertices of specific sub graph in T_6 and in that case the value of maximum color which must be used in the specific sub graph will determine the required minimum span.

In chapter 5, we focus on determining the exact values of $\lambda'_{2,1}(T_6)$. It was known that $15 \leq \lambda'_{2,1}(T_6) \leq 16$ and was conjecture that $\lambda'_{2,1}(T_6) = 16$ [15]. Deepthy and Joseph [69] claimed to have proved the conjecture but their proof is found to be incorrect and we will discuss it in chapter 5. In our work, we prove the conjecture. To prove the conjecture, we first identify a sub graph G' in T_6 and examine minimum how many colors are required to color the edges of the sub graph. Next, we identify another sub graph G'' in T_6 where G' is a proper sub graph of G''. First, for every color used in G' we examine that how many times at maximum the color can be reused in G''. By doing so, we conclude that at least a new color apart from the colors used in G' is needed to color the edges of G''. Later by analysing the reusability condition of the colors including the new color in G'', we find at least how many colors are required to color the edges of G'' and that leads us to settle the conjecture. Later in this chapter, we work on circular $L(2,1)$-edge labeling for T_6. Values of upper and lower bounds for $\sigma'_{2,1}(T_6)$ are given in [15]. But no labeling function was given there for the upper bound. Here we give a labeling function for circular $L(2,1)$-edge labeling in T_6 and the labeling function proposed by us give the same upper bound as mentioned in [15].

In chapter 6, we study the $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling problem for T_3 for even $\ell \geq 8$ and $k_1 = k_2 = \cdots, k_\ell = 1$. In [18], Jacko and Jendrol determined the exact value of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ for all odd ℓ. They also determined the exact value of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ for all even $\ell < 8$. For all even $\ell \geq 8$, they derived the upper bound and conjectured the exact value of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$. In our work, we determine the exact values of $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ for even $\ell \geq 8$. Our calculated values exactly coincide with the corresponding conjectured values and hence the conjecture is settled. In our approach, first we prove the conjecture when $\ell = 8$. In this case we choose any vertex in T_3 and consider two sub graphs G' and G'' which are induced by all the vertices at distance four and distance eight apart respectively from the chosen vertex. For $L(\underbrace{1,1,\ldots,1}_{8})$-vertex labeling of T_3, it can be shown that 31 distinct colors are required to color the vertices of G'. Next we examine how many times at maximum the colors of G' can be reused in G'' individually and all together. It is observed that some colors loose the potential of maximum re-usability if all the colors used in G' are to be reused in G'' and thereby we conclude that new color/s is/are to be introduced in G''. By further investigating the problem, least number of new colors required to color the vertices of G'' are determined and proceeding in this way, we prove the conjecture for $\ell = 8$. Next we prove the conjecture for all even $\ell > 8$. Here also we choose any vertex $v \in T_3$. Then we consider a sub graph (say G') induced by the vertices which are at distance $\frac{l}{2}$ from v. Again we consider an another sub graph (say G'') induced by the vertices which are at distance $(\frac{\ell}{2} + 2\lfloor\frac{\ell}{4}\rfloor - 1)$ from v. Here we observe that the colors used in G' are all distinct for ℓ distance coloring as distance between any pair of vertices in G' is at most ℓ. So minimum number of colors required to color the vertices of G' is $\lambda_\ell(G')$. Next we show that at least $(\frac{\ell}{4})$ no of extra colors apart from the colors used in G' must be required to color the vertices of G''. So, $\lambda_\ell(G'') \geq \lambda_\ell(G') + \frac{\ell}{4}$. From this and the value of the upper bound of $\lambda_\ell(T_3)$ derived in literature, we finally prove the conjecture.

The works presented in the thesis are summarized in the following table 1.1.

1.4 Organization of thesis

We organize the thesis in following manner. In *2nd* chapter, description of work for improving upper and lower bounds of $\lambda'_{1,2}(T_\Delta)$ for $\Delta = 3,4,6$ have been presented. In chapter 3, the proof of $\lambda'_{1,2}(T_8) = 28$ have been presented. In chapter 4, a theoretical approach to find out $L(k_1,k_2)$-vertex labeling for T_6 when $k_1 \leq k_2$ has been described. In chapter 5, we describe the work regarding $L(2,1)$-edge labeling and circular $L(2,1)$-edge labeling for T_6. In chapter 6, $L(\underbrace{1,1,\ldots,1}_{\ell})$-vertex labeling of T_3 has been studied and the corresponding conjecture in literature for $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3)$ has been proved. In last chapter, concluding remarks as well as relevant future research scope have been mentioned.

Table 1.1: Existing results and our results

<table>
<tr><th colspan="3">Existing and our results depicted chapterwise</th></tr>
<tr><th>Chapter</th><th>Existing results</th><th>Ours</th></tr>
<tr><td rowspan="3">Ch. 2</td><td>$7 \leq \lambda'_{1,2}(T_3) \leq 8$ [16].</td><td>$\lambda'_{1,2}(T_3) = 7$.</td></tr>
<tr><td>$10 \leq \lambda'_{1,2}(T_4) \leq 11$ [16].</td><td>$\lambda'_{1,2}(T_4) = 11$.</td></tr>
<tr><td>$16 \leq \lambda'_{1,2}(T_6) \leq 20$ [17].</td><td>$18 \leq \lambda'_{1,2}(T_6) \leq 20$.</td></tr>
<tr><td>Ch. 3</td><td>$25 \leq \lambda'_{1,2}(T_8) \leq 28$ [17].</td><td>$\lambda'_{1,2}(T_8) = 28$.</td></tr>
<tr><td rowspan="2">Ch. 4</td><td>(Through computer simulation [12])</td><td>(Theoretically)</td></tr>
<tr><td>$$\lambda_{h,1}(T_6) = \begin{cases} 3+2h, & 0 \leq h \leq 1/3 \\ 11h, & 1/3 \leq h \leq 3/8 \\ 3+3h, & 3/8 \leq h \leq 2/5 \\ 1+8h, & 2/5 \leq h \leq 3/7 \\ 4+h, & 3/7 \leq h \leq 1/2 \\ 9h, & 1/2 \leq h \leq 4/7 \\ 4+2h, & 4/7 \leq h \leq 2/3 \\ 8h, & 2/3 \leq h \leq 5/7 \\ 5+h, & 5/7 \leq h \leq 3/4 \\ 2+5h, & 3/4 \leq h \leq 4/5 \\ 1, & 4/5 \leq h \leq 1 \end{cases}$$</td><td>$$\lambda_{h,1}(T_6) = \begin{cases} \geq 3+2h, & 0 \leq h \leq 1/2 \\ \geq 4, & 1/2 \leq h \leq 1 \end{cases}$$
Lower bound coincides with [12]
when $0 \leq h \leq 1/3$
but provides loose bound
when $1/3 \leq h \leq 1$.</td></tr>
<tr><td rowspan="2">Ch. 5</td><td>$15 \leq \lambda'_{2,1}(T_6) \leq 16$ [15].</td><td>$\lambda'_{2,1}(T_6) = 16$.</td></tr>
<tr><td>$16 \leq \sigma'_{2,1}(T_6) \leq 18$,
No labeling function
to show $\sigma'_{2,1}(T_6) \leq 18$.</td><td>Shown $\sigma'_{2,1}(T_6) \leq 18$ by
giving a labeling function.</td></tr>
<tr><td rowspan="2">Ch. 6</td><td>Conjecture for even $\ell \geq 8$ [18].</td><td>Proved for $\ell \geq 8$.</td></tr>
<tr><td>$$\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3) + 1 = \left[\frac{3}{8}\left(\ell + \frac{4}{3}\right)^2\right]$$
($[x] \in \mathbb{Z}, x \in \mathbb{R}$
and $x - \frac{1}{2} < [x] \leq x + \frac{1}{2}$)</td><td>$$\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3) + 1 = \left[\frac{3}{8}\left(\ell + \frac{4}{3}\right)^2\right]$$
($[x] \in \mathbb{Z}, x \in \mathbb{R}$
and $x - \frac{1}{2} < [x] \leq x + \frac{1}{2}$)</td></tr>
</table>

Chapter 2

Improved bounds on $L(1,2)$-edge labeling for T_3, T_4 and T_6

2.1 Introduction

$L(h,k)$-edge labeling problem for infinite regular grids has been studied by various authors for specific values of h and k [14, 15, 16, 17]. The exact values of the minimum spans for $L(1,1)$, $L(2,1)$-edge labeling for T_3 and $L(1,1)$-edge labeling for T_6 have already been determined in [15]. But the exact values of the minimum spans for some of the cases for $L(h,k)$-edge labeling for T_3, T_4 and T_6 when $h,k \in \{1,2\}$ have not been determined yet rather upper and lower bounds were determined for the corresponding cases [16, 17]. In this chapter, we investigate $L(1,2)$-edge labeling problem for T_3, T_4 and T_6. and determined the exact values of the spans for T_3, T_4 and improved the lower bounds on the minimum spans for T_6. The rest of the chapter is organized as follows. In section 2.2, we present the basic definitions and the approach we have taken to obtain the results. In section 2.3, we state and prove the main results. Concluding remarks has been stated in section 2.4.

2.2 Basic definitions and our approach

Definition 1 *For two non-negative integers h and k, an $L(h,k)$-vertex labeling of a graph $G(V,E)$ is a function $\mathbf{f} : V \rightarrow \{0,1,\ldots,n\}, \forall v \in V$ such that $|\mathbf{f}(u) - \mathbf{f}(v)| \geq h$ when $d(u,v) = 1$ and $|\mathbf{f}(u) - \mathbf{f}(v)| \geq k$ when $d(u,v) = 2$.*

Here $d(u,v)$ is the distance between two vertices u and v and its value is the minimum number of edges in E that connect u and v.

Definition 2 *$\lambda_{h,k}(G)$ of $L(h,k)$-vertex labeling of a graph G is the minimum n such that G admits an $L(h,k)$-vertex labeling.*

Definition 3 *For two non-negative integers h and k, an $L(h,k)$-edge labeling of a graph $G(V,E)$ is a function $\mathbf{f'} : E \rightarrow \{0, 1, \ldots, n\}, \forall e \in E$ such that $|\mathbf{f'}(e_1) - \mathbf{f'}(e_2)| \geq h$ when $d'(e_1, e_2) = 1$ and $|\mathbf{f'}(e_1) - \mathbf{f'}(e_2)| \geq k$ when $d'(e_1, e_2) = 2$.*

Here distance between any two edges e_1 and e_2 is denoted as $d'(e_1, e_2)$ and $d'(e_1, e_2) = k$ represents that minimum number of edges in E that connect e_1 and e_2 is $k - 1$, where $k \geq 1$.

Definition 4 *$\lambda'_{h,k}(G)$ of $L(h,k)$-edge labeling of a graph G is the minimum n such that G admits an $L(h,k)$-edge labeling.*

Definition 5 *Given a graph $G(V,E)$, its line graph $L(G)(V', E')$ is a graph such that each vertex of $L(G)$ represents an edge of G and two vertices of $L(G)$ have an edge if and only if their corresponding edges share a common vertex in G.*

Figs. 2.1, 2.2 and 2.3 show portions of T_3, T_4 and T_6 respectively.

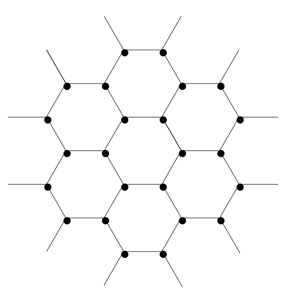


Figure 2.1: Part of T_3

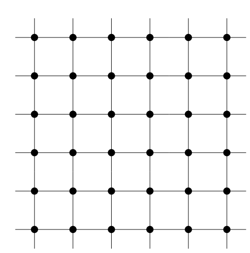
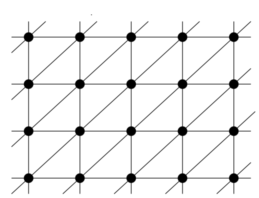


Figure 2.2: Part of T_4



T_6

Figure 2.3: Part of T_6

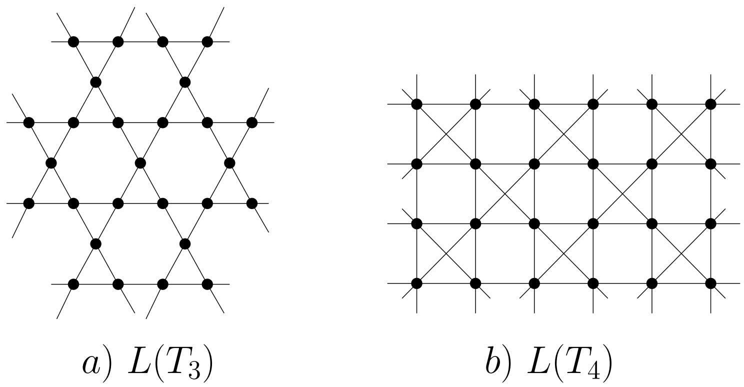


Figure 2.4: Part of $L(T_3)$ and $L(T_3)$

It is well-known that if G is d-regular then $L(G)$ is $2(d-1)$-regular. It is also well-known that edge labeling of G is equivalent to vertex labeling of $L(G)$. That is, $\lambda'_{h,k}(G) = \lambda_{h,k}(L(G))$. In our approach, instead of $L(1,2)$-edge labeling of T_3 and T_4, we use $L(1,2)$-vertex labeling of $L(T_3)$ and $L(T_4)$. In Figure 2.4, $L(T_3)$ and $L(T_4)$ are shown. Note that, $L(T_6)$ is 10-regular. Because of this high degree, we consider $L(1,2)$-edge labeling of T_6 directly. In all the cases, we choose proper sub graphs of $L(T_3)$, $L(T_4)$, T_6 and analyze the structural properties of the respective sub graphs to obtain the results. Our results on $\lambda'_{1,2}(G)$ for T_3, T_4 and T_6 are summarized in Table 2.1. In this table, $a-b$ represents that $a \leq \lambda'_{1,2}(G) \leq b$. Here, we use coloring and labeling interchangeably.

Table 2.1: The main results obtained for $L(1,2)$-edge labeling for T_3, T_4 and T_6.

	$\lambda'_{1,2}(G)$	
	Known	Ours
T_3	7-8 [16]	7-7
T_4	10-11 [16]	11-11
T_6	16-20 [17]	18-20

2.3 Our results

2.3.1 Hexagonal grid

Consider $L(T_3)$ and the three co-ordinate axes **X**, **Y** and **Z** as shown in Figure 2.5. Each vertex is an intersection of two of the three axes. The vertices of $L(T_3)$ can be

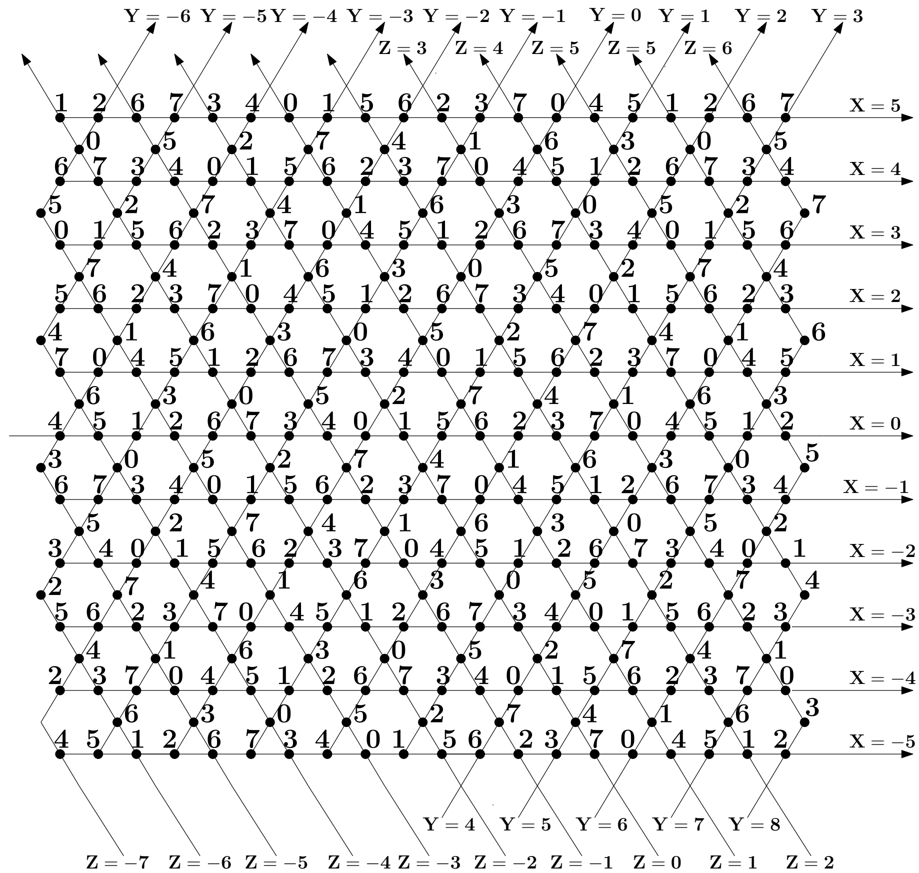


Figure 2.5: Sub graph G_S of $L(T_3)$ and its $L(1,2)$-vertex labeling

partitioned into three disjoint sets U_{xy}, V_{yz} and W_{zx} as defined bellow:
$U_{xy} = \{u_{xy} : u_{xy}\ \textit{is an intersection of}\ \mathbf{X} = \mathbf{x}\ \textit{and}\ \mathbf{Y} = \mathbf{y}\}$,
$V_{yz} = \{v_{yz} : v_{yz}\ \textit{is an intersection of}\ \mathbf{Y} = \mathbf{y}\ \textit{and}\ \mathbf{Z} = \mathbf{z}\}$,
$W_{zx} = \{w_{zx} : w_{zx}\ \textit{is an intersection of}\ \mathbf{Z} = \mathbf{z}\ \textit{and}\ \mathbf{X} = \mathbf{x}\}$.

Theorem 2.3.1 $\lambda'_{1,2}(T_3) = 7$.

Proof:

The coloring functions of vertices of $L(T_3)$ are defined as follows.
$f(u_{xy}) = \big((4 \times \lceil \frac{x}{2} \rceil + 2 \times \lfloor \frac{x}{2} \rfloor) \mod 8 + (5 \times y) \mod 8\big) \mod 8, \forall u_{xy} \in U_{xy}$.
$g(v_{yz}) = ((2 + 3 \times z) \mod 8 + (2 \times y) \mod 8) \mod 8, \forall v_{yz} \in V_{yz}$.
$h(w_{zx}) = ((1 + 5 \times z) \mod 8 + (2 \times x) \mod 8) \mod 8, \forall w_{zx} \in W_{zx}$.

Here we consider that $0 \le (x \mod y) < y$ where $x \in \mathbb{Z}$ and $y \in \mathbb{Z} \setminus \{0\}$.

The colors of the vertices of a finite sub graph G_S of $L(T_3)$ are shown in Figure 2.5. It can be verified that colors of every pair of vertices satisfy all the $L(1,2)$-vertex labeling constraints. It is also evident that the colors obey a regular modulo pattern which can be extended up to infinity and there will be no color conflict between any pair of vertices of $L(T_3)$ if the assigned colors satisfy the coloring functions. The minimum and maximum color used here are 0 and 7 respectively. Hence $\lambda_{1,2}(L(T_3)) \leq 7$. It has been shown in [16] that $\lambda_{1,2}(L(T_3)) \geq 7$. Hence $\lambda'_{1,2}(T_3) = \lambda_{1,2}(L(T_3)) = 7$. □

2.3.2 Square grid

Let us consider the induced sub graph G_{S_1} of $L(T_4)$ as shown in Fig 2.6 where all vertices are at mutual distance at most three. Let $S_1 = \{a,b\}$, $S_2 = \{k,l\}$, $S_3 = \{c,g\}$, $S_4 = \{f,j\}$ and $S_5 = \{d,\ e,\ h,\ i\}$.

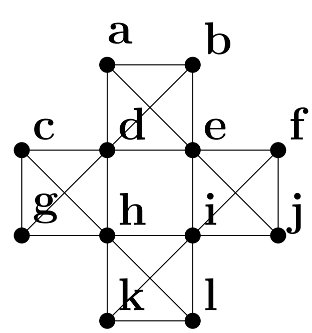


Figure 2.6: A sub graph G_{S_1} of $L(T_4)$

Definition 6 *The set of vertices in S_5 are termed as* ***central vertices*** *in G_{S_1}.*

Definition 7 *The set of vertices in $S_1 \cup S_2 \cup S_3 \cup S_4$ are termed as* ***peripheral vertices*** *in G_{S_1}.*

Now we have the following observations in G_{S_1}. Here the color of vertex a is denoted by $\mathbf{f}(a)$.

Observation 1 *: If colors of vertices of G_{S_1} are all distinct then $\lambda_{1,2}(G_{S_1}) \geq 11$.*

Proof: As G_{S_1} has 12 vertices, if all of them get distinct colors then $\lambda_{1,2}(G_{S_1}) \geq 11$. $\square$

Observation 2 *: No color can be used thrice in G_{S_1}. Colors used at the central vertices in S_5 can not be reused in G_{S_1}. Colors used at the peripheral vertices in S_1 can be reused only at the peripheral vertices in S_2. Similarly, colors used at the peripheral vertices in S_3 can be reused only at the peripheral vertices in S_4.*

Proof: No three vertices are mutually distant three apart. Hence no color can be used thrice in G_{S_1}. For any central vertex in S_5 there does not exist any vertex in G_{S_1} which is distance three apart from it. So colors used in the central vertices in S_5 can not be reused in G_{S_1}. For all peripheral vertices in $S_1 \cup S_2$, $d(x, y) = 3$ only when $x \in S_1$ and $y \in S_2$. Hence color used at peripheral vertex in S_1 can only be reused in S_2. Similarly, color used at peripheral vertex in S_3 can only be reused in S_4. $\square$

Observation 3 *: If $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$ where $x \in S_1$ and $y \in S_2$ then either $\mathbf{c} \pm 1$ is to be used in $(S_1 \cup S_2) \setminus \{x, y\}$ or it should remain unused in G_{S_1}. Similarly, if $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$ where $x \in S_3$ and $y \in S_4$ then either $\mathbf{c} \pm 1$ is to be used in $(S_3 \cup S_4) \setminus \{x, y\}$ or it should remain unused in G_{S_1}.*

Proof: Note that for all vertices $z \in V(G_{S_1}) \setminus (S_1 \cup S_2)$, either $d(z, x) = 2$ or $d(z, y) = 2$, where $x \in S_1$ and $y \in S_2$. Hence $\mathbf{c} \pm 1$ can not be used in $V(G_{S_1}) \setminus (S_1 \cup S_2)$. So $\mathbf{c} \pm 1$ can only be used in $(S_1 \cup S_2) \setminus \{x, y\}$ or it should remain unused in G_{S_1}. Similarly, if $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$, where $x \in S_3$ and $y \in S_4$, then $\mathbf{c} \pm 1$ can only be used in $(S_3 \cup S_4) \setminus \{x, y\}$ or it should remain unused in G_{S_1}. $\square$

Observation 4 *: Let $\mathbf{f}(x) = \mathbf{f}(y) = \mathbf{c}$ where $x \in S_1$ and $y \in S_2$. If $|\mathbf{f}(x) - \mathbf{f}(x')| \geq 2$, where $x' \in S_1 \setminus \{x\}$, then one of $\mathbf{c} \pm 1$ must remain unused in G_{S_1}. Similarly if $|\mathbf{f}(y) - \mathbf{f}(y')| \geq 2$, where $y' \in S_2 \setminus \{y\}$, then one of $\mathbf{c} \pm 1$ must remain unused in G_{S_1}. Similar facts hold when $x \in S_3$, $x' \in S_3 \setminus \{x\}$, $y \in S_4$ and $y' \in S_4 \setminus \{y\}$.*

Proof: Since $|\mathbf{f}(x) - \mathbf{f}(x')| \geq 2$, $\mathbf{f}(x') \neq \mathbf{c} \pm 1$. Hence from observation 3, one of $\mathbf{c} \pm 1$ must remain unused in G_{S_1}. $\square$

If no color is reused in G_{S_1}, then $\lambda_{1,2}(G_{S_1}) \geq 11$ from observation 1. To make $\lambda_{1,2}(G_{S_1}) < 11$, at least one color must be reused in G_{S_1}. From observation 2, there are at most 4 distinct pairs of peripheral vertices in G_{S_1} where a pair can have the same color. Now consider the sub graph G_1 of $L(T_4)$ as shown in Fig. 2.7.a. Note that G_1 consists of 5 sub graphs G', G'_1, G'_2, G'_3 and G'_4 which all are isomorphic to G_{S_1} having central vertices $\{d, h, i, e\}$, $\{t_1, c, d, a\}$, $\{b, e, f, t_2\}$, $\{i, l, t_3, j\}$ and $\{g, t_4, k, h\}$ respectively. Based on the span requirements of coloring G_1, we derive the following theorem.

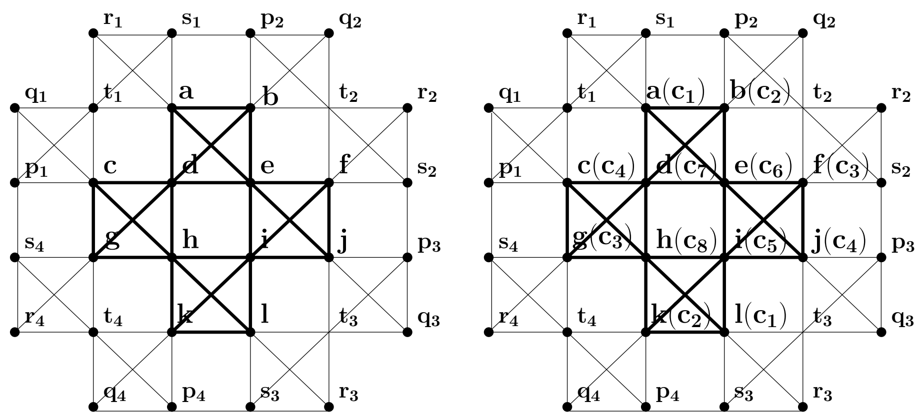


Figure 2.7: A sub graph G_1 of $L(T_4)$ and assignment of colors to some of its vertices.

Theorem 2.3.2 $\lambda_{1,2}(L(T_4)) \geq \lambda_{1,2}(G_1) \geq 11$.

Proof:

Case 1: When at most one pair of peripheral vertices use the same color in any sub graph of $L(T_4)$ isomorphic to G_{S_1}.

If no color is reused in G', then $\lambda_{1,2}(G') \geq 11$ from observation 1. We now consider the case when exactly one pair reuse a color in G'. Without loss of generality, consider $\mathbf{f}(a) = \mathbf{f}(l) = c_1$. From Observation 3, $c_1 \pm 1$ can only be put in $\{b, k\}$. Let $\mathbf{f}(k) = c_1 - 1$ and $\mathbf{f}(b) = c_1 + 1$. We assume that $c_1 - 1$ is the minimum color. Let us consider $\mathbf{f}(d) = c_1 + n$ where $n \in \mathbb{N}$ and $n \geq 2$. From observation 4, $x \in \{c_1, c_1 + n\}$ can be reused in G'_2 only if one of $x \pm 1$ remains unused in G'_2. In either case, $\lambda_{1,2}(G'_2) \geq 11$. So x can not be reused in G'_2. Since $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $c_1 - 1$ can only be put in $\{r_2, s_2\}$ as vertex b is already colored and for all other vertices $z \in V(G'_2) \setminus \{r_2, s_2\}$, either $d(z, a) = 2$ or $d(z, l) = 2$. Without loss of generality, let $\mathbf{f}(r_2) = c_1 - 1$.

In that case, $c_1 + n \pm 1$ can only be put in $\{e, s_2\}$. Without loss of generality, let $\mathbf{f}(e) = c_1 + n - 1$ and $\mathbf{f}(s_2) = c_1 + n + 1$. Since $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $\mathbf{f}(i) \neq c_1 \pm 1$ and hence $|\mathbf{f}(l) - \mathbf{f}(i)| \geq 2$. Now if $|\mathbf{f}(d) - \mathbf{f}(c)| \geq 2$, then from observation 4, one of $\mathbf{f}(c) \pm 1$, $\mathbf{f}(d) \pm 1$ and $\mathbf{f}(i) \pm 1$ remains unused in G'_4 if $\mathbf{f}(c)$ or $\mathbf{f}(d)$ or $\mathbf{f}(i)$ is reused in G'_4 respectively. In either case, this implies $\lambda_{1,2}(G'_4) \geq 11$. So $|\mathbf{f}(d) - \mathbf{f}(c)| = 1$ and $\mathbf{f}(c) = c_1 + n + 1$. There are 5 more vertices $\{g, h, i, j, f\}$ in G' which are to be colored with 5 distinct colors. Hence at least color $c_1 + n + 6$ must be used. Observe that if $\mathbf{f}(f) = c_1 + n + 2$ then $|\mathbf{f}(e) - \mathbf{f}(f)| = 3$ and $|\mathbf{f}(k) - \mathbf{f}(h)| \geq 3$ implying $\lambda_{1,2}(G'_3) \geq 11$ from observation 4. As $d(s_2, i) = d(s_2, j) = 2$ and $\mathbf{f}(s_2) = c_1 + n + 1$, we get $\mathbf{f}(i) \neq c_1 + n + 2$ and $\mathbf{f}(j) \neq c_1 + n + 2$. Therefore, either $\mathbf{f}(g) = c_1 + n + 2$ or $\mathbf{f}(h) = c_1 + n + 2$. So, $\mathbf{f}(p_4) \neq c_1 + n + 1$ and $\mathbf{f}(q_4) \neq c_1 + n + 1$. In that case, $\mathbf{f}(p_4)$ and $\mathbf{f}(q_4)$ must be in $\{c_1 + n, c_1 + n - 1\}$ if color $c_1 + n$ is to be reused in G'_4, otherwise, $\lambda_{1,2}(G_1) \geq 11$. As c_1 can not be reused in G'_4, either $\mathbf{f}(r_4) = c_1 + 1$ or $\mathbf{f}(s_4) = c_1 + 1$. Let $\mathbf{f}(r_4) = c_1 + 1$. When $n = 2$, $c_1 + n - 1 = c_1 + 1$ and when $n = 3$, $c_1 + n - 1 = c_1 + 2$. As $d(p_4, l) = d(p_4, r_4) = d(q_4, l) = d(q_4, r_4) = 2$, $\mathbf{f}(p_4), \mathbf{f}(q_4) \notin \{c_1 + 1, c_1 + 2\}$. So, $n \geq 4$ and hence $c_1 + n + 6 \geq c_1 + 10$. So at least 12 color are required in G_1 including $c_1 - 1$ and $c_1 + 10$. Hence $\lambda_{1,2}(G_2) \geq 11$.

Case 2: There exists at least one sub graph of $L(T_4)$ isomorphic to G_{S_1} where two pairs of peripheral vertices use a color each.

There are two different ways of reusing two colors in G'.

Case 2.1: First consider the case when $\mathbf{f}(a) = \mathbf{f}(l) = c_1$ and $\mathbf{f}(c) = \mathbf{f}(j) = c_2$. From observation 3, $c_1 \pm 1$ and $c_2 \pm 1$ must be used in $\{b, k\}$ and $\{g, f\}$ respectively. From observation 2, c_1 can only be reused in $\{r_2, s_2\}$ in G'_2. But $\mathbf{f}(r_2) \neq c_1$ and $\mathbf{f}(s_2) \neq c_1$ as $|\mathbf{f}(b) - c_1| = 1$ and $d(b, r_2) = d(b, s_2) = 2$. Again, from observation 2, c_2 can only be reused in $\{p_2, q_2\}$. But $\mathbf{f}(p_2) \neq c_2$ and $\mathbf{f}(q_2) \neq c_2$ as $|\mathbf{f}(f) - c_2| = 1$ and $d(f, p_2) = d(f, q_2) = 2$. From observation 3, if $\mathbf{f}(i)$ is to be reused in G'_2, then $|\mathbf{f}(i) - c_2| = 1$. But $\mathbf{f}(i) \neq c_2 \pm 1$ as $d(c, i) = 2$ and $\mathbf{f}(c) = c_2$. If $\mathbf{f}(d)$ is to be reused in G'_2, then $|\mathbf{f}(d) - c_1| = 1$. But $\mathbf{f}(d) \neq c_1 \pm 1$ as $d(d, l) = 2$ and $\mathbf{f}(l) = c_1$. Therefore, no color can be reused in G'_2 and hence $\lambda_{1,2}(G_1) \geq 11$.

Case 2.2: Consider the case when $\mathbf{f}(a) = \mathbf{f}(l) = c_1$ and $\mathbf{f}(b) = \mathbf{f}(k) = c_2$. Without loss of generality, assume $c_2 > c_1$. From observation 3, $c_1 \pm 1$ and $c_2 \pm 1$ must be used in $\{b, k\}$ and $\{a, l\}$ respectively. Even if we set $c_2 = c_1 + 1$, at least one of $c_1 - 1$ and $c_2 + 1$ must remain unused in G'. So the 8 vertices in $V(G') \setminus (\{a, l\} \cup \{b, k\})$ must get 8 distinct colors other than c_1 and c_2. So, $\lambda_{1,2}(G') \geq 10$. Note that $\lambda_{1,2}(G') = 10$ only if $c_2 = c_1 + 1$, c_1 is minimum color ($c_1 - 1$ does not exists) or c_2 is maximum color ($c_2 + 1$ does not exists). If both c_1 and c_2 are non-extreme color, then $\lambda_{1,2}(G') \geq 11$ and we are done. So, we consider $c_1 = 0$, $c_2 = c_1 + 1 = 1$ and $c_2 + 1 = 2$ as unused in G'. In that case, $\mathbf{f}(d) = x \geq 3$ and hence $|\mathbf{f}(d) - \mathbf{f}(a)| \geq 3$. From observation 4, if x is reused in G'_2, then one of $x \pm 1$ can not be used in G'_2. If only x is reused in G'_2, then $\lambda_{1,2}(G'_2) \geq 11$. If x and one of $\{\mathbf{f}(i), \mathbf{f}(j)\}$ are reused in G'_2, then from Case 2.1 above, $\lambda_{1,2}(G_1) \geq 11$. If x and both of $\{\mathbf{f}(i), \mathbf{f}(j)\}$ are reused in G'_2, from Case 3 below, we will see that $\lambda_{1,2}(G_1) \geq 11$. So, to keep $\lambda_{1,2}(G_1) < 11$, x should not be reused in G'_2. In that case, $x - 1$ must be used at one of $\{c, g, h, e\}$ in G'. Now arguing similarly as stated in case **1**, we can conclude that $x + 7$ must be used in G'_1 or G'_2. If $x = 3$, then $x - 1 = 2$ must be used in G' which is a contradiction, as 2 must remain unused in G'. Hence $x \geq 4$ implying $x + 7 = 11$. Hence $\lambda_{1,2}(G_1) \geq 11$.

Case 3: The exists at least one sub graph of $L(T_4)$ isomorphic to G_{S_1} where three pairs of peripheral vertices use a color each.

Without loss of generality, let us consider $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $\mathbf{f}(b) = \mathbf{f}(k) = c_2$ and $\mathbf{f}(c) = \mathbf{f}(j) = c_3$. From observation 3, $c_1 \pm 1$ and $c_2 \pm 1$ must be used in $\{b, k\}$ and $\{a, l\}$ respectively. It can be observed that $\lambda_{1,2}(G') = 9$ only if $|c_1 - c_2| = 1$, $|c_3 - \mathbf{f}(g)| = 1$, $|c_3 - \mathbf{f}(f)| = 1$ and any one of $\{c_1, c_2\}$ is one extreme color. Without loss of generality consider $\mathbf{f}(g) = c_3 + 1$, $\mathbf{f}(f) = c_3 - 1$, c_1 is minimum color and $c_2 = c_1 + 1$. From observation 2, c_3 can only be reused in $\{p_2, q_2\}$. But $\mathbf{f}(p_2) \neq c_3$ and $\mathbf{f}(q_2) \neq c_3$ as $\mathbf{f}(f) = c_3 - 1$ and $d(f, p_2) = d(f, q_2) = 2$. From observation 3, if $\mathbf{f}(i)$ is to be reused in G'_2, then $|\mathbf{f}(i) - c_3| = 1$. But $\mathbf{f}(i) \neq c_3 \pm 1$ as $d(c, i) = 2$ and $\mathbf{f}(c) = c_3$. From observation 2, c_1 can only be reused in $\{r_2, s_2\}$. But $\mathbf{f}(r_2) \neq c_1$ and $\mathbf{f}(s_2) \neq c_1$ as $\mathbf{f}(b) = c_2 = c_1 + 1$ and $d(b, r_2) = d(b, s_2) = 2$. Now arguing similarly as stated in case **2.2** above, we can conclude that $c_2 + 1$ must remain unused in

G'. So, $(c_1 - \mathbf{f}(d)) \geq 3$. Now from observation 4, if $\mathbf{f}(d)$ is reused in G'_2 then any one of $\mathbf{f}(d) \pm 1$ must remain unused in G'_2. Thus in G'_2, only $\mathbf{f}(d)$ can be reused by keeping one of $\mathbf{f}(d) \pm 1$ as unused. Hence $\lambda_{1,2}(G_1) \geq 11$. If we consider $\lambda_{1,2}(G') = 10$, the same result can be obtained by considering the corresponding G'_i, $1 \leq i \leq 4$.

Case 4: The exists at least one sub graph of $L(T_4)$ isomorphic to G_{S_1} where all four pairs of peripheral vertices use a color each.
Let us consider $\mathbf{f}(a) = \mathbf{f}(l) = c_1$, $\mathbf{f}(b) = \mathbf{f}(k) = c_2$, $\mathbf{f}(g) = \mathbf{f}(f) = c_3$ and $\mathbf{f}(c) = \mathbf{f}(j) = c_4$. From observation 3, $c_1 \pm 1$, $c_2 \pm 1$, $c_3 \pm 1$ and $c_4 \pm 1$ must be used in $\{b,k\}$, $\{a,l\}$, $\{c,j\}$ and $\{g,f\}$ respectively. It can be observed that $\lambda_{1,2}(G') = 9$ only if $|c_1 - c_2| = 1$, $|c_3 - c_4| = 1$, one of $\{c_1, c_2\}$ is an extreme color and one of $\{c_3, c_4\}$ is the other extreme color. Without loss of generality, consider $c_1 = 0$, $c_4 = 9$, $c_2 = c_1 + 1 = 1$ and $c_3 = c_4 - 1 = 8$. So $c_2 + 1 = 2$ and $c_3 - 1 = 7$ are two distinct unused colors. Without loss of generality, consider $c_8 = c_2 + 2$, $c_5 = c_8 + 1$, $c_6 = c_5 + 1$ and $c_7 = c_6 + 1$. Since $|c_3 - c_4| = 1$ and $d(g, p_4) = d(g, q_4) = 2$, we get $\mathbf{f}(p_4) \neq c_4$ and $\mathbf{f}(q_4) \neq c_4$. Similarly, $\mathbf{f}(r_4) \neq c_1$ and $\mathbf{f}(s_4) \neq c_1$. From observation 2, c_5 can only be reused at $\{s_4, r_4\}$ in G'_4 but $\mathbf{f}(s_4) \neq c_5$ and $\mathbf{f}(r_4) \neq c_5$ as $d(h, s_4) = d(h, r_4) = 2$ and $\mathbf{f}(h) = c_8 = c_5 - 1$. Therefore, only c_7 can be reused in $\{p_4, q_4\}$. From observation 4, one of $c_7 \pm 1$ must remain unused in G'_4 as $(c_4 - c_7) = 3$. Hence $\lambda_{1,2}(G_1) \geq 11$. For other assignment of central vertices and for the case when $\lambda_{1,2}(G') = 10$, we can obtain the same result by considering the corresponding G'_i, $1 \leq i \leq 4$. □

2.3.3 Triangular grid

For any vertex u, the set of vertices which are adjacent to u is called $N(u)$. Let us define $N(S) = \{\cup_{u \in S} N(u) : u \in S\}$. Let v be any vertex in T_6. Consider the sub graph $G_v(V, E)$ of T_6 centering v as shown in Figure 2.8, where $V = N(v) \cup N(N(v))$ and E is set of all the edges which are incident to u where $u \in N(v)$. Observe that in G_v, for any two edges e_1 and e_2, $d(e_1, e_2) \leq 3$. Now we define the following three sets of edges S_1, S_2 and S_3:

S_1: Edges of G_v incident to v.
S_2: Edges of G_v whose both end points incident to e_1 and e_2 where $e_1,\ e_2 \in S_1$.
S_3: $E \setminus (S_1 \cup S_2)$.

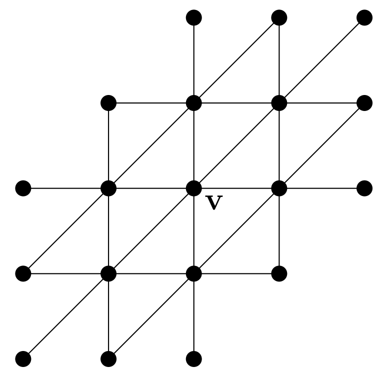


Figure 2.8: A sub graph G_v of T_6

Consider the 6-cycle, H_v formed with the edges of S_2 in G_v. We say e and e_1 as a pair of *opposite edges* in H_v iff $d(e, e_1) = 3$. This implies that the same color can be used at a pair of opposite edges in $L(1,2)$-edge labeling. An edge $e(v, w)$ *covers* the set of edges E' if for every $e' \in E'$, $d(e, e') \leq 2$. This implies that a color used at e can not be used at any edge $e' \in E'$ in $L(1,2)$-edge labeling. Now we have the following lemmas.

Lemma 2.3.1 *If c be a color used to color an edge e in S_1, then c can not be used in $E \setminus e$.*

Proof: Since e is incident to v, for any other edge $e_1 \in E$, $d(e, e_1) \leq 2$. Hence $f'(e_1) \neq c$ for $L(1,2)$-edge labeling, where $f'(e_1)$ denotes the color of e_1. □

Lemma 2.3.2 *If c be a color used to color an edge in S_1, then $c+1$ and $c-1$ both can be used at most once in G_v.*

Proof: Let e be an edge in S_1 such that $f'(e) = c$. Since e is incident to v, for any other edge $e_1 \in E$, $d(e, e_1) \leq 2$. Let $S_e = \{e_1 : d(e, e_1) = 1\}$. For $L(1,2)$-edge labeling, $c+1$ can only be used in an edge e_1 in S_e. It can be noted that for any two edges $e_1, e_2 \in S_e$, $d(e_1, e_2) \leq 2$. Hence $c+1$ can be used at most once. Proof for $c-1$ can be done in similar manner. □

Lemma 2.3.3 *If c be a color used to color an edge e in S_2, then c can be used at most one edge in $E \setminus e$ in G_v.*

Proof: Note that c can not be used at any edge in S_1. Here c can be used at the *opposite edge* e_1 of e in S_2 or at an edge e_2 in S_3, which is adjacent to e_1. When c is used at e and e_1, then c can not be used again in G_v as e and e_1 together cover all the edges of G_v. When c is used at e and e_2, c can not be used again in G_v as e and e_2 together also cover all the edges of G_v. □

Lemma 2.3.4 *If c be a color used to color an edge e in S_2, then $c+1$ and $c-1$ both can be used at most twice in G_v.*

Proof: Suppose e_1 be an edge colored with $c+1$. If e_1 is not adjacent to e then $d(e_1, e) = 3$. From statement of lemma 2.3.3, it follows that there does not exist two edges along with e in G_v which are mutually distance 3 apart, otherwise c would have been used for three times. Hence $c+1$ can be used at most once.

When e_1 is adjacent to e, e_2 can be colored with $c+1$ if e_2 is at distance 3 apart from both e_1 and e. Again from the statement of lemma 2.3.3, it follows that there does not exist two edges along with e in G_v which are mutually distance 3 apart, otherwise c would have been used for three times. So, $c+1$ can be used at most twice, one in one of the edges adjacent to e and other in one of the edges which are at distance 3 apart from e. Proof for $c-1$ can be done in similar manner. □

Lemma 2.3.5 *If c be a color used to color an edge e in S_3, then c can be used at most twice in $E \setminus e$.*

Proof: It follows from Figure 2.8 that exactly one end point of e is incident to a vertex in H_v. Note that for any walk through H_v, every third vertex is distance 2 apart. So edges incident to those vertices are distance 3 apart. Since the order of H_v is 6, there can be at most $6/2 = 3$ vertices which are mutually distance 2 apart. Hence c can be used thrice. □

Lemma 2.3.6 *If c be a color used to color an edge e in S_3, then $c+1$ and $c-1$ both can be used at most thrice in G_v.*

Proof:

We know that $c+1$ can be used at an edge adjacent to e. From lemma 2.3.5 it is clear that c can be used at most thrice. So, $c+1$ can also be used at most thrice, where each such edge is adjacent to one of the three edges colored with c. It can be proved similarly for $c-1$. □

Lemma 2.3.7 i. *To color the edges of S_1, at least 6 colors are required.*

ii. To color the edges of S_2, at least 3 colors are required.

iii. To color the edges of S_3, at least 6 colors are required.

Proof: i. From lemma 2.3.1, every edge of S_1 has an unique color. As there are 6 edges in S_1, 6 distinct colors are required here.

ii. In S_2, there are 3 pairs of *opposite edges*. Each pair of opposite edges requires at least one unique color. So at least 3 colors are required.

iii. A color can be used thrice in S_3 by lemma 2.3.5. In S_3, there are 18 edges. So, at least 6 colors are required. $\square$

Theorem 2.3.3 *For any optimal labeling of G_v, 6 consecutive colors including either the minimum color or the maximum color must be used in S_1.*

Proof: It is clear from lemma 2.3.7.i that S_1 needs at least 6 colors to color its edges. From lemma 2.3.2, note that if c be a color used in an edge of S_1 then both $c+1$ and $c-1$ can be used at most once in G_v. Whereas a color can be used twice in S_2 and thrice in S_3. Thus our aim should be to minimize the number of colors which can be used only once in G_v. This implies that consecutive colors should be used in S_1 for optimal coloring. If the minimum color (*min*) or the maximum color (*max*) is used in S_1 then further benefit can be achieve as $min-1$ or $max+1$ does not exist. Therefore, optimal span can be achieved only when the colors of S_1 are consecutive including either *min* or *max*. $\square$

Lemma 2.3.8 *If three consecutive colors c, $c+1$, $c+2$ are used thrice each in S_3 then neither $c-1$ nor $c+3$ can be used in S_3.*

Proof: Observe that there are exactly 2 sets of three alternating vertices in H_v where a color can be used thrice at edges incident to any set of alternating vertices. If $c-1$ would have been used in S_3 then either it was used at an edge adjacent to the edges colored with c or at an edge distance 3 apart from the edge colored with c. Now observe that c and $c-1$ are used at two edges of S_3 which form a triangle with one edge of S_2. Suppose c, $c-1$ be the colors used at those two edges $e, e_1 \in S_3$ respectively, where e is incident to u and e_1 is incident to w where $uw \in S_2$. Note that c is used thrice in S_3. Then c must be reused at an edge incident to x, and $xw \in S_2$. So c and $c-1$ are used at two edges at distance 2 apart, which violets the condition of $L(1,2)$-edge labeling. Hence $c-1$ can not be used in G_v. Similarly it can be shown that $c+3$ can also not be used in G_v. This implies that no 4 consecutive colors can be used thrice each in G_v. $\square$

Theorem 2.3.4 $\lambda'_{1,2}(G_v) \geq 16$.

Proof:

By Theorem 2.3.3, 6 consecutive colors must be used to color the edges of S_1. Recall that, we assume the minimum color is used at S_1. Let c_1 be the maximum color used in S_1 and c be the minimum color used in S_2. Now we consider the coloring of the edges in $S_2 \cup S_3$. If the edges of S_2 have all consecutive colors $c, c+1, \ldots, c+5$ then 6 colors are needed for S_2. From Lemma 2.3.3, any color c', used in S_2 can be reused at most once more in G_v unless $c' = c_1 + 1$ (Lemma 2.3.2). Thus we can color at most 6 edges in S_3 using those colors. From Lemma 2.3.4, the color $c+6$ can be used at most twice in G_v. Note that color c_1+1 can be used at most once in G_v. So far, at most 9 edges in S_3 are colored. So at least 9 edges are left to be colored in S_3. So at least 3 more colors are needed for S_3, as any color can be used at most thrice in S_3. However, from Lemma 2.3.8, in that case, all of $c+7$, $c+8$ and $c+9$ can not be used thrice each in S_3. Thus at least $c+10$ is needed for G_v. Since $c-6$ is used in S_1, we get $\lambda'_{1,2}(G_v) \geq (c+10)-(c-6) = 16$. Now, consider the case when 4 colors are used in S_2. Observe that then in any possible coloring, at least 3 colors can be used at most twice in S_3. Therefore, at most 9 edges of S_3 can be colored. Hence here too, we can argue that $c+10$ must be used in G_v. In a similar manner, we can argue that when five colors are used in S_2, the color $c+10$ must be used in G_v. Hence in all the cases discussed above, we get $\lambda'_{1,2}(G_v) \geq 16$. Now consider the case when only three colors say c, c', c'' are used in S_2. Without loss of generality assume $c'-c \geq 2$ and $c''-c' \geq 2$. First consider the cases assuming $c_1+1 = c-1$. Observe that if $c+1 \neq c'-1$ and $c'+1 \neq c''-1$, then $c \pm 1, c' \pm 1, c'' \pm 1$ can color at most $(1+5\times 2) = 11$ edges of S_3. So at least 7 edges are left to be colored in S_3 requiring at least 3 more colors. So at least 9 colors are required for S_3 and hence $\lambda'_{1,2}(G_v) \geq 17$. If c'' is maximum color, then $c''+1$ can not be used and in this case too, at least 17 colors are required for G_v and hence $\lambda'_{1,2}(G_v) \geq 16$. Now we consider the case when $c+1 = c'-1$ but $c'+1 \neq c''-1$. Here $c+1, c'+1, c'' \pm 1$ can color at most 8 edges of S_3. Thus using $c-1$ also we are able to color at most 9 edges in S_3. Therefore at least 9 edges are left to be colored in S_3 requiring at least 3 more colors. So at least 8 colors are required for S_3 and hence $\lambda'_{1,2}(G_v) \geq 16$. Similarly we can argue that $\lambda'_{1,2}(G_v) \geq 16$ when $c+1 \neq c'-1$ $c'+1 = c''-1$. If $c+1 = c'-1$ and $c'+1 = c''-1$, then c, $c+2$ and $c+4$ are used in S_2. Here $c \pm 1, c+3, c+5$ can color at most 7 edges of S_3. So at least 11 edges are left to be colored in S_3 requiring at least 4 more colors. So at least 8 colors are required for S_3 and hence $\lambda'_{1,2}(G_v) \geq 16$. Similarly, one can verify that when $c_1 + 1 \neq c - 1$ the bound remains the same for all the cases. Thus considering all cases, $\lambda'_{1,2}(G_v) \geq 16$. □

We assume that the minimum color is used in S_1. The maximum color can be used at most thrice in S_3 and at most twice in S_2. In all cases, there exists a vertex say v' in H_v such that color of any edge incident to v' is neither minimum nor maximum. Now we consider the sub graph $G_{v'}$ of T_6 centering v' and isomorphic to G_v. Let min_1 and max_1 be the minimum and maximum colors used to color the edges of S'_1 in $G_{v'}$.

Lemma 2.3.9 *If $max_1 - min_1 \geq 7$, i.e., there exists at least two intermediate colors between min_1 and max_1 which are not used in S'_1, then $\lambda'_{1,2}(G_{v'}) \geq 18$.*

Proof:

At least two unused colors c_1, c_2 are there in S'_1 such that $\forall c \in \{c_1,\ c_2\}$, either $c+1$ or $c-1$ is used in S'_1. From lemma 2.3.2, c_1, c_2, $min_1 - 1$ and $max_1 + 1$ can be used at most once each in $G_{v'}$. Let x' be the number of colors required for $G_{v'}$. Let us first consider that 6 colors are used in S'_2. These colors can be reused at most once each in S'_3. Consider that all of $c_1,\ c_2,\ min_1 - 1$ and $max_1 + 1$ are also used once each in S'_3. So at most $(6+4) = 10$ edges of S'_3 have been colored so far. So at least $(18-10) = 8$ edges are left to be colored in S'_3 requiring at least 3 more colors. So, $x' \geq (6(for\ S'_1) + 6(for\ S'_2) + 7(for\ S'_3)) = 19$ and hence $\lambda'_{1,2}(G_{v'}) \geq 18$. Now assume 5 colors are used in S'_2. The only possibility is that 4 colors must be used once each in S'_2 and 1 color should be used two times in S'_2. These 4 colors can be reused at most once each in S'_3. So at most $(4+4) = 8$ edges of S'_3 have been colored so far. So at least 4 more colors are required to color the remaining edges of S'_3. So, $x' \geq (6+5+8) = 19$ and hence $\lambda'_{1,2}(G_{v'}) \geq 18$. If 4 colors are used in S'_2, similarly we can show that $\lambda'_{1,2}(G_{v'}) \geq 18$.

Now consider that 3 colors are used two times each in S'_2. These 3 colors can not be used anymore in S'_3. So at least $(18-4) = 14$ edges are left to be colored in S'_3, requiring at least 5 more colors. At least 4 out of these 5 colors must be used three times each in S'_3. From Lemma 2.3.8, if 3 consecutive colors are used 3 times each in S'_3, then at least one color can not be used in S'_3 resulting $\lambda'_{1,2}(G_{v'}) \geq 18$. Otherwise, at most two sets of two consecutive colors can be reused three times each in S'_3. In that case, at least three colors $x,\ y,\ z$ are there such that for all $w \in \{x,\ y,\ z\}$ either $w+1$ or $w-1$ is in one of S'_1 or S'_2. So each of them can not be used three times each. So these 5 colors can color at most $(3 \times 2 + 2 \times 3) = 12$ edges in S'_3, requiring at least 1 more color for S'_3. So at least $(4+6) = 10$ colors are required for S'_3 and hence $\lambda'_{1,2}(G_{v'}) \geq 18$. For all other cases we also get $\lambda'_{1,2}(G_{v'}) \geq 18$ using similar argument. □

Theorem 2.3.5 $\lambda'_{1,2}(T_6) \geq 18$.

Proof: Assume that x be a vertex which is not adjacent to edges colored with maximum and minimum colors used in G_x. Let us consider G_x is not colored and u, w be two vertices of H_x in G_x. Let us define S_{x1} as the set of edges adjacent to x. We consider the following two cases.

When $w \in N(u)$**:** u and w are connected by an edge e. Let $\{c_1, \ldots, c_6\}$ and $\{c'_1, \ldots, c'_6\}$ be two sequences consisting of consecutive colors are used at the edges incident to u and w respectively. It is possible to assign consecutive colors at those edges when e is colored with either $c_6 = c'_1$ or $c_1 = c'_6$. Now observe two edges e' and e'_1 of S_{x1} are already colored and those are not consecutive. Note that $|f'(e') - f'(e'_1)| \geq 2$. If $|f'(e') - f'(e'_1)| = 2$ then $f'(e')$ and $f'(e'_1)$ is neither minimum nor maximum color used in u and w. Then any color of any other edge in S_{x1} is neither consecutive to $f'(e')$ nor $f'(e'_1)$. So $max - min \geq 7$ where min and max be the minimum and maximum colors used in S_{x1}. If $|f'(e') - f'(e'_1)| > 2$, then also $max - min \geq 7$. Hence from lemma 2.3.9, $\lambda'_{1,2}(G_x) \geq 18$.

When $w \notin N(u)$**:** Note that $x \in \{N(u) \cap N(w)\}$. Let two sequences $\{c_1, \ldots, c_6\}$ and $\{c'_1, \ldots, c'_6\}$ consisting of consecutive colors are used at the edges incident to u and w respectively. Let uv and wv are e' and e'_1 respectively. If $f'(e')$ and $f'(e'_1)$ are consecutive then either $f'(e') = c_6,\ f'(e'_1) = c'_1$ or $f'(e') = c_1,\ f'(e'_1) = c'_6$. Now observe that for any other edge e in S_{x1}, $|f'(e) - f'(e')| > 2$ implying $max - min \geq 7$ where min and max be the minimum and maximum colors used inc S_{x1}. If $f'(e')$ and $f'(e'_1)$ are not consecutive then $|f'(e') - f'(e'_1)| \geq 2$. If $|f'(e') - f'(e'_1)| = 2$ then the intermediate color must be used at an edge $e \in S_{x1}$. There are still 4 edges remain uncolored. It can be checked that for any coloring of the rest of the graph, there exists a vertex $y \in H_x$ or $y \in N(z), z \in H_x$, for which $max - min \geq 7$ where min and max be the minimum and maximum colors used to color the edges incident to y and they are neither maximum nor minimum color used in G_x. Hence from lemma 2.3.9, $\lambda'_{1,2}(G_x) \geq 18$. Hence the proof. □

2.4 Conclusions

Here, we improve lower and upper bounds of the minimum spans of $L(1,2)$-edge labeling for infinite regular hexagonal, square and triangular using structural properties of those graphs. An interesting problem will be to improve or introduce new bounds on those graphs for other values of h and k for which the exact values of the minimum span is not known. One can try this problem for other infinite grids.

Chapter 3

Improved lower bound for $L(1,2)$-edge-labeling of Infinite 8-regular grid

3.1 Introduction

$L(h,k)$-edge labeling problem for infinite regular grids has been studied by various authors for specific values of h and k [14, 15, 16, 17]. Among them, $L(1,2)$-edge labeling problem for infinite 8-regular grid T_8 was studied in [17]. In [17], it was shown that $25 \leq \lambda'_{1,2}(T_8) \leq 28$. Note that there is a gap between lower and upper bounds. In this chapter, we prove that $\lambda'_{1,2}(T_8) \geq 28$. As in [17], $\lambda'_{1,2}(T_8) \leq 28$, it is concluded that $\lambda'_{1,2}(T_8) = 28$. The rest of the chapter is organized as follows. In section 3.2, we state some preliminaries of T_8 and prove our results. In section 3.3, concluding remarks have been drawn.

3.2 Preliminaries and results

Fig. 3.1 shows a portion of infinite 8-regular grid T_8. Observe that there are four types of edges in T_8. The edges which are along or parallel to the X axis and the edges which are along or parallel to the Y axis are said to be *horizontal* edges and *vertical* edges respectively. The edges which are at 45° to any of a horizontal edge and the edges which are at 135° to any of a horizontal edge are said to be *right slanting* edges and *left slanting* edges respectively. Here the angular distance between two adjacent edges e_1 and e_2 represents the smaller of the two angles measured

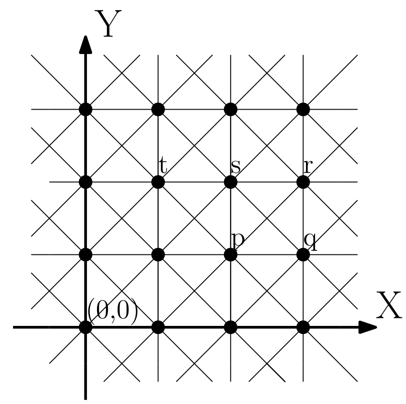


Figure 3.1: A part of infinite 8-regular grid T_8.

in anticlockwise from e_1 to e_2 and from e_2 to e_1. In Fig. 3.1, the edges (p,q), (p,r), (p,s) and (p,t) are a horizontal, a right slanting, a vertical and a slanting edges respectively. By *slanting* edges, we mean all the left and right slanting edges and by *non slanting* edges, we mean all the horizontal and vertical edges.

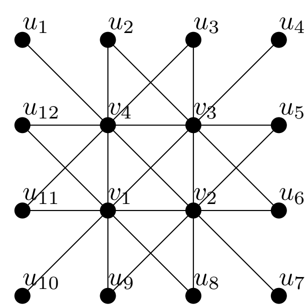


Figure 3.2: The G_S corresponding to the K_4 having vertex set $S = \{v_1, v_2, v_3, v_4\}$.

Consider any K_4 (complete graph of 4 vertices) in T_8 and let $S = V(K_4)$ be the vertex set of the K_4. Let v be a vertex in T_8 and N_v be the set of vertices in T_8 which are adjacent to v. Let $N(S) = \bigcup_{v \in S} N_v$ be the set of all vertices in T_8 which are adjacent to at least one vertex in S. Let us define G_S as the sub graph of T_8 such that $V(G_S) = S \cup N(S)$ and $E(G_S)$ is the set of all edges of T_8 which are incident to at least one vertex in S. Fig. 3.2 shows the the K_4 with vertex set $S = \{v_1, v_2, v_3, v_4\}$ and the corresponding G_S. It is evident that any two edges of G_S are at distance at most 2. Hence no two edges of G_S can be given the same color for $L(1,2)$-edge labeling of G_S. As $|E(G_S)| = 26$ and $d'(e_1, e_2) \leq 2$, $\forall e_1, e_2 \in E(G_S)$, $\lambda'_{1,2}(G_S) \geq 25$ for $L(1,2)$ edge labeling of G_S.

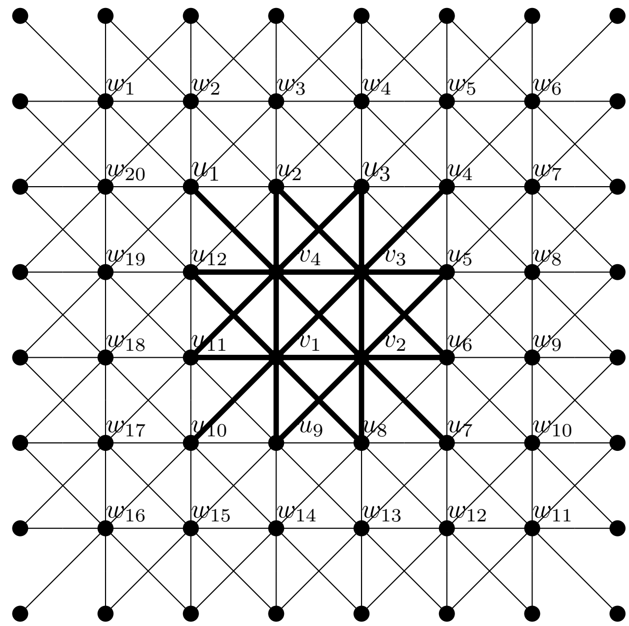


Figure 3.3: A sub graph G of an infinite 8-regular grid T_8.

Consider two edges $e_1, e_2 \in E(G_S)$ such that $d'(e_1, e_2) = 2$. Let the color c be assigned to e_1. Clearly, the colors $c \pm 1$ can not be used at e_2 for $L(1,2)$-edge labeling. As there exists no pair of edges e_1 and e_2 such that $d'(e_1, e_2) \geq 3$ in G_S, both $c \pm 1$ must be used at the adjacent edges of e_1 in G_S.

Consider the K_4 having vertex set $S = \{v_1, v_2, v_3, v_4\}$ as shown in Fig. 3.3. Now define $S' = S \cup N(S) \cup N(N(S))$. Fig. 3.3 shows the graph G such that $V(G) = S' \cup N(S')$ and $E(G)$ is the set of all edges in T_8 which are incident to at least one vertex in S'. Note that there are total 25 distinct K_4s including the K_4 having vertex set S in G. Note that here G is built over the G_S. Consider a G_S and the corresponding G. Let two consecutive colors c and $c+1$ be used in G_S. In the following lemma we identify all the K_4s having vertex sets $S_1, S_2, \ldots, S_m$ such that c and $c+1$ can not be used at $G_{S_1}, G_{S_2}, \ldots, G_{S_m}$ when c and $c+1$ are used in G_S.

Lemma 3.2.1 *For every pair of consecutive colors c and $c+1$ used in two adjacent edges e_1 and e_2 in G_S, except when e_1 and e_2 form an angle of $45°$ at their common incident vertex, there exists at least 4 different K_4s having vertex sets S_1, S_2, S_3 and S_4 other than S such that 1) c can not be used at G_{S_1} and G_{S_2} and 2) $c+1$ can not be used at G_{S_3} and G_{S_4}. When e_1 and e_2 are at angle $45°$, there exists 3 different K_4s having vertex sets S_1, S_2 and S_3 other than S such that either 1) c can not be used at G_{S_1} and G_{S_2} and $c+1$ can not be used at G_{S_3} or 2) $c+1$ can not be used at G_{S_1} and G_{S_2} and c can not be used at G_{S_3}.*

Proof:

Note that the angular distance between any two adjacent edges in T_8 can be any one of $45°$, $90°$, $135°$, $180°$, $225°$, $270°$ and $315°$. From symmetry, it is suffice to consider the cases where the angular distance between two adjacent edges in T_8 is $180°$ or $135°$ or $90°$ or $45°$.

- When angular distance between two adjacent edges is $180°$: Observe that two adjacent horizontal edges or two adjacent vertical edges or two adjacent left slanting edges or two adjacent right slanting edges can be at $180°$. Clearly the cases where two horizontal edges and two vertical edges forming $180°$ are symmetric. Similarly the cases where two left slanting edges and two right slanting edges forming $180°$ are also symmetric. So we need to consider the following two cases.

  Case 1) When two adjacent horizontal edges $e_1 = (v_1, v_2)$ and $e_2 = (v_2, u_6)$ form $180°$ (Fig. 3.3). Let $f'(e_1) = c$ and $f'(e_2) = c+1$. Note that any edge e incident to any of the vertices in $\{u_5, u_6, u_7\}$ is at distance at most two from e_1. As $f'(e_1) = c$, the color c can not be used at any of those edges for $L(1,2)$ edge labeling. Similarly observe that any edge e incident to any of the vertices in $\{w_8, w_9, w_{10}\}$ but not incident to any of the vertices in $\{u_5, u_6, u_7\}$ is at distance two from e_2. As $f'(e_2) = c+1$, the color c can not be used at any of those edges for $L(1,2)$ edge labeling. Hence there exists 2 different K_4s having vertex sets $S_1 = \{u_5, u_6, w_8, w_9\}$ and $S_2 = \{u_6, u_7, w_9, w_{10}\}$ such that c can not be used at G_{S_1} and G_{S_2}. With similar argument, it can be shown that there exists 2 different K_4s having vertex sets $S_3 = \{v_1, v_4, u_{11}, u_{12}\}$ and $S_4 = \{v_1, u_9, u_{10}, u_{11}\}$ such that $c+1$ can not be used at G_{S_3} and G_{S_4}.

Case 2) When two adjacent right slanting edges $e_1 = (v_1, v_3)$ and $e_2 = (v_1, u_{10})$ form 180° (Fig. 3.3). Let $f'(e_1) = c$ and $f'(e_2) = c+1$. In this case, there exists 3 different K_4s having vertex sets $S_1 = \{u_{10}, w_{15}, w_{16}, w_{17}\}$, $S_2 = \{u_9, u_{10}, w_{14}, w_{15}\}$ and $S_3 = \{u_{10}, u_{11}, w_{17}, w_{18}\}$ such that c can not be used at G_{S_1}, G_{S_2} and G_{S_3}. Similarly, there exists 3 different K_4s having vertex sets $S_3 = \{v_3, v_4, u_2, u_3\}$, $S_4 = \{v_3, u_3, u_4, , u_5\}$ and $S_5 = \{v_2, v_3, u_5, u_6\}$ such that $c+1$ can not be used at G_{S_3}, G_{S_4} and G_{S_5}.

- When angular distance between two adjacent edges is 135°: Observe that a horizontal edge and its adjacent left slanting edge or a horizontal edge and its adjacent right slanting edge or a vertical edge and its adjacent left slanting edge or a vertical edge and its adjacent right slanting edge may be at 135°. Clearly all the cases are symmetric. So we need to consider the following case only. Let us consider the horizontal edge $e_1 = (v_1, v_2)$ and the right slanting edge $e_2 = (v_2, u_5)$. Let $f'(e_1) = c$ and $f'(e_2) = c+1$. From similar discussion stated in the previous case, there exists 3 different K_4s having vertex sets $S_1 = \{v_3, u_3, u_4, u_5\}$, $S_2 = \{u_4, u_5, w_7, w_8\}$ and $S_3 = \{u_5, u_6, w_8, w_9\}$ such that c can not be used at G_{S_1}, G_{S_2} and G_{S_3}. Similarly, there exists 2 different K_4s having vertex sets $S_4 = \{v_1, v_4, u_{11}, u_{12}\}$ and $S_5 = \{v_1, u_9, u_{10}, u_{11}\}$ such that $c+1$ can not be used at G_{S_4} and G_{S_5}.

- When angular distance between two adjacent edges is 90°: Observe that a horizontal edge and its adjacent vertical edge or a vertical edge and its adjacent horizontal edge or a right slanting edge and its adjacent left slanting edge may be at 90°. Clearly the first two cases are symmetric. So we need to consider the following two cases.

Case 1) Consider the horizontal edge $e_1 = (v_1, v_2)$ and the vertical edge $e_2 = (v_2, v_3)$. Let $f'(e_1) = c$ and $f'(e_2) = c+1$. As similar argument stated in the previous cases, there exists 2 different K_4s having vertex sets $S_1 = \{v_3, v_4, u_2, u_3\}$ and $S_2 = \{v_3, u_3, u_4, u_5\}$ such that c can not be used at G_{S_1} and G_{S_2}. Similarly, there exists 2 different K_4s having vertex sets $S_3 = \{v_1, v_4, u_{11}, u_{12}\}$ and $S_4 = \{v_1, u_9, u_{10}, u_{11}\}$ such that $c+1$ can not be used at G_{S_3} and G_{S_4}.

Case 2) Consider the right slanting edge $e_1 = (v_1, v_3)$ and the left slanting edge $e_2 = (v_1, u_{12})$. Let $f'(e_1) = c$ and $f'(e_2) = c + 1$. As similar argument stated in the previous cases, there exists 3 different K_4s having vertex sets $S_1 = \{u_1, u_{12}, w_{19}, w_{20}\}$, $S_2 = \{u_1, u_2, u_{12}, v_4\}$ and $S_3 = \{u_{11}, u_{12}, w_{18}, w_{19}\}$ such that c can not be used at G_{S_1}, G_{S_2} and G_{S_3}. Similarly, there exists 3 different K_4s having vertex sets $S_4 = \{v_3, v_4, u_2, u_3\}$, $S_5 = \{v_3, u_3, u_4, u_5\}$ and $S_6 = \{v_2, v_3, u_5, u_6\}$ such that $c + 1$ can not be used at G_{S_4}, G_{S_5} and G_{S_6}.

- When angular distance between two edges is 45°: Observe that a horizontal edge and its adjacent left slanting edge or a horizontal edge and its adjacent right slanting edge or a vertical edge and its adjacent left slanting edge or a vertical edge and its adjacent right slanting edge may be at 45°. Clearly all the cases are symmetric. So we need to consider the following case only. Consider the horizontal edge $e_1 = (v_1, v_2)$ and the left slanting edge $e_2 = (v_2, v_4)$. Let $f'(e_1) = c$ and $f'(e_2) = c + 1$. As similar argument stated in the previous cases, there exists 2 different K_4s having vertex sets $S_1 = \{v_4, u_1, u_2, u_{12}\}$ and $S_2 = \{v_3, v_4, u_2, u_3\}$ such that c can not be used at G_{S_1} and G_{S_2}. Similarly, there exists a K_4s having vertex set $S_3 = \{v_1, u_9, u_{10}, u_{11}\}$ such that $c + 1$ can not be used at G_{S_3}. If $f'(e_1) = c + 1$ and $f'(e_2) = c$, then there exists 2 different K_4s having vertex sets $S_1 = \{v_4, u_1, u_2, u_{12}\}$ and $S_2 = \{v_3, v_4, u_2, u_3\}$ such that $c + 1$ can not be used at G_{S_1} and G_{S_2}. Similarly, there exists a K_4s having vertex set $S_3 = \{v_1, u_9, u_{10}, u_{11}\}$ such that c can not be used at G_{S_3}. $\square$

Now we state and prove the following Theorems.

Theorem 3.2.1 $\lambda'_{1,2}(T_8) \geq 26$.

Proof: Let us consider the K_4 with vertex set $S = \{v_1, v_2, v_3, v_4\}$ and the corresponding G_S as shown in Fig. 3.2. Observe that there are 26 edges in G_S. Note that there are no two edges at more than distance two apart in G_S. Hence all the colors used in G_S must be distinct for $L(1, 2)$-edge labeling. Hence 26 consecutive colors $\{0, 1, \ldots, 25\}$ are to be used in G_S, otherwise $\lambda'_{1,2}(G_S) \geq 26$. In that case, any two consecutive colors c and $c + 1$ must be used at two adjacent edges in G_S.

Therefore, from Lemma 3.2.1, there exists at least a K_4 having vertex set S_1 such that c can not be used at G_{S_1}. But in G_{S_1}, 26 distinct colors must be used. So if we do not use c in G_{S_1}, at least a new color which is not used in G_S must be used in G_{S_1}. So at least the color 26 must be introduced in G_{S_1}. Hence $\lambda'_{1,2}(G_{S_1}) \geq 26$ implying $\lambda'_{1,2}(T_8) \geq 26$. $\square$

Theorem 3.2.2 $\lambda'_{1,2}(T_8) \geq 27$.

Proof:

Consider $S = \{v_1, v_2, v_3, v_4\}$ and the corresponding G_S as shown in Fig. 3.3. Note that $\lambda'_{1,2}(G_S) \leq 26$ only if 1) 26 consecutive colors $\{0, 1, \ldots, 25\}$ or $\{1, 2, \ldots, 26\}$ are used in G_S, or 2) the colors $\{0, 1, \ldots, 26\} \setminus \{c'\}$ are used in G_S, where $1 \leq c' \leq 25$. The cases of $\{0, 1, \ldots, 25\}$ and $\{1, 2, \ldots, 26\}$ are clearly symmetric. Hence we need to consider only the following two cases.

Consider the first case. Observe that the set of colors $\{0, 1, \ldots, 25\}$ can be partitioned into 13 disjoint consecutive pairs of colors $(0, 1)$, $(2, 3)$, $\ldots$, $(24, 25)$. Consider a pair of consecutive colors $(c, c+1)$, where $0 \leq c \leq 24$. From Lemma 3.2.1, it follows that for the pair $(c, c+1)$, there exists two distinct S_1 and S_2 other that S such that either c or $c+1$ can not be used in G_{S_1} and G_{S_2}. So for the 13 disjoint pairs mentioned above, there must be 26 such $S_1, S_2, \ldots, S_{26}$ other than S. Note that there are total 25 S_js including S in G. Therefore, there exists only 24 such S_js other than S in G. Hence from pigeon hole (26 pigeons and 24 holes) principle there must be at least one G_{S_j} where two different colors which are used in G_S can not be used there and hence $\lambda'_{1,2}(G_{S_j}) \geq 27$ implying $\lambda'_{1,2}(T_8) \geq 27$.

Now we consider the second case. Consider that the colors $\{0, 1, \ldots, 26\} \setminus \{c'\}$ have been used in G_S, where $1 \leq c' \leq 25$. First let us consider the case when c' is even. In that case the set of colors $\{0, \ldots, c'-1, c'+1, \ldots 26\}$ can be partitioned into 13 disjoint consecutive pairs of colors $(0, 1), \ldots, (c'-2, c'-1), (c'+1, c'+2), \ldots, (25, 26)$. Hence proceeding similarly as above case, from pigeon hole principle, we get that $\lambda'_{1,2}(T_8) \geq 27$. Now consider the case when c' is odd. Let us first consider $c' \neq 1, 25$. Note that the set of colors $\{0, \ldots, c'-1, c'+1, \ldots 26\}$ can be partitioned into 12 disjoint consecutive pairs of colors $(0, 1), \ldots, (c'-3, c'-2), (c'+1, c'+2), \ldots, (24, 25)$. So there must be 24 K_4s having vertex sets $S_1, S_2, \ldots, S_{24}$ other than S in G. Now consider the pair of consecutive colors $(25, 26)$. Now, from Lemma 3.2.1, there must exists at least one S_{25} other than S such that color 26 can not be used in $G_{S_{25}}$. So we need 25 such $S_1, S_2, \ldots, S_{25}$ other than S. But there exists only 24 such distinct S_js other than S in G. Hence from pigeon hole (25 pigeons and 24 holes) principle there must be at least one G_{S_j} where two different colors used in G_S can not be used there and hence $\lambda'_{1,2}(G_{S_j}) \geq 27$ implying $\lambda'_{1,2}(T_8) \geq 27$. When $c' = 1$ the set of colors $\{0, 2, \ldots 26\}$ can be partitioned into 12 disjoint consecutive pairs of colors $(2, 3), \ldots, (24, 25)$. Considering these 12 pairs and the pair of consecutive colors $(25, 26)$, we get $\lambda'_{1,2}(T_8) \geq 27$ by proceeding similarly as above. When $c' = 25$ the set of colors $\{0, \ldots 24, 26\}$ can be partitioned into 12 disjoint consecutive pairs of colors $(1, 2,), \ldots, (23, 24)$. Considering these 12 pairs and the pair of consecutive colors $(0, 1)$, we get $\lambda'_{1,2}(T_8) \geq 27$ by proceeding similarly as above. $\square$

Observe that a K_3 contains one horizontal, one vertical and one slanting edge. Note that three consecutive colors $c-1, c, c+1$ can be used at three edges with or without forming a K_3 (complete graph of 3 vertices) in G_S, where c being used at a slanting or non slanting edge. Accordingly, we now have the following two Lemmas.

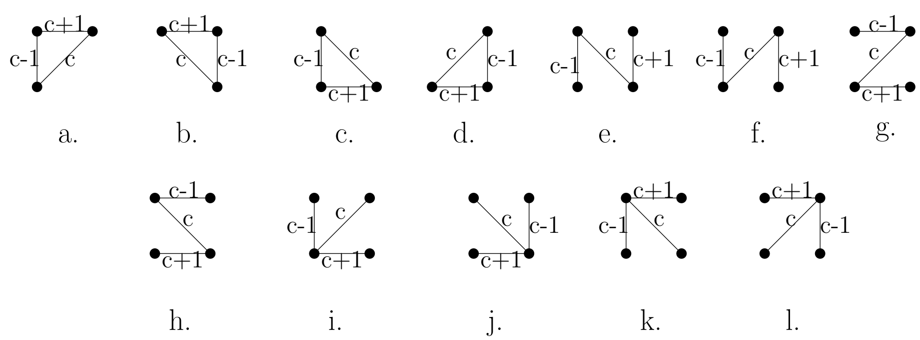


Figure 3.4: The all possible cases where c is used in a slanting edge e and both $c \pm 1$ are used at edges forming 45° with e.

Lemma 3.2.2 *When three consecutive colors $c-1$, c and $c+1$ are used at three edges forming a K_3 in G_S with c being used at the slanting edge, then there exists a K_4 with vertex set S_1 in G such that c can not be used in G_{S_1}. When three consecutive colors $c-1$, c and $c+1$ are used at three edges without forming a K_3 in G_S with c being used at the slanting edge, then there exists at least 2 different K_4s with vertex sets S_1 and S_2 in G such that c can not be used at G_{S_1} and G_{S_2}.*

Proof:

Let us consider $c-1$, c and $c+1$ are used at three edges which are forming a K_3 with c being used at the slanting edge. The different cases where such scenarios occur are shown in Fig. 3.4 a. to Fig. 3.4 d. Note that all the cases shown in Fig. 3.4 a. to Fig. 3.4 d. are symmetric. So we consider any one of the cases. Consider the case where the colors $c-1$, c and $c+1$ are used at three edges (v_1, v_2), (v_1, v_3) and (v_2, v_3) respectively (Fig. 3.3). Now consider the K_4 with vertex set $S_1 = \{v_2, u_6, u_7, u_8\}$ and observe that any edge incident to any of the vertices in $S_1 \setminus \{v_2\}$ is at distance two from (v_1, v_2). As $f'(v_1, v_2) = c-1$, c can not be used at those edges for $L(1,2)$ edge labeling. As any edge incident to v_2 is at distance at most 2 from (v_1, v_3) and $f'(v_1, v_3) = c$, the color c can not be used there as well. So c can not be used at G_{S_1}.

Now we consider the case where $c-1$, c and $c+1$ are used at three edges which are not forming a K_3 with c being used at the slanting edge. Note that the colors $c-1$ and $c+1$ can be used at two edges both of which are at 45° with the edge having color c. The different cases where such scenarios arrive are shown in Fig. 3.4 e. to Fig. 3.4 l. Note that the cases shown in Fig. 3.4 e. to Fig. 3.4 h. are symmetric and the cases shown in Fig. 3.4 i. to Fig. 3.4 l. are also symmetric. So here we need to consider only two cases.

- Consider the case where the colors $c-1$, c and $c+1$ are used at three edges (v_1, v_4), (v_1, v_3) and (v_2, v_3) respectively (Fig. 3.3). In this case, the said three edges form a structure isomorphic to Fig.3.4 e. to Fig.3.4 h. Here note that there exists two K_4s with vertex sets $S_1 = \{v_4, u_1, u_2, u_{12}\}$ and $S_2 = \{v_2, u_6, u_7, u_8\}$ such that c can not be used in G_{S_1} and G_{S_2}.
- Consider the case where the colors $c-1$, c and $c+1$ are used at three edges (v_1, v_4), (v_1, v_3) and (v_1, v_2) respectively (Fig. 3.3). In this case the said three edges form a structure isomorphic to Fig. 3.4 i. to Fig. 3.4 l. Here also, there exists two K_4s with vertex sets $S_1 = \{v_4, u_1, u_2, u_{12}\}$ and $S_2 = \{v_2, u_6, u_7, u_8\}$ such that c can not in G_{S_1} and G_{S_2}.

Now we consider the case where the edge having color $c+1$ (or $c-1$) is not forming 45° with the slanting edge having color c and the other edge having color $c-1$ (or $c+1$) forming 45° with the slanting edge. In that case, from Lemma 3.2.1, it follows that there exists at least 2 distinct K_4s having vertex sets S_1 and S_2 other than S such that c can not be used in G_{S_1} and G_{S_2}. As the other color $c-1$ (or $c+1$) is used at an edge forming 45° with the slanting edge, there also exists at least another K_4 having vertex set S_3 other than S such that c can not be used in G_{S_3}. So in this case there are at least 3 distinct K_4s.

We now consider the case where both the colors $c-1$ and $c+1$ both are used at edges not forming 45° with the slanting edge with color c. In this case, from Lemma 3.2.1, there exists at least 4 different K_4s having vertex sets S_1, S_2, S_3 and S_4 other than S such that c can not be used in G_{S_1}, G_{S_2}, G_{S_3} and G_{S_4}. So in this case there are at least 4 distinct K_4s. $\square$

Lemma 3.2.3 *If three consecutive colors $c-1$, c and $c+1$ are used at three edges of G_S forming a K_3 with c being used at a non slanting edge, then there exists two K_4s with vertex sets S_1 and S_2 in G such that c can not be used in G_{S_1} and G_{S_2}. If three consecutive colors $c-1$, c and $c+1$ are used at three edges of G_S without forming a K_3 with c being used at a non slanting edge, then there exists at least 3 K_4s with vertex sets S_1, S_2 and S_3 in G such that c can not be used in G_{S_1}, G_{S_2} and G_{S_2}.*

Proof:

- Let us first consider the case where $c-1$, c and $c+1$ are used at three edges which are forming a K_3 with c being used at the non slanting edge. Consider the K_3 with vertex set $\{v_1, v_2, v_3\}$ in G_S as shown in Fig. 3.3. Suppose the colors $c-1$, c and $c+1$ are used at (v_1, v_2), (v_2, v_3) and (v_1, v_3) respectively. In this case, there exists two K_4s with vertex sets $S_1 = \{v_1, u_9, u_{10}, u_{11}\}$ and $S_2 = \{v_1, v_4, u_{11}, u_{12}\}$ such that c can not be used at G_{S_1} and G_{S_2}.

- Now we consider the case where $c-1$, c and $c+1$ are used at three edges which are not forming a K_3 with c being used at the non slanting edge. In the previous case, as the two edges with colors $c-1$ and $c+1$ have a common vertex other than the end vertices of the edge with color c, we get only two different K_4s. Note that if three edges do not form a K_3, there does not exists a common vertex of the two edges with colors $c-1$ and $c+1$, other than an end vertex of the edge with color c. In that case, there exists at least 3 different K_4s having vertex sets S_1, S_2 and S_3 such that c can not be used at G_{S_1}, G_{S_2} and G_{S_3}. $\square$

We now consider how many K_3S can be there in G_S such that for each such K_3 three consecutive colors can be used. For this, we have the following Observation.

Observation 5 *Consider the K_4 having vertex set $S = \{v_1, v_2, v_3, v_4\}$ and the sub graph G_S. There can be at most 8 K_3s in G_S such that in each of the K_3, three consecutive colors can be used.*

Proof: Consider the edges $e_1 = (v_1, v_2)$, $e_2 = (v_2, v_3)$, $e_3 = (v_3, v_4)$ and $e_4 = (v_4, v_1)$ of the K_4 having vertex set $S = \{v_1, v_2, v_3, v_4\}$ (Fig. 3.2). Observe that there are total 12 K_3s in G_S and each K_3 has at least one edge in $\{e_1, e_2, e_3, e_4\}$. Each edge $e \in \{e_1, e_2, e_3, e_4\}$ is a common edge of 4 different K_3s. Let a color c be used in an edge $e \in \{e_1, e_2, e_3, e_4\}$. In order to form a K_3 with three consecutive colors with e, either $c + 1$ or $c - 1$ must be used in that K_3. So, out of the 4 K_3s that include e, at most two of then can have 3 consecutive colors. As there are 4 edges in $\{e_1, e_2, e_3, e_4\}$, we can have at most 8 K_3s where each of them has 3 consecutive colors. $\square$

Now we state and prove the following Theorem.

Theorem 3.2.3 $\lambda'_{1,2}(T_8) \geq 28$.

Proof:

Consider the graph G, the K_4 with vertex set $S = \{v_1, v_2, v_3, v_4\}$ and the sub graph G_S as shown in Fig. 3.3. Note that 26 distinct colors from $\{0, 1, \ldots, 27\}$ must be used in G_S. In other words, there must be 2 colors in $\{0, 1, \ldots, 27\}$ which should remain unused. Let these two colors be c_1 and c_2 where $c_1 < c_2$. Let us now consider the 6 colors 0, $c_1 - 1$, $c_1 + 1$, $c_2 - 1$, $c_2 + 1$ and 27.

We first consider the case when all these 6 colors are distinct and denote $X = \{0, c_1 - 1, c_1 + 1, c_2 - 1, c_2 + 127\}$. In this case, for each color $c \in X$, only one of $c \pm 1$ is used and the other is not used in G_S. From Lemma 3.2.1, for each $c \in X$, there exists at least one K_4 having vertex set S_1 other than S in G such that c can not be used at G_{S_1}. Moreover, c must be used at a slanting edge in G_S in this case. Considering all 6 colors in X are used in 6 slanting edges, we get at least 6 such K_4s. For each c among the remaining $26 - 6 = 20$ colors, both $c \pm 1$ are used in G_S. There are total 26 edges in G_S among which 14 are slanting edges and 12 are non-slanting edges. Therefore, we are yet to consider the colors used in the remaining $14 - 6 = 8$ slanting edges. Note that for each such color c, both $c \pm 1$ are used in G_S. Assume these 8 colors, x many colors $c'_1, c'_2, \ldots, c'_x$ are there such that for each

c_i', three consecutive colors $c_i' - 1$, c_i', $c_i' + 1$ can be used in a K_3. From Lemma 3.2.2, for each such color c_i', there exists at least one K_4 having vertex set S_1 other than S in G such that c_i' can not be used at G_{S_1}. Considering all these x colors, we get at least x such K_4s in G. Now consider the remaining $8 - x$ colors $c'_{x+1}, c'_{x+2}, \dots, c'_8$ used in slanting edges. Note that for each such c_i', three consecutive colors $c_i' - 1$, c_i' and $c_i' + 1$ can not be used in a K_3. From Lemma 3.2.2, for each such c_i', there exists at least 2 different K_4s having vertex sets S_1 and S_2 other than S in G such that c_i' can not be used at G_{S_1} and G_{S_2}. Considering all these $8 - x$ colors, we get at least $2(8 - x)$ K_4s in G.

Now consider the 12 non slanting edges in G_S. Assume that among them, y many colors $c_1'', c_2'', \dots, c_y''$ are there such that for each c_i'', three consecutive colors $c_i'' - 1$, c_i'', $c_i'' + 1$ can be used in a K_3. Clearly all those y many K_3s must be different from those x many K_3 considered for slanting edges. From Observation 5, there exists at most 8 K_3s in G_S such that for each of them, three consecutive colors can be used. As x many K_3s have already been considered for slanting edges, y can be at most $8 - x$. From Lemma3.2.3, for each such c_i'', there exists at least 2 different K_4s having vertex sets S_1 and S_2 other than S such that c_i'' can not be used at G_{S_1} and G_{S_2}. Considering all these $8 - x$ colors, we get at least $2(8 - x)$ K_4s. We are yet to consider the remaining $z = 12 - (8 - x) = 4 + x$ non slanting edges. For each such color c'', the colors $c'' - 1$, c'' and $c'' + 1$ can not be used in a K_3 in G_S. So from Lemma3.2.3, for each such c'', there exists at least 3 different K_4s having vertex sets S_1, S_2 and S_3 other than S in G such that c'' can not be used at G_{S_1}, G_{S_2} and G_{S_3}. Considering all these $4 + x$ colors, we get at least $3(4 + x)$ K_4s. In total we get at least $6 + x + 2(8 - x) + 2(8 - x) + 3(4 + x) = 50$ K_4s in G. But there are only 24 distinct K_4s in G other than S. From pigeon hole principle (50 pigeons and 24 holes), there exists at least one S_i in G such that at least 3 colors which are used in G_S can not be used in G_{S_i} and hence $\lambda'_{1,2}(G_{S_i}) \geq 28$ implying $\lambda'_{1,2}(T_8) \geq 28$.

If the colors 0, $c_1 - 1$, $c_1 + 1$, $c_2 - 1$, $c_2 + 1$ and 27 are not distinct, proceeding similarly, we can show that there are a need of more that 50 K_4s in G and hence from pigeon hole principle (more than 50 pigeons and 24 holes) $\lambda'_{1,2}(T_8) \geq 28$. $\square$

3.3 Conclusion

It was proved in [17] that $25 \leq \lambda'_{1,2}(T_8) \leq 28$ with a gap between the lower and upper bounds. In this chapter, we filled the gap and proved that $\lambda'_{1,2}(T_8) \geq 28$. This essentially implies $\lambda'_{1,2}(T_8) = 28$.

Chapter 4

The span of $L(k_1, k_2)$-vertex labeling for T_6

In this chapter we study the lower bound of $\lambda_{k_1,k_2}(T_6)$ when $k_1 \leq k_2$. For a graph G, it can be shown from the scaling lemma in [13], $\lambda_{k_1,k_2}(G) = k_2 * \lambda_{\frac{k_1}{k_2},1}(G) = k_2 * \lambda_{h,1}(G)$, where $h = \frac{k_1}{k_2}$. Determining $\lambda_{k_1,k_2}(G)$ is equivalent to evaluating $\lambda_{h,1}(G)$ and multiplying it with k_2.

$L(h, 1)$-vertex labeling problem when $0 \leq h \leq 1$ when has been studied by several researchers [13, 12]. In [12], exact values of $\lambda_{h,1}(T_6)$ when $0 \leq h \leq 1$ have been determined for different values of h . But the values of $\lambda_{h,1}(T_6)$ obtained here are partly based on computer simulations. In this chapter, we introduce an approach where we determine the lower bounds of $\lambda_{h,1}(T_6)$ theoretically by exploring underlined graph structure for the intervals when $0 \leq h \leq \dfrac{1}{2}$ and when $h \geq \dfrac{1}{2}$. H Our obtained results are $\lambda_{h,1}(T_6) \geq 3 + 2h$ when $0 \leq h \leq \dfrac{1}{2}$ and $\lambda_{h,1}(T_6) \geq 4$ when $h \geq \dfrac{1}{2}$. Our result exactly coincides with the result obtained in [13, 12] (as stated in equations (4.1) and (4.2)) through computer simulations when $0 \leq h \leq \dfrac{1}{3}$. When $h > \dfrac{1}{3}$, results obtained in [13, 12] are finer than ours. In the rest the chapter, our discussion will based on $L(h, 1)$-vertex labeling for T_6 when $0 \leq h \leq 1$.

We organize the remaining of the chapter in following manner. The previous result for $\lambda_{h,1}(T_6)$ has been explicitly stated in section 4.1. In, section 4.2, we define and explain some terms that we use to prove our results. We will state and prove our main result in section 4.3. Section 4.4 concludes this chapter.

4.1 Previous results

$$\lambda_{h,1}(T_6) = \begin{cases} = 2h+3, & if\ 0 \le h \le 1/3, \\ \in [2h+3, 11h], & if\ 1/3 \le h \le 9/22, \\ \in [2h+3, 9/2], & if\ 9/22 \le h \le 3/7, \\ \in [9h, 9/2], & if\ 3/7 \le h \le 1/2, \\ \in [9/2, 16/3], & if\ 1/2 \le h \le 2/3, \\ \in [16/3, 23/4], & if\ 2/3 \le h \le 3/4, \\ \in [23/4, 6], & if\ 3/4 \le h \le 4/5, \\ = 6 & if\ 4/5 \le h \le 1. \end{cases} \tag{4.1}$$

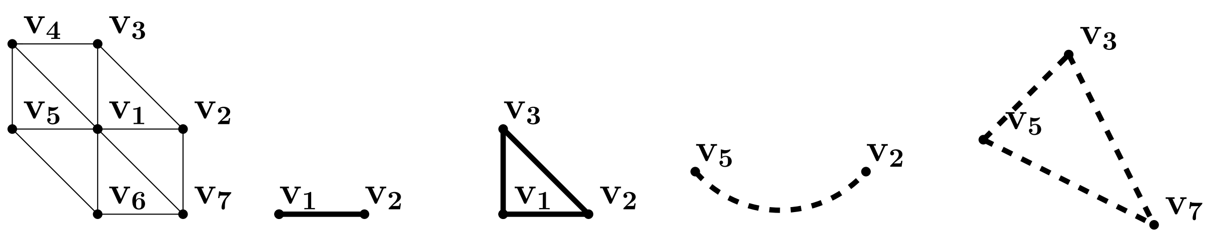
$$\lambda_{h,1}(T_6) = \begin{cases} 2h+3, & if\ 0 \le h \le 1/3, \\ 11h, & if\ 1/3 \le h \le 3/8, \\ 3h+3, & if\ 3/8 \le h \le 2/5, \\ 8h+1, & if\ 2/5 \le h \le 3/7, \\ h+4, & if\ 3/7 \le h \le 1/2, \\ 9h, & if\ 1/2 \le h \le 4/7, \\ 2h+4, & if\ 4/7 \le h \le 2/3, \\ 8h, & if\ 2/3 \le h \le 5/7, \\ h+5, & if\ 5/7 \le h \le 3/4, \\ 5h+2, & if\ 3/4 \le h \le 4/5, \\ 6 & if\ 4/5 \le h \le 1. \end{cases} \tag{4.2}$$

The above two sets of equations (4.1) and (4.2) present the results obtained in [13] and [12] respectively. In [13], authors determined the bounds of $\lambda_{h,1}(T_6)$ for $0 \le h \le 1$. But some of the bounds were not tight and moreover, the bounds were obtained based on computer simulation by considering all possible $L(h,1)$ labeling of three induced sub graphs of T_6 having 7, 19 and 37 vertices using computer simulation. Later, Král and Skoda [12] gave exact values of $\lambda_{h,1}(T_6)$ for different sub intervals for $0 \le h \le 1$. But here also, the bounds are obtained through brute force computer simulations on the induced sub graphs of T_6 having 81, 100, 169 and 225 nodes. In following sections, after depicting the key ideas we will present our theoretically obtained results.

4.2 Key ideas

Definition 8 *Suppose $G(V,E)$ is a graph. A subset V' of V is said to constitute a D_k^n, a distance k clique of size n, if $|V'| = n$ and every pair of vertices in V' is at distance k from each other in G, where $k \geq 1$ and $n \geq 2$ are two integers.*

We have shown a graph G and its different D_k^n in Figure. 4.1. In this figure, a dotted edge between two vertices represents that they are at distance 2 apart from each other in G.



a. $G(V,E)$ *b.* $D_1^2 = \{v_1, v_2\}$. *c.* $D_1^3 = \{v_1, v_2, v_3\}$. *d.* $D_2^2 = \{v_5, v_2\}$. *e.* $D_2^3 = \{v_3, v_5, v_7\}$.

Figure 4.1: A graph $G(V,E)$ and its different D_k^ns.

We introduce the notion of *color class* to partition the available colors into a set of disjoint color classes. Let us formally define the color class as follows.

Definition 9 *A color class $C_k = \{i : k \leq i < k+1, i \in \mathbb{R}, k \in \mathbb{N}\}$.*

Note that the absolute difference between any two colors in a particular color class is strictly less than 1. Let f be an $L(h,1)$-vertex labeling of G where $0 \leq h \leq 1$. Now the following facts are immediate for any pair of vertices u and v in G.

- When $d(u,v) = 2$ then $|f(u) - f(v)| \geq 1$, so $f(u)$ and $f(v)$ must belong to two different color classes.
- When $d(u,v) = 1$ then $|f(u) - f(v)| \geq h$, so $f(u)$ and $f(v)$ may or may not belong to the same class class.

Let u and v be two vertices in G and f be an $L(h,1)$-vertex labeling of G. We say that $\mathbf{u} \sim \mathbf{v}$ if $f(u)$ and $f(v)$ belong to same color class and $\mathbf{u} \nsim \mathbf{v}$ if $f(u)$ and $f(v)$ do not belong to same color class.

Definition 10 *Let $G(V, E)$ be a graph and f be a $L(h,1)$-vertex labeling of it. Let $\{v_1, v_2, v_3\} \subseteq V$ be a D_1^3. We term D_1^3 as a monochromatic triangle if $f(v_1)$, $f(v_2)$ and $f(v_3)$ belong to the same color class.*

Suppose an $L(h,1)$-vertex labeling f of the D_1^2 shown in Figure. 4.1.b produces $f(v_1) = 0$ and $f(v_2) = h$. If $h < 1$ then the edge (v_1, v_2) is said to be a *monochromatic edge* with color class C_0 as 0, $h \in C_0$. Similarly, suppose an $L(h,1)$-vertex labeling f of the D_1^3 shown in Figure. 4.1.c produces $f(v_1) = 0$, $f(v_2) = h$ and $f(v_3) = 2h$. If $h < 1/2$ then the triangle D_1^3 is said to be a *monochromatic triangle* with color class C_0 as 0, h, $2h \in C_0$.

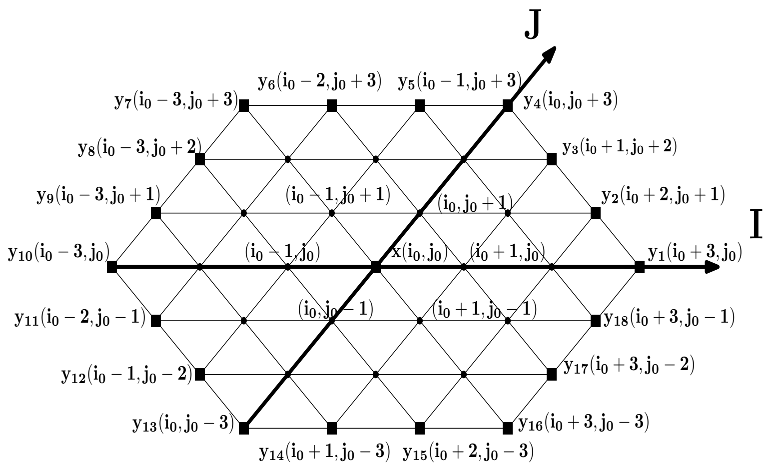


Figure 4.2: Representation of vertices of T_6 with co-ordinates.

As we are dealing with $L(h,1)$-vertex labeling of T_6, we now introduce co-ordinate system to represent the vertices of T_6. For any vertex in T_6 it has 6 neighbors at distance 1 and hence 6 edges incident to v. Angle between any two consecutive edges in anti-clockwise direction incident to any vertex v in T_6 is $60°$. Taking any vertex v as origin and two consecutive edges in anti-clockwise direction incident to v as two co-ordinate axes, a co-ordinate system can be constructed in T_6. We follow one such co-ordinate system with axes (I, J) as shown in Figure. 4.2. In this coordinate system, for any vertex x with co-ordinate (i_0, j_0), the co-ordinates of its six neighbors are (i_0+1, j_0), (i_0, j_0+1), (i_0-1, j_0+1), (i_0-1, j_0), (i_0, j_0-1) and (i_0+1, j_0-1) as shown in Figure. 4.2. The 18 vertices $y_i \in T_6$, $i \in \{1, 2, \ldots, 18\}$ at distance 3 from x are also shown in Figure. 4.2. Among them, the 6 vertices y_1, y_4, y_7, y_{10}, y_{13} and y_{16} are termed *corner* vertices of x. The remaining 12 vertices y_2, y_3, y_5, y_6, y_8, y_9, y_{11}, y_{12}, y_{14}, y_{15}, y_{17} and y_{18} are termed *non-corner* vertices of x.

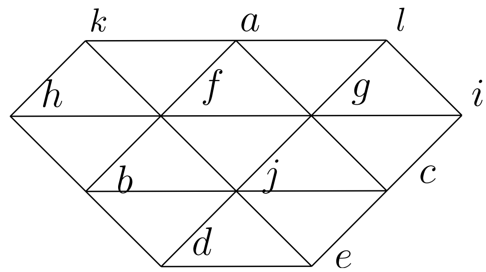


Figure 4.3: A sub graph G_3 of T_6.

4.3 Our results

For any two vertices $x, y \in T_6$ with $d(x,y) = 3$, either $x \sim y$ or $x \nsim y$. In subsection 4.3.1 we will determine $\lambda_{h,1}(T_6)$ when $\mathbf{x} \nsim \mathbf{y}$ for all $x,\ y \in T_6$ with $d(x,y) = 3$ and in subsection 4.3.2 we will determine $\lambda_{h,1}(T_6)$ when $\mathbf{x} \sim \mathbf{y}$ for at least a pair of vertices $x,\ y \in T_6$ with $d(x,y) = 3$. Then we will state and prove our main result in theorem 4.3.2.

4.3.1 Determining $\lambda_{h,1}(T_6)$ when $x \nsim y$ for all $x,\ y \in T_6$ with $d(x,y) = 3$.

Figure. 4.3 shows a sub graph G_3 of T_6 such that $\forall x, y \in G_3$, $d(x,y) \leq 3$. Such a sub graph can be constructed as follows: consider any D_1^3 in T_6 and then G_3 is the sub graph induced by the vertices of the D_1^3 and all its neighbors. Figure. 4.3 shows the G_3 constructed from $D_1^3 = \{f, g, j\}$.

Lemma 4.3.1 *In G_3, at most 3 vertices can be assigned colors from the same color class and these three vertices must form a D_1^3 when $\mathbf{x} \nsim \mathbf{y}$ for all $x,\ y \in G_3$ with $d(x,y) = 3$.*

Proof: Two vertices at distance 2 in T_6 can not be assigned with the colors from the same color class. Therefore for two vertices $v_1,\ v_2 \in G_3$, $v_1 \sim v_2$ only when $d(v_1, v_2) = 1$, as in the case under consideration, $\mathbf{x} \nsim \mathbf{y}$ for all $x,\ y \in T_6$ with $d(x,y) = 3$. Let us consider $v_1 \sim v_2$ and color class of it is C_k. Any vertex in G_3 other than the vertices at distance 1 from both of v_1 and v_2 in G_3 are at distance 2 or 3 from either v_1 or v_2, hence these vertices can not be assigned with the colors from the color class C_k. Therefore C_k can be used in v_1, v_2 and the vertex v_3 which is at distance 1 from both of v_1 and v_2. Thus v_1, v_2 and v_3 form a D_1^3. Any vertex other than v_1, v_2 and v_3 in G_3 is at distance 2 from any one of v_1, v_2 and v_3. Hence except v_1, v_2 and v_3 any other vertex can not be assigned with the colors from the color class C_k. Thereby Lemma 4.3.1 is proved. $\square$

Theorem 4.3.1 *$\lambda_{h,1}(T_6) \geq 4$ if $\mathbf{x} \nsim \mathbf{y}$ for all $x,\ y \in T_6$ with $d(x,y) = 3$.*

Proof: Consider the G_3 constructed from $\{f, g, j\}$ as shown in Figure. 4.3. It is evident that $\lambda_{h,1}(T_6) \geq \lambda_{h,1}(G_3)$ as G_3 is a sub graph of T_6. From lemma 4.3.1, at most three vertices can have colors from the same color class and they must form a D_1^3 when $x \nsim y$ for all $x,\ y \in T_6$ with $d(x,y) = 3$. Note that any D_1^3 in G_3 must include at least one vertex from $\{f, g, j\}$. So at most three disjoint D_1^3s can be constructed in G_3. Let M_f be a D_1^3 which includes f. There are six such D_1^3s are possible. Consider M_f as any one of them. Any vertex in G_3 other than those belong to M_f is at distance 2 from at least one of the three vertices of M_f. Same thing holds for M_g and M_j as well. Therefore M_f, M_g and M_j must get colors from 3 different color classes. Let C_{k_1}, C_{k_2} and C_{k_3} be the color classes of M_f, M_g and M_j respectively, where $k_1 \neq k_2 \neq k_3$. So at most 9 vertices can be colored with three color classes. The rest three vertices other than the vertices of M_f, M_g and M_j can not form a D_1^3. Hence at least two of them are at distance 2 or 3 from each other. Two vertices x and y with $d(x,y) = 2$ can not be colored with the same color class. For the case under consideration, $\mathbf{x} \nsim \mathbf{y}$ for all $x,\ y \in T_6$ with $d(x,y) = 3$. So these two vertices must have colors from two different color classes. Let C_{k_4} and C_{k_5} be the color classes of those two vertices, where $k_4 \neq k_5$. As any vertex other than M_f is at distance 2 from at least one of the three vertices of M_f, we get $k_4 \neq k_1$ and $k_5 \neq k_1$. The same argument holds for M_g and M_j as well and hence $k_4 \neq k_2$, $k_4 \neq k_3$, $k_5 \neq k_2$ and $k_5 \neq k_3$. Therefore at least 5 different color classes are required to color the vertices of G_3. Hence $\lambda_{h,1}(T_6) \geq 4$. $\square$

It is now evident that $\lambda_{h,1}(T_6) \geq 4$ if $\mathbf{x} \nsim \mathbf{y}$ for all $x,\ y \in T_6$ with $d(x,y) = 3$. So we will investigate whether $\lambda_{h,1}(T_6)$ can be kept below 4 by making $\mathbf{x} \sim \mathbf{y}$ for at least a pair of vertices $x,\ y \in T_6$ with $d(x,y) = 3$.

4.3.2 Determining $\lambda_{h,1}(T_6)$ when $x \sim y$ for at least a pair of vertices $x,\ y \in T_6$ with $d(x,y) = 3$.

Lemma 4.3.2 *For two vertices x and y in T_6 where $d(x,y) = 3$, if $x \sim y$ then at least C_3 must be used to color the vertices of T_6.*

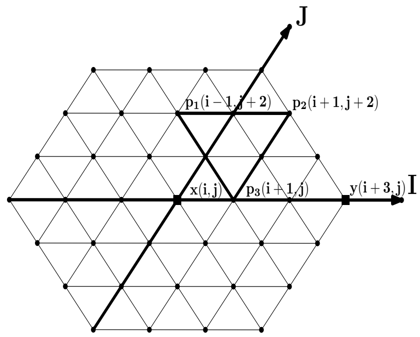


Figure 4.4: A D_2^3 which is formed by three vertices located at a specific relative position in T_6.

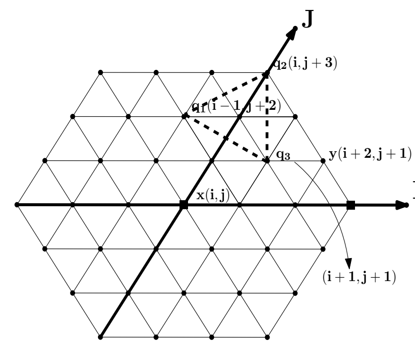


Figure 4.5: A D_2^3 which is formed by three vertices located at another specific relative position in T_6.

Proof: Let $x(i, j)$ be any vertex in T_6 and $f(x) \in C_k$. Now we consider the following two cases.

When y is a corner vertex of x: Let us consider the corner vertex $y(i+3, j)$ with axes (I, J) as shown in Figure. 4.4 and $f(y) \in C_k$. The three vertices $p_1(i-1, j+2)$, $p_2(i+1, j+2)$ and $p_3(i+1, j)$ form a D_2^3 as shown in Figure. 4.4. It can be verified that $d(p_1, x) = 2$, $d(p_2, y) = d(p_3, y) = 2$. Therefore three distinct color classes other than C_k must be required to color the mentioned D_2^3. Hence 4 distinct color classes must be required to color x, y and the D_2^3 implying at least C_3 must be used here. For other corner vertices of x, co-ordinates of corresponding D_2^3 can be found and same result can be obtained there also.

When y is a non-corner vertex of x: Let us consider the non-corner vertex $y(i+2, j+1)$ with axes (I, J) as shown in Figure. 4.5 and $f(y) \in C_k$. The three vertices $q_1(i-1, j+2)$, $q_2(i, j+3)$ and $q_3(i+1, j+1)$ form a D_2^3 as shown in Figure. 4.5. It can be verified that $d(q_1, x) = d(q_3, x) = 2$, $d(q_2, y) = 2$. Therefore three distinct color classes other than C_k must be required to color the mentioned D_2^3. Hence 4 distinct color classes must be required to color x, y and the D_2^3 implying at least C_3 must be used here. For other non-corner vertices of x, co-ordinates of corresponding D_2^3 can be found and same result can be obtained there also. Hence the proof. □

From Lemma 4.3.2, C_3 must be used in a vertex if $x \sim y$ for a pair of vertices x and y in T_6 with $d(x, y) = 3$. Therefore, let us consider a vertex $u \in T_6$ where $f(u) \in C_3$.

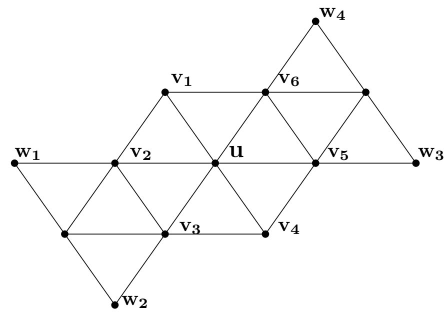


Figure 4.6: A sub graph G_6 of T_6.

Lemma 4.3.3 *Let us consider a vertex $u \in T_6$ with $f(u) \in C_3$. If $v \nsim u$, $\forall v \in T_6$ where $d(u,v) = 1$, then $\lambda_{h,1}(T_6) \geq 4$.*

Proof: Consider the sub graph G_6 of T_6 as shown in Figure. 4.6. Assume $f(u) \in C_3$ and $f(v_i) \notin C_3$ for all vertices v_i, $i \in \{1, 2, \ldots, 6\}$ with $d(u, v_i) = 1$. Assume $f(v_1) \in C_{k_1}$, $f(v_3) \in C_{k_2}$ and $f(v_5) \in C_{k_3}$. As v_1, v_3, v_5 form a D_2^3, k_1, k_2 and k_3 are all distinct and none of them are 3. Note that $f(w_1) \notin C_3$ as $d(w_1, u) = 2$ and $f(u) \in C_3$. Consider $f(w_1) \in C_k$. Note that $k \notin \{k_1, k_2\}$ as w_1, v_1, v_3 form a D_2^3 and $f(v_1) \in C_{k_1}$, $f(v_3) \in C_{k_2}$. Therefore $f(w_1) \in C_{k_3}$. So either $f(v_2) \in C_{k_1}$ or $f(v_2) \in C_{k_2}$, otherwise, $\lambda_{h,1}(G_6) \geq 4$. When $f(v_2) \in C_{k_1}$, we get $f(v_6) \in C_{k_3}$ and $f(v_4) \in C_{k_2}$. Note that $d(w_2, u) = d(w_2, v_2) = d(w_2, v_4) = d(w_2, w_1) = 2$. Therefore $f(w_2) \in C_k$, where $k \notin \{3, k_1, k_2, k_3\}$. Hence a fifth color class is required for w_2. Hence $\lambda_{h,1}(G_6) \geq 4$ in this case. When $f(v_2) \in C_{k_2}$, arguing similarly it can be shown that a fifth color class is required for w_4 resulting $\lambda_{h,1}(G_6) \geq 4$. Hence $\lambda_{h,1}(T_6) \geq 4$. □

From Lemma 4.3.3 we can conclude that there must exists a monochromatic edge with color class C_3 to keep $\lambda_{h,1}(T_6) < 4$. So we now consider the case when for two vertices $u, v \in T_6$ with $d(u,v) = 1$, $u \sim v$ and $f(u) \in C_3$.

Lemma 4.3.4 *Let us consider four vertices u,v, w_1 and w_2 in T_6 where $d(u,v) = 1$, $u \sim v$, $f(u) \in C_3$, $f(v) \in C_3$, $d(w_1, u) = d(w_1, v) = 1$ and $d(w_2, u) = d(w_2, v) = 1$. If neither $w_1 \nsim u$, $w_1 \nsim v$ nor $w_2 \nsim u$, $w_2 \nsim v$ then $\lambda_{h,1}(T_6) \geq 4$.*

Proof: Consider the sub graph G_7 of T_6 as shown in Figure. 4.7. Assume $f(u) \in C_3$ and $f(v) \in C_3$. As $d(r_3, u) = d(r_3, v) = 1$ and $d(r_4, u) = d(r_4, v) = 1$ we assume $f(r_3) \notin C_3$ and $f(r_4) \notin C_3$. Note that $f(r_1) \notin C_3$ and $f(r_2) \notin C_3$ as $d(r_1, v) = 2$ and

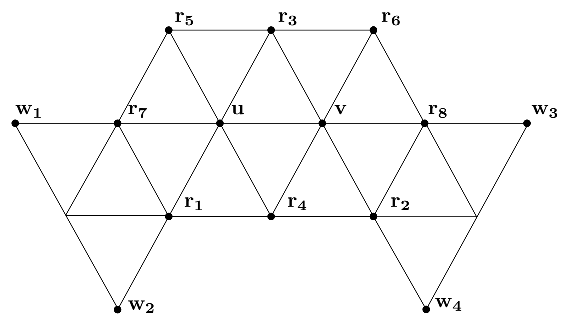


Figure 4.7: A sub graph G_7 of T_6.

$d(r_2, u) = 2$ respectively. Assume $f(r_1) \in C_{k_1}$, $f(r_2) \in C_{k_2}$ and $f(r_3) \in C_{k_3}$. As r_1, r_2 and r_3 form a D_2^3, k_1, k_2 and k_3 are all distinct and none of them are 3. Note that $f(r_4) \notin C_3$ by our assumption. Again, $f(r_4) \notin C_{k_3}$ as $d(r_4, r_3) = 2$ and $f(r_3) \in C_{k_3}$. So, either $f(r_4) \in C_{k_1}$ or $f(r_4) \in C_{k_2}$, otherwise, $\lambda_{h,1}(G_7) \geq 4$. First consider $f(r_4) \in C_{k_1}$. In that case $f(r_5) \in C_{k_2}$ and $f(r_6) \in C_{k_3}$ as r_4, r_5 and r_6 form a D_2^3 and $f(r_2) \in C_{k_2}$. Now $f(w_1) \notin C_3$ as $d(w_1, u) = 2$ and $f(u) \in C_3$. Again, $f(w_1) \notin C_{k_1}$ and $f(w_1) \notin C_{k_2}$, as w_1, r_1, r_5 form a D_2^3 with $f(r_1) \in C_{k_1}$ and $f(r_5) \in C_{k_2}$. Therefore $f(w_1) \in C_{k_3}$. Observe that $f(r_7) \in C_{k_2}$ because $f(r_7) \notin C_3$ as $d(r_7, v) = 2$, $f(v) \in C_3$; $f(r_7) \notin C_{k_1}$ as $d(r_7, r_4) = 2$, $f(r_4) \in C_{k_1}$; $f(r_7) \notin C_{k_3}$ as $d(r_7, r_3) = 2$, $f(r_3) \in C_{k_3}$. Note that $d(w_2, u) = d(w_2, r_4) = d(w_2, r_7) = d(w_2, w_1) = 2$. Therefore $f(w_2) \in C_k$, where $k \notin \{3, k_1, k_2, k_3\}$. Hence a fifth color class is required for w_2. Hence $\lambda_{h,1}(G_7) \geq 4$ in this case. When $f(r_4) \in C_{k_2}$, arguing similarly it can be shown that a fifth color class is required for w_4 resulting $\lambda_{h,1}(G_7) \geq 4$. Hence $\lambda_{h,1}(T_6) \geq 4$. $\square$

From Lemma 4.3.4 we can conclude that there must exists a monochromatic triangle with color class C_3 to keep $\lambda_{h,1}(T_6) < 4$. Now we have the following theorem.

Theorem 4.3.2 *$\lambda_{h,1}(T_6) \geq 3 + 2h$ when $h < 1/2$ and $\lambda_{h,1}(T_6) \geq 4$ when $h \geq 1/2$.*

Proof: If $x \nsim y$ for all pair of vertices x, $y \in T_6$ with $d(x, y) = 3$, then from theorem 4.3.1, we get $\lambda_{h,1}(T_6) \geq 4$. If $x \sim y$ for at least one pair of vertices x, $y \in T_6$ with $d(x, y) = 3$, from lemma 4.3.2 it follows that C_3 must be used in T_6 to keep $\lambda_{h,1}(T_6; f) < 4$. Lemmas 4.3.3 and 4.3.4 state that there must exit a monochromatic edge and a monochromatic tringle with color class C_3 to keep $\lambda_{h,1}(T_6) < 4$. Hence $\lambda_{h,1}(T_6) \geq 3 + 2h$ when $h < 1/2$ and $\lambda_{h,1}(T_6) \geq 4$ when $h \geq 1/2$. $\square$

4.4 Conclusions

In this work, we derived $\lambda_{h,1}(T_6)$ theoretically by exploring the underlined graph structures without any computer simulation. We introduced the notion of color class which essentially eliminates the need for enumerations and leads us to arrive at few cases. Our bound exactly coincides with that of the known bound obtained through computer simulations when $0 \leq h \leq \frac{1}{3}$. For $h > \frac{1}{3}$, existing bounds obtained through simulation are finer than ours. Our future research scope includes identifying appropriate sub graphs and accordingly obtaining tighter bounds by dividing the intervals into finer sub-intervals, when $h > \frac{1}{3}$.

Chapter 5

Improved bounds on $L(2,1)$-edge labeling for T_6

$L(2,1)$-edge labeling and circular $L(2,1)$-edge labeling of T_6 have been studied in [15]. They gave the bounds on $\lambda'_{2,1}(T_6)$ of $L(2,1)$-edge labeling and conjectured that $\lambda'_{2,1}(T_6) = 16$. In this chapter we prove this conjecture. Regarding $\sigma'_{2,1}(T_6)$ of circular $L(2,1)$-edge labeling of T_6, authors in [15] proved that $16 \leq \sigma'_{2,1}(T_6) \leq 18$ but for the upper bound, they did not give any labeling function for it. In this chapter, we also give a labeling function for circular $L(2,1)$-edge labeling of T_6, where the maximum color used in the labeling function is 17 which shows that $\sigma'_{2,1}(T_6) \leq 18$.

The rest of the chapter is organized as follows. In section 5.1, we will present previous results and our improved results in tabular form. In section 5.2, we will present the proofs of the results we obtained. In section 5.3 concluding remarks will be stated.

5.1 Previous results and our results

Lin and Wu [15] gave the following conjecture and the bound $16 \leq \sigma'_{2,1}(T_6) \leq 18$.

Conjecture 5.1.1 *$\lambda'_{2,1}(T_6) = 16$.*

First we prove that $\lambda'_{2,1}(T_6) \geq 16$ and as $\lambda'_{2,1}(T_6) \leq 16$ [14], it will immediately follow that $\lambda'_{2,1}(T_6) = 16$. Then we derive a labeling function for the circular $L(2,1)$-edge labeling of T_6 such that $\sigma'_{2,1}(T_6) \leq 18$ as no labeling function was given by Lin and Wu [15].

Table 5.1: Previous result and our improved result for $\lambda'_{2,1}(T_6)$.

	$\lambda'_{2,1}(G)$	
Grid	Known	Ours
T_6	15-16 [15], (16-16) [69]*	16-16

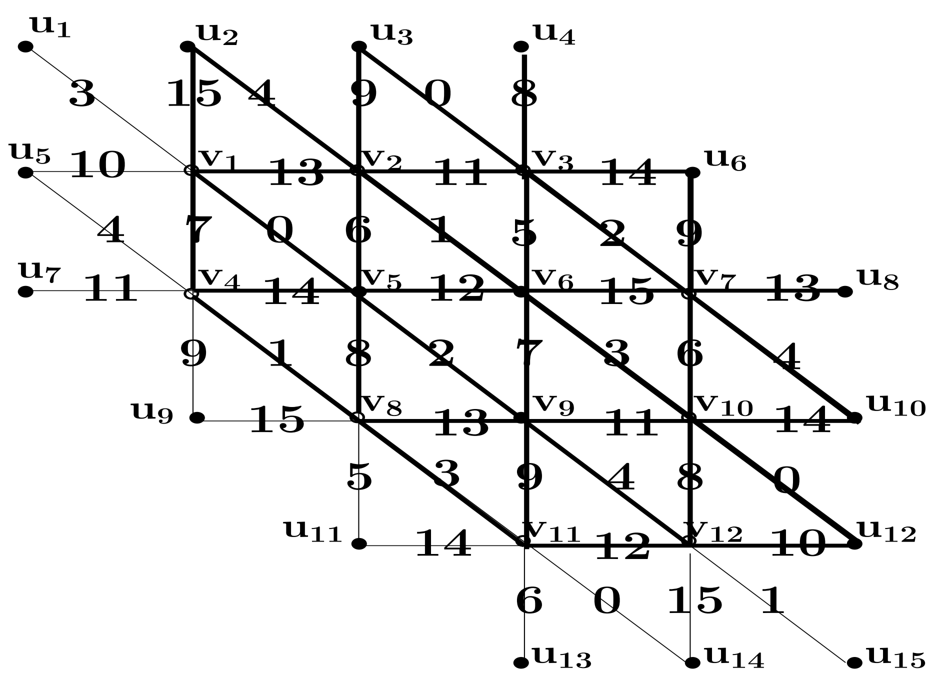


Figure 5.1: $L(2,1)$-edge coloring of the sub graph G of T_6 with colors $\{0, 1, \ldots, 15\}$.

In Table 5.1, *the proof of lower bound for $\lambda'_{2,1}(T_6)$ stated in [69] is found to be incorrect. They proved $\lambda'_{2,1}(H'') \geq 16$ where H'' is the sub graph of T_6 as shown by the bold edges in Figure 5.1. Now consider the sub graph G of T_6 and its labeling as shown in Figure 5.1. From this labeling it can be concluded that $\lambda'_{2,1}(G) \leq 15$. Note that H'' is a sub graph of G.

5.2 Our results

5.2.1 $L(2,1)$-edge labeling

For a vertex v, let $N(v)$ denotes the set of neighbors of v and for a set of vertices S, let $N(S) = \underset{v \in S}{\cup} N(v)$. Consider the sub graph $G(V, E)$ of T_6 centering the triangle formed by $S = \{v_5, v_6, v_9\}$ as shown in Figure 5.2, where $V = S \cup N(S) \cup N(N(S))$ and E is the set of all edges which are incident to u where $u \in S \cup N(S)$. Now, we

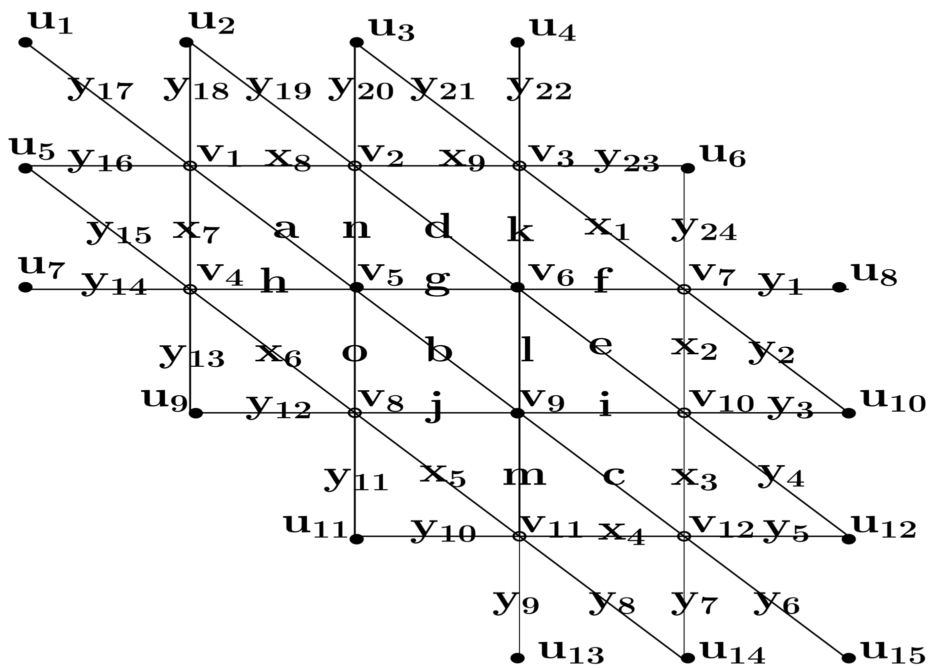


Figure 5.2: A sub graph G of T_6

define the set of edges in G into five subsets of edges as following:

$S_1 = \{g, b, l\}$
$S_2 = \{d, e, n, o, i, j\}$
$S_3 = \{k, f, a, h, m, c\}$
$S_4 = \{x_1, x_2, x_3, \ldots, x_9\}$
$S_5 = \{y_1, y_2, y_3, \ldots, y_{24}\}$

Our proof approach is as follows. In first few Observations and Lemma (Observation 6, Observations 7, Observations 8, Observation 9 and Lemma 5.2.1) we investigate if a color c' is used in S_1 or in S_2 or in S_3, then how many times at maximum it can be reused in G and repetition pattern of the colors such that edges of G can be colored with $15(0, 1, \ldots, 14)$ colors. In Observation 10, we determine the maximum number of times a color c' can be used in G if it is not used in $S1 \cup S2 \cup S3$. In subsequent Observation and Lemmas (Observation 11, Lemma 5.2.2, Lemma 5.2.3) we discuss the scenario when a color $c' \notin \{5, 10\}$ is unused in $S_1 \cup S_2 \cup S_3$ and two consecutive colors c' and $c' + 1$ are used in different types of edges. Based on the discussion and results of the mentioned Observations and Lemmas, we determine the lower bound of $\lambda'_{2,1}(T_6)$ when a color $c_u \notin \{5, 10\}$ is unused in $S_1 \cup S_2 \cup S_3$ in Theorem 5.2.1, Theorem 5.2.2 and Theorem 5.2.3. In Theorem 5.2.4, we determine the lower bound of $\lambda'_{2,1}(T_6)$ when a color $c_u \in \{5, 10\}$ is unused in $S_1 \cup S_2 \cup S_3$. In Theorem 5.2.5, we determine the lower bound of $\lambda'_{2,1}(T_6)$ when all colors in $\{0, 1, \ldots, 14\}$ are used in $S_1 \cup S_2 \cup S_3$. In all the cases, derived lower bounds of $\lambda'_{2,1}(T_6)$ are identical.

Observation 6 *Let c' be any color used at an edge in S_1, then c' can be used in at most once more in G.*

Proof: Without loss of generality assume, c' is used at the edge g. Now, it is clear that c' can only be used at some edges incident to v_{11} and v_{12}. Since, edges incident to v_{11} and v_{12} are not mutually three distance apart, c' can only be used at most one edge. $\square$

Observation 7 *Let c' be any color used at an edge in S_2, then c' can be used in at most two more times in G.*

Proof: As all the edges in S_2 are in symmetric position, without loss of generality we can assume that, c' is used at the edge d. One can verify that, some edges incident to v_4, v_8, v_{11}, v_{12} only are distant three from d. Note that v_4 and v_8 are adjacent to each other. Similarly v_{11} and v_{12} are also mutually adjacent. Hence c' can at most be used twice, one in an edge incident to v_4, v_8 and the other in an edge incident to v_{11}, v_{12}. But if c' is used at x_5 then c' can not be used once more. $\square$

Observation 8 *Let c' be any color used at an edge in S_3, then c' can be used in at most three more edges in G.*

Proof: Observe that, here also all the edges are symmetric, so, without loss of generality we can assume that c' is used at the edge k. Some edges adjacent to the vertices $v_1, v_4, v_8, v_{11}, v_{12}$ only are three distance apart from k. It is clear that, c' can be used at the edges adjacent to alternate vertices in the sequence $v_1, v_4, v_8, v_{11}, v_{12}$. Moreover, it can be observed that each edge in S_4 where c' can be used is adjacent to two vertices in $v_1, v_4, v_8, v_{11}, v_{12}$. So, if we want to use c' three more times in G then c' must be used at edges adjacent to v_1, v_8, v_{12}, and the color must be used at edges in S_5 only. $\square$

From Observations 6, 7 and 8 it is clear that colors used to color the edges in S_1, S_2 and S_3 can be used at most two, three and four times in G, respectively. Note that, G has 48 edges in total. We need all distinct color to color the edges in $S_1 \cup S_2 \cup S_3$ as they are mutually at most two distance apart. So, we need 3, 6 and 6 distinct colors to color the edges in S_1, S_2 and S_3, respectively. If we can repeat all these color with their maximum potential then it is possible to color the graph G using 15 colors. Here we state Observation 9 and give the unique repetition pattern of the colors, to color G using 15 colors in Lemma 5.2.1. Let H be the sub graph of G induced by the edges in $S_1 \cup S_2 \cup S_3$. We say two edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$ as a pair of *opposite edges* iff $d(e_1, e_2) = 3$, $d(u_1, u_2) = 2$, $d(u_1, v_2) = 2$, $d(v_1, u_2) = 2$ and $d(v_1, v_2) = 2$.

Observation 9 *If a color c' used at an edge e_1 in T_6 and is not used at the opposite edge of e_1 then there exists a sub graph H' isomorphic to H where c' can not be used.*

Proof: Without loss of generality let us consider the pair of opposite edges n and x_2 (Figure 5.2). Let $\mathbf{f}'(n) = c'$ and $\mathbf{f}'(x_2) \neq c'$. Consider the two triangles $\{v_3, v_6, v_7\}$ and $\{v_6, v_9, v_{10}\}$. Clearly, c' can not be used in any edge incident to v_3, v_6 and v_9. Therefore c' can be assigned to either an edge incident to v_7 or an edge incident to v_{10} but not both. So, c' can not be used either in the sub-graph H' isomorphic to H with $S'_1 = \{v_3, v_6, v_7\}$ or in the sub-graph H'' isomorphic to H with $S''_1 = \{v_6, v_9, v_{10}\}$. Hence the proof. □

Lemma 5.2.1 *If G is colored with 15 colors only then there is the following unique repetition of the colors used in sub-graph H: (a) each color used at the edges in S_3 must be repeated thrice more in the edges of S_5, (b) each color used at the edges in S_2 must be repeated twice more, once in its opposite edge in S_4 and the other in an edge in S_5, (c) each color used at the edges in S_1 must be repeated once more in its opposite edge in S_4.*

Proof: Observe that, the first condition for G to be colored with 15 colors is the colors used in S_1, S_2 and S_3 have to be used twice, thrice and four times in G, respectively. From the proof of Observation 8 it directly follows that if we want to reuse three times a color c_3 that has been used in an edge of S_3 then c_3 has to be reused at the edges of S_5 only. Now assume that, c_2 be a color used in an edge of S_2 then c_2 can not be reused at two edges in S_4 as there does not exists two mutually three distant edges in S_4 such that c_2 can be used. From Observation 9 it follows that c_2 should be reused at the corresponding opposite edge in S_4 and another edge in S_5. Observe that, only three edges in S_4 remain uncolored now. Again from Observation 9 we can say that if c_1 be a color used in an edge of S_1 then c_1 has to be reused at the corresponding opposite edge in S_4 only. $\square$

We aim to prove $\lambda'_{2,1}(T_6) = 16$. We already know that, $\lambda'_{2,1}(H) = 14$ and $\lambda'_{2,1}(T_6) \geq 15$ [14]. So, without loss of generality, we can assume that, one color is *unused* at H. In Observations 6, 7, 8 we showed the re-usability of the colors used at S_1, S_2 and S_3, respectively. Now we focus on the color which is not used in H.

Observation 10 *Let c' be any color not used in H, then c' can be used at most four edges in G.*

Proof: Clearly, c' is used at edges in $S = S_4 \cup S_5$. All edges in S are incident to one or two vertices in $Q = \{v_1, v_2, v_3, v_7, v_{10}, v_{12}, v_{11}, v_8, v_4\}$. So, c' can be used at edges adjacent to alternative vertices in the sequence Q. There are nine vertices in Q. If we pick every alternate vertices in the sequence starting from v_1 then we end up with a set Q_1 of five vertices $\{v_1, v_3, v_{10}, v_{11}, v_4\}$. Since, v_1 and v_4 are adjacent in G, we can give the color c' to edges incident to either v_1 or v_4. So, we have only four vertices whose adjacent edges can be colored with c', and note that that no two edge adjacent to same vertices can be colored with same color. Hence there are at most four edges which can be colored with the unused color. Similarly, if we pick alternate vertices in the sequence starting from v_2, then we end up with picking a set Q_2 of four vertices $\{v_2, v_7, v_{12}, v_8\}$. Again, we similarly argue that, at most in four edges we can use the unused color in G. $\square$

From Figure 5.2 it is clear that, the sub-graph H consists of five vertical edges, five horizontal edges and five slanting edges. Let us denote these three types of edges as T_v, T_h and T_s, respectively. Now, we prove some properties of colors used in these three types of edges.

Observation 11 *Let $\mathbf{f}'(p) = c'$ and $\mathbf{f}'(q) = c' + 1$ where $p, q \in E(H)$ and they are different types of edges. Then there exists a H' isomorphic to H in T_6 where either c' or $c' + 1$ can not be used.*

Proof: Let us consider $p = f, q = a$ (Figure 5.2). Note that $\mathbf{f}'(f) = c'$ can not be used at any edge incident to v_2 as $d(v_2, v_6) = 1$. Again, c' can not be used at any edge incident to v_1 or v_5 as $\mathbf{f}'(a) = c' + 1$. Hence c' can not be used in the sub graph H' isomorphic to H centering the triangle $S_1' = \{v_1, v_2, v_5\}$. In general, for any (p, q) pair in H, such a triangle can be found by considering the two triangles with common edge p and the two triangles with common edge q. $\square$

Lemma 5.2.2 *In the sub graph H, let the unused color $c' \notin \{5, 10\}$. Then, there must exist two disjoint pair of different typed edges (p, q) and (r, s) such that consecutive colors have been used in each pair of edges and $p \in T_h, q \in T_s$; $r \in T_s, s \in T_v$.*

Proof: Since, distance between any two edges in H is at most two, no color can be repeated. We need three sets of five different colors to color each type of edges in H. That means we need to divide the set of colors into three sets of equal size (i.e., five colors in each set) in such a way that we can maximize the number of sets that contain consecutive colors. We can say that the unused color divides the color set into two parts. By pigeon hole principle, we can show that either one part contains more than 10 consecutive colors or both of them contains more than five consecutive colors as the unused color c' is neither 5 nor 10. In the former case, all three types of edges get color from the larger part of the colors as it contains more than 10 colors. In the later case, at least one type of edges get color from both the parts. So, in both the cases we get at least two such pairs. $\square$

Lemma 5.2.3 *For every (p, q) pair of edges in H where p and q are different types of edges and consecutive colors are assigned to p and q, at least 2 edges of $E(G) \setminus E(H)$ cannot be colored with the colors used in H.*

Proof: We prove this Lemma using case analysis and the cases are based on where the edges p and q are present. Without loss of generality, we assume $p \in T_h$ and $q \in T_s$ are colored with z and $z + 1$ respectively. We can have the following cases:

- $p, q \in S_3$: Without loss of generality, let $p = f$ and $q = c$. From Lemma 5.2.1 it follows that if z is to be reused for 3 more times then it has to be reused at edges of S_5 incident to v_1, v_8 and v_{12}, which is not possible since c is adjacent to v_{12}. Similarly, if $z + 1$ is to be used for 3 more times then it has to be reused at edges of S_5 incident to v_7, v_2 and v_4, which is also not possible since f is adjacent to v_7. So, there are two edges in $G \setminus H$ which remain uncolored with the colors used in H.

- $p \in S_3, q \in S_2$: Without loss of generality, let $p = h$ and $q = e$. From Lemma 5.2.1, if z is to be used for 3 more times then it has to be reused at edges of S_5 incident to v_3, v_{10} and v_{11}. But it is not possible since e is adjacent to v_{10}. Similarly, if $z+1$ is to be used for 2 more times then it has to be reused at $x_5 = (v_8, v_{11})$ of S_4 and an edge of S_5 incident to v_1. It is not possible to use $z+1$ at x_5 as z is reused at an edge of S_5 adjacent to v_{11}. So, there are two edges in $G \setminus H$ which can not be colored with the colors used in H.

- $p, q \in S_2$: Without loss of generality, let $p = j$ and $q = d$. From Lemma 5.2.1, z is to be reused at $x_8 = (v_1, v_2)$ which is not possible here as edge d is adjacent to (v_1, v_2). Similarly, $z+1$ is to be reused at $x_6 = (v_4, v_8)$ which is also not possible as edge j is adjacent to (v_4, v_8). Therefore, two edges at $G \setminus H$ can not be colored with the colors used in H.

- $p \in S_1, q \in S_3$: Without loss of generality, let $p = g$ and $q = c$. From Lemma 5.2.1 it follows that z has to be reused at x_4. But here it is not possible as $z+1$ used at c.

  If x_4 is not left uncolored, then a color of H in $\{h, a, n, d, k, f\}$ can only be used at that edge. If $\mathbf{f}'(h)$ is used at x_4, then $\mathbf{f}'(h)$ can only be used at an edge of S_5 incident to v_3 but not at the edges of S_5 incident to v_{10} and v_{11}. So, in that case, two edges will remain uncolored. If $\mathbf{f}'(n)$ is used at x_4, then $\mathbf{f}'(n)$ can neither be used at x_2 nor be used at an edge of S_5 incident to v_{11}. So, in that case too, two edges will remain uncolored. All other possibilities are symmetric to one of these two cases.

  We now consider the case when the unused color, say u of H is used at x_4. Since u can not be used at v_8 and v_{10}, colors of $\{d, k, f, e\}$ must be reused at the edges $\{x_6, y_{12}, y_{11}, x_5\}$ and the colors of $\{n, a, h, o\}$ must be reused at the edges $\{x_2, y_3, y_4, x_3\}$. If $\mathbf{f}'(i)$ and $\mathbf{f}'(j)$ are reused at x_9 and x_8 respectively, then the edges where colors $\mathbf{f}'(i) \pm 1$ and $\mathbf{f}'(j) \pm 1$ can be used are $\{f, g\}$ and $\{g, h\}$ respectively. Therefore, if both $\mathbf{f}'(i)$ and $\mathbf{f}'(j)$ are reused at x_9 and x_8 respectively, then $\mathbf{f}'(g) \pm 1 = z \pm 1$ can only be used at $\{i, j\}$. Since $z+1$ is used at c, either $\mathbf{f}'(i)$ can not be reused at x_9 or $\mathbf{f}'(j)$ can not be reused at x_8. Hence the unused color u of H has to be used at x_9 or x_8. Hence two edges at $G \setminus H$ can not be colored with the colors used in H.

$\square$

Now we will look at the difference of colors in the edges incident to same vertex. Initially we investigate the case when the difference is at least three for every pair. Then we consider the case when there exists at least a pair of edges with difference exactly two. We classify the six edges incident to the same vertex as follows. We say two such edges are at 60° if one is the immediate next edge of the other in clockwise or anticlockwise direction. They are said to be at 120° and 180° if exactly one and two edges respectively is/are there in between them. Now we subdivide the second case into three more cases depending on the angle between them.

Lemma 5.2.4 *If $|\mathbf{f}'(e_1) - \mathbf{f}'(e_2)| \geq 3$ for every pair of adjacent edges $e_1, e_2 \in E(T_6)$, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof: Let us consider the edge $g = (v_5, v_6)$ and without loss of generality assume $\mathbf{f}'(g) = 0$. To keep $\lambda'_{2,1}(T_6)$ below 16, the colors that can be used at the remaining five incident edges of v_5 are 3, 6, 9, 12 and 15. Now the least color that can be used at any of the five edges incident to v_6 is 4. Therefore, the colors that can be used to the remaining four edges incident to v_6 are 7, 10, 13 and 16 respectively. Hence $\lambda'_{2,1}(T_6) \geq 16$. $\square$

Therefore there exists at least two adjacent edges in T_6 having color c_1 and c_2 with $|c_1 - c_2| = 2$ otherwise $\lambda'_{2,1}(T_6) \geq 16$.

Theorem 5.2.1 *If two colors c' and $c' + 2$ have been assigned in any two adjacent edges at an angle 60° in T_6, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof: Without loss of generality, assume $\mathbf{f}'(b) = c'$ and $\mathbf{f}'(g) = c' + 2$. Observe that $c' + 1$ must remain unused in H as $\forall e_1 \in E(H) \setminus \{b, g\}$ either $d(e_1, b) = 1$ or $d(e_1, g) = 1$. From Lemma 5.2.2 and Lemma 5.2.3, there are 4 edges in $G \setminus H$ which can not be colored with colors used in H. To make $\lambda'_{2,1}(G)$ below 16, $c' + 1$ is to be used in those 4 edges. Without loss of generality, assume $c' + 1$ has been used at the edges incident to v_1, v_3, v_8 and v_{12} respectively. Note that $\mathbf{f}'(g) = c' + 2$ can not be used at x_4 as $c' + 1$ is used at an edge incident to v_{12}.

So, $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n), \mathbf{f}'(d), \mathbf{f}'(f), \mathbf{f}'(k)\}$. Consider the case when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n)\}$ as the case when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(d), \mathbf{f}'(f), \mathbf{f}'(k)\}$ can be proved similarly. Assume $\mathbf{f}'(x_4) = \mathbf{f}'(a)$. There are four edges of $S_4 \cup S_5$ incident to v_{10} where $\mathbf{f}'(a)$, $\mathbf{f}'(h)$, $\mathbf{f}'(n)$ and $\mathbf{f}'(o)$ can only be used. But $\mathbf{f}'(a)$ can not be used there as $\mathbf{f}'(x_4) = \mathbf{f}'(a)$. To make $\lambda'_{2,1}(G)$ below 16, $\mathbf{f}'(x_3) = c' + 1$ and $\mathbf{f}'(n)$, $\mathbf{f}'(h)$ and $\mathbf{f}'(o)$ are assigned to the other three edges. Similarly, there are four edges of $S_4 \cup S_5$ incident to v_2 where $\mathbf{f}'(i)$, $\mathbf{f}'(c)$, $\mathbf{f}'(m)$ and $\mathbf{f}'(j)$ can only be used. Now observe that $\mathbf{f}'(b) = c'$ can not be used at x_1 as $c' + 1$ is used at an edge incident to v_3. Again, $\{\mathbf{f}'(n), \mathbf{f}'(h), \mathbf{f}'(o)\}$ and $\{\mathbf{f}'(i), \mathbf{f}'(c), \mathbf{f}'(m), \mathbf{f}'(j)\}$ can not be used at x_1 as they are used at edges incident to v_{10} and v_2 respectively. Hence $\mathbf{f}'(x_1) = \mathbf{f}'(a)$. Proceeding similarly we can show that $\mathbf{f}'(x_9) = \mathbf{f}'(i)$, $\mathbf{f}'(x_8) = \mathbf{f}'(j)$, $\mathbf{f}'(x_7) = \mathbf{f}'(l)$ and $\mathbf{f}'(x_5) = \mathbf{f}'(e)$. Now observe that $\mathbf{f}'(a)$ and $\mathbf{f}'(x)$ are used at two adjacent vertices for all $x \in E(H) \setminus \{d\}$. Therefore one of $\mathbf{f}'(a) \pm 1$ can not be used in H. Moreover, none of $\mathbf{f}'(a) \pm 1$ can be the unused color $c' + 1$ as c' and $c' + 2$ are used at b and g respectively. Hence $\lambda'_{2,1}(G) \geq 16$. $\square$

Theorem 5.2.2 *If two colors c' and $c' + 2$ have been assigned in any two adjacent edges at an angle 120° in T_6, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof: Without loss of generality, assume $\mathbf{f}'(g) = c'$ and $\mathbf{f}'(e) = c' + 2$ (Figure 5.2). There may be two cases, when $c' + 1$ is used in H and when $c' + 1$ is not used in H. First consider the second case. From Lemma 5.2.2, there are 4 edges in $G \setminus H$ which can not be colored with the colors used in H. To make $\lambda'_{2,1}(G)$ below 16, $c' + 1$ is to be used in those 4 edges. Without loss of generality, assume $c' + 1$ is used in edges in $G \setminus H$ adjacent to vertices v_1, v_3, v_8 and v_{12} respectively. Note that $\mathbf{f}'(x_4) \neq \mathbf{f}'(g) = c'$ as $c' + 1$ is used in an edge adjacent to v_{12}. Therefore $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n), \mathbf{f}'(d), \mathbf{f}'(f), \mathbf{f}'(k)\}$. Let $\mathbf{f}'(x_4) = \mathbf{f}'(d)$. The color of four edges in $S_4 \cup S_5$ incident to v_8 are $c' + 1$ and any three of $\mathbf{f}'(d), \mathbf{f}'(e), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$. But $\mathbf{f}'(e)$ can not be used there as $\mathbf{f}'(e) = c' + 2$; $\mathbf{f}'(d)$ can not be used there as $\mathbf{f}'(x_4) = \mathbf{f}'(d)$. Hence a new color must be introduced here resulting $\lambda'_{2,1}(T_6) \geq \lambda'_{2,1}(G) \geq 16$. Similar result holds when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(f), \mathbf{f}'(k)\}$. Now let us consider the case when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n)\}$. Let us consider $\mathbf{f}'(x_4) = \mathbf{f}'(a)$. Here the colors of four edges in $S_4 \cup S_5$ incident to v_{12} are $c' + 1, \mathbf{f}'(h), \mathbf{f}'(n)$ and $\mathbf{f}'(o)$. Therefore $\mathbf{f}'(d), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ must be used at three edges incident to v_{12}. As, $\mathbf{f}'(e) = c' + 2$ can not be used at any edge incident to v_8 due to usage of $c' + 1$ at v_8, any one of $\mathbf{f}'(d), \mathbf{f}'(f)$

and $\mathbf{f}'(k)$ must be used in x_5. But it is not possible as $\mathbf{f}'(d), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ are used in edges adjacent to v_{12}. Hence another color must be introduced here resulting $\lambda'_{2,1}(T_6) \geq \lambda'_{2,1}(G) \geq 16$. Similar argument holds when $\mathbf{f}'(x_4) \in \{\mathbf{f}'(h), \mathbf{f}'(n)\}$. Hence the proof for this case.

Now we consider the case when $\mathbf{f}'(g) = c'$, $\mathbf{f}'(e) = c' + 2$ and $c' + 1$ is used in H. Let us consider a color $c'' \in C = \{0, 1, \ldots, 15\} \setminus \{c', c' + 1, c' + 2\}$ is unused in H. Without loss of generality, say $c'' = c' + 5$. For any other color c'' in C we can prove the same result with similar arguments. From Lemma 5.2.2, $c' + 5$ must be used in 4 edges of $S_4 \cup S_5$ in $G \setminus H$. Here we assume $c' + 5$ is used in 4 edges adjacent to the vertices v_1, v_3, v_8 and v_{12} respectively. Since $f(g) = c'$, $c' \pm 1$ can only be used in $\{c, i, j, m\}$. Let us consider $\mathbf{f}'(j) = c' - 1$. The three edges h, f and i in T_h are to be colored. Here we assume exactly two distinct pair of different types of edges (p, q) and (r, s) are assigned consecutive colors such that $p \in T_h, q \in T_s$ and $r \in T_h, s \in T_v$. In that case $c' - 2$, $c' - 3$ and $c' - 4$ must be assigned to edges in T_h in H. Let us consider $\mathbf{f}'(j) = c' - 1$. Now if $\mathbf{f}'(m) = c' + 1$, then two edges j and m at 60° have colors $c' - 1$ and $c' + 1$ and from theorem 5.2.1, $\lambda'_{2,1}(T_6) \geq 16$. Again $\mathbf{f}'(i) \neq c' + 1$ as $\mathbf{f}'(e) = c' + 2$. Therefore $\mathbf{f}'(c) = c' + 1$. If $\mathbf{f}'(b) = c' + 3$ then neither $c' + 4$ nor $c' + 6$ can be used at the edge a as $c' + 5$ is used at an edge incident to v_1. Hence $\mathbf{f}'(a) = c' + 3$. Similarly we can show that $\mathbf{f}'(d) = c' + 4$ and $\mathbf{f}'(b) = c' + 6$. As $c' + 5$ is used at an edge in $S_4 \cup S_5$ incident to v_{12}, at least any three among $\mathbf{f}'(a), \mathbf{f}'(h), \mathbf{f}'(n)$ and $\mathbf{f}'(o)$ must be assigned to the edges of $S_4 \cup S_5$ incident to v_{10}. Note that $\mathbf{f}'(a) = c' + 3$ and $\mathbf{f}'(e) = c' + 2$. Therefore $\mathbf{f}'(a)$ can not be assigned to an edge incident to v_{10}. In that case $\mathbf{f}'(h)$ and $\mathbf{f}'(i)$ can not be consecutive. So we get $\mathbf{f}'(h) = c' - 2$, $\mathbf{f}'(f) = c' - 3$ and $\mathbf{f}'(i) = c' - 4$. With similar argument we can show that $\mathbf{f}'(o) = c' - 5, \mathbf{f}'(m) = c' - 6, \mathbf{f}'(k) = c' - 7, \mathbf{f}'(n) = c' - 8$ and $\mathbf{f}'(l) = c' - 9$. Now consider the edges of $S_4 \cup S_5$ incident to v_8. Only $\mathbf{f}'(d), \mathbf{f}'(e), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ can be used at edges of $S_4 \cup S_5$ incident to v_8. Note that $\mathbf{f}'(d) = c' + 4$ can not be used there because $c' + 5$ is used at an edge of S_5 incident to v_8. Therefore $\mathbf{f}'(k) = c' - 7, \mathbf{f}'(f) = c' - 3$ and $\mathbf{f}'(e) = c' + 2$ must be used at edges in $S_4 \cup S_5$ incident to v_8. Now $\mathbf{f}'(x_5) \neq \mathbf{f}'(k) = c' - 7$ as $\mathbf{f}'(m) = c' - 6$. If $\mathbf{f}'(x_5) = \mathbf{f}'(f) = c' - 3$ then two edges j and x_5 at 60° have colors $c' - 1$ and $c' - 3$ and from theorem 5.2.1, $\lambda'_{2,1}(T_6) \geq 16$. Hence $\mathbf{f}'(x_5) = \mathbf{f}'(e) = c' + 2$. Since either $\mathbf{f}'(y_{11}) = c' + 5$ or $\mathbf{f}'(y_{12}) = c' + 5$ we get either $\mathbf{f}'(x_6) = c' - 3$ or $\mathbf{f}'(x_6) = c' - 7$. In both cases two edges o and x_6 residing at an angle 60° have colors $(c' - 5, c' - 3)$

or $(c' - 5, c' - 7)$. Hence from theorem 5.2.1 $\lambda'_{2,1}(T_6) \geq 16$. When more than two distinct pair of different types of edges are assigned consecutive colors, arguing similarly, we can prove the same result. Hence the proof. $\square$

Theorem 5.2.3 *If two colors c' and $c' + 2$ have been assigned in any two adjacent edges at an angle $180°$ in T_6, then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof: Without loss of generality, assume $\mathbf{f}'(g) = c'$ and $\mathbf{f}'(h) = c' + 2$ (Figure 5.2). There may be two cases, when $c' + 1$ is used in H and when $c' + 1$ is not used in H. First assume $c' + 1$ is used in H. Let us consider a color $c'' \in C = \{0, 1, \ldots, 15\} \setminus \{c', c' + 1, c' + 2\}$ is unused in H and it is used at the edges of $S_4 \cup S_5$ in $G \setminus H$ adjacent to v_1, v_3, v_8 and v_{12}. Without loss of generality, say $c'' = c' + 7$. For any other color c'' in C we can prove the same result with similar arguments. Both the colors $c' \pm 1$ must be used at edges in H incident to v_9. To make $\lambda'_{2,1}(G)$ less than 16, the two colors must be used at two edges at $180°$ incident to v_9 otherwise from theorem 5.2.1 or theorem 5.2.2, $\lambda'_{2,1}(T_6) \geq 16$. So, we assume $\mathbf{f}'(j) = c' + 1$ and $\mathbf{f}'(i) = c' - 1$. Similarly, $\mathbf{f}'(f) = c' - 2$. The color $c' - 3$ can be used at an edge adjacent to v_5. Let us consider $\mathbf{f}'(n) = c' - 3$. Therefore, $\mathbf{f}'(m) = c' - 4, \mathbf{f}'(k) = c' - 5, \mathbf{f}'(o) = c' - 6$ and $\mathbf{f}'(l) = c' - 7$. Similarly, it can be shown that $\mathbf{f}'(d) = c' + 3, \mathbf{f}'(c) = c' + 4, \mathbf{f}'(a) = c' + 5, \mathbf{f}'(e) = c' + 6$ and $\mathbf{f}'(b) = c' + 8$. Now observe that $\mathbf{f}'(c), \mathbf{f}'(m), \mathbf{f}'(i) = c' - 1$ and $\mathbf{f}'(j) = c' + 1$ must be used at the edges at $S_4 \cup S_5$ incident to v_2 as the unused color c'' is not used here. This implies $\mathbf{f}'(j) = c' + 1$ and $\mathbf{f}'(i) = c' - 1$ must be used at two edges in $S_4 \cup S_5$ incident to v_2 which are at $180°$, as otherwise from theorem 5.2.1 or theorem 5.2.2, $\lambda'_{2,1}(T_6) \geq 16$. Hence $c' + 1$ and $c' - 1$ must be used at x_8 and x_9. Now notice that $\mathbf{f}'(d) = c' + 3$ and the edge d is at $60°$ and $120°$ with x_9 and x_8 respectively. Hence at v_2, $(c' + 1, c' + 3)$ must be used at two edges either at $60°$ or at $120°$ resulting $\lambda'_{2,1}(T_6) \geq 16$ from theorem 5.2.1 or theorem 5.2.2.

Now consider the case when $\mathbf{f}'(g) = c', \mathbf{f}'(h) = c' + 2$ and $c' + 1$ is not used in H. Assume $c' + 1$ is used at four edges incident to v_1, v_3, v_8 and v_{12}. Note that $c' - 1$ must be used at an edge incident to v_9 in H. Here we assume exactly two distinct pair of different types of edges (p, q) and (r, s) are assigned consecutive colors such that $p \in T_h, q \in T_s$ and $r \in T_h, s \in T_v$. Consider $\mathbf{f}'(j) = c' - 1$. So, $\mathbf{f}'(f) = c' - 2$ otherwise c' and $c' - 2$ must be at two adjacent edges at an angle $60°$ or $120°$. Note that either $\mathbf{f}'(i) = c' - 3$ or $\mathbf{f}'(i) = c' + 3$. First we assume $\mathbf{f}'(i) = c' + 3$.

In that case $c' + 4$ can only be used at an edge incident to v_6 in H, as otherwise $c' + 2$ and $c' + 4$ will be at two edges at an angle 60° or 120°. So, $\mathbf{f}'(d) = c' + 4$. Therefore, $\mathbf{f}'(a) = c' + 5, \mathbf{f}'(c) = c' + 6, \mathbf{f}'(e) = c' + 7$ and $\mathbf{f}'(b) = c' + 8$. Similarly, $\mathbf{f}'(n) = c' - 3, \mathbf{f}'(m) = c' - 4, \mathbf{f}'(k) = c' - 5, \mathbf{f}'(o) = c' - 6$ and $\mathbf{f}'(l) = c' - 7$. Note that any three among $\mathbf{f}'(d), \mathbf{f}'(e), \mathbf{f}'(f)$ and $\mathbf{f}'(k)$ must be used at $S_4 \cup S_5$ incident to v_8 as $c' + 1$ is used here. But $\mathbf{f}'(f) = c' - 2$ and $\mathbf{f}'(k) = c' - 5$ can not be used there as $\mathbf{f}'(j) = c' - 1$ and $\mathbf{f}'(o) = c' - 6$. Hence one more color must be introduced here resulting $\lambda'_{2,1}(T_6) \geq \lambda'_{2,1}(G) \geq 16$. Similar argument holds when $\mathbf{f}'(i) = c' - 3$. Hence the proof. $\square$

Till now we have considered the case when in H, the unused color $c_u \notin \{5, 10\}$. Now we consider the case when $c_u \in \{5, 10\}$ and the case when there is no unused color in H.

Theorem 5.2.4 *If a color $c_u \in \{5, 10\}$ is unused at H then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof: Without loss of generality let us assume color 5 is unused in H. The set of colors $\{0, 1, \ldots, 4\}$ must be used in same type of edges and same holds for the sets $\{6, 7, \ldots, 10\}$ and $\{11, 12, \ldots, 15\}$ otherwise there exists at least two disjoint pair of edges (p, q) and (r, s) where consecutive colors are used in each pair and there we can prove $\lambda'_{2,1}(T_6) \geq 16$ using similar argument as depicted in Lemma 5.2.4 and Theorem 5.2.1 or 5.2.2 or 5.2.3. Let us consider $p = f, q = a, \mathbf{f}'(p) = c'$ and $\mathbf{f}'(q) = c' + 1$. From Observation 11, c' can not be used in the sub graph H' isomorphic to H centering $S'_1 = \{v_1, v_2, v_5\}$. If $c' \neq 10$, then from Theorem 5.2.1 or 5.2.2 or 5.2.3 $\lambda'_{2,1}(T_6) \geq 16$. If $c' = 10$ then $\mathbf{f}'(q) = c' + 1 = 11$. In that case the colors $\{0, 1, \ldots, 4\}$ must be used in T_v, $\{6, 7, \ldots, 10\}$ must be used in T_h and $\{11, 12, \ldots, 15\}$ must be used in T_s. From Observation 9, the edges for reusing $\mathbf{f}'(f) = c' = 10$ are y_5, y_{12}, y_{16} and (u_3, u_4). If $\mathbf{f}'(f) = c' = 10$ is used at (u_3, u_4) then $\mathbf{f}'(a) = c' + 1 = 11$ can not be used at its opposite edge y_{21} and hence from Observation 9, Lemma 5.2.4 and Theorem 5.2.1 or 5.2.2 or 5.2.3 $\lambda'_{2,1}(T_6) \geq 16$. If $\mathbf{f}'(f) = c' = 10$ is not used at (u_3, u_4) then either color 5 or a color $c'' \neq \{5, 10\}$ used in H must be used there. If a color c'' is used there, then there exists a pair of opposite edges where c'' can not be used and again from Observation 9, Lemma 5.2.4 and Theorem 5.2.1 or 5.2.2 or 5.2.3 $\lambda'_{2,1}(T_6) \geq 16$. So, to keep $\lambda'_{2,1}(T_6)$ below 16, color 5 must be used at (u_3, u_4). With exactly same argument we can show that color 5 must also be used at y_5, y_{12}, y_{16} otherwise $\lambda'_{2,1}(T_6) \geq 16$. Remember that the color 4 is used in T_v and it must also be used at its opposite edges. Note that for any edges $e_1 \in T_v \setminus \{m\}$, either e_1 or its opposite edge is adjacent to an edge e_2 where $f(e_2) = 5$. Hence $\mathbf{f}'(m) = 4$. But y_{13}, the opposite edge of m, can not have color 4 as color 5 is used at y_{12}. Hence from Observation 9, Lemma 5.2.4 and Theorem 5.2.1 or 5.2.2 or 5.2.3 $\lambda'_{2,1}(T_6) \geq 16$. □

Theorem 5.2.5 *If all colors $c_1 \in \{0, \ldots, 14\}$ are used at H then $\lambda'_{2,1}(T_6) \geq 16$.*

Proof: In this case there exists a pair of different types of edges in H where $(c', c'+1)$ are used. From Observation 11, there exists a H' isomorphic to H where c' or $c'+1$ can not be used. Hence either from Theorem 5.2.4 or from Lemma 5.2.4 and Theorem 5.2.1 or 5.2.2 or 5.2.3, it follows that $\lambda'_{2,1}(T_6) \geq 16$. □

5.2.2 Circular $L(2,1)$-edge labeling

Lemma 5.2.5 $\sigma'_{2,1}(T_6) \leq 18$

Proof: In this grid there are three types of edges- horizontal, vertical and slanted.

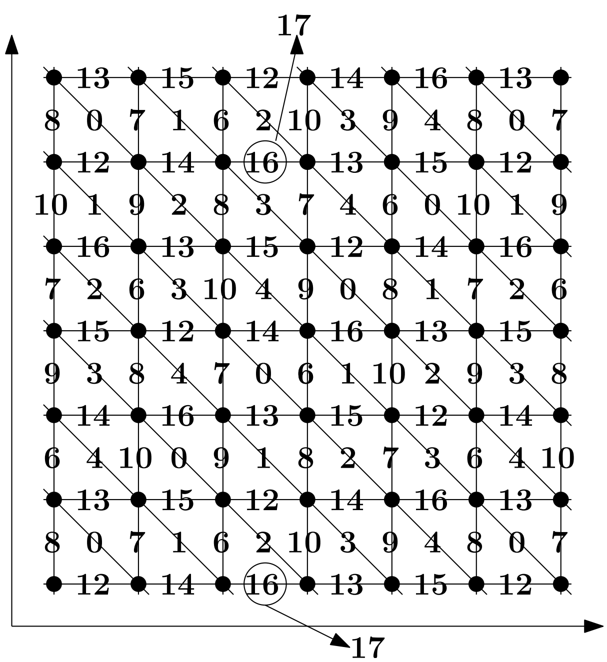


Figure 5.3: An assignment of colors of the edges of T_6 satisfying circular $L(2,1)$-edge labeling.

In order to prove the Lemma, we now consider the labeling shown in Figure 5.3.

Assuming left bottom corner point as origin, the labeling functions corresponding to horizontal, vertical and slanted edges can be stated as:

$$\mathbf{f}'((x,y),(x+1,y)) = (7x+y) \; mod \; 5 + 12$$

$$\mathbf{f}'((x,y),(x,y+1)) = (3y-x+2) \; mod \; 5 + 6$$

$$\mathbf{f}'((x-1,y+1),(x,y)) = (x+4y-1) \; mod \; 5$$

. Observe that, different types of edges get different colors in the coloring. Horizontal edges get color $12, 13, 14, 15, 16$; vertical edges get $6, 7, 8, 9, 10$ and slanted edges get $0, 1, 2, 3, 4$. By the pattern it is clear that, two adjacent edges of same type do not get two consecutive colors. By the definition of n-circular $L(2,1)$-edge labeling 0 and n are two consecutive colors. In our coloring there are many places where two adjacent edges get 0 and 16. So, the color 16 can not be the circular span of the grid. That's why we introduce a new color 17, which is used at edges as a replacement for 16 such that 0 and 16 become two non-consecutive colors. Two such 16 colored edges are shown in Figure 5.3 whose colors can be replaced by 17. Putting 17 at any one such edge is sufficient. The main goal behind introducing a new color 17 was to make 0 and 16 non-consecutive. As the colors 5 and 11 are unused in the graph, we can conclude that no two adjacent edges of different types get two consecutive colors. Now we have to show that no two edges at distance two get the same color. For the same type of edges it can be easily followed from the pattern of repetition. In case of different type of edges, observe that, distant two edges of two different types get color with difference at least two. Hence this labeling can be extended to infinite grid. □

5.3 Conclusions

Here we prove the conjecture $\lambda'_{2,1}(T_6) = 16$ given by Lin and Wu [14]. We prove that $\lambda'_{2,1}(T_6) \geq 16$ and as $\lambda'_{2,1}(T_6) \leq 16$ [14], it immediately follows that $\lambda'_{2,1}(T_6) = 16$. We also show that $\sigma'_{2,1}(T_6) \leq 18$ by giving a labeling function. Determining the value of $\sigma'_{2,1}(T_6)$ is an open problem and can be done as a future work.

Chapter 6

Proving a conjecture on $L(1, 1, \ldots, 1)$-vertex labeling for T_3

Studies of $L(k_1, k_2, \ldots, k_\ell)$-vertex labeling for T_3 for $\ell > 2$ have been made by several authors [64, 65, 18, 50]. Let $\lambda_\ell(T_3)$ is the minimum number of colors required for $L(\underbrace{1, 1, \ldots, 1}_{\ell})$-vertex labeling for T_3. For all odd $\ell > 2$ and all even $\ell < 8$, exact values of $\lambda_\ell(T_3)$ have been determined in [18]. For even $\ell \geq 8$, values of $\lambda_\ell(T_3)$ was conjectured in [18]. For even $\ell \geq 8$, the conjecture is stated as follows.

Conjecture 1.
$$\lambda_\ell(T_3) = \left[\frac{3}{8} \left(\ell + \frac{4}{3} \right)^2 \right] \tag{6.1}$$

Here $[x]$ is an integer, $x \in \mathbb{R}$ and $x - \dfrac{1}{2} < [x] \leq x + \dfrac{1}{2}$. Here color starts from 1 and λ_ℓ represents the highest color used.

In this chapter we prove the conjecture for every even $\ell \geq 8$. First we study the problem when $\ell = 8$. In first subsection (section 6.1) we prove the conjecture when $\ell = 8$. In the next subsection (section 6.2) we prove the conjecture for all even $\ell \geq 10$. We deal the cases of $\ell = 8$ and even $\ell \geq 10$ separately as some of the results that hold for even $\ell \geq 10$ does not hold for $\ell = 8$, as mentioned in the proof of Theorem 6.2.2. In subsection 6.3, concluding remarks have been stated.

6.1 Determining $\lambda_\ell(T_3)$ when $\ell = 8$

In [18], it was shown that when $\ell = 4m$ where m is a positive integer,

$$\lambda_\ell(T_3) \leq \frac{3}{8}\left(\ell + \frac{4}{3}\right)^2 + \frac{1}{3}. \tag{6.2}$$

Equation (6.2) implies that $\lambda_8(T_3) \leq 33$. Moreover, using a computer routine that explores all possible colorings of a sub-graph of T_3 with 109 vertices, authors in [18] found that 33 colors are required for the sub-graph for the 8 distance labeling $L(\underbrace{1,1,\ldots,1}_{8})$. In our approach, we prove that $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3) \geq 32$. Note that $\lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3)$ represents the minimum span, i.e., the difference between the maximum color and the minimum color used in the optimal labeling. Hence by definition, $\lambda_\ell(T_3) = \lambda_{\underbrace{1,1,\ldots,1}_{\ell}}(T_3) + 1$ and $\lambda_8(T_3) = \lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3) + 1 \geq (32+1)$=33. So, minimum number of colors obtained through our approach exactly coincides with the value obtained from the conjecture stated in equation (6.1) as well as with the upper bound stated in equation (6.2). In this section, we prove that $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3) \geq 32$ which implies $\lambda_8(T_3) \geq 33$. Since $\lambda_8(T_3) \leq 33$ [18], we conclude $\lambda_8(T_3) = 33$, which exactly coincides with the value obtained from the conjecture stated in equation (6.1).

The rest of the section is organized as follows. In subsection 6.1.1, we state some preliminary ideas which will be used to establish our results. In subsection 6.1.2, we will discuss about the obtained result when $\ell = 8$.

6.1.1 Preliminaries

Infinite hexagonal grid T_3 is alternatively termed as Infinite honeycomb grid. Sometimes T_3 is alternatively represented as infinite brick structure grid. In brick structure representation of T_3, coordinates of the vertices can be represented more conveniently than that of T_3. For this reason, now onwards we use brick structure representation of T_3. In Figure 6.1, a part of brick structure representation of T_3 along with the coordinates of the corresponding vertices are shown.

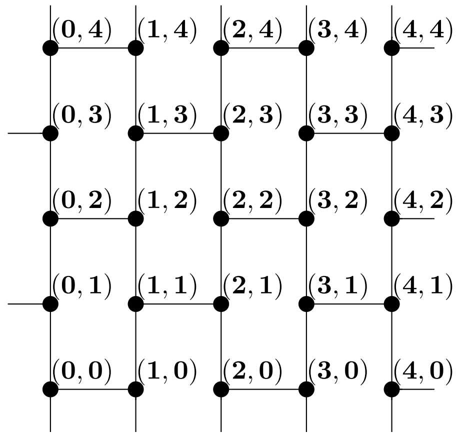


Figure 6.1: Brick structure representation of T_3 and coordinates of its vertices.

Definition 11 *A vertex with co-ordinates (i, j) in brick structure representation of T_3 is said to be a right vertex or x_r if it is connected to the vertex with co-ordinates $(i+1, j)$ by an edge. A vertex with co-ordinates (i, j) in brick structure representation of T_3 is said to be a left vertex or x_l if it is connected to the vertex with co-ordinates $(i-1, j)$ by an edge.*

A right vertex $x_r(i, j)$ is adjacent to the vertices having co-ordinates $(i+1, j)$, $(i, j+1)$ and $(i, j-1)$ but not adjacent to the vertex with co-ordinates $(i-1, j)$. A left vertex x_l is adjacent to the vertices having co-ordinates $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$ but not adjacent to the vertex with co-ordinates $(i+1, j)$.

In Figure 6.1, the vertex with coordinates $(1, 2)$ is a left vertex and the vertex with coordinates $(2, 2)$ is a right vertex.

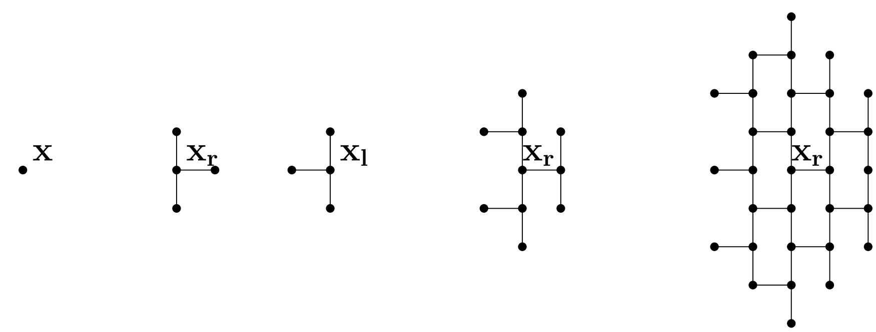
Definition 12 *A maximum distance $2p$ clique D_x^{2p} ($p \in \mathbb{N}$) centring at vertex x in T_3 is the maximum cardinality vertex induced sub-graph of T_3 where for each pair of vertices u and v in D_x^{2p}, $d(u, v) \leq 2p$ and for every vertex w in D_x^{2p}, $d(w, x) \leq p$. Here $d(u, v)$, the distance between vertices u and v, denotes the minimum number of edges that connect u and v.*

In Figure 6.2 different D_x^{2p} in T_3 are shown.

Note that for any right vertex x_r and left vertex x_l, $D_{x_r}^{2p}$ and $D_{x_l}^{2p}$ are isomorphic. So any property that holds for $D_{x_r}^{2p}$ also holds for $D_{x_l}^{2p}$. Therefore, we will state and prove our results for $D_{x_r}^{2p}$ and these also hold for $D_{x_l}^{2p}$.

Consider a right vertex $x_r(0, 0)$ and the sub-graph $D_{x_r}^{16}$ centering x_r as shown in Figure 6.3. Note that there are $k_j = 3j$ vertices which are at distance j from x_r, where $1 \leq j \leq 8$. In Figure 6.3, the $k_j = 3j$ vertices are denoted as $v_1^j, v_2^j, \ldots, v_{k_j}^j$ where $1 \leq j \leq 8$. We now define $\mathcal{F}_j = \bigcup_{1 \leq i \leq k_j} \{v_i^j\}$ where $1 \leq j \leq 8$. For $5 \leq q \leq 8$, we define the following sets of vertices, where $k_q = 3q$.

$$V_{i-j}^q = \begin{cases} \{v_l^q : i \leq l \leq j\} & \text{when } i \leq j \\ V_{i-k_q}^q \cup V_{1-j}^q & \text{when } i > j \end{cases}$$



$a.\ D_x^0$ $b.\ D_{x_r}^2$ $c.\ D_{x_l}^2$ $d.\ D_{x_r}^4$ $e.\ D_{x_r}^8$

Figure 6.2: Different D_x^{2p}.

6.1.2 Results

In the following discussion, we will first investigate that at most how many times and where the colors of $D_{x_r}^8$ can be reused in V' where $V' = V(D_{x_r}^{16}) \setminus V(D_{x_r}^8) = \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8$. Then we will state that how many times a new color may be used in V' if a new color (a color not used in $D_{x_r}^8$) is required to be introduced at all. Finally, we will state and prove our main theorem by finding the least number of new color/s which are necessary for the coloring of the vertices of V'.

As $|V(D_{x_r}^8)| = 1 + |\mathcal{F}_1| + |\mathcal{F}_2| + |\mathcal{F}_3| + |\mathcal{F}_4| = 31$, we need 31 distinct colors to color the vertices of $D_{x_r}^8$ and hence $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(D_{x_r}^8) \geq 30$. In Figure 6.3, we denote the $k_j = 3j$ colors of the vertices in $\mathcal{F}_j$ as $c_1^j, c_2^j, \ldots, c_{k_j}^j$, where $1 \leq j \leq 4$. It is evident that color c assigned at x_r can not be reused at all at V' due to the reuse distance. In subsequent Observations we will state and prove at most how many times the colors $c_1^j, c_2^j, \ldots, c_{k_j}^j$ can be repeated in V'.

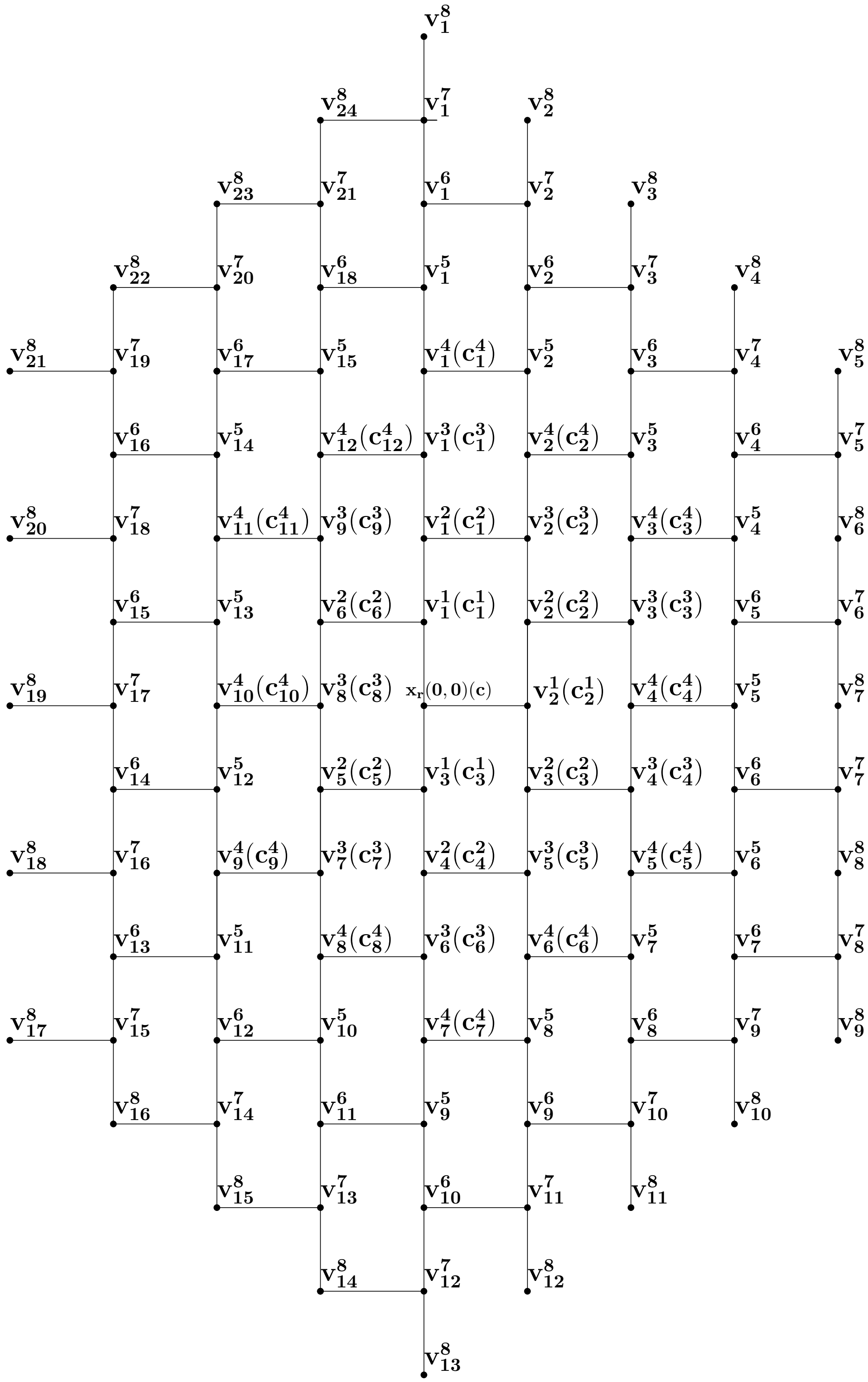


Figure 6.3: All the vertices v with $1 \leq d(v, x_r) \leq 8$ and colors of the all vertices u with $d(u, x_r) \leq 4$ (Color is mentioned within brackets beside the corresponding vertex).

Observation 12 *Each color c_i^1 with $i \in \{1,2,3\}$ can be reused at most thrice at V'. For maximum re-usability, each color c_i^1 must be reused thrice at $\mathcal{F}_8$.*

Proof: There are three vertices at $\mathcal{F}_1$ where the colors c_i^1 with $i \in \{1,\ldots,3\}$ are used. We will first consider the color c_1^1 used at vertex v_1^1. Observe that c_1^1 can be reused at $R = V_{5-17}^8 = R_1 \cup R_2 \cup R_3$ where $R_1 = V_{5-8}^8$, $R_2 = V_{9-12}^8$ and $R_3 = V_{13-17}^8$ are three disjoint subsets of vertices. Note that every pair of vertices in R_1 are at distance at most 8, and the same is true for R_2 and R_3. Hence c_1^1 can be reused at most thrice in V', once at R_1, once at R_2 and once at R_3. Note that v_1^1 and v_3^1 are symmetric with respect to x_r where c_1^1 and c_3^1 are used respectively. Hence result obtained regarding how many times the color c_1^1 can be reused at V' also holds for c_3^1. So we are remaining to consider the color c_2^1 used at vertex v_2^1. For c_2^1, the corresponding sets R, R_1, R_2 and R_3 can easily be obtained as $R = V_{13-1}^8$, $R_1 = V_{13-16}^8$, $R_2 = V_{17-21}^8$ and $R_3 = V_{22-1}^8$ respectively. Hence c_2^1 can also be reused at most thrice in V'.

It is evident that each c_i^1 with $i \in \{1,2,3\}$ can only be reused at $\mathcal{F}_8$. Hence for maximum re-usability, each color c_i^1 must be reused thrice at $\mathcal{F}_8$.

Observation 13 *Each color c_i^2 with $i \in \{1,\ldots,6\}$ can be reused at most twice at V'. For maximum re-usability, each color c_i^2 must be reused twice at $\mathcal{F}_7 \cup \mathcal{F}_8$.*

Proof: There are six vertices at $\mathcal{F}_2$ where the colors c_i^2 with $i \in \{1,\ldots,6\}$ are used. Observe that the vertices v_1^2 and v_4^2 are symmetric with respect to x_r where c_1^2 and c_4^2 are used respectively. Similar fact holds for c_2^2 and c_3^2; c_5^2 and c_6^2. Hence results obtained regarding how many times the colors c_1^2, c_2^2 and c_5^2 can be reused in V' also hold for c_4^2, c_3^2 and c_6^2 respectively. Therefore we need to consider the colors c_1^2, c_2^2 and c_5^2 only.

- We will first consider the color c_1^2. It can be reused at $R = V_{8-15}^7 \cup V_{9-17}^8$. Observe that $R = R_1 \cup R_2$ where $R_1 = V_{8-11}^7 \cup V_{9-12}^8$ and $R_2 = V_{12-15}^7 \cup V_{13-17}^8$ are two vertex disjoint subsets. Note that there does not exist any pair of vertices at R_1 at distance 9 or more. Same fact holds for R_2. Hence c_1^2 can be reused at most twice in V', once at R_1 and once at R_2. It is evident that each c_i^2 with $i \in \{1,\ldots,6\}$ can only be reused at $\mathcal{F}_7 \cup \mathcal{F}_8$. Hence c_1^2 must be reused twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum re-usability. Similar result holds for c_4^2 also.

For the colors c_2^2 and c_5^2 the sets R, R_1 and R_2 are stated below.

- c_2^2: $R = V^7_{12-19} \cup V^8_{13-21} = R_1 \cup R_2$; $R_1 = V^7_{12-15} \cup V^8_{13-17}$; $R_2 = V^7_{16-19} \cup V^8_{18-21}$.
- c_5^2: $R = V^7_{1-8} \cup V^8_{1-9} = R_1 \cup R_2$; $R_1 = V^7_{1-4} \cup V^8_{1-4}$; $R_2 = V^7_{5-8} \cup V^8_{5-9}$.

$\square$

Observation 14 *Each color c_i^3 with $i \in \{1,\ldots,9\}$ can be reused at most thrice at V'. For maximum re-usability, each c_i^3 with $i \in \{2,5,8\}$ must be reused at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ and each c_i^3 with $i \in \{1,\ldots,9\} \setminus \{2,5,8\}$ must be reused at least once in $\mathcal{F}_7 \cup \mathcal{F}_8$.*

Proof: Note that the pair of colors c_1^3 and c_6^3 are assigned to the vertices v_1^3 and v_6^3 which are symmetric with respect to x_r. Similar fact holds for c_2^3 and c_5^3; c_3^3 and c_4^3; c_9^3 and c_7^3. Hence results obtained regarding how many times the colors c_1^3, c_2^3, c_3^3 and c_9^3 can be reused in V' also hold for c_6^3, c_5^3, c_4^3 and c_7^3 respectively. Therefore, we need to consider the colors c_1^3, c_2^3, c_3^3, c_9^3 only. We will consider the case for the color c_8^3 separately.

- First consider the color c_1^3. It can be reused at $R = V^6_{7-13} \cup V^7_{8-15} \cup V^8_{5-18} = R_1 \cup R_2 \cup R_3$ where $R_1 = V^7_{8-8} \cup V^8_{5-9}$, $R_2 = V^6_{7-10} \cup V^7_{9-12} \cup V^8_{10-13}$ and $R_3 = V^6_{11-13} \cup V^7_{13-15} \cup V^8_{14-18}$ are three disjoint subsets of vertices. Observe that every pair of vertices belonging to the same subset are at distance at most 8. Hence c_1^3 can be reused at most thrice, once at $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at R_2 and once at R_3. To reuse c_1^3 thrice in V', it must be reused at least once at $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_6^3 also.

  For each of the colors c_2^3, c_3^3, c_9^3 and c_8^3 the set R and its corresponding partitions R_1, R_2, R_3 are as stated below.

- c_2^3: $R = V^6_{10-13} \cup V^7_{12-15} \cup V^8_{9-21} = R_1 \cup R_2 \cup R_3$; $R_1 = V^8_{9-12}$; $R_2 = V^6_{10-13} \cup V^7_{12-15} \cup V^8_{13-16}$; $R_3 = V^8_{17-21}$; To reuse c_2^3 thrice, it must be used once at $R_1 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at R_2 and once at $R_3 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_2^3 thrice, it must be used at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_5^3 also.
- c_3^3: $R = V^6_{10-16} \cup V^7_{12-19} \cup V^8_{12-1} = R_1 \cup R_2 \cup R_3$; $R_1 = V^6_{10-12} \cup V^7_{12-14} \cup V^8_{12-16}$; $R_2 = V^6_{13-16} \cup V^7_{15-18} \cup V^8_{17-20}$; $R_3 = V^7_{19-19} \cup V^8_{21-1}$; To reuse c_3^3 thrice, it must be used once at R_1, once at R_2 and once at $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to

reuse c_3^3 thrice, it must be used at least once at $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_4^3 also.

- c_9^3: $R = V_{4-10}^6 \cup V_{5-12}^7 \cup V_{4-17}^8 = R_1 \cup R_2 \cup R_3$; $R_1 = V_{4-6}^6 \cup V_{5-7}^7 \cup V_{4-8}^8$; $R_2 = V_{7-10}^6 \cup V_{8-11}^7 \cup V_{9-12}^8$; $R_3 = V_{12-12}^7 \cup V_{13-17}^8$; To reuse c_9^3 thrice, it must be used once at R_1, once at R_2 and once at $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_9^3 thrice, it must be used at least once at $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_7^3 also.
- c_8^3: at $R = V_{4-7}^6 \cup V_{5-8}^7 \cup V_{1-13}^8 = R_1 \cup R_2 \cup R_3$; $R_1 = V_{1-4}^8$; $R_2 = V_{4-7}^6 \cup V_{5-8}^7 \cup V_{5-9}^8$; $R_3 = V_{10-13}^8$; To reuse c_8^3 thrice, it must be used once at $R_1 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at R_2 and once at $R_3 \subset \mathcal{F}_8 \subset \mathcal{F}_7 \cup \mathcal{F}_8$. Hence to reuse c_8^3 thrice, it must be used at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$.

$\square$

Observation 15 *Each color c_i^4 with $i \in \{1, \ldots, 12\}$ can be reused at most thrice in V'. For maximum re-usability, each c_i^4 with $i \in \{1, \ldots, 12\}$ must be reused at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$.*

Proof: Note that the colors c_1^4 and c_7^4 are used at v_1^4 and v_7^4 respectively which are symmetric with respect to x_r. Similar fact also hold for c_2^4 and c_6^4; c_3^4 and c_5^4; c_8^4 and c_{12}^4; c_9^4 and c_{11}^4. Hence result obtained regarding how many times the colors $c_1^4, c_2^4, c_3^4, c_8^4$ and c_9^4 can be reused in V' are same for the colors $c_7^4, c_6^4, c_5^4, c_{12}^4$ and c_{11}^4 respectively. Therefore we need to consider the colors $c_1^4, c_2^4, c_3^4, c_8^4$ and c_9^4 only. We will consider the remaining two colors c_4^4 and c_{10}^4 separately.

- We first consider the color c_1^4. It can be reused at $R = V_{6-11}^5 \cup V_{7-13}^6 \cup V_{7-16}^7 \cup V_{8-18}^8 = R_1 \cup R_2 \cup R_3$ where $R_1 = V_{6-8}^5 \cup V_{7-9}^6 \cup V_{7-10}^7 \cup V_{8-11}^8$, $R_2 = V_{9-11}^5 \cup V_{10-13}^6 \cup V_{11-14}^7 \cup V_{12-16}^8$ and $R_3 = V_{15-16}^7 \cup V_{17-18}^8$ are three disjoint subsets. Observe that every pair of vertices belonging to the same subset are at distance at most 8. Hence c_1^4 can be reused at most thrice, once at R_1, once at R_2 and once at R_3.

  Observe that c_1^4 can be reused twice in $\mathcal{F}_5 \cup \mathcal{F}_6$ only when c_1^4 is used once at $u \in V_{6-8}^5 \cup V_{7-9}^6 \subset R_1$ and once at $v \in V_{9-11}^5 \cup V_{10-13}^6 \subset R_2$ such that $d(u,v) \geq 9$. Note that for any such (u,v) pair, there does not exist any $w \in R_3$ such that $d(u,w) \geq 9$ and $d(v,w) \geq 9$. That is, in that case, c_1^4 can not be reused once more in R_3. This implies that for maximum re-usability, c_1^4 can be reused at most once in $\mathcal{F}_5 \cup \mathcal{F}_6$. Hence c_1^4 must be used at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum re-usability. Similar result holds for c_7^4 also.

For each of the colors c_2^4, c_3^4, c_4^4, c_8^4, c_9^4 and c_{10}^4 the set R and its corresponding partitions R_1, R_2, R_3 are stated below.

- c_2^4: $R = V_{9-11}^5 \cup V_{10-13}^6 \cup V_{8-19}^7 \cup V_{9-21}^8 = R_1 \cup R_2 \cup R_3$;

  $R_1 = V_{8-11}^7 \cup V_{9-12}^8$; $R_2 = V_{9-11}^5 \cup V_{10-13}^6 \cup V_{12-15}^7 \cup V_{13-17}^8$; $R_3 = V_{16-19}^7 \cup V_{18-21}^8$;

  To reuse c_2^4 thrice, it must be reused once at $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at R_2 and once at $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and hence it must be reused at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_6^4 also.

- c_3^4: $R = V_{9-14}^5 \cup V_{10-16}^6 \cup V_{11-20}^7 \cup V_{12-22}^8 = R_1 \cup R_2 \cup R_3$;

  $R_1 = V_{9-11}^5 \cup V_{10-13}^6 \cup V_{11-14}^7 \cup V_{12-16}^8$; $R_2 = V_{15-16}^7 \cup V_{17-18}^8$; $R_3 = V_{12-14}^5 \cup V_{14-16}^6 \cup V_{17-20}^7 \cup V_{19-22}^8$;

  Observe that c_3^4 can be reused twice in $\mathcal{F}_5 \cup \mathcal{F}_6$ only when c_3^4 is used once at $u \in V_{9-11}^5 \cup V_{10-13}^6 \subset R_1$ and once at $v \in V_{12-14}^5 \cup V_{14-16}^6 \subset R_3$ such that $d(u, v) \geq 9$. Note that for any such (u, v) pair, there does not exist any $w \in R_2$ such that $d(u, w) \geq 9$ and $d(v, w) \geq 9$. That is, in that case, c_3^4 can not be reused once more in R_2. This implies that for maximum re-usability, c_1^4 can be reused at most once in $\mathcal{F}_5 \cup \mathcal{F}_6$. Hence c_3^4 must be used at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum re-usability. Similar result holds for c_5^4 also.

- c_4^4: $R = V_{11-14}^5 \cup V_{13-16}^6 \cup V_{12-1}^7 \cup V_{13-1}^8 = R_1 \cup R_2 \cup R_3$.

  $R_1 = V_{12-15}^7 \cup V_{13-17}^8$; $R_2 = V_{11-14}^5 \cup V_{13-16}^6 \cup V_{16-18}^7 \cup V_{18-20}^8$; $R_3 = V_{19-1}^7 \cup V_{21-1}^8$;

  To reuse c_4^4 thrice, it must be reused once at $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at R_2 and once at $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and hence it must be reused at least twice at $\mathcal{F}_7 \cup \mathcal{F}_8$.

- c_8^4: $R = V_{1-4}^5 \cup V_{1-4}^6 \cup V_{19-8}^7 \cup V_{21-9}^8 = R_1 \cup R_2 \cup R_3$;

  $R_1 = V_{19-1}^7 \cup V_{21-1}^8$; $R_2 = V_{1-4}^5 \cup V_{1-4}^6 \cup V_{2-4}^7 \cup V_{2-4}^8$; $R_3 = V_{5-8}^7 \cup V_{5-9}^8$;

To reuse c_8^4 thrice, it must be reused once at $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and once at R_2 and hence it must be reused at least twice at $\mathcal{F}_7 \cup \mathcal{F}_8$. Similar result holds for c_{12}^4 also.

- c_9^4: $R = V_{1-6}^5 \cup V_{1-7}^6 \cup V_{21-9}^7 \cup V_{24-10}^8 = R_1 \cup R_2 \cup R_3$;

  $R_1 = V_{1-3}^5 \cup V_{1-3}^6 \cup V_{21-4}^7 \cup V_{24-3}^8$; $R_2 = V_{5-5}^7 \cup V_{4-5}^8$; $R_3 = V_{4-6}^5 \cup V_{4-7}^6 \cup V_{6-9}^7 \cup V_{6-10}^8$;

  Observe that c_9^4 can be reused twice in $\mathcal{F}_5 \cup \mathcal{F}_6$ only when c_9^4 is used once at $u \in V_{1-3}^5 \cup V_{1-3}^6 \subset R_1$ and once at $v \in V_{4-6}^5 \cup V_{4-7}^6 \subset R_3$ such that $d(u,v) \geq 9$. Note that for any such (u,v) pair, there does not exist any $w \in R_2$ such that $d(u,w) \geq 9$ and $d(v,w) \geq 9$. That is, in that case, c_9^4 can not be reused once more in R_2. This implies that for maximum re-usability, c_9^4 can be reused at most once in $\mathcal{F}_5 \cup \mathcal{F}_6$. Hence c_9^4 must be used at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain its maximum re-usability. Similar result holds for c_{11}^4 also.

- c_{10}^4: $R = V_{4-6}^5 \cup V_{4-7}^6 \cup V_{1-12}^7 \cup V_{1-13}^8 = R_1 \cup R_2 \cup R_3$;

  $R_1 = V_{1-4}^7 \cup V_{1-4}^8$; $R_2 = V_{4-6}^5 \cup V_{4-7}^6 \cup V_{5-8}^7 \cup V_{5-9}^8$; $R_3 = V_{9-12}^7 \cup V_{10-13}^8$;

  To reuse c_{10}^4 thrice, it must be reused once at $R_1 \subset \mathcal{F}_7 \cup \mathcal{F}_8$, once at R_2 and once at $R_3 \subset \mathcal{F}_7 \cup \mathcal{F}_8$ and hence it must be reused at least twice at $\mathcal{F}_7 \cup \mathcal{F}_8$.

□

We will now see whether the colors used in $D_{x_r}^8$ are sufficient to color the vertices of V' or new color/s is/are to be introduced to color them. Note that if new color/s is/are necessary then the required number of new color/s will depend on the number of vertices of V' which can not be colored with the colors used in $D_{x_r}^8$ and how many times at maximum a new color can be used in V'. In following Observation, we will first state how many times at maximum a new color can be used in V' and then based on this result, in Theorem 6.1.1, we will finally state the minimum number of colors required to color the vertices of $D_{x_r}^{16}$.

Observation 16 *A new color c_n can be used at most five times in V'.*

Proof: A new color c_n can be used at $V' = \mathcal{F}_5 \cup \mathcal{F}_6 \cup \mathcal{F}_7 \cup \mathcal{F}_8 = V_{1-24}^8 \cup V_{1-21}^7 \cup V_{1-18}^6 \cup V_{1-15}^5$. Observe that V' can be partitioned into six disjoint subsets $R_1 = V_{21-1}^8 \cup V_{19-1}^7 \cup V_{16-1}^6 \cup V_{14-1}^5$, $R_2 = V_{17-20}^8 \cup V_{15-18}^7 \cup V_{13-15}^6 \cup V_{11-13}^5$,

$R_3 = V^8_{12-16} \cup V^7_{11-14} \cup V^6_{9-12} \cup V^5_{8-10}$, $R_4 = V^8_{7-11} \cup V^7_{7-10} \cup V^6_{5-8} \cup V^5_{5-7}$, $R_5 = V^8_{2-6} \cup V^7_{2-5} \cup V^6_{2-4} \cup V^5_{2-4}$ and $R_6 = V^7_{6-6}$ where every pair of vertices in a subset is at most distance 8 apart. If c_n is not used at v^7_6, the only vertex in R_6, then c_n can be used at most five times in V'. In other words, to use c_n six times in V', c_n must be used at v^7_6. If c_n is used at v^7_6 then the set of vertices where c_n can be reused is $R' = V^8_{11-2} \cup V^7_{11-1} \cup V^6_{9-1} \cup V^5_{9-15}$. Now observe that R' can be partitioned into four disjoint subsets $R'_1 = V^8_{11-15} \cup V^7_{11-13} \cup V^6_{9-12} \cup V^5_{9-10}$, $R'_2 = V^8_{16-19} \cup V^7_{14-18} \cup V^6_{13-15} \cup V^5_{11-13}$, $R'_3 = V^8_{20-24} \cup V^7_{19-21} \cup V^6_{16-1} \cup V^5_{14-15}$ and $R'_4 = V^8_{1-2} \cup V^7_{1-1}$ where every pair of vertices in a subset is at most distance 8 apart. This implies that c_n can be used at most five times in V' regardless of whether c_n is used or not used at v^7_6. $\square$

Now we state and prove the following theorem.

Theorem 6.1.1 $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3) \geq 32$.

Proof: As $|V(D^8_{x_r})| = 31$ and $|V(D^{16}_{x_r})| = 109$, $(109 - 31) = 78$ vertices are to be colored with the colors from $c^j_1, c^j_2, \ldots, c^j_{k_j}$, where $1 \leq j \leq 4$ and $k_j = 3j$ (Note that color c of x_r can not be reused in $D^{16}_{x_r}$ as any vertex in $D^{16}_{x_r}$ is at distance at most 8 from x_r). From Observation 12, Observation 13, Observation 14 and Observation 15, using these colors we can color at most $(\underbrace{9(3\times 3)}_{c^1_i, i\in\{1,2,3\}} + \underbrace{12(6\times 2)}_{c^2_i, i\in\{1,2,\ldots,6\}} + \underbrace{27(9\times 3)}_{c^3_i, i\in\{1,2,\ldots,9\}} + \underbrace{36(12\times 3)}_{c^4_i, i\in\{1,2,\ldots,12\}}) = 84$ vertices in V' if each of them are reused with their maximum potential of re-usability.

As discussed in Observation 12, all $c^1_i, i \in \{1,2,3\}$ must be reused thrice in $\mathcal{F}_8$ to attain their maximum re-usability in V'; Hence all c^1_is, $i \in \{1,2,3\}$ together must occupy 9 vertices in $\mathcal{F}_8$ here. As discussed in Observations 13, all $c^2_i, i \in \{1,2,\ldots,6\}$ must be reused twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum re-usability; Hence all c^2_is, $i \in \{1,2,\ldots,6\}$ together must occupy 12 vertices in $\mathcal{F}_8 \cup \mathcal{F}_7$ here. As discussed in Observations 14, each of c^3_2, c^3_5 and c^3_8 must be used at least twice in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum re-usability and each of $c^3_i, i \in \{1,2,\ldots,9\} \setminus \{2,5,8\}$ must be reused at least once in $\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum re-usability. So, all c^3_is, $i \in \{1,2,\ldots,9\}$ together must occupy at least 12 vertices in $\mathcal{F}_7 \cup \mathcal{F}_8$ here. As discussed in Observations 15, all $c^4_i, i \in \{1,2,\ldots,12\}$ must be reused at least twice in

$\mathcal{F}_7 \cup \mathcal{F}_8$ to attain their maximum re-usability. So, all c_i^4s, $i \in \{1, 2, \ldots, 12\}$ together must occupy at least 24 vertices in $\mathcal{F}_7 \cup \mathcal{F}_8$ here. Therefore, to satisfy the maximum re-usability of each color used in $D^8_{x_r}$, total positions required at $\mathcal{F}_7 \cup \mathcal{F}_8$ is at least $9 + 12 + 12 + 24 = 57$. However, total positions available at $\mathcal{F}_7 \cup \mathcal{F}_8$ is $21 + 24 = 45$ only. Since two or more colors can not be given at the same vertex, these colors together must loose the potential of maximum re-usability by at least $57 - 45 = 12$ in V'. Hence they together can color maximum $(84 - 12) = 72$ vertices in V'. Since V' has 78 vertices, new color/s must be needed to color at least $(78 - 72) = 6$ vertices of V'. From observation 16, a new color can color at most five vertices in V'. So at least two new colors are required to color these six vertices. Since 31 distinct colors are required for $D^8_{x_r}$, at least $(31 + 2) = 33$ colors are required for $D^{16}_{x_r}$. Hence $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(D^{16}_{x_r}) \geq 32$.

As for any left vertex x_l, $D^{16}_{x_r}$ and $D^{16}_{x_l}$ are isomorphic, so $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3)(D^{16}_{x_l}) \geq 32$. As $D^{16}_{x_r}$ and $D^{16}_{x_l}$ are sub graphs of T_3, we conclude that $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3) \geq 32$. Hence the proof. $\square$

In the following section we will derive the proof of the conjecture for all even $\ell > 8$.

6.2 Deriving the span of l distance coloring T_3 for even $\ell > 8$

In this section first we discuss some preliminaries and then we prove the conjecture for even $\ell > 8$.

6.2.1 Preliminaries

As discussed in [18], T_3 is a bipartite graph, where there exists two disjoint sets V_0, V_1 such that $V_0 \cup V_1 = V(T_3)$ and for each edge $(u, v) \in E(T_3)$, $u \in V_0$, $v \in V_1$ or otherwise. For any vertex $v(i, j) \in T_3$, $\tau(v)$ is defined as $\tau(v) = ((i + j) \mod 2)$ [18]. For a vertex $v(i, j) \in T_3$, if $\tau(v) = 0$, then we consider $v \in V_0$ otherwise $v \in V_1$ [18]. Here we assume the coordinates of the origin is $(0, 0)$ and it belongs to V_0. Assuming $i_1 \leq i_2$, for two vertices $v_1(i_1, j_1), v_2(i_2, j_2) \in T_3$, $d(v_1, v_2)$ can be defined as follows.

$$d(v_1, v_2) = \begin{cases} |i_1 - i_2| + |j_1 - j_2| & if\ |i_1 - i_2| \leq |j_1 - j_2|, \\ 2|i_1 - i_2| + \tau(v_1) - \tau(v_2), & if\ |i_i - i_2| > |j_1 - j_2|. \end{cases} \tag{6.3}$$

Equation (6.3) can be proved similarly as that of Lemma 3.2 in [18].

Note that there are $3k$ vertices which are at distance k from x where $k \in \mathbb{Z}^+$ [18]. In the following definitions we assume the coordinates of x as (i, j). In the proofs, for the sake of simplicity, we will assume the coordinates of x as $(0, 0)$.

Let $\mathcal{F}_{x,k} = \{v \in V(T_3) : d(x, v) = k\}$ be the set of those $3k$ vertices. We denote these $3k$ vertices as $v^1_{x,k}, v^2_{x,k}, \ldots, v^{3k}_{x,k}$. Note that the coordinates of the vertex $v^1_{x,k}$ is $(i, j + k)$. The vertices $v^2_{x,k}, \ldots, v^{3k}_{x,k}$ are labeled anti-clockwise starting from $v^1_{x,k}$ in T_3. Therefore the coordinates of these vertices should be as follows: $v^1_{x,k}(i, j + k), v^2_{x,k}(i + 1, j + k - 1), \ldots, v^{\lceil \frac{k}{2} \rceil + 1}_{x,k}(i + \lceil \frac{k}{2} \rceil, j + \lfloor \frac{k}{2} \rfloor) \ldots, v^{k+1}_{x,k}(i + \lceil \frac{k}{2} \rceil, j - \lfloor \frac{k}{2} \rfloor), \ldots, v^{k + \lceil \frac{k}{2} \rceil + 1}_{x,k}(i, j - k)\ , \ldots, v^{2k+1}_{x,k}(i - \lfloor \frac{k}{2} \rfloor, j - \lceil \frac{k}{2} \rceil), \ldots, v^{2k + \lceil \frac{k}{2} \rceil + 1}_{x,k}(i - \lfloor \frac{k}{2} \rfloor, j + \lceil \frac{k}{2} \rceil), \ldots, v^{3k}_{x,k}(i - 1, j + k - 1)$. When $k = 7$, the vertices of $\mathcal{F}_{x,7}$ are shown in Figure 6.4.

Observe that for $k = 1$, the number of vertices in $\mathcal{F}_{x,k}$ are 3. For $k = 2, 3, \ldots$, the number of vertices of $\mathcal{F}_{x,k}$ are $6, 9, \ldots$. Therefore for $k \geq 2$, these $3k$ vertices of $\mathcal{F}_{x,k}$ can be partitioned into 6 disjoint sets $G^1_{x,k}$, $G^2_{x,k}$, …, $G^6_{x,k}$. The vertices belong to the sets $G^1_{x,k}, G^2_{x,k}, \ldots, G^6_{x,k}$ are mentioned below. The coordinates of the corresponding vertices are also mentioned there.

$G^1_{x,k} = \{v^m_{x,k}(i + m - 1, j + k - m + 1) : 1 \leq m \leq \lceil \frac{k}{2} \rceil\}$

$G^2_{x,k} = \{v^{m + \lceil \frac{k}{2} \rceil}_{x,k}(i + \lceil \frac{k}{2} \rceil, j + \lfloor \frac{k}{2} \rfloor - 2m + 2) : 1 \leq m \leq \lfloor \frac{k}{2} \rfloor\}$,

$G^3_{x,k} = \{v^{m+k}_{x,k}(i + \lceil \frac{k}{2} \rceil - m + 1, j - \lfloor \frac{k}{2} \rfloor - m + 1) : 1 \leq m \leq \lceil \frac{k}{2} \rceil\}$,

$G^4_{x,k} = \{v^{m + k + \lceil \frac{k}{2} \rceil}_{x,k}(i - m + 1, j - k + m - 1) : 1 \leq m \leq \lfloor \frac{k}{2} \rfloor\}$,

$G^5_{x,k} = \{v^{m+2k}_{x,k}(i - \lfloor \frac{k}{2} \rfloor, j - \lceil \dfrac{k}{2} \rceil + 2m - 2) : 1 \leq m \leq \lceil \frac{k}{2} \rceil\}$,

$G^6_{x,k} = \{v^{m + 2k + \lceil \frac{k}{2} \rceil}_{x,k}(i - \lfloor \frac{k}{2} \rfloor + m - 1, j + \lceil \frac{k}{2} \rceil + m - 1) : 1 \leq m \leq \lfloor \frac{k}{2} \rfloor\}$.

For illustration, Figure 6.4 shows the $3k$ vertices for $k = 7$ and the corresponding $G^r_{x,k}$ where $r = 1, 2, \ldots, 6$.

Definition 13 *A vertex $v(p,q) \in \mathcal{F}_{x,k}$, $k \geq 2, k \in \mathbb{Z}^+$ is said to be a corner vertex with respect to $x(i,j)$ in $\mathcal{F}_{x,k}$ if one of the following six conditions holds.*

1. $p = i, q = j + k$.
2. $p = i + \lceil \frac{k}{2} \rceil, q = j + \lfloor \frac{k}{2} \rfloor$.
3. $p = i + \lceil \frac{k}{2} \rceil, q = j - \lfloor \frac{k}{2} \rfloor$.
4. $p = i, q = j - k$.
5. $p = i - \lfloor \frac{k}{2} \rfloor, q = j - \lceil \frac{k}{2} \rceil$.
6. $p = i - \lfloor \frac{k}{2} \rfloor, q = j + \lceil \frac{k}{2} \rceil$.

The set of vertices in $\mathcal{F}_{x,k}$, $k \geq 2, k \in \mathbb{Z}^+$ which are not corner vertices of $\mathcal{F}_{x,k}$ are said to be *non corner* vertices of $\mathcal{F}_{x,k}$.

Note that there are six corner vertices in $\mathcal{F}_{x,k}$, $k \geq 2$ and they are $v^1_{x,k}$, $v^{\lceil \frac{k}{2} \rceil+1}_{x,k}$, $v^{k+1}_{x,k}$, $v^{k+\lceil \frac{k}{2} \rceil+1}_{x,k}$, $v^{2k+1}_{x,k}$ and $v^{2k+\lceil \frac{k}{2} \rceil+1}_{x,k}$. For simplicity, we denote the six corner vertices $v^1_{x,k}$, $v^{\lceil \frac{k}{2} \rceil+1}_{x,k}$, $v^{k+1}_{x,k}$, $v^{k+\lceil \frac{k}{2} \rceil+1}_{x,k}$, $v^{2k+1}_{x,k}$ and $v^{2k+\lceil \frac{k}{2} \rceil+1}_{x,k}$ of $\mathcal{F}_{x,k}$ as $v^{c_1}_{x,k}$ $v^{c_2}_{x,k}$, $v^{c_3}_{x,k}$, $v^{c_4}_{x,k}$, $v^{c_5}_{x,k}$ and $v^{c_6}_{x,k}$ respectively. The set of six corner vertices of $\mathcal{F}_{x,k}$ are denoted as $\mathcal{F}^c_{x,k}$. From Definition 13, we get the coordinates of those six corner vertices. For an example, if we consider $\mathcal{F}_{x,7}$ ($k = 7$) and coordinates of x is $(0,0)$, the coordinates of the corresponding six corner vertices $v^1_{x,7}$, $v^5_{x,7}$, $v^8_{x,7}$, $v^{12}_{x,7}$, $v^{15}_{x,7}$ and $v^{19}_{x,7}$ are $(0,7)$, $(4,3)$, $(4,-3)$, $(0,-7)$, $(-3,-4)$ and $(-3,4)$ respectively. The set of non corner vertices of $\mathcal{F}_{x,k}$ is denoted as $\mathcal{F}^{nc}_{x,k}$.

Here, we are going to find the minimum number of colors required for $2p$ distance coloring of T_3. To determine that, it is required to discuss the reusability of a color of D^{2p}_x in T_3. In the context of reusability of a color of D^{2p}_x, the following definition is given.

Definition 14 *For a vertex $v \in V(D^{2p}_x)$ and a set $S \subseteq V(T_3) \setminus V(D^{2p}_x)$, $R^S_v = \{u : d(u,v) \geq 2p+1, u \in S\}$ denotes the subset of vertices of S where $f(v)$ can be reused in $2p$ distance coloring.*

For example, consider Figure. 6.4 and the vertex $v = v^{c_1}_{x,7}$ in $\mathcal{F}^c_{x,6}$. Using equation (6.3), it can be shown that the set of vertices in $\mathcal{F}_{x,7}$ where $f(v)$ can be reused is $G^3_{x,7} \cup G^4_{x,7} \cup \{v^{c_5}_{x,7}\}$ for 12 distance coloring. Therefore $R^{\mathcal{F}_{x,7}}_v = G^3_{x,7} \cup G^4_{x,7} \cup \{v^{c_5}_{x,7}\}$.

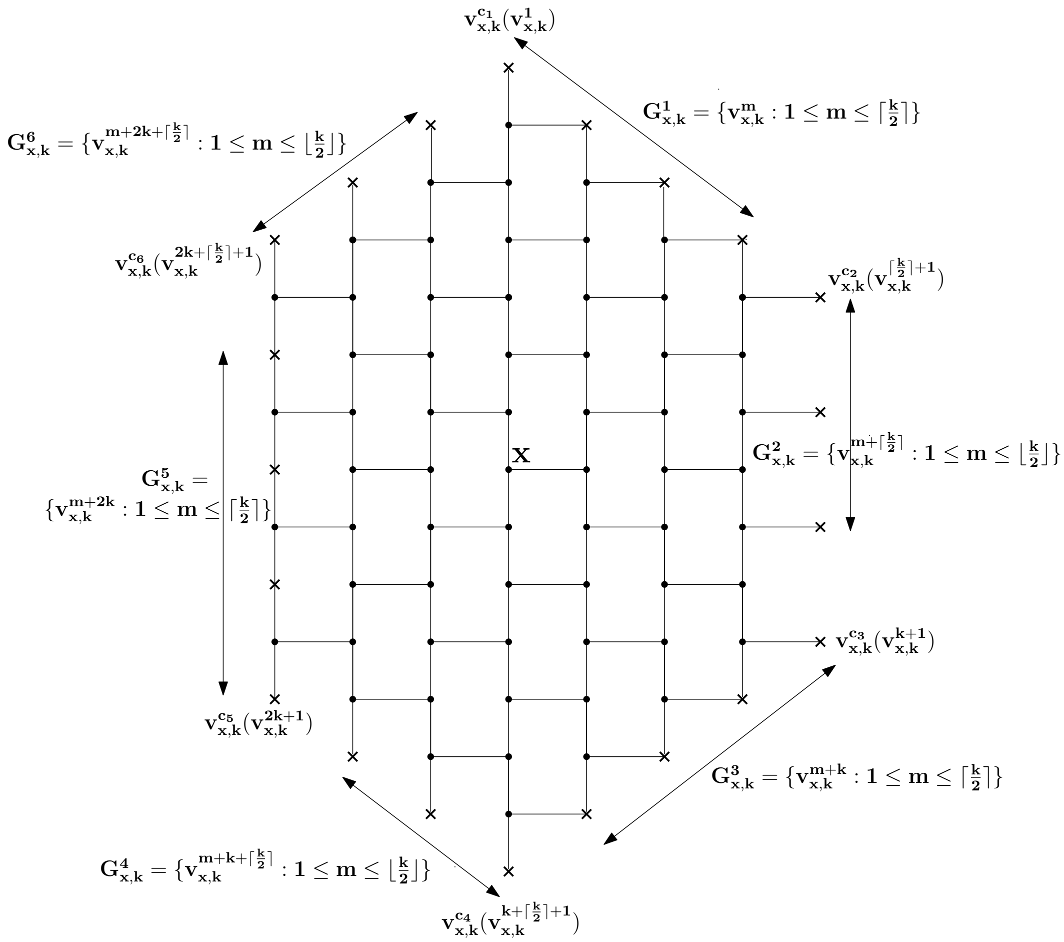


Figure 6.4: The vertices (Marked as $\times$) at distance $k = 7$ from x.

6.2.2 Results

In the following Observations and Theorems, we calculate the distance between two vertices by using equation (6.3). But in most of the cases the corresponding calculations have not been shown here for better readability. For the sake of simplicity we will assume the coordinates of x is $(0, 0)$. Below we state Observations 17 and 18 whose proofs are trivial and hence we omit them.

Observation 17 *The number of vertices in D_x^{2p} is $|D_x^{2p}| = 1 + \frac{3p(p+1)}{2}$.*

Observation 18 $1 + \frac{3p(p+1)}{2} + \lfloor \frac{p}{2} \rfloor = \left\lceil \frac{3}{8} \left(2p + \frac{4}{3} \right)^2 \right\rceil$.

Our proof technique is based on examining the reusability of the colors of the vertices of D_x^{2p} in $\bigcup_{q=1}^{2\lfloor \frac{p}{2} \rfloor -1} \mathcal{F}_{x,p+q}$ and then by checking the minimum number of vertices in $\bigcup_{q=1}^{2\lfloor \frac{p}{2} \rfloor -1} \mathcal{F}_{x,p+q}$ which can not be colored by the colors used in D_x^{2p}. Therefore some colors which are not used in D_x^{2p} must be used in $\bigcup_{q=1}^{2\lfloor \frac{p}{2} \rfloor -1} \mathcal{F}_{x,p+q}$. The minimum number of such extra colors which are not used in D_x^{2p} can be determined by evaluating maximum number of times a color can be used in $\bigcup_{q=1}^{2\lfloor \frac{p}{2} \rfloor -1} \mathcal{F}_{x,p+q}$. To do so, in Observation 19, we show how many times at maximum the color of a corner vertex in $\mathcal{F}^c_{x,p-q}$ can be reused in $\mathcal{F}_{x,p+q+1}$, where $q = 0, 1, \ldots, p-2$.

Observation 19 *For any vertex $v \in \mathcal{F}^c_{x,p-q}$, $f(v)$ can be reused at most twice in $\mathcal{F}_{x,p+q+1}$, where $q = 0, 1, \ldots, p-2$.*

Proof: Consider the vertex $v = v^{c_1}_{x,p-q}$ in $\mathcal{F}^c_{x,p-q}$. Let $k = p + q + 1$. Using equation (6.3), it can be shown that the set of vertices in $\mathcal{F}_{x,k}$ where $f(v)$ can be reused is given by $R_v^{\mathcal{F}_{x,k}} = G^3_{x,k} \cup G^4_{x,k} \cup \{v^{c_5}_{x,k}\}$. Let $R_v^{\mathcal{F}_{x,k}} = X \cup Y$ where $X = G^3_{x,k}$ and $Y = G^4_{x,k} \cup v^{c_5}_{x,k}$. Observe that $\nexists u_1,\ u_2 \in X$ such that $d(u_1, u_2) \geq 2p+1$ and $\nexists w_1,\ w_2 \in Y$ such that $d(w_1, w_2) \geq 2p+1$. Hence $f(v)$ can be reused at most twice in $X \cup Y$, once at a vertex u in X and once at a vertex v in Y, provided there exists $u \in X$ and $v \in Y$ such that $d(u, v) \geq 2p+1$. Similarly we can prove that for any other corner vertex $w \in \mathcal{F}^c_{x,p-q}$, $f(w)$ can be reused at most twice in $\mathcal{F}_{x,k}$. This completes the proof of Observation 19. □

In Observation 20, we will show how many times at maximum the color of a non corner vertex in $\mathcal{F}^{nc}_{x,p-q}$ can be reused in $\mathcal{F}_{x,p+q+1}$, where $q = 0, 1, \ldots, p-3$.

Observation 20 *For any vertex $v \in \mathcal{F}^{nc}_{x,p-q}$, $f(v)$ can be reused at most once in $\mathcal{F}_{x,p+q+1}$, where $q = 0, 1, \ldots, p-3$.*

Proof: Let us consider any non corner vertex $v \in G^6_{x,p-q}$. Let $k = p + q + 1$. Using equation (6.3), we can show that the set of vertices in $\mathcal{F}_{x,k}$ where $f(v)$ can be reused is given by $R_v^{\mathcal{F}_{x,k}} = G^3_{x,k} \cup \{v^{c_4}_{x,k}\}$. It can now be observed that $\nexists\ u_1, u_2 \in R_v^{\mathcal{F}_{x,k}}$

such that $d(u_1, u_2) \geq 2p+1$. That is, $f(v)$ can be reused only once in $\mathcal{F}_{x,k}$. For other non corner vertices in $G^1_{x,p-q}, G^2_{x,p-q}, G^3_{x,p-q}, G^4_{x,p-q}, G^5_{x,p-q}$ the same result can be proved considering $G^4_{x,k} \cup \{v^{c_5}_{x,k}\}, G^5_{x,k} \cup \{v^{c_6}_{x,k}\}, G^6_{x,k} \cup \{v^{c_1}_{x,k}\}, G^1_{x,k} \cup \{v^{c_2}_{x,k}\}, G^2_{x,k} \cup \{v^{c_3}_{x,k}\}$, respectively. This completes the proof of Observation 20. $\square$

Now we will check the reusability of the colors of D^{2p}_x in $\mathcal{F}_{x,p+1}$. In Theorem 6.2.1, we will show that reusing the colors of D^{2p}_x in $\mathcal{F}_{x,p+1}$, all the vertices of $\mathcal{F}_{x,p+1}$ can not be colored and a *new color* which is not used in D^{2p}_x is to be used in $\mathcal{F}_{x,p+1}$. Before going to Theorem 6.2.1, we formally state the notion of new color.

Consider two sets of vertices V_1 and V_2, where the colors used in V_1 are reused to color the vertices of V_2. Suppose there exists at least one vertex in V_2 which can not be colored by any of the colors used in V_1. In such case, we say that a new color which is not used in V_1 must be introduced to color the vertices of V_2.

Theorem 6.2.1 *At least a new color which is not used in D^{2p}_x must be introduced to color the vertices of $\mathcal{F}_{x,p+1}$ for $2p$ distance coloring.*

Proof: Note that $f(v_1) \neq f(v_2)$ where $v_1 \in \mathcal{F}_{x,p+1}, v_2 \in \mathcal{F}_{x,q}$ and $q = 0, 1, \ldots p-1$ for $2p$ distance coloring. So, colors of vertices of $\mathcal{F}_{x,p}$ can only be reused at the vertices of $\mathcal{F}_{x,p+1}$. Note that $|\mathcal{F}_{x,p}| = 3p$ and $|\mathcal{F}_{x,p+1}| = 3(p+1)$.

So colors of some vertices of $\mathcal{F}_{x,p}$ must be reused more than once in $\mathcal{F}_{x,p+1}$.

From Observation 19, we get that only the colors of the six corner vertices $v^{c_1}_{x,p}, v^{c_2}_{x,p}, \ldots, v^{c_6}_{x,p}$ of $\mathcal{F}_{x,p}$ can be reused twice in $\mathcal{F}_{x,p+1}$. The only possibility to reuse $f(v^{c_1}_{x,p})$ twice in $\mathcal{F}_{x,p+1}$ is to use $f(v^{c_1}_{x,p})$ in $v^{c_3}_{x,p+1}$ and $v^{c_5}_{x,p+1}$. That is, $f(v^{c_3}_{x,p+1}) = f(v^{c_1}_{x,p})$ and $f(v^{c_5}_{x,p+1}) = f(v^{c_1}_{x,p})$. Similarly, if $f(v^{c_2}_{x,p}), f(v^{c_3}_{x,p}), \ldots, f(v^{c_6}_{x,p})$ are to be reused twice in $\mathcal{F}_{x,p+1}$, then $f(v^{c_4}_{x,p+1}) = f(v^{c_6}_{x,p+1}) = f(v^{c_2}_{x,p})$; $f(v^{c_5}_{x,p+1}) = f(v^{c_1}_{x,p+1}) = f(v^{c_3}_{x,p})$; $f(v^{c_6}_{x,p+1}) = f(v^{c_2}_{x,p+1}) = f(v^{c_4}_{x,p})$; $f(v^{c_1}_{x,p+1}) = f(v^{c_3}_{x,p+1}) = f(v^{c_5}_{x,p})$; and $f(v^{c_2}_{x,p+1}) = f(v^{c_4}_{x,p+1}) = f(v^{c_6}_{x,p})$.

From Observation 20, we get that the color of any non corner vertex of $\mathcal{F}_{x,p}$ can be reused at most once in $\mathcal{F}_{x,p+1}$. So if the vertices of $\mathcal{F}_{x,p+1}$ are to be colored only with the colors used in $\mathcal{F}_{x,p}$, the colors of at least three corner vertices of $\mathcal{F}_{x,p}$ must be reused twice each in $\mathcal{F}_{x,p+1}$.

Note that any two of $f(v^{c_1}_{x,p})$, $f(v^{c_3}_{x,p})$ and $f(v^{c_5}_{x,p})$ can not be reused twice each simultaneously in $\mathcal{F}_{x,p+1}$ as in that case they must be reused in a common vertex in $\mathcal{F}_{x,p+1}$ which is not possible. Similarly, any two of $f(v^{c_2}_{x,p})$, $f(v^{c_4}_{x,p})$ and $f(v^{c_6}_{x,p})$ can not be reused twice each simultaneously in $\mathcal{F}_{x,p+1}$ as in that case too they must be reused in a common vertex in $\mathcal{F}_{x,p+1}$ which is also not possible. So the colors of at most two corner vertices of $\mathcal{F}_{x,p}$, one from $\{v^{c_1}_{x,p}, v^{c_3}_{x,p}, v^{c_5}_{x,p}\}$ and one from $\{v^{c_2}_{x,p}, v^{c_4}_{x,p}, v^{c_6}_{x,p}\}$ can be reused twice each in $\mathcal{F}_{x,p+1}$. In other words, at least one vertex remains uncolored in $\mathcal{F}_{x,p+1}$. Hence at least a new color which is not used in $V(D^{2p}_x)$ must be introduced in $\mathcal{F}_{x,p+1}$. $\square$

Now we will check the reusability of the colors of the vertices of D^{2p}_x in those vertices which are at distance $p+1$, $p+2$ and $p+3$ from x. In Theorem 6.2.2, we will show that reusing the colors of $D^{2p}_x \cup \mathcal{F}_{x,p+1}$ in $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$, all the vertices of $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$ can not be colored and a second new color which is not used in $D^{2p}_x \cup \mathcal{F}_{x,p+1}$ must be required to color the vertices of $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$. To prove the Theorem 6.2.2, we will consider vertices defined in Definition 15 and their reusability. In Lemma 6.2.1 and Lemma 6.2.2, we will give results of the reusability of such vertices and with the help of the results of these two Lemmas, we will prove Theorem 6.2.2. Now we formally state Definition 15.

Definition 15 *We define $\mathcal{S}^{2r}_{x,k}$ as the set of non corner vertices in $\mathcal{F}^{nc}_{x,k}$ which are at distance k from x and at distance $2r$ from a corner vertex in $\mathcal{F}^{c}_{x,k}$. That is, $\mathcal{S}^{2r}_{x,k} = \{v : v \in \mathcal{F}^{nc}_{x,k} \cap (G^1_{x,k} \cup G^3_{x,k} \cup G^5_{x,k}), \exists u \in \mathcal{F}^c_{x,k}, d(v,u) = 2r, 1 \leq r \leq \lceil \frac{k}{2} \rceil - 1\} \cup \{v : v \in \mathcal{F}^{nc}_{x,k} \cap (G^2_{x,k} \cup G^4_{x,k} \cup G^6_{x,k}), \exists u \in \mathcal{F}^c_{x,k}, d(v,u) = 2r, 1 \leq r \leq \lfloor \frac{k}{2} \rfloor - 1\}$, where $k \geq 5$.*

For $k = 4$, the conjecture has already been proved in 6.1, so we consider $k \geq 5$.

As an example, for $k = 7$, from Fig. 6.4, we get that $\mathcal{S}^2_{x,k} = \{ v^{3k}_{x,k}, v^2_{x,k}, v^{\lceil \frac{k}{2} \rceil}_{x,k}, v^{\lceil \frac{k}{2} \rceil+2}_{x,k}, v^k_{x,k}, v^{k+2}_{x,k}, v^{k+\lceil \frac{k}{2} \rceil}_{x,k}, v^{k+\lceil \frac{k}{2} \rceil+2}_{x,k}, v^{2k}_{x,k}, v^{2k+2}_{x,k}, v^{2k+\lceil \frac{k}{2} \rceil}_{x,k}, v^{2k+\lceil \frac{k}{2} \rceil+2}_{x,k} \}$.

Observe that a vertex $v \in \mathcal{S}^{2r}_{x,k}$ with respect to a corner vertex in $\mathcal{F}^c_{x,k}$ is also in $\mathcal{S}^{2(\frac{k}{2}-r)}_{x,k}$ with respect to another corner vertex in $\mathcal{F}^c_{x,k}$ if k is even and $r \in \{1, 2, \ldots, \frac{k}{2} - 1\}$. When k is odd, a vertex $v \in \mathcal{S}^{2r}_{x,k}$ with respect to a corner vertex in $\mathcal{F}^c_{x,k}$ is also in $\mathcal{S}^{2(\lceil \frac{k}{2} \rceil - r)}_{x,k}$, $r \in \{1, 2, \ldots, \lceil \frac{k}{2} \rceil - 1\}$ with respect to another corner vertex in $\mathcal{F}^c_{x,k}$ when $v \in G^1_{x,k} \cup G^3_{x,k} \cup G^5_{x,k}$. When $v \in G^2_{x,k} \cup G^4_{x,k} \cup G^6_{x,k}$ then $f(v) \in \mathcal{S}^{2(\lfloor \frac{k}{2} \rfloor - r)}_{x,k}$ and $r \in \{1, 2, \ldots, \lfloor \frac{k}{2} \rfloor - 1\}$. Here we consider x is a right vertex. For x to be a left vertex, corresponding fact can be concluded. For example, consider the Fig. 6.4 where $k = 7$. Now $v^2_{x,k} \in \mathcal{S}^2_{x,k}$ with respect to the corner vertex $v^{c_1}_{x,k}$ (here $r = 1$). Again $v^2_{x,k} \in \mathcal{S}^{2(\lceil \frac{k}{2} \rceil - r)}_{x,k} = \mathcal{S}^6_{x,k}$ with respect to the corner vertex $v^{c_2}_{x,k}$. Note that $v^2_{x,k} \in G^1_{x,k}$. Now consider the vertex $v^{3k}_{x,k} \in \mathcal{S}^2_{x,k}$. It is in $\mathcal{S}^2_{x,k}$ with respect to $v^{c_1}_{x,k}$ (here $r = 1$).

Again $v_{x,k}^{3k} \in \mathcal{S}_{x,k}^{2(\lfloor \frac{k}{2} \rfloor - r)} = \mathcal{S}_{x,k}^{4}$ with respect to the corner vertex $v_{x,k}^{c_6}$. Note that here $v_{x,k}^{3k} \in G_{x,k}^{6}$.

So, a vertex $v \in \mathcal{S}_{x,k}^{2r}$ with respect to a corner vertex in $\mathcal{F}_{x,k}^{c}$ is also in $\mathcal{S}_{x,k}^{2r'}$ where $r' = \frac{k}{2} - r$ for even k and $r' = \lfloor \frac{k}{2} \rfloor - r$ or $r' = \lceil \frac{k}{2} \rceil - r$ for odd k.

Now we have the following Lemma when $r \in \{1, 2, \ldots, \lfloor \frac{k}{4} \rfloor\}$.

Lemma 6.2.1 *For any $v \in \mathcal{S}_{x,p-q}^{2r}$, if $f(v)$ is reused once in $\mathcal{F}_{p+q+1}$ then $f(v)$ can be reused at most once more in $\mathcal{F}_{x,p+q+2r+1}$, where $q = 0, 1, \ldots, p-4$ and $r = 1, 2, \ldots, \lfloor \frac{p-q}{4} \rfloor$.*

Proof: Let us consider the vertex $v \in \mathcal{S}_{x,p-q}^{2r}$. For the sake of simplicity of argument, here we consider the vertex $v \in G_{x,p-q}^{6}$ and it is at distance $2r$ from $v_{x,p-q}^{c_1}$. We consider the coordinates of the vertex x as $(0,0)$. So the coordinates of the vertex v are $(-r, (p-q)-r) = (i_1, j_1)$. Here first we are going to determine in which set of vertices in $\mathcal{F}_{x,p+1} \cup \mathcal{F}_{x,p+2} \cup \cdots \cup \mathcal{F}_{x,p+q+2r+1}$, $f(v)$ can be reused. For this purpose, first we will consider $\mathcal{F}_{x,p+q+2r+1}$. Note that $\mathcal{F}_{x,p+q+2r+1} = G_{x,p+q+2r+1}^{1} \cup G_{x,p+q+2r+1}^{2} \cup \cdots \cup G_{x,p+q+2r+1}^{6}$. Now we will check the reusability of $f(v)$ in $G_{x,p+q+2r+1}^{1} \cup G_{x,p+q+2r+1}^{2} \cup \cdots \cup G_{x,p+q+2r+1}^{6}$.

Consider a vertex $u \in G_{x,p+q+2r+1}^{1}$. The coordinates of the vertex u is $(m_1 - 1, (p+q+2r+1) - m_1 + 1) = (i_2, j_2)$ where $1 \le m_1 \le \lceil \frac{p+q+2r+1}{2} \rceil$. Observe that $i_1 < i_2$. Therefore $i_1 - i_2 = (-r) - (m_1 - 1) = -(m_1 + r - 1)$. Note that $m_1 + r - 1 \ge 1$ as $m_1 \ge 1$ and $r \ge 1$. So, $|i_1 - i_2| = m_1 - 1 + r$. Observe that $j_1 - j_2 = ((p-q) - r) - ((p+q+2r+1) - m_1 + 1) = m_1 - 2q - 3r - 2$. Now $m_1 - 2q - 3r - 2 \le 0$ or $m_1 - 2q - 3r - 2 > 0$. So we have the following two cases.

- Case 1: $j_1 - j_2 = m_1 - 2q - 3r - 2 \le 0$: In this case $|j_1 - j_2| = -(j_1 - j_2) = (2q + 3r + 2 - m_1)$. One of the following two cases may happen.

  - **When $|\mathbf{i_1} - \mathbf{i_2}| \le |\mathbf{j_1} - \mathbf{j_2}|$**: From equation (6.3), $d(v,u) = |i_1 - i_2| + |j_1 - j_2|$.

    $d(v,u) = |i_1 - i_2| + |j_1 - j_2|$
    $\Rightarrow d(v,u) = (m_1 - 1 + r) + (2q + 3r + 2 - m_1) = 2q + 4r + 1$
    $\Rightarrow d(v,u) = 2q + 4r + 1 \le 2q + 4\lfloor \frac{p-q}{4} \rfloor + 1$ as $r \le \lfloor \frac{p-q}{4} \rfloor$
    $\Rightarrow d(v,u) \le 2q + 4\lfloor \frac{p-q}{4} \rfloor + 1 \le 2q + 4\frac{(p-q)}{4} + 1 = p + q + 1$
    $\Rightarrow d(v,u) \le p + q + 1 < 2p + 1$ as $q < p$

- **When $|\mathbf{i_1} - \mathbf{i_2}| > |\mathbf{j_1} - \mathbf{j_2}|$:** From equation (6.3), $d(v,u) = 2|i_1 - i_2| + \tau(v) - \tau(u)$.

  $d(v,u) = 2|i_1 - i_2| + \tau(v) - \tau(u)$
  $\Rightarrow d(v,u) = 2(m_1 - 1 + r) + \tau(v) - \tau(u)$ as $|i_1 - i_2| = (m_1 - 1 + r)$
  $\Rightarrow d(v,u) \leq 2(\lceil \frac{p+q+2r+1}{2} \rceil - 1 + r) + \tau(v) - \tau(u)$ as $m_1 \leq \lceil \frac{p+q+2r+1}{2} \rceil$

  There are the following two sub cases.

  1. **When $\mathbf{p + q + 2r + 1}$ is even:**

     $\Rightarrow d(v,u) \leq 2(\frac{p+q+2r+1}{2} - 1 + r) + \tau(v) - \tau(u)$
     $\Rightarrow d(v,u) \leq (p + q + 2r + 1 - 2 + 2r) + 1 \; as \; \tau(v) - \tau(u) \leq 1$
     $\Rightarrow d(v,u) \leq p + q + 4r$
     $\Rightarrow d(v,u) \leq p + q + 4(\lceil \frac{p-q}{4} \rceil)$ as $r \leq \lceil \frac{p-q}{4} \rceil$
     $\Rightarrow d(v,u) \leq p + q + 4\frac{(p-q)}{4} < 2p$
  2. **When $\mathbf{p + q + 2r + 1}$ is odd:**

     $p + q$ is even and $p - q$ is also even
     $\tau(v) = (i_1 + j_1) \bmod 2 = (p - q - 2r) \bmod 2 = 0$
     $\tau(u) = (i_2 + j_2) \bmod 2 = (p + q + 2r + 1) \bmod 2 = 1$
     $d(v,u) = 2|i_1 - i_2| + \tau(v) - \tau(u)$
     $\Rightarrow d(v,u) = 2(m_1 - 1 + r) - \tau(u) + \tau(v) = 2(m_1 - 1 + r) - 1 = 2m_1 + 2r - 3$
     $\Rightarrow d(v,u) = 2(\frac{p+q+2r+1}{2} + \frac{1}{2}) + 2r - 3 = p + q + 4r - 1$
     $\Rightarrow d(v,u) \leq p + q + 4(\lfloor \frac{p-q}{4} \rfloor) - 1$
     $\Rightarrow d(v,u) \leq p + q + 4(\frac{p-q}{4}) - 1 = 2p - 1 < 2p$

- Case 2. When $j_1 - j_2 = m_1 - 2q - 3r - 2 > 0$: same result can be concluded.

So, $f(v)$ can not be used at any vertex $u \in G^1_{x,p+q+2r+1}$.

Now we will consider a vertex $u \in G^2_{x,p+q+2r+1}$. The coordinates of u is $(\lceil \frac{p+q+2r+1}{2} \rceil, \lfloor \frac{p+q+2r+1}{2} \rfloor - 2m_1 + 2) = (i_2, j_2)$ where $1 \leq m_1 \leq \lfloor \frac{p+q+2r+1}{2} \rfloor$. Proceeding similar algebraic methods, we can show that for vertices $u \in G^2_{x,p+q+2r+1}$ with $\frac{p+q+1}{2} \leq m_1 \leq \lfloor \frac{p+q+2r+1}{2} \rfloor$, $f(v)$ can be reused at a vertex in $G^2_{x,p+q+2r+1}$.

With similar algebraic argument, it can be shown that $f(v)$ can be reused in $G^3_{x,p+q+2r+1} \cup G^4_{x,p+q+2r+1}$. In $G^5_{x,p+q+2r+1}$, the only vertex where $f(v)$ can be reused is $v^{c_5}_{x,p+q+2r+1}$. Note that $f(v)$ can not be reused in $G^6_{x,p+q+2r+1}$.

With the similar methods using equation (6.3), we can show that in $\mathcal{F}_{x,p+q+2r}$, $f(v)$ can be reused in $\{v^{m_1+\lceil\frac{p+q+2r}{2}\rceil}_{x,p+q+2r} : \frac{p+q+3}{2} \le m_1 \le \lfloor\frac{p+q+2r}{2}\rfloor\} \cup G^3_{x,p+q+2r} \cup \{v^{m_2+p+q+2r+\lceil\frac{p+q+2r}{2}\rceil}_{x,p+q+2r} : 1 \le m_2 \le \lfloor\frac{p+q+2r}{2}\rfloor - \lceil\frac{p+q+3}{2}\rceil + 1\}$. In $\mathcal{F}_{x,p+q+2r-1}$, $f(v)$ can be reused in $\{v^{m_1+\lceil\frac{p+q+2r-1}{2}\rceil}_{x,p+q+2r-1} : \frac{p+q+1}{2} \le m_1 \le \lfloor\frac{p+q+2r-1}{2}\rfloor\} \cup G^3_{x,p+q+2r-1} \cup \{v^{m_2+p+q+2r-1+\lceil\frac{p+q+2r-1}{2}\rceil}_{x,p+q+2r-1} : 1 \le m_2 \le \lfloor\frac{p+q+2r-1}{2}\rfloor - \lceil\frac{p+q+1}{2}\rceil + 1\}$ and so on. Note that in $\mathcal{F}_{x,p+q+1}$, $f(v)$ can be reused in $G^3_{x,p+q+1} \cup \{v^{c_4}_{x,p+q+1}\}$ provided the vertex is at least at distance $2p+1$ from v.

It has been assumed that $f(v)$ is used in $F_{x,p+q+1}$. Therefore, $f(v)$ must be reused in $G^3_{x,p+q+1} \cup \{v^{c_4}_{x,p+q+1}\}$. Using similar algebraic process using equation (6.3), it can be shown that $f(v)$ can be reused at most once again at $u_1 \in G^4_{x,p+q+2r+1} \cup \{v^{c_5}_{x,p+q+2r+1}\} \setminus \{v^{c_4}_{x,p+q+2r+1}\}$.

The statement of Lemma 6.2.1 can be proved similarly for any other vertex $v \in \mathcal{S}^{2r}_{x,p-q}$ where $q = 0, 1, \ldots, p-4$ and $r = 1, 2, \ldots, \lfloor\frac{p-q}{4}\rfloor$. This completes the proof of Lemma 6.2.1. $\square$

Before going to next Lemma, we will now state the notion that the colors used in V_1 lose their reusability in V_2 by n_a times. Let $V_1 = \{v_1, v_2, \ldots, v_p\}$. Suppose that individually $f(v_1), f(v_2), \ldots, f(v_p)$ can be reused at most $n_1, n_2, \ldots, n_p$ times in V_2 respectively. If altogether the colors of V_1 are reused at most n_b times in V_2 then we say that the colors of V_1 lose their reusability by at least $(n_1 + n_2 + \cdots + n_p) - n_b = n_a$ times.

Lemma 6.2.2 *If the colors used in $\mathcal{F}_{x,p}$, $\mathcal{F}_{x,p-1}$ and $\mathcal{F}_{x,p-2}$ is reused in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$, $\bigcup_{q=2}^{3} \mathcal{F}_{x,p+q}$ and $\mathcal{F}_{x,p+3}$ respectively then colors of at least 6 vertices of $\mathcal{F}^c_{x,p-2} \cup \mathcal{S}^2_{x,p}$ lose their reusability in $\mathcal{F}_{x,p+3}$ by at least 6.*

Proof: We consider the reusability of the colors of $\bigcup_{r=0}^{p} \mathcal{F}_{x,r}$ in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$. Note that the colors used in $\bigcup_{r=0}^{p-3} \mathcal{F}_{x,r}$ can not be reused in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$ as $d(u_1, u_2) < 2p+1$

$\forall u_1 \in \bigcup\limits_{r=0}^{p-3} \mathcal{F}_{x,r}$ and $\forall u_2 \in \bigcup\limits_{q=1}^{3} \mathcal{F}_{x,p+q}$. Again note that the colors used in $\mathcal{F}_{x,p}$, $\mathcal{F}_{x,p-1}$ and $\mathcal{F}_{x,p-2}$ can be reused in $\bigcup\limits_{q=1}^{3} \mathcal{F}_{x,p+q}$, $\bigcup\limits_{q=2}^{3} \mathcal{F}_{x,p+q}$ and $\mathcal{F}_{x,p+3}$ respectively.

First consider that the colors of 2 corner vertices of $\mathcal{F}^c_{x,p}$ are reused twice each simultaneously in $\mathcal{F}_{x,p+1}$. From the proof of Lemma 6.2.1, the color of each vertex of $\mathcal{S}^2_{x,p}$ may be reused once more in $\mathcal{F}_{x,p+3}$ even after using once in $\mathcal{F}_{x,p+1}$. Again observe that color of any vertex of $\bigcup\limits_{q=0}^{2} \mathcal{F}^{nc}_{x,p-q} \setminus \mathcal{S}^2_{x,p}$ can not be reused any more in $\bigcup\limits_{r=1}^{3} \mathcal{F}_{x,p+r}$ when colors of vertices of $\mathcal{F}^{nc}_{x,p}$, $\mathcal{F}^{nc}_{x,p-1}$ and $\mathcal{F}^{nc}_{x,p-2}$ are reused in $\mathcal{F}_{x,p+1}$, $\bigcup\limits_{q=2}^{3} \mathcal{F}_{x,p+q}$ and $\mathcal{F}_{x,p+3}$ respectively.

From Lemma 6.2.1, if $f(v)$ ($v \in \mathcal{S}^2_{x,p}$), is reused once in $u_1 \in \mathcal{F}_{x,p+1}$ then it can be reused once more in $u_2 \in \mathcal{F}_{x,p+3}$. If $u_1 \in \mathcal{F}^c_{x,p+1}$ then $u_2 \in \mathcal{F}^c_{x,p+3} \cup \mathcal{S}^2_{x,p+3}$ and if $u_1 \in \mathcal{S}^2_{x,p+1}$ then $u_2 \in \mathcal{F}^c_{x,p+3}$ (It can be proved by equation (6.3)). From the discussion of Theorem 6.2.1, 4 vertices of $\mathcal{F}^c_{x,p+1}$ are already colored by the colors of 2 corner vertices of $\mathcal{F}^c_{x,p}$ as we consider colors of 2 corner vertices of $\mathcal{F}^c_{x,p}$ is reused twice each simultaneously in $\mathcal{F}_{x,p+1}$. Hence in the remaining 2 corner vertices of $\mathcal{F}^c_{x,p+1}$, colors of at most 2 vertices of $\mathcal{S}^2_{x,p}$ may be reused. Colors of these two vertices may be reused once each in two vertices in $\mathcal{S}^2_{x,p+3}$. But the colors of the at least 10 remaining vertices of $\mathcal{S}^2_{x,p}$ must be reused once each in $\mathcal{S}^2_{x,p+1}$ and once each in $\mathcal{F}^c_{x,p+3}$. But there are only 6 vertices in $\mathcal{F}^c_{x,p+3}$. So colors of at least $(10-6) = 4$ vertices of $\mathcal{S}^2_{x,p}$ can not be reused each in $\mathcal{F}_{x,p+3}$. In this case, 2 vertices of $\mathcal{S}^2_{x,p+3}$ and 6 vertices of $\mathcal{F}^c_{x,p+3}$ may be colored by the colors of the vertices of $\mathcal{S}^2_{x,p}$. Hence the remaining 10 vertices of $\mathcal{S}^2_{x,p+3}$ are yet to be considered.

Note that, if the color of a corner vertex of $\mathcal{F}^c_{x,p-2}$ is to be reused in two vertices $u_1, u_2 \in \mathcal{F}_{x,p+3}$, then there are two possible cases to consider. *a*. color of a corner vertex of $\mathcal{F}^c_{x,p-2}$ is reused at $u_1 \in \mathcal{S}^4_{x,p+3}$ and in that case $u_2 \in \mathcal{F}^c_{x,p+3}$ and *b*. color of a corner vertex of $\mathcal{F}^c_{x,p-2}$ is not reused at $u_1 \in \mathcal{S}^4_{x,p+3}$ and in that case $u_1 \in \mathcal{F}^c_{x,p+3} \cup \mathcal{S}^2_{x,p+3}$ and $u_2 \in \mathcal{F}^c_{x,p+3} \cup \mathcal{S}^2_{x,p+3}$. But the only possibility that remains here is to use the color of a vertex of $\mathcal{F}^c_{x,p-2}$ in $\mathcal{S}^2_{x,p+3}$ as the vertices of $\mathcal{F}^c_{x,p+3}$ have already been colored by the colors of the vertices of $\mathcal{S}^2_{x,p}$. Since there are at most 10 vertices left in $\mathcal{S}^2_{x,p+3}$ for coloring, only colors of at most 5 vertices of $\mathcal{F}^c_{x,p-2}$ can be reused twice each in $\mathcal{F}_{x,p+3}$ and in that case color of a vertex of $\mathcal{F}^c_{x,p-2}$ can not be reused in $\mathcal{F}_{x,p+3}$. So, color of a corner vertex in $\mathcal{F}^c_{x,p-2}$ lose the reusability in $\mathcal{F}_{x,p+3}$ by 2 and colors of 4 vertices in $\mathcal{S}^2_{x,p}$ lose all together the reusability by 4 in $\mathcal{F}_{x,p+3}$.

Proceeding similarly, in all other cases, when the color of a corner vertex in $\mathcal{F}^c_{x,p}$ has been used two times in $\mathcal{F}_{x,p+1}$ or color of no corner vertex in $\mathcal{F}^c_{x,p}$ has been used two times in $\mathcal{F}_{x,p+1}$, it can be shown that colors of 6 vertices of $\mathcal{F}^c_{x,p-2} \cup \mathcal{S}^2_{x,p}$ lose their reusability in $\mathcal{F}_{x,p+3}$ by at least 6. This completes the proof of Lemma 6.2.2. $\square$

Now we will prove the following Theorem.

Theorem 6.2.2 *A second new color which is not used in $V(D^{2p}_x) \cup \mathcal{F}_{x,p+1}$ must be required to color the vertices of $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$ for $2p$ distance coloring.*

Proof: Here we consider the reuse of the colors of $\mathcal{F}_{x,p}$, $\mathcal{F}_{x,p-1}$ and $\mathcal{F}_{x,p-2}$ in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$, $\bigcup_{q=2}^{3} \mathcal{F}_{x,p+q}$ and $\mathcal{F}_{x,p+3}$ respectively.

From the discussion of Theorem 6.2.1, colors of at most 2 corner vertices of $\mathcal{F}^c_{x,p}$ can be reused twice each simultaneously in $\mathcal{F}_{x,p+1}$. From Observation 20, the color of a non corner vertex in $\mathcal{F}_{x,p}$ can be reused at most once in $\mathcal{F}_{x,p+1}$. Note that there are 3 more vertices in $\mathcal{F}_{x,p+1}$ than that of in $\mathcal{F}_{x,p}$ and colors of at most 2 corner vertices may be reused twice each in $\mathcal{F}_{x,p+1}$. So, there exists at least a vertex $u \in \mathcal{F}_{x,p+1}$ where a color which is not used in D^{2p}_x must be used. Since there are 3 more vertices in $\mathcal{F}_{x,p+1}$ than that of in $\mathcal{F}_{x,p}$ and colors of 2 corner vertices are reused twice each in $\mathcal{F}_{x,p+1}$, if color of any non corner vertex in $\mathcal{F}_{x,p}$ is not reused in $\mathcal{F}_{x,p+1}$, then there must exist another vertex other than u which can not be colored with the colors used in $V(D^{2p}_x)$. In that case there exists at least two vertices in $\mathcal{F}_{x,p+1}$ where the colors used in D^{2p}_x can not be used.

Note that there are various ways of reusing the colors used in $\mathcal{F}_{x,p}$ into $\mathcal{F}_{x,p+1}$. Among them, the only way where the new color is used exactly once in $\mathcal{F}_{x,p+1}$ is when the colors of two corner vertices of $\mathcal{F}_{x,p}$ are reused twice each in $\mathcal{F}_{x,p+1}$ and the colors of the other vertices of $\mathcal{F}_{x,p}$ are reused once each in $\mathcal{F}_{x,p+1}$. Note that the new color can be used at a corner vertex or at a non corner vertex in $\mathcal{F}_{x,p+1}$. First assume it is used at a corner vertex in $\mathcal{F}_{x,p+1}$ and without loss of generality consider it is used at $v^{c_1}_{x,p+1}$. Then by equation (6.3), for $p \geq 5$, it can be shown that the new color can be reused again in $(G^3_{x,p+1} \cup G^4_{x,p+1} \cup \{v^{c_5}_{x,p+1}\}) \cup (\{v^{p+2}_{x,p+2}\} \cup G^3_{x,p+2} \cup$

$G^4_{x,p+2} \cup \{v^{c_5}_{x,p+2}, v^{2(p+2)+2}_{x,p+2}\}) \cup (\{v^{p+3}_{x,p+3}\} \cup G^3_{x,p+3} \cup G^4_{x,p+3} \cup \{v^{c_5}_{x,p+3}, v^{2(p+3)+2}_{x,p+3}\}) = X \cup Y$, where $X = G^3_{x,p+1} \cup (\{v^{p+2}_{x,p+2}\} \cup G^3_{x,p+2}) \cup (\{v^{p+3}_{x,p+3}\} \cup G^3_{x,p+3})$ and $Y = (G^4_{x,p+1} \cup \{v^{c_5}_{x,p+1} \cup G^4_{x,p+2} \cup \{v^{c_5}_{x,p+2}, v^{2(p+2)+2}_{x,p+2}\}) \cup (G^4_{x,p+3} \cup \{v^{c_5}_{x,p+3}, v^{2(p+3)+2}_{x,p+3}\})$ are two disjoint sets of vertices. It can be shown by equation (6.3) that the new color can be reused again at most once in X and at most once in Y. Similarly if the new color is used in a non corner vertex in $\mathcal{F}_{x,p+1}$, then with similar argument stated in Lemma 6.2.1, it can be shown that the new color can be reused again at most 2 times in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$. So a new color can be used at most 3 times in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$ provided it has been used once in $\mathcal{F}_{x,p+1}$. Note that this result does not hold for $p = 4$ as the new color can be reused more that 3 times in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$ even if it has been used once in $\mathcal{F}_{x,p+1}$.

Since our proof is based on the number of vertices of $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$ which can not be colored by the colors used in $V(D^{2p}_x)$, we can assume that color of each non corner vertex of $\mathcal{F}_{x,p}$ is reused once in $\mathcal{F}_{x,p+1}$, colors of 2 corner vertices of $\mathcal{F}_{x,p}$ are reused twice each in $\mathcal{F}_{x,p+1}$ and colors of each of the remaining 4 corner vertices of $\mathcal{F}_{x,p}$ are used only once each in $\mathcal{F}_{x,p+1}$. If any of them is not reused in $\mathcal{F}_{x,p+1}$, it may color one more vertex in $\bigcup_{q=2}^{3} \mathcal{F}_{x,p+q}$, but at the same time one more vertex in $\mathcal{F}_{x,p+1}$ must be colored with a new color. As we are counting uncolored vertices in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$, there is no benefit of not reusing any of them in $\mathcal{F}_{x,p+1}$.

From the proof of Lemma 6.2.1, the color of each vertex of $\mathcal{S}^2_{x,p}$ may be reused once more in $\mathcal{F}_{x,p+3}$ even after using once in $\mathcal{F}_{x,p+1}$. Again observe that color of any vertex of $\bigcup_{q=0}^{2} \mathcal{F}^{nc}_{x,p-q} \setminus \mathcal{S}^2_{x,p}$ can not be reused any more in $\bigcup_{r=1}^{3} \mathcal{F}_{x,p+r}$ when colors of vertices of $\mathcal{F}^{nc}_{x,p}$, $\mathcal{F}^{nc}_{x,p-1}$ and $\mathcal{F}^{nc}_{x,p-2}$ are reused in $\mathcal{F}_{x,p+1}$, $\bigcup_{q=2}^{3} \mathcal{F}_{x,p+q}$ and $\mathcal{F}_{x,p+3}$ respectively.

Note that $\mathcal{F}^c_{x,p}$ has 6 vertices. Let S be the subset of vertices of $\mathcal{F}^c_{x,p}$ such that for every $v \in S$, $f(v)$ has not been reused twice in $\mathcal{F}_{x,p+1}$. Since as per our assumption, colors of 2 corner vertices have already been reused twice each in $\mathcal{F}_{x,p+1}$, there are $(6-2)=4$ vertices in S, each of which may be reused once more in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$. Note that there are $3(p+2)+3(p+3)=6p+15$ many vertices in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$ which may be colored by the colors used in $\mathcal{F}_{x,p-1}\cup\mathcal{F}_{x,p-2}\cup\mathcal{S}^2_{x,p}\cup S$.

Observe that the colors of the $(3(p-1)-6)+(3(p-2)-6)=6p-21$ many non corner vertices in $\mathcal{F}_{x,p-1}\cup\mathcal{F}_{x,p-2}$ may be reused at most once each in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$. So, using the colors of them together, we can color at most $6p-21$ vertices of $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$. From Lemma 6.2.2, colors of 6 vertices of $\mathcal{F}^c_{x,p-2}\cup\mathcal{S}^2_{x,p}$ lose their reusability in $\mathcal{F}_{x,p+3}$ by at least 6.

Again the colors of 6 corner vertices of $\mathcal{F}^c_{x,p-1}$ can be reused at most twice each in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$. So their colors can be reused together to color at most 12 vertices in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$. Therefore all together using the colors used in $\mathcal{F}_{x,p-1}\cup\mathcal{F}_{x,p-2}\cup\mathcal{S}^2_{x,p}$, at most $(3(p-1)-6)\times 1+6\times 2+(3(p-2)-6)\times 1+6\times 2+12\times 1-6=6p+9$ vertices in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$ can be colored. Now there are $(6p+15)-(6p+9)=6$ more vertices of $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$ which are yet to be colored. These 6 vertices may be colored by the colors used in the 4 vertices of S.

Since the colors of two corner vertices of $\mathcal{F}^c_{x,p}$ have already been reused in the 4 corner vertices of $\mathcal{F}_{x,p+1}$, there are only 2 remaining corner vertices in $\mathcal{F}_{x,p+1}$ where colors of S may be reused. If $f(v)$ ($v\in S$) is reused in a corner vertex of $\mathcal{F}^c_{x,p+1}$ then $f(v)$ may be reused once again in $\mathcal{F}^c_{x,p+3}\cup\mathcal{S}^2_{x,p+3}$ or $\mathcal{F}^c_{x,p+2}\cup\mathcal{S}^2_{x,p+2}$. But the vertices of $\mathcal{F}^c_{x,p+3}\cup\mathcal{S}^2_{x,p+3}$ have already been colored. So here we consider the reusability of $f(v)$ in $\mathcal{F}^c_{x,p+2}\cup\mathcal{S}^2_{x,p+2}$. If $f(v)$ is reused in a non corner vertex in $\mathcal{F}_{x,p+1}$ then it may only be reused in a corner vertex of $\mathcal{F}_{x,p+2}$. But if the colors of the six vertices of $\mathcal{F}^c_{x,p-1}$ are to be reused twice each in $\mathcal{F}_{x,p+2}$, then each of them have to be reused once in a corner vertex of $\mathcal{F}^c_{x,p+2}$. So using the colors of the 6 corner vertices of $\mathcal{F}^c_{x,p-1}$, 6 corner vertices of $\mathcal{F}^c_{x,p+2}$ have been colored. So the only possibility remaining here is to reuse $f(v)$ in a corner vertex of $\mathcal{F}^c_{x,p+1}$. But as per above discussion, there may be at most 2 corner vertices remaining in $\mathcal{F}_{x,p+1}$ where $f(v)$ can be reused. So, at most 2 of the 4 vertices of S can be reused once each in $\mathcal{F}_{x,p+2}$. Therefore at least $6-2=4$ vertices in $\mathcal{F}_{p+2}\cup\mathcal{F}_{p+3}$ can not be colored by the colors used in D^{2p}_x. Note that the new color used in $\mathcal{F}_{x,p+1}$ can be reused at most twice more in $\mathcal{F}_{x,p+1}\cup\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$. Hence to color the remaining $(4-2)=2$ vertices in $\mathcal{F}_{x,p+2}\cup\mathcal{F}_{x,p+3}$, a second new color must be used. It completes the proof of the Theorem 6.2.2. $\square$

From the discussion of Theorem 6.2.2, it is observed that the vertices of $\mathcal{F}^c_{x,p+1} \cup \mathcal{S}^2_{x,p+1} \cup \mathcal{F}^c_{x,p+3} \cup \mathcal{S}^2_{x,p+3}$ have been colored by the colors of the vertices of $\mathcal{F}^c_{x,p-2} \cup \mathcal{S}^2_{x,p} \cup \mathcal{F}^c_{x,p}$.

Let us consider colors used in $\mathcal{F}_{x,p}$, $\mathcal{F}_{x,p-1}$ and $\mathcal{F}_{x,p-2}$ have been reused in $\bigcup_{q=1}^{3} \mathcal{F}_{x,p+q}$, $\bigcup_{q=2}^{3} \mathcal{F}_{x,p+q}$ and $\mathcal{F}_{x,p+3}$ respectively with their maximum reusability. Then the vertices of $\mathcal{F}^c_{x,p+1}$, $\mathcal{S}^2_{x,p+1}$, $\mathcal{F}^c_{x,p+3}$ and $\mathcal{S}^2_{x,p+3}$ have been colored as the way discussed in Lemma 6.2.2 and Theorem 6.2.2. Vertices of $\mathcal{F}^c_{x,p+1} \cup \mathcal{S}^2_{x,p+1}$ have been colored by the colors of the vertices of $\mathcal{F}^c_{x,p} \cup \mathcal{S}^2_{x,p}$. Vertices of $\mathcal{F}^c_{x,p+3} \cup \mathcal{S}^2_{x,p+3}$ have been colored with the colors of the vertices $\mathcal{F}^c_{x,p} \cup \mathcal{S}^2_{x,p} \cup \mathcal{F}^c_{x,p-2}$. Now consider the coloring of the vertices of $\mathcal{F}_{x,p+5}$. From Lemma 6.2.1, note that the colors used in $\mathcal{S}^4_{x,p}$ and $\mathcal{S}^2_{x,p-2}$, if reused once each in $\mathcal{F}_{x,p+1}$ and in $\mathcal{F}_{x,p+3}$ respectively, then can be reused again in $\mathcal{F}_{x,p+5}$. As the colors of the vertices of $\mathcal{S}^4_{x,p}$ can not be reused in $\mathcal{F}^c_{x,p+1} \cup \mathcal{S}^2_{x,p+1}$, they are to be reused once in $\mathcal{S}^4_{x,p+1}$ and once in $\mathcal{F}^c_{x,p+5}$. As there are 12 vertices in $\mathcal{S}^4_{x,p}$ and only 6 vertices are there in $\mathcal{F}^c_{x,p+5}$, colors of at least 6 vertices of $\mathcal{S}^4_{x,p}$ can not reused in $\mathcal{F}^c_{x,p+5}$. The colors of the vertices of $\mathcal{S}^2_{x,p-2}$ may be reused at most once more in $\mathcal{F}_{x,p+5}$ if they are used in $\mathcal{F}_{x,p+3}$. But their colors can not be used in $\mathcal{F}^c_{x,p+3} \cup \mathcal{S}^2_{x,p+3}$. In that case if the color of such a vertex, if is to be reused once in $\mathcal{F}_{x,p+3}$ and once in $\mathcal{F}_{x,p+5}$, then it must be reused once in $\mathcal{S}^4_{x,p+3}$ and once in $\mathcal{S}^2_{x,p+5}$ (it can be proved using equation (6.3)). Considering this, if color of a vertex of $\mathcal{F}^c_{x,p-4}$ is to be reused twice in $\mathcal{F}_{x,p+5}$, then it must be reused twice in $\mathcal{S}^4_{x,p+5}$. So reusing the colors of 6 vertices of $\mathcal{F}^c_{x,p-4}$ twice each in $\mathcal{F}_{x,p+5}$, all the 12 vertices of $\mathcal{S}^4_{x,p+5}$ have been colored. Proceeding similarly, it can be shown that for the coloring of the vertices of $\mathcal{F}_{x,p+1} \cup \mathcal{F}_{x,p+2} \cup \cdots \cup \mathcal{F}_{x,h}$ for each of $h = 3, 5, \ldots, 2r+1$ and $r \leq \lfloor \frac{p}{2} \rfloor - 1$, the vertices of $\mathcal{F}^c_{x,p+3} \cup \{ \bigcup_{h'=1}^{(3-1)/2} \mathcal{S}^{2h'}_{x,p+3} \}$, $\mathcal{F}^c_{x,p+5} \cup \{ \bigcup_{h'=1}^{(5-1)/2} \mathcal{S}^{2h'}_{x,p+5} \}, \cdots,$ $\mathcal{F}^c_{x,p+h} \cup \{ \bigcup_{h'=1}^{(h-1)/2} \mathcal{S}^{2h'}_{x,p+h} \}$ have been colored. Proceeding similarly, it can be shown also that colors of some vertices of D^{2p}_x lose the reusability by at least 6 vertices in $\mathcal{F}_{x,p+1} \cup \mathcal{F}_{x,p+2} \cup \cdots \cup \mathcal{F}_{x,h}$ for each of $h = 3, 5, \ldots, 2r+1$ and $r \leq \lfloor \frac{p}{2} \rfloor - 1$. This discussion will be used to prove the next Theorem. In next Theorem, we will prove that $\lfloor \frac{p}{2} \rfloor$ th new color must be used to color the vertices of $\mathcal{F}_{x,p+2\lfloor \frac{p}{2} \rfloor - 1}$ for $2p$ distance coloring.

From the discussion of the proof of Lemma 6.2.1 it follows that for a vertex $v \in \mathcal{S}_{x,p}^{2}$, if $f(v)$ is reused once in $\mathcal{F}_{x,p+1}$, then it can be reused at most once more in $\mathcal{F}_{x,p+3}$ and it can not be reused in $\mathcal{F}_{x,p+2}$ any more. In other words, $f(v)$ can be reused at most once more in $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$ in this case. Now if $f(v)$ is not reused in $\mathcal{F}_{x,p+1}$, then it can be shown using similar argument stated in Lemma 6.2.1 that $f(v)$ can be reused at most twice in $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$. So regardless of whether $f(v)$ is reused in $\mathcal{F}_{x,p+1}$ or not, $f(v)$ can reused at most twice in $\mathcal{F}_{x,p+1} \cup \mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$. Similar result holds in all possible cases of reusing of the colors of $\mathcal{F}_{x,p-h} \cup \mathcal{F}_{x,p-h+1} \cup \cdots \cup \mathcal{F}_{x,p}$ in $\mathcal{F}_{x,p+1} \cup \mathcal{F}_{x,p+2} \cup \cdots \cup \mathcal{F}_{x,p+h+1}$. So minimum number of colors required remain unchanged no matter $f(v)$, where $v \in \mathcal{F}_{x,p-h}$, has been used in $\mathcal{F}_{x,p+h+1}$ or not. So for the sake of simplicity we assumed that $f(v)$ where $v \in \mathcal{F}_{x,p-h}$ has been used in $\mathcal{F}_{x,p+h+1}$ with its maximum possible reusabilty in order to find the minimum number of colors required globally.

Theorem 6.2.3 *$\lfloor \frac{p}{2} \rfloor$-th new color must be used to color the vertices of $\mathcal{F}_{x,p+2\lfloor \frac{p}{2} \rfloor -1}$ for $2p$ distance coloring.*

Proof: From Theorem 6.2.1, we get that a color which is not used in D_x^{2p} must be required to color the vertices of $\mathcal{F}_{x,p+1}$. Again from Theorem 6.2.2, another color is required to color the vertices of $\mathcal{F}_{x,p+2} \cup \mathcal{F}_{x,p+3}$.

Next we will show that if r many colors which are not used in D_x^{2p} are to be required to color the vertices of $\bigcup_{q=1}^{2r-1} \mathcal{F}_{x,p+q}$ then an $r+1$ th color which is not used in $\bigcup_{q=1}^{2r-1} \mathcal{F}_{x,p+q}$ will be required to color the vertices of $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$. After that we will justify why r many colors which are not used in D_x^{2p} are to be required to color the vertices of $\bigcup_{q=1}^{2r-1} \mathcal{F}_{x,p+q}$. At the end we will conclude the result of our theorem. We assume that the colors of the vertices in $\mathcal{S}_{x,p}^{2r}$, $\mathcal{S}_{x,p-2}^{2(r-1)}, \ldots,$ $\mathcal{S}_{x,p-2(r-1)}^{2(r-(r-1))}$ are used once each in $\mathcal{F}_{x,p+1}$, $\mathcal{F}_{x,p+3}$, $\ldots$, $\mathcal{F}_{x,p+2(r-1)+1}$ respectively.

As similar discussion stated in Observation 20 and Lemma 6.2.1, we get that the colors used in the vertices $\mathcal{S}_{x,p}^{2r} \cup \mathcal{S}_{x,p-2}^{2(r-1)} \cup \cdots \cup \mathcal{S}_{x,p-2(r-1)}^{2(r-(r-1))}$ can be reused once more each in $\mathcal{F}_{x,p+2r+1}$, where $r \leq \lfloor \frac{p}{4} \rfloor$. When $r > \lfloor \frac{p}{4} \rfloor$ and $r \leq \lceil \frac{p}{2} \rceil - 1$ or $r \leq \lfloor \frac{p}{2} \rfloor - 1$ (depending on whether p is even or odd), there exists $r' < \lfloor \frac{p}{4} \rfloor$ such that the chosen vertex $v \in \mathcal{F}_{x,p}$ is at distance $2r$ from one corner vertex in $\mathcal{F}^c_{x,p}$ and is at distance $2r'$ from another corner vertex in $\mathcal{F}^c_{x,p}$. Therefore it is evident that $v \in \mathcal{S}^{2r}_{x,p}$ as well as $v \in \mathcal{S}^{2r'}_{x,p}$. As per our assumption, $f(v)$ is reused once in $\mathcal{F}_{x,p+1}$. Then from Lemma 6.2.1, $f(v)$ can be reused once again in $\mathcal{F}_{x,p+2r'+1}$. As $v \in \mathcal{S}^{2r}_{x,p}$, from Lemma 6.2.1, it follows that $f(v)$ may be reused once more in $\mathcal{F}_{x,p+2r+1}$. As $r' < r$, so $p + 2r' + 1 < p + 2r + 1$. Since we are assuming the reusing of colors in the lower layer around x before the higher layer around x, each $f(v)$ is reused once in $\mathcal{F}_{x,p}$ and may be reused again in $\mathcal{F}_{x,p+2r'+1}$. Depending on whether $f(v)$ is used in $F_{x,p+2r'+1}$, $f(v)$ can be reused at most once again in $F_{x,p+2r+1}$.

Let $S_1 = \mathcal{S}^{2r}_{x,p} \cup \mathcal{S}^{2(r-1)}_{x,p-2} \cup \cdots \cup \mathcal{S}^{2(r-(r-1))}_{x,p-2(r-1)}$. Now from Observation 19 and Observation 20, we get that the colors of the vertices in $\mathcal{F}_{x,p-2r}$ can be reused in $\mathcal{F}_{x,p+2r+1}$ where $r \leq \lfloor \frac{p}{2} \rfloor - 1$.

Observe that $3(p-2r)-6$ and 6 many non corner and corner vertices are there in $\mathcal{F}_{x,p-2r}$ respectively. From Observation 19 and Observation 20, the color of a corner and a non corner vertex of $\mathcal{F}_{x,p-2r}$ can be reused twice and once in $\mathcal{F}_{x,p+2r+1}$ respectively. Since each $\mathcal{S}^{2(r-k)}_{x,p-2k}$, $0 \leq k \leq r-1$, contains 12 vertices (for $p - 2k \geq 4$), there are total $12r$ vertices, all of which may be reused in $\mathcal{F}_{x,p+2r+1}$.

As the coloring of the vertices of lower layer around x have been done before the coloring of the vertices of higher layer around x, the coloring of the vertices of $\bigcup_{q=1}^{2r-1} \mathcal{F}_{x,p+q}$ have been done. From the discussion of the paragraph after Theorem 6.2.2, note that the vertices of $\mathcal{F}^c_{x,p+3} \cup \{\bigcup_{h'=1}^{(3-1)/2} \mathcal{S}^{2h'}_{x,p+3}\}$, $\mathcal{F}^c_{x,p+5} \cup \{\bigcup_{h'=1}^{(5-1)/2} \mathcal{S}^{2h'}_{x,p+5}\}$, $\cdots$, $\mathcal{F}^c_{x,p+2r-1} \cup \{\bigcup_{h'=1}^{(2r-1-1)/2} \mathcal{S}^{2h'}_{x,p+2r-1}\}$ have been colored. Due to that, if the colors of all the corner vertices of $\mathcal{F}_{x,p-2r}$ are to be reused twice each in $\mathcal{F}_{x,p+2r+1}$ and colors of all of the vertices of $\mathcal{S}^{2(r-k)}_{x,p-2k}$, $0 \leq k \leq r-1$, are to be reused once each in $\mathcal{F}_{x,p+2k+1}$ and once each in $\mathcal{F}_{x,p+2r+1}$ then as similar discussion stated in Theorem 6.2.2 and using equation (6.3), at least 12 of them must be reused in corner vertices of $\mathcal{F}_{x,p+2r+1}$. But there are only 6 corner vertices in $\mathcal{F}_{x,p+2r+1}$. So colors of at least 6 of them can not be reused once each in $\mathcal{F}_{x,p+2r+1}$. Therefore, all of them together may be reused in $12r + 6 \times 2 - 6 = 12r + 6$ vertices in $\mathcal{F}_{x,p+2r+1}$. As there are $(3(p-2r)-6)$ many non corner vertices in $\mathcal{F}_{x,p-2r}$, the colors of the vertices of $\mathcal{F}_{x,p-2r} \cup S_1$ can be reused in $(3(p-2r)-6) \times 1 + 12r + 6 = 3p + 6r$ vertices of $\mathcal{F}_{x,p+2r+1}$. But there are $3(p+2r+1) = 3p+6r+3$ vertices in $\mathcal{F}_{x,p+2r+1}$. So, at least 3 vertices in $\mathcal{F}_{x,p+2r+1}$ are to be colored.

Now consider the reusing of the colors of the vertices in $\mathcal{F}_{x,p-2r+1} \cup \mathcal{S}^{2(r-1)}_{x,p-1} \cup \mathcal{S}^{2(r-2)}_{x,p-3} \cup \cdots \cup \mathcal{S}^{2((r-1)-(r-2))}_{x,p-1-2(r-2)}$. The colors used in the vertices of $\mathcal{F}_{x,p-2r+1} \cup \mathcal{S}^{2(r-1)}_{x,p-1} \cup \mathcal{S}^{2(r-2)}_{x,p-3} \cup \cdots \cup \mathcal{S}^{2((r-1)-(r-2))}_{x,p-1-2(r-2)}$ can be reused in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$ where $r \leq \lfloor \frac{p}{2} \rfloor - 1$. Let $S_2 = \mathcal{S}^{2(r-1)}_{x,p-1} \cup \mathcal{S}^{2(r-2)}_{x,p-3} \cup \cdots \cup \mathcal{S}^{2((r-1)-(r-2))}_{x,p-1-2(r-2)}$. Observe that there are $3(p-2r+1)-6$ and 6 non corner and corner vertices in $\mathcal{F}_{x,p-2r+1}$ respectively. From Observations 19 and Observation 20, color of a corner and a non corner vertex of $\mathcal{F}_{x,p-2r+1}$ can be reused twice and once in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$ respectively. Here also we assume that colors of all the vertices of $\mathcal{S}^{2(r-1-k)}_{x,p-1-2k}$, $0 \leq k \leq r-2$, have been reused in $\mathcal{F}_{x,p+2k+2}$. As discussed in the previous case, color of a vertex in $\mathcal{S}^{2(r-1-k)}_{x,p-1-2k}$, $0 \leq k \leq r-2$, if reused in $\mathcal{F}_{x,p+2k+2}$ may be reused once more in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$. Since each $\mathcal{S}^{2(r-1-k)}_{x,p-1-2k}$, $0 \leq k \leq r-2$, contains 12 vertices ($p-1-2k \geq 4$), there are total $12(r-1)$ vertices in S_2 and colors of all of which may be reused in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$. So the colors of the vertices of $\mathcal{F}_{x,p-2r+1} \cup S_2$ together can be reused in $(3(p-2r+1)-6) \times 1 + 6 \times 2 + 12(r-1) = 3p+6r-3$ vertices of $\mathcal{F}_{x,p+2r+1} \cup \mathcal{F}_{x,p+2r}$.

Therefore using the colors of the vertices of $(\mathcal{F}_{x,p-2r} \cup \mathcal{F}_{x,p-2r+1}) \cup S_1 \cup S_2$ together, we can color at most $(3p+6r) + (3p+6r-3) = 6p+12r-3$ vertices of $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$. But there are $3(p+2r) + 3(p+2r+1) = 6p+12r+3$ vertices in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$. So $(6p+12r+3) - (6p+12r-3) = 6$ vertices are to be colored.

As discussed in the next paragraph after the proof of Theorem 6.2.2, colors of some vertices of D^{2p}_x lose the reusability by least 6 in $\mathcal{F}_{x,p+1} \cup \mathcal{F}_{x,p+2} \cup \cdots \cup \mathcal{F}_{x,p+2r-1}$ when $r \leq \lfloor \frac{p}{2} \rfloor - 1$. Hence their colors may be reused in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{x,p+2r+1}$ and in that case, if they are to be reused in $\mathcal{F}_{x,p+2r+1}$, they must be reused in $\mathcal{F}^c_{x,p+2r+1} \cup \mathcal{S}^2_{x,p+2r+1} \cup \mathcal{S}^4_{x,p+2r+1} \cup \cdots \cup \mathcal{S}^{2r}_{x,p+2r+1}$ (This can be shown by equation (6.3)). Note that the vertices of $\mathcal{F}^c_{x,p+2r+1} \cup \mathcal{S}^2_{x,p+2r+1} \cup \mathcal{S}^4_{x,p+2r+1} \cup \cdots \cup \mathcal{S}^{2r}_{x,p+2r+1}$ have already been colored by the colors of the vertices of $\mathcal{F}_{x,p-2r} \cup S_1$. So these 6 colors can not be reused in $\mathcal{F}_{x,p+2r+1}$. So they must be reused in $\mathcal{F}_{x,p+2r}$. If they are to be reused in $\mathcal{F}_{x,p+2r}$, then those 6 colors must be reused in $\mathcal{F}^c_{x,p+2r} \cup \mathcal{S}^2_{x,p+2r} \cup \mathcal{S}^4_{x,p+2r} \cup \cdots \cup \mathcal{S}^{2r}_{x,p+2r}$ (This can be shown by equation (6.3)). Again, from similar discussion in the proof of Theorem 6.2.2 and its next paragraph, if all the colors used in $\mathcal{F}_{x,p-2r+1} \cup \mathcal{F}_{x,p-2r} \cup S_1 \cup S_2$ are to be reused in $\mathcal{F}_{p+2r} \cup \mathcal{F}_{p+2r+1}$, then at most 3 vertices remain uncolored in $\mathcal{F}^c_{x,p+2r} \cup \mathcal{S}^2_{x,p+2r} \cup \mathcal{S}^4_{x,p+2r} \cup \cdots \cup \mathcal{S}^{2r}_{x,p+2r}$. So out of the 6 colors, at most 3 can be reused in $\mathcal{F}_{x,p+2r}$ and hence at least $(6-3) = 3$ vertices in $\mathcal{F}_{p+2r} \cup \mathcal{F}_{p+2r+1}$ are to be colored.

Note that the r th new color used twice in $\mathcal{F}_{x,p+2r-2} \cup \mathcal{F}_{p+2r-1}$ can be reused at most twice more in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{p+2r+1}$. Hence to color the remaining $(3-2) = 1$ vertices in $\mathcal{F}_{x,p+2r} \cup \mathcal{F}_{p+2r+1}$ the $r+1$ th new color must be used.

Now observe that from Theorem 6.2.1, the first new color is required to color the vertices of $\mathcal{F}_{x,p+1}$. From Theorem 6.2.2, the 2nd new color is required to color the vertices of $\bigcup_{q=2}^{3} \mathcal{F}_{x,p+q}$. Therefore from similar discussion stated above, 3-rd new color is required to color the vertices of $\bigcup_{q=4}^{5} \mathcal{F}_{x,p+q}$. Proceeding similarly, the rth new color is required to color the vertices of $\bigcup_{q=2r-2}^{2r-1} \mathcal{F}_{x,p+q}$. Thus we justify why r many new colors are required to color the vertices of $\bigcup_{q=1}^{2r-1} \mathcal{F}_{x,p+q}$.

In our discussion, we considered $r \leq \lfloor \frac{p}{2} \rfloor - 1$. If $r = \lfloor \frac{p}{2} \rfloor$, then the vertices of $\mathcal{S}^{2r}_{x,p}$ coincides with the corner vertices of $\mathcal{F}^c_{x,p}$. Therefore the iteration terminates when $r = \lfloor \frac{p}{2} \rfloor - 1$. Hence total number of new colors required, other than the colors used in D^{2p}_x, is $1 + (\lfloor \frac{p}{2} \rfloor - 1) = \lfloor \frac{p}{2} \rfloor$. $\square$

Theorem 6.2.4 $\lambda_\ell(T_3) = \left[\frac{3}{8} \left(\ell + \frac{4}{3} \right)^2 \right]$ *where* $[x]$ *is an integer,* $x \in \mathbb{R}$ *and* $x - \frac{1}{2} < [x] \leq x + \frac{1}{2}$ *for even* $\ell \geq 10$.

Proof Note that all colors used in D^{2p}_x must be distinct. Now from Theorem 6.2.3 and Observations 17 and 18, we get that $\lambda_{2p}(T_3) \geq |D^{2p}_x| + \lfloor \frac{p}{2} \rfloor = \left[\frac{3}{8} \left(2p + \frac{4}{3} \right)^2 \right]$. It has been shown in [18] that $\lambda_{2p}(T_3) \leq \left[\frac{3}{8} \left(2p + \frac{4}{3} \right)^2 \right]$. This completes the proof of Theorem 6.2.4. $\square$

6.3 Conclusion

Jacko and Jendrol [18], determined the exact value of $\lambda_\ell(T_3)$ for any odd ℓ and for even $\ell \geq 8$, it was conjectured that $\lambda_\ell(T_3) = \left[\frac{3}{8}\left(\ell + \frac{4}{3}\right)^2\right]$, where $[x]$ is an integer, $x \in \mathbb{R}$ and $x - \frac{1}{2} < [x] \leq x + \frac{1}{2}$. In this chapter, we prove the conjecture for any even $\ell \geq 8$.

Chapter 7

Conclusions and future directions

In this thesis we have studied some of the $L(k_1, k_2, \ldots, k_\ell)$ labeling problems for infinite hexagonal, square, triangular and 8-regular grids when $\ell \geq 2$. The problems have practical relevance because some special cases of the channel assignment problem can be modelled theoretically as $L(k_1, k_2, \ldots, k_\ell)$ labeling of infinite regular grids. In this thesis, we have analyzed the underlined graph structure and obtained some new results as well as improved some existing results. We have derived exact values of $\lambda'_{1,2}(T_3)$, $\lambda'_{1,2}(T_4)$, $\lambda'_{2,1}(T_6)$ and $\lambda'_{1,2}(T_8)$. Besides it, we have also improved the lower bounds of $\lambda'_{1,2}(T_6)$. We have derived a labeling function for circular $L(2,1)$-edge labeling of T_6 for which no such labeling function was known. For vertex labeling, we have theoretically determined the lower bounds of $\lambda_{k_1,k_2}(T_6)$ when $k_1 \leq k_2$ for some sub intervals, whereas, the corresponding values were determined previously by partial computer simulation. We have also determined the values of $\lambda_{\underbrace{1,1,\ldots,1}_{8}}(T_3)$ which is the settlement of a conjecture posed by Jacko and jendrol. The methods we have adopted here to solve the various problems stated above may be extended for solving the general cases.

We have improved the lower bounds of $\lambda'_{1,2}(T_6)$. But yet there is gap between them. So, attempts may be taken to improve the bounds. The upper and lower bounds of $\sigma'_{2,1}(T_6)$ are also not identical. So, scope of future research exists here also. We have theoretically determined the lower bounds of $\lambda_{k_1,k_2}(T_6)$ when $k_1 \leq k_2$ but in some sub intervals of $\dfrac{1}{3} \leq \dfrac{k_1}{k_2} \leq 1$, finer results are present (though obtained through computer assistance). So, we can extend our method to obtain the finer results theoretically for those intervals.

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