Homotopical computations for projective Stiefel manifolds and related quotients

Debanil Dasgupta

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Author:
Debanil Dasgupta

Supervisor:
Samik Basu

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Chapter 1

Introduction

This thesis explores certain topological results on quotients of Stiefel manifolds. The perspective behind these results are cohomology calculations which in turn lead to geometric consequences. The quotients of Stiefel manifolds form a nice collection of homogeneous spaces which are amenable to computational techniques. We consider

$$V_{n,k} = \text{ real Stiefel manifold of } k\text{-orthonormal vectors in } \mathbb{R}^n \cong O(n)/O(n-k),$$

$$W_{n,k} = \text{ complex Stiefel manifold of } k\text{-orthonormal vectors in } \mathbb{C}^n \cong U(n)/U(n-k).$$

These manifolds are ubiquitous in the literature, and many of their features have been extensively studied.

The Stiefel manifolds have an action by the orthogonal group $O(k)$ in the real case, and $U(k)$ in the complex case, whose quotients are the Grassmann manifolds. This is in a sense the quotient of a Lie group by a parabolic subgroup. One may instead take the quotient by a Borel subgroup and end up with flag manifolds. These manifolds are also widely studied in topology and algebraic geometry.

A different viewpoint is provided when we consider actions by cyclic subgroups of the orthogonal group. As an example, one has the action of S^1 on $W_{n,k}$ by diagonal matrices in $U(k)$. The quotient space is called the projective Stiefel manifold $PW_{n,k}$. The real version of this is the quotient $PV_{n,k} = V_{n,k}/C_2$, where the cyclic group C_2 of order 2 acts via ± 1. One may also consider a cyclic group action on $W_{n,k}$ by m^{th} roots of unity, and call the quotient $W_{n,k;m}$.

The projective Stiefel manifolds have been of interest in connection with a varied spectrum of topological questions. On one hand, they are useful in studying equivariant maps between the Stiefel manifolds [33], and on the other, they form a part of an obstruction theory for constructing sections of multiples of a given line bundle [6]. In the real case, they play an important role in the immersion problem for real projective spaces [37].

1.1 BP-cohomology of complex projective Stiefel manifolds

The cohomology of $PW_{n,k}$ with $\mathbb{Z}/p$-coefficients was computed in [6], which is analogous to the $\mathbb{Z}/2$-computation for real projective Stiefel manifolds in [19]. Among other applications, this has been used to prove the non-existence of S^1-equivariant maps between complex Stiefel manifolds [33].

A natural idea here is that extending the computations to generalized cohomology theories would yield further results about equivariant maps. We follow through along these lines and compute the BP-cohomology as (Theorem 3.3.7)

Theorem 1.1.1. The BP-cohomology of $PW_{n,k}$ is described as

$$BP^*(PW_{n,k}) \cong \Lambda_{BP^*(pt)}(\gamma_{n-k+2}, \cdots, \gamma_n) \otimes_{BP^*(pt)} BP^*(pt)[[x]]/I$$

where γ_j's are of degree $2j-1$, x is of degree 2, and I is the ideal generated by $\{\binom{n}{j}x^j | n-k < j \leq n\}$.

The method used to compute the BP-cohomology is the homotopy fixed point spectral sequence. This works for any complex oriented cohomology theory, where the class x comes from the choice of complex orientation. Consequently, the K-theory of the complex projective Stiefel manifold has an analogous formula, which was computed in [20] using the Hodgkin spectral sequence for the cohomology of homogeneous spaces. The same method is also likely to work for $P_\ell W_{n,k}$, the quotient by a variant of the S^1-action, whose cohomology was computed in [12]. Here, ℓ refers to a tuple of integers $(l_1, \cdots, l_k)$ and the action of S^1 is given by $z \cdot (v_1, \cdots, v_k) = (z^{l_1}v_1, \cdots, z^{l_k}v_k)$.

We observe that the BP-cohomology ring of $PW_{n,k}$ is just the extension of coefficients from $\mathbb{Z}_{(p)}$ in ordinary cohomology to $\mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ in BP-cohomology. Therefore, the primary multiplicative structure does not yield new results for equivariant maps between Stiefel manifolds. However, BP has the action of Adams operations [5], which yield the following new result on equivariant maps. (see Theorem 3.4.8)

Theorem 1.1.2. Suppose that m, n, l, k are positive integers satisfying

1) $n-k < m-l$ and there is an s such that $m < 2^s + m - l \leq n$.

2) 2 divides all the binomial coefficients $\binom{n}{n-k+1}, \cdots, \binom{n}{m-l}$.

3) 2 does not divide $\binom{m}{m-l+1}$ and $2 \nmid m-l$.

Then, there is no S^1-equivariant map from $W_{n,k}$ to $W_{m,l}$.

We also obtain some new results using the action of Steenrod operations on $H^*PW_{n,k}$. We point out that the analysis of equivariant maps on Stiefel manifolds also leads to results in topological combinatorics [11].

1.2 p-local decompositions of projective Stiefel manifolds

We explore homotopical results about the quotients of the Stiefel manifolds after localization at primes. The cohomology of the Stiefel manifold is an exterior algebra, so a natural question is how much homotopically alike it is to a product of spheres. If it is so, the fibration $W_{n,k} \to S^{2n-1}$ must have a section $S^{2n-1} \to W_{n,k}$, the existence of which has a precise answer [4], [7]. On the other hand, one may try to write down conditions under which $W_{n,k}$ is p-regular, that is, when does its p-localization become equivalent to a product of spheres? This has been studied by Yamaguchi [39]. His results imply that if $p > n$, $W_{n,k}$ is p-regular. We consider the analogous question for the quotient $PW_{n,k}$.

The cohomology of $PW_{n,k}$ matches that of $\mathbb{C}P^{n-k} \times \prod_{j=n-k+2}^{n} S^{2j-1}$ if $p > n$. We explore whether, after localization at p, the two spaces are homotopy equivalent. Elementary arguments imply that if p is more than half the dimension of $PW_{n,k}$, this result holds. More precisely, (see Theorem 4.2.7)

Theorem 1.2.1. If $p > \frac{2nk-k^2-1}{2} + k - n$,

$$\left(PW_{n,k}\right)_{(p)} \simeq \left[\mathbb{C}P^{n-k} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\right]_{(p)}.$$

For a significantly better bound on p in the equivalence above, we look at the obstructions to forming a map from $PW_{n,k}$ to the product $\mathbb{C}P^{n-k} \times \prod_{j=n-k+2}^{n} S^{2j-1}$. We see that if $p > n$, these obstructions (localized at p) belong to the "stable range". This leads us to consider the stable homotopy type of the projective Stiefel manifold, for which we prove the following result using certain calculations with the Chern character. (see Theorem 4.3.15)

Theorem 1.2.2. If $p > n$, the projective Stiefel manifold $PW_{n,k}$ stably splits into a wedge of spheres in the p-local category.

The path to proving the theorem above goes via a homotopy theoretic result which says that for a CW-complex of dimension $\leq 2p^2 - 2p$, if p-local cohomology is torsion-free, and the Chern character takes values in $\mathbb{Z}_{(p)}$, the space stably splits into a wedge of spheres in the p-local category (see Theorem 4.3.7). The stable splitting of the projective Stiefel manifold allows us to construct stable maps to the desired space. The obstruction theory to lift this to the unstable category involves another intricate argument with the Chern character that introduces a new bound on k for which the p-local decomposition result holds. (see Theorem 4.4.3)

Theorem 1.2.3. Suppose $p > n+1$ and $k \leq p + n - \sqrt{p^2 + n^2 - 4p + 2}$. Then,

$$\left(PW_{n,k}\right)_{(p)} \simeq \left[\mathbb{C}P^{n-k} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\right]_{(p)}.$$

The homotopical decomposition results for $PW_{n,k}$ imply that with the bound above, the p-regularity result for the Stiefel manifold is indeed S^1-equivariant (see Theorem 4.4.4). The techniques also imply decomposition results (see Theorem 4.4.5) for the finite cyclic quotients of Stiefel manifolds $W_{n,k;m}$, and that of $P_\ell W_{n,k}$ for $\ell = (l_1, \cdots, l_k)$, $l_i \in \mathbb{Z}$, which is $W_{n,k}/S^1$ with the action on the i^{th} vector by z^{l_i}.

1.3 Characteristic classes for quotients of Stiefel manifolds

The Stiefel manifolds are parallelizable except for the spheres. For the quotients of Stiefel manifolds one has a nice approach towards calculation of their tangent bundle [27]. There is also a direct method to compute their cohomology via the Serre spectral sequence [19],[6]. This allows us to calculate explicitly characteristic classes for these manifolds, such as the Stiefel-Whitney classes, or the Pontrjagin classes. These computations may then be used to deduce non-embedding and non-immersion results into Euclidean spaces. Further, we may explore the question whether these manifolds are parallelizable, and obtain bounds on the number of linearly independent vector fields. Partial answers to these questions are provided in [35].

One may also consider skew embeddings of these manifolds, that is, embeddings in which the affine spaces corresponding to the tangent spaces in the embedding are skew. This is related to an embedding of the tangent bundle of the ordered configuration space. Bounds on such embeddings are also related to Stiefel-Whitney classes in [8]. This allows us to compute bounds on skew embeddings of $PV_{n,k}$ (5.1.2).

Recently in [26], in an attempt to compute the cohomology of the oriented Grassmannian, the authors considered the question of representing cohomology classes via Stiefel-Whitney classes of bundles. More precisely, they consider the subalgebra of $H^*(X;\mathbb{Z}_2)$ generated by Stiefel-Whitney classes of vector bundles. The question posed for a space X is the highest degree d such that every class of degree $\leq d$ lies in this subalgebra. Such a d is called the ucharrank(X). This was computed for the Stiefel manifolds in [25]. We compute the same for $PV_{n,k}$ in a significant number of cases (5.1.4).

We also consider the diagonal inclusion of S^1 inside a maximal torus of $O(k)$, and quotient the Stiefel manifold with the associated action. This is the space $Y_{n,k} = V_{n,2k}/S^1$ with the action via $S^1 \cong SO(2)$ included in $O(2k)$ as blocks. The manifolds $Y_{n,k}$ are called circle quotient Stiefel manifolds. We compute it's tangent bundle (5.2.1), and show that it is not parallelizable if $n-2k \geq 3$ (5.3.1). Further, we compute it's cohomology (5.2.3), skew embedding dimension (5.3.3), and upper characteristic rank (5.3.4).

1.4 Organization

In Chapter 2, we recall methods to compute the cohomology of certain quotients of Stiefel manifolds, via the Serre spectral sequence. In Chapter 3, we produce the calculation of BP-cohomology for the complex projective Stiefel manifolds and explore the applications for such a computation. In Chapter 4, we discuss product splittings of projective Stiefel manifolds and similar quotients after localization at a prime. In chapter 5, we discuss certain characteristic class computations for these quotients of Stiefel manifolds.

Chapter 2

Cohomology of certain quotients of Stiefel manifolds

In this chapter, we recall well known computations for the cohomology of Stiefel manifolds and certain quotients of them. The main technique used is the Serre spectral sequence. All the spaces considered are homogeneous, so any such G/H fits into a fibration $G/H \to BH \to BG$. The associated Serre spectral sequence is then computed to derive the cohomology of G/H with suitable coefficients.

2.1 Stiefel manifolds and related homogeneous spaces

We begin with the definitions of some homogeneous spaces we are going to deal with.

Definition 2.1.1.
- The manifold consisting of all orthonormal k-frames in $\mathbb{C}^n$ will be called the **complex Stiefel manifold** $W_{n,k}$.
- The manifold consisting of all orthonormal k-frames in $\mathbb{R}^n$ will be called the **real Stiefel manifold** $V_{n,k}$.

$W_{n,k}$ can be realized as the orbit space $U(n)/U(n-k)$ and $V_{n,k}$ can be realized as the orbit space $O(n)/O(n-k)$. If $n > k$, $V_{n,k}$ can also be realized as the orbit space $SO(n)/SO(n-k)$. Hence we have the following fibrations:

- $W_{n,k} \longrightarrow BU(n) \longrightarrow BU(n-k)$
- $V_{n,k} \longrightarrow BO(n) \longrightarrow BO(n-k)$
- $V_{n,k} \longrightarrow BSO(n) \longrightarrow BSO(n-k)$, for $n > k$.

Definition 2.1.2.
- A **complex projective Stiefel manifold** $PW_{n,k}$ is defined to be the orbit space of the free S^1-action on $W_{n,k}$ described as follows:

$$z \cdot (w_1, \cdots, w_k) = (zw_1, \cdots, zw_k).$$

- A **real projective Stiefel manifold** $PV_{n,k}$ is defined to be the orbit space of the free C_2-action on $V_{n,k}$ described as follows:

$$-1 \cdot (v_1, \cdots, v_k) = (-v_1, \cdots, -v_k).$$

There is a canonical complex (resp. real) line bundle $\zeta_{n,k}$ over $PW_{n,k}$ (resp. $PV_{n,k}$) associated to the principal S^1 (resp. C_2) bundle mentioned in the above definition. We also have the following fibrations:

- $W_{n,k} \longrightarrow PW_{n,k} \longrightarrow \mathbb{C}P^\infty$
- $V_{n,k} \longrightarrow PV_{n,k} \longrightarrow \mathbb{R}P^\infty$.

The bundles $\zeta_{n,k}$ over projective Stiefel manifolds ($PV_{n,k}$ or $PW_{n,k}$) enjoy a certain universal property. They classify the line bundles (real or complex) with the property that their n-fold Whitney sum has a (real or complex) rank k trivial subbundle. This leads to the following pullback diagrams:

-
$$\begin{tikzcd} PV_{n,k} \arrow[r] \arrow[d] & BO(n-k) \arrow[d] \\ \mathbb{R}P^\infty \arrow[r] & BO(n), \end{tikzcd} \tag{2.1.1}$$

  where the map at the bottom is classifying map of n-fold Whitney sum of canonical real line bundle. [19]

-
$$\begin{tikzcd} PW_{n,k} \arrow[r] \arrow[d] & BU(n-k) \arrow[d] \\ \mathbb{C}P^\infty \arrow[r] & BU(n), \end{tikzcd} \tag{2.1.2}$$

  where the map at the bottom is classifying map of n-fold Whitney sum of canonical complex line bundle. [6]

The complex projective Stiefel manifolds can be realized as special cases of a more general class of manifolds known as generalized complex projective Stiefel manifolds.

Definition 2.1.3. A **generalized complex projective Stiefel manifold** $P_\ell W_{n,k}$ is defined to be the orbit space of the free S^1 action on $W_{n,k}$ described as follows:

$$z \cdot (w_1, \cdots, w_k) = (z^{l_1} w_1, \cdots, z^{l_k} w_k),$$

for a primitive tuple $\ell = (l_1, \cdots, l_k) \in \mathbb{Z}^k$.

These generalized complex projective Stiefel manifolds also satisfy a universal property as described in the following theorem,[12].

Theorem 2.1.4. The space $P_\ell W_{n,k}$ classifies the line bundles L for which there exists an $(n-k)$-bundle E such that $E \oplus_j L^j$ is a trivial bundle.

Definition 2.1.5. An m-**projective Stiefel manifold** $W_{n,k;m}$ is defined to be the quotient space of $W_{n,k}$ under the free action of the cyclic group of order m, C_m considered as a subgroup of the circle group.

2.2 The Serre spectral sequence for quotients of Stiefel manifolds

Our aim is now to compute the ordinary cohomology with suitable coefficients of the spaces described in the previous section. In order to doing so our main computational tool will be the Serre spectral sequence.

Theorem 2.2.1. For a fibration $F \longrightarrow E \longrightarrow B$ with the action of $\pi_1(B)$ on $H^*(F)$ being trivial, there is an associated multiplicative spectral sequence called the **Serre spectral sequence** converging to $H^*(E)$, whose E_2-page is given as follows:

$$E_2^{p,q} = H^p(B; H^q(F)) \implies H^{p+q}(E).$$

In most of the cases the space B turns out to be simply connected and hence we won't have to bother with the condition of triviality of the action of $\pi_1(B)$ on $H^*(F)$.

We shall consider appropriate fibrations involving the homogeneous spaces of our interest and calculate differentials of the associated Serre spectral sequences. The general pattern of our cases is as follows: G be a subgroup of $U(k)$ (resp. $O(k)$). Then we consider the free action of G on $W_{n,k}$ (resp. $V_{n,k}$) as restriction the free action of $U(k)$ (resp. $O(k)$). Orbit space under the free action of $U(k)$ (resp. $O(k)$) is the Grassmann manifold $Gr_k(\mathbb{C}^n)$ (resp.

$Gr_k(\mathbb{R}^n)$). The homogeneous space we are interested in are $W_{n,k}/G$ (resp. $V_{n,k}/G$). Hence we may consider the following comparisons of fibrations:

$$\begin{tikzcd} W_{n,k} \arrow[r, equal] \arrow[d] & W_{n,k} \arrow[d] \\ W_{n,k}/G \arrow[r] \arrow[d] & G_k(\mathbb{C}^n) \arrow[d] \\ BG \arrow[r] & BU(k) \end{tikzcd} \qquad \begin{tikzcd} V_{n,k} \arrow[r, equal] \arrow[d] & V_{n,k} \arrow[d] \\ V_{n,k}/G \arrow[r] \arrow[d] & G_k(\mathbb{R}^n) \arrow[d] \\ BG \arrow[r] & BO(k) \end{tikzcd} \qquad \begin{tikzcd} V_{n,k} \arrow[r, equal] \arrow[d] & V_{n,k} \arrow[d] \\ V_{n,k}/G \arrow[r] \arrow[d] & \widetilde{G}_k(\mathbb{R}^n) \arrow[d] \\ BG \arrow[r] & BSO(k). \end{tikzcd} \tag{2.2.1}$$

Now our job is to calculate the Serre spectral sequence associated to the fibration on left of each diagram by doing so for the fibrations on right and then comparing spectral sequences. But for that we need the knowledge of the cohomology of $W_{n,k}$ (resp. $V_{n,k}$) and Borel's theorem stated below provides us that [14]. Recall that

- $H^*(BU(n);\mathbb{Z}) \cong \mathbb{Z}[c_1, \cdots, c_n]$, where c_i's are universal Chern classes.
- $H^*(BO(n);\mathbb{Z}_2) \cong \mathbb{Z}_2[w_1, \cdots, w_n]$, where w_i's are universal Stiefel Whitney classes.
- $H^*(BSO(n);\mathbb{Z}_p) \cong \begin{cases} \mathbb{Z}_p[p_1, \cdots, p_{\frac{n-1}{2}}] & \text{if } n \text{ is odd} \\ \mathbb{Z}_p[p_1, \cdots, p_{\frac{n-2}{2}}, e_n] & \text{if } n \text{ is even,} \end{cases}$

  where p_i's are mod p reduction of Pontrjagin classes and e_n is the Euler class.

Theorem 2.2.2.
- $H^*(W_{n,k};\mathbb{Z}) = \Lambda_{\mathbb{Z}}(z_{n-k+1}, \cdots, z_n)$, with $deg(z_i) = 2i-1$ and z_i's are characterized by the property that in the Serre spectral sequence associated to

$$W_{n,k} \longrightarrow BU(n-k) \longrightarrow BU(n)$$

  they are transgressive and transgress to c_i's, the universal Chern classes.

- $H^*(V_{n,k};\mathbb{Z}_2) = V_{\mathbb{Z}_2}(\omega_{n-k}, \cdots, \omega_{n-1})$, with $deg(\omega_i) = i$ and ω_i's are characterized by the property that in the Serre spectral sequence associated to

$$V_{n,k} \longrightarrow BO(n-k) \longrightarrow BO(n)$$

  they are transgressive and transgress to w_{i+1}'s, the universal Stiefel-Whitney classes.

- $$H^*(V_{n,k};\mathbb{Z}_p) \cong \begin{cases} \Lambda(x_{\frac{n-k+2}{2}},\cdots,x_{\frac{n-1}{2}},\sigma_{n-k}) & \text{if } k \text{ is odd}, n \text{ is odd} \\ \Lambda(x_{\frac{n-k+1}{2}},\cdots,x_{\frac{n-1}{2}}) & \text{if } k \text{ is even}, n \text{ is odd} \\ \Lambda(x_{\frac{n-k+1}{2}},\cdots,x_{\frac{n-2}{2}},\epsilon_n) & \text{if } k \text{ is odd}, n \text{ is even} \\ \Lambda(x_{\frac{n-k+2}{2}},\cdots,x_{\frac{n-2}{2}},\sigma_{n-k},\epsilon_n) & \text{if } k \text{ is even}, n \text{ is even} \end{cases} \tag{2.2.3}$$

  where $deg(x_i) = 4i-1, deg(\sigma_j) = j, deg(\epsilon_n) = n-1$.

Now calculation of the Serre spectral sequences associated the fibrations on right of (2.2.1) is done via the following commutative diagrams of fibrations

$$\begin{tikzcd} U(n) \arrow[r] \arrow[d] & W_{n,k} \arrow[d] \\ G_k(\mathbb{C}^n) \arrow[r, equal] \arrow[d] & G_k(\mathbb{C}^n) \arrow[d] \\ B(U(k)\times U(n-k)) \arrow[r] & BU(k) \end{tikzcd} \qquad \begin{tikzcd} O(n) \arrow[r] \arrow[d] & V_{n,k} \arrow[d] \\ G_k(\mathbb{R}^n) \arrow[r, equal] \arrow[d] & G_k(\mathbb{R}^n) \arrow[d] \\ B(O(k)\times O(n-k)) \arrow[r] & BO(k) \end{tikzcd}$$

$$\begin{tikzcd} SO(n) \arrow[r] \arrow[d] & V_{n,k} \arrow[d] \\ \widetilde{G}_k(\mathbb{R}^n) \arrow[r, equal] \arrow[d] & \widetilde{G}_k(\mathbb{R}^n) \arrow[d] \\ B(SO(k)\times SO(n-k)) \arrow[r] & BSO(k). \end{tikzcd} \tag{2.2.2}$$

The top rows of these diagrams above are given by the quotient maps, and the bottom rows by the projections onto first factors. Then for complex case computation of the differentials of spectral sequence associated to the right column in (2.2.2) was done in [12] and as a consequence the following theorem was obtained.

Proposition 2.2.4. The cohomology ring $H^*(W_{n,k};\mathbb{Z})$ is the exterior algebra $\Lambda(z_{n-k+1},\cdots,z_n)$, with $|z_j| = 2j-1$. For the spectral sequence associated to $W_{n,k} \to G_k(\mathbb{C}^n) \to BU(k)$, the classes z_j are transgressive with $d_{2j}(z_j) = -c'_j$, where c'_j are defined by the equation $(1+c'_1+\cdots)(1+c_1+\cdots+c_k) = 1$.

Similar arguments work for the real case and the following analogous results are obtained in [11].

Proposition 2.2.5. With $\mathbb{Z}_2$-coefficients, the cohomology ring $H^*(V_{n,k})$ has a basis given by the square-free monomials in $\omega_{n-k}, \cdots, \omega_{n-1}$, with $|\omega_j| = j$. For the spectral sequence associated to $V_{n,k} \to G_k(\mathbb{R}^n) \to BO(k)$, the classes ω_j are transgressive with $d_j(\omega_j) = -w'_{j+1}$, where w'_j are defined by the equation $(1 + w'_1 + \cdots)(1 + w_1 + \cdots + w_k) = 1$.

Proposition 2.2.6. Assume that k is odd. With $\mathbb{Z}_p$-coefficient ($p > 2$) in the spectral sequence associated to $V_{n,k} \to \tilde{Gr}_k(\mathbb{R}^n) \to BSO(k)$, the classes x_i are transgressive and $d_{4i}(x_i) = -p'_i$ where p'_i are defined by the equation $(1+p'_1+\cdots)(1+p_1+\cdots p_k) = 1$. The classes σ_{n-k} (for n odd) and ϵ_n (for n even) (2.2.3) are permanent cycles.

2.3 Cohomology of projective Stiefel manifolds

Cohomology of projective Stiefel manifolds with various coefficients are calculated by Comparing spectral sequences associated to the columns of the pullback diagrams in (2.1.1) and (2.1.2).

2.3.1 Real projective Stiefel manifolds

In the real case the map in the bottom row of (2.1.1) is the classifying map of k-fold Whitney sum of canonical real line bundle and hence it pulls back w_i to $\binom{n}{i}x^i$. Comparing spectral sequences associated to the columns of (2.1.1) the following theorem was obtained in [19].

Theorem 2.3.1. Suppose $k < n$. Let $N = \min\{j \mid n-k+1 \leqslant j \leqslant n \text{ and } \binom{n}{j} \text{ is odd}\}$. Then $H^*(PV_{n,k}; \mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^N) \otimes V(A)$ additively, where $deg(x) = 1$ and $V(A)$ is a $\mathbb{Z}_2$-algebra with simple system of generators $\{y_{n-k}, \cdots, y_{N-2}, y_N \cdots, y_{n-1}\}$ and $deg(y_i) = i$. That is, $V(A)$ has a basis given by the square-free monomials in $\{y_{n-k}, \cdots, y_{N-2}, y_N \cdots, y_{n-1}\}$.

Also note that, whenever 2 is invertible in the coefficient ring R, $H^*(\mathbb{R}P^\infty; R) = R$ and hence $H^*(PV_{n,k}; R) = H^*(V_{n,k}; R)$.

2.3.2 Complex projective Stiefel manifolds

By an entirely similar method used for real projective Stiefel manifolds, the $\mathbb{Z}_p$-cohomology of complex projective Stiefel manifolds was determined in [6].

Theorem 2.3.2. Let $p > 2$ be prime and $N = \min\{j \mid n-k+1 \leqslant j \leqslant n \text{ and } p \nmid \binom{n}{j}\}$ There exist classes $y_j \in H^{2j-1}(PW_{n,k}; \mathbb{Z}_p)$ for $n - k < j \leqslant n$ such that

$$H^*(PW_{n,k}; \mathbb{Z}_p) = \mathbb{Z}_p[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \cdots, \widehat{y_N}, \cdots, y_n),$$

and $deg(x) = 2$.

Theorem 2.3.3. Let $N = \min\{j \mid n-k+1 \leqslant j \leqslant n \text{ and } \binom{n}{j} \text{ is odd}\}$. There exist classes $y_j \in H^{2j-1}(PW_{n,k};\mathbb{Z}_2)$ for $n-k < j \leqslant n$ and $x \in H^2(PW_{n,k};\mathbb{Z}_2)$ such that

- If $n \not\equiv 2 \pmod 4$ or $k < n$, then

$$H^*(PW_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \cdots, \widehat{y_N}, \cdots, y_n).$$

- If $n \equiv 2 \pmod 4$ and $k = n$, then

$$H^*(PW_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2[y_1]/(y_1^4) \otimes \Lambda(y_3, \cdots, y_n)$$

  and $x = y_1^2$.

Computation of rational cohomology for complex projective Stiefel manifolds is done by computing the transgressions in the spectral sequence associated to $W_{n,k} \longrightarrow PW_{n,k} \longrightarrow \mathbb{C}P^\infty$. The algebra generators of $H^*(W_{n,k})$, y_j's transgress to $\binom{n}{j}x^j$. So $\frac{1}{\binom{n}{k-1}}y_{n-k+1}$ kills x^{n-k+1} on $2(n-k+1)$-th page and one obtains the following:

Theorem 2.3.4. The rational cohomology of $PW_{n,k}$ is

$$H^*(PW_{n,k};\mathbb{Q}) = \mathbb{Q}[x]/(x^{n-k+1}) \otimes \Lambda_{\mathbb{Q}}(y_{n-k+2}, \cdots, y_n),$$

where $deg(y_j) = 2j-1$ and $deg(x) = 2$.

Calculation of $\mathbb{Z}_{(p)}$-cohomology of $PW_{n,k}$ requires little extra care.

Theorem 2.3.5. The $\mathbb{Z}_{(p)}$-cohomology of $PW_{n,k}$ is

$$H^*(PW_{n,k};\mathbb{Z}_{(p)}) = \mathbb{Z}_{(p)}[x]/I \otimes \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2}, \cdots, \gamma_n),$$

where I is the ideal of $\mathbb{Z}_{(p)}[x]$ generated by the set $\{\binom{n}{j}x^j \mid n-k < j \leqslant n\}$, $deg(\gamma_j) = 2j-1, deg(x) = 2$.

Proof. In the spectral sequence associated to $W_{n,k} \longrightarrow PW_{n,k} \longrightarrow \mathbb{C}P^\infty$, the exterior algebra generators $y_j \in H^*(W_{n,k};\mathbb{Z}_{(p)}) = \Lambda_{\mathbb{Z}_{(p)}}(y_{n-k+1}, \cdots, y_n)$ transgress to $\binom{n}{j}x^j \in H^*(\mathbb{C}P^\infty) = \mathbb{Z}_{(p)}[x]$. We see that the first non-trivial differential is $d_{2(n-k+1)}$ and the generator y_{n-k+1} and all its multiples do not survive in the next page since

$$d_{2(n-k+1)}(y_{n-k+1}) = \binom{n}{n-k+1}x^{n-k+1}.$$

For the y_j of higher degree, it may happen that the only non-trivial differential on it

$$d_{2j}y_j = \binom{n}{j}x^j$$

may be zero. This precisely happens when $\binom{n}{j}$ lies in the ideal generated by $\binom{n}{i}$ for $n-k+1 \leq i < j$ inside $\mathbb{Z}_{(p)}$. This condition may be interpreted in terms of p-adic valuations of these numbers. For $j > n-k+1$, even if $d_{2j}(y_j) \neq 0$, we still obtain a multiple $p^s y_j$ on which the differential is 0, determined by the formula $s + v_p(\binom{n}{j}) = \min_{n-k+1 \leq i < j} v_p(\binom{n}{i})$. So for $j > n-k+1$, even if y_j doesn't survive in the E_{2j+1} page, some p-power multiple of it would and that multiple becomes a permanent cycle detecting an element $\gamma_j \in H^{2j-1}(PW_{n,k};\mathbb{Z}_{(p)})$. □

2.4 Cohomology of other quotients of Stiefel manifolds

2.4.1 Generalized complex projective Stiefel manifolds

The cohomology of $P_\ell W_{n,k}$ with $\mathbb{Z}_p$-coefficients was computed in [12] by the method described in the last section. One calculates the differentials in the Serre spectral sequence associated to the fibration $W_{n,k} \longrightarrow P_\ell W_{n,k} \longrightarrow \mathbb{C}P^\infty$. It turns out that the exterior algebra generators z_j's of $H^*(W_{n,k};\mathbb{Z}_{(p)})$ are transgressive and $d_{2j}(z_j) = \sum_{|I|=j}(-1)^j l^I x^j$, where $\ell^I := \prod \ell_j^{i_j}$ for $\ell = (\ell_1, \cdots, \ell_k)$ and $I = (i_1, \cdots, i_k)$.

Theorem 2.4.1. For an odd prime p,

$$H^*(P_\ell W_{n,k};\mathbb{Z}_p) \cong \mathbb{Z}_p[x]/(x^N) \otimes \Lambda(y_{n-k+1}, \cdots, \widehat{y}_N, \cdots, y_n),$$

where $N = \min\{j \mid j > n-k \text{ and } \sum_{|I|=j} \ell^I \not\equiv 0 \pmod{p}\}$.

The similar calculation for $PW_{n,k}$ with $\mathbb{Z}_{(p)}$-coefficients yields the formula

$$H^*(P_\ell W_{n,k};\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2}, \cdots, \gamma_n) \otimes \mathbb{Z}_{(p)}[x]/J,$$

where $|\gamma_j| = 2j-1$, $|x| = 2$ and J is the ideal of $\mathbb{Z}_{(p)}[x]$ generated by the set $\{\sum_{|I|=j}(-1)^j l^I x^j \mid n-k < j \leq n\}$. Note that for a prime p not dividing $\sum_{|I|=n-k+1} l^I$, we have the following reduction

$$H^*(P_\ell W_{n,k};\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2}, \cdots, \gamma_n) \otimes \mathbb{Z}_{(p)}[x]/(x^{n-k+1}), \tag{2.4.2}$$

which is exactly same as rational cohomology except the coefficients are now replaced by $\mathbb{Q}$.

2.4.2 m-projective Stiefel manifolds

The cohomology calculation of $W_{n,k;m}$ is done by considering the fibration $S^1 \longrightarrow W_{n,k;m} \longrightarrow PW_{n,k}$ and the associated complex line bundle is shown to be $\zeta_{n,k}^m$ [21]. Its $\mathbb{Z}_p$-cohomology was computed in [21]. In the associated Serre spectral sequence the cohomology generator of S^1 transgresses to mx, where x is the Euler class of $\zeta_{n,k}$ and d_2 is the only non-trivial differential.

Theorem 2.4.3. Suppose that $2 \leqslant k < n$, $m \geqslant 2$ and $N = \min\{j \mid p \nmid \binom{n}{j}$ and $n-k+1 \leqslant j \leqslant n\}$,

- If $p \nmid m$, $H^*(W_{n,k;m};\mathbb{Z}_p) \cong \Lambda_{\mathbb{Z}_p}(x_{n-k+1},\cdots,x_n)$, where $|x_i| = 2i-1$.
- If $p \mid m$ and p is odd, $H^*(W_{n,k;m};\mathbb{Z}_p) \cong \mathbb{Z}_p[x]/(x^N) \otimes \Lambda_{\mathbb{Z}_p}(x_1, x_{n-k+1},\cdots,\widehat{x}_N,\cdots,x_n)$, where $|x| = 2, |x_i| = 2i-1$.
-
  1. If $4 \mid m-2$, then $H^*(W_{n,k;m};\mathbb{Z}_2) \cong \mathbb{Z}_2[x_1]/(x_1^{2N}) \otimes \Lambda_{\mathbb{Z}_p}(x_1, x_{n-k+1},\cdots,\widehat{x}_N,\cdots,x_n)$, where $|x_i| = 2i-1$.
  2. If $4 \mid m$, then $H^*(W_{n,k;m};\mathbb{Z}_2) \cong \mathbb{Z}_2[x]/(x^N) \otimes \Lambda_{\mathbb{Z}_2}(x_1, x_{n-k+1},\cdots,\widehat{x}_N,\cdots,x_n)$, where $|x| = 2, |x_i| = 2i-1$.

Following the same method, the cohomology with $\mathbb{Z}_{(p)}$ coefficients (for $p \mid m$) may be computed. For $p > n$, the formula takes the following form

$$H^*(W_{n,k;m};\mathbb{Z}_{(p)}) \cong (\Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+1},\gamma_{n-k+2},\cdots,\gamma_n) \otimes \mathbb{Z}_{(p)}[x])/(mx, x^{n-k+1}, \gamma_{n-k+1}x), \tag{2.4.4}$$

where $|\gamma_j| = 2j-1$, and $|x| = 2$. Again we compute the Serre spectral sequence associated to the fibration $S^1 \longrightarrow W_{n,k;m} \longrightarrow PW_{n,k}$ and the only non-trivial differential d_2 sends the degree 1 class e generating $H^*(S^1;\mathbb{Z}_{(p)})$ to mx. Note that the class $e \otimes x^{n-k}$ survives in the E_∞-page detecting the degree $2n-2k+1$ class γ_{n-k+1}.

Chapter 3

BP-cohomology of the projective Stiefel manifolds

In this chapter, we compute the BP-cohomology of the complex projective Stiefel manifold $PW_{n,k}$ using the homotopy fixed point spectral sequence. Following this computation we discuss applications to equivariant maps between complex Stiefel manifolds. Most of the arguments carry over to any complex oriented cohomology theory. The same method is also likely to work for $P_\ell W_{n,k}$, the quotient by a variant of the S^1-action, whose cohomology was computed in [12]. Here, ℓ refers to a tuple of integers $(l_1, \cdots, l_k)$ and the action of S^1 is given by $z \cdot (v_1, \cdots, v_k) = (z^{l_1} v_1, \cdots, z^{l_k} v_k)$.

We observe that the BP-cohomology ring of $PW_{n,k}$ is just the extension of coefficients from $\mathbb{Z}_{(p)}$ in ordinary cohomology to $\mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ in BP-cohomology. Therefore, the primary multiplicative structure does not yield new results for equivariant maps between Stiefel manifolds. However, BP has the action of Adams operations [5], which yields new results on equivariant maps. (see Theorem 3.4.8)

We also obtain some new results using the action of Steenrod operations on $H^*PW_{n,k}$. We point out that the analysis of equivariant maps on Stiefel manifolds also leads to results in topological combinatorics [11]. The results in this chapter are written up in the paper [9].

3.1 Homotopy fixed point spectral sequence

The purpose of this section is to set up the computational tools for the following sections. The main idea here is the homotopy fixed point spectral sequence for (naive) G-equivariant spectra: for a spectrum Z with a G-action, there is a spectral sequence with E_2-page $H^s(G; \pi_t Z)$ which converges to $\pi_{t-s} Z^{hG}$ [17] (see also [22]).

The principal example for our paper is when G acts on a function spectrum $F(X, E)$ for a spectrum E and a based G-space X. Let us make this more precise. Let E be a spectrum so that the reduced E-cohomology of based spaces is computed as (note here that the cohomological grading is negative of the usual homotopy grading)

$$\tilde{E}^n(X) \cong [X, \Sigma^n E] \cong \pi_{-n} F(X, E).$$

Here we use the notation $[-,-]$ for the homotopy classes of maps between spectra. We follow the construction of function spectra in [28]. If X has a G-action, the function spectrum $F(X, E)$ is a spectrum with G-action (that is, a G-spectrum indexed over a trivial G-universe). We write $[-,-]^G_{tr}$ for the equivariant homotopy classes in the category of spectra with G-action, and $F^G_{tr}(-,-)$ for the equivariant function spectrum with G-action. We have the following result regarding this construction.

Proposition 3.1.1. [28, Ch. XVI, §1, (1.9)] Let X be a based G-space, and E a spectrum. Then,

$$\pi^G_{-n} F^G_{tr}(X, E) \cong [X, \Sigma^n E]^G_{tr} \cong [X/G, \Sigma^n E] \cong \tilde{E}^n(X/G).$$

For a free G-space X, we may apply Proposition 3.1.1 by adding a disjoint base-point. The homotopy fixed points of a spectrum Z with G-action are $Z^{hG} = F^G_{tr}(EG_+, Z)^G$, where EG is the contractible space with free G-action. We know that for a free G-space X, the projection $X \times EG \to X$ is a G-equivalence. Therefore, we have the following equivalence of spectra.

Proposition 3.1.2. Let X be a free G-space, and E a spectrum. Then

$$F^G_{tr}(X_+, E)^{hG} \simeq F^G_{tr}(X_+, E)^G \simeq F(X/G_+, E).$$

In this paper, we apply Corollary 3.1.2 to the case $X = W_{n,k}$, the Stiefel manifold of k-orthogonal vectors in $\mathbb{C}^n$. This action is free and the quotient space is the projective Stiefel manifold $PW_{n,k}$.

Corollary 3.1.3. Let E be a spectrum. There is an equivalence of spectra

$$F(PW_{n,k_+}, E) \simeq F^{S^1}_{tr}(W_{n,k_+}, E)^{hS^1}.$$

We attempt to understand $F^{S^1}_{tr}(W_{n,k_+}, E)^{hS^1}$ via the homotopy fixed point spectral sequence. For a spectrum Z with S^1-action, we follow the exposition in [15] replacing homology with homotopy groups. We have a S^1-equivariant filtration of ES^1 given by

$$\varnothing \subset S(\mathbb{C}) \subset S(\mathbb{C}^2) \subset \cdots \subset S(\mathbb{C}^r) \subset S(\mathbb{C}^{r+1}) \subset \cdots,$$

so that

$$Z^{hS^1} \simeq \varprojlim_r F_{tr}^{S^1}(S(\mathbb{C}^r)_+, Z)^{S^1}.$$

We index the filtration of ES^1 as

$$E^{(r)}S^1 = \begin{cases} S(\mathbb{C}^{\frac{r}{2}+1}) & \text{if } r \text{ is even} \\ E^{(r-1)}S^1 & \text{if } r \text{ is odd,} \end{cases}$$

so that

$$E^{(2r)}S^1/E^{(2r-1)}S^1 \simeq S^1_+ \wedge S^{2r}, \quad E^{(2r+1)}S^1/E^{(2r)}S^1 \simeq *,$$

where the action of S^1 on S^{2r} is the trivial action. The filtration on the induced tower of fibrations is written as

$$Z^{hS^1}_{(r)} = F_{tr}^{S^1}(E^{(r)}S^1_+, Z)^{S^1},$$

so that

$$Z^{hS^1}_{(r)}/Z^{hS^1}_{(r-1)} \simeq \begin{cases} F_{tr}^{S^1}(S^1_+ \wedge S^r, Z)^{S^1} \simeq \Sigma^{-r}Z & \text{if } r \text{ is even} \\ * & \text{if } r \text{ is odd.} \end{cases}$$

We may now follow [15] to obtain a conditionally convergent spectral sequence [13].

Proposition 3.1.4. Let Z be a homotopy commutative ring spectrum with S^1-action. There is a conditionally convergent multiplicative spectral sequence

$$E_2^{s,t} = H^s(S^1; \pi_t(Z)) \implies \pi_{t-s}(Z^{hS^1}).$$

In this expression, the group cohomology $H^*(S^1; \pi_t Z)$ of S^1 with coefficients in the discrete group $\pi_t Z$ equals $\mathbb{Z}[y] \otimes \pi_t Z$ with $|y| = (2, 0)$.

Example 3.1.5. If $Z = E$ with trivial S^1-action, the homotopy fixed point spectrum $Z^{hS^1} \simeq F(BS^1_+, E)$. In this case, the homotopy fixed point spectral sequence becomes

$$E_2^{s,t} = H^s(\mathbb{C}P^\infty) \otimes \pi_t E \implies \pi_{t-s} F(\mathbb{C}P^\infty_+, E).$$

Making identifications $E^n(\mathbb{C}P^\infty) \cong \pi_{-n}F(\mathbb{C}P^\infty_+, E)$, we observe that this reduces to the Atiyah-Hirzebruch spectral sequence for $\mathbb{C}P^\infty$. If E is complex orientable, the class y becomes a permanent cycle.

Next we specialize to the case $Z = F^{S^1}_{tr}(X_+, E)$ where X is a free S^1-space, and E is a spectrum. The homotopy groups of $F^{S^1}_{tr}(X_+, E)$ in Proposition 3.1.4 are computed by forgetting the S^1 action, and thus we have,

$$\pi_t F^{S^1}_{tr}(X_+, E) \cong \pi_t F(X_+, E) \cong E^{-t}(X).$$

On the other hand, we apply Corollary 3.1.2 to deduce

$$\pi_t F^{S^1}_{tr}(X_+, E)^{hS^1} \cong \pi_t F^{S^1}_{tr}(X_+, E)^{S^1} \cong \pi_t F(X/S^1_+, E) \cong E^{-t}(X/S^1).$$

We now switch the sign of the t-grading in the spectral sequence of Proposition 3.1.4 to obtain a conditionally convergent multiplicative spectral sequence

$$E_2^{s,t} = H^s(S^1; E^t(X)) \cong \mathbb{Z}[y] \otimes E^t(X) \implies E^{s+t}(X/S^1).$$

We summarize these facts together in Proposition 3.1.6. For the rest of the section, X is a free S^1-space and E a homotopy commutative ring spectrum.

Proposition 3.1.6. There is a conditionally convergent multiplicative spectral sequence

$$E_2^{s,t} = H^s(S^1; E^t(X)) \cong \mathbb{Z}[y] \otimes E^t(X) \implies E^{s+t}(X/S^1).$$

1) If E is complex orientable, the class y is a permanent cycle.

2) The differential d_r changes the grading by $(s,t) \mapsto (s+r, t-r+1)$.

3) If X, X/S^1 are finite CW complexes, and E is complex orientable, the spectral sequence is strongly convergent.

Proof. The degree of the differentials follow from the construction of the exact couple for the spectral sequence. We also have the map $X_+ \to S^0$ which gives a map $E \to F^{S^1}_{tr}(X_+, E)$ which is S^1-equivariant. Thus we have a map between the homotopy fixed point spectral sequences which maps the classes y to one another, and so, 1) follows from the identification in Example 3.1.5.

It remains to prove 3). For this, we show that for k sufficiently large, y^k lies in the image of a differential. It will then follow that for r sufficiently large the classes y^m and their $\pi_* E$

multiples are 0 in the E_r-page for $m \geq k$. Therefore, the E_r-page will be concentrated in the columns between 1 and k, and $E_\infty = E_r$ by increasing r further if necessary. Hence, the spectral sequence converges strongly [13, Theorem 7.4].

The space X/S^1 being finite dimensional implies that the classifying map $X/S^1 \to BS^1$ (for the S^1-bundle $X \to X/S^1$) factors through a finite skeleton. Hence, we have an equivariant map $X \to S(\mathbb{C}^{k+1})$ for some k, and thus a map $F_{tr}^{S^1}(S(\mathbb{C}^{k+1})_+, E) \to F_{tr}^{S^1}(X_+, E)$. As E is complex orientable,

$$\pi_* F_{tr}^{S^1}(S(\mathbb{C}^{k+1})_+, E) \cong \pi_* F(\mathbb{C}P^k, E) \cong E^{-*}(\mathbb{C}P^k) \cong \pi_* E[y]/(y^{k+1})$$

for some choice of complex orientation y. Observe that the homotopy fixed point spectral sequence for the space ES^1 as in Example 3.1.5 matches with the Atiyah-Hirzebruch spectral sequence for $\mathbb{C}P^\infty$. It follows that the class y represents the complex orientation in the E_2-page. For $S(\mathbb{C}^{k+1})$ and hence also for X via the equivariant map $X \to S(\mathbb{C}^{k+1})$, the class y represents a nilpotent class whose $k+1$-power is 0. Therefore, y^{k+1} must lie in the image of a differential, and 3) follows. $\square$

Example 3.1.7. Suppose that $E = HR$ for a commutative ring R, the Eilenberg-MacLane spectrum with $\pi_0 HR = R$. In this case the spectral sequence in Proposition 3.1.6 matches the Serre spectral sequence (from the E^2-page onwards) associated to the fibration

$$X \to X/S^1 \to \mathbb{C}P^\infty$$

obtained by identifying the homotopy orbits space $X_{hS^1} \simeq X/S^1$, and the classifying space $BS^1 \simeq \mathbb{C}P^\infty$. In this case, the spectral sequence is strongly convergent from the corresponding result for the Serre spectral sequence. Moreover, due to the fact that X/S^1 is a finite complex, the E_∞-page vanishes beyond the dimension of X/S^1.

Next we provide a method to compute the differentials in the spectral sequence of Proposition 3.1.6. In the tower of fibrations used to construct the spectral sequence, the spectrum at the bottom of the tower is $F_{tr}^{S^1}(S^1 \times X_+, E)^{S^1} \simeq F(X_+, E)$.

$$\begin{tikzcd}
& F_{tr}^{S^1}(S(\mathbb{C}^r) \times X_+, E)^{S^1} \arrow[d] \\
& \vdots \\
& F_{tr}^{S^1}(S(\mathbb{C}^2) \times X_+, E)^{S^1} \arrow[d] \\
F(X/S^1_+, E) \simeq F_{tr}^{S^1}(X_+, E)^{hS^1} \arrow[r] \arrow[ur] \arrow[uuur] & F_{tr}^{S^1}(S^1 \times X_+, E)^{S^1} \simeq F(X_+, E).
\end{tikzcd}$$

Let Q denote the homotopy cofibre of the map $X \to X/S^1$. In the category of spectra, $F(\Sigma^{-1}Q, E) \simeq \Sigma F(Q, E)$ is the homotopy cofibre of the map $F(X/S^1_+, E) \to F(X_+, E)$. In view of the commutative square

$$\begin{tikzcd}
F(X/S^1_+, E) \arrow[r, "\simeq"] & F_{tr}^{S^1}(ES^1 \times X_+, E)^{S^1} \arrow[r] \arrow[d] & F(X_+, E) \arrow[d, equal] \\
& F_{tr}^{S^1}(S(\mathbb{C}^{k+1}) \times X_+, E)^{S^1} \arrow[r] & F(X_+, E),
\end{tikzcd}$$

we obtain coherent maps $P_k : \Sigma F(Q, E) \to \Sigma F_{tr}^{S^1}(Q(k), E)^{S^1}$, where $Q(k) = [S(\mathbb{C}^{k+1})/S(\mathbb{C})] \wedge X_+$ is the S^1-equivariant homotopy cofibre of $X \times S(\mathbb{C}) \to X \times S(\mathbb{C}^{k+1})$. Projecting onto the first factor gives a map $Q(k) \to S(\mathbb{C}^{k+1})/S(\mathbb{C})$ which gives a map

$$F(\mathbb{C}P^k, E) \simeq F_{tr}^{S^1}(S(\mathbb{C}^{k+1})/S(\mathbb{C}), E)^{S^1} \to F_{tr}^{S^1}(Q(k), E)^{S^1}.$$

Let an element $x \in E^n(X)$ be represented by the map $S^{-n} \xrightarrow{x} F(X_+, E)$. Our hypothesis about such an x is a factorization in the following commutative diagram for $0 \leq k \leq \infty$.

$$\begin{tikzcd}
S^{-n} \arrow[r, "y"] \arrow[d, "x"] & \Sigma F(\mathbb{C}P^k, E) \arrow[d] \\
F(X_+, E) \arrow[r] & \Sigma F_{tr}^{S^1}(Q(k), E)^{S^1}.
\end{tikzcd} \tag{3.1.8}$$

Before applying this hypothesis we note

Proposition 3.1.9. Suppose that the composite $S^{-n} \xrightarrow{x} F(X_+, E) \to \Sigma F_{tr}^{S^1}(Q(k), E)^{S^1}$ is null-homotopic. Then, $d_r(x) = 0$ for $r \leq 2k + 1$.

Proof. The statement follows from the fact that the composite being null-homotopic implies that x lifts in the tower of fibrations to $F_{tr}^{S^1}(S(\mathbb{C}^{k+1}) \times X_+, E)^{S^1}$. $\square$

Example 3.1.10. In the case $E = HR$, the spectral sequence is the Serre spectral sequence of the fibration $X \to X/S^1 \to \mathbb{C}P^\infty$ according to Example 3.1.7. Note that

$$F(\mathbb{C}P^k, HR) \simeq \bigvee_{1 \le i \le k} \Sigma^{-2i} HR,$$

so in the diagram (3.1.8) y may be non-trivial only when n is odd, and $k \ge \frac{n+1}{2}$. If n is odd and (3.1.8) holds for $k = \frac{n+1}{2}$, the class x is transgressive, and $d_{n+1}(x)$ is the composite

$$S^{-n} \xrightarrow{y} \Sigma F(\mathbb{C}P^{\frac{n+1}{2}}, HR) \simeq \bigvee_{1 \le i \le \frac{n+1}{2}} \Sigma^{-2i+1} HR \to \Sigma^{-n} HR.$$

We assume now that E is connective, and that (3.1.8) holds for $k = \infty$. In this case we have

Proposition 3.1.11. Suppose that (3.1.8) holds for $k = \infty$ and that E is connective. Then, $d_r(x) = 0$ if $r \le n$. Further, $d_{n+1}(x) = d^H_{n+1}(q_H(x))$, where q_H is the map $E \to H\pi_0 E$, and d^H_{n+1} is the $(n+1)^{th}$ differential for the spectral sequence of Proposition 3.1.6 for $H\pi_0 E$. (Here we observe that the spectral sequence is one of $\pi_0 E$-modules, so this allows us to interpret the last statement.)

Proof. We observe that E is connective implies that $\Sigma F(\mathbb{C}P^k, E)$ is $(-2k+1)$-connective (that is, the homotopy groups are 0 in degree $\le -2k$). Therefore, the composite

$$S^{-n} \xrightarrow{y} \Sigma F(\mathbb{C}P^\infty, E) \to \Sigma F(\mathbb{C}P^k, E)$$

is trivial for degree reasons if $-n \le -2k$. From the commutative square

$$\begin{tikzcd}
S^{-n} \arrow[r, "y"] \arrow[d, "x"] & \Sigma F(\mathbb{C}P^\infty, E) \arrow[r] \arrow[d] & \Sigma F(\mathbb{C}P^k, E) \arrow[d] \\
F(X_+, E) \arrow[r] & \Sigma F(Q, E) \arrow[r, "P_k"] & \Sigma F^{S^1}_{tr}(Q(k), E)^{S^1}.
\end{tikzcd}$$

we deduce that the composite map from S^{-n} to $\Sigma F^{S^1}_{tr}(Q(k), E)^{S^1}$ along the lower row is trivial. Hence, from Proposition 3.1.9 we get that $d_r(x) = 0$ if $r \le n$.

Via the map $q_H : E \to H\pi_0 E$, we observe that (3.1.8) also holds for $q_H(x)$ when we replace E by $H\pi_0 E$. Therefore, in the associated spectral sequence $d^H_r(q_H(x)) = 0$ if $r \le n$ and $d^H_{n+1}(q_H(x))$ is described in the formula in Example 3.1.10. Also we need only assume n is odd, as the result is vacuously true in the other case. We fix $k = \frac{n+1}{2}$ so that $n = 2k - 1$.

With this choice of n and k,

$$\pi_{-n}\Sigma F(\mathbb{C}P^k, E) \cong \pi_0 E,\ \pi_{-n}\Sigma F(\mathbb{C}P^{k-1}, E) = 0.$$

It follows that the composite of y to $\Sigma F(\mathbb{C}P^k, E)$ lifts to $y_k : S^{-n} \to \Sigma F(\mathbb{C}P^k/\mathbb{C}P^{k-1}, E)$.

The differential $d_{n+1}(x)$ may be described as the composite

$$S^{-n} \xrightarrow{\chi} F_{tr}^{S^1}(S(\mathbb{C}^k) \times X_+, E)^{S^1} \to \Sigma F_{tr}^{S^1}([S(\mathbb{C}^{k+1})/S(\mathbb{C}^k)] \wedge X_+, E)^{S^1}$$

where χ is a lift of x along the map $F_{tr}^{S^1}(S(\mathbb{C}^k) \times X_+, E)^{S^1} \to F(X_+, E)$. We expand this in the diagram below

$$\begin{tikzcd}
S^{-n} \arrow[r, "d_{n+1}(x)"] \arrow[d, "\chi"] \arrow[dd, bend right, "x"'] & \Sigma F_{tr}^{S^1}([S(\mathbb{C}^{k+1})/S(\mathbb{C}^k)] \wedge X_+, E)^{S^1} \\
F_{tr}^{S^1}(S(\mathbb{C}^k) \times X_+, E)^{S^1} \arrow[r] \arrow[d] & \Sigma F_{tr}^{S^1}([S(\mathbb{C}^\infty)/S(\mathbb{C}^k)] \wedge X_+, E)^{S^1} \arrow[u] \arrow[d] \\
F(X_+, E) \arrow[r] & \Sigma F_{tr}^{S^1}(Q, E)^{S^1}.
\end{tikzcd}$$

Observe that the bottom square is a homotopy pullback square of spectra as the homotopy fibre of both the vertical maps are $F_{tr}^{S^1}(Q(k-1), E)^{S^1}$. Therefore, the map χ is determined from x and the map $S^{-n} \to \Sigma F_{tr}^{S^1}([S(\mathbb{C}^\infty)/S(\mathbb{C}^k)] \wedge X_+, E)^{S^1}$. This may now be computed using the lift of y to $\Sigma F(\mathbb{C}P^\infty/\mathbb{C}P^{k-1}, E)^{S^1}$ as it's restriction to $\mathbb{C}P^{k-1}$ is 0. We may now compute $d_{n+1}(x)$ via the following commutative diagram

$$\begin{tikzcd}
S^{-n} \arrow[r] \arrow[rr, bend left, "y_k"] \arrow[d, "\chi"] \arrow[drr, "d_{n+1}(x)"] & \Sigma F(\mathbb{C}P^\infty/\mathbb{C}P^{k-1}, E)^{S^1} \arrow[r] \arrow[d] & \Sigma F(\mathbb{C}P^k/\mathbb{C}P^{k-1}, E) \arrow[d] \\
F_{tr}^{S^1}(S(\mathbb{C}^k) \times X_+, E)^{S^1} \arrow[r] & \Sigma F_{tr}^{S^1}([S(\mathbb{C}^\infty)/S(\mathbb{C}^k)] \wedge X_+, E)^{S^1} \arrow[r] & \Sigma F_{tr}^{S^1}([S(\mathbb{C}^{k+1})/S(\mathbb{C}^k)] \wedge X_+, E)^{S^1}.
\end{tikzcd}$$

The middle vertical map is the one which quotients out the factor X, and this also induces the right vertical map. Under the identification $\Sigma F(\mathbb{C}P^k/\mathbb{C}P^{k-1}, E) \simeq \Sigma^{-2k+1}E$, and $\pi_{-n}\Sigma^{-2k+1}E \cong \pi_0 E$, we identify y_k with $d_{n+1}^H(x)$. □

3.2 The cohomology of $W_{n,k}$

In this section, we calculate the generalized cohomology of $W_{n,k}$ with respect to a complex oriented spectrum E. Later in the section, we specialize to $E = BP$, the spectrum for Brown-Peterson cohomology.

Recall that a complex orientation for a homotopy commutative ring spectrum E is a class $x \in \tilde{E}^2(\mathbb{C}P^\infty)$, which restricts to a generator of the free rank one $\pi_0 E$-module $\tilde{E}^2(S^2) \cong E^0(pt)$. For a complex oriented spectrum E, we have [3].

$$E^*(\mathbb{C}P^n) \cong E^*(pt)[x]/(x^{n+1}), \quad E^*(\mathbb{C}P^\infty) \cong E^*(pt)[[x]].$$

For the complex Stiefel manifold, the classical computations of their cohomology [31] proceeds using the Serre spectral sequence as for other homogeneous spaces. It is proved that the cohomology of $W_{n,k}$ is an exterior algebra with generators in degrees $2n-2k+1, 2n-2k+3, \cdots, 2n-1$. The Stiefel manifold also has a filtration

$$\begin{tikzcd} W_{n-k+1,1} \arrow[r, hook] \arrow[d, equal] & W_{n-k+2,2} \arrow[r, hook] & \cdots W_{n-1,k-1} \arrow[r, hook] & W_{n,k}, \\ S^{2n-2k+1} & & & \end{tikzcd}$$

where the inclusion $W_{n-1,k-1} \hookrightarrow W_{n,k}$ is given by adding the last vector e_n. The filtration quotients are computed using the following homotopy pushout ([38, Chapter IV] defines the maps in the diagram below, and [32, Ch. 5, Proposition 2] proves the entirely analogous result in the real case)

$$\begin{tikzcd} \Sigma(\mathbb{C}P_+^{n-2}) \times W_{n-1,k-1} \arrow[r, "\mu_{n-1}"] \arrow[d, hook] & W_{n-1,k-1} \arrow[d, hook] \\ \Sigma(\mathbb{C}P_+^{n-1}) \times W_{n-1,k-1} \arrow[r, "\mu_n"] & W_{n,k}. \end{tikzcd} \tag{3.2.1}$$

In order to construct μ_n one defines

$$S^1 \times \mathbb{C}P^{n-1} \to U(n)$$

by $(z, L) \mapsto A(z, L)$, where $A(z, L) : \mathbb{C}^n \to \mathbb{C}^n$ is the unitary transformation which multiplies the elements of L by z and fixes the orthogonal complement. The map μ_n is induced by matrix multiplication in $U(n)$ and the left action on $W_{n,k}$. From the construction of μ and the fact that $W_{n,k} \cong U(n)/U(n-k)$, one obtains the induced map

$$\mu_{n,k} : \Sigma[\mathbb{C}P^{n-1}/\mathbb{C}P^{n-k-1}] \to W_{n,k}.$$

It follows from (3.2.1) that

$$W_{n,k}/W_{n-1,k-1} \simeq \Sigma[\mathbb{C}P^{n-1}/\mathbb{C}P^{n-2}] \wedge W_{n-1,k-1_+} \simeq \Sigma^{2n-1}(W_{n-1,k-1_+}).$$

In the case of ordinary cohomology, the exterior algebra generators for the cohomology of $W_{n,k}$ pull back under $\mu_{n,k}$ to Σx^{i-1} ($\Sigma : H^*X \to H^*\Sigma X$ is the suspension isomorphism). We now use this filtration to prove analogous results for the E-cohomology of $W_{n,k}$.

Proposition 3.2.2. Let E be a complex oriented cohomology theory, such that there is no 2-torsion in $E^*(pt)$. Then,

$$E^*(W_{n,k}) \cong \Lambda_{E^*(pt)}(z_{n-k+1}, \cdots, z_n),$$

is an exterior algebra with $|z_i| = 2i - 1$. These generators satisfy

1) The inclusion $W_{n-1,k-1} \hookrightarrow W_{n,k}$ sends z_i to z_i if $n - k + 1 \leq i \leq n - 1$ and sends z_n to 0.

2) $\mu^*_{n,k}(z_i) = \Sigma x^{i-1}$.

Proof. We prove the results by induction on k, constructing the generators z_i along the way. For $k = 1$, the Stiefel manifold is the sphere S^{2n-1}, and in this case, we know that the E-cohomology is the exterior algebra on one generator. This starts the induction.

In the induction step, we know that $E^*(W_{n-1,k-1})$ is as described in this Proposition, and attempt to derive the same for $E^*(W_{n,k})$ via the pushout (3.2.1). This gives us the following maps between long exact sequences corresponding to the columns of (3.2.1).

$$\begin{tikzcd}
{} \arrow[r, "0"] & \widetilde{E}^r(\Sigma^{2n-1}(W_{n-1,k-1_+})) \arrow[r, "j^*"] \arrow[d, "id"] & E^r(W_{n,k}) \arrow[r, two heads, "i^*"] \arrow[d, "\mu^*"] & E^r(W_{n-1,k-1}) \arrow[r, "0"] \arrow[d] & {} \\
{} \arrow[r, "0"] & \widetilde{E}^r(\Sigma^{2n-1}(W_{n-1,k-1_+})) \arrow[r] & E^r(\Sigma(\mathbb{C}P^{n-1}_+) \times W_{n-1,k-1}) \arrow[r, two heads] & E^r(\Sigma(\mathbb{C}P^{n-2}_+) \times W_{n-1,k-1}) \arrow[r, "0"] & {}
\end{tikzcd} \tag{3.2.3}$$

We now justify the various identifications described in (3.2.3). The fact that E is complex oriented implies $E^*(\mathbb{C}P^{n-1}) \to E^*(\mathbb{C}P^{n-2})$ is surjective, and the induction hypothesis gives us that $E^*(W_{n-1,k-1})$ is a free $E^*(pt)$-module. This implies that

$$E^r(\Sigma(\mathbb{C}P^{n-1}_+) \times W_{n-1,k-1}) \to E^r(\Sigma(\mathbb{C}P^{n-2}_+) \times W_{n-1,k-1})$$

is surjective. This implies the identifications in the bottow row of (3.2.3). Note that the map $\Sigma(\mathbb{C}P^{n-2}_+) \times W_{n-1,k-1} \to W_{n-1,k-1}$ has a section corresponding to the inclusion of the base-point of $\Sigma(\mathbb{C}P^{n-2}_+)$, and thus, the map induced on E-cohomology is injective. The identifications on the top follow from the ones of the bottom row, and the fact that $E^r(W_{n-1,k-1}) \to E^r(\Sigma(\mathbb{C}P^{n-2}_+) \times W_{n-1,k-1})$ is injective. It follows that we have short exact sequences

$$0 \to E^*(\Sigma^{2n-1} W_{n-1,k-1_+}) \xrightarrow{j^*} E^*(W_{n,k}) \xrightarrow{i^*} E^*(W_{n-1,k-1}) \to 0.$$

For $n-k+1 \leq i \leq n-1$, we choose $z_i \in E^{2i-1}(W_{n,k})$ so that they map to z_i under i^*. The class z_n is chosen so that it maps to Σx^{n-1} under $\mu_{n,n-1}$. From (3.2.3), it follows that z_n is a generator for the ideal of $E^*(W_{n,k})$ given by image of j^*. By the construction 1) and 2) follow. As the classes z_i are in odd degree and $E^*(pt)$ has no 2-torsion, we have $z_i^2 = 0$, and (3.2.3) implies that $E^*(W_{n,k})$ additively matches with the exterior algebra on the z_i. The result now follows by induction on k. □

We now proceed to define the generators of the exterior algebra $E^*(W_{n,k})$ in a strict fashion which will satisfy 1) and 2) of Proposition 3.2.2. From the proof, we note that for any classes z_i satisfying 2), $E^*(W_{n,k}) \cong \Lambda_{E^*(pt)}(z_{n-k+1}, \cdots, z_n)$. Although the results in the following will have analogous consequences for any complex oriented E, we fix our attention to the case $E = BP$, which will be used in the following sections. Recall [34]

$$BP^*(pt) \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots],$$

where v_i denotes the Araki generators [34, A2.2.2] that lie in degree $-2(p^i-1)$. We also fix from now on $x \in \widetilde{BP}^2(\mathbb{C}P^\infty)$ to denote the fixed orientation for a p-typical formal group law over $BP^*(pt)$. We also assume that x is such that it maps to the first Chern class under the map $\lambda : BP \to H\mathbb{Z}_{(p)}$.

The method of choosing the generators y_j for $BP^*(W_{n,k})$ is by relating them to the BP-Chern classes c_j^{BP} [16]. We start with the case $k = n$, when $W_{n,n} = U(n)$. Recall that $H^*(U(n);\mathbb{Z}_{(p)}) = \Lambda_{\mathbb{Z}_{(p)}}(y_1^H, \cdots, y_n^H)$ with $|y_j^H| = 2j-1$, and in Serre spectral sequence for the fibration

$$U(n) \to EU(n) \to BU(n),$$

y_j^H transgresses to the j^{th}-Chern class c_j^H. This follows from [14] which identifies the transgression for the above spectral sequence. We also know that $A^*(y_j^H) = \Sigma x_H^{j-1}$, where $A : \Sigma(\mathbb{C}P_+^{n-1}) \to U(n)$ is induced from $(z, L) \mapsto A(z, L)$, and x_H is the first H-Chern class of the canonical line bundle over $\mathbb{C}P^\infty$. Write $\sigma : \Sigma U(n) \to BU(n)$ for the adjoint of the equivalence $U(n) \simeq \Omega BU(n)$, and form the composite diagram

$$\begin{tikzcd} \Sigma^2(\mathbb{C}P_+^{n-1}) \arrow[r, "\Sigma A"] \arrow[rr, dashed, bend right, "\phi"'] & \Sigma U(n) \arrow[r, "\sigma"] & BU(n). \end{tikzcd}$$

For a cohomology theory E, denote by ϕ_E^* (respectively σ_E^*) the map induced by ϕ (respectively σ) on E-cohomology. We have $\phi_H^*(c_j) = \Sigma^2 x_H^{j-1}$ as $\sigma_H^*(c_j) = \Sigma y_j^H$.

Proposition 3.2.4. There are classes $\tau_j \in BP^{2j}(BU(n))$ of the form

$$\tau_n = c_n^{BP}, \quad \text{and } \forall\, 1 \leq j \leq n, \tau_j = c_j^{BP} + \sum_{k>j} \nu_k c_k^{BP}$$

for $\nu_k \in BP^*(pt)$, such that

$$\phi_{BP}^* \tau_j = \Sigma^2 x^{j-1}.$$

The standard map $BU(n) \to BU(n+1)$ classifying the canonical bundle plus a trivial bundle sends τ_j to τ_j for $j \leq n$, and τ_{n+1} to 0. Define $y_j^{BP} \in BP^{2j-1}(U(n))$ by the formula $\Sigma y_j^{BP} = \sigma_{BP}^* \tau_j$. Then,

1) The classes $y_1^{BP}, \cdots, y_n^{BP}$ are generators for the exterior algebra $BP^*(U(n))$.

2) $\lambda(y_j^{BP}) = y_j^H$. (That is, the classes y_j^{BP} are lifts of the cohomology classes y_j^H to BP.)

Proof. We note that using Proposition 3.2.2, it suffices to prove the statements about τ_j. Consider the following commutative diagram

$$\begin{tikzcd}
BP^*(\Sigma^2(\mathbb{C}P_+^{n-1})) \arrow[d, "\lambda"'] & BP^*(\Sigma U(n)) \arrow[l] \arrow[d, "\lambda"'] & BP^*(BU(n)) \arrow[l, "\sigma_{BP}^*"'] \arrow[d, "\lambda"] \arrow[ll, bend right, dashed, "\phi_{BP}^*"'] \\
H^*(\Sigma^2(\mathbb{C}P_+^{n-1}); \mathbb{Z}_{(p)}) & H^*(\Sigma U(n); \mathbb{Z}_{(p)}) \arrow[l] & H^*(BU(n); \mathbb{Z}_{(p)}). \arrow[l, "\sigma_H^*"] \arrow[ll, bend left, dashed, "\phi_H^*"]
\end{tikzcd} \tag{3.2.5}$$

We have $\lambda(c_j^{BP}) = c_j$, and also that λ maps the complex orientation of BP to that of H. It readily follows that $\phi_{BP}^*(c_j^{BP}) - \Sigma^2 x^{j-1}$ lies in the kernel of λ, which is the ideal $(v_1, v_2, \cdots)$. As

$$BP^*(\Sigma^2(\mathbb{C}P_+^{n-1})) \cong \mathbb{Z}_{(p)}[v_1, v_2, \cdots]\{\Sigma^2 1, \Sigma^2 x, \cdots, \Sigma^2 x^{n-1}\},$$

the left vertical arrow of (3.2.5) is an isomorphism in degree $2n$. It follows that $\phi_{BP}^*(c_n^{BP}) = \Sigma^2 x^{n-1}$, and so, $\tau_n^{BP} = c_n^{BP}$ maps to the element of $BP^*(\Sigma^2(\mathbb{C}P_+^{n-1}))$ required by the Proposition.

We proceed to construct the τ_j such that $\phi_{BP}^* \tau_j = \Sigma^2 x^{j-1}$. Starting from $j = n$, suppose that τ_{j+1} has already been defined. We now have $\phi_{BP}^*(c_j^{BP}) - \Sigma^2 x^{j-1} \in (v_1, v_2, \cdots)$. For degree reasons we have,

$$\phi_{BP}^*(c_j^{BP}) - \Sigma^2 x^{j-1} = \sum_{k>j} \rho_k \Sigma^2 x^{k-1} = \sum_{k>j} \rho_k \phi_{BP}^*(\tau_k),$$

for some $\rho_k \in (v_1, v_2, \cdots)$. Rearranging terms and substituting the formula for τ_k, we obtain an equation

$$\Sigma^2 x^{j-1} = \phi^*_{BP}(c_j^{BP} + \sum_{k>j} \nu_k c_k^{BP}),$$

where ν_k is a $BP^*(pt)$-linear combination of the classes ρ_k in the preceeding equation. It follows that $\tau_j = c_j^{BP} + \sum_{k>j} \nu_k c_k^{BP}$ satisfies the required criteria. We note that ϕ^*_{BP} has image in $BP^*(\Sigma^2(\mathbb{C}P^{n-1}_+))$ which is a suspension. It follows that the decomposable elements over $BP^*(pt)$ map to 0 under ϕ^*_{BP}. Also the formula $\phi^*_{BP}(\tau_j) = \Sigma^2 x^{j-1}$ implies that ϕ^*_{BP} induces an isomorphism when restricted to the module of indecomposables. This shows that the elements ν_k are unique, and so the classes τ_k are coherently defined over n as required in the Proposition. $\square$

We now provide a strict definition for the generators of $BP^*(W_{n,k})$ following Proposition 3.2.4. Recall that there are maps

$$i : W_{n-1,k-1} \to W_{n,k}, \quad q : W_{n,k} \to W_{n,k-1},$$

where i adds the vector e_n at the end, and q forgets the last vector. We have already seen in (3.2.3) that i^* is surjective in BP-cohomology. We also note that q^* is injective. For, q^* applied to the generators of $BP^*(W_{n,k-1})$ as in Proposition 3.2.2 together with a generator of $BP^*(S^{2n-2k+1}) = BP^*(W_{n-k+1,1})$ satisfies 2) of Proposition 3.2.2. This provides a tuple of exterior algebra generators for $BP^*(W_{n,k})$. Therefore, the quotient map $\pi : U(n) \to W_{n,k}$ is injective in BP-cohomology.

Proposition 3.2.6. With notations as above, π^* maps $BP^*(W_{n,k})$ to the subalgebra of $BP^*(U(n))$ generated by the classes $y^{BP}_{n-k+1}, \cdots, y^{BP}_n$.

Proof. We have a diagram of fibrations

$$\begin{tikzcd}
U(n) \arrow[r] \arrow[d] & W_{n,k} \arrow[d] \\
EU(n) \arrow[r] \arrow[d] & BU(n-k) \arrow[d] \\
BU(n) \arrow[r, equal] & BU(n),
\end{tikzcd}$$

which induces the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & BP^i(U(n)) \arrow[r, "\delta"] & BP^{i+1}(EU(n),U(n)) \arrow[r] & BP^{i+1}(EU(n)) \arrow[r] & \cdots \\
\cdots \arrow[r] & BP^i(W_{n,k}) \arrow[r, "\delta"] \arrow[u, "\pi^*"] & BP^{i+1}(BU(n-k),W_{n,k}) \arrow[r] \arrow[u, "\pi^*"] & BP^{i+1}(BU(n-k)) \arrow[r] \arrow[u] & \cdots \\
& & \widetilde{BP}^{i+1}(BU(n)). \arrow[u, "\alpha"] & &
\end{tikzcd} \tag{3.2.7}$$

From the construction of the classes y_j^{BP} we have, $\pi^*\alpha(\tau_j) = \delta(y_j^{BP})$. On the other hand, if $j > n-k$, the class $\alpha(\tau_j)$ maps to 0 in $BP^*(BU(n-k))$. The map $BP^*BU(n) \to BP^*BU(n-k)$ is the map on BP-cohomology associated to the standard inclusion $BU(n-k) \to BU(n)$ classifying the sum of the canonical bundle with k-copies of a trivial bundle. This maps c_j^{BP} to 0 for $j > k$, and hence, from the formula in Proposition 3.2.4, the classes τ_j to 0 if $j > k$. It follows that for $j > k$, there are classes $y_j \in BP^*(W_{n,k})$ such that $\alpha(\tau_j) = \delta(y_j)$, and from (3.2.7) that $\pi^*(y_j) = y_j^{BP}$. Also the property $\phi_{BP}^*(\tau_j) = \Sigma^2 x^{j-1}$ implies that the classes y_j satisfies 2) of Proposition 3.2.2. The result follows readily. $\square$

3.3 BP-cohomology of $PW_{n,k}$

In this section, we describe the BP-cohomology ring of $PW_{n,k}$ using the homotopy fixed point spectral sequence (Proposition 3.1.6). This is a strongly convergent spectral sequence

$$E_2^{s,t} = \mathbb{Z}[x] \otimes BP^t(W_{n,k}) \implies BP^{s+t}(PW_{n,k}) \tag{3.3.1}$$

Recall that

$$BP^*(W_{n,k}) \cong \Lambda_{BP^*(pt)}(y_{n-k+1}, \cdots, y_n)$$

by Proposition 3.2.6. We start with a proposition describing the initial differential on the classes y_j.

Proposition 3.3.2. In the spectral sequence (3.3.1), the differentials on y_j are described by

$$d_r(y_j) = \begin{cases} 0 & \text{if } r < 2j \\ \binom{n}{j} x^j & \text{if } r = 2j. \end{cases}$$

Proof. The proof will follow from the existence of a diagram as in (3.1.8). We have the commutative diagram

$$\begin{tikzcd}
W_{n,k} \arrow[r] \arrow[d] & PW_{n,k} \arrow[r] \arrow[d] & \mathbb{C}P^{\infty} \arrow[d, "f"] \\
W_{n,k} \arrow[r] & BU(n-k) \arrow[r] & BU(n),
\end{tikzcd}$$

in which the rows are fibrations [6]. The map f classifies the n-fold Whitney sum of universal canonical complex line bundle. It leads to the following commutative diagram

$$\begin{tikzcd}
BP^{2j-1}(W_{n,k}) \arrow[r] \arrow[d] & BP^{2j}(BU(n-k), W_{n,k}) \arrow[d] & BP^{2j}(BU(n)) \arrow[l] \arrow[d, "f^*"] \\
BP^{2j-1}(W_{n,k}) \arrow[r] & BP^{2j}(PW_{n,k}, W_{n,k}) & BP^{2j}(\mathbb{C}P^{\infty}). \arrow[l]
\end{tikzcd}$$

Suppose that the class τ_j of Proposition 3.2.4 is mapped to ψ_j under f^*. In the first row, the image of y_j and image of τ_j coincide (Proposition 3.2.4 and (3.2.7)), hence, the same must happen in the bottom row leading to the following homotopy commutative diagram

$$\begin{tikzcd}
S^{-(2j-1)} \arrow[r, "\psi_j"] \arrow[d, "y_j"] & \Sigma F(\mathbb{C}P^{\infty}, BP) \arrow[d] \\
F(W_{n,k_+}, BP) \arrow[r] & \Sigma F(PW_{n,k}/W_{n,k}, BP).
\end{tikzcd} \tag{3.3.3}$$

The Whitney sum formula for Chern classes over BP-cohomology implies that $f^*(c_j^{BP}) = \binom{n}{j}x^j$. The description of τ_j in Proposition 3.2.4 now leads to the following form for ψ_j

$$\psi_j = \binom{n}{j}x^j + \sum_{k>j} \nu_k \binom{n}{k} x^k. \tag{3.3.4}$$

Now apply Proposition 3.1.11 to get $d_r(y_j) = 0$ if $r < 2j$, and $d_{2j}y_j$ is determined from the corresponding spectral sequence over $H\mathbb{Z}_{(p)}$. This may be computed as in [6] to be $d_{2j}y_j = \binom{n}{j}x^j$. Hence the result follows. □

We now proceed to compute the E_∞-page of the spectral sequence. The main idea here is that (3.3.3) may be used to determine all the differentials on the classes y_j.

Proposition 3.3.5. The E_∞-page of the spectral sequence (3.3.1) is given by

$$E_\infty = \Lambda_{BP^*(pt)}(\gamma_{n-k+2}, \cdots, \gamma_n) \otimes_{BP^*(pt)} BP^*(pt)[[x]]/I$$

where γ_j are certain elements in $BP^*(W_{n,k})$ with $\deg(\gamma_j) = 2j-1$, and I is the ideal of $BP^*[[x]]$ generated by the set $\{\binom{n}{j}x^j | n-k < j \leq n\}$.

Proof. The class x is a permanent cycle by Proposition 3.1.6. The multiplicative structure determines all the differentials once they are known on the classes y_j. We notice that E_{2n+1} is the E_∞-page because $d_{2n}(y_n) = x^n$ (Proposition 3.3.2) and so all the higher powers of x are killed in the E_{2n}-page.

From Proposition 3.3.2, we see that the first non-trivial differential is $d_{2(n-k+1)}$ and the generator y_{n-k+1} and all its multiples do not survive to the next page since

$$d_{2(n-k+1)}(y_{n-k+1}) = \binom{n}{n-k+1} x^{n-k+1}.$$

For the y_j of higher degree, it may happen that the first non-trivial differential on it

$$d_{2j}y_j = \binom{n}{j} x^{n-k+1}$$

may be zero. This precisely happens when $\binom{n}{j}$ lies in the ideal generated by $\binom{n}{i}$ for $n-k+1 \leq i < j$ inside $\mathbb{Z}_{(p)}$. This condition may be interpreted in terms of p-adic valuations of these numbers. We then obtain a multiple $p^s y_j$ on which the differential is 0, determined by the formula $s + v_p(\binom{n}{j}) = \min_{n-k+1 \leq i < j} v_p(\binom{n}{i})$. The class $p^s y_j$ may now support higher order differentials. Their formula is determined by computing $p^s\psi_j$ using (3.3.3) in the form of (3.1.8)

$$\begin{tikzcd}
S^{-j} \arrow[r, "p^s\psi_j"] \arrow[d, "p^s y_j"] & \Sigma F(\mathbb{C}P^N, BP) \arrow[d] \\
F(W_{n,k}, BP) \arrow[r] & \Sigma F_{tr}^{S^1}([S(\mathbb{C}^{N+1})/S^1] \wedge W_{n,k_+}, BP)^{S^1}
\end{tikzcd}$$

for $N > 2j$. According to the formula (3.3.4), the next possible differential is

$$d_{2j+2}(p^s y_j) = p^s \nu_{j+1} \binom{n}{j+1} x^{j+1} = p^s \nu_{j+1} d_{2j+2}(y_{j+1}).$$

We now rectify this class as $p^s y_j - p^s \nu_{j+1} y_{j+1}$ and obtain a cycle. This process continues until we reach the E_{2n+1}-page following which there are no further non-zero differentials.

We now formalize the above process by writing down a series of modifications to produce the element γ_j. Starting with $\gamma_j^{(2)} := y_j$, in r-th step of the modification, the modified element will be denoted by $\gamma_j^{(r)}$. Below we describe transformations, exactly one of which will be performed to produce $\gamma_j^{(r+1)}$ from $\gamma_j^{(r)}$.

1. If $d_r\gamma_j^{(r)} = 0$, then it survives to the next page and we call that element $\gamma_j^{(r+1)}$.

2. If $r = 2j$ and $\gamma_j^{(2j)} = y_j$ and $d_r(\gamma_j^{(r)}) = \binom{n}{j}x^j$. Define s by the formula $s + v_p(\binom{n}{j}) = \min_{n-k+1\leq i<j} v_p(\binom{n}{i})$, and declare $\gamma_j^{(r+1)} = p^s\gamma_j^{(r)}$.

3. If $r > 2j$, and $d_r(\gamma_j^{(r)}) \neq 0$, then we know r is even, and there is a BP^*-multiple of $y_{\frac{r}{2}}$ mapped by d_r onto the same class (this follows from the formula for ψ_j in the same way as demonstrated for $p^s y_j$ above). That is, $d_r(\gamma_j^{(r)}) = \lambda d_r(y_{\frac{r}{2}})$ for some $\lambda \in BP^*(pt)$. We declare $\gamma_j^{(r+1)} = \gamma_j^{(r)} - \lambda y_{\frac{r}{2}}$.

We finally write $\gamma_j = \gamma_j^{(2n+1)}$ which survives to the E_∞-page. Hence, we have shown that the 0-th column of the E_∞-page is $\Lambda_{BP^*}(\gamma_{n-k+2}, \cdots, \gamma_n)$. Also on the E_∞-page the ideal generated by $\{\binom{n}{j}x^j | n-k < j \leq n\}$ goes to 0, as each of the generators are hit by the differentials $d_{2j}(y_j)$. This completes the proof. $\square$

It remains now to solve the additive and multiplicative extension problems to obtain $BP^*PW_{n,k}$ from the expression in Proposition 3.3.5. In the following lemma, we show that the part $BP^*(pt)[x]/I$ forms a subalgebra of $BP^*PW_{n,k}$. Recall the fibration $W_{n,k} \to PW_{n,k} \overset{p_k}{\to} \mathbb{C}P^\infty$. We prove

Lemma 3.3.6. The kernel of the map $p^* : BP^*(\mathbb{C}P^\infty) \to BP^*(PW_{n,k})$ contains the ideal I generated by $\{\binom{n}{j}x^j | n-k < j \leq n\}$ in $BP^*(pt)[[x]]$.

Proof. The proof goes by induction on k. For $k = 1$, the fibration is up to homotopy the following sequence

$$W_{n,1} = S^{2n-1} \longrightarrow PW_{n,1} = \mathbb{C}P^{n-1} \overset{p_1}{\hookrightarrow} \mathbb{C}P^\infty,$$

so that the kernel of p_1^* is the ideal generated by x^n, satisfying the statement of the lemma. Suppose that the lemma is true for $PW_{n,k-1}$. To show the result for $PW_{n,k}$, we consider the diagram

$$\begin{tikzcd}
PW_{n,k} \arrow[rr, "T_{n,k}"] \arrow[dr, "p_k"] \arrow[dd, "q"'] & & BU(n-k) \arrow[dr, hook] \arrow[dd, hook] & \\
& \mathbb{C}P^\infty \arrow[rr, "f"] & & BU(n). \\
PW_{n,k-1} \arrow[ur, "p_{k-1}"] \arrow[rr, "T_{n,k-1}"'] & & BU(n-k+1) \arrow[ur, hook] &
\end{tikzcd}$$

In the above diagram, f classifies the bundle $n\gamma$ where γ is the canonical line bundle over $\mathbb{C}P^\infty$, and q is induced by the S^1-equivariant projection $W_{n,k} \to W_{n,k-1}$. The three squares in the diagram are homotopy pullbacks. Our aim is to understand the kernel of q^*. We see that $PW_{n,k} \to PW_{n,k-1}$ is, up to homotopy, the sphere bundle associated to the complex bundle classified by the map $T_{n,k-1}$. This is because $BU(n-k)$ is (up to homotopy) the sphere bundle of the canonical $n-k+1$-plane bundle over $BU(n-k+1)$. As BP is complex oriented, we obtain a Gysin sequence

$$\cdots \longrightarrow BP^*PW_{n,k-1} \xrightarrow{e^{BP}(T_{n,k-1})} BP^*PW_{n,k-1} \xrightarrow{q^*} BP^*PW_{n,k} \longrightarrow \cdots$$

It follows that the kernel of q^* is the ideal generated by $e^{BP}(T_{n,k-1})$ in $BP^*PW_{n,k-1}$. The bundle $T_{n,k-1}$ is obtained by lifting the composite $PW_{n,k-1} \stackrel{p_{k-1}}{\to} \mathbb{C}P^\infty \stackrel{n\gamma}{\to} BU(n)$ to $BU(n-k)$ so that $T_{n,k-1} + (k-1)\epsilon = np_{k-1}^*\gamma$. We readily compute e^{BP} as the top BP-Chern class

$$e^{BP}(T_{n,k-1}) = p_{k-1}^* c_{n-k+1}(n\gamma) = \binom{n}{n-k+1} x^{n-k+1}.$$

Therefore, $\binom{n}{n-k+1}x^{n-k+1}$ lies in the kernel of $p_k^* : BP^*\mathbb{C}P^\infty \to BP^*PW_{n,k}$. By the inductive formula for the kernel of $p_{k-1}^* : BP^*\mathbb{C}P^\infty \to BP^*PW_{n,k-1}$, the proof is now complete. □

We now apply Lemma 3.3.6 and Proposition 3.3.5 to complete the calculation of $BP^*PW_{n,k}$.

Theorem 3.3.7. For every prime p, the BP-cohomology algebra of $PW_{n,k}$ is described additively by $BP^*(pt)$-module

$$BP^*(PW_{n,k}) \cong \Lambda_{BP^*(pt)}(\gamma_{n-k+2}, \cdots, \gamma_n) \otimes_{BP^*(pt)} BP^*(pt)[[x]]/I$$

where γ_j's are of degree $2j-1$, x is of degree 2, and I is the ideal generated by $\{\binom{n}{j}x^j | n-k < j \leq n\}$. This isomorphism is also multiplicative if $p \neq 2$.

Proof. Lemma 3.3.6 implies that p^* induces a ring map of BP^*-modules $BP^*(pt)[[x]]/I \to BP^*(PW_{n,k})$. Choosing representatives for generators γ_j of Proposition 3.3.5 in the E_∞-page we obtain a $BP^*(pt)$-module map $\Lambda_{BP^*(pt)}(\gamma_{n-k+2}, \cdots, \gamma_n)$ to $BP^*PW_{n,k}$. The multiplication as a bilinear map on these factors gives a map

$$\Lambda_{BP^*(pt)}(\gamma_{n-k+2}, \cdots, \gamma_n) \otimes_{BP^*(pt)} BP^*(pt)[[x]]/I \to BP^*PW_{n,k}$$

of $BP^*(pt)$-modules. This is an isomorphism by Proposition 3.3.5 and the multiplicative structure of the spectral sequence (3.3.1). Further if $p \neq 2$, we have $\gamma_j^2 = 0$ as γ_j lies in odd degree. Therefore, the isomorphism is also multiplicative. □

We observe that in Theorem 3.3.7, we do not expect the isomorphism to be multiplicative when $p = 2$, as it does not even hold over $H\mathbb{Z}/p$ [6].

3.4 Equivariant maps between Stiefel manifolds

In this section, we demonstrate how the computations of BP-cohomology operations may be used to rule out S^1-equivariant maps between the Stiefel manifolds. The results of [33] can be improved in this way.

3.4.1. Applications using Steenrod operations. We start with an example using Steenrod operations in $\mathbb{Z}/2$-cohomology. The Steenrod operations on $H^*(PW_{n,k}; \mathbb{Z}/2)$ are described in [6, Theorem 1.2]. We have from [33] that if there is an S^1-equivariant map from $W_{n,k}$ to $W_{m,l}$ with $n - k = m - l$, then

$$\binom{n}{n-k+1} \text{ divides } \binom{m}{m-l+1},$$

which is then used to rule out such equivariant maps in many cases when $n - k = m - l$ and $n > m$ [33, Theorem 3.10]. The Steenrod operations allow us to rule out equivariant maps for cases where the above divisibility is valid. An example is given in the Theorem below.

Theorem 3.4.2. Suppose $r \equiv -1,\ -2,$ or $3 \pmod 9$ and $r \equiv 2,\ 1,$ or $-2 \pmod 7$, and $m = 16r - 2$. Then, there is no S^1-equivariant map from $W_{m-3,7}$ to $W_{m,10}$.

Proof. Write $n = m - 3$, $k = 7$ and $l = 10$. Observe that the following are satisfied by these integers

1. m, l even and n, k odd, and $m - l = n - k$.
2. 2 divides both $\binom{m}{m-l+1}$, $\binom{n}{n-k+1}$ but 4 does not divide either.
3. $\binom{n}{n-k+1} | \binom{m}{m-l+1}$.

An S^1-equivariant map f from $W_{n,k}$ to $W_{m,l}$ induces a map of fibration sequences

$$\begin{tikzcd}
W_{n,k} \arrow[r] \arrow[d, "f"'] & PW_{n,k} \arrow[r] \arrow[d] & \mathbb{C}P^\infty \arrow[d] \\
W_{m,l} \arrow[r] & PW_{m,l} \arrow[r] & \mathbb{C}P^\infty .
\end{tikzcd}$$

We compare the associated Serre spectral sequences with $\mathbb{Z}$-coefficients in the case $n - k = m - l$. The condition (2) implies that $f^*(y_{m-l+1}) = cy_{n-k+1}$, where c is odd. This is because in those spectral sequences y_j transgresses to $\binom{n}{j}x^j$ and $\binom{m}{j}x^j$ respectively. The classes y_{n-k+1} and y_{m-l+1} also survive in the $\mathbb{Z}/2$-cohomology spectral sequence by (2), and we have $f^*(y_{m-l+1}) = y_{n-k+1}$. [6, Theorem 1.2] implies

$$Sq^2(y_{m-l+1}) = (m-l)y_{m-l+2} + mxy_{m-l+1} = 0,$$

and

$$Sq^2(y_{n-k+1}) = (n-k)y_{n-k+2} + nxy_{n-k+1} = xy_{n-k+1}.$$

This is a contradiction. □

3.4.3. Results using BP-operations. We have seen how Steenrod squares yield some results on non-existence of S^1-equivariant maps between complex Stielfel manifolds. We now derive stronger results using BP-theory and cohomology operations associated to it. The operations we use here are the Adams operations defined via [5, 2.4]. These are multiplicative, stable operations with the formula

$$\Psi^a_{BP}(x) = a^{-1}[a]_{BP}(x), \tag{3.4.4}$$

where $a \in \mathbb{Z}^{\times}_{(p)}$, and $[a]_{BP}$ denotes the a-series using the BP-formal group law. These operations act on the coefficient ring via $\Psi^a_{BP}(v_i) = a^{p^i-1}v_i$.

Denote the ideal $(v_1, v_2, \cdots)$ in $BP^*(pt) = \mathbb{Z}_{(p)}[v_1, v_2, \cdots]$ by J. We fix the $\{v_i \mid i \geq 1\}$ to be the Araki generators [34, A2.2.2]. The formal group law μ_{BP} , associated to BP with respect to our chosen orientation is strictly isomorphic to the additive formal group law over $BP^*(pt) \otimes \mathbb{Q}$ and the isomorphism is given by BP-log series. The choice of generators imply that the BP-log series has the form

$$\log_{BP}(x) = x + \sum_{i\geq 1} l_i x^{p^i},$$

where l_i are determined by the relations

$$pl_n = \sum_{0\leq i\leq n} l_i v_{n-i}^{p^i}$$

with $l_0 = 1$ and $v_0 = p$. This implies the formula

$$l_n = \frac{v_n}{p - p^{p^n}} \pmod{J^2}.$$

Now we consider the expression of [3, Part II, Proposition 7.5] $\pmod{J^2}$ to obtain the following relation for the $\exp_{BP}$-series

$$\exp_{BP}(x) = x - \sum_{i\geq 1} l_i x^{p^i} = x - \sum_{i\geq 1} \frac{v_i}{p - p^{p^i}} x^{p^i} \pmod{J^2}.$$

This implies

$$\begin{aligned}
x +_{BP} y &= \exp_{BP}(\log_{BP} x + \log_{BP} y) \\
&= \log_{BP} x + \log_{BP} y - \sum_{i\geq 1} l_i (\log_{BP} x + \log_{BP} y)^{p^i} \pmod{J^2} \\
&= x + y + \sum_{i\geq 1} l_i (x^{p^i} + y^{p^i}) - \sum_{i\geq 1} l_i (x + y + \sum_{j\geq 1} l_j (x^{p^j} + y^{p^j}))^{p^i} \pmod{J^2} \\
&= x + y + \sum_{i\geq 1} l_i (x^{p^i} + y^{p^i} - (x+y)^{p^i}) \pmod{J^2} \\
&= x + y + \sum_{i\geq 1} \frac{v_i}{p - p^{p^i}} (x^{p^i} + y^{p^i} - (x+y)^{p^i}) \pmod{J^2},
\end{aligned} \tag{3.4.5}$$

where by $+_{BP}$ we mean the formal sum under the formal group law μ_{BP}.

We now restrict our attention to $p = 2$, and obtain the following reduction for Ψ^3_{BP} (3.4.4) by applying (3.4.5) multiple times.

$$\begin{aligned}
\Psi^3_{BP}(x) &= \frac{1}{3}[3]_{BP}(x) \\
&= \frac{1}{3}(x +_{BP} [2]_{BP}(x)) \\
&= \frac{1}{3}(x +_{BP} (2x +_{BP} v_1 x^2 +_{BP} v_2 x^4 +_{BP} \cdots +_{BP} v_i x^{2^i} +_{BP} \cdots)) \\
&= \frac{1}{3}(x +_{BP} 2x +_{BP} (v_1 x^2 + \cdots + v_i x^{2^i} + \cdots)) \pmod{J^2} \\
&= \frac{1}{3}((3x + \sum_{i\geq 1} \frac{v_i}{2 - 2^{2^i}} (x^{2^i} + (2x)^{2^i} - (3x)^{2^i})) +_{BP} (v_1 x^2 + \cdots + v_i x^{2^i} + \cdots)) \pmod{J^2} \\
&= \frac{1}{3}\left[3x + \sum_{i\geq 1} \frac{v_i}{2 - 2^{2^i}} \left(x^{2^i} + (2x)^{2^i} - (3x)^{2^i}\right)\right] + \frac{1}{3}\left[v_1 x^2 + \cdots + v_i x^{2^i} + \cdots\right] \pmod{J^2} \\
&= x + \sum_{i\geq 1} \frac{1 - 3^{2^i - 1}}{2(1 - 2^{2^i - 1})} v_i x^{2^i} \pmod{J^2}
\end{aligned}$$

We note that $\frac{1-3^{2^i-1}}{2(1-2^{2^i-1})} = \alpha_i$ lies in $\mathbb{Z}^\times_{(2)}$, and in this notation we have

$$\Psi^3_{BP}(x) = x + \sum_{i\geq 1} \alpha_i v_i x^{2^i} \pmod{J^2} \tag{3.4.6}$$

We shall now determine the action of Ψ^3_{BP} on $BP^*(W_{n,k}) = \Lambda_{BP^*}(y_{n-k+1}, \cdots, y_n)$ modulo the ideal I^2, where I is the ideal of $BP^*(W_{n,k})$ generated by $y_{n-k+1}, \cdots, y_n$.

Proposition 3.4.7.

$$\Psi^3_{BP}(y_j) = y_j + (j-1) \sum_{i \geq 1,\ 2^i+j-1 \leq n} \alpha_i v_i y_{2^i+j-1} \pmod{I^2 + J^2}$$

Proof. Recall the map $\mu_{n,k} : \Sigma[\mathbb{C}P^{n-1}/\mathbb{C}P^{n-k+1}] = \Sigma P_{n,k} \longrightarrow W_{n,k}$, for which we had $\mu^*_{n,k}(y_j) = \Sigma x^{j-1}$. Hence this will give us the isomorphism

$$\mu^*_{n,k} : BP^*(W_{n,k})/I^2 \longrightarrow \Sigma BP^*(P_{n,k})$$

By naturality of the Adams operations, Ψ^3_{BP} commutes with $\mu^*_{n,k}$. The action of Ψ^3_{BP} on y_j is determined up to I^2 from the computation for Σx^{j-1}. The Adams operation being stable, commutes with the suspension, so it is enough to compute the action of Ψ^3_{BP} on x^{j-1}, which comes from the multiplicative structure and the formulas above.

$$\begin{aligned}\Psi^3_{BP}(x^{j-1}) &= (\Psi^3_{BP}(x))^{j-1} \\ &= (x + \sum_{i\geq 1} \alpha_i v_i x^{2^i})^{j-1} \pmod{J^2} \text{ (using 3.4.6)} \\ &= x^{j-1} + \sum_{i \geq 1} (j-1)\alpha_i v_i x^{2^i+j-2} \pmod{J^2}.\end{aligned}$$

Hence the proposition follows. □

We now use the action of BP-Adams operations to prove new results about equivariant maps between complex Stiefel manifolds. We note from [33] that the existence of a S^1-equivariant map $W_{n,k} \to W_{m,l}$ implies that $n-k \leq m-l$. It states a number of hypotheses on n, k, m, l in the case $n-k = m-l$ for which equivariant maps do not exist. Proposition 3.4.2 proves some further results for this case. We use BP-operations to rule out equivariant maps in some cases where $n-k < m-l$.

Theorem 3.4.8. Suppose that m, n, l, k are positive integers satisfying
1) $n-k < m-l$ and there is an s such that $m < 2^s + m - l \leq n$.
2) 2 divides all the binomial coefficients $\binom{n}{n-k+1}, \cdots, \binom{n}{m-l}$.
3) 2 does not divide $\binom{m}{m-l+1}$ and $2 \nmid m-l$.
Then, there is no S^1-equivariant map from $W_{n,k}$ to $W_{m,l}$.

Proof. We assume the contrary that $g : W_{n,k} \to W_{m,l}$ is an S^1-equivariant map. This induces a map of homotopy fixed point spectral sequences, and also a compatible map between the associated projective Stiefel manifolds. The formula for the differentials in the homotopy fixed point spectral sequence (Proposition 3.3.2) and the fact that $\binom{n}{m-l+1}$ must be odd due to the hypotheses 2) and 3), implies that the pullback satisfies

$$g^*(y_{m-l+1}) = \beta y_{m-l+1} + \sum_{j>m-l+1} p_j y_j \pmod{I^2 + J^2}.$$

for some $\beta \in \mathbb{Z}_{(2)}^{\times}$ and $p_j \in BP^*(pt)$. Note that $|y_j| = 2j-1$ and $|v_j| = 2 - 2^{j+1}$. For degree reasons, the second term in the above expression will be of the form

$$\sum_{j\geq 1,\ n\geq 2^j+m-l} k_j v_j y_{2^j+m-l}$$

where $k_j \in \mathbb{Z}_{(2)}$.

Now we shall compute $\Psi^3_{BP}(g^*(y_{m-l+1}))$ and $g^*(\Psi^3_{BP}(y_{m-l+1}))$ modulo the ideal $I^2 + J^2$.

$$\begin{aligned}
\Psi^3_{BP}(g^*(y_{m-l+1})) &= \Psi^3_{BP}(\beta y_{m-l+1} + \sum_{j\geq 1,\ n\geq 2^j+m-l} k_j v_j y_{2^j+m-l}) \pmod{I^2+J^2} \\
&= \beta(y_{m-l+1} + (m-l)\sum_{i\geq 1,\ 2^i+m-l\leq n} \alpha_i v_i y_{2^i+m-l}) + \\
&\quad \sum_{j\geq 1,\ n\geq 2^j+m-l} k_j \Psi^3_{BP}(v_j)\Psi^3_{BP}(y_{2^j+m-l}) \pmod{I^2+J^2} \\
&= \beta(y_{m-l+1} + (m-l)\sum_{i\geq 1,\ n\geq 2^i+m-l} \alpha_i v_i y_{2^i+m-l}) + \\
&\quad \sum_{j\geq 1,\ n\geq 2^j+m-l} k_j \cdot 3^{2^j-1} v_j \cdot y_{2^j+m-l} \pmod{I^2+J^2}.
\end{aligned} \tag{3.4.9}$$

On the other hand, we have

$$\begin{aligned}
g^*(\Psi^3_{BP}(y_{m-l+1})) &= g^*(y_{m-l+1} + (m-l)\sum_{i\geq 1,\ m\geq 2^i+m-l} \alpha_i v_i y_{2^i+m-l}) \pmod{I^2+J^2} \\
&= \beta y_{m-l+1} + \sum_{j\geq 1,\ n\geq 2^j+m-l} k_j v_j y_{2^j+m-l} \\
&\quad + \sum_{i\geq 1,\ m\geq 2^i+m-l} \alpha_i v_i g^*(y_{2^i+m-l}) \pmod{I^2+J^2}.
\end{aligned} \tag{3.4.10}$$

Note that for degree reasons,

$$\alpha_i v_i g^*(y_{2^i+m-l}) = \nu\alpha_i v_i y_{2^i+m-l} \pmod{I^2+J^2},$$

for some $\nu \in \mathbb{Z}_{(2)}$. Since $\Psi^3_{BP}(g^*(y_{m-l+1})) = g^*(\Psi^3_{BP}(y_{m-l+1}))$, the coefficients for y_{2^s+m-l} (for s as in 1)) in the expressions (3.4.9) and (3.4.10) must be the same modulo the ideal I^2+J^2. This implies

$$\begin{aligned} \beta(m-l)\alpha_s + 3^{2^s-1}k_s &= k_s \\ \implies \beta(m-l) &= 2(1-2^{2^s-1})k_s. \end{aligned}$$

This contradicts the fact that $\beta(m-l) \in \mathbb{Z}^{\times}_{(2)}$. Hence no such S^1-equivariant map g can exist. □

Example 3.4.11. One may easily figure out values of m, n, l, k for which the hypothesis of Theorem 3.4.8 are satisfied. For example putting $k = n$ and $m - l + 1 = 2$, we obtain : If n is even and $\binom{m}{2}$ odd, and there is some s such that $m < 2^s + 1 \leq n$, then, there is no S^1-equivariant map from $W_{n,n}$ to $W_{m,m-1}$.

Chapter 4

p-local decomposition of projective Stiefel manifolds

The main objective of this chapter is to analyze the p-local homotopy type of the complex projective Stiefel manifolds, and other analogous quotients of Stiefel manifolds. We take the cue from a result of Yamaguchi about the p-regularity of the complex Stiefel manifolds [39] which lays down some hypotheses under which the Stiefel manifold is p-locally a product of odd dimensional spheres. We show that in many cases, the projective Stiefel manifolds are p-locally a product of a complex projective space and some odd dimensional spheres. As an application, we prove that in these cases, the p-regularity result of Yamaguchi is also S^1-equivariant. These results appear in the paper [10].

4.1 Cohomology of projective Stiefel manifolds

We recall the cohomology of projective Stiefel manifolds and other associated quotients of Stiefel manifolds pointed out in Chapter 2.

$$H^*(PW_{n,k};\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2},\cdots,\gamma_n) \otimes_{\mathbb{Z}_{(p)}} \frac{\mathbb{Z}_{(p)}[x]}{(\binom{n}{j}x^j \mid n-k+1 \leq j \leq n)}, \tag{4.1.1}$$

with $|\gamma_i| = 2i-1$ and $|x| = 2$. The computation is carried out using the Serre spectral sequence for the fibration $W_{n,k} \to PW_{n,k} \to \mathbb{C}P^\infty$ [6, (2.1)]. One identifies the cohomology of $W_{n,k}$ as the exterior algebra $\Lambda(z_{n-k+1},\cdots,z_n)$ with $|z_j| = 2j-1$, and computes the differentials by the fact that the z_j are transgressive, and the equation $d_{2j}(z_j) = \binom{n}{j}x^j$. If $p > n$, one observes that the binomial coefficients $\binom{n}{j}$ are units in $\mathbb{Z}_{(p)}$, so we have the following reduction of (4.1.1)

$$H^*(PW_{n,k};\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2},\cdots,\gamma_n) \otimes \frac{\mathbb{Z}_{(p)}[x]}{(x^{n-k+1})}. \tag{4.1.1}$$

In this case, observe that the cohomology of $PW_{n,k}$ matches with that of the product $M_{n,k} = \mathbb{C}P^{n-k} \times \prod_{i=n-k+2}^{n} S^{2i-1}$. We now follow [6, §3] to identify the generators γ_j of (4.1.1). Let E be a contractible space with free $U(n)$-action. We have the following homotopy commutative diagram in which all the squares are homotopy pullbacks,

$$\begin{tikzcd} W_{n,k} \arrow[r, "\pi"] \arrow[d] & PW_{n,k} \arrow[r, "f"] \arrow[d] & BU(n-k) \arrow[d] \\ E \arrow[r] & \mathbb{C}P^\infty \arrow[r, "f_0"'] & BU(n), \end{tikzcd} \tag{4.1.2}$$

where f_0 classifies the bundle $n\gamma$. This gives rise to the following diagram

$$\begin{tikzcd} 0 \arrow[d] & \\ H^{2j-1}(PW_{n,k};\mathbb{Z}_{(p)}) \arrow[d, "\delta"] & 0 \arrow[d] \\ H^{2j}(\mathbb{C}P^\infty, PW_{n,k};\mathbb{Z}_{(p)}) \arrow[d, "i^*"] & H^{2j}(BU(n), BU(n-k);\mathbb{Z}_{(p)}) \arrow[l, "f^*"'] \arrow[d] \\ H^{2j}(\mathbb{C}P^\infty;\mathbb{Z}_{(p)}) & H^{2j}(BU(n);\mathbb{Z}_{(p)}). \arrow[l, "f_0^*"'] \end{tikzcd} \tag{4.1.3}$$

We identify $H^*(BU(n), BU(n-k);\mathbb{Z}_{(p)})$ with the ideal of $H^*(BU(n);\mathbb{Z}_{(p)})$ generated by the universal Chern classes c_j for $n-k < j \leq n$, and write $u_j^H = f^*c_j$. In this notation, $\gamma_j^H \in H^{2j-1}(PW_{n,k};\mathbb{Z}_{(p)})$ is defined by the equation

$$\delta\gamma_j^H = \rho_j = u_j^H - x^{j-(n-k+1)}\frac{\mu_j}{\mu_{n-k+1}}u_{n-k+1}^H, \tag{4.1.4}$$

where $\mu_j = \binom{n}{j}$. In order to observe how this formula makes sense, one should note that $f_0^*(c_j) = \binom{n}{j}x^j$.

4.1.1 Other quotients of Stiefel manifolds

One may proceed in an analogous manner to the above to write down the cohomology ring structure for the other quotients of Stiefel manifolds. Let $\ell = (l_1,\cdots,l_k)$ such that the $\gcd$ of the l_i is 1. The cohomology of $P_\ell W_{n,k}$ with $\mathbb{Z}/p$-coefficients was computed in [12]. The similar calculation with $\mathbb{Z}_{(p)}$-coefficients yields the formula

$$H^*(P_\ell W_{n,k};\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2},\cdots,\gamma_n)\otimes \mathbb{Z}_{(p)}[x]/J,$$

where $|\gamma_j| = 2j-1$, $|x| = 2$ and J is the ideal of $\mathbb{Z}_{(p)}[x]$ generated by the set $\{\sum_{|I|=j}(-1)^j\ell^I x^j \mid n-k<j\leq n\}$. This formula is obtained by calculating the differentials in the Serre spectral sequence associated to the fibration $W_{n,k} \longrightarrow P_\ell W_{n,k} \longrightarrow \mathbb{C}P^\infty$. It turns out that the exterior algebra generators z_j's of $H^*(W_{n,k};\mathbb{Z}_{(p)})$ are transgressive and $d_{2j}(z_j) = \sum_{|I|=j}(-1)^j\ell^I x^j$. Note that for a prime p not dividing $\sum_{|I|=n-k+1}\ell^I$, we have the following reduction

$$H^*(P_\ell W_{n,k};\mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+2},\cdots,\gamma_n)\otimes \mathbb{Z}_{(p)}[x]/(x^{n-k+1}). \tag{4.1.5}$$

In this case $P_\ell W_{n,k}$ and $M_{n,k}$ have isomorphic cohomology rings. The analogue of the pullback (4.1.2) is the diagram [12, (2.1)]

$$\begin{tikzcd} W_{n,k} \arrow[r, "\pi"] \arrow[d] & P_\ell W_{n,k} \arrow[r, "\phi"] \arrow[d] & Gr_k(\mathbb{C}^n) \arrow[d] \\ E \arrow[r] & \mathbb{C}P^\infty \arrow[r, "\phi_0"'] & BU(k), \end{tikzcd} \tag{4.1.6}$$

where ϕ_0 classifies the bundle $\sum \gamma^{l_j}$. One may now consider a diagram similar to (4.1.3) by working with the pair $(BU(k), Gr_k(\mathbb{C}^n))$ to identify the cohomology generators for $P_\ell W_{n,k}$.

The cohomology of $W_{n,k;m}$ with $\mathbb{Z}/p$ coefficients was computed in [21]. For $p \nmid m$, this is equivalent to the cohomology of $W_{n,k}$, so the interesting case is when $p \mid m$. Following the same method, the cohomology with $\mathbb{Z}_{(p)}$ coefficients (for $p \mid m$) may be computed. For $p > n$, the formula takes the following form

$$H^*(W_{n,k;m};\mathbb{Z}_{(p)}) \cong (\Lambda_{\mathbb{Z}_{(p)}}(\gamma_{n-k+1},\gamma_{n-k+2},\cdots,\gamma_n)\otimes \mathbb{Z}_{(p)}[x])/(mx, x^{n-k+1}, \gamma_{n-k+1}x), \tag{4.1.7}$$

where $|\gamma_j| = 2j-1$, and $|x| = 2$. The method requires determining the Serre spectral sequence associated to the fibration $S^1 \longrightarrow W_{n,k;m} \longrightarrow PW_{n,k}$ and the only differential d_2 sends the degree 1 class e generating $H^*(S^1;\mathbb{Z}_{(p)})$ to mx. Note that the class $e\otimes x^{n-k}$ survives in the E_∞-page detecting the degree $2n-2k+1$ class γ_{n-k+1}.

4.2 Decomposition results at very large primes

In §4.1, the expressions for the cohomology of the various quotients of $W_{n,k}$ say that for large primes, the cohomology of $PW_{n,k}$ and $P_\ell W_{n,k}$ matches that of $M_{n,k}$. For $W_{n,k;m}$, the expression matches the cohomology of a product of the lens space $L_m(2n-2k+1)$ and a bunch of odd dimensional spheres. Using elementary arguments, in this section we observe that these isomorphisms may be lifted to p-local homotopy equivalences for a sufficiently large lower bound on p.

The first step towards a homotopical result starting from a cohomology isomorphism is a rational homotopy calculation. Note that the Serre spectral sequence for the fibration $W_{n,k} \longrightarrow PW_{n,k} \longrightarrow \mathbb{C}P^\infty$ with rational coefficients tell us that

$$H^*(PW_{n,k};\mathbb{Q}) = \mathbb{Q}[x]/(x^{n-k+1}) \otimes \Lambda_{\mathbb{Q}}(\gamma_{n-k+2},\cdots,\gamma_n),$$

where $|x|=2, |\gamma_j| = 2j-1$.

4.2.1. Rational splittings for $PW_{n,k}$. The rational homotopy type of simply connected spaces are determined by its minimal model [18]. For a space X, we denote its minimal model by m_X. We shall show that $m_{PW_{n,k}}$ is isomorphic to $m_{M_{n,k}}$, which will tell us that they are rational homotopy equivalent.

We know that

$$m_{M_{n,k}} = m_{\mathbb{C}P^{n-k}} \otimes m_{S^{2n-2k+3}} \otimes \cdots \otimes m_{S^{2n-1}}.$$

Also $m_{S^{2j-1}} = (\Lambda(y_j), d=0)$, where $|y_j| = 2j-1$, and $m_{\mathbb{C}P^{n-k}} = (P(x)\otimes\Lambda(y_{n-k+1}), d)$, where $|x|=2, |y_{n-k+1}| = 2n-2k+1$ and $d(x)=0, d(y_{n-k+1}) = x^{n-k+1}$ [18]. In this expression, $P(x)$ stands for the polynomial algebra on the generator x.

Proposition 4.2.2. The minimal model for $PW_{n,k}$ is given by

$$m_{PW_{n,k}} = P(\tilde{x}) \otimes \Lambda(\tilde{y}_{n-k+1},\cdots,\tilde{y}_n),$$

where $|\tilde{x}| = 2, |\tilde{y}_j| = 2j-1$ and the action of differential is determined by the following: $d(\tilde{x}) = 0, d(\tilde{y}_{n-k+1}) = \tilde{x}^{n-k+1}, d(\tilde{y}_j) = 0, \forall n-k+1 < j \leq n$.

Proof. We only need to construct a differential graded algebra (DGA) morphism φ from the the minimal Sullivan DGA given in the statement to the DGA $A^*_{PL}(PW_{n,k})$ (following the same notation as [18]) which will also be a quasi-isomorphism. Actually it turns out that $PW_{n,k}$ is a formal space ie. its minimal model is determined by its cohomology ring which we will see below. Define φ by the following action on generators

$$\varphi(\tilde{x}) = \omega, \varphi(\tilde{y}_{n-k+1}) = \theta, \varphi(\tilde{y}_j) = \sigma_j, \forall n-k+1 < j \leq n \tag{4.2.3}$$

where ω and σ_j are cocycles in $A^*_{PL}(PW_{n,k})$ such that $[\omega] = x \in H^*(PW_{n,k};\mathbb{Q})$, $[\sigma_j] = \gamma_j \in H^*(PW_{n,k};\mathbb{Q})$. Since $x^{n-k+1} = 0$ in $H^*(PW_{n,k};\mathbb{Q})$, there will be some element $\theta \in A^{2n-2k+1}_{PL}(PW_{n,k})$ whose image under the differential in $A^*_{PL}(PW_{n,k})$ will be ω^{n-k+1}. This is the θ used in 4.2.3. With this definition, φ is clearly a DGA quasi-isomorphism. □

Now it is evident that both $PW_{n,k}$ and $M_{n,k}$ have the same minimal model. Therefore, $PW_{n,k\mathbb{Q}} \simeq M_{n,k\mathbb{Q}}$.

4.2.4. Splitting at large primes. We now replicate the rational result of §4.2.1 in the p-local homotopy category. The rough estimate is so that $2p-3$ is larger than the dimension of the manifold, for which we obtain a result by elementary means. We note

$$\dim(PW_{n,k}) = 2nk - k^2 - 1 = \dim(P_\ell W_{n,k}), \quad \dim(W_{n,k;m}) = 2nk - k^2.$$

In the following we use the class $x \in H^2(PW_{n,k})$ to obtain the map $PW_{n,k} \to \mathbb{C}P^\infty$.

Proposition 4.2.5. For all primes $p > \frac{2nk-k^2-1}{2} + k - n$, there is a map $(PW_{n,k})_{(p)} \longrightarrow \mathbb{C}P^{n-k}_{(p)}$ which is a lift of the map $(PW_{n,k})_{(p)} \to \mathbb{C}P^\infty_{(p)}$ up to homotopy. The same conclusion holds for $P_\ell W_{n,k}$ under the additional assumption that $p \nmid \sum_{|I|=n-k+1} \ell^I$.

Proof. The homotopy fibre of the inclusion $\mathbb{C}P^{n-k} \to \mathbb{C}P^\infty$ is $S^{2n-2k+1}$. Therefore, obstructions for lifting the map $PW_{n,k} \to \mathbb{C}P^\infty$ on p-localizations lie in $H^r(PW_{n,k}; \pi_{r-1}(S^{2n-2k+1}_{(p)}))$, which are 0 under the given hypothesis. As $\dim(PW_{n,k}) < \dim(S^{2n-2k+1}) + 2p - 3$, the only obstruction that may arise is if $r = 2n-2k+2$, in which degree the cohomology of $PW_{n,k}$ is 0 from (4.1.1). The same argument also works for $P_\ell W_{n,k}$ under the additional hypothesis. □

Let $Z_{n,k} = \left(S^{2n-2k+3} \times S^{2n-2k+5} \times \cdots \times S^{2n-1}\right)_{(p)}$. The cohomology of $Z_{n,k}$ is the exterior algebra

$$H^*(Z_{n,k}) \cong \Lambda(\epsilon_{2n-2k+3}, \epsilon_{2n-2k+5}, \cdots, \epsilon_{2n-1}),$$

where the classes ϵ_{2j-1} are pullbacks of the generators of $H^*(S^{2j-1})$ via the corresponding projection. We prove

Proposition 4.2.6. Suppose p is as in the hypothesis of Proposition 4.2.5. There is a map $\rho : (PW_{n,k})_{(p)} \to Z_{n,k}$ such that $\rho^*(\epsilon_{2j-1}) = \gamma_j$ for all $n-k+2 \leq j \leq n$, with γ_j as in (4.1.4). The same result holds for $P_\ell W_{n,k}$ if $p \nmid \sum_{|I|=n-k+1} \ell^I$.

Proof. Suppose $L = K(\mathbb{Z}_{(p)}, 2n-2k+3)\times\cdots\times K(\mathbb{Z}_{(p)}, 2n-1)$, then we get a map $\nu : Z_{n,k} \to L$ which classifies the cohomology generators of $Z_{n,k}$. Now,

$$\pi_r(Z_{n,k}) \cong \pi_r(S^{2n-2k+3}_{(p)}) \oplus \cdots \oplus \pi_r(S^{2n-1}_{(p)}),$$

and $\pi_r(S^{2n-2k+j}_{(p)}) = 0$ for $2n-2k+j < r < 2n-2k+j+2p-3$, imply that $\pi_r(Z_{n,k}) = \mathbb{Z}_{(p)}$, for r odd and $r = 2n-2k+3, \cdots, 2n-1$ and $\pi_r(Z_{n,k}) = 0$, for all other $r \leq \dim(PW_{n,k})$. So ν must be a $\dim(PW_{n,k})$-equivalence.

From the conclusion of the paragraph above, we will get the following isomorphism

$$[(PW_{n,k})_{(p)}, Z_{n,k}] \xrightarrow{\nu_*} [(PW_{n,k})_{(p)}, L] \cong H^{2n-2k+3}(PW_{n,k}; \mathbb{Z}_{(p)}) \oplus \cdots \oplus H^{2n-1}(PW_{n,k}; \mathbb{Z}_{(p)}).$$

Hence we get a map $\rho : (PW_{n,k})_{(p)} \to Z_{n,k}$ that pulls back the cohomology generators $\epsilon_{2j-1} \in H^*(Z_{n,k})$ corresponding to the generators of $H^*(S^{2j-1}_{(p)})$ to $\gamma_j \in H^*(PW_{n,k}, \mathbb{Z}_{(p)})$. This argument works entirely analogously for $P_\ell W_{n,k}$. □

We may now assemble the two results from Propositions 4.2.5 and 4.2.6 to get maps for p large

$$\Big(PW_{n,k}\Big)_{(p)} \to \Big(M_{n,k}\Big)_{(p)}, \quad \Big(P_\ell W_{n,k}\Big)_{(p)} \to \Big(M_{n,k}\Big)_{(p)}$$

which are cohomology isomorphisms. Thus, we have proved the following result.

Theorem 4.2.7. Suppose $p > \frac{2nk-k^2-1}{2} + k - n$. Then we have,

$$\Big(PW_{n,k}\Big)_{(p)} \simeq \Big[\mathbb{C}P^{n-k} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\Big]_{(p)}.$$

If further $p \nmid \sum_{|I|=n-k+1} \ell^I$,

$$\Big(P_\ell W_{n,k}\Big)_{(p)} \simeq \Big[\mathbb{C}P^{n-k} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\Big]_{(p)}.$$

Theorem 4.2.7 may be used to provide an S^1-equivariant decomposition of the Stiefel manifold.

Proposition 4.2.8. For $p > \frac{2nk-k^2-1}{2} + k - n$, we have the following splitting as S^1-spaces.

$$\Big(W_{n,k}\Big)_{(p)} \simeq [S^{2n-2k+1} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}]_{(p)}.$$

Proof. For the bound on p stated in the proposition, there is a map $\phi_k : (PW_{n,k})_{(p)} \to \mathbb{C}P^{n-k}_{(p)}$ lifting the classifying map of the S^1-bundle $W_{n,k} \to PW_{n,k}$. Hence we have the following homotopy pullback diagram,

$$\begin{tikzcd}
(W_{n,k})_{(p)} \arrow[r] \arrow[d] & S^{2n-2k+1}_{(p)} \arrow[d] \\
(PW_{n,k})_{(p)} \arrow[r] & \mathbb{C}P^{n-k}_{(p)}
\end{tikzcd}$$

leading to an S^1-equivariant map $(W_{n,k})_{(p)} \to S^{2n-2k+1}_{(p)}$. We also have an S^1-equivariant map $(W_{n,k})_{(p)} \to (W_{n,k-1})_{(p)}$ arising from the S^1-equivariant projection $W_{n,k} \to W_{n,k-1}$. The product of these two maps is evidently a cohomology isomorphism. So we obtain an S^1-equivariant equivalence

$$(W_{n,k})_{(p)} \simeq S^{2n-2k+1}_{(p)} \times (W_{n,k-1})_{(p)}.$$

And by induction on k, we arrive at the result stated in the proposition. □

Now observe that both $W_{n,k;m}$ and $L_m(2n-2k+1) \times S^{2n-2k+3} \times \cdots \times S^{2n-1}$ have the same $\mathbb{Z}_{(p)}$-cohomology whenever $p > n$.

Proposition 4.2.9. For $p > \frac{2nk-k^2-1}{2} + k - n$ we have the the following splitting

$$\Big(W_{n,k;m}\Big)_{(p)} \simeq [L_m(2n-2k+1) \times S^{2n-2k+3} \times \cdots \times S^{2n-1}]_{(p)}.$$

Proof. For $p > \frac{2nk-k^2-1}{2} + k - n$, we have the S^1-equivariant map $W_{n,k} \to S^{2n-2k+1}$ as shown in Proposition 4.2.8. So we get a map $W_{n,k;m} \to L_m(2n-2k+1)$ by considering the C_m-orbit spaces. Now comparing the spectral sequences associated to the fibrations

$$S^1 \to W_{n,k;m} \to PW_{n,k}, \text{ and } S^1 \to L_m(2n-2k+1) \to \mathbb{C}P^{n-k},$$

we can see that the map $W_{n,k;m} \to L_m(2n-2k+1)$ induces an isomorphism on $H^j(-;\mathbb{Z}_{(p)})$ for $j \leq 2n-2k+1$. We also have the maps $W_{n,k;m} \to PW_{n,k} \to S^{2n-2k+2r+1}$ for $k > r > 0$ and they map the cohomology generators of S^{2j-1} to the corresponding generator of $H^*(W_{n,k;m})$ in (4.1.7). Hence we have a map from $W_{n,k;m}$ to $L_m(2n-2k+1) \times S^{2n-2k+3} \times \cdots \times S^{2n-1}$ that induces an isomorphism on cohomology, and both spaces being simple, we get the desired equivalence. □

4.3 Stable decompositions of projective Stiefel manifolds

In this section, we prove stable decomposition results for the projective Stiefel manifold $PW_{n,k}$ at primes greater than n. The crucial observation here is that if $p > n$, we have the stable equivalence

$$\mathbb{C}P^n_{(p)} \simeq S^2_{(p)} \vee S^4_{(p)} \vee \cdots \vee S^{2n}_{(p)}.$$

This fact may be proved by showing that the attaching maps of the cells of the p-local complex projective space are stably trivial if $p > n$. The reason is that the first non-trivial p-torsion in the stable homotopy groups of S^0 occurs in degree $2p-3$, which is greater than the degrees of the attaching maps of $\mathbb{C}P^n$ if $p > n$. However one requires a more delicate argument to make this work for $PW_{n,k}$, because there are cell attachments of degree greater than $2p-3$.

4.3.1. Minimal cell structures. We recall the minimal cell structures for CW complexes from [23, §4.C]. For a simply connected CW complex with finitely generated homology groups, we have a CW complex structure with "minimum number of cells". More precisely, writing the homology groups using generators and relations, we have one "generator" n-cell for every generator of a cyclic copy of $H_n(X)$, and one "relator" $n+1$-cell whose boundary is k times a "generator" n-cell in cellular homology whenever the cyclic copy in $H_n(X)$ corresponding to the generator is $\mathbb{Z}/k$.

We will use a version of the above result for p-local spaces. In this case we use p-local cells $\mathcal{D}^n_{(p)}$ which is the cone of $S^{n-1}_{(p)}$, and a p-local n-cell attachment to a p-local space X is the attachment of $\mathcal{D}^n_{(p)}$ along a map $S^{n-1}_{(p)} \to X$. A p-local finite CW complex is one which is obtained by attaching finitely many p-local cells in increasing dimensions. The following proposition follows directly using the arguments of [23, Proposition 4.C.1].

Proposition 4.3.2. Suppose that X is a simply connected p-local space such that $H_*(X;\mathbb{Z}_{(p)})$ is finitely generated and torsion free. Then, X has a p-local CW complex structure with one p-local n-cell for each basis element of $H_n(X;\mathbb{Z}_{(p)}) \cong \mathbb{Z}^l_{(p)}$.

Remark 4.3.3. Suppose that X is as in Proposition 4.3.2. We observe that as the boundary map in cellular homology is 0, the attaching map of such a cell factors through the $(n-2)$-skeleton.

4.3.4. The Chern character. Recall the Chern character [24, Chapter 5]

$$ch : K^*(X) \to H^*(X;\mathbb{Q})[u^{\pm}]$$

where u is a degree 2 class inducing the 2-periodicity in the second factor. We restrict our attention to p-local finite CW complexes of Proposition 4.3.2, and note that the arguments in page 73 of [24] implies the following result.

Proposition 4.3.5. Suppose that X is a simply connected p-local space such that $H_*(X;\mathbb{Z}_{(p)})$ is finitely generated and torsion free. Then, $K^*(X)\otimes\mathbb{Z}_{(p)}$ is torsion-free and the Chern character induces an isomorphism $K^*(X)\otimes\mathbb{Q}\cong H^*(X;\mathbb{Q})[u^{\pm}]$.

4.3.6. A criterion for stable splitting. We now formulate a general criterion to have a p-local stable decomposition into a wedge of spheres. This involves ruling out attaching maps in the image of the J-homomorphism using an assumption on the Chern character. For the remainder of this section, we work in the category $\mathcal{S}$ which stands for the stable homotopy category, whose objects are sequential spectra and morphisms are the homotopy classes of maps between them. For a space X, we use the same notation for the suspension spectrum as an object of $\mathcal{S}$. We also use the Chern character for cellular spectra with finitely many cells. Note from [34, Theorem 1.1.14] that elements in $\pi_k^s(S^0)$ for $k>0$ lie in the image of the J-homomorphism if $k<2p^2-2p$.

Theorem 4.3.7. Let X be a simply connected p-local finite CW-complex satisfying the following conditions

1. $\dim X < 2p^2-2p$.
2. $H^*(X;\mathbb{Z}_{(p)})$ is free as a $\mathbb{Z}_{(p)}$-module.
3. The Chern character map for X has image in $H^*(X;\mathbb{Z}_{(p)})\subset H^*(X;\mathbb{Q})$.

Then, $X\simeq$ a wedge of p-local spheres.

Proof. We consider the minimal cell structure of X of Proposition 4.3.2 and prove that the attaching maps are 0 in the stable homotopy category. We show this by induction over k for $X^{(k)}$, which is the k^{th}-skeleton of X. The properties of the minimal cell structure, the naturality of the Chern character together with Proposition 4.3.5, implies that any sub-complex A of X satisfies the three hypotheses stated in the theorem.

We assume that $X^{(k)}$ splits as a wedge of p-local cells, and we want to conclude the same for $X^{(k+1)}$. Since there are finitely many cells, it suffices to prove that every attaching map $\phi : S^k_{(p)} \longrightarrow X^{(k)}$ is trivial. We write Y for the mapping cone on ϕ. From Remark 4.3.3, this attaching map will actually land in $X^{(k-1)}_{(p)}$. Writing

$$X^{(k)}\simeq S^{n_1}_{(p)}\vee\cdots\vee S^{n_r}_{(p)},$$

the attaching map of the p-local $(k+1)$-cell in $X_{(p)}$ must factor through $\bigvee_{n_j \neq k} S^{n_j}_{(p)}$, and is therefore a sum of maps of the form $f_j : S^k_{(p)} \to S^{n_j}_{(p)}$. Note that $k - n_j < 2p^2 - 2p$ by condition 1), so that the homotopy class of these maps lie in the image of J homomorphism. Since these classes are in odd degree, we may assume $k - n_j$ is odd. Now we have the following homotopy commutative diagram

$$\begin{tikzcd}
S^k_{(p)} \arrow[r, "\phi"] \arrow[d] & X^{(k)}_{(p)} \arrow[r] \arrow[d] & Y_{(p)} \arrow[r] \arrow[d] & S^{k+1}_{(p)} \arrow[d, "="] \\
S^k_{(p)} \arrow[r, "f_j"'] & S^{n_j}_{(p)} \arrow[r] & C(f_j) \arrow[r] & S^{k+1}_{(p)}
\end{tikzcd} \tag{4.3.8}$$

where the second vertical map from left is induced by the retraction

$$S^{n_1}_{(p)} \vee \cdots \vee S^{n_r}_{(p)} \longrightarrow S^{n_j}_{(p)}.$$

To show that f_j is trivial it suffices to show that the K-theoretic e-invariant [2, §7] of f_j, $e_K(f_j)$ vanishes. Note that [2, Proposition 7.14] implies that we may compute this using complex K-theory as $p \neq 2$. We also notice that since we are dealing with p-local spheres, the e-invariant will take values in $\mathbb{Q}/\mathbb{Z}_{(p)}$.

We look at the following diagram of short exact sequences induced by the Chern character:

$$\begin{tikzcd}
0 & K^*_{(p)}(X^{(k)}) \arrow[l] \arrow[d, "ch"'] & K^*_{(p)}(Y) \arrow[l] \arrow[d, "ch"'] & K^*_{(p)}(S^{k+1}) \arrow[l] \arrow[d, "ch"] & 0 \arrow[l] \\
0 & H^*(X^{(k)};\mathbb{Q})[u^{\pm}] \arrow[l] & H^*(Y;\mathbb{Q})[u^{\pm}] \arrow[l] & H^*(S^{k+1};\mathbb{Q})[u^{\pm}] \arrow[l] & 0 \arrow[l]
\end{tikzcd}$$

where the short exactness of the first row follows from the short exactness of the bottom row along with the injectivity of the Chern character map for p-local sphere. As Y is a subcomplex of X, the image of the Chern character lies in $H^*(Y;\mathbb{Z}_{(p)})$ by condition 3). Now from the following diagram (the injectivity of the horizontal arrows follow from (4.3.8))

$$\begin{tikzcd}
K_{(p)}(C(f_j)) \arrow[r, hook] \arrow[d, "ch"'] & K_{(p)}(Y) \arrow[d, "ch"] \\
H^*(C(f_j);\mathbb{Q})[u^{\pm}] \arrow[r, hook] & H^*(Y;\mathbb{Q})[u^{\pm}],
\end{tikzcd}$$

we conclude that the image of the Chern character for $C(f_j)$ also lies in $\mathbb{Z}_{(p)}$-cohomology. Applying the reformulation of the e-invariant in [2, Proposition 7.8], we deduce that $e(f_j) = 0$, and hence $f_j \simeq 0$. ∎

4.3.9. Algebra generators for $K^*(W_{n,k})$ and $K^*(PW_{n,k})$. We now write down the structure of $K^*PW_{n,k}$ (and $K^*W_{n,k}$), identifying algebra generators as was done in (4.1.4) for ordinary cohomology. The coefficient ring for p-local complex K-theory is $\mathbb{Z}_{(p)}[\beta^{\pm}]$ where β stands for the Bott element lying in degree 2. Recall that complex K-theory is complex oriented, and $x_K = \beta^{-1}(L-1)$ is a choice of complex orientation for the canonical line bundle L over $\mathbb{C}P^\infty$. This implies that every complex vector bundle is K-orientable, and defines universal Chern classes c_j^K such that $K^*_{(p)}(BU(n)) \cong \mathbb{Z}_{(p)}[\beta^{\pm}][[c_1^K, \cdots, c_n^K]]$.

We note that the complex orientation expressed as a map

$$x_K : \mathbb{C}P^\infty \to \Sigma^2 K_{(p)}$$

may be expressed as a composite via the connective cover $ku_{(p)}$ of $K_{(p)}$

$$\mathbb{C}P^\infty \stackrel{x_{ku}}{\to} \Sigma^2 ku_{(p)} \to \Sigma^2 K_{(p)}$$

as $\mathbb{C}P^\infty$ is 1-connected. This implies that the universal Chern classes c_j^K are the image of $ku_{(p)}$-Chern classes c_j^{ku}. Following the method described in [9, Proposition 3.4 and Proposition 3.6] we obtain classes $\tau_j^{ku} \in ku^*_{(p)}(BU(n))$ such that $\tau_j^{ku} = c_j^{ku} + \sum_{k>j} \nu_k c_k^{ku}$, where $\nu_k \in ku^*_{(p)}(pt)$. We use the map $\alpha : (BU(n-k), W_{n,k}) \to (BU(n), *)$ arising from the fibration $W_{n,k} \to BU(n-k) \to BU(n)$, and the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & ku^i_{(p)}(W_{n,k}) \arrow[r, "\delta"] & ku^{i+1}_{(p)}(BU(n-k), W_{n,k}) \arrow[r] & ku^{i+1}_{(p)}(BU(n-k)) \arrow[r] & \cdots \\
 & & \widetilde{ku}^{i+1}_{(p)}(BU(n)). \arrow[u, "\alpha^*"] & &
\end{tikzcd}$$

to define y_j^{ku} by the equation $\delta(y_j^{ku}) = \alpha^*(\tau_j^{ku})$ for $j \geq n-k+1$. The classes y_j^{ku} serve as generators of $ku^*_{(p)}(W_{n,k})$, that is,

$$ku^*_{(p)}(W_{n,k}) \cong \Lambda_{ku^*_{(p)}}(y^{ku}_{n-k+1}, \cdots, y^{ku}_n).$$

Pushing forward to $K^*_{(p)}$, by the map $ku_{(p)} \to K_{(p)}$, we obtain classes $\tau_j^K \in K^*_{(p)}(BU(n))$ such that $\tau_j^K = c_j^K + \sum_{k>j} \nu_k c_k^K$, and $y_j^K \in K^*_{(p)}(W_{n,k})$ such that in the diagram

$$\begin{tikzcd} \cdots \arrow[r] & K^i_{(p)}(W_{n,k}) \arrow[r, "\delta"] & K^{i+1}_{(p)}(BU(n-k), W_{n,k}) \arrow[r] & K^{i+1}_{(p)}(BU(n-k)) \arrow[r] & \cdots \\ & & \widetilde{K_{(p)}}^{i+1}(BU(n)), \arrow[u, "\alpha^*"] & & \end{tikzcd}$$

$\delta(y_j^K) = \alpha^*(\tau_j^K)$, and one can show y_j^K give a set of generators for $K^*_{(p)}(W_{n,k})$. We now multiply y_j^K by a power of β so that $y_j^K \in K^{-1}_{(p)}(W_{n,k})$. For the projective Stiefel manifold we have

$$K^*_{(p)}(PW_{n,k}) \cong K^*_{(p)} \otimes_{BP^*} BP^*(PW_{n,k}) \cong \Lambda_{K^*_{(p)}(pt)}(\gamma_{n-k+3}, \cdots, \gamma_n) \otimes_{K^*_{(p)}(pt)} K^*_{(p)}(pt)[[x]]/I$$

where $|\gamma_i| = -1$ and $|x| = 0$ and I is the ideal generated by $\{\binom{n}{j}x^j | n-k < j \leq n\}$ [9]. We are assuming $p > n$, so that

$$K^*_{(p)}(PW_{n,k}) \cong \Lambda_{K^*_{(p)}(pt)}(\gamma_{n-k+3}, \cdots, \gamma_n) \otimes_{K^*_{(p)}(pt)} K^*_{(p)}(pt)[[x]]/(x^{n-k+1})$$

where $|\gamma_i| = -1$ and $|x| = 0$. We may construct the K-theoretic algebra generators γ_j^K via the following analogue of (4.1.3)

$$\begin{tikzcd} 0 \arrow[d] & \\ K^{-1}_{(p)}(PW_{n,k}) \arrow[d, "\delta"] & 0 \arrow[d] \\ K^0_{(p)}(\mathbb{C}P^\infty, PW_{n,k}) \arrow[d, "i^*"] & K^0_{(p)}(BU(n), BU(n-k)) \arrow[l, "f^*"'] \arrow[d] \\ K^0_{(p)}(\mathbb{C}P^\infty) & K^0_{(p)}(BU(n)) \arrow[l, "f_0^*"'] \end{tikzcd} \tag{4.3.10}$$

Identify $K^0_{(p)}(BU(n), BU(n-k))$ with the ideal in $K^0_{(p)}(BU(n))$ generated by c_j^K for $n-k < j \leq n$, and write $u_j^K = f^*\tau_j^K$. Since

$$f_0^*\tau_j^K = \binom{n}{j} x_K^j + \sum_{k>j} \binom{n}{k} \nu_k x_K^k = x_K^j \mu_j^K,$$

where μ_j^K is a unit in $K^*_{(p)}(\mathbb{C}P^\infty)$, we see that

$$i^*(u_j^K - x_K^{j-(n-k+1)}\frac{\mu_j^K}{\mu_{n-k+1}^K}u_{n-k+1}^K) = 0.$$

Let $\gamma_j^K \in K^{-1}_{(p)}(PW_{n,k})$ be defined by

$$\delta\gamma_j^K = u_j^K - x_K^{j-(n-k+1)}\frac{\mu_j^K}{\mu_{n-k+1}^K}u_{n-k+1}^K. \tag{4.3.11}$$

In the following proposition, we show that γ_j^K generates the exterior algebra part of $K^*_{(p)}(PW_{n,k})$.

Proposition 4.3.12. With notations as above,

$$K^*_{(p)}(PW_{n,k}) \cong \Lambda_{K^*_{(p)}(pt)}(\gamma^K_{n-k+3}, \cdots, \gamma^K_n) \otimes_{K^*_{(p)}(pt)} K^*_{(p)}(pt)[x]/(x^{n-k+1}).$$

Proof. Consider the homotopy fixed point spectral sequence [9, Proposition 2.1]

$$E_2^{s,t} = \mathbb{Z}[y] \otimes K^t_{(p)}(W_{n,k}) \implies K^{s+t}_{(p)}(PW_{n,k}),$$

and from the analogous computation of [9, Proposition 4.5] deduce that a set of elements form generators of the exterior algebra part if they pullback to corresponding exterior algebra generators of $K^*_{(p)}(W_{n,k})$. Consequently, it suffices to prove that $\pi^*\gamma_j^K = y_j^K$ in $K^*_{(p)}(W_{n,k})$. From the diagram (4.1.2) we get the following commutative diagram extending (4.3.10)

$$\begin{tikzcd}
K^{-1}_{(p)}(W_{n,k}) \arrow[d, "\delta"', "\cong"] & K^{-1}_{(p)}(PW_{n,k}) \arrow[l, "\pi^*"'] \arrow[d, "\delta"] & \\
K^0_{(p)}(E, W_{n,k}) & K^0_{(p)}(\mathbb{C}P^\infty, PW_{n,k}) \arrow[l, "\pi^*"] & K^0_{(p)}(BU(n), BU(n-k)). \arrow[l, "f^*"]
\end{tikzcd}$$

Here we compute for $j > n-k+1$,

$$\begin{aligned}
\delta\pi^*(\gamma_j^K) &= \pi^*\delta(\gamma_j^K) \\
&= \pi^*(u_j^K - x_K^{j-(n-k+1)}\frac{\mu_j^K}{\mu_{n-k+1}^K}u_{n-k+1}^K) \\
&= \pi^* u_j^K \text{ (as } \pi^*(x) = 0) \\
&= \pi^* f^*(\tau_j^K).
\end{aligned}$$

On the other hand, we have the following diagram

$$\begin{tikzcd}[row sep=small, column sep=small]
 & U(n) \arrow[rr] \arrow[dl] \arrow[ddl] & & * \arrow[dl] \arrow[ddl] \\
W_{n,k} \arrow[rr] \arrow[d] & & BU(n-k) \arrow[d] & \\
E \arrow[rr] & & BU(n) &
\end{tikzcd}$$

which leads to the diagram below

$$\begin{tikzcd}
K^{-1}_{(p)}(W_{n,k}) \arrow[r, "\cong"] \arrow[d, hook] & K^0_{(p)}(E, W_{n,k}) \arrow[d] & K^0_{(p)}(BU(n), BU(n-k)) \arrow[l] \arrow[d] \\
K^{-1}_{(p)}(U(n)) \arrow[r, "\cong"'] & K^0_{(p)}(E, U(n)) & K^0_{(p)}(BU(n), *) \arrow[l]
\end{tikzcd}$$

In the bottom row of above diagram image of y_j^K and image of τ_j^K coincide by the construction of y_j^K. So in the first row the same will happen for $j > n-k$. Hence we must have $\delta\pi^*\gamma_j^K = \delta y_j^K$, and δ being injective, we get our desired result. $\square$

4.3.13. Stable decompositions for $PW_{n,k(p)}$. We verify the hypothesis of Theorem 4.3.7 for $PW_{n,k}$ if $n < p$. First, we prove a lemma regarding the image of the Chern character. Recall that

$$H^*(PW_{n,k}; \mathbb{Z}_{(p)}) \cong \Lambda_{\mathbb{Z}}(\gamma^H_{n-k+2}, \cdots, \gamma^H_n) \otimes \mathbb{Z}_{(p)}[x]/(x^{n-k+1})$$

with $|\gamma_j^H| = 2j-1$ as in (4.1.4), $|x| = 2$. In the following lemma, we use the construction of γ_j^K of (4.3.11) which generate the exterior algebra part of the K-theory of $PW_{n,k(p)}$ by Proposition 4.3.12.

Lemma 4.3.14. For $n-k+2 \leq j \leq n$, $ch(\gamma_j^K)$ is a $\mathbb{Q}$-linear combination of $\{\gamma_s^H x^t \mid n-k+2 \leq s \leq n,\ 0 \leq t \leq n-k\}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
K^{-1}(PW_{n,k}) \arrow[r, hook, "\delta"] \arrow[d, "ch"] & K(\mathbb{C}P^\infty, PW_{n,k}) \arrow[d, "ch"] & K(BU(n), BU(n-k)) \arrow[l, "(f_0,f)^*"'] \arrow[d, "ch"] \\
H^{odd}(PW_{n,k}; \mathbb{Q}) \arrow[r, hook, "\delta"] & H^{ev}(\mathbb{C}P^\infty, PW_{n,k}; \mathbb{Q}) & H^{ev}(BU(n), BU(n-k); \mathbb{Q}) \arrow[l, "(f_0,f)^*"]
\end{tikzcd}$$

with f, f_0 as defined in (4.1.2). Now $H^*(BU(n), BU(n-k); \mathbb{Q})$ is the ideal $(c_{n-k+1}, \cdots, c_n)$ of $H^*(BU(n); \mathbb{Q}) = \mathbb{Q}[c_1, \cdots, c_n]$. In these terms, we must have an equation of the form

$$ch(\tau_j^K) = c_{n-k+1} \cdot P_{n-k+1,j} + c_{n-k+2} \cdot P_{n-k+2,j} \cdots + c_n \cdot P_{n,j}$$

for some power series $P_{i,j} \in \mathbb{Q}[[c_1, \cdots, c_n]]$. The right commutative square in the above implies (as $(f_0, f)^*$ preserves the multiplication of relative cohomology classes)

$$ch(u_j^K) = ch(f^*\tau_j^K) = f^*(c_{n-k+1}) \cdot f_0^*(P_{n-k+1,j}) + f^*(c_{n-k+2}) \cdot f_0^*(P_{n-k+2,j}) + \cdots + f^*(c_n) \cdot f_0^*(P_{n,j})$$

with $f_0^*(P_{i,j}) \in H^*(\mathbb{C}P^\infty; \mathbb{Q}) = \mathbb{Q}[[x]]$. We may now compute from (4.3.11)

$$\begin{aligned}
\delta ch(\gamma_j^K) &= ch(\delta\gamma_j^K) \\
&= ch(u_j^K - x_K^{j-(n-k+1)} \frac{\mu_j^K}{\mu_{n-k+1}^K} u_{n-k+1}^K) \\
&= ch(f^*\tau_j^K - x_K^{j-(n-k+1)} \frac{\mu_j^K}{\mu_{n-k+1}^K} f^*\tau_{n-k+1}^K) \\
&= f^*(c_{n-k+1}) \cdot f_0^*(Q_{n-k+1,j}) + \rho_{n-k+2} \cdot f_0^*(Q_{n,j}) + \cdots + \rho_n \cdot f_0^*(Q_{n,j}),
\end{aligned}$$

where ρ_j is defined in (4.1.4) as,

$$\rho_j = u_j^H - x^{j-(n-k+1)} \frac{\mu_j}{\mu_{n-k+1}} u_{n-k+1}^H = f^*(c_j) - x^{j-(n-k+1)} \frac{\mu_j}{\mu_{n-k+1}} f^*(c_{n-k+1}),$$

and for some polynomials $Q_{i,j}$ in the c_i. The element $\delta ch(\gamma_j^K)$ maps to 0 in $H^*(\mathbb{C}P^\infty)$ from which it follows that $f_0^* Q_{n-k+1,j}$ must be 0, so that

$$\delta ch(\gamma_j^K) = \rho_{n-k+2} \cdot f_0^*(Q_{n-k+2,j}) + \cdots + \rho_n \cdot f_0^*(Q_{n,j}).$$

We now write $f_0^*(Q_{i,j}) = \phi_{i,j}(x)$, and use the fact that $\delta : H^*(PW_{n,k}) \to H^{*+1}(\mathbb{C}P^\infty, PW_{n,k})$ is a map of $H^*(\mathbb{C}P^\infty)$-modules, which implies

$$\begin{aligned}
\delta ch(\gamma_j^K) &= \rho_{n-k+2} \cdot f_0^*(Q_{n-k+2,j}) + \cdots + \rho_n \cdot f_0^*(Q_{n,j}) \\
&= \delta(\gamma_{n-k+2}^H) \cdot \phi_{n-k+2,j}(x) + \cdots + \delta(\gamma_n) \cdot \phi_{n,j}(x) \\
&= \delta(\gamma_{n-k+2}^H) \cdot \phi_{n-k+2,j}(x) + \cdots + \gamma_n^H \cdot \phi_{n,j}(x)).
\end{aligned}$$

As δ is injective, we have that $ch(\gamma_j)$ is a linear combination of $\gamma_j^H x^r$ where $n-k+2 \leq j \leq n$ and $r \leq n-1$. Further as x^{n-k+1} maps to 0 in $H^*(PW_{n,k}; \mathbb{Q})$, $ch(\gamma_j^K)$ must be a $\mathbb{Q}$-linear combination of the set described in the statement of the lemma. □

We now have all the ingredients in place to prove that the projective Stiefel manifold splits into a wedge of spheres in the stable homotopy category.

Theorem 4.3.15. Let p be a prime $> n$. Then, the p-localization of the projective Stiefel manifold $PW_{n,k(p)}$ stably splits as a wedge of p-local spheres.

Proof. We verify that $PW_{n,k}$ satisfies the conditions of Theorem 4.3.7. The condition (2) already follows from the expression of (4.1.1). We also have

$$\begin{aligned}\dim(PW_{n,k}) &= 2nk - k^2 - 1 = 2n(k - \frac{k^2}{2n}) - 1\\ &< 2n(n-1) \text{ as } k \leq n-1\\ &< 2p(p-1) \text{ as } n < p,\end{aligned}$$

which verifies the condition (1). Finally, we have to verify that the Chern character ch has image in $H^*(PW_{n,k};\mathbb{Z}_{(p)})$. As ch is a map of rings, it suffices to verify this on the generators x and γ_j^K of Proposition 4.3.12. The class x is the pullback of the complex orientation class via $PW_{n,k} \to \mathbb{C}P^\infty$, so we have

$$ch(x) = e^x - 1 = \sum_{i=1}^{n-k} \frac{x^i}{i!} \text{ as } x^{n-k+1} = 0,$$

which clearly lies in $H^*(PW_{n,k};\mathbb{Z}_{(p)})$ as $p > n-k$. The Chern character on the classes γ_j^K is described by Lemma 4.3.14 on which we now apply the integrality result of Adams [1, Theorem 1]. As $W_{n,k}$ is $2(n-k)$-connected, from the fibration

$$W_{n,k} \to PW_{n,k} \to \mathbb{C}P^\infty$$

we obtain that there is a cell structure on $PW_{n,k}$ whose $2(n-k)$-skeleton is homotopy equivalent to $\mathbb{C}P^{n-k}$. It follows that the restriction of the classes γ_j^K to $PW_{n,k}^{(2(n-k))}$ is 0 with respect to this cell structure. The expression of Lemma 4.3.14 implies that the highest degree term which may occur in the expression of $ch(\gamma_j^K)$ is $\gamma_n^H x^{n-k}$, and this lies in degree $2n-1+2(n-k)$. Therefore, following [1], we have $ch_{2(n-k)+1+2r}(\gamma_j^K) = 0$ if $r \geq n-1$. For $r < n-1 < (p-1)$, the $m(r)$ appearing in [1] is not divisible by p; so by [1, Theorem 1], the proof is complete. □

Remark 4.3.16. The argument above may be easily modified to deduce that the spaces $P_\ell W_{n,k}$ have a p-local stable decomposition into a wedge of spheres under the condition $p > n$, and

$\sum_{|I|=n-k+1} \ell^I$ is not divisible by p. The latter condition implies that the cohomology of $P_\ell W_{n,k}$ with $\mathbb{Z}_{(p)}$-coefficients is torsion-free by (4.1.5). The K-theory is also torsion-free, and the generators may be chosen via the diagram (4.1.6) using the pair $(BU(k), Gr_k(\mathbb{C}^n))$ in place of $(BU(n), BU(n-k))$ in the calculations above. An analogous calculation implies that the Chern character has image in $\mathbb{Z}_{(p)}$ and then, we realize the p-local decomposition using Theorem 4.3.7.

4.4 Unstable p-local decompositions

In this section, we use the stable decomposition of the p-local $PW_{n,k}$ of §4.3 for $p > n$, to deduce an unstable product decomposition. Throughout this section, we work in the category of p-local spaces. As in §4.2, we compare $PW_{n,k}$ with the product $M_{n,k} = \mathbb{C}P^{n-k} \times S^{2(n-k)+3} \times S^{2(n-k)+5} \times \cdots \times S^{2n-1}$, which also stably decomposes into a wedge of spheres for $p > n$. Note that if $p > n$, $PW_{n,k}$ and Y both have the same cohomology with $\mathbb{Z}_{(p)}$-coefficients.

In order to show the equivalence between $PW_{n,k}$ and $M_{n,k}$, we construct maps from $PW_{n,k}$ to the individual factors of $M_{n,k}$ in the p-local category. This is possible for small values of k in comparison to p and n. We first construct maps from $PW_{n,k}$ to $\mathbb{C}P^{n-k}$ which lift the map $PW_{n,k} \to \mathbb{C}P^\infty \simeq K(\mathbb{Z}, 2)$ classifying the class $x \in H^2(PW_{n,k})$.

Proposition 4.4.1. Suppose that $p > n+1$ and $k \leq \min\{n, p+n-\sqrt{p^2+n^2-2p+1}\}$. Then, the map $PW_{n,k} \to \mathbb{C}P^\infty$ lifts to $\mathbb{C}P^{n-k}$.

Proof. For $n = k$, there is nothing to prove. We assume $n > k$, and that we have a p-local minimal cell structure on $PW_{n,k}$ via Proposition 4.3.2. The $\big(2(n-k)+2\big)$-skeleton of $PW_{n,k}$ is homotopy equivalent to $\mathbb{C}P^{n-k}$. It suffices to prove that this equivalence extends all the way to a map $PW_{n,k} \to \mathbb{C}P^{n-k}$.

We prove the extension by considering one cell attachment at a time. Suppose we have an extension $PW \to \mathbb{C}P^{n-k}$ for a subcomplex PW of $PW_{n,k}$. We consider the following diagram which attaches a single cell to PW

$$\begin{tikzcd}
S^q \arrow[d, "\phi"'] & \\
PW \arrow[r] \arrow[d] & \mathbb{C}P^{n-k}, \\
C(\phi) \arrow[ur, dashed] &
\end{tikzcd}$$

and prove that the dashed arrow exists. This will be ensured if the composite $S^q \to \mathbb{C}P^{n-k}$ is trivial. Up to homotopy, this lifts to $S^q \xrightarrow{\alpha} S^{2(n-k)+1}$, and it's enough to show that this lifted map is null-homotopic. We first check that α belongs to the stable range using [39, Corollary 3.2], which happens if $q \leq 2(n-k)p + 2p - 3$. Observe that

$$q \leq \dim(PW_{n,k}) - 1 = 2nk - k^2 - 2,$$

and that the quadratic equation in k

$$2nk - k^2 - 2 = 2(n-k)p + 2p - 3$$

has the positive zero (over $\mathbb{R}$) as the second bound in the Proposition. Therefore, we are in the stable range, and further $\dim(PW_{n,k}) < 2p^2 - 2p$ (which holds for $p > n$) guarantees that it is in fact in the image of J-homomorphism as in the proof of Theorem 4.3.7. We prove that the e-invariant of this map vanishes.

Note that α induces a map $\kappa : C(\phi) \to \mathbb{C}P^{n-k+1}$ on mapping cones. We consider the diagram

$$\begin{tikzcd}
PW \arrow[r] \arrow[d] & \mathbb{C}P^{n-k} \arrow[d] & \\
C(\phi) \arrow[r, "\kappa"] \arrow[d] & \mathbb{C}P^{n-k+1} \arrow[r] \arrow[d] & C \arrow[d] \\
S^{q+1} \arrow[r, "\Sigma\alpha"] & S^{2n-2k+2} \arrow[r] & C'.
\end{tikzcd}$$

In this diagram the squares placed vertically come from the map between cofibre sequences and so do the squares placed horizontally. It suffices to show $e(\Sigma\alpha)$ vanishes. If not, there exists $\tau \in K^*(C')$ such that $ch(\tau) \notin H^*(C'; \mathbb{Z}_{(p)})[u^{\pm}]$ from [2, Proposition 7.8]. Note that this implies also that the Chern character of the image of τ in $K^*(C)$ goes outside of the image of $H^*(C; \mathbb{Z}_{(p)})[u^{\pm}] \subset H^*(C; \mathbb{Q})[u^{\pm}]$. Our task boils down to checking that the Chern character map for C takes value in the image of $H^*(C; \mathbb{Z}_{(p)})[u^{\pm}] \subset H^*(C; \mathbb{Q})[u^{\pm}]$.

We note that the diagram

$$\begin{tikzcd}
PW \arrow[r] \arrow[d] & \mathbb{C}P^{n-k} \arrow[d] \\
C(\phi) \arrow[r, "\kappa"] \arrow[d] & \mathbb{C}P^{n-k+1} \arrow[d] \\
PW_{n,k} \arrow[r] & \mathbb{C}P^{\infty}
\end{tikzcd}$$

commutes, where the bottom row is the classifying map of the S^1-bundle $S^1 \to W_{n,k} \to PW_{n,k}$. Hence we get the following map between cofibrations,

$$\begin{tikzcd} C(\phi) \arrow[r, "\kappa"] \arrow[d] & \mathbb{C}P^{n-k+1} \arrow[r] \arrow[d] & C \arrow[d, "\omega"] \\ PW_{n,k} \arrow[r] & \mathbb{C}P^{\infty} \arrow[r] & D \end{tikzcd}$$

with D being the homotopy cofiber of $PW_{n,k} \to \mathbb{C}P^{\infty}$. Applying $K_{(p)}$, we obtain the diagram

$$\begin{tikzcd} K^{-1}_{(p)}(C(\phi)) \arrow[r, hook] & K_{(p)}(C) \arrow[r] & K_{(p)}(\mathbb{C}P^{n-k+1}) \arrow[r, "\kappa^*"] & K_{(p)}(C(\phi)) \\ K^{-1}_{(p)}(PW_{n,k}) \arrow[r, hook] \arrow[u, two heads] & K_{(p)}(D) \arrow[r] \arrow[u] & K_{(p)}(\mathbb{C}P^{\infty}) \arrow[r] \arrow[u, two heads] & K_{(p)}(PW_{n,k}). \arrow[u, two heads] \end{tikzcd}$$

Now surjectivity of the two terminal vertical arrows follow from the fact that $C(\phi)$ is subcomplex of $PW_{n,k}$ under the minimal p-local CW structure (Proposition 4.3.2). In the proof of Theorem 4.3.15, we have checked that the image of the Chern character for $PW_{n,k}$ lies inside cohomology with $\mathbb{Z}_{(p)}$-coefficients, and so, the same is true for the image of $K^{-1}_{(p)}(C(\phi))$ inside $K_{(p)}(C)$.

To complete the proof, we show that the Chern character carries $ker(\kappa^*)$ to $H^{ev}(C;\mathbb{Z}_{(p)})[u^{\pm}]$. The composition $\mathbb{C}P^{n-k} \longrightarrow C(\phi) \xrightarrow{\kappa} \mathbb{C}P^{n-k+1}$ is homotopic to the inclusion, and so, $ker(\kappa^*)$ must be equal to the $\mathbb{Z}_{(p)}$-module generated by x^{n-k+1}, where x is the complex orientation of K-theory. Now consider the diagram obtained from (4.3.10),

$$\begin{tikzcd} K_{(p)}(BU(n), BU(n-k)) \arrow[r, "f^*"] \arrow[d] & K_{(p)}(D) \arrow[r, "\omega^*"] \arrow[d] & K_{(p)}(C) \arrow[d] \\ K_{(p)}(BU(n)) \arrow[r] & K_{(p)}(\mathbb{C}P^{\infty}) \arrow[r] & K_{(p)}(\mathbb{C}P^{n-k+1}). \end{tikzcd}$$

We see that $\omega^* f^* c^K_{n-k+1} \in K_{(p)}(C)$ is mapped to $\binom{n}{n-k+1} x^{n-k+1} \in K_{(p)}(\mathbb{C}P^{n-k+1})$. As $\binom{n}{n-k+1}$ is a unit in $\mathbb{Z}_{(p)}$, it suffices to show that $ch(\omega^* f^* c^K_{n-k+1})$ lies in $H^{ev}(\mathbb{C}P^{n-k+1};\mathbb{Z}_{(p)})[v^{\pm}]$. We also notice that using computations analogous to Lemma 4.3.14

$$\begin{aligned} ch(\omega^* f^* c^K_{n-k+1}) &= \omega^* f^*(ch(c^K_{n-k+1})) \\ &= \omega^* f^*(c_{n-k+1} \cdot P_{n-k+1} + \cdots + c_n \cdot P_n) \text{ [where } P_j \in \mathbb{Q}[[c_1, \cdots, c_n]] \text{]} \\ &= \omega^* f^* c_{n-k+1} \cdot Q_{n-k+1} + \cdots + \omega^* f^* c_n \cdot Q_n \text{ [where } Q_j \in \mathbb{Q}[x]/(x^{n-k+2}) \text{]} \end{aligned}$$

This computation shows that the maximum degree of the homogeneous parts of $ch(\omega^* f^* c^K_{n-k+1})$ is

$$|c_n| + |x^{n-k+1}| = 2n + 2(n-k+1).$$

As $\kappa : C(\phi) \to \mathbb{C}P^{n-k+1}$ is a $(2n-2k+1)$-equivalence, C is $(2n-2k)$-connected. Therefore, as $p > n+1$, we apply [1, Theorem 1] to deduce the result. □

Proposition 4.4.1 constructs for us the map from $PW_{n,k} \to \mathbb{C}P^{n-k}$. The remaining factors in the product decomposition of $M_{n,k}$ are spherical and maps are constructed via a connectivity argument.

Proposition 4.4.2. If $p > n$ and $k \leq \min\{n, (p+n) - \sqrt{p^2+n^2-4p+2}\}$, then for $1 \leq r \leq k-1$, there is a map from $PW_{n,k}$ to $S^{2(n-k)+2r+1}$ which pulls back the standard $\mathbb{Z}_{(p)}$-cohomology generator of $S^{2(n-k+r)+1}$ to the class $\gamma_{n-k+r} \in H^*(PW_{n,k}; \mathbb{Z}_{(p)})$.

Proof. From the equivalence of $\Sigma^\infty PW_{n,k}$ and $\Sigma^\infty M_{n,k}$ we get maps

$$\nu_r : \Sigma^\infty PW_{n,k} \to \Sigma^\infty S^{2(n-k+r)+1}$$

for $1 \leq r \leq k-1$, which satisfy $\nu_r^*(\epsilon_{2(n-k+r)+1}) = \gamma_{n-k+r}$. By the usual $\Sigma^\infty - \Omega^\infty$ adjunction, we obtain a map of spaces

$$\tilde{\nu}_r : PW_{n,k} \to QS^{2(n-k+r)+1}.$$

Now as we are in the p-local category, the fibre $F(2(n-k+r)+1)$ of the natural map $S^{2(n-k+r)+1} \to QS^{2(n-k+r)+1}$ is $(2p(n-k+r)+2p-4)$-connected [39, corollary 3.2]. The given bounds on k imply that $\dim(PW_{n,k}) \leq 2p(n-k+r)+2p-4$. Hence, the map $\tilde{\nu}_r$ lifts to $S^{2(n-k+r)+1}$, and we are done. □

We let $M(n,p) = p + n - \sqrt{p^2+n^2-4p+2}$, and summarize the results of Propositions 4.4.1 and 4.4.2 in the following theorem.

Theorem 4.4.3. Let $p > n+1$ and $k \leq \min(n, M(n,p))$. Then, in the p-local category

$$PW_{n,k} \simeq \mathbb{C}P^{n-k} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}.$$

Observe that the bound on k holds whenever $k < n/2$. As in §4.2, the product decomposition of the projective Stiefel manifold implies an S^1-equivariant decomposition of the complex Stiefel manifold.

Theorem 4.4.4. Let $p > n+1$, and $k \leq \min(n, M(n,p))$. Then, we have an equivalence of S^1-spaces

$$W_{n,k(p)} \simeq \left[S^{2(n-k)+1} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\right]_{(p)}.$$

Proof. As both sides of the equivalence possess a free S^1-action, it suffices to exhibit an S^1-equivariant map which is a weak equivalence. We use the map

$$\phi_k : PW_{n,k} \to \mathbb{C}P^{n-k}$$

from Proposition 4.4.1 and pullback the circle bundle $q : S^{2(n-k)+1} \to \mathbb{C}P^{n-k}$ via ϕ_k. As ϕ_k is a lift of the classifying map of the circle bundle $W_{n,k} \to PW_{n,k}$, we have a commutative square

$$\begin{tikzcd} W_{n,k} \arrow[r, "\hat{\phi}_k"] \arrow[d] & S^{2n-2k+1} \arrow[d] \\ PW_{n,k} \arrow[r] & \mathbb{C}P^{n-k}. \end{tikzcd}$$

We now observe that $\hat{\phi}_k$ is S^1-equivariant as it fits in a pullback diagram of S^1-bundles. We now form the equivariant map

$$W_{n,k} \to S^{2n-2k+1} \times W_{n,k-1}.$$

Now $k-1 \leq M(n,p)$, and we proceed by induction assuming that $W_{n,k-1}$ supports the splitting stated in the theorem, to deduce the result. □

There are also analogous splittings for the spaces $W_{n,k;m}$ and $P_\ell W_{n,k}$.

Theorem 4.4.5. Let $p > n+1$, and $k \leq \min(n, M(n,p))$. Then, we have equivalences

$$\left(W_{n,k;m}\right)_{(p)} \simeq \left[L_m(2n-2k+1) \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\right]_{(p)} \quad \text{if } p \mid m,$$
$$\left(P_\ell W_{n,k}\right)_{(p)} \simeq \left[\mathbb{C}P^{n-k} \times S^{2n-2k+3} \times \cdots \times S^{2n-1}\right]_{(p)} \quad \text{if } p \nmid \sum_{|I|=n-k+1} \ell^I.$$

Proof. The S^1-equivariant map $W_{n,k} \to S^{2n-2k+1}$ in Theorem 4.4.4 yields the map

$$W_{n,k;m} \to L_m(2n-2k+1)$$

on C_m-orbits. The other factors receive maps via the composite

$$W_{n,k;m} \to PW_{n,k} \to S^{2n-2k+2r+1}$$

where the latter map is defined by the equivalence of Theorem 4.4.3. The product of these two maps imply the first p-local equivalence in the theorem. For the second equivalence, one applies the stable decomposition for $P_\ell W_{n,k}$ outlined in Remark 4.3.16. After that, observe that Proposition 4.4.2 works verbatim for $P_\ell W_{n,k}$ and Proposition 4.4.1 works analogously by using the pair $(BU(k), Gr_k(\mathbb{C}^n))$ instead of $(BU(n), BU(n-k))$. $\square$

Chapter 5

Characteristic classes on certain quotients of Stiefel manifolds

In this chapter, we compute some topological invariants of projective Stiefel manifolds and similar quotients of Stiefel manifols. In particular, we consider the circle quotient manifolds which is defined as the quotient $V_{n,2k}/S^1$ with the circle acting via the diagonal embedding of $SO(2)$ inside $SO(2k)$. The tangent bundle for these manifolds are computable, and together with the cohomology calculation, we determine the characteristic classes of these manifolds. We then discuss topological consequences for these manifolds.

5.1 Computations for projective Stiefel manifolds

In this section, we describe some geometric consequences for the real projective Stiefel manifolds, which are derivable from the cohomology. We only use the cohomology with $\mathbb{Z}_2$-coefficients in this section. Computations with $\mathbb{Z}$-coefficients or $\mathbb{Z}_{(2)}$-coefficients is more involved even in the case of Stiefel manifolds particularly when $n < 2k$.

Recall that the $\mathbb{Z}_2$-cohomology of $PV_{n,k}$ was determined in [19].

$$H^*(PV_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^N) \otimes V(A), \text{ for } k < n,$$

where $N = \min\{j \mid n-k < j \leq n, \text{ and } \binom{n}{j} \text{ is odd }\}$, $A = \{y_j \mid n-k \leq j < n\} - \{y_{N-1}\}$ and $|x| = 1, |y_j| = j$. This is computed using the Serre spectral sequence for the fibration $V_{n,k} \to PV_{n,k} \to \mathbb{R}P^\infty$. The differentials are computed using the commutative diagram of fibrations

$$\begin{tikzcd}
V_{n,k} \arrow[r] \arrow[d] & V_{n,k} \arrow[d] \\
PV_{n,k} \arrow[r] \arrow[d] & BO(n-k) \arrow[d] \\
\mathbb{R}P^\infty \arrow[r] & BO(n).
\end{tikzcd}$$

In fact, the bottom square is a homotopy pullback diagram, which may be used to describe homotopy classes of maps into $PV_{n,k}$. Proceeding in this direction, one discovers that the $PV_{n,k}$ classifies line bundles L such that nL has k linearly independent sections.

The tangent bundle of $PV_{n,k}$ was determined in [27] and satisfies the following relation

$$T(PV_{n,k}) \oplus \binom{k+1}{2}\epsilon = nk\zeta_{n,k}, \tag{5.1.1}$$

where $\zeta_{n,k}$ is the Hopf bundle over $PV_{n,k}$. The computation of the tangent bundle is done via the 2^{k-1}-sheeted covering space $PV_{n,k} \to F_k(\mathbb{R}^n)$, where the latter is the space of flags $V_0 \subset V_1 \subset \cdots \subset V_k$ of subspaces of $\mathbb{R}^n$ with $\dim(V_i) = i$.

The computation of the tangent bundle (5.1.1) allows us to calculate the Stiefel Whitney classes of $PV_{n,k}$. These Stiefel-Whitney classes may also be calculated using the cohomology and Steenrod operations via Wu's formula. Note that the total Stiefel Whitney class of $\zeta_{n,k}$ is described by

$$w(\zeta_{n,k}) = 1 + x.$$

By the Whitney sum formula, we obtain,

$$w(T(PV_{n,k})) = (1+x)^{nk}. \tag{5.1.2}$$

5.1.1 Skew embeddings of $PV_{n,k}$

The Stiefel Whitney classes for a manifold may also be viewed as obstructions to trivializing vector bundles, and also to constructing linearly independent sections therein. The span of a manifold is the maximum number of linearly independent sections. For the projective Stiefel manifold, there are bounds on the span proved using the Stiefel Whitney classes and also by K-theory calculations [35].

A general embedding problem for a manifold M^n of dimension n seeks to find the precise k such that M^n embeds in $\mathbb{R}^{n+k}$. An analogous statement may also be formulated for immersions. In this case, the Stiefel Whitney class of the stable normal bundle provides an obstruction to the immersion dimension. More precisely, consider $\bar{w}(M) = w(M)^{-1}$, and suppose $\bar{w}_k(M) \neq 0$. Then, the manifold does not immerse in $\mathbb{R}^{n+k-1}$.

An embedding of a manifold inside the Euclidean space $\mathbb{R}^N$ is called totally skew if the affine subspaces of $\mathbb{R}^N$ associated to the tangent space at different points are skew. Two affine subspaces V, W of $\mathbb{R}^N$ are called skew, if their affine span has dimension $\dim(V)+\dim(W)+1$. For a smooth manifold M, we define

$$N(M) = \min\{n \mid M \text{ admits a skew embedding in } \mathbb{R}^n\}$$

Suppose an n dimensional manifold M admits a skew embedding inside R^N. Then the "skewness" condition ensures that there is a vector bundle monomorphism

$$T(F_2(M)) \oplus \epsilon \longrightarrow F_2(M) \times R^N,$$

where $F_2(M) = M \times M - \Delta(M)$, and this in turns implies that if $\bar{w}_k(T(F_2(M)) \neq 0$ then $N \geq 2N + k + 1$. In particular, this gives us a lower bound for $N(M)$:

$$N(M) \geq 2n + k + 1.$$

The authors in [8] then found a condition in terms of the Stiefel Whitney classes of M to produce a k for which $\bar{w}_k(T(F_2(M)) \neq 0$.

Theorem 5.1.1. [8] If $k := \max\{i \mid \bar{w}_i(M) \neq 0\}$, then $\bar{w}_{2k}(T(F_2(M)) \neq 0$ and hence $N(M) \geq 2n + 2k + 1$.

Then we have a direct consequence of the above result for $M = PV_{n,k}$.

Theorem 5.1.2. $N(PV_{n,k})$ satisfies the following inequality

$$N(PV_{n,k}) \geq 2\dim(PV_{n,k}) + 2m + 1,$$

where $m = \max\{j \mid \binom{nk+j-1}{nk-1} \neq 0 \text{ and } 0 \leqslant j \leqslant N-1\}$ and $N = \min\{j \mid \binom{n}{j} \text{ odd and } n-k < j \leq n\}$.

Proof. From the description of the stable tangent bundle of $PV_{n,k}$, we obtain the total Stiefel-Whitney class of $PV_{n,k}$ as follows

$$w(PV_{n,k}) = (1+x)^{nk}.$$

From this we can see

$$m = \max\{j \mid \binom{nk+j-1}{nk-1} \neq 0 \text{ and } 0 \leqslant j \leqslant N-1\} = \max\{j \mid \bar{w}_j(PV_{n,k}) \neq 0\}.$$

Applying 5.1.1 we get $N(PV_{n,k}) \geqslant 2\dim(PV_{n,k}) + 2m + 1$. □

5.1.2 Characteristic rank of projective Stiefel manifolds

Let X be a connected finite CW-complex and ξ a real vector bundle over X. The **characteristic rank** of ξ over X, denoted by $charrank_X(\xi)$, is by definition the largest integer $k, 0 \leqslant k \leqslant dim(X)$, such that every cohomology class $x \in H^j(X;\mathbb{Z}/2), 0 \leqslant j \leqslant k$, is a polynomial in the Stiefel-Whitney classes $w_i(\xi)$.The **upper characteristic rank** of X, denoted by $ucharrank(X)$, is the maximum of $charrank_X(\xi)$ as ξ varies over all vector bundles over X.

The main result regarding the characteristic rank for Stiefel manifolds shows that the lowest degree non-zero class in cohomology usually does not arise as a Stiefel-Whitney class of a vector bundle, [25]. On the other hand, when we consider a quotient of a Stiefel manifold $V_{n,k}/G$, the classes in low degrees naturally arise from characteristic classes of G-representations. In the case of the projective Stiefel manifold with $\mathbb{Z}/2$-coefficients, the class x and it's powers are expressable in terms of Stiefel-Whitney classes of bundles. The remaining part which is additively an exterior algebra pulls back non-trivially to the Stiefel manifold, and this is usually not representable in terms of Stiefel-Whitney classes of bundles. We elaborate this in the next few results.

We start with a lemma which will be used later.

Lemma 5.1.3. For any real vector bundle ξ over $V_{8r+1,2}$ with $r > 1$, the class $w_{8r}(\xi) = 0$.

Proof. We have the following pushout diagram

$$\begin{tikzcd}
\mathbb{R}P^{n-2} \times V_{n-1,k-1} \arrow[r] \arrow[d] & V_{n-1,k-1} \arrow[d] \\
\mathbb{R}P^{n-1} \times V_{n-1,k-1} \arrow[r] & V_{n,k}
\end{tikzcd}$$

This tells us that the cofiber of the map $V_{n-1,k-1} \longrightarrow V_{n,k}$ is $\Sigma^{n-1}(V_{n-1,k-1})_+$.

Now taking $(n,k) = (8r+1,2)$ gives the following cofiber sequence

$$S^{8r-1} \longrightarrow V_{8r+1,2} \longrightarrow \Sigma^{8r} S^{8r-1}_+.$$

Applying $\widetilde{KO}$ on it we get the following exact sequence

$$\widetilde{KO}(\Sigma^{8r}S_+^{8r-1}) \longrightarrow \widetilde{KO}(V_{8r+1,2}) \longrightarrow \widetilde{KO}(S^{8r-1}).$$

Now we know $\widetilde{KO}(S^{8r-1}) = \pi_{8r-1}(BO) = \pi_7(BO) = 0$. So $\widetilde{KO}(\Sigma^{8r}S_+^{8r-1}) \longrightarrow \widetilde{KO}(V_{8r+1,2})$ must be a surjection. But we have

$$\begin{aligned}\widetilde{KO}(\Sigma^{8r}S_+^{8r-1}) &= \widetilde{KO}(S^{16r-1} \vee S^{8r}) \\ &= \widetilde{KO}(S^{16r-1}) \oplus \widetilde{KO}(S^{8r}) \\ &= \pi_7(BO) \oplus \widetilde{KO}(S^{8r}) \\ &= \widetilde{KO}(S^{8r})\end{aligned}$$

So we have a surjection $\widetilde{KO}(S^{8r}) \longrightarrow \widetilde{KO}(V_{8r+1,2})$. This means any stable bundle over $V_{8r+1,2}$ is a pullback of a stable bundle over S^{8r} by this composed map. But $w_n(\xi) = 0$ for any vector bundle ξ over S^n whenever $n \neq 1, 2, 4, 8$ [29]. So for $r > 1$, $w_{8r}(\xi) = 0$ for any vector bundle ξ over S^{8r} and hence same is true for $V_{8r+1,2}$ as well. $\square$

Theorem 5.1.4. If $n - k = 5$, 6 or ≥ 9, the upper characteristic rank of $PV_{n,k}$ is given by

$$\text{ucharrank}(PV_{n,k}) = \begin{cases} n-k-1 & \text{if } 2 \mid \binom{n}{k-1} \\ n-k & \text{if } 2 \nmid \binom{n}{k-1}. \end{cases}$$

Proof. First we consider the case $n - k \neq N - 1$, then ucharrank$(PV_{n,k}) < n - k$, since $y_{n-k} \in H^{n-k}(PV_{n,k})$ can not be a polynomial combination of Stiefel-Whitney classes of some bundle ξ over $PV_{n,k}$. Because otherwise the pullback of y_{n-k} via the map $V_{n,k} \longrightarrow PV_{n,k}$ would be a polynomial combination of Stiefel-Whitney classes of the pullback bundle of ξ over $V_{n,k}$ contradicting [25]. Moreover we can easily see charrank$(\zeta_{n,k}) = n - k - 1$. So in this case ucharrank$(PV_{n,k}) = n - k - 1$.

On the other hand for $n - k = N - 1$, $H^{n-k}(PV_{n,k}) = \mathbb{Z}_2\langle x^{n-k}\rangle$ and $H^{n-k+1}(PV_{n,k}) = \mathbb{Z}_2\langle y_{n-k+1}\rangle$. If $n-k+1$ is not a power of 2, then y_{n-k+1} cannot be a polynomial combination of Stiefel-Whitney classes because of the cohomology ring structure of $H^*(PV_{n,k}; \mathbb{Z}_2)$ and from the fact that the Stiefel-Whitney classes of a bundle are generated by the Stiefel-Whitney classes of degree powers of 2 of that bundle over the Steenrod algebra (this fact is a consequence of Wu's formula) [30]. So in this case ucharrank$(PV_{n,k}) = n - k$ since charrank$(\zeta_{n,k}) = n - k$.

Now we consider the case where $n-k = N-1$, and $n-k+1 = 2^s$, for some $s > 3$. In this case the only way y_{n-k+1} can be a polynomial in Stiefel-Whitney classes is if $y_{n-k+1} = w_{n-k+1}$. But the lemma 5.1.3 rules out this possibility and we get ucharrank$(PV_{n,k}) = n-k$. □

We also note that one may carry the argument above forward to even obtain partial results in the cases $n-k = 1,2,3,4,7,8$. For example, if $n-k=1$, we observe that if $n \equiv 2,3 \pmod 4$, then $N=2$, so the first class in the exterior algebra part is y_2. This pulls back non-trivially to $V_{n,k}$, where $y_2 = y_1^2$, and y_1 is a first Stiefel-Whitney class. Thus, y_2 is expressible using Stiefel-Whitney classes over $V_{n,k}$. However, this does not allow us to deduce that y_2 is thus, expressible over $PV_{n,k}$. On the other hand if $n \equiv 0,1 \pmod 4$, the first possible value of N is 4, and if we further assume that $\binom{n}{4} \equiv 0 \pmod 2$, then y_3 survives in the cohomology of $PV_{n,k}$. From [25], we know that y_3 is not expressible in terms of Stiefel-Whitney classes in $V_{n,k}$, so we have ucharrank$(PV_{n,k}) = 2$ in this case.

5.1.3 The complex case

For the complex projective Stiefel manifold $PW_{n,k} = W_{n,k}/S^1$, recall that the $\mathbb{Z}_2$-cohomology is determined additively in [6].

$$H^*(PW_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^N) \otimes \Lambda_{\mathbb{Z}_2}(A), \text{ for } k < n,$$

where $N = \min\{j \mid n-k < j \leqslant n, \text{ and } \binom{n}{j} \text{ is odd }\}$, $A = \{y_j \mid n-k < j \leqslant n\} - \{y_{N-1}\}$ and $|x| = 2, |y_j| = 2j-1$. We compute the characteristic rank in the proposition below, which turns out to be much easier in this case.

Proposition 5.1.5. The upper characteristic rank of $PW_{n,k}$ is given by

$$\text{ucharrank}(PW_{n,k}) = \begin{cases} 2(n-k) & \text{if } 2 \mid \binom{n}{k-1} \\ 2(n-k)+2 & \text{if } 2 \nmid \binom{n}{k-1}. \end{cases}$$

Proof. Here the first class which appears in the exterior algebra part of the cohomology is in odd degree. The lower classes belong to the algebra generated by x which is closed under products and Steenrod operations. By Wu's formula, a Stiefel-Whitney class in degree which is not a power of 2, is expressible in terms of lower degree Stiefel-Whitney classes via the multiplication and Steenrod operations. This implies that the first generator of the exterior algebra part of the cohomology cannot be expressible in terms of Stiefel-Whitney classes. The result follows. □

5.2 The circle quotient Stiefel manifolds

There may be many circle actions defined on a real Stiefel manifold $V_{n,k}$. The ones that give us a homogeneous space act via a homomorphism $S^1 \to O(k)$, where the latter acts on the Stiefel manifold by an orthogonal transformation on the vectors. This comes from a homomorphism into a maximal torus of $O(k)$, and up to conjugation they are a block diagonal inclusion of products of $SO(2)$. As in the case of projective Stiefel manifolds, we look at the diagonal inclusion of S^1 in the maximal torus. This gives us a manifold that we call the circle quotient Stiefel manifold, which is more precisely defined below.

We define the space $Y_{n,k}$ as the orbit space of the S^1-action on $V_{n,2k}$ defined as follows: We consider S^1 as $SO(2)$ embedded inside the maximal torus $SO(2)^k \subset SO(2k)$ by diagonal map. Then the action of our interest is the restriction of the action of $SO(2k)$ on $V_{n,2k}$ from right by matrix multiplication.

The construction gives a principal fiber bundle

$$S^1 \longrightarrow V_{n,2k} \longrightarrow Y_{n,k}.$$

The complex line bundle associated to this principal bundle over $Y_{n,k}$ will be denoted by ζ. We denote the realification of ζ by ζ_r.

From the definition of $Y_{n,k}$, it is clear that we have the following diagram of fibrations:

$$\begin{tikzcd}
SO(2) \arrow[r] \arrow[d, "\Delta"'] & V_{n,2k} \arrow[r] \arrow[d, equal] & Y_{n,k} \arrow[d, "\pi"'] \arrow[dd, bend left, "q"] \\
SO(2)^k \arrow[r] \arrow[d] & V_{n,2k} \arrow[r] \arrow[d, equal] & \widetilde{F} \arrow[d, "p"'] \\
O(2)^k \arrow[r] & V_{n,2k} \arrow[r] & F,
\end{tikzcd} \tag{5.2.1}$$

where $\widetilde{F}$ is the 2^k-sheeted cover of F, the space of flags $V_0 \subset V_1 \subset \cdots \subset V_k$ of subspaces of $\mathbb{R}^n$ with $\dim(V_i) = 2i$. The canonical real vector bundles over F will be denoted by ξ_j, for $1 \leqslant j \leqslant k+1$ and rank of $\xi_j = 2$, for $1 \leqslant j \leqslant k$. Recall that the tangent bundle of F, $TF \cong \bigoplus_{1 \leqslant i < j \leqslant k+1} \xi_i \otimes_{\mathbb{R}} \xi_j$, [27].

5.2.1 The tangent bundle of $Y_{n,k}$

We now identify the tangent bundle of $Y_{n,k}$ in following theorem.

Theorem 5.2.1. The tangent bundle of $Y_{n,k}$ is described as

$$TY_{n,k} = \binom{k}{2}\zeta_r \otimes_{\mathbb{R}} \zeta_r \oplus (k\zeta_r \otimes_{\mathbb{R}} q^*\xi_{k+1}) \oplus (k-1)\epsilon,$$

and additionally it satisfies the following relation

$$TY_{n,k} \oplus \binom{k+1}{2}\zeta_r \otimes_{\mathbb{R}} \zeta_r = nk\zeta_r \oplus (k-1)\epsilon.$$

Proof. From 5.2.1 we obtain the principal bundle

$$\frac{SO(2)^k}{\Delta(SO(2))} \longrightarrow Y_{n,k} \xrightarrow{\pi} \widetilde{F}.$$

The tangent bundle is a direct sum of pullback of the tangent bundle of the base and the vector bundle which restricts to the tangents along each fibre, with the latter being a trivial bundle. Now we can determine the tangent bundle of $Y_{n,k}$ as follows:

$$\begin{aligned} TY_{n,k} &\cong \pi^* T\widetilde{F} \oplus (k-1)\epsilon \\ &\cong q^* TF \oplus (k-1)\epsilon \\ &\cong q^*\Big(\bigoplus_{1\leqslant i<j\leqslant k+1} \xi_i \otimes_{\mathbb{R}} \xi_j\Big) \oplus (k-1)\epsilon \\ &\cong \binom{k}{2}\zeta_r \otimes_{\mathbb{R}} \zeta_r \oplus (k\zeta_r \otimes_{\mathbb{R}} q^*\xi_{k+1}) \oplus (k-1)\epsilon \end{aligned}$$

Since $\bigoplus_{1\leqslant j\leqslant k+1} \xi_j = n\epsilon$, by pulling back both sides via q^* we have $k\zeta_r \oplus q^*\xi_{k+1} = n\epsilon$. So from the previous formula for $TY_{n,k}$, we have

$$TY_{n,k} \oplus \binom{k+1}{2}\zeta_r \otimes_{\mathbb{R}} \zeta_r = nk\zeta_r \oplus (k-1)\epsilon.$$

□

Once we have the formula for the tangent bundle of $Y_{n,k}$, we may start analyzing questions regarding the number of linearly independent vector fields, restrictions on immersions into Euclidean space, skew embeddings, and characteristic rank. The necessary ingredient in the entire matter is the cohomology calculation for $Y_{n,k}$.

5.2.2 $\mathbb{Z}_2$-cohomology of $Y_{n,k}$

As the action of S^1 on $V_{n,2k}$ occurs via a homomorphism $D : S^1 \to O(2k)$, we have the following commutative diagram

$$\begin{tikzcd}
S^1 \arrow[r] \arrow[d] & O(2k) \arrow[d] \\
V_{n,2k} \arrow[r] \arrow[d] & V_{n,2k} \arrow[d] \\
Y_{n,k} \arrow[r] \arrow[d] & Gr_{2k}(\mathbb{R}^n) \arrow[d] \\
\mathbb{C}P^\infty \arrow[r] & BO(2k),
\end{tikzcd}$$

where the rows are part of a homotopy fibration sequence. As a consequence the bottom square gives a homotopy pullback diagram

$$\begin{tikzcd}
Y_{n,k} \arrow[r] \arrow[d] & Gr_{2k}(\mathbb{R}^n) \arrow[d] \\
\mathbb{C}P^\infty \arrow[r] & BO(2k).
\end{tikzcd}$$

This implies that the circle quotient Stiefel manifold $Y_{n,2k}$ has the following universal property.

Proposition 5.2.2. Up to homotopy, the space $Y_{n,k}$ classifies complex line bundles ξ such that for the realification $r(\xi)$, $kr(\xi)$ has a complimentary bundle μ of dimension $n-2k$. (That is, $\mu \oplus kr(\xi) = n\epsilon$.)

The fibrations above allow us to compute the cohomology of $Y_{n,k}$.

Theorem 5.2.3. The $\mathbb{Z}_2$-cohomology of $Y_{n,k}$ is (additively)

$$H^*(Y_{n,k};\mathbb{Z}_2) \cong \Lambda_{\mathbb{Z}_2}(y_{n-2k}, \cdots, \widehat{y}_{2J-1}, \cdots, y_{n-1}) \otimes \mathbb{Z}_2[x]/(x^J),$$

where $deg(y_j) = j$ and x is the mod 2 Euler class of the bundle ζ, and $J = \min\{r \mid \binom{k+r-1}{k-1}$ is odd and $n-2k \leqslant 2r \leqslant n-1\}$.

Proof. We consider the following commutative diagram of fibrations

$$\begin{tikzcd} V_{n,2k} \arrow[r] \arrow[d, equal] & Y_{n,k} \arrow[r] \arrow[d] & \mathbb{C}P^{\infty} \arrow[d] \\ V_{n,2k} \arrow[r] & Gr_{2k}(\mathbb{R}^n) \arrow[r] & BO(2k). \end{tikzcd} \tag{5.2.2}$$

It is known that $H^*(V_{n,k};\mathbb{Z}_2)$ is additively exterior algebra on generators $\{y_j \mid n-k \leqslant j < n\}$, with $deg(y_j) = j$, [14]. We want to compute the action of differentials on y_j's in the spectral sequence associated to first row of 5.2.2. This is done by comparing the spectral sequences associated to the diagram 5.2.2. But the spectral sequence for the bottom row of 5.2.2 was done in [11] and says that y_j's are transgressive with $\tau(y_j) = \bar{w}_{j+1}$, where $\bar{w}_{j+1}$ is the $(j+1)$-th inverse universal Stiefel-Whitney class. So we know y_j's are transgressive in the spectral sequence of our interest and image of the transgression is given below.

The map $\mathbb{C}P^{\infty} \longrightarrow BO(2k)$ is the classifying map for the k-fold Whitney sum of the underlying real 2-plane bundle $r(\gamma^1)$ of the canonical complex line bundle γ^1. So we can conclude $w(kr(\gamma^1)) = (1+x)^k$ and hence $\overline{w}(kr(\gamma^1)) = (1+x)^{-k}$. So

$$\tau(y_j) = \overline{w}_{j+1}(kr(\gamma^1)) = \begin{cases} \binom{k+\frac{j+1}{2}-1}{k-1} x^{\frac{j+1}{2}} & \text{if } j \text{ is odd} \\ 0 & \text{if } j \text{ is even.} \end{cases}$$

This calculation enables us to determine the E_∞-page of the spectral sequence associated to the first row of the diagram and will be equal to

$$E_\infty = V(S) \otimes \mathbb{Z}_2[x]/(x^J),$$

where $S = \{y_j \mid n-2k \leqslant j \leqslant n-1\} - \{y_{2J-1}\}$ and $J = \min\{r \mid \binom{k+r-1}{k-1}$ is odd and $n-2k \leqslant 2r \leqslant n-1\}$. Now choosing lifts of the elements in S we can conclude that additively

$$H^*(Y_{n,k};\mathbb{Z}_2) \cong \Lambda_{\mathbb{Z}_2}(y_{n-2k}, \cdots, \widehat{y}_{2J-1}, \cdots, y_{n-1}) \otimes \mathbb{Z}_2[x]/(x^J),$$

where $deg(y_j) = j$ and x is the mod 2 Euler class of the bundle ζ, and $J = \min\{r \mid \binom{k+r-1}{k-1}$ is odd and $n-2k \leqslant 2r \leqslant n-1\}$. ∎

5.3 Computations for the circle quotient manifolds

Computations described in section 3 will allow us to obtain numerical information for certain topological invariants for spaces $Y_{n,k}$.

5.3.1 Stable span and paralleizability of $Y_{n,k}$

We shall check the parallelizability of $Y_{n,k}$ using the relation obtained for $TY_{n,k}$ in 5.2.1.

The total Pontryagin class for ζ_r is $p(\zeta_r) = 1 + x_0^2$, where $x_0 \in H^2(Y_{n,k};\mathbb{Z})$ is the Euler class of ζ and whose mod 2 reduction is x. To determine the total Pontryagin class for $\zeta_r \otimes_{\mathbb{R}} \zeta_r$, we note that the complexification of $\zeta_r \otimes_{\mathbb{R}} \zeta_r$ is $(\zeta \otimes_{\mathbb{C}} \zeta) \oplus (\zeta^* \otimes_{\mathbb{C}} \zeta^*) \oplus 2\epsilon$. Hence $p(\zeta_r \otimes_{\mathbb{R}} \zeta_r) = 1 + 4x_0^2$.

So from the following relation relating the first Pontryagin classes:

$$p_1(TY_{n,k}) = p_1(nk\zeta_r) - p_1(\binom{k+1}{2}\zeta_r \otimes_{\mathbb{R}} \zeta_r),$$

we get $p_1(TY_{n,k}) = (nk - 2k^2 - 2k)x_0^2$. Considering $n - 2k \geq 4$, if we look at the Gysin sequence for the circle bundle $V_{n,2k} \longrightarrow Y_{n,k}$, we have

$$0 = H^3(V_{n,2k}) \longrightarrow H^2(Y_{n,k}) \xrightarrow{\cup x_0} H^4(Y_{n,k}) \longrightarrow \cdots .$$

This ensures x_0^2 is non-zero and hence $p_1(TY_{n,k}) \neq 0$. Hence we obtain

Theorem 5.3.1. If $n - 2k \geqslant 4$, $Y_{n,k}$ is not parallelizable.

The above theorem is a special case of a very general result due to Singhof and Wemmer [36], which further guarantees that for $n - 2k \geqslant 4$, $Y_{n,k}$ is not even stably parallelizable. Their theorem also implies for $n - 2k = 1$ or 2, $Y_{n,k}$ is stably parallelizable whereas for $n - 2k = 3$ it is not.

Recall that span of a vector bundle is its maximum number of linearly independent sections. We know Stiefel Whitney classes for a manifold M provide an upper bound for its stable span, $\text{span}^0(M) = \text{span}(TM \oplus \epsilon) - 1$. That bound for $Y_{n,k}$ is described in the theorem below.

Theorem 5.3.2. If

$$m := \max\{j \mid \binom{nk+j-1}{nk-1} \not\equiv 0 \pmod 2, 0 \leqslant j \leqslant J-1\},$$

then $2m$-th inverse Stiefel-Whitney class of $Y_{n,k}$, $\bar{w}_{2m}(Y_{n,k})$ is non-zero and hence stable span of $Y_{n,k}$ satisfies the following inequality

$$\mathrm{span}^0(Y_{n,k}) \leqslant \dim(Y_{n,k}) - 2m.$$

Proof. The formula 5.2.1 for tangent bundle of $Y_{n,k}$ allows us to calculate its total Stiefel-Whitney class.

$$\begin{aligned} w(Y_{n,k}) &= w(nk\zeta_r) \cdot w\Big(\binom{k+1}{2}\zeta_r \otimes_{\mathbb{R}} \zeta_r\Big)^{-1} \\ &= (1+x)^{nk} \cdot w\Big(\zeta_r \otimes_{\mathbb{R}} \zeta_r\Big)^{-\binom{k+1}{2}}. \end{aligned} \tag{5.3.1}$$

To determine the total Stiefel-Whitney class of $\zeta_r \otimes_{\mathbb{R}} \zeta_r$, we apply splitting principle for this bundle. Suppose the bundle ζ_r splits as $L_1 \oplus L_2$ over Y'. Then $\zeta_r \otimes_{\mathbb{R}} \zeta_r$ splits as $\bigoplus_{1\leqslant i,j\leqslant 2} L_i \otimes_{\mathbb{R}} L_j$ over Y'. We calculate the total Stiefel-Whitney class of $\bigoplus_{1\leqslant i,j\leqslant 2} L_i \otimes_{\mathbb{R}} L_j$ below:

$$\begin{aligned} w(\bigoplus_{1\leqslant i,j\leqslant 2} L_i \otimes_{\mathbb{R}} L_j) &= \prod_{1\leqslant i,j\leqslant 2} w(L_i \otimes_{\mathbb{R}} L_j) \\ &= \prod_{1\leqslant i,j\leqslant 2} (1 + w_1(L_i) + w_1(L_j)) \\ &= (1 + w_1(L_1) + w_1(L_2))^2 \\ &= (1 + w_1(L_1 \oplus L_2))^2 = 1. \end{aligned}$$

Hence we must have $w\Big(\zeta_r \otimes_{\mathbb{R}} \zeta_r\Big) = 1$. So from 5.3.1, we get

$$w(Y_{n,k}) = (1+x)^{nk} \implies \bar{w}(Y_{n,k}) = (1+x)^{-nk}. \tag{5.3.2}$$

Then from the definition of m, we see that $\bar{w}_{2m}(Y_{n,k}) \neq 0$ and the theorem follows. □

5.3.2 Skew embedding and immersion dimensions of $Y_{n,k}$

The result concerning non-vanishing of inverse Stiefel-Whitney class stated in theorem 5.3.2 immediately produces lower bounds of skew embedding and immersion dimension of $Y_{n,k}$.

Theorem 5.3.3. $Y_{n,k}$ does not admit an immersion in $\mathbb{R}^{dim(Y_{n,k})+2m-1}$ and a skew embedding in $\mathbb{R}^{2dim(Y_{n,k})+4m}$, where

$$m = \max\{j \mid \binom{nk+j-1}{nk-1} \not\equiv 0 \pmod 2, 0 \le j \le J-1\}.$$

Proof. The statement concerning immersion follows since Stiefel-Whitney classes of the normal bundle of an immersion in an Euclidean space are same as the inverse Stiefel-Whitney classes of tangent bundle.

The statement about skew embedding follows directly by combining theorems 5.1.1 and 5.3.2. $\square$

5.3.3 Characteristic rank of $Y_{n,k}$

We shall determine the upper characteristic rank of $Y_{n,k}$ for a large number of cases.

Theorem 5.3.4. If $n-2k=5$, 6 or ≥ 9, the upper characteristic rank of $Y_{n,k}$ is given by

$$\text{ucharrank}(Y_{n,k}) = \begin{cases} n-2k-1 & \text{if } 2 \mid \binom{n}{2k-1} \\ n-2k & \text{if } 2 \nmid \binom{n}{2k-1}. \end{cases}$$

Proof. The proof is similar to the proof of theorem 5.1.4. First we consider the case $n-2k \neq 2J-1$. Then there is a non-zero class $y_{n-2k} \in H^{n-2k}(Y_{n,k})$ which is pulled back to the generator of $H^{n-2k}(V_{n,2k};\mathbb{Z}_2) = \mathbb{Z}_2$ and which is not expressible as a polynomial in Stiefel-Whitney classes of some bundle over $Y_{n,k}$ because that would contradict the fact that $\text{ucharrank}(V_{n,2k}) = n-2k-1$, [25] otherwise. So in this case we must have $\text{ucharrank}(Y_{n,k}) = n-2k-1$ since $\text{charrank}(\zeta) = n-2k-1$.

Next we consider the case $n-2k = 2J-1$. Then $\bigoplus_{0 \leqslant i \leqslant n-2k} H^i(Y_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2[x]/(x^J)$ and $H^{n-2k+1}(Y_{n,k};\mathbb{Z}_2) = \mathbb{Z}_2\{y_{n-2k+1}\}$. So for $n-2k+1 \neq$ a power of 2, Wu's formula rules out the possibility of y_{n-2k+1} being Stiefel-Whitney class of a bundle over $Y_{n,k}$ and we get $\text{ucharrank}(Y_{n,k}) = n-2k$. And for $n-2k+1 =$ a power of 2 greater than 8, invoking the lemma 5.1.3 one again obtains $\text{ucharrank}(Y_{n,k}) = n-2k$.

$\square$

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