UNITARY CONNECTIONS AND Q-SYSTEMS

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To my parents ....

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Contents

Chapter 1

Introduction

V. Jones' pathbreaking results on the index for subfactors gave rise to the modern theory of subfactors [J83]. S. Popa proved that amenable finite index subfactors of II_1 factors are completely classified by their standard invariant [P94], which are axiomatized in general by standard λ-lattices [P95] or planar algebras [J21]. This has led to remarkable progress in the classification of finite index hyperfinite subfactors, by transforming a large part of this fundamentally analytic problem to the (essentially) algebraic problem of classifying abstract standard invariants [JMS14, AMP15].

In the finite depth setting, A. Ocneanu introduced and established the theory of biunitary connections on finite 4-partite graphs as an essential tool for constructing hyperfinite subfactors, which he called *paragroups* [O88]. The *paragroups* can also be used to construct infinite depth hyperfinite subfactors from finite graphs (see [P97, S90]). While the other approaches to standard invariants are now more common, the theory of biunitary connections remain an important ingredient in the construction and classification of hyperfinite subfactors [EK98, JMS14]. Many features of subfactor theory now have a clear higher-categorical interpretation [M03, B97, CPJP22, JMS14], and while there is some work investigating biunitary connections from a categorical viewpoint [C20], the general theory of biunitary connections and particularly their role in hyperfinite subfactor construction has remained mysterious from the categorical viewpoint.

We shed some light on this problem by showing that graphs and bi-unitary connections can be viewed naturally as part of a larger W* 2-category $\mathbf{UC}^{\mathrm{tr}}$ (see Section 3.2). We then build a 2-functor to the 2-category of tracial von Neumann algebras, which puts the hyperfinite subfactor construction from biunitary connections into a larger categorical context. The 0-cells (or objects) of the 2-category $\mathbf{UC}^{\mathrm{tr}}$ are Bratteli diagrams equipped with tracial weighting data. These generalize the Bratteli diagrams arising from taking the tower of relative commutants of a finite index subfactor. 1-cells in our 2-category are *unitary connections* between Bratteli diagrams which are compatible with the tracial data. These naturally generalize Ocneanu's biunitary connections from subfactor theory to our Bratteli diagram setting. Finally, the 2-cells of our category are built as certain fixed points under a ucp (unital completely positive) map, strongly resembling a noncommutative Poisson boundary as in [I04, NY17].

We prove the existence of a W*-2-functor $\mathcal{PB} : \mathbf{UC}^{\text{tr}} \to \mathbf{vNAlg}$ which is fully faithful at the level of 2-cells. The von Neumann algebras in the image of $\mathcal{PB}$ are always hyperfinite by construction. To see how the usual subfactor theory construction fits into this story, from a 4-partite graph and a biunitary connection, we build a pair of tracial Bratteli diagrams by repeatedly reflecting the "vertical" bipartite graphs, and taking the Markov trace as data. The horizontal graphs and the biunitary connection assemble into a 1-morphism in $\mathbf{UC}^{\text{tr}}$ from this pair of tracial Bratteli diagrams. By carefully choosing the initial vertex data, we can build a unital inclusion of hyperfinite von Neumann algebras from this data, which we will see is just a special case of our construction Section 4.1. One way of looking at our result is that we are generalizing connections to be 1-morphisms between graphs that can be composed. Our main result is that a "compositional" version of the subfactor construction holds, and many of the results from the subfactor setting are true for bimodules as well. For example, it is well known that the relative commutants of the subfactor constructed as above can be computed as the "flat part" of the initial biunitary connection. We prove a generalization of this in Theorem 3.4.6.

To motivate our definitions in $\mathbf{UC}^{\text{tr}}$, we first consider a purely algebraic category $\mathbf{UC}$ consisting of Bratteli diagrams (without tracial data), unitary connections between them, and natural intertwiners between connections which we call flat sequences in Section 3.1. This 2-category is essentially equivalent to the 2-category studied in [CPJ22] in the context of fusion category actions on AF-C*-algebras, with only minor differences at the level of 0-cells and 2-cells only. As in [CPJ22], from a 0-cell we define an AF-algebra[1]. We see that the 1-morphisms in $\mathbf{UC}$ are precisely the data we need to define inductive limit bimodules between the AF-algebras built from the 0-cells; the 2-cells in $\mathbf{UC}$ are precisely the intertwiners between the resulting bimodules. Then picking a tracial state on the AF-algebras, we ask which 1-cell bimodules extend to the corresponding von Neumann completion, and if they do, what are the morphisms between them? This consideration leads us precisely to our definitions of 1-cell and 2-cell in $\mathbf{UC}^{\text{tr}}$, which answers this question and proves the existence of the W*-2-functor $\mathcal{PB} : \mathbf{UC}^{\text{tr}} \to \mathbf{vNAlg}$. We explore several examples, including the relationship between our work and classical subfactor constructions. We also discuss various examples arising out of directed graphs and vertex models and investigate the relationship between our work with that of M. Izumi [I04].

The second part of my thesis is about Q-systems in $\mathbf{UC}$. The standard invariant of a finite index subfactor of a II_1 factor was first defined as a λ-lattice [P95]. In [M03], a Q-system which is a unitary version of a Frobenius algebra object in a C*-tensor category or C*-2-category, is exhibited as an alternative axiomatization of the standard invariant of a finite index subfactor [O88, P95, J21]. This further fostered classification of small index subfactors [JMS14, AMP15] . Q-systems were first introduced in [L94], to characterize

[1] we are slightly abusing terminology: by AF-algebra we mean inductive limit of finite dimensional C*-algebras in the category of *-algebras, so we do not complete in norm

canonical endomorphism associated to a finite index subfactor of a Type III factor.

The study of Q-systems in the context of C*-tensor categories or C*-2-categories is an active area of research. Given any rigid, semisimple, C*-tensor category $\mathcal{C}$ with simple tensor unit $\mathbb{1}$, an indecomposable Q-system $Q \in \mathcal{C}$ $(\mathrm{End}_{Q-Q}(Q) \simeq \mathbb{C})$ and a fully-faithful unitary tensor functor $H : \mathcal{C} \to \mathrm{Bim}(N)$ for some II_1 factor N, we can apply realization procedure [JP19, JP20] to construct a II_1 factor M containing N as a generalized crossed product $N \rtimes_H Q$. Also, every irreducible finite index extension of N is of this form.

Recently [CPJP22] introduced the notion of *Q-system completion* for C*/W*-2-categories which is another version of a higher idempotent completion for C*/W*-2-categories in comparison with 2-categories of separable monads [DR18] and condensation monads in [GJF19]. Given a C*/W*-2-category $\mathcal{C}$, its *Q-system completion* is the 2-category $\mathbf{QSys}(\mathcal{C})$ of Q-systems, bimodules and intertwiners in $\mathcal{C}$. There is a canonical inclusion *-2-functor $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathbf{QSys}(\mathcal{C})$ which is always an equivalence on all hom categories. $\mathcal{C}$ is said to be *Q-system complete* if $\iota_{\mathcal{C}}$ is a *-equivalence of *-2-categories. In this thesis, we explore *Q-system completeness* in the context of pre-C*-2-categories. We call a pre-C*-2-category $\mathcal{C}$ to be *Q-system complete* if every Q-system in $\mathcal{C}$ 'splits'.

We prove Q-system completeness of **UC**. The task of proving this, comprises of the following parts:

(i) Building a new suitable 0-cell in **UC**. We achieve this by applying the realization of the given Q-system as in Section 3.1 and using ideas from [CPJP22].

(ii) Constructing a dualizable 1-cell $(X_\bullet, W_\bullet)$ in **UC** having a unitarily separable dual in **UC**. We use subfactor theoretic ideas [B97, EK98, P89, P94] to build our appropriate 1-cell in **UC**.

(iii) Setting up natural isomorphisms $\gamma^{(k)}$'s between $\overline{X}_k X_k$ and functors apearing in the given Q-system, so that $\gamma^{(k)}$'s intertwine the algebra maps and satisfy exchange relations eventually for all k.

Chapter 2

Preliminaries

We recall basic results and definitions pertaining to finite, C*-semisimple categories, 2-categories and Q-systems which are the essential building blocks of this thesis . Most of the results discussed in this chapter can be found in [CPJP22, DGGJ22, EGNO, M03, JY21, NT]. All categories considered in this thesis are *essentially small*, that is, its class of objects as well as its isomorphism classes of objects form a set.

2.1 C*-semisimple categories

Definition 2.1.1. A category $\mathcal{C}$ is called a *C*-category* if:

(i) each morphism space $\mathcal{C}(U, V)$ is a Banach space for all objects U , V such that the map
$$\mathcal{C}(V, W) \times \mathcal{C}(U, V) \ni (S, T) \overset{\circ}{\longmapsto} S \circ T \in \mathcal{C}(U, W)$$
is bilinear and satisfies $\|S \circ T\| \leq \|S\| \, \|T\|$.

(ii) there exists a 'zero' object in $\mathcal{C}$ such that all morphisms from or to it, is zero.

(iii) there exists a contravariant, conjugate-linear involutive functor $* : \mathcal{C} \to \mathcal{C}$ satisfying the following properties:

 (a) it fixes every object, that is, $U^* = U$ for any object $U \in \mathcal{C}$.

 (b) $\|T^* \circ T\| = \|T \circ T^*\| = \|T\|^2$ for every $T \in \mathcal{C}(U, V)$. Thus,in particular, $\mathrm{End}(U)$ is a C*-algebra for every $U \in \mathcal{C}$.

 (c) $T^* \circ T$ is a positive element in the C*-algebra $\mathrm{End}(U)$, for every $T \in \mathcal{C}(U, V)$.

An easy example is the category of Hilbert spaces $\mathcal{H}ilb$.

Subobjects and direct sums. In a C*-category $\mathcal{C}$, $Y \in \mathrm{Ob}(\mathcal{C})$ is said to be a *subobject* of $X \in \mathrm{Ob}(\mathcal{C})$ if there is an isometry $u \in \mathcal{C}(Y, X)$, that is, $u^* \circ u = 1_Y$. Observe that if u is an isometry then $u \circ u^*$ is a projection in $\mathrm{End}(X)$. Sometimes, Y will be referred to as the subobject corresponding to the projection $u \circ u^*$.

Given objects $X_1, \cdots, X_n \in \mathrm{Ob}(\mathcal{C})$, their *direct sum* is defined as the object $Z \in \mathrm{Ob}(\mathcal{C})$ alongwith isometries $u_i : X_i \to Z$ for each $i = 1, \cdots n$, such that $\sum_{i=1}^{n} u_i \circ u_i^* = 1_Z$ and $u_i^* u_j = \delta_{i,j} 1_{X_i}$. Usually, Z is denoted by $\bigoplus_{i=1}^{n} X_i$.

Definition 2.1.2. A C*-category $\mathcal{C}$ is called *semisimple* if:

(i) all morphism spaces are finite-dimensional.

(ii) all finite direct sums exist.

(iii) subobject with respect to any projection in any endomorphism space exists.

Definition 2.1.3. Let $\mathcal{C}$ be a C*-category. An object $X \in \mathcal{C}$ is called simple if every suboject is unitarily equivalent to X.

For an object $X \in \mathrm{Ob}(\mathcal{C})$, suppose $\mathrm{End}(X)$ is given to have dimension 1. It is straightforward to see that X is simple. The converse is known as *Schur's lemma* and it holds for C*-semisimple categories. Moreover, observe that in a C*-semisimple category, every object is a direct sum of simple objects. For two objects $X, Y \in \mathrm{Ob}(\mathcal{C})$, we say that Y has multiplicity m in X if m is the maximum number of isometries from Y to X such that their corresponding projections in $\mathrm{End}(X)$ are orthogonal; if Y is simple, then observe that $m = \dim_{\mathbb{C}}(\mathcal{C}(Y, X))$.

Definition 2.1.4. A C*-semisimple category $\mathcal{C}$ is said to be *finite* if there are finitely many isomorphism classes of simple objects.

The category of finite-dimensional Hilbert spaces $\mathcal{H}ilb_{fd}$, the category of finite-dimensional (as a vector space) right-A-correspondences $\mathcal{R}_A$ for a given finite-dimensional C*-algebra A, are some of the examples of finite, C*-semisimple categories.

Definition 2.1.5. A functor $F : \mathcal{C} \to \mathcal{D}$ between C*-semisimple categories, is called $*$-*linear* if the map $F : \mathcal{C}(X, Y) \to \mathcal{D}(F(X), F(Y))$ is a linear map and $F(f)^* = F(f^*)$ for every $X, Y \in \mathrm{Ob}(\mathcal{C})$ and for every $f \in \mathcal{C}(X, Y)$.

As an example, consider the functor $F \coloneqq \mathcal{C}(s, \bullet) : \mathcal{C} \to \mathcal{H}ilb_{fd}$, where $\mathcal{C}$ is a C*-semisimple category and s is a simple object in $\mathcal{C}$. Then, F is a $*$-linear functor.

2.2 2-categories

In this thesis by a 2-*category* we will mean a weak 2-category or a bicategory.

Definition 2.2.1. A bicategory is a tuple $(B, 1, c, a, l, r)$ consisting of the following data

Objects : B is equipped with a class $\mathrm{Ob}(B) = B_0$, whose elements are called objects or 0-cells in B . If $X \in B_0$, we also write $X \in B$.

Hom categories : For each pair of objects $X, Y \in B$, B is equipped with a category $B(X,Y)$, called a hom category.

1. Its objects are called 1-cells in B. The collection of all the 1-cells in B is denoted by B_1.
2. Its morphisms are called 2-cells in B. The collection of all the 2-cells in B is denoted by B_2.
3. Composition and identity morphisms in the category $B(X,Y)$ are called vertical composition and identity 2-cells, respectively.
4. An isomorphism in $B(X,Y)$ is called an invertible 2-cell, and its inverse is called a vertical inverse.
5. For a 1-cell f , its identity 2-cell is denoted by 1_f

Identity 1**-Cells** : For each object $X \in B$, there is a distinguished 1-cell $1_X \in B(X,X)$, called the identity 1-cell of X.

Horizontal Composition: For each triple of objects $X, Y, Z \in B$, $\boxtimes : B(Y,Z) \times B(X,Y) \to B(X,Z)$ is a functor, called the horizontal composition.

Associator: For objects $W, X, Y, Z \in B$, an associator natural isomorphism

$$\begin{tikzcd}
B(Y,Z) \times B(X,Y) \times B(W,X) \arrow[r, "(\bullet\boxtimes\bullet)\times\mathrm{Id}"] \arrow[d, "\mathrm{Id}\times(\bullet\boxtimes\bullet)"'] \arrow[dr, phantom, "\overset{\alpha}{\Longrightarrow}"] & B(X,Z) \times B(W,X) \arrow[d, "\bullet\boxtimes\bullet"] \\
B(Y,Z) \times B(W,Y) \arrow[r, "\bullet\boxtimes\bullet"'] & B(W,Z)
\end{tikzcd}$$

This means that, for all $P \in B(Y,Z), Q \in B(X,Y), R \in B(W,X)$, we have an isomorphism $\alpha_{P,Q,R} : P \boxtimes (Q \boxtimes R) \to (P \boxtimes Q) \boxtimes R$ which is natural in all three variables.

Unitor: For each pair of objects $X, Y \in B$ and for every $A \in B(X,Y)$ we have

$$1_Y \boxtimes A \xrightarrow{l_{X,Y}} A \xleftarrow{r_{X,Y}} A \boxtimes 1_X$$

are natural isomorphisms called *left unitor* and *right unitor* respectively.

Triangle axiom: For objects V, W, $X \in B$ and for every $f \in B(V,W)$, $g \in B(W,X)$, the diagram in $B(V,X)$,

$$\begin{tikzcd}
(g \boxtimes 1_W) \boxtimes f \arrow[rr, "\alpha_{g,1_W,f}"] \arrow[dr, "r_{g,1_W}\boxtimes\mathrm{Id}"'] & & g \boxtimes (1_W \boxtimes f) \arrow[dl, "\mathrm{Id}\boxtimes l_{1_W,f}"] \\
& g \boxtimes f &
\end{tikzcd}$$

is commutative.

Pentagon axiom: For objects $V, W, X, Y \in B$ and for every $f \in B(V,W)$, $g \in B(W,X)$, $h \in B(X,Y)$, $k \in B(Y,Z)$ the diagram in $B(V,Z)$,

$$\begin{tikzcd}
 & (k \boxtimes h) \boxtimes (g \boxtimes f) \arrow[dr, "\alpha_{k,h,gh}"] & \\
((k \boxtimes h) \boxtimes g) \boxtimes f \arrow[ur, "\alpha_{kh,g,f}"] \arrow[d, "\alpha_{k,h,g}\boxtimes 1_f"] & & k \boxtimes (h \boxtimes (g \boxtimes f)) \\
(k \boxtimes (h \boxtimes g)) \boxtimes f \arrow[rr, "\alpha_{k,hg,f}"] & & k \boxtimes ((h \boxtimes g) \boxtimes f) \arrow[u, "1_k \boxtimes \alpha_{h,g,f}"']
\end{tikzcd}$$

is commutative.

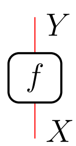
Remark 2.2.2. The horizontal composition $\boxtimes$ (in Definition 2.2.1) is a functor means:

(i) it preserves identity 2-cells, that is, $1_g \boxtimes 1_f = 1_{g\boxtimes f}$ in $B(X,Z)(g \boxtimes f, g \boxtimes f)$, for $f \in B(X,Y)$, $g \in B(Y,Z)$.

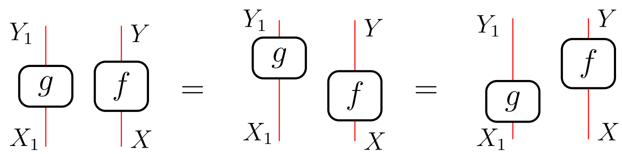
(ii) it preserves vertical composition, that is, $(\beta'\beta)\boxtimes(\alpha'\alpha) = (\beta'\boxtimes\alpha')(\beta\boxtimes\alpha)$ in $B(X,Z)(gf, g''f'')$ for 1-cells $f, f', f'' \in B(X,Y)$, $g, g', g'' \in B(Y,Z)$ and 2-cells $\alpha : f \to f'$, $\alpha' : f' \to f''$, $\beta : g \to g'$, $\beta' : g' \to g''$.

An example of a 2-category is the 2-category of *Categories* whose 0-cells are categories, 1-cells are functors and 2-cells are natural transformations.

Pictorial notations related to 2-categories. Suppose $\mathcal{C}$ is a 2-category and $a, b \in \mathcal{C}_0$ be two 0-cells. A 1-cell from $a \xrightarrow{X} b$ is denoted by ${}_bX_a$. Pictorially, a 1-cell will be denoted by a red strand and a 2-cell will be denoted by a box with strings with passing through it. Suppose we have two 1-cells $X, Y \in \mathcal{C}_1(a,b)$ and $f \in \mathcal{C}_2(X,Y)$ be a 2-cell. Then we will denote f as



. We write tensor product $\boxtimes$ of 1-cells from right to left ${}_cY \underset{b}{\boxtimes} X_a$ for $X \in \mathcal{C}_1(a,b)$ and $Y \in \mathcal{C}_1(b,c)$. We refer the reader to [HV19], for a detailed study of graphical calculus. Pictorially, Remark 2.2.2 says that, for $X, Y \in \mathcal{C}_1(a,b)$ and $X_1, Y_1 \in \mathcal{C}_1(b,c)$



for $f \in \mathcal{C}_2(X,Y)$, $g \in \mathcal{C}_2(X_1,Y_1)$

.

All 2-categories $\mathcal{C}$ in this thesis are *small*, that is, each hom category is a *essentially small* category and $\mathcal{C}_0$ is a set.

2.3 Trace on natural transformations

2.3.1 Categorical trace

Let $\mathcal{M}$ be a semisimple C*-category and V be a maximal set of mutually non-isomorphic simple objects in $\mathcal{M}$. For all $v \in V$, $x \in \mathrm{Ob}(\mathcal{M})$, consider the inner product $\langle \cdot, \cdot \rangle_{v,x}$ on $\mathcal{M}(v,x)$ defined by $\tau^*\sigma = \langle \sigma, \tau \rangle_{v,x} \, 1_v$. An orthonormal basis for such spaces is basically a maximal orthogonal family of isometries in $\mathcal{M}(v,x)$.
Convention. If a statement is independent of the choice of orthonormal basis for $\mathcal{M}(v,x)$, then we denote it by $\mathrm{ONB}(v,x)$. For instance, $1_x = \sum\limits_{v\in V} \sum\limits_{\sigma \in \mathrm{ONB}(v,x)} \sigma\sigma^*$.

Given a map $\mu : V \to (0,\infty)$ (referred as a *weight function on* $\mathcal{M}$), consider the linear functional

$$\mathrm{End}(x) \ni \alpha \overset{\mathrm{Tr}_x}{\longmapsto} \sum_{v\in V} \sum_{\sigma\in \mathrm{ONB}(v,x)} \mu_v \ \left\langle \alpha\sigma, \sigma \right\rangle_{v,x} \ \in \mathbb{C} \ .$$

Clearly, Tr_x is a faithful, positive functional. Moreover, $\mathrm{Tr} = (\mathrm{Tr}_x)_{x\in \mathrm{ob}(\mathcal{M})}$ is a 'categorical' trace, namely it satisfies $\mathrm{Tr}_x(\alpha\beta) = \mathrm{Tr}_y(\beta\alpha)$ for all $\alpha \in \mathcal{M}(y,x)$, $\beta \in \mathcal{M}(x,y)$. We refer Tr as the *categorical trace associated to the weight function* μ.

2.3.2 Graphs and functors

Let $\Gamma = (V_\pm, E)$ (also denoted by $V_- \xrightarrow{\Gamma} V_+$) be a bipartite graph with vertex sets $V_\pm$ and edge sets E_{v_+,v_-} for $(v_+,v_-) \in V_+ \times V_-$, such that the set of edges attached to any vertex is non-empty and finite. Consider the semisimple C*-category $\mathcal{M}_\pm$ whose objects consists of finitely supported $V_\pm$-graded finite dimensional Hilbert spaces. Note that Γ induces the following pair of faithful functors

$$\mathcal{M}_- \ni \left(H_{v_-}\right)_{v_-\in V_-} \overset{F_+}{\longmapsto} \left(\bigoplus_{v_-\in V_-} H_{v_-} \otimes \ell^2(E_{v_+,v_-}) \right)_{v_+\in V_+} \in \mathcal{M}_+$$

$$\mathcal{M}_+ \ni \left(H_{v_+}\right)_{v_+\in V_+} \overset{F_-}{\longmapsto} \left(\bigoplus_{v_+\in V_+} H_{v_+} \otimes \ell^2(E_{v_+,v_-}) \right)_{v_-\in V_-} \in \mathcal{M}_-$$

where the action of each of the functors on a morphism is obtained by distributing it over the direct sum and tensor product keeping the edge vectors (in $\ell^2(E_{v_+,v_-})$'s) fixed. One can easily show that such $F_\pm$ is $*$-linear, *bi-faithful* (that is, both itself and it adjoint are faithful), and F_+ and F_- are adjoints of each other. Conversely, every adjoint pair of $*$-linear faithful functors $F_\pm : \mathcal{M}_\mp \to \mathcal{M}_\pm$ between semisimple C*-categories $\mathcal{M}_\pm$, gives rise to such a bipartite graph by setting the vertex set $V_\pm$ as a maximal set of mutually non-isomorphic simple objects in $\mathcal{M}_\pm$, and edge set E_{v_+,v_-} as a choice of orthonormal basis in $\mathcal{M}_+(v_+, F_+v_-)$ with respect to $\langle \cdot, \cdot \rangle_{v_+,F_+v_-}$ (defined in Section 2.3.1).

For $F_\pm$, $\mathcal{M}_\pm$, $V_\pm$ as before, we will try to characterize the set of solutions to conjugate equations implementing the duality of $F_\pm$. At this point, it will be useful for us to introduce some pictorial notation for morphisms and natural transformations which are quite standard in articles appearing in category theory.

Pictorial notation

(i) A morphism $f : C \to D$ will be denoted by $\overset{D}{\underset{C}{\boxed{f}}}$, and composition of two morphisms will be represented by two vertically stacked labelled boxes.

(ii) Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F, G : \mathcal{C} \to \mathcal{D}$ be two functors. Then a natural transformation $\eta : F \to G$ will be denoted by $\overset{G}{\underset{F}{\boxed{\eta}}}$. For an object x in $\mathcal{C}$, the morphism $\eta_x : Fx \to Gx$ will be denoted by $\overset{G \quad x}{\underset{F \quad x}{\boxed{\eta_x}}} = \overset{G}{\underset{F}{\boxed{\eta}}}\ x$.

(iii) For a $*$-linear functor $F : \mathcal{C} \to \mathcal{D}$ between two semisimple C*-categories categories, we will denote a solution to conjugate equation by

$$\rho = F \smile F' : \mathrm{id}_{\mathcal{D}} \longrightarrow FF' \quad \text{and} \quad \rho' = F' \smile F : \mathrm{id}_{\mathcal{C}} \longrightarrow F'F$$

$$\rho^* = F \frown F' : FF' \longrightarrow \mathrm{id}_{\mathcal{D}} \quad \text{and} \quad [\rho']^* = F' \frown F : F'F \longrightarrow \mathrm{id}_{\mathcal{C}}$$

where $F' : \mathcal{D} \to \mathcal{C}$ is an adjoint functor of F.

We will extend the above dictionary (between things appearing in the category world and pictures) in an obvious way. For instance, composition of morphisms and natural transformations will be pictorially represented by stacking the boxes vertically whereas tensor product (resp., composition) of objects (resp., functors) by parallel vertical strings. For simplicity, sometimes we will not label all of the strings (with any object or functor) emanating from a box (labelled with a morphism or a natural transformation) when it can be read off from the context. We urge the reader to get used to the various picture moves which are induced by relations satisfied by operations, such as, composition, tensor product, etc. between objects, morphisms, functors and natural transformations. In fact, the main purpose of using this graphical calculus is because of the ease of working with these moves instead of long equations.

Fact 2.3.1. *If $\rho^{\pm} : id_{\mathcal{M}_{\pm}} \to F_{\pm} F_{\mp}$ is a solution to the conjugate equation for $F_{\pm}$, then for each $(v_+, v_-) \in V_+ \times V_-$, there exists an orthonormal basis E_{v_+,v_-} of $\mathcal{M}_+(v_+, F_+ v_-)$ and a 'weight' function $\kappa_{v_+,v_-} : E_{v_+,v_-} \to (0, \infty)$ and satisfying the following:*

$$\left(\rho^-_{v_-}\right)^* \ F_- \left(\sigma \ \tau^*\right) \ \rho^-_{v_-} \ = \ \delta_{\sigma=\tau} \ \kappa_{v_+,v_-}(\sigma) \ 1_{v_-} \ \text{ for all } \ \sigma, \tau \in E_{v_+,v_-}, \ v_{\pm} \in V_{\pm}. \tag{2.1}$$

Conversely, to every such family of orthonormal bases and weight functions, one can associate a solution to the conjugate equations implementing the duality of $F_{\pm}$ satisfying Equation (2.1).(cf. [C20, Proposition 4.3.2]).

The above easily follows from the spectral decomposition of the faithful positive functional $\left[\left(\rho^-_{v_-}\right)^* \ F_-(\bullet) \ \rho^-_{v_-}\right] : \mathrm{End}(F_+v_-) \longmapsto \mathrm{End}(v_-) = \mathbb{C}1_{v_-}$. Thus, from the adjoint pair of $*$-linear faithful functors, we not only get a bipartite graph, the solution to the conjugate equations puts a positive scalar weight on each edge. Further, the set $E_{v_-,v_+} :=$ $\left\{\left(\kappa_{v_+,v_-}(\sigma)\right)^{-\frac{1}{2}} \ [F_- \ \sigma^*] \ \rho^-_{v_-} \ : \ \sigma \in E_{v_+,v_-}\right\}$ turns out to be an orthonormal basis of $\mathcal{M}_-(v_-, F_-v_+)$ and satisfies an equation analogous to Equation (2.1) with weight function $\kappa_{v_-,v_+} := \dfrac{1}{\kappa_{v_+,v_-}}$.

The solution $\rho^\pm$ will be called '*tracial*' if the weight function is constant on edges for every fixed pair of vertices. Indeed, for tracial solution $\rho^\pm$, Equation (2.1) becomes

$$\begin{aligned}
\left(\rho^-_{v_-}\right)^* \ F_-(\sigma\ \tau^*) \ \rho^-_{v_-} &= \kappa_{v_+,v_-} \ \langle\sigma,\tau\rangle_{v_+,F_+v_-} \ 1_{v_-} \quad \text{for all} \quad \sigma,\tau \in \mathcal{M}_+(v_+, F_+v_-) \\
\left(\rho^+_{v_+}\right)^* \ F_+(\sigma\ \tau^*) \ \rho^+_{v_+} &= \kappa_{v_-,v_+} \ \langle\sigma,\tau\rangle_{v_-,F_-v_+} \ 1_{v_+} \quad \text{for all} \quad \sigma,\tau \in \mathcal{M}_-(v_-, F_-v_+)
\end{aligned} \tag{2.2}$$

and the map $\left[\left(\rho^-_{v_-}\right)^* \ F_-(\bullet) \ \rho^-_{v_-}\right]$ is tracial and so is $\left[\left(\rho^+_{v_+}\right)^* \ F_+(\bullet) \ \rho^+_{v_+}\right]$. We also get a conjugate linear unitaries

$$\begin{aligned}
\mathcal{M}_+(v_+, F_+v_-) \ni \sigma &\overset{J_{v_+,v_-}}{\longmapsto} \sqrt{\kappa_{v_-,v_+}} \ [F_- \ \sigma^*] \ \rho^-_{v_-} \in \mathcal{M}_-(v_-, F_-v_+) \ , \\
\mathcal{M}_-(v_-, F_-v_+) \ni \sigma &\overset{J_{v_-,v_+}}{\longmapsto} \sqrt{\kappa_{v_+,v_-}} \ [F_+ \ \sigma^*] \ \rho^+_{v_+} \in \mathcal{M}_+(v_+, F_+v_-) \ .
\end{aligned}$$

The two 'loops' are given by:

$$\left(\rho^+_\bullet\right)^* \circ \rho^+_\bullet = \left(\left\{\sum_{v_-\in V_-} N_{v_+,v_-} \ \kappa_{v_-,v_+}\right\} 1_{v_+}\right)_{v_+\in V_+} \in \mathrm{End}(\mathrm{id}_{\mathcal{M}_+}), \tag{2.3}$$

$$\left(\rho^-_\bullet\right)^* \circ \rho^-_\bullet = \left(\left\{\sum_{v_+\in V_+} N_{v_+,v_-} \ \kappa_{v_+,v_-}\right\} 1_{v_-}\right)_{v_-\in V_-} \in \mathrm{End}(\mathrm{id}_{\mathcal{M}_-}), \tag{2.4}$$

where $N_{v_+,v_-} := \dim_{\mathbb{C}}\left(\mathcal{M}_+(v_+, F_+v_-)\right) = \dim_{\mathbb{C}}\left(\mathcal{M}_-(v_-, F_-v_+)\right)$ (that is, the number of edges between v_+ and v_- in the bipartite graph). (Note that a natural linear transformation between *-linear functors from one semisimple C*-category to another, is captured fully by its components corresponding to the simple objects.)

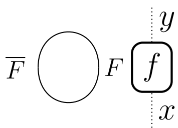
Bi-faithfulness of functors. We now state a result concerning bi-faithfulness of functors which is known to experts. Nevertheless, we give a proof.

Lemma 2.3.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be C*-semisimple categories and $F : \mathcal{C} \to \mathcal{D}$ be a $*$-linear functor with adjoint $\left(\overline{F}, \rho, \overline{\rho}\right)$. The following are equivalent:*

(1) F is bi-faithful.

(2) $\rho^*\rho$ and $\overline{\rho}^*\overline{\rho}$ are invertible elements in the C-algebras* $\mathrm{End}(\mathrm{Id}_{\mathcal{D}})$ *and* $\mathrm{End}(\mathrm{Id}_{\mathcal{C}})$ *respectively.*

Proof. $\underline{(1) \implies (2)}$: Suppose $F : \mathcal{C} \to \mathcal{D}$ and $\overline{F} : \mathcal{D} \to \mathcal{C}$ are both faithful. Since $\mathcal{D}$ has only finitely many isomorphism classes simple objects, to prove that $\rho^*\rho$ is invertible it is enough to show that, for every simple object t in $\mathcal{D}$, the scalar in $\rho_t^*\rho_t = \lambda_t 1_t$ is positive. Clearly, $\lambda_t \geq 0$. If $\lambda_t = 0$ then this will imply that $\rho_t = 0$, hence $F(1_t) = 0$, a contradiction to faithfulness of F . Thus, $\rho^*\rho$ is invertible. Similarly, one can prove that $\overline{\rho}^*\overline{\rho}$ is also invertible.

$\underline{(2) \implies (1)}$: Suppose $\rho^*\rho$ and $\overline{\rho}^*\overline{\rho}$ are invertible. For $f \in \mathcal{C}(x, y)$, suppose $F(f) = 0$. Then,  $= 0$. By invertibility of $\overline{\rho}^*\overline{\rho}$ we get that $f = 0$. Thus, F is faithful. In a similar way, we can prove that $\overline{F}$ is also faithful. $\square$

2.3.3 Trace on natural transformations

In this thesis, we will be working with the 2-category of *weighted semisimple C*-categories*, denoted by WSSC*Cat, whose 0-cells are finite semisimple C*-categories along with a weight function on it (that is, a positive real valued map from the isomorphism classes of simple objects, as considered in Section 2.3.1), 1-cells are $*$-linear bi-faithful functors and 2-cells are natural linear transformations. Further, for the duality of the adjoint pair of 1-cells $\left(\mathcal{M}_-, \underline{\mu}^-\right) \underset{F_-}{\overset{F_+}{\rightleftarrows}} \left(\mathcal{M}_+, \underline{\mu}^+\right)$, we will consider tracial solution $\rho^\pm : \mathrm{id}_{\mathcal{M}_\pm} \to F_\pm F_\mp$ to the conjugate equations associated to the constant weight on edges given by $\kappa_{v_+, v_-} = \dfrac{\mu^+_{v_+}}{\mu^-_{v_-}}$ for $(v_+, v_-) \in V_+ \times V_-$; we refer such a solution to be *commensurate with the weight functions (on the simple objects)* $\left(\underline{\mu}^-, \underline{\mu}^+\right)$. Using the categorical trace Tr associated to the weight function $\underline{\mu}^\pm$, one may obtain the relation:

$$\mathrm{Tr}_x \left(\left(\rho^\pm_x\right)^* F_\pm(\alpha)\rho^\pm_x\right) \;=\; \mathrm{Tr}_{F_\mp(x)}(\alpha) \;\; \text{for all } x \in \mathrm{ob}(\mathcal{M}_\pm), \alpha \in \mathrm{End}(F_\mp(x)). \tag{2.5}$$

We will now exhibit a similar categorial trace on the endomorphism space of every 1-cell between two 'finite' 0-cells (that is, there is finitely many isomorphism classes of simple objects in the semisimple C*-category of the 0-cell); further, this trace will be compatible with the tracial solution commensurate with the weight function in the 0-cells.

Proposition 2.3.3. *Let* $\left(\mathcal{M}, \underline{\mu}\right)$, $(\mathcal{N}, \underline{\nu})$, $(\mathcal{Q}, \underline{\pi})$ *be finite* 0*-cells in* WSSC*Cat*, and* $\Lambda : \mathcal{M} \to \mathcal{N}$, $\Sigma : \mathcal{N} \to \mathcal{Q}$ *be $*$-linear bi-faithful functors. Suppose* $V_\mathcal{M}$, $V_\mathcal{N}$, $V_\mathcal{Q}$ *are maximal sets of mutually non-isomorphic simple objects in* $\mathcal{M}$, $\mathcal{N}$, $\mathcal{Q}$ *respectively.*

(a) The map

$$\mathrm{End}(\Lambda) \ni \eta \xmapsto{\mathrm{Tr}^\Lambda} \sum_{u \in V_\mathcal{M}} \mu_u \, \mathrm{Tr}^{\nu}_{\Lambda u}\left(\eta_u\right) \; \in \mathbb{C}$$

is a positive faithful trace.
We will refer Tr^Λ *as the 'trace on* $\mathrm{End}(\Lambda)$ *commensurate with* $\left(\underline{\mu}, \underline{\nu}\right)$*'.*

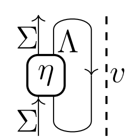
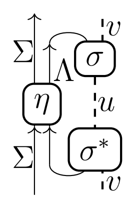
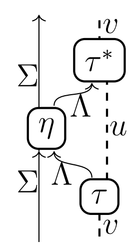
(b) If $\left(id_{\mathcal{M}} \xrightarrow{\rho} \overline{\Lambda}\Lambda\,,\ id_{\mathcal{N}} \xrightarrow{\overline{\rho}} \Lambda\overline{\Lambda}\right)$ *(resp.,* $\left(id_{\mathcal{N}} \xrightarrow{\beta} \overline{\Sigma}\Sigma\,,\ id_{\mathcal{Q}} \xrightarrow{\overline{\beta}} \Sigma\overline{\Sigma}\right)$*) is a solution to conjugate equations for the duality of* Λ *(resp.,* Σ*) commensurate with* $(\underline{\mu}, \underline{\nu})$ *(resp.,* $(\underline{\nu}, \underline{\pi})$*), then*

$$\mathrm{Tr}^{\Lambda}\left(\beta^*_{\Lambda}\ \overline{\Sigma}(\eta)\ \beta_{\Lambda}\right) = \mathrm{Tr}^{\Sigma\Lambda}(\eta) = \mathrm{Tr}^{\Sigma}\left(\Sigma\left(\overline{\rho}^*\right)\ \eta_{\overline{\Lambda}}\ \Sigma\left(\overline{\rho}\right)\right) \quad \text{for} \quad \eta \in \mathrm{End}(\Sigma\Lambda)\,.$$

Proof. (a) To each $\alpha \in \mathrm{End}(\Lambda x)$ for $x \in \mathrm{Ob}(\mathcal{M})$, we associate the natural transformation $[\alpha] := \left(\sum\limits_{u\in V_{\mathcal{M}}} \sum\limits_{\substack{\sigma\in \mathrm{ONB}(u,x)\\ \tau\in\mathrm{ONB}(u,y)}} \Lambda\left(\tau\,\sigma^*\right)\,\alpha\,\Lambda\left(\sigma\,\tau^*\right)\right)_{y\in\mathrm{Ob}(\mathcal{M})}$. In terms of this association, we may express $\mathrm{End}(\Lambda)$ as a direct sum of full matrix algebras indexed by $V_{\mathcal{M}} \times V_{\mathcal{N}}$, and a system of matrix units of the summand corresponding to $(u, v) \in V_{\mathcal{M}} \times V_{\mathcal{N}}$ is given by $\{[\sigma\tau^*] : \sigma, \tau \in \mathrm{ONB}(v, \Lambda u)\}$. Note that $\mathrm{Tr}^{\Lambda}\left([\sigma\tau^*]\right) = \delta_{\sigma=\tau}\,\mu_u\,\nu_v$ which is positive and independent of the choice of σ and τ.

(b) From the definition, the left side turns out to be $\sum\limits_{u\in V_{\mathcal{M}}} \mu_u\ \mathrm{Tr}^{\underline{\nu}}_{\Lambda u}\left(\beta^*_{\Lambda u}\ \overline{\Sigma}(\eta_u)\ \beta_{\Lambda u}\right)$ which is equal to (applying Equation (2.5)) $\sum\limits_{u\in V_{\mathcal{M}}} \mu_u\ \mathrm{Tr}^{\underline{\pi}}_{\Sigma\Lambda u}\left(\eta_u\right) = \mathrm{Tr}^{\Sigma\Lambda}(\eta)$.

Pictorially the right side can be expressed as

$$\sum_{v\in V_{\mathcal{N}}} \nu_v\ \mathrm{Tr}^{\underline{\pi}}_{\Sigma v}$$

$$= \sum_{\substack{u\in V_{\mathcal{M}}\\ v\in V_{\mathcal{N}}\\ \sigma\in\mathrm{ONB}(u,\overline{\Lambda}v)}} \nu_v\ \mathrm{Tr}^{\underline{\pi}}_{\Sigma v}$$

$$= \sum_{\substack{u\in V_{\mathcal{M}}\\ v\in V_{\mathcal{N}}\\ \tau\in\mathrm{ONB}(v,\Lambda u)}} \mu_u\ \mathrm{Tr}^{\underline{\pi}}_{\Sigma v}$$

$$= \sum_{\substack{u\in V_{\mathcal{M}}\\ v\in V_{\mathcal{N}}\\ \sigma\in\mathrm{ONB}(v,\Lambda u)}} \mu_u\ \mathrm{Tr}^{\underline{\pi}}_{\Sigma\Lambda u}\left(\eta\ \Sigma\left(\tau\,\tau^*\right)\right) = \mathrm{Tr}^{\Sigma\Lambda}(\eta)\,.$$

□

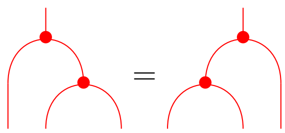
Remark 2.3.4. The trace in Proposition 2.3.3 (a), is 'categorical', that is, $\tilde{\Lambda} : \mathcal{M} \to \mathcal{N}$ is another functor with the same PF vectors as that of Λ, then $\mathrm{Tr}^{\Lambda}\left(\gamma\,\eta\right) = \mathrm{Tr}^{\tilde{\Lambda}}\left(\eta\,\gamma\right)$ for $\eta \in \mathrm{NT}(\Lambda, \tilde{\Lambda})$, $\gamma \in \mathrm{NT}(\tilde{\Lambda}, \Lambda)$.

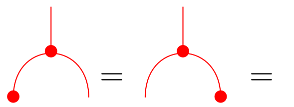
2.4 Q-system completion

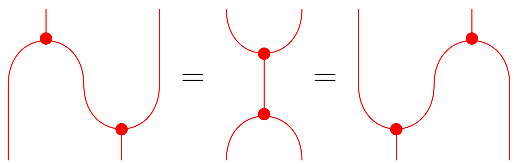
Definition 2.4.1. A pre-C*-2-category is a 2-category such that the hom-1-categories satisfies all the conditions of a C*-category except that the 2-cell spaces need not be complete with repsect to the given norm.

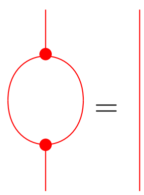
Let $\mathcal{C}$ be a pre-C*-2-category.

Definition 2.4.2. A Q-system in $\mathcal{C}$ is a 1-cell ${}_bQ_b \in \mathcal{C}_1(b,b)$ along with multiplication map $m \in \mathcal{C}_2(Q \boxtimes_b Q, Q)$ and unit map $i \in \mathcal{C}_2(1_b, Q)$, as mentioned in section 1, satisfying the following properties:

(Q1)  (associativity)

(Q2)  (unitality)

(Q3)  (Frobenius condition)

(Q4)  (Separability)

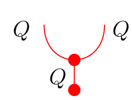
Definition 2.4.3. [CPJP22] Given a Q-system (Q, m, i), we define

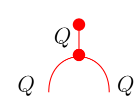
$d_Q :=$  $\in \mathrm{End}_{\mathcal{C}}(1_b)^+$

.

- If d_Q is invertible, we call Q *non-degenerate* or an *extension* of 1_b.
- If d_Q is an idempotent, we call Q a *summand* of 1_b.

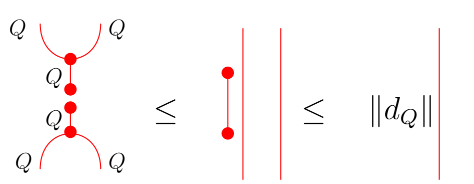
We recall some facts about Q-systems in C*-tensor categories already mentioned in [CPJP22, Z07].

Fact 2.4.4. *(F1) Q is a self-dual 1-cell with $ev_Q :=$* 

and $coev_Q :=$ 

.

(F2) Using (F1) and [Z07, Lemma 1.16] we have the following inequalitites:

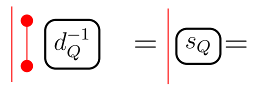


(F3) By [[Z07](#), Corollary 1.19] either d_Q is invertible, or zero is an isolated point in Spec(d_Q). Define , $f : \text{Spec}(d_Q) \to \mathbb{C}$ by

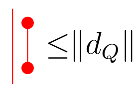
$$f(x) = \begin{cases} 0 & x = 0 \\ x^{-1} & x \neq 0 \end{cases}$$

By abuse of notation, set $d_Q^{-1} := f(d_Q)$. By continuous functional calculus, set $s_Q := d_Q d_Q^{-1}$. Then we have the following :

(a)



(b)



Definition 2.4.5. Suppose $\mathcal{C}$ is a pre-C*-2-category and ${}_bX_a \in \mathcal{C}_1(a,b)$. A *unitarily separable left dual* for ${}_bX_a$ is a dual $\left({}_a\overline{X}_b, ev_X, coev_X\right)$ such that $ev_X \circ ev_X^* = \text{id}_{1_a}$ (cf. [CPJP22, Example 3.9]).

Given a unitarily separable left dual for ${}_bX_a \in \mathcal{C}_1(a,b)$, ${}_bX \underset{a}{\boxtimes} \overline{X}_b \in \mathcal{C}_1(b,b)$ is a Q-system with multiplication map $m := \text{id}_X \boxtimes ev_X \boxtimes \text{id}_{\overline{X}}$ and unit map $i := coev_X$.

Given a Q-system $Q \in \mathcal{C}_1(b,b)$, if it is of the above form then we say that the Q-system Q 'splits'.

Definition 2.4.6. A pre-C*-2-category $\mathcal{C}$ is said to be *Q-system complete* if every Q-system in $\mathcal{C}$ 'splits', that is, given a Q-system $Q \in \mathcal{C}_1(b,b)$ there is an object $c \in \mathcal{C}_0$ and a dualizable 1-cell $X \in \mathcal{C}_1(c,b)$ which admits a unitary separable dual $\left(\overline{X}, ev_X, coev_X\right)$ such that (Q,m,i) is isomorphic to ${}_bX \underset{c}{\boxtimes} \overline{X}_b$ as Q-systems.

Remark 2.4.7. In [CPJP22], Q-system completion has been treated in the context of C*/W*-2-categories. It has been proved that Definition 2.4.6 is equivalent to their definition of *Q-system completeness* (see [CPJP22, Theorem 3.36]) of C*/W*-2-categories.

Chapter 3

Unitary connections on Bratteli diagrams

3.1 Unitary connections and right correspondences

Bratteli diagrams are incredibly useful tools for studying inductive limits of semisimple algebras (also called locally semisimple algebras). In this section we introduce a combinatorial 2-category whose objects are Bratteli diagrams and 1-cells are generalizations of Ocneanu's connections. Our perspective is that our 1-cells can naturally be viewed as "Bratteli diagrams for bimodules" between locally semisimple algebras. Thus as we describe our 2-category **UC**, we will explain its relationship to algebras and bimodules. As a consequence, we build a fully faithful 2-functor **UC** into the 2-category of algebras, bimodules, and intertwiners.

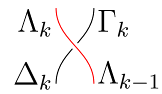
3.1.1 The 0-cells

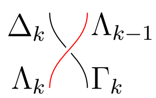
These are sequences consisting of finite bipartite graphs $V_0 \xrightarrow{\Gamma_1} V_1 \xrightarrow{\Gamma_2} V_2 \xrightarrow{\Gamma_3} V_3 \cdots$ (where V_j's are the vertex sets) such that none of the vertices is isolated. As described in the Section 2.3.2, given such a data, we will often work with the corresponding $*$-linear, bi-faithful functor $\Gamma_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ (where $\mathcal{M}_k$ is a semisimple C*-category whose isomorphism classes of the simple objects are indexed by the vertex set V_k). We will denote such a 0-cell by $\left\{\mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k\right\}_{k\geq 1}$ or sometimes simply $\Gamma_\bullet$.

Given such a 0-cell, we fix an object $m_0 := \bigoplus\limits_{v\in V_0} v \in \mathrm{ob}(\mathcal{M}_0)$. Consider the sequence of finite dimensional C*-algebras $\{A_k := \mathrm{End}(\Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 0}$ (assuming $A_0 = \mathrm{End}(m_0)$) along with the unital $*$-algebra inclusions given by $A_{k-1} \ni \alpha \hookrightarrow \Gamma_k\, \alpha \in A_k$. Indeed, the Bratteli diagram of A_{k-1} inside A_k is given by the graph Γ_k. To the 0-cell $\Gamma_\bullet$, we associate the $*$-algebra $A_\infty := \bigcup\limits_{k\geq 0} A_k$.

3.1.2 The 1-cells

Definition 3.1.1. A 1-cell from the 0-cell $\left\{\mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k\right\}_{k\geq 1}$ to the 0-cell $\left\{\mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k\right\}_{k\geq 1}$ consists of a sequence of $*$-linear bi-faithful functors $\{\Lambda_k : \mathcal{M}_k \to \mathcal{N}_k\}_{k\geq 0}$ and natural unitaries $W_k : \Delta_k\Lambda_{k-1} \to \Lambda_k\Gamma_k$ for $k \geq 1$. Such a 1-cell will be denoted by $(\Lambda_\bullet, W_\bullet)$ or simply by $\Lambda_\bullet$, and $W_\bullet$ will be referred as a *unitary connection associated to* $\Lambda_\bullet$. Denote the set of 1-cells from $\Gamma_\bullet$ to $\Delta_\bullet$ by $\mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$.

We will abuse the notation Λ_k to denote the functor $\Lambda_k : \mathcal{M}_k \to \mathcal{N}_k$ as well as its associated adjacency matrix $(V_{\mathcal{N}_k} \times V_{\mathcal{M}_k})$, and the same will be done for Γ_k's and Δ_k's. From the context, it will be clear whether we are using it as a functor or a matrix. Pictorially, the natural unitary W_k appearing in the 1-cell will be represented by  and W_k^* by

 .

To each such 1-cell $(\Lambda_\bullet, W_\bullet)$, we will associate an A_∞-B_∞ right correspondence where n_0 and B_k's are related to $\left\{\mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k\right\}_{k\geq 1}$ exactly the way m_0 and A_k's are related to $\left\{\mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k\right\}_{k\geq 1}$ respectively. For $k \geq 0$, set $H_k := \mathcal{N}_k\left(\Delta_k \cdots \Delta_1 n_0, \Lambda_k\Gamma_k \cdots \Gamma_1 m_0\right)$. We have an obvious A_k-B_k-bimodule structure on H_k in the following way:

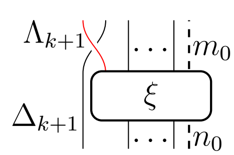
$$A_k \times H_k \times B_k \ni (\alpha, \xi, \beta) \longmapsto \Lambda_k(\alpha) \circ \xi \circ \beta \ \in H_k \tag{3.1}$$

Again, there is a B_k-valued inner product on H_k given by

$$H_k \times H_k \ni (\xi, \zeta) \overset{\langle\cdot,\cdot\rangle_{B_k}}{\longmapsto} \langle \xi, \zeta\rangle_{B_k} := \zeta^* \circ \xi \ \in B_k \tag{3.2}$$

Next, observe that H_k sits inside H_{k+1} via the map

$$H_k \ni \xi \overset{I_{k+1}}{\longmapsto} \left[(W_{k+1})_{\Gamma_k\cdots\Gamma_1 m_0}\right] \circ \left[\Delta_{k+1}\xi\right] \ = \tag{3.3}$$



$\in H_{k+1}$

Lemma 3.1.2. *The inclusions $H_k \hookrightarrow H_{k+1}$, $A_k \hookrightarrow A_{k+1}$, $B_k \hookrightarrow B_{k+1}$ and the corresponding actions are compatible in the obvious sense.*

Proof. Naturality of W implies

$$\begin{aligned} &\Lambda_{k+1}\Gamma_{k+1}\alpha \ \circ \ \left[(W_{k+1})_{\Gamma_k\cdots\Gamma_1 m_0} \ \circ \ \Delta_{k+1}\xi\right] \ \circ \ \Delta_{k+1}\beta \\ &= \left[(W_{k+1})_{\Gamma_k\cdots\Gamma_1 m_0}\right] \ \circ \ \Delta_{k+1}\left(\Lambda_k\alpha \circ \xi \circ \beta\right) \end{aligned}$$

for all $\xi \in H_k$, $\alpha \in A_k$, $\beta \in B_k$. $\square$

Set $H_\infty := \bigcup\limits_{k\geq 0} H_k$ which clearly becomes an A_∞-B_∞ right correspondence. Further, we will exhibit a Pimsner-Popa (PP) basis of the right-B_∞-module H_∞ with respect to the B_∞-valued inner product.

Lemma 3.1.3. *There exists a finite subset $\mathscr{S}$ of H_0 such that $\sum\limits_{\sigma \in \mathscr{S}} \sigma \circ \sigma^* = 1_{\Lambda_0 m_0}$; moreover, any such $\mathscr{S}$ is a PP-basis for the right B_∞-module H_∞.*

Proof. Since n_0 contains every simple object of $\mathcal{N}_0$ as a subobject, therefore expressing the identity of $\operatorname{End}(\Lambda_0 m_0)$ as a sum of minimal projections, we have a resolution of identity $1_{\Lambda_0 m_0}$ factoring through n_0, that is, there exists a subset $\mathscr{S}$ of $\mathcal{N}(n_0, \Lambda_0 m_0) = H_0$ satisfying:

(i) $\sigma^* \sigma$ is a minimal projection of $\operatorname{End}(n_0)$, and

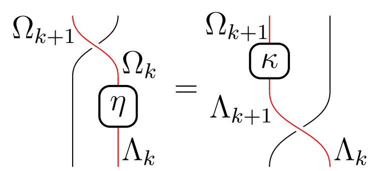
(ii) $\sum\limits_{\sigma \in \mathscr{S}} \sigma \sigma^* = 1_{\Lambda_0 m_0}$.

Condition (ii) and the definition of B_∞-valued inner product directly imply that $\mathscr{S}$ is indeed a PP-basis for the right B_∞-module H_∞. $\square$

3.1.3 The 2-cells

Let $\Lambda_\bullet$ and $\Omega_\bullet$ be two 1-cells from the 0-cell $\left\{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \right\}_{k\geq 1}$ to $\left\{ \mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k \right\}_{k\geq 1}$. The natural way to define a 2-cell will be considering a sequence of natural linear transformations from Λ_k to Ω_k which are compatible with the natural unitaries W_k^Γ and W_k^Ω for $k \geq 1$. We define such compatibility in the following way.

Definition 3.1.4. A pair $(\eta, \kappa) \in \operatorname{NT}(\Lambda_k, \Omega_k) \times \operatorname{NT}(\Lambda_{k+1}, \Omega_{k+1})$ is said to satisfy *exchange relation* if the condition



holds.

Remark 3.1.5. The exchange relation pair is unique separately in each variable, that is, if (η, κ_1) and (η, κ_2) (resp., (η_1, κ) and (η_2, κ)) both satisfy exchange relation, then $\kappa_1 = \kappa_2$ (resp., $\eta_1 = \eta_2$); this is because the connections are unitary and the functors Γ_k and Δ_k are bi-faithful.

We only require that the 2-cells satisfy this exchange relation *eventually.* To make this precise, we let

$$\operatorname{Ex}(\Lambda_\bullet, \Omega_\bullet)$$

denote the space of sequences $\{\eta^{(k)} \in \operatorname{NT}(\Lambda_k, \Omega_k)\}_{k\geq 0}$ such that there exists an N such that (η_k, η_{k+1}) satifies the exchange relation for all $k \geq N$. Consider the subspace

$$\operatorname{Ex}_0(\Lambda_\bullet, \Omega_\bullet) := \{\{\eta_k\}_{k\geq 0} \in \operatorname{Ex}(\Lambda_\bullet, \Omega_\bullet) : \ \eta_k = 0 \ \text{ for all } \ k \geq N \ \text{ for some } \ N \in \mathbb{N}\}$$

Definition 3.1.6. Let $\Lambda_\bullet, \Omega_\bullet \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$. We define the space of 2-cells

$$\mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet) := \frac{\mathrm{Ex}(\Lambda_\bullet, \Omega_\bullet)}{\mathrm{Ex}_0(\Lambda_\bullet, \Omega_\bullet)}$$

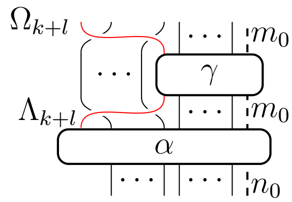
For notational convenience, instead of denoting a 2-cell by an equivalence class of sequences, we simply use a sequence in the class and truncate upto a level after which the exchange relation holds for every consecutive pair, namely, $\left\{\eta^{(k)}\right\}_{k\geq N} \in \mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$ where (η_k, η_{k+1}) satifies the exchange relation for all $k \geq N$.

If $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k\geq K} \in \mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$ and $\underline{\kappa} = \left\{\kappa^{(k)}\right\}_{k\geq L} \in \mathbf{UC}_2(\Omega_\bullet, \Xi_\bullet)$, then define the '*vertical*' composition of 2-cells by $\underline{\kappa} \cdot \underline{\eta} := \left\{\left(\kappa^{(k)} \circ \eta^{(k)}\right)\right\}_{k\geq \max\{K,L\}}$. It is easy to check that $\underline{\kappa} \cdot \underline{\eta} \in \mathbf{UC}_2(\Lambda_\bullet, \Xi_\bullet)$ is well defined and the composition is associative.

Given two 0-cells $\Gamma_\bullet$ and $\Delta_\bullet$, we have obtained a category whose object space consists of 1-cells $\Lambda_\bullet$, and morphisms are given by 2-cells. We call this the *category of unitary connections from* $\Gamma_\bullet$ *to* $\Delta_\bullet$ and denote by $\mathbf{UC}_{\Gamma_\bullet, \Delta_\bullet}$.

Following with the structure in the previous subsections, we will see that 2-cells uniquely define bimodule intertwiners between the bimodules associated to the 1-cells. We will borrow the notations H_k, H_∞ $\mathscr{S}$, etc. (arising out of $\Lambda_\bullet$) from previous subsections, and for those arising out of $\Omega_\bullet$, we will use the notation G_k, G_∞, $\mathscr{T}$, etc. and we will also work with the pictures as before. For each $k \geq 0$, we define $\mathcal{N}_k(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0 \ , \ \Omega_k\Gamma_k \cdots \Gamma_1 m_0) \ni \gamma \overset{\Phi}{\longmapsto} \Phi_\gamma \in \mathcal{L}(H_\infty, G_\infty)$ (the space of adjointable operators with respect to the B_∞-valued inner product) in the following way

$$H_\infty \supset H_{k+l} \ni \alpha \overset{\Phi_\gamma}{\longmapsto}$$



$$\in G_{k+l} \subset G_\infty \tag{3.4}$$

for $l \geq 0$. It is easy to check that Φ_γ is well-defined and adjointable. We list a few basic properties of Φ in the following lemma.

Lemma 3.1.7. *For all $k \geq 0$ and $\gamma \in \mathcal{N}_k(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0 \ , \ \Omega_k\Gamma_k \cdots \Gamma_1 m_0)$, the following conditions hold*

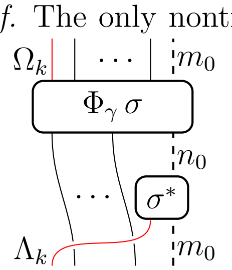
(i) $\Phi_{\gamma^*} = (\Phi_\gamma)^*$,

(ii) $\Phi_\gamma(H_l) \subset G_l$ *for all* $l \geq k$,

(iii) the map $\gamma \longmapsto \Phi_\gamma\big|_{H_k}$ *is one-to-one, and*

(iv) $\Phi_\gamma \in \mathcal{L}_{B_\infty}(H_\infty, G_\infty)$.

Proof. The only nontrivial part is to prove (iii). This easily follows from the equality $\gamma = \sum_{\sigma\in\mathscr{S}}$



. In fact, we have deduced a stronger statement, namely, γ is nonzero if and only if $\Phi_\gamma\big|_{H_0}$ is nonzero. ∎

Lemma 3.1.8. *For each $k \geq 0$, the space $\mathcal{N}_k\left(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0\ ,\ \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$ gets an A_k-A_k-bimodule structure via*

$$(\alpha_1, \gamma, \alpha_2) \in A_k \times \mathcal{N}_k\left(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0\ ,\ \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right) \times A_k$$
$$\longmapsto$$
$$\Omega_k\alpha_1 \circ \gamma \circ \Lambda_k\alpha_2 \in \mathcal{N}_k\left(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0\ ,\ \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$$

and the space $\mathrm{NT}(\Lambda_k, \Omega_k)$ of natural linear transformations is isomorphic to the space of A_k-central vectors in $\mathcal{N}_k\left(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0\ ,\ \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$ via $\eta \longmapsto \eta_{\Gamma_k\cdots\Gamma_1 m_0}$.

Proof. The map

$$\gamma \in \mathcal{N}_k\left(\Lambda_k\Gamma_k \cdots \Gamma_1 m_0\ ,\ \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$$
$$\longmapsto$$
$$\left(\sum_{v \in V_{\mathcal{M}_k}} \left[\dim_{\mathbb{C}}\left(\mathcal{M}_k(v, \Gamma_k\cdots\Gamma_1 m_0)\right)\right]^{-1} \sum_{\substack{\sigma \in \mathrm{ONB}(v,x) \\ \tau \in \mathrm{ONB}(v, \Gamma_k \cdots \Gamma_1 m_0)}} \Omega_k(\sigma\tau^*) \circ \gamma \circ \Lambda_k(\tau\sigma^*)\right)_{x \in \mathrm{Ob}(\mathcal{M}_k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)$$

when restricted to the A_k-central vectors, turns out to be the inverse of the map in the statement of the lemma (since m_0 contains every simple as a subobject and Γ_j's are bi-faithful). $\square$

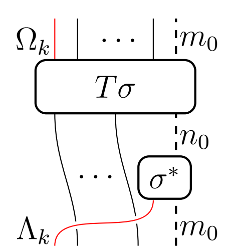
Lemma 3.1.9. *The pair $(\eta, \kappa) \in \mathrm{NT}(\Lambda_k, \Omega_k) \times \mathrm{NT}(\Lambda_{k+1}, \Omega_{k+1})$ satisfies exchange relation if and only if $\Phi_{\eta_{\Gamma_k\cdots\Gamma_1 m_0}} = \Phi_{\kappa_{\Gamma_{k+1}\cdots\Gamma_1 m_0}}$*

Proof. The 'only if' part direct follows from the definitions.
For the 'if' part, let ${}^\times\eta$ and $\kappa_\times$ denote the left and the right sides of the exchange relation equation. Applying Lemma 3.1.7 (iii) on the equation in our hypothesis, we deduce $({}^\times\eta)_{\Gamma_k\cdots\Gamma_1 m_0} = (\kappa_\times)_{\Gamma_k\cdots\Gamma_1 m_0}$. Now, by bi-faithfulness, $\Gamma_k\cdots\Gamma_1 m_0$ contains all the simples of $\mathcal{M}_k$ as subobjects, and thereby ${}^\times\eta = \kappa_\times$. $\square$

Theorem 3.1.10. *Starting from a 2-cell $\left\{\eta^{(k)} \in \mathrm{NT}\left(\Lambda_k, \Omega_k\right)\right\}_{k \geq K}$, we have an intertwiner $\Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}} \in {}_{A_\infty}\mathcal{L}_{B_\infty}(H_\infty, G_\infty)$ which is independent of $k \geq K$.*

Conversely, for every $T \in {}_{A_\infty}\mathcal{L}_{B_\infty}(H_\infty, G_\infty)$ (= the space of A_∞-B_∞-linear adjointable operator) and for all $k \geq K_T := \min\left\{l : TH_0 \subset G_l\right\}$, there exists unique $\eta^{(k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)$ such that $T = \Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}}$. Further, $\left(\eta^{(k)}, \eta^{(k+1)}\right)$ satisfies exchange relation for all $k \geq K_T$.

Proof. The forward direction trivially follows from Lemma 3.1.9 and the A_∞-B_∞-linearity is obvious. For the converse, set

$\zeta_k := \sum_{\sigma \in \mathscr{S}}$  $\in \mathcal{N}\left(\Lambda_k\Gamma_k\cdots\Gamma_1 m_0, \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$ for $k \geq K_T$

where $T\sigma$ is treated as an element of G_k and $\mathscr{S}$ $(\subset H_0)$ is a PP-basis for the right B_∞-module H_∞. Using the right B_∞-valued inner product, the PP-basis and right B_k-linearity of T, one can easily conclude $T\xi = \Phi_{\zeta_k}\xi$ for all $\xi \in H_k$; moreover, this equation uniquely determines ζ_k by Lemma 3.1.7 (iii). Further, the left side of the equation is A_k-linear; then so is the right side. Again by Lemma 3.1.7 (iii), ζ_k becomes A_k-central. Applying Lemma 3.1.8, we get a unique $\eta^{(k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)$ satisfying $\zeta_k = \eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}$. The rest of the proof is straight forward. □

Remark 3.1.11. If $\mathcal{C}_{B_\infty, A_\infty}$ denotes the category of A_∞-B_∞ right correspondences where A_∞ and B_∞ are the unital filtration of finite dimesnional C*-algebras associated to the 0-cells $\Gamma_\bullet$ and $\Delta_\bullet$ respectively, then combining Theorem 3.1.10, Definition 3.1.6 and the definition of the vertical composition of 2-cells, we have a fully faithful $*$-linear functor from

$$\Psi_{\Gamma_\bullet, \Delta_\bullet} : \mathbf{UC}_{\Gamma_\bullet, \Delta_\bullet} \longrightarrow \mathcal{C}_{B_\infty, A_\infty} .$$

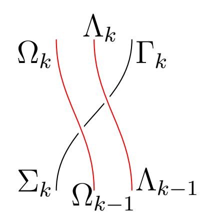
3.1.4 The horizontal structure

This is the final step of constructing a $*$-linear 2-category of unitary connections, denoted by $\mathbf{UC}$ whose 0-, 1- and 2-cells are already defined in Sections 3.1.1, 3.1.2 and 3.1.3 respectively. For 0-cells $\Gamma_\bullet, \Delta_\bullet, \Sigma_\bullet$, we will define a bifunctor

$$\boxtimes : \mathbf{UC}_{\Delta_\bullet, \Sigma_\bullet} \times \mathbf{UC}_{\Gamma_\bullet, \Delta_\bullet} \longrightarrow \mathbf{UC}_{\Gamma_\bullet, \Sigma_\bullet}$$

in such a way that it corresponds to the reverse relative tensor product of the associated right correspondences. For $\Omega_\bullet \in \mathbf{UC}_1(\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda_\bullet \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$, define

$$\Omega_\bullet \boxtimes \Lambda_\bullet := \left(\{\Omega_k \Lambda_k\}_{k \geq 0} \, , \, \left\{ \quad \right\}_{k \geq 1} \right) . \tag{3.5}$$



Proposition 3.1.12. *The bimodule $\Psi_{\Gamma_\bullet, \Sigma_\bullet}(\Omega_\bullet \boxtimes \Lambda_\bullet)$ is isomorphic to the relative tensor product of the right correspondences $\Psi_{\Gamma_\bullet, \Delta_\bullet}(\Lambda_\bullet)$ and $\Psi_{\Delta_\bullet, \Sigma_\bullet}(\Omega_\bullet)$.*

Proof. We first consider the following notations:

$$\left\{ \mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k \right\}_{k \geq 1} \rightsquigarrow \quad \cdots \subset A_k = \mathrm{End}(\Gamma_k \cdots \Gamma_1 m_0) \subset \cdots \subset \bigcup_{k \geq 0} A_k = A_\infty$$

$$\left\{ \mathcal{N}_{k-1} \xrightarrow{\Delta_k} \mathcal{N}_k \right\}_{k \geq 1} \rightsquigarrow \quad \cdots \subset B_k = \mathrm{End}(\Delta_k \cdots \Delta_1 n_0) \subset \cdots \subset \bigcup_{k \geq 0} B_k = B_\infty$$

$$\left\{ \mathcal{Q}_{k-1} \xrightarrow{\Sigma_k} \mathcal{Q}_k \right\}_{k \geq 1} \rightsquigarrow \quad \cdots \subset C_k = \mathrm{End}(\Sigma_k \cdots \Sigma_1 q_0) \subset \cdots \subset \bigcup_{k \geq 0} C_k = C_\infty$$

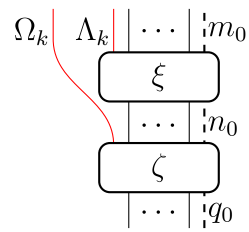
$$\Lambda_\bullet \rightsquigarrow \cdots \subset H_k = \mathcal{N}_k(\Delta_k \cdots \Delta_1 n_0, \Lambda_k \Gamma_k \cdots \Gamma_1 m_0) \subset \cdots \subset \bigcup_{k \geq 0} H_k = H_\infty$$

$\Omega_\bullet \rightsquigarrow \cdots \subset G_k = \mathcal{Q}_k(\Sigma_k \cdots \Sigma_1 q_0, \Omega_k \Delta_k \cdots \Delta_1 n_0) \subset \cdots \subset \bigcup\limits_{k \geq 0} G_k = G_\infty$

$\Omega_\bullet \boxtimes \Lambda_\bullet \rightsquigarrow \cdots \subset F_k = \mathcal{Q}(\Sigma_k \cdots \Sigma_1 q_0, \Omega_k \Lambda_k \Gamma_k \cdots \Gamma_1 m_0) \subset \cdots \subset \bigcup\limits_{k \geq 0} F_k = F_\infty$

Consider the linear map

$$H_k \otimes G_k \ni \xi \otimes \zeta \longrightarrow \Omega_k(\xi) \circ \zeta =$$



$$\in F_k.$$

It follows directly from the definition that the above map is A_k-C_k-linear and compatible with the inclusions $H_k \hookrightarrow H_{k+1}$, $G_k \hookrightarrow G_{k+1}$ and $F_k \hookrightarrow F_{k+1}$; as a result, it extends to a A_∞-C_∞-linear map $f : H_\infty \otimes G_\infty \longrightarrow F_\infty$. Consider the C_∞-valued sesquilinear form on $H_\infty \otimes G_\infty$ defined by

$$(H_\infty \otimes G_\infty) \times (H_\infty \otimes G_\infty) \ni (\xi_1 \otimes \zeta_1 \ , \ \xi_2 \otimes \zeta_2) \longmapsto \left\langle \langle \xi_1, \xi_2 \rangle_{B_\infty} \zeta_1, \zeta_2 \right\rangle_{C_\infty} \in C_\infty$$

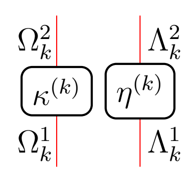
which after applying f, clearly goes to the desired one on the right correspondence G_∞. Moreover, kernel of f matches exactly with the null space of the form (using non-degeneracy of the C_∞-valued inner product on F_∞). Thus f factors through the relative tensor product and induces an injective A_∞-linear map.

Finally, we need to show that it is surjective as well. For this, consider the PP-basis $\mathscr{S}$ (resp., $\mathscr{T}$) sitting inside H_0 (resp., G_0) for the right B_∞- (resp., C_∞-) module H_∞ (resp., G_∞) considered in Lemma 3.1.3. Note that $\sum\limits_{\sigma \in \mathscr{S}} \sum\limits_{\tau \in \mathscr{T}} \Omega_0(\sigma) \circ \tau \circ \tau^* \circ \Omega_0(\sigma^*) = 1_{\Omega_0 \Lambda_0 m_0}$. Thus by Lemma 3.1.3, $\{\Omega_0(\sigma) \circ \tau\}_{(\sigma,\tau) \in \mathscr{S} \times \mathscr{T}}$ turns out to be a PP-basis for the right C_∞-module F_∞. $\square$

We next proceed towards defining $\boxtimes$ at the level of 2-cells.

Definition 3.1.13. For $\Omega^i_\bullet \in \mathbf{UC}_1(\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda^i_\bullet \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$ where $i = 1, 2$, and 2-cells $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k \geq K} \in \mathbf{UC}_2(\Lambda^1_\bullet, \Lambda^2_\bullet)$ and $\underline{\kappa} = \left\{\kappa^{(k)}\right\}_{k \geq L} \in \mathbf{UC}_2(\Omega^1_\bullet, \Omega^2_\bullet)$, define $\underline{\kappa} \boxtimes \underline{\eta} \in \mathbf{UC}_2(\Omega^1_\bullet \boxtimes \Lambda^1_\bullet \ , \ \Omega^2_\bullet \boxtimes \Lambda^2_\bullet)$ by

$$\left(\underline{\kappa} \boxtimes \underline{\eta}\right)_k := \ \Omega^2_k\left(\eta^{(k)}\right) \circ \kappa^{(k)}_{\Lambda^1_k} \ = \ \kappa^{(k)}_{\Lambda^2_k} \circ \Omega^1_k\left(\eta^{(k)}\right) \ =$$



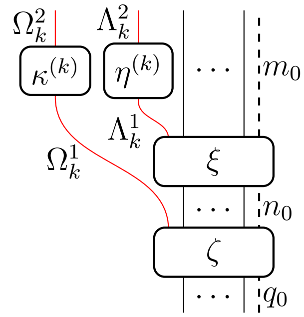
for $k \geq \max\{K, L\}$. (It is easy to check that every pair of consecutive terms in $\underline{\kappa} \boxtimes \underline{\eta}$ satisfies the exchange relation, which is a requirement for being a 2-cell.)

The compatibility of the vertical and horizontal compositions $\cdot$ and $\boxtimes$ between 2-cells follows easily from the pictures.

Proposition 3.1.14. *Continuing with the above set up, $\Psi_{\Gamma_\bullet, \Sigma_\bullet}\left(\underline{\kappa} \boxtimes \underline{\eta}\right)$ corresponds to the operator $\Psi_{\Gamma_\bullet, \Delta_\bullet}\left(\underline{\eta}\right) \underset{B_\infty}{\otimes} \Psi_{\Delta_\bullet, \Sigma_\bullet}\left(\underline{\kappa}\right)$ via the isomorphism of bimodules in Proposition 3.1.12.*

Proof. Let H^i_∞, G^i_∞ and F^i_∞ denote the A_∞-B_∞-, B_∞-C_∞- and A_∞-C_∞-right correspondences $\Psi_{\Gamma_\bullet,\Delta_\bullet}(\Lambda^i_\bullet)$, $\Psi_{\Delta_\bullet,\Sigma_\bullet}(\Omega^i_\bullet)$ and $\Psi_{\Gamma_\bullet,\Sigma_\bullet}(\Omega^i_\bullet \boxtimes \Lambda^i_\bullet)$ for $i = 1, 2$ respectively.
Set $T := \Psi_{\Gamma_\bullet,\Delta_\bullet}(\underline{\eta}) \in {}_{A_\infty}\mathcal{L}_{B_\infty}(H^1_\infty, H^2_\infty)$ and $S := \Psi_{\Delta_\bullet,\Sigma_\bullet}(\underline{\kappa}) \in {}_{B_\infty}\mathcal{L}_{C_\infty}(G^1_\infty, G^2_\infty)$. Suppose X denote the intertwiner in ${}_{A_\infty}\mathcal{L}_{C_\infty}(F^1_\infty, F^2_\infty)$ induced by $T \underset{B_\infty}{\otimes} S$ under the isomorphism in Proposition 3.1.12. For $k \geq \max\{K, L\}$ and $\xi \in H^1_k$, $\zeta \in G^1_k$, applying X on the element corresponding to the basic tensor $\xi \underset{B_\infty}{\otimes} \zeta$ (via Proposition 3.1.12), we get

$$X\left(\Omega^1_k(\xi) \circ \zeta\right) = \Omega^2_k\left(\Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}}(\xi)\right) \circ \left[\Phi_{\kappa^{(k)}_{\Delta_k\cdots\Delta_1 q_0}}(\zeta)\right] =$$



$= \Psi_{\Gamma_\bullet,\Sigma_\bullet}\left(\underline{\kappa} \boxtimes \underline{\eta}\right)\left(\Omega^1_k(\xi) \circ \zeta\right).$ □

We summarize the above finding in the following theroem.

Theorem 3.1.15. *Ψ is a $*$-linear, fully faithful, tensor-reversing 2-functor from the 2-category of unitary connections* $\mathbf{UC} = \{\mathbf{UC}_{\Gamma_\bullet,\Delta_\bullet} : \Gamma_\bullet, \Delta_\bullet$ are 0-cells$\}$ *to the 2-category of right correspondence over pairs of AFD pre-C*-algebras.*

Remark 3.1.16. The $*$-algebras A_∞, B_∞ associated to 0-cells $\Gamma_\bullet$, $\Delta_\bullet$ can be completed using their unique C*-norm, and obtain the C*-algebras A, B respectively. Then, the A-B right correspondence associated to the 1-cell $\Lambda_\bullet$ will be the completion H of the space H_∞ with respect to the norm $\|\xi\|_{C^*} := \sqrt{\|\langle\xi,\xi\rangle_B\|}$. The PP-basis $\mathscr{S}$ for the right B_∞-module H_∞ continue to be so for the right B-module H. As a result, H as a B-module becomes isomorphic to $q\left[\mathbb{C}^{\mathscr{S}} \otimes B\right]$ where the right B-action on the latter module is the diagonal one and q is the projection $\sum\limits_{\sigma_1,\sigma_2\in\mathscr{S}} E_{\sigma_1,\sigma_2} \otimes \langle\sigma_2,\sigma_1\rangle_B$ in the C*-algebra $M_{\mathscr{S}\times\mathscr{S}} \otimes B$. The left A-action on H will translate into a $*$-homomorphism $\Pi : A \longrightarrow q\left[M_{\mathscr{S}\times\mathscr{S}} \otimes B\right]q$ giving rise to an A-action on $q\left[\mathbb{C}^{\mathscr{S}} \otimes B\right]$. Now, at the level of 2-cells from the 1-cell $\Lambda_\bullet$ to $\Omega_\bullet$, the obvious candidates that come up are the adjointable A-B intertwiners; these are in one-to-one correspondence with elements in $s\left[M_{\mathscr{T}\times\mathscr{S}} \otimes B\right]q$ which intertwines $\Pi(a)$ and $\widetilde{\Pi}(a)$ for $a \in A$ (where $\mathscr{T}, s, \widetilde{\Pi}$ are related to the 1-cell $\Omega_\bullet$ in the same way as $\mathscr{S}, q, \Pi$ are related to $\Lambda_\bullet$ respectively). However, there is no apparent interpretation of such 2-cells in terms of natural transformations between Λ_k's and Ω_k's compatible with the W_k's as shown in Theorem 3.1.10.

Remark 3.1.17. From the definition of $\mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$, we observe that two 2-cells $\left\{\eta^{(k)}\right\}_{k\geq N}$, $\left\{\tau^{(k)}\right\}_{k\geq L} \in \mathbf{UC}_2(\Lambda_\bullet, \Omega_\bullet)$ are equal if and only if $\eta^{(k)} = \tau^{(k)}$ eventually. So, two 1-cells $\Lambda_\bullet$ and $\Omega_\bullet$ are isomorphic in $\mathbf{UC}$ if there is a sequence of natural transformations $U_k : \Lambda_k \to \Omega_k$ which satisfies exchange relation from some level l and which implements isomorphism between Λ_k and Ω_k eventually.

3.2 The tracial case

Locally semisimple algebras equipped with a tracial state, extend to finite von Neumann algebras. Hyperfinite subfactor reconstruction works by passing from the algebraic category described in the previous section to von Neumann algebras, and showing that for special cases arising in finite index subfactor theory, it is fully faithful. A natural question is to figure out what happens in our more general setting.

To make this question precise, we consider modifications of the 2-category **UC** at every level. At the level of 0-cells, we will be considering Bratteli diagrams with extra tracial data, and make necessary adjustments to the 1- and 2-cells in order to "preserve" this extra structure. Our particular choice of adjustments admittedly appears ad hoc, but it is the condition that was required to make our functor work in the forthcoming proofs. This new modified 2-category will be denoted by $\mathbf{UC}^{\mathrm{tr}}$. In this section, we will only start laying out the definition of the main ingredients of $\mathbf{UC}^{\mathrm{tr}}$. At the end of next section, we will prove all the necessary conditions to turn it into an honest 2-category.

- **0-cells.** The 0-cells are given by pairs $\left(\Gamma_\bullet, \underline{\mu}^\bullet\right)$ where $\Gamma_\bullet$ is a 0-cell $\left\{\mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k\right\}_{k\geq 0}$ in **UC**, and $\underline{\mu}^\bullet$ denotes weight vectors $\underline{\mu}^k = (\mu^k_v)_{v\in V_{\mathcal{M}_k}}$ with positive entries satisfying $\sum\limits_{v\in V_{\mathcal{M}_0}} \mu^0_v = 1$ and $(\Gamma_k)' \, \underline{\mu}^k = \underline{\mu}^{k-1}$ all $k \geq 1$ (where we use the same symbol for the functor and its adjacency matrix). In other words (recasting in terms of semisimple categories and functors), the data of a 0-cell in $\mathbf{UC}^{\mathrm{tr}}$ is a sequence of weighted finitely semisimple C*-categories $\left\{\left(\mathcal{M}_k, \underline{\mu}^k\right)\right\}_{k\geq 0}$ along with a sequence of $*$-linear, bi-faithful functors $\Gamma_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ such that the tracial solution (say $\left(\rho_k : \mathrm{id}_{\mathcal{M}_k} \to \Gamma_k \, \Gamma'_k \ , \ \rho'_k : \mathrm{id}_{\mathcal{M}_{k-1}} \to \Gamma'_k \, \Gamma_k\right)$ where Γ'_k is adjoint to Γ_k) to the conjugate equations commensurate with the weight functions $\left(\underline{\mu}^{k-1}, \underline{\mu}^k\right)$ satisfies
$$(\rho'_k)^*_\bullet \circ (\rho'_k)_\bullet = 1_\bullet \tag{3.6}$$
which is equivalent to the matrix equation $(\Gamma_k)' \, \underline{\mu}^k = \underline{\mu}^{k-1}$ (via Equation (2.3)). The purpose of the equation $\sum\limits_{v\in V_{\mathcal{M}_0}} \mu^0_v = 1$ is to normalize scaling. For simplicity, we will denote the 0-cell of $\mathbf{UC}^{\mathrm{tr}}$ by $\Gamma_\bullet$.

- **1-cells.** A 1-cell in $\mathbf{UC}^{\mathrm{tr}}$ from the 0-cell $\left\{\left(\mathcal{M}_{k-1}, \underline{\mu}^{k-1}\right) \xrightarrow{\Gamma_k} \left(\mathcal{M}_k, \underline{\mu}^k\right)\right\}_{k\geq 0}$ to $\left\{\left(\mathcal{N}_{k-1}, \underline{\nu}^{k-1}\right) \xrightarrow{\Delta_k} (\mathcal{N}\right.$ is given by a 1-cell $\Lambda_\bullet$ in $\mathbf{UC}_1\left(\Gamma_\bullet, \Delta_\bullet\right)$ such that there exists $\epsilon, M > 0$ satisfying the boundedness condition:
$$\epsilon \ \mu^k_v \ \leq \ \left[\Lambda'_k \ \underline{\nu}^k\right]_v \ \leq \ M \ \mu^k_v \ \text{ for all } \ k \geq 0, \ v \in V_{\mathcal{M}_k} \ . \tag{3.7}$$

- **Tensor of 1-cells.** For 0-cells $\Gamma_\bullet, \Delta_\bullet, \Sigma_\bullet$, we will define a map
$$\boxtimes \ : \ \mathbf{UC}^{\mathrm{tr}}_1\left(\Delta_\bullet, \Sigma_\bullet\right) \times \mathbf{UC}^{\mathrm{tr}}_1\left(\Gamma_\bullet, \Delta_\bullet\right) \longrightarrow \mathbf{UC}^{\mathrm{tr}}_1\left(\Gamma_\bullet, \Sigma_\bullet\right)$$

exactly the same as that for $\mathbf{UC}_1$ given by Equation (3.5); however, we have to check whether Equation (3.7) is satisfied by $\Omega_\bullet \boxtimes \Lambda_\bullet$, where $\Omega_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}(\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}(\Gamma_\bullet, \Delta_\bullet)$. Suppose we have,

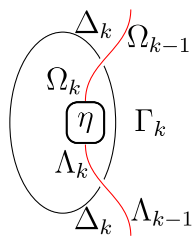
$$\epsilon \mu_u^k \leq [\Lambda'_k \, \underline{\nu}^k]_u \leq M \mu_u^k \quad \text{and} \quad \delta \nu_v^k \leq [\Omega'_k \, \underline{\pi}^k]_v \leq N \nu_v^k \quad \text{for each} \quad k \geq 0, u \in V_{\mathcal{M}_k}, v \in V_{\mathcal{N}_k}.$$

Applying Λ'_k on the second set of inequalities, we get $\epsilon\delta\mu_u^k \leq [\Lambda'_k \, \Omega'_k \, \underline{\pi}^k]_u \leq MN\mu_u^k$ for each $k \geq 0, u \in V_{\mathcal{M}_k}$. Thus, Equation (3.7) is satisfied for $\Omega_\bullet \boxtimes \Lambda_\bullet$.

- **2-cells.** Consider two 1-cells $\Lambda_\bullet, \Omega_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}\left(\left(\Gamma_\bullet, \underline{\mu}^\bullet\right), \left(\Delta_\bullet, \underline{\nu}^\bullet\right)\right)$. The 2-cells in $\mathbf{UC}_2\left(\Lambda_\bullet, \Omega_\bullet\right)$, that is, the sequences eventually satisfying the exchange relation at every level, do not use the extra data of $\underline{\mu}^\bullet$ and $\underline{\nu}^\bullet$. We introduce the following tool which will generalize the exchange relation.

Definition 3.2.1. The *loop operator from* $\Lambda_\bullet$ *to* $\Omega_\bullet$ is the sequence of linear maps $\{S_k : \mathrm{NT}(\Lambda_k, \Omega_k) \to \mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1})\}_{k\geq 1}$ defined by

$\mathrm{NT}(\Lambda_k, \Omega_k) \ni \eta \xmapsto{S_k}$



$\in \mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1})$.

where the cap and the cup come from tracial solution to the conjugate equation for the duality of $\Delta_k : \mathcal{N}_{k-1} \to \mathcal{N}_k$ commensurate with $\left(\underline{\nu}^{k-1}, \underline{\nu}^k\right)$. (Such pictorial operators did appear in the literature, more specifically in planar algebras, cf. [J21, Theorem 2.11.8]).

We will encounter equations and inequalities involving multiple loop operators all of which might not have the same source 1-cell or the same target 1-cell in $\mathbf{UC}_1^{\mathbf{tr}}$; for notational convenience, we will simply use $S_\bullet$, and from the context, it will be clear what the source and the targets are.

Remark 3.2.2. The loop operator satisfies the following properties which are easy to derive:

(i) S_k is unital when $\Lambda_\bullet = \Omega_\bullet$, (which follows from Equation (3.6)),

(ii) $S_k \eta^* = (S_k \eta)^*$,

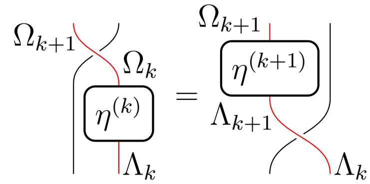
(iii) $S_k \eta^* \circ S_k \eta \leq S_k(\eta^* \eta)$ in the C*-algebra $\mathrm{NT}(\Lambda_{k-1}, \Lambda_{k-1})$ and the loop operator is a contraction.

Definition 3.2.3. A sequence $\underline{\eta} = \left\{\eta^{(k)} \in \mathrm{NT}\left(\Lambda_k, \Omega_k\right)\right\}_{k\geq 0}$ will be referred as:

(a) *quasi-flat sequence from* $\Lambda_\bullet$ *to* $\Omega_\bullet$ if it satisfies $S_{k+1}\eta^{(k+1)} = \eta^{(k)}$ for all $k \geq 0$,

(b) *flat sequence from* $\Lambda_\bullet$ *to* $\Omega_\bullet$ if it is quasi-flat and there exists $K \in \mathbb{N}$ such that $\left(\eta^{(k)}, \eta^{(k+1)}\right)$ satisfies the exchange relation (Definition 3.1.4) for every $k \geq K$.

Remark 3.2.4. There is a one-to-one correspondence between flat sequences from $\Lambda_\bullet$ to $\Omega_\bullet$, and the 2-cells in $\mathbf{UC}_2\left(\Lambda_\bullet, \Omega_\bullet\right)$. Note that the exchange relation



, unitarity of the connection and Equation (3.6) yield the equation $S_{k+1}\eta^{(k+1)} = \eta^{(k)}$. Thus every 2-cell $\left\{\eta^{(k)}\right\}_{k\geq K} \in \mathbf{UC}_2\left(\Lambda_\bullet, \Omega_\bullet\right)$ extends to a unique quasi-flat sequence from $\Lambda_\bullet$ to $\Omega_\bullet$ by setting $\eta^{(k)} := S_{k+1}\cdots S_K\ \eta^{(K)}$ for $k < K$. Further, a flat sequence is bounded in C*-norm.

Definition 3.2.5. A 2-cell in $\mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda_\bullet, \Omega_\bullet\right)$ is given by a bounded (in C*-norm) quasi-flat sequence (abbreviated as 'BQFS') from $\Lambda_\bullet$ to $\Omega_\bullet$.

- **Horizontal and vertical compositions of 2-cells.**

  Definition 3.2.6. (a) The *vertical composition of the 2-cells* $\underline{\kappa} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Omega_\bullet, \Xi_\bullet\right)$ and $\underline{\eta} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda_\bullet, \Omega_\bullet\right)$ is defined as

$$\left(\underline{\kappa}\cdot\underline{\eta}\right) := \left\{\left(\underline{\kappa}\cdot\underline{\eta}\right)^{(k)} := \lim_{l\to\infty} S_{k+1}\cdots S_{k+l}\left(\kappa^{(k+l)}\circ\,\eta^{(k+l)}\right)\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda_\bullet, \Xi_\bullet\right)\ .$$

  (b) Let $\Lambda^i_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}(\Gamma_\bullet, \Delta_\bullet)$ and $\Omega^i_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}(\Delta_\bullet, \Sigma_\bullet))$ for $i = 1, 2$. Then, the *horizontal composition* (or the *tensor product*) *of the 2-cells* $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda^1_\bullet, \Lambda^2_\bullet\right)$ and $\underline{\kappa} = \left\{\kappa^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Omega^1_\bullet, \Omega^2_\bullet\right)$ is given by

$$\underline{\kappa}\boxtimes\underline{\eta} := \left\{\left(\underline{\kappa}\boxtimes\underline{\eta}\right)^{(k)} := \lim_{l\to\infty} S_{k+1}\cdots S_{k+l}\left(\Omega^2_{k+l}\left(\eta^{(k+l)}\right)\circ\kappa^{(k+l)}_{\Lambda^1_{k+l}}\right)\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Omega^1_\bullet\boxtimes\Lambda^1_\bullet, \Omega^2_\bullet\boxtimes\Lambda^2_\bullet\right)\ .$$

Remark 3.2.7. At this point, several questions remain unanswered, namely, (a) well-definedness - why the limits in Definition 3.2.6 exists and even if they all exist, why the sequences built by these limit will yield a 2-cell in $\mathbf{UC}^{\mathrm{tr}}$, (b) why the two compositions are associative and compatible with each other, etc. One way to settle this issue is by viewing the loop operator S as a UCP operator and express the compositions as a certain Chois-Effros product (along the lines of Izumi's Poisson boundary approach in [I04]). However, we will not take this route. Instead, in Section 3.3, we will make $\mathbf{UC}^{\mathrm{tr}}$ sit inside the 2-category of von Neumann algebras, bimodules and intertwiners in a fully faithful way, and then use it to answer the questions towards the end in Corollary 3.3.14. The reason for this nonlinear approach of defining $\mathbf{UC}^{\mathrm{tr}}$, is that it looks compact and becomes easy for future references.

Remark 3.2.8. In the passage from $\mathbf{UC}$ to $\mathbf{UC}^{\mathrm{tr}}$, we are imposing restriction at the level of 1-cells but the 2-cell spaces have been generalized. So, on the nose, neither we have a forgetful functor nor one turns out as a subcategory of the other. However, we do have a subcategory of $\mathbf{UC}^{\mathrm{tr}}$ which we call its *flat part* and denote by $\mathbf{UC}^{\mathrm{flat}}$ where everything is the same as that of $\mathbf{UC}^{\mathrm{tr}}$ at the level of 0- and 1-cells but the 2-cells in $\mathbf{UC}^{\mathrm{flat}}$ are only flat sequences (and not all BQFS). Indeed compositions of the 2-cells in $\mathbf{UC}^{\mathrm{flat}}$ correspond to exactly to those in $\mathbf{UC}$; this easily follows from Remark 3.2.4.

3.3 A concrete realization of $\mathbf{UC}^{\mathrm{tr}}$

The goal of this section is to complete the unfinished work of turning $\mathbf{UC}^{\mathrm{tr}}$ into a 2-category along with building fully faithful 2-functor $\mathcal{PB} : \mathbf{UC}^{\mathrm{tr}} \to \mathbf{vNAlg}$ where $\mathbf{vNAlg}$ is the 2-category of von Neumann algebras, bimodules and intertwiners. Our starting point will be the pre-C* algebras and right correspondences produced from 0- and 1-cells in $\mathbf{UC}^{\mathrm{tr}}$ viewed as those in $\mathbf{UC}$ as descibed in Section 3.1, and then take their appropriate completions. At this point, it might seem it is enough to build the functor starting from the flat part $\mathbf{UC}^{\mathrm{flat}}$; however, in that case, the functor may not be fully faithful at the level of 2-cells (which are only flat sequences). In this section, we will be analyzing "the kernel" of the 2-functor from $\mathbf{UC}^{\mathrm{flat}}$; as a consequence, we justify the need of generalizing the 2-cells in $\mathbf{UC}^{\mathrm{flat}}$ to those in $\mathbf{UC}^{\mathrm{tr}}$.

3.3.1 $\mathcal{PB}$ on 0-cells

Given a 0-cell $\left(\Gamma_\bullet, \underline{\mu}^\bullet\right)$ in $\mathbf{UC}^{\mathrm{tr}}$, we consider m_0, A_k's and their inclusions as in the non-tracial case Section 3.1. Using the categorical trace $\mathrm{Tr} = (\mathrm{Tr}_x)_{x \in \mathrm{ob}(\mathcal{M}_k)}$ associated to the weight vector $\underline{\mu}^k$, we define $\mathrm{Tr}_{A_k} \coloneqq \mathrm{Tr}_{\Gamma_k \cdots \Gamma_1 m_0} : A_k \to \mathbb{C}$ which turns out to be a faithful tracial state which by Equation (2.5), turns out to be compatible with the inclusion. Thus, we have a faithful tracial state Tr_{A_∞} on the $*$-algebra A_∞. Note that the action of an element of A_∞ on the GNS Hilbert $L^2(A_\infty, \mathrm{Tr}_{A_\infty})$ is bounded. Let A denote the type II_1 von Neumann algebra obtained by taking the WOT closure of A_∞ acting on $L^2(A_\infty, \mathrm{Tr}_{A_\infty})$.

Definition 3.3.1. We define $\mathcal{PB}\left(\Gamma_\bullet, \underline{\mu}^\bullet\right) \coloneqq A = A_\infty'' \subseteq \mathcal{L}(L^2(A, \mathrm{Tr}(A_\infty))$.

3.3.2 $\mathcal{PB}$ on 1-cells

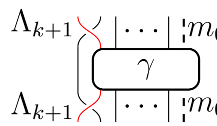
Let $\Lambda_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}\left(\left(\Gamma_\bullet, \underline{\mu}^\bullet\right), \left(\Delta_\bullet, \underline{\nu}^\bullet\right)\right)$. Consider the A_∞-B_∞ right correspondence H_∞ associated to the 1-cell $\Lambda_\bullet$ treated as a 1-cell in $\mathbf{UC}$. Let H be the completion of H_∞ with respect to the scalar inner product $\langle \xi, \zeta \rangle \coloneqq \mathrm{Tr}_{B_\infty}\left(\langle \xi, \zeta \rangle_{B_\infty}\right)$ for $\xi, \zeta \in H_\infty$. A_∞, B_∞ being locally semisimple $*$-algebras, must have the action of their elements on H_∞ bounded, and hence extend to action on H.

To obtain a right B-action on H, we work with the Pimsner-Popa basis $\mathscr{S}$ for the right-B_∞-module H_∞ with respect to the B_∞-valued inner product obtained in Lemma 3.1.3. Observe that the map

$$H \supset H_\infty \ni \xi \longmapsto \sum_{\sigma \in \mathscr{S}} \sigma \otimes \langle \xi, \sigma \rangle_{B_\infty} \in q\left[\ell^2(\mathscr{S}) \otimes L^2(B_\infty, \mathrm{Tr}_{B_\infty})\right] =: K \quad \text{(say)}$$

extends to an isometric isomorphism preserving the right B_∞-action where q is the projection $\sum\limits_{\sigma, \tau \in \mathscr{S}} E_{\sigma,\tau} \otimes \langle \tau, \sigma \rangle_{B_\infty}$. Clearly, the B_∞ action on K extends to a normal action of B and hence, the same holds for the Hilbert space H.

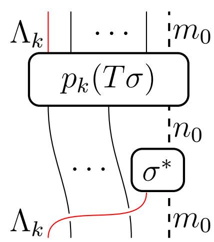
In order to extend the A_∞-action on H (which is clearly bounded) to a normal action of A, we first analyse the commutant of B in $\mathcal{L}(H)$. For $k \geq 0$, define $C_k := \mathrm{End}(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0)$. The $*$-homomorphism $C_k \ni \gamma \longmapsto \Phi_\gamma|_{H_k} = \gamma \circ \bullet \in \mathcal{L}(H_k)$ is faithful by Lemma 3.1.7, and hence an isometry. Thus, Φ_γ extends to the whole of H as a bounded operator commuting with the right action of B_∞ (and thereby B). Consider the unital $*$-algebra inclusion

$$C_k \ni \gamma \longmapsto (W_k)_{\Gamma_k \cdots \Gamma_1 m_0} \circ \Delta_{k+1}\gamma \circ (W_k)^*_{\Gamma_k \cdots \Gamma_1 m_0} =$$  $$\in C_{k+1} \; .$$

Note that Φ_γ is compatible with the above inclusion. Indeed, $C_\infty \ni \gamma \overset{\Phi}{\longmapsto} \Phi_\gamma \in \mathcal{L}_B(H)$ becomes a unital faithful $*$-algebra homomorphism where $C_\infty := \bigcup\limits_{k \geq 0} C_k$.

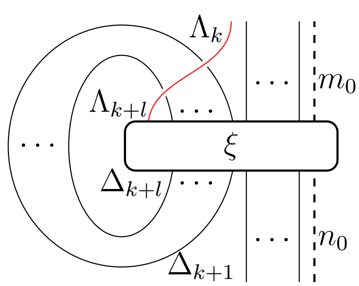
Proposition 3.3.2. $\mathcal{L}_B(H) = \{\Phi_\gamma : \gamma \in C_\infty\}''$.

Proof. Consider the projection $p_k \in \mathcal{L}(H)$ such that $\mathrm{Range}(p_k) = H_k$. Since $A_k \cdot H_k \cdot B_k = H_k$, therefore p_k must be A_k-B_k-linear.

Let $T \in \mathcal{L}_B(H)$. Set $\zeta_k := \sum\limits_{\sigma \in \mathscr{S}}$  $\in C_k$ where $\mathscr{S}$ $(\subset H_0)$ is a PP-basis

of the right B-module H as constructed in Lemma 3.1.3. Since T (resp. p_k) is right B- (resp. B_k-) linear, one may deduce the relation $p_k T p_k = \Phi_{\zeta_k} p_k$. Clearly, p_k converges to id_H in SOT as k goes to ∞, and $\{\Phi_{\zeta_k}\}_{k \geq 0}$ is a norm bounded subset of $\mathcal{L}(H)$. Hence, $T \in \overline{\{\Phi_\gamma : \gamma \in C_\infty\}}^{\mathrm{SOT}} = \{\Phi_\gamma : \gamma \in C_\infty\}''$. ☐

Remark 3.3.3. Using Equation (3.6), we may represent the projection p_k in the following way

$$H \supset H_{k+l} \ni \xi \overset{p_k}{\longmapsto}$$  $$\in H_k \subset H$$

where the local maxima and minima are given by the natural transformation appearing in the tracial solution to conjugate equation for the duality of the functors Δ_i's associated to the positive weights $\underline{\nu}^{i-1}$ and $\underline{\nu}^i$ on the vertices (that is, the simple objects of $\mathcal{N}_{i-1}$ and $\mathcal{N}_i$). Clearly, p_k is A_k-B_k-linear.

Consider the unital $*$-algebra inclusion $A_k \ni \alpha \overset{\Lambda_k}{\longmapsto} \Lambda_k \alpha \in C_k$. Again, this inclusion is compatible with $C_k \hookrightarrow C_{k+1}$ and $A_k \hookrightarrow A_{k+1}$; thus, A_∞ sits as a unital $*$-subalgebra inside

C_∞. Observe that if $\gamma \in C_k$ comes from A_k, that is, $\gamma = \Lambda_k \alpha$ for some $\alpha \in A_k$, then Φ_γ matches exactly with the action of α on H_∞. Now, the functional

$$\mathrm{Tr}' := [d_B(H)]^{-1} \sum_{\sigma \in \mathscr{S}} \langle \bullet\, \sigma, \sigma \rangle : \mathcal{L}_B(H) \to \mathbb{C}$$

is a faithful normal tracial state where $\mathscr{S}$ is a PP-basis for the module H_B and $d_B(H) := \sum\limits_{\sigma \in \mathscr{S}} \|\sigma\|^2$; however, its restriction on A_∞ may not match with that of Tr_{A_∞}.

Proposition 3.3.4. *The above inclusion of A_∞ inside C_∞ extends to a normal inclusion of A inside $\mathcal{L}_B(H)$, and thereby H becomes a 'von Neumann' A-B-bimodule.*

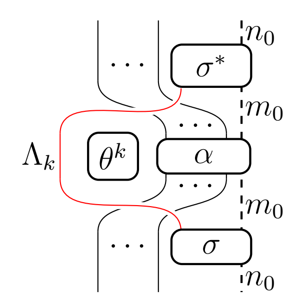
Proof. By construction, A is the von Neumann algebra obtained from the GNS of A_∞ with respect to Tr_{A_∞}. Let A''_∞ denote the double commutant of A_∞ sitting inside $\mathcal{L}(H)$ via the inclusions $A_\infty \ni \alpha \xmapsto{\Lambda_\bullet} \Lambda_\bullet \alpha \in C_\infty$ and $C_\infty \overset{\Phi}{\hookrightarrow} \mathcal{L}(H)$. It is enough to produce a central positive invertible element T in A''_∞ satisfying $\mathrm{Tr}'\left(\Phi_{\Lambda_\bullet \alpha}\, T\right) = \mathrm{Tr}_{A_\infty}(\alpha)$ for $\alpha \in A_\infty$ (that is, Tr_{A_∞} extends to a faithful normal trace on A''_∞). To justify this, let $\mathrm{Tr}_{A''_\infty} := \mathrm{Tr}'\left(\bullet\, T_0\right)\big|_{A''_\infty}$ which is clearly a normal faithful tracial state on A''_∞. Thus, A''_∞ is isomorphic as a von Neumann algebra to its image under GNS representation on $L^2\left(A''_\infty, \mathrm{Tr}_{A''_\infty}\right)$. Now, the map $L^2\left(A_\infty, \mathrm{Tr}_{A_\infty}\right) \ni \alpha \xmapsto{u} \Phi_{\Lambda_\bullet \alpha} \in L^2\left(A''_\infty, \mathrm{Tr}_{A''_\infty}\right)$ extends to a unitary preserving the left A_∞-action. Since $\mathrm{Tr}'\left(\bullet\, T_0\right)$ is WOT-continuous, by double commutant theorem, uAu^* turns out to be isomorphic as a von Neumann algebra to A''_∞.

Consider the natural transformation $\theta^k := \left(\dfrac{\mu^k_v}{[\Lambda'_k\ \underline{\nu}^k]_v} 1_v \right)_{v \in V_{\mathcal{M}_k}} \in \mathrm{End}(\mathrm{id}_{\mathcal{M}_k})$. Set $T_k := \Phi_{\Lambda_k\left(\theta^k_{\Gamma_k \cdots \Gamma_1 m_0}\right)} \in A''_\infty$ and $\psi := \sum\limits_{\sigma \in \mathscr{S}} \langle \bullet\, \sigma, \sigma \rangle = d_B(H)\ \mathrm{Tr}'$.

Assertion: $\psi\left(\Phi_{\Lambda_k(\bullet)}\, T_k\right) = \mathrm{Tr}_{A_k}$ for all $k \geq 0$.

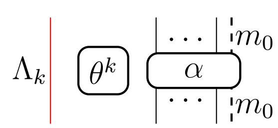
Proof of the assertion. Let $\alpha \in A_k$. Then, $\psi\left(\Phi_{\Lambda_k(\alpha)}\, T_k\right) = \sum\limits_{\sigma \in \mathscr{S}} \left\langle \Phi_{\Lambda_k\left(\alpha\ \theta^k_{\Gamma_k \cdots \Gamma_1 m_0}\right)}\, \sigma, \sigma \right\rangle$

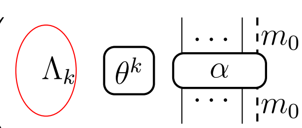
$$= \sum_{\sigma \in \mathscr{S}} \mathrm{Tr}_{B_k}\left(\left\langle \Phi_{\Lambda_k\left(\alpha\ \theta^k_{\Gamma_k \cdots \Gamma_1 m_0}\right)}\, \sigma, \sigma \right\rangle_{B_k} \right) = \sum_{\sigma \in \mathscr{S}} \mathrm{Tr}_{\Delta_k \cdots \Delta_1 m_0}\left(\quad \right) .$$



Using the property of the categorical trace, the equation in Lemma 3.1.3 satisfied by the set $\mathscr{S}$ and the natural unitaries (namely, the crossings), we may rewrite the last expression as

$$\mathrm{Tr}_{\Lambda_k \Gamma_k \cdots \Gamma_1 m_0}\left(\quad \right) = \mathrm{Tr}_{\Gamma_k \cdots \Gamma_1 m_0}\left(\quad \right)$$





by Equation (2.5) where the red cap and cup correspond to tracial solution to conjugate equation for the duality of the functor Λ_k with respect to weight vectors $\underline{\mu}^k$ and $\underline{\nu}^k$ on the vertex sets $V_{\mathcal{M}_k}$ and $V_{\mathcal{N}_k}$ respectively. Now, it is a matter of routine verification that the red loop appearing above is indeed the inverse of θ^k in the algebra $\mathrm{End}(\mathrm{id}_{\mathcal{M}_k})$. Cancelling the two, we get $\mathrm{Tr}_{A_k}(\alpha)$.

Equation (3.7) implies that C*-norm of θ^k is uniformly bounded by ϵ^{-1} for $k \geq 0$, and thereby $\{T_k\}_{k\geq 0}$ is norm-bounded sequence in $A''_\infty \subset \mathcal{L}(H)$. By compactness, there exists a subsequence $\{T_{k_l}\}_l$ which converges in WOT to $T_0 \in A''_\infty$ (say). Clearly, $\psi(\Phi_{\Lambda_\bullet}(\alpha)\, T_0) = \mathrm{Tr}_{A_\infty}(\alpha)$ for all $\alpha \in A_\infty$. Observe that T_k commutes with $\Phi_{\Lambda_k \alpha}$ for all $\alpha \in A_k$, $k \geq 0$; this implies T_0 must be central in A''_∞. Again, T_k is a positive element in A''_∞ satisfying $T_k \geq M^{-1}$ (using Equation (3.7)); thus, the subsequential WOT-limit T_0 (hence, positive) also satisfies the same, and thereby, becomes invertible. □

Definition 3.3.5. Define

$$\mathbf{UC}_1^{\mathrm{tr}}\left(\left(\Gamma_\bullet, \underline{\mu}^\bullet\right), \left(\Delta_\bullet, \underline{\nu}^\bullet\right)\right) \ni \Lambda_\bullet \overset{\mathcal{PB}}{\longmapsto} \mathcal{PB}\left(\Lambda_\bullet\right) := {}_A H_B \in \mathbf{vNalg_1}\left(\mathcal{PB}\left(\Delta_\bullet, \underline{\nu}^\bullet\right), \mathcal{PB}\left(\Gamma_\bullet, \underline{\mu}^\bullet\right)\right) \ .$$

3.3.3 $\mathcal{PB}$ on 2-cells

Let $\Lambda_\bullet, \Omega_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}\left(\left(\Gamma_\bullet, \underline{\mu}^\bullet\right), \left(\Delta_\bullet, \underline{\nu}^\bullet\right)\right)$ and $\mathcal{PB}\left(\Gamma_\bullet, \underline{\mu}^\bullet\right) = A$, $\mathcal{PB}\left(\Delta_\bullet, \underline{\nu}^\bullet\right) = B$. We will borrow the notations H_k, H, p_k, $\mathscr{S}$, etc. (arising out of $\Lambda_\bullet$) from previous subsections, and for those arising out of $\Omega_\bullet$, we will use G_k, G, q_k, $\mathscr{T}$, etc. respectively, and we will also work with the pictures as before. For $\gamma \in \mathcal{N}_k\left(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0\ ,\ \Omega_k \Gamma_k \cdots \Gamma_1 m_0\right)$, we will consider the unique bounded extension of $\Phi_\gamma \in \mathcal{L}_{B_\infty}\left(H_\infty, G_\infty\right)$ (defined in Equation (3.4)) and denote it with the same symbol $\Phi_\gamma \in \mathcal{L}_B\left(H, G\right)$.

Proposition 3.3.6. *(a)* $q_{k-1}\ \Phi_{\eta_{\Gamma_k \cdots \Gamma_1 m_0}}\ p_{k-1} = \Phi_{(S_k \eta)_{\Gamma_{k-1} \cdots \Gamma_1 m_0}}\ p_{k-1}$ *for all* $\eta \in \mathrm{NT}(\Lambda_k, \Omega_k)$ *and* $k \geq 0$.

(b) If $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda_\bullet, \Omega_\bullet\right)$ *(that is, a BQFS from* $\Lambda_\bullet$ *to* $\Omega_\bullet$*), then it gives rise to a unique bounded operator* $T \in {}_A\mathcal{L}_B\left(H, G\right)$ *satisfying*

$$q_k\ T\ p_k = \Phi_{\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}}\ p_k \ \ \textit{for all} \ \ k \geq 0 \ . \tag{3.8}$$

Proof. Part (a) directly follows from Remark 3.3.3.

For (b), set $T_k := \Phi_{\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}}$ for each $k \geq 0$; it is easy to see that $T_k \in {}_{A_k}\mathcal{L}_B(H, G)$ and $\|T_k\| = \left\|\eta^{(k)}\right\|$. For all $\gamma \in H_k$, using part (a) followed by quasi-flat condition of $\underline{\eta}$, we have

$$q_k T_{k+1}\, \gamma = \Phi_{\left(S_{k+1}\ \eta^{(k+1)}\right)_{\Gamma_k \cdots \Gamma_1 m_0}}\, \gamma = \Phi_{\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}}\, \gamma = T_k\, \gamma \ .$$

In other words, $q_k\, T_{k+1}\, p_k = T_k\, p_k$. Applying this iteratively, we get $q_k\, T_{k+l}\, p_k = T_k\, p_k = q_k\, T_k\, p_k$ for all $k, l \geq 0$. This again implies

$$\left\|T_{l+m}\, \gamma - T_l\, \gamma\right\|^2 = \left\|T_{l+m}\, \gamma\right\|^2 - \left\|T_l\, \gamma\right\|^2 \quad \text{for all } k \leq l,\ \gamma \in H_k \ . \tag{3.9}$$

Now, fix $\gamma \in H_k$. Equation (3.9) tell us that the sequence $\{\|T_l\,\gamma\|\}_{l\geq k}$ must be increasing; also, it is bounded by $\left[\sup_{m\geq 0}\left\|\eta^{(m)}\right\|\right]\|\gamma\|$ and hence convergent. Letting l tend towards ∞ in Equation (3.9), we find that $\{T_k\}_{k\geq 0}$ converges pointwise on H_∞ (because of completeness of G). Since H_∞ is dense in H and $\{T_k\}_{k\geq 0}$ is norm-bounded by $\sup_{k\geq 0}\left\|\eta^{(k)}\right\|$, we may conclude that $\{T_k\}_{k\geq 0}$ converges in SOT to some $T \in \mathcal{L}(H,G)$.

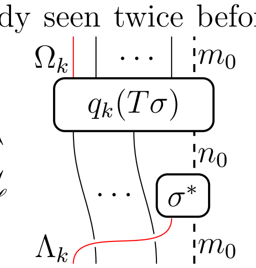
To prove Equation (3.8), consider $q_k\,T\,p_k = \text{SOT-}\lim\limits_{l\to\infty} q_k\,T_{k+l}\,p_k = q_k\,T_k\,p_k = \Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}}\,p_k$. Since the right side of condition (b) is A_k-B_k-linear, so is the other side namely, q_kTp_k. Since $\{q_kTp_k\}_{k\geq 0}$ converges in SOT to T, therefore, T must be A_k-B_k-linear, and therby A_∞-B_∞-linear, and finally A-B-linear.

If T_1 is any other operator satisfying Equation (3.8), then $q_k\,(T - T_1)\,p_k = 0$. Now, p_k and q_k increase to id_H and id_G respectively. This forces T and T_1 to be identical. $\square$

Definition 3.3.7. For $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda_\bullet, \Omega_\bullet\right)$, define

$$\mathcal{PB}\left(\underline{\eta}\right) \coloneqq \text{SOT-}\lim_{k\to\infty} \Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}} \in {}_A\mathcal{L}_B\left(H,G\right) = \mathbf{vNAlg_2}\left(\mathcal{PB}\left(\Lambda_\bullet\right), \mathcal{PB}\left(\Omega_\bullet\right)\right)\ .$$

Proposition 3.3.8. *For every $T \in {}_A\mathcal{L}_B(H,G)$ and $k \geq 0$, there exists unique $\eta^{(k)} \in NT(\Lambda_k, \Omega_k)$ such that $q_k\,T\,p_k = \Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}}\,p_k$ (which is the same as Equation (3.8)).*

Proof. We will use a modified version of a trick which we have already seen twice before, namely, in the proofs of Lemma 3.1.7 and Theorem 3.1.10. Set $\zeta_k \coloneqq \sum\limits_{\sigma\in\mathscr{S}}$  $\in$

$\mathcal{N}_k\left(\Lambda_k\Gamma_k\cdots\Gamma_1 m_0, \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$. With similar reasoning as before, one can easily conclude $q_k\,T\,p_k = \Phi_{\zeta_k}\,p_k$; moreover, this equation uniquely determines ζ_k by Lemma 3.1.7 (iii). Further, the left side of the equation is A_k-linear; then so is the right side. Again by Lemma 3.1.7 (iii), ζ_k becomes an A_k-central vector of $\mathcal{N}_k\left(\Lambda_k\Gamma_k\cdots\Gamma_1 m_0, \Omega_k\Gamma_k\cdots\Gamma_1 m_0\right)$. Applying Lemma 3.1.8, we get a unique $\eta^{(k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)$ satisfying $\zeta_k = \eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}$. This completes the proof. $\square$

Theorem 3.3.9. *The following is an isomorphism*

$$\mathbf{UC}_2^{\mathrm{tr}}\left(\Lambda_\bullet, \Omega_\bullet\right) \ni \underline{\eta} \overset{\mathcal{PB}}{\longmapsto} \mathcal{PB}\left(\underline{\eta}\right) \in \mathbf{vNAlg_2}\left(\mathcal{PB}\left(\Lambda_\bullet\right), \mathcal{PB}\left(\Omega_\bullet\right)\right)\ .$$

(This will eventually imply that the 2-functor $\mathcal{PB}$ is fully faithful.)

Proof. Suppose $\mathcal{PB}\left(\underline{\eta}\right) = 0$. Then, by Equation (3.8), we have $\left.\Phi_{\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}}\right|_{H_k} = 0$ which (by Lemma 3.1.7 (iii)) implies $\eta^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0} = 0$. Now, Lemma 3.1.8 ensures that $\eta^{(k)}$ must be zero for all k.

For surjectivity, pick $T \in {}_A\mathcal{L}_B(H,G)$. We only need to show that the unique sequence $\left\{\eta^{(k)} \in \mathrm{NT}(\Lambda_k, \Omega_k)\right\}_{k\geq 0}$ associated to T obtained in Proposition 3.3.8, is quasi-flat and bounded in C*-norm. Note that for all $\gamma \in H_k = \mathcal{N}_k(\Delta_k \cdots \Delta_1 n_0, \Lambda_k \Gamma_k \cdots \Gamma_1 m_0)$, we apply Equation (3.8) twice and obtain

$$\Phi_{\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}} \gamma = q_k T p_k \ \gamma = q_k q_{k+1} T p_{k+1} \ \gamma = q_k \Phi_{\eta^{(k+1)}_{\Gamma_{k+1} \cdots \Gamma_1 m_0}} p_{k+1} \gamma = \Phi_{\left(S_{k+1} \ \eta^{(k+1)}\right)_{\Gamma_k \cdots \Gamma_1 m_0}} \gamma$$

where the last equality follows from Proposition 3.3.6 (a). By Lemma 3.1.7 (iii), we must have $\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0} = \left[S \ \eta^{(k+1)}\right]_{\Gamma_k \cdots \Gamma_1 m_0}$ which via the isomorphism in Lemma 3.1.8, implies $\eta^{(k)} = S_{k+1} \ \eta^{(k+1)}$. For boundedness, we apply the norm on both sides of Equation (3.8); note that the map in Lemma 3.1.7 (iii) is actually an isometry (with respect to the C*-norms) which yeilds the inequality $\|T\| \geq \left\|\eta^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}\right\| = \left\|\eta^{(k)}\right\|$ where the last equality holds because $\Gamma_k \cdots \Gamma_1 m_0$ contains every simple of $\mathcal{M}_k$ as a subobject. □

3.3.4 $\mathcal{PB}$ preserves tensor product of 1-cells and compostions of 2-cells

Our goal here is clear from the title of this section. In addition, we will answer the questions in Remark 3.2.7. Note that vertical composition of 2-cells in $\mathbf{UC}^{\text{tr}}$ is denoted by $\cdot$, while that in **vNAlg**, is denoted by $\circ$.

Proposition 3.3.10. *For $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}^{\text{tr}}_2 \left(\Lambda_\bullet, \Omega_\bullet\right)$ and $\underline{\kappa} = \left\{\kappa^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}^{\text{tr}}_2 \left(\Omega_\bullet, \Xi_\bullet\right)$,*

(a) the sequence $\left\{S_{k+1} \cdots S_{k+l} \left(\kappa^{(k+l)} \circ \eta^{(k+l)}\right)\right\}_{l\geq 0}$ converges (in $NT(\Lambda_k, \Xi_k)$) for every $k \geq 0$, and

(b) $\mathcal{PB}\left(\underline{\kappa} \cdot \underline{\eta}\right) = \mathcal{PB}\left(\underline{\kappa}\right) \circ \mathcal{PB}\left(\underline{\eta}\right)$.

Proof. We continue using the previous notations and let us denote the Hilbert spaces and the projection corresponding to $\Xi_\bullet$ by F_k, F and s_k; the intertwiners corresponding to $\underline{\eta}$ and $\underline{\kappa}$ will be denoted by $X \in {}_A\mathcal{L}_B(H, G)$ and $Y \in {}_A\mathcal{L}_B(G, F)$ respectively. Set $Z := YX \in {}_A\mathcal{L}_B(H, F)$ whose corresponding BQFS will be $\left\{\psi^{(k)}\right\}_{k\geq 0}$.

For fixed $k \geq 0$, using Proposition 3.3.8 and Proposition 3.3.6 (a), we obtain

$$\begin{aligned}
&\Phi_{\psi^{(k)}_{\Gamma_k \cdots \Gamma_1 m_0}} p_k = s_k Y X p_k = \text{SOT-}\lim_{l\to\infty} s_k Y q_{k+l} X p_k \\
&= \text{SOT-}\lim_{l\to\infty} s_k \Phi_{\kappa^{(k+l)}_{\Gamma_{k+l} \cdots \Gamma_1 m_0}} \Phi_{\eta^{(k+l)}_{\Gamma_{k+l} \cdots \Gamma_1 m_0}} p_k = \text{SOT-}\lim_{l\to\infty} \Phi_{\left[S_{k+1} \cdots S_{k+l}\left(\kappa^{(k+l)} \circ \eta^{(k+l)}\right)\right]_{\Gamma_k \cdots \Gamma_1 m_0}} p_k
\end{aligned}$$

Since $\mathcal{N}_k \left(\Lambda_k \Gamma_k \cdots \Gamma_1 m_0, \Xi_k \Gamma_k \cdots \Gamma_1 m_0\right)$ is finite dimensional, by the isometry in Lemma 3.1.7(iii), we may conclude that $\left[S_{k+1} \cdots S_{k+l} \left(\kappa^{(k+l)} \circ \eta^{(k+l)}\right)\right]_{\Gamma_k \cdots \Gamma_1 m_0}$ converges as l approaches ∞ which again implies convergence of $\left\{S_{k+1} \cdots S_{k+l} \left(\kappa^{(k+l)} \circ \eta^{(k+l)}\right)\right\}_{l\geq 0}$ via Lemma 3.1.8. □

Remark 3.3.11. Proposition 3.3.10 (a) immediately implies that the limits appearing in Definition 3.2.6 (a) exists and the sequence is indeed a BQFS. Further, by Proposition 3.3.10 (b) and Theorem 3.3.9, we get associativity of vertical composition of 2-cells in $\mathbf{UC}^{\text{tr}}$.

Next, we deal with tensor product of 1-cells. We will show that $\mathcal{PB}$ preserves it in the reverse order. Note that the tensor product of 1-cells and horizontal composition of 2-cells in $\mathbf{UC}^{\mathrm{tr}}$ is denoted by $\boxtimes$, while that in **vNAlg** (namely, the Connes fusion), will be denoted by $\otimes$.

Proposition 3.3.12. *For 0-cells $\Gamma_\bullet, \Delta_\bullet, \Sigma_\bullet$ in $\mathbf{UC}_0^{\mathrm{tr}}$, and $\Omega_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}(\Delta_\bullet, \Sigma_\bullet)$ and $\Lambda_\bullet \in \mathbf{UC}_1^{\mathrm{tr}}(\Gamma_\bullet, \Delta_\bullet)$, the bimodule $\mathcal{PB}(\Omega_\bullet \boxtimes \Lambda_\bullet)$ is isomorphic to the Connes fusion $\mathcal{PB}(\Lambda_\bullet) \otimes \mathcal{PB}(\Omega_\bullet)$.*

Proof. We borrow the notations in Proposition 3.1.12 where we have already obtained an A_∞-C_∞-linear isomorphism from the relative tensor product of the dense subspace of $\mathcal{PB}(\Lambda_\bullet)$ and $\mathcal{PB}(\Omega_\bullet)$ to that of $\mathcal{PB}(\Omega_\bullet \boxtimes \Lambda_\bullet)$. Moreover, this isomorphism preserves the right C_∞-valued inner product; composing with Tr_{C_∞}, it preserves the scalar inner product as well, and hence extends to a A-C-linear unitary. $\square$

Let us denote this unitary implementing the isomorphism in Proposition 3.3.12 by $U_{\Omega_\bullet,\Lambda_\bullet} : \mathcal{PB}(\Lambda_\bullet) \otimes \mathcal{PB}(\Omega_\bullet) \longrightarrow \mathcal{PB}(\Omega_\bullet \boxtimes \Lambda_\bullet)$. We next proceed towards the horizontal composition of 2-cells.

Proposition 3.3.13. *Let $\Lambda_\bullet^i \in \mathbf{UC}_1^{\mathrm{tr}}(\Gamma_\bullet, \Delta_\bullet)$ and $\Omega_\bullet^i \in \mathbf{UC}_1^{\mathrm{tr}}(\Delta_\bullet, \Sigma_\bullet))$ for $i = 1, 2$, and $\underline{\eta} = \left\{\eta^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}(\Lambda_\bullet^1, \Lambda_\bullet^2)$ and $\underline{\kappa} = \left\{\kappa^{(k)}\right\}_{k\geq 0} \in \mathbf{UC}_2^{\mathrm{tr}}(\Omega_\bullet^1, \Omega_\bullet^2)$. Then,*

(a) for each $k \geq 0$, the sequence $\left\{S_{k+1} \cdots S_{k+l}\left(\Omega_{k+l}^2\left(\eta^{(k+l)}\right) \circ \kappa^{(k+l)}_{\Lambda^1_{k+l}}\right)\right\}_{l\geq 0}$ converges in $NT\left(\Omega_k^1\Lambda_k^1, \Omega_k^2\Lambda_k^2\right)$ where S is the loop operator from $\Omega_\bullet^1 \boxtimes \Lambda_\bullet^1$ to $\Omega_\bullet^2 \boxtimes \Lambda_\bullet^2$, and indeed the sequence $\underline{\kappa} \boxtimes \underline{\eta} := \left\{\left(\underline{\kappa} \boxtimes \underline{\eta}\right)^{(k)} := \lim\limits_{l\to\infty} S_{k+1} \cdots S_{k+l}\left(\Omega_{k+l}^2\left(\eta^{(k+l)}\right) \circ \kappa^{(k+l)}_{\Lambda^1_{k+l}}\right)\right\}_{k\geq 0}$ is a BQFS from $\Omega_\bullet^1 \boxtimes \Lambda_\bullet^1$ to $\Omega_\bullet^2 \boxtimes \Lambda_\bullet^2$,

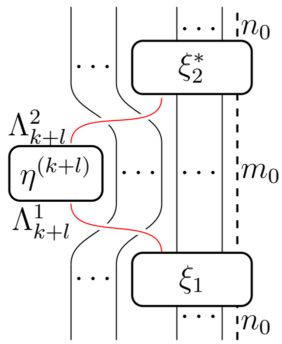
(b) $\mathcal{PB}\left(\underline{\kappa} \boxtimes \underline{\eta}\right) = U_{\Omega_\bullet^2,\Lambda_\bullet^2}\left[\mathcal{PB}\left(\underline{\eta}\right) \otimes \mathcal{PB}\left(\underline{\kappa}\right)\right] U^*_{\Omega_\bullet^1,\Lambda_\bullet^1}$.

Proof. Continuing with the notations used in Proposition 3.1.12, we set ${}_AH_B^i := \mathcal{PB}(\Lambda_\bullet^i)$, ${}_BG_C^i := \mathcal{PB}(\Omega_\bullet^i)$, ${}_AF_C^i := \mathcal{PB}(\Omega_\bullet^i \boxtimes \Lambda_\bullet^i)$ for $i = 1, 2$.
Set $T := \mathcal{PB}(\underline{\eta}) \in {}_A\mathcal{L}_B(H^1, H^2)$ and $T' := \mathcal{PB}(\underline{\kappa}) \in {}_B\mathcal{L}_C(G^1, G^2)$. Suppose X denote the intertwiner in ${}_A\mathcal{L}_C(F^1, F^2)$ induced by $T \underset{B}{\otimes} T'$ under the isomorphism in Proposition 3.3.12. We need to prove that $\underline{\kappa} \boxtimes \underline{\eta}$ is the unique BQFS which gets mapped to X under the functor $\mathcal{PB}$. For $\xi_i \in H_k^i$, $\zeta_i \in G_k^i$, $i = 1, 2$, $k \geq 0$, applying Proposition 3.3.8 and using the isomorphism of bimodules in Proposition 3.3.12 (in fact, Proposition 3.1.12), we get

$$I := \left\langle \Phi_{\left(\underline{\kappa}\boxtimes\underline{\eta}\right)^{(k)}_{\Gamma_k\cdots\Gamma_1 m_0}} \left(\Omega_k^1(\xi_1) \circ \zeta_1\right) \ , \ \Omega_k^2(\xi_2) \circ \zeta_2 \right\rangle_{F_k^2} = \left\langle X\left(\Omega_k^1(\xi_1) \circ \zeta_1\right) \ , \ \Omega_k^2(\xi_2) \circ \zeta_2 \right\rangle_{F^2}$$
$$= \left\langle \langle T\xi_1, \xi_2\rangle_B \ T'\zeta_1 \ , \ \zeta_2 \right\rangle_{G^2} \ .$$

We will now express $\langle T\xi_1, \xi_2\rangle_B$ as a limit. Consider the sequence

$$\left\{ b_l := \right.$$

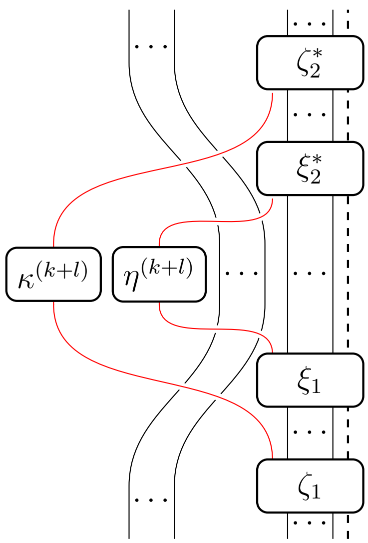


$$\left. \in B_{k+l} \subset B \right\}_{l\geq 0} .$$

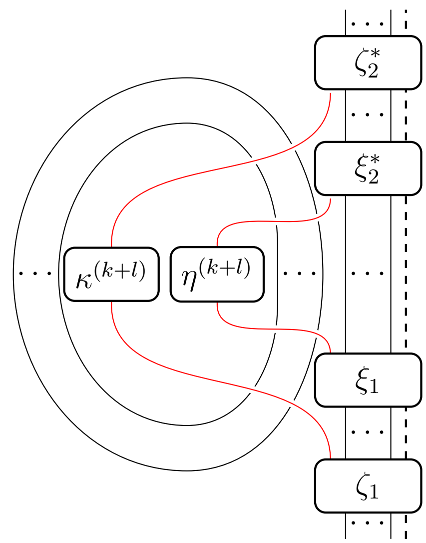
Observe that $\langle b_l\, b' \ ,\ b''\rangle_{L^2(B)}$ eventually becomes $\langle\langle T\xi_1, \xi_2\rangle_B\ b' \ ,\ b''\rangle_{L^2(B)}$ as l grows bigger for $b', b'' \in B_\infty$. Since $\{b_l\}_{l\geq 0}$ is bounded, it converges ultraweakly to $\langle T\xi_1, \xi_2\rangle_B$. Thus,

$$I = \lim_{l\to\infty} \langle T'(b_l\,\zeta_1)\ ,\ \zeta_2\rangle_{G^2} = \lim_{l\to\infty} \left\langle \tilde{\Phi}_{\kappa^{(k+l)}_{\Sigma_{k+l}\cdots\Sigma_1 q_0}} (b_l\,\zeta_1)\ ,\ \zeta_2 \right\rangle_{G^2_{k+l}}$$

$$= \lim_{l\to\infty} \mathrm{Tr}_{C_{k+l}}$$



$$= \lim_{l\to\infty} \mathrm{Tr}_{C_k}$$



$$= \lim_{l\to\infty} \left\langle \Phi_{\left[S_{k+1}\cdots S_{k+l}\left(\Omega^2_{k+l}\left(\eta^{(k+l)}\right)\circ\kappa^{(k+l)}_{\Lambda^1_{k+l}}\right)\right]_{\Gamma_k\cdots\Gamma_1 m_0}} \left(\Omega^1_k(\xi_1)\circ\zeta_1\right)\ ,\ \Omega^2_k(\xi_2)\circ\zeta_2 \right\rangle_{F^2_k} .$$

Since $NT\left(\Omega^1_k\Lambda^1_k, \Omega^2_k\Lambda^2_k\right)$ has finite dimension and sits injectively in $\mathcal{L}(F^1_k, F^2_k)$ via $\Phi_\bullet|_{F^1_k}$, the limit in part (a) indeed converges. The rest is already taken care by the construction. $\square$

Corollary 3.3.14. $\mathbf{UC}^{\mathrm{tr}}$ *is a 2-category.*

Proof. Proposition 3.3.13 (a) implies that the limit in Definition 3.2.6 (b) exists, and thereby, the corresponding sequence turns out to be a BQFS. Next, we need to establish the compatibility of horizontal and vertical compositions of 2-cells in $\mathbf{UC}^{\mathrm{tr}}$. For a 1-cell $\Lambda_\bullet$, it is easy to detect that the identity 2-cell $1_{\Lambda_\bullet} \in \mathbf{UC}^{\mathrm{tr}}_2(\Lambda_\bullet, \Lambda_\bullet)$ is given by the sequence $\{1_{\Lambda_k}\}_{k\geq 0}$.

By the proof of Proposition 3.3.6, observe that $\mathcal{PB}(1_{\Lambda_\bullet}) = \mathrm{id}_{\mathcal{PB}(\Lambda_\bullet)}$. Given composable 1-cells $\Lambda^1_\bullet, \Lambda^2_\bullet, \Omega^1_\bullet, \Omega^2_\bullet$ and 2-cells $\underline{\eta} \in \mathbf{UC}^{\mathrm{tr}}_2(\Lambda^1_\bullet, \Lambda^2_\bullet)$, $\underline{\kappa} \in \mathbf{UC}^{\mathrm{tr}}_2(\Omega^1_\bullet, \Omega^2_\bullet)$ as considered in the hypothesis of Proposition 3.3.13, we have

$$\begin{aligned}
\mathcal{PB}\left(\underline{\kappa} \boxtimes \underline{\eta}\right) &= U_{\Omega^2_\bullet,\Lambda^2_\bullet}\left[\mathcal{PB}\left(\underline{\eta}\right) \otimes \mathcal{PB}\left(\underline{\kappa}\right)\right] U^*_{\Omega^1_\bullet,\Lambda^1_\bullet} \quad \text{(by Proposition 3.3.13 (b))}\\
&= U_{\Omega^2_\bullet,\Lambda^2_\bullet}\left[\left(\mathcal{PB}\left(\underline{\eta}\right) \otimes \mathrm{id}_{\mathcal{PB}(\Omega^2_\bullet)}\right) \circ \left(\mathrm{id}_{\mathcal{PB}(\Lambda^1_\bullet)} \otimes \mathcal{PB}\left(\underline{\kappa}\right)\right)\right] U^*_{\Omega^1_\bullet,\Lambda^1_\bullet}\\
&= U_{\Omega^2_\bullet,\Lambda^2_\bullet}\left[\left(\mathcal{PB}\left(\underline{\eta}\right) \otimes \mathcal{PB}\left(1_{\Omega^2_\bullet}\right)\right)\right] U^*_{\Omega^2_\bullet,\Lambda^1_\bullet}\, U_{\Omega^2_\bullet,\Lambda^1_\bullet}\left[\left(\mathcal{PB}\left(1_{\Lambda^1_\bullet}\right) \otimes \mathcal{PB}\left(\underline{\kappa}\right)\right)\right] U^*_{\Omega^1_\bullet,\Lambda^1_\bullet}\\
&= \mathcal{PB}(1_{\Omega^2_\bullet} \boxtimes \underline{\eta}) \circ \mathcal{PB}(\underline{\kappa} \boxtimes 1_{\Lambda^1_\bullet}) \quad \text{(using Proposition 3.3.13 (b) again)}\\
&= \mathcal{PB}\left((1_{\Omega^2_\bullet} \boxtimes \underline{\eta}) \cdot (\underline{\kappa} \boxtimes 1_{\Lambda^1_\bullet})\right) \quad \text{(by Proposition 3.3.10 (b))}
\end{aligned}$$

Applying Theorem 3.3.9, we get $\underline{\kappa} \boxtimes \underline{\eta} = (1_{\Omega^2_\bullet} \boxtimes \underline{\eta}) \cdot (\underline{\kappa} \boxtimes 1_{\Lambda^1_\bullet})$, and similarly, $\underline{\kappa} \boxtimes \underline{\eta} = (\underline{\kappa} \boxtimes 1_{\Lambda^2_\bullet}) \cdot (1_{\Omega^1_\bullet} \boxtimes \underline{\eta})$.

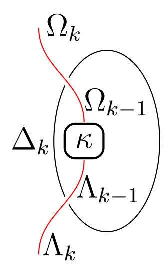
Finally, (a) the unit object corresponding to each 0-cell in $\mathbf{UC}^{\mathrm{tr}}$ is induced by the identity functor and trivial unitary connection at each level, and (b) the associativity and unit constraints are both trivial, that is, strictness is built-in. This concludes the corollary. $\square$

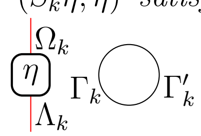
3.4 Flatness

We have seen that a BQFS depends solely on the loop operator. In order to understand when a BQFS turns out to be flat, analyzing the loop operator becomes crucial. We take on this job next.

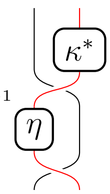
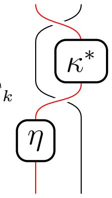
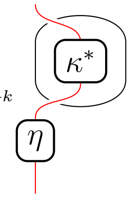
Let $\Lambda_\bullet$ and $\Omega_\bullet$ be two 1-cells from the 0-cell $\Gamma_\bullet$ to $\Delta_\bullet$ in $\mathbf{UC}^{\mathrm{tr}}$. We will work with the adjoints of the functors Γ_k's, Δ_k's, Λ_k's, and solution to conjugate equations commensurate with the given weight functions associated to the objects in WSSC*Cat (defined in Section 2.3.3).

Proposition 3.4.1. *(a) If the spaces* $\mathrm{NT}\left(\Lambda_k, \Omega_k\right)$ *and* $\mathrm{NT}\left(\Lambda_{k-1}, \Omega_{k-1}\right)$ *are equipped with the inner product induced by the categorical traces* Tr^{Λ_k} *and* $\mathrm{Tr}^{\Lambda_{k-1}}$ *(as defined in Proposition 2.3.3(a)) respectively, then the adjoint of the loop operator* $S_k : \mathrm{NT}\left(\Lambda_k, \Omega_k\right) \longrightarrow \mathrm{NT}\left(\Lambda_{k-1}, \Omega_{k-1}\right)$ *is given by*

$\mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1}) \ni \kappa \xmapsto{S^*_k}$  $\in \mathrm{NT}(\Lambda_k, \Omega_k)$.

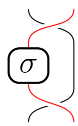
(b) For $\eta \in \mathrm{NT}\left(\Lambda_k, \Omega_k\right)$*, the pair* $(S_k\eta, \eta)$ *satisfy the exchange relation (as in Definition 3.1.4) if and only if* $S^*_k S_k \eta =$  *where* Γ'_k *is an adjoint of* Γ_k *and the loop is the natural transformation from* $\mathrm{id}_{\mathcal{M}_k}$ *to* $\mathrm{id}_{\mathcal{M}_k}$ *coming from the solution to the conjugate equation commensurate with* $\left(\underline{\mu}^{k-1}, \underline{\mu}^k\right)$*. (cf. [J21, Theorem 2.11.8])*

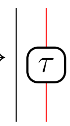
Proof. (a) Using Proposition 2.3.3 multiple times, the inner product $\langle S_k\eta, \kappa\rangle$ turns out to be

$= \mathrm{Tr}^{\Delta_{k-1}\Lambda_{k-1}}$  $= \mathrm{Tr}^{\Lambda_k\Gamma_k}$  $= \mathrm{Tr}^{\Lambda_k}$  $= \langle \eta, S_k^*\kappa\rangle$

where $\eta \in \mathrm{NT}(\Lambda_k, \Omega_k)$ and $\kappa \in \mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1})$.

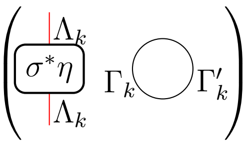
(b) The 'only if' part easily follows from the pictorial relations.
if part: Consider the maps

$\mathrm{NT}(\Lambda_k, \Omega_k) \ni \sigma \xmapsto{f}$  $\in \mathrm{NT}(\Delta_k\Lambda_{k-1}, \Delta_k\Omega_{k-1})$

$\mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1}) \ni \tau \xmapsto{g}$  $\in \mathrm{NT}(\Delta_k\Lambda_{k-1}, \Delta_k\Omega_{k-1})$

and the subspace Q of $\mathrm{NT}(\Delta_k\Lambda_{k-1}, \Delta_k\Omega_{k-1})$ generated by the ranges of f and g. Let η satisfy the hypothesis. We need to establish the equation $f(\eta) = g(S_k\eta)$. It is enough to show that $\langle f(\eta), \chi\rangle = \langle g(S_k\eta), \chi\rangle$ for all $\chi \in Q$ where the inner product is induced by $\mathrm{Tr}^{\Delta_k\Lambda_{k-1}}$.

For $\sigma \in \mathrm{NT}(\Lambda_k, \Omega_k)$, we get (from the 'categorical trace' property) $\langle f(\eta), f(\sigma)\rangle = \mathrm{Tr}^{\Lambda_k\Gamma_k}([\sigma^*\eta]_{\Gamma_k})$ which by Proposition 2.3.3(b) becomes

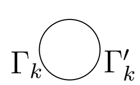
$\mathrm{Tr}^{\Lambda_k}\Bigg($  $\Bigg) = \mathrm{Tr}^{\Lambda_k}(\sigma^* \ S_k^* S_k(\eta)) = \mathrm{Tr}^{\Lambda_{k-1}}(S_k(\sigma^*) \ \ S_k(\eta))$

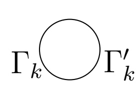
where the last equality follows from part (a). Applying Proposition 2.3.3(b) and categorical trace property again on the the last expression, we get $\mathrm{Tr}^{\Delta_k\Lambda_{k-1}}([f(\sigma)]^* \ g(S_k(\eta))) = \langle g(S_k(\eta)), f(\sigma)\rangle$.

For $\tau \in \mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1})$, we use Proposition 2.3.3(b) and deduce

$$\langle f(\eta), g(\tau)\rangle = \mathrm{Tr}^{\Delta_k\Lambda_k}([g(\tau)]^* \ f(\eta)) = \mathrm{Tr}^{\Lambda_k}(\tau^* \ S_k(\eta)) \ .$$

which by Equation (3.6) along with Proposition 2.3.3(b) turns out to be $\langle g(S_k(\eta)), g(\tau)\rangle$. $\square$

Remark 3.4.2. Similar to Proposition 3.4.1, one can prove that for $\kappa \in \mathrm{NT}(\Lambda_{k-1}, \Omega_{k-1})$, $\Bigg(\kappa \ , \ S_k^*\kappa \odot \Big[$  $\Big]^{-1}\Bigg)$ satisfy exchange relation if and only if

$S_k\Bigg(S_k^*\kappa \ \odot \Big[$  $\Big]^{-1}\Bigg) = \kappa$

where $\odot$ stands for tensor product of natural transformations.

Remark 3.4.3. If the 0-cell $\left(\Gamma_\bullet, \underline{\mu}^\bullet\right)$ satisfy an extra condition that $\Gamma_k \bigcirc \Gamma'_k$ is trivial (that is, $\Gamma_k \underline{\mu}^{k-1} = d_k\ \underline{\mu}^k$ for some $d_k > 0$) eventually for all k, then a BQFS $\left\{\eta^{(k)}\right\}_{k\geq 0}$ is flat if and only if $\eta^{(k)}$ is an eigenvector of $S_k^* S_k$ with respect to the eigenvalue d_k eventually for all k.

3.4.1 Periodic case

In this section, we focus on a particular case where the 0- and the 1-cells are periodic in nature, and investigate whether the 2-cells are flat. To do that, we need a fact in linear algebra which could very well be standard; we give a proof nevertheless.

Definition 3.4.4. For $X \in M_n(\mathbb{C})$, a sequence $\left\{\underline{x}^{(k)}\right\}_{k\geq 0}$ in $\mathbb{C}^n$ is called *X-harmonic* if $X\underline{x}^{(k+1)} = \underline{x}^{(k)}$ for all $k \geq 0$.

Proposition 3.4.5. *If spectral radius of $X \in M_n(\mathbb{C})$ is at most 1, then the space of bounded X-harmonic sequences are spanned by elements of the form $\left\{\lambda^{-k}\underline{a}\right\}_{k\geq 0}$ where λ is an eigenvalue of X and $\underline{a}$ is a corresponding eigenvector such that $|\lambda| = 1$.*

Proof. Let $X = TYT^{-1}$ where Y is in Jordan canonical form. Note that $\left\{\underline{y}^{(k)}\right\}_{k\geq 0}$ is bounded Y-harmonic if and only if $\left\{T\underline{y}^{(k)}\right\}_{k\geq 0}$ is bounded X-harmonic. Suppose $\{p_s : 1 \leq s \leq t\}$ be projections in M_n such that $I_n = \sum\limits_{1\leq s\leq t} p_s$, and for each s, $p_s Y = Y p_s$ has exactly one nonzero Jordan block corresponding an eigen value, say, λ_s. As a result, any bounded Y-harmonic sequence $\left\{\underline{y}^{(k)}\right\}_{k\geq 0}$ splits into the sum of $\left\{p_s\, \underline{y}^{(k)}\right\}_{k\geq 0}$ which is bounded Y-harmonic as well as $(p_s\, Y)$-harmonic. So, it becomes essential to find bounded harmonic sequences for a Jordan block.

Suppose $J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda & 1 & \cdots & 0 & 0 \\ 0 & 0 & \lambda & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & 0 & \cdots & 0 & \lambda \end{bmatrix}_{m\times m} = \lambda\, I_m + N$ where $|\lambda| \leq 1$. If $\lambda = 0$, then the only J-harmonic sequence would be the zero sequence. So, let us assume $\lambda \neq 0$. Any nonzero J-harmonic sequence is of the form $\left\{J^{-k}\,\underline{x}\right\}_{k\geq 0}$ for some nonzero vector $\underline{x} \in \mathbb{C}^m$; however, it may not always be bounded as k varies. Now, $J^{-k} = \lambda^{-k} \sum\limits_{l=0}^{m-1} \binom{k+l-1}{l} \left[-\lambda^{-1}N\right]^l$ for $k \geq 1$. Fix nonzero $\underline{x} \in \mathbb{C}^m$. Set $t \coloneqq \max\{l : x_l \neq 0\}$ and $C = \max\{|x_l| : 1 \leq l \leq m\}$. Hence

$$\left[J^{-k}\underline{x}\right]_1 = \lambda^{-k} \sum_{l=0}^{t-1} \binom{k+l-1}{l} \left(-\lambda^{-1}\right)^l x_{l+1}\,.$$

Since $0 < |\lambda| \leq 1$ and $\binom{k+l-1}{l}$ increase as l increases, we have the following inequality if $t > 1$

$$\left|\left[J^{-k}\underline{x}\right]_1\right| \;\geq\; |\lambda|^{-k} \left[\binom{k+t-2}{t-1} |\lambda|^{-(t-1)}\, |x_t| \;-\; \binom{k+t-3}{t-2} |\lambda|^{-(t-2)}\, Ct\right].$$

If $t > 1$, then

$$\left|\left[J^{-k}\underline{x}\right]_1\right| \;\geq\; |\lambda|^{-(k+t-1)} \binom{k+t-3}{t-2} \left[\frac{k+t-2}{t-1}\,|x_t| \;-\; |\lambda|\,Ct\right] \to \infty \text{ as } k \to \infty\,.$$

Thus, in order to have a **bounded** J-harmonic sequence, $\underline{x}$ must be in the unique one-dimensional eigen space of J, that is, $\mathbb{C}\,\underline{e}_1$. On the other hand, if $|\lambda| < 1$ and $t = 1$, then

$$\left|\left[J^{-k}\underline{x}\right]_1\right| \;=\; |\lambda|^{-k}\,|x_1| \to \infty \text{ as } k \to \infty\,.$$

So, a nonzero bounded J-harmonic sequence exists only when $|\lambda| = 1$, and then it is a scalar multiple of $\left\{\lambda^{-k}\underline{e}_1\right\}_{k\geq 0}$.

Getting back to the matrix Y (which is in Jordan canonical form) and applying the above result, we may conclude that all bounded Y-harmonic sequences are linear combination of sequences of the form

$$\left\{\lambda^{-k}\underline{y}\right\}_{k\geq 0}$$

where λ is an eigenvalue with absolute value 1 and $\underline{y}$ is a corresponding eigenvector. By similarity, the same result holds for X too. $\square$

We are now ready to prove the main theorem in this section which is a generalized version of Ocneanu's compactness (see [EK98])

Theorem 3.4.6. *Let* $\left(\Lambda_\bullet, W^\Lambda_\bullet\right)$ *and* $\left(\Omega_\bullet, W^\Omega_\bullet\right)$ *be two* 1*-cells from the* 0*-cell* $\left(\Gamma_\bullet, \underline{\mu}^\bullet\right)$ *to* $\left(\Delta_\bullet, \underline{\nu}^\bullet\right)$ *in* $\mathbf{UC}^{\mathrm{tr}}$ *such that there exists a 'period'* $K \in \mathbb{N}$ *and a 'Perron-Frobenius (PF) value'* $d > 0$ *satisfying:*

(i) the periodic condition: Γ_k*'s,* Δ_k*'s,* Λ_k*'s,* Ω_k*'s,* W^Λ_k*'s and* W^Ω_k*'s repeat with a periodicity* K *eventually for all* k*,*

(ii) the PF condition:

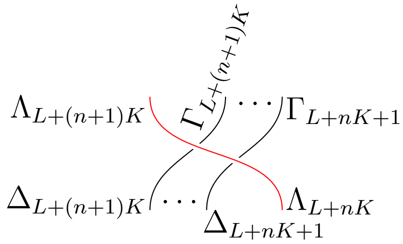
$$d^{-1}\;\Gamma_{k+K}\cdots\Gamma_{k+1}\;\underline{\mu}^k = \underline{\mu}^k = d\;\underline{\mu}^{k+K}$$
$$d^{-1}\;\Delta_{k+K}\cdots\Delta_{k+1}\;\underline{\nu}^k = \underline{\nu}^k = d\;\underline{\nu}^{k+K}$$

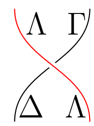
eventually for all k*.*

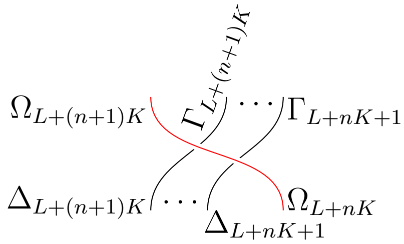
Then, all BQFS *from* $\left(\Lambda_\bullet, W^\Lambda_\bullet\right)$ *to* $\left(\Omega_\bullet, W^\Omega_\bullet\right)$ *are flat.*

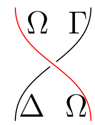
Proof. Choose a level $L \in \mathbb{N}$ large enough after which the ingredients in (i) keep repeating with periodicity K, and the equations in (ii) hold. Set
$\mathcal{M} := \mathcal{M}_L = \mathcal{M}_{L+nK}$
$\mathcal{N} := \mathcal{N}_L = \mathcal{N}_{L+nK}$
$\Gamma := \Gamma_{L+K}\cdots\Gamma_{L+1} = \Gamma_{L+(n+1)K}\cdots\Gamma_{L+nK+1} : \mathcal{M} \longrightarrow \mathcal{M}$
$\Delta := \Delta_{L+K}\cdots\Delta_{L+1} = \Delta_{L+(n+1)K}\cdots\Delta_{L+nK+1} : \mathcal{N} \longrightarrow \mathcal{N}$
$\Lambda := \Lambda_L = \Lambda_{L+nK} : \mathcal{M} \longrightarrow \mathcal{N}$

$\Omega \coloneqq \Omega_L = \Omega_{L+nK} : \mathcal{M} \longrightarrow \mathcal{N}$

$W^\Lambda \coloneqq$ 

and also denoted by 

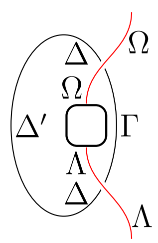
$W^\Omega \coloneqq$ 

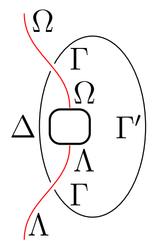
and also denoted by 

$\underline{\mu} \coloneqq \underline{\mu}^L = d^n \ \underline{\mu}^{L+nK}$
$\underline{\nu} \coloneqq \underline{\nu}^L = d^n \ \underline{\nu}^{L+nK}$
for any $n \geq 0$. Now condition (ii) and Equation (3.6) imply the following relations:

$$\Gamma \underline{\mu} = d \ \underline{\mu} = \Gamma' \underline{\mu} \ \text{ and } \ \Delta \underline{\nu} = d \ \underline{\nu} = \Delta' \underline{\nu} \ .$$

Consider the loop operators given by

$S \coloneqq d^{-1}$ 

and $S^* \coloneqq d^{-1}$ 

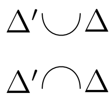
$: \mathrm{NT}(\Lambda, \Omega) \longrightarrow \mathrm{NT}(\Lambda, \Omega)$

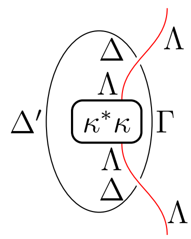
where we use tracial solution to the conjugate equation for $\Gamma \in \mathrm{End}\,(\mathcal{M})$ (resp., $\Delta \in \mathrm{End}\,(\mathcal{N})$) commensurate with the weight function $\underline{\mu}$ on $\mathcal{M}$ (resp., $\underline{\nu}$ on $\mathcal{N}$) for both source and target, and the crossings are given by $W^\Lambda, W^{\Lambda^*}, W^\Omega, W^{\Omega^*}$.

Observe that $S = S_{L+1} \cdots S_{L+K} = S_{L+nK+1} \cdots S_{L+(n+1)K}$ for all $n \geq 0$. To see this, note that an S_k in composition $[S_{L+1} \cdots S_{L+K}]$ is defined using tracial solution to conjugate equation for Δ_k commensurate with $\underline{\nu}^{k-1}$ and $\underline{\nu}^k$. So, for the composition $[S_{L+1} \cdots S_{L+K}]$, we are effectively using tracial solution to the conjugate equation for $\Delta_{L+K} \cdots \Delta_{L+1} = \Delta$ commensurate with the $\underline{\nu}^L = \underline{\nu}$ and $\underline{\nu}^{L+K} = d^{-1} \ \underline{\nu}$; let us denote this solution by $\left(\mathrm{id}_{\mathcal{N}} \overset{\rho}{\longrightarrow} \Delta\Delta' \ , \ \mathrm{id}_{\mathcal{N}} \overset{\rho'}{\longrightarrow} \Delta'\Delta \right)$. The solution to the conjugate equation for Δ commensurate with $\underline{\nu}$ for both source and target, is given by $\left(d^{-\frac{1}{2}} \rho, d^{\frac{1}{2}} \rho' \right)$. Only $d^{\frac{1}{2}} \rho'$ is used while defining S. Replacing the cap and the cup by $[d^{\frac{1}{2}} \rho']^*$ and $d^{\frac{1}{2}} \rho'$, we get the desired equation.

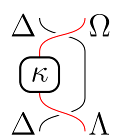
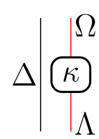
We next prove a one-to-one correspondence between bounded S-harmonic sequnces and BQFS's. Let $\left\{ \eta^{(k)} \right\}_{k \geq 0}$ be a BQFS. Clearly, $\left\{ \eta^{(L+nK)} \right\}_{n \in \mathbb{N}}$ becomes and S-harmonic sequence. Equip $\mathrm{NT}\,(\Lambda, \Omega)$ with the inner product induced by the trace Tr^Λ commensurate with $(\underline{\mu}, \underline{\nu})$. Finite dimensionality of $\mathrm{NT}(\Lambda, \Omega)$ implies that boundedness of a subset in C*-norm is equivalent to that of the 2-norm.

Conversely, let $\{\kappa_n\}_{n\in\mathbb{N}}$ be a bounded S-harmonic sequence. Set $\eta^{(k)} := S_{k+1} \cdots S_{L+nK}(\kappa_n)$ for any n such that $L + nK > k$. Indeed $\eta^{(k)}$ is well-defined and by construction $\{\eta^{(k)}\}_{k\geq 0}$ is quasi-flat. Again by finite dimensionality of $\mathrm{NT}(\Lambda, \Omega)$, $\{\kappa_n\}_{n\in\mathbb{N}}$ is bounded in C*-norm, and by Remark 3.2.2(iii), $\eta^{(k)}$'s become uniformly bounded as well.

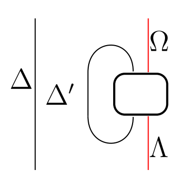
In order to apply Proposition 3.4.5 on S, it is enough to show that operator norm of S (acting on the finite dimensional Hilbert space $\mathrm{NT}(\Lambda, \Omega)$) is at most 1. Note that d^{-1}  is a projection in $\mathrm{End}(\Delta'\Delta)$ and hence less than $1_{\Delta'\Delta}$. Using this and applying Equation (2.2) multiple times, we have

$$\|S\kappa\|^2 = \mathrm{Tr}^\Lambda\left((S\kappa)^* S\kappa\right) \leq d^{-1}\mathrm{Tr}^\Lambda\left(\right.$$

$$\left.\right) = \|\kappa\|^2 .$$

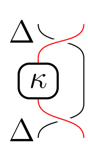
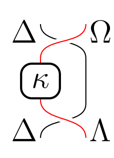
Thus, by Proposition 3.4.5, every bounded S-harmonic sequence turns out to be a linear combination of sequences of the form $\{\lambda^{-n}\kappa\}_{n\geq 0}$ where $|\lambda| = 1$ and κ is an eigenvector of S with respect to the eigenvalue λ. We will call such sequences *elementary*. We will also borrow the notion of flatness for sequences in $\mathrm{NT}(\Lambda, \Omega)$ from previous the section when every consecutive pair satisfy the exchange relation with respect to $\Gamma, \Delta, \Lambda, \Omega, W^\Lambda$ and W^Ω. Since flat sequences form a vector space, we may conclude every bounded S-harmonic sequence will be flat if all elementary ones are so. Consider the above elementary S-harmonic sequence given by λ and κ. It is enough to show


$$= \lambda$$

.

Note that both sides of the above equation belongs to the space $\mathrm{NT}(\Delta\Lambda, \Delta\Omega)$. We equip this space with inner product induced by $\mathrm{Tr}^{\Delta\Lambda}$ commensurate with $(\underline{\mu}, \underline{\nu})$. Consider the subspace $\Delta(\mathrm{NT}(\Lambda, \Omega)) \subset \mathrm{NT}(\Delta\Lambda, \Delta\Omega)$. It is routine to check that the orthogonal projection onto this subspace is given by

$$E := d^{-1}$$

$$: \mathrm{NT}(\Delta\Lambda, \Delta\Omega) \longrightarrow \Delta(\mathrm{NT}(\Lambda, \Omega))$$

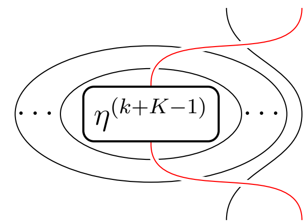
Now,

$$E\left(\right.$$

$$\left.\right) = \Delta(S\kappa) = \lambda\Delta\kappa \quad \text{and} \quad \left\|\right.$$

$$\left.\right\|_2^2 = \|\lambda\Delta\kappa\|_2^2 .$$

So, the above equation must hold.

From correspondence between bounded S-harmonic sequences and BQFS's and the flatness of the former, we may conculde that in a BQFS $\left\{\eta^{(k)}\right\}_{k\geq 0}$, two terms which are K steps apart, must satisfy exchange relation starting from level L (namely, $\eta^{(L)}, \eta^{(L+K)}, \eta^{(L+2K)}, \ldots$). Now, we have the freedom of choosing higher L's; as a result, we obtain exchange relation of any two terms which are K steps apart after level L. To establish exchange relation for consecutive terms, pick a $k > L$ and recall the maps f and g defined in the proof of Proposition 3.4.1(b). By quasi-flat property, we get

$$f\left(\eta^{(k)}\right) = f\left(S_{k+1} \cdots S_{k+K-1}\left(\eta^{(k+K-1)}\right)\right) =$$



which (by Equation (3.6), unitarity of the connection and the K-step exchange relation) turns out to be $g\left(\eta^{(k-1)}\right)$. Hence, $\left(\eta^{(k-1)}, \eta^{(k)}\right)$ satisfies the exchange relation. This ends the proof. $\square$

Chapter 4

Examples

4.1 Subfactors

Subfactors, more specifically, their standard invariants constitute the initial source of examples generalizing which we arrived at our objects of interest, namely, **UC** and $\mathbf{UC}^{\text{tr}}$. Basically, we associate a 1-cell in $\mathbf{UC}^{\text{tr}}$ to the subfactor which captures all the information of the associated planar algebra.

Let ${}_C X_D$ be an extremal bifinite bimodule over II_1 factors C, D. For any bifinite bimodule ${}_A Y_B$, let $\langle {}_A Y_B \rangle$ denote the category of bifinite A-B-bimodules which are direct sum of irreducibles appearing in ${}_A Y_B$. Set $\mathcal{M}_0 \coloneqq \mathcal{H}ilb_{fd}$ (the category of finite dimesnional Hilbert spaces), and for $k \geq 0$, let

$$\mathcal{M}_{2k+1} \coloneqq \left\langle {}_D \left(\overline{X} \underset{C}{\otimes} X \right)^{\underset{D}{\otimes} k}_D \right\rangle \text{ and } \mathcal{M}_{2k+2} \coloneqq \left\langle {}_D \left(\overline{X} \underset{C}{\otimes} X \right)^{\underset{D}{\otimes} k} \underset{D}{\otimes} \overline{X}_C \right\rangle$$

and functors $\Gamma_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ be defined by

$$\text{ob}(\mathcal{M}_0) \ni \mathbb{C} \overset{\Gamma_1}{\longmapsto} {}_D L^2(D)_D \in \text{ob}(\mathcal{M}_1), \quad \Gamma_{2k+2} \coloneqq \bullet \underset{D}{\otimes} \overline{X} \Big|_{\mathcal{M}_{2k+1}} \text{ and } \Gamma_{2k+3} \coloneqq \bullet \underset{C}{\otimes} X \Big|_{\mathcal{M}_{2k+2}}$$

. The sequence $\Gamma_\bullet$ will serve as the source 0-cell in **UC**. For the target 0-cell in **UC**, define

$$\mathcal{N}_{2k} \coloneqq \left\langle {}_C \left(X \underset{D}{\otimes} \overline{X} \right)^{\underset{C}{\otimes} k}_C \right\rangle \text{ and } \mathcal{N}_{2k+1} \coloneqq \left\langle {}_C \left(X \underset{D}{\otimes} \overline{X} \right)^{\underset{C}{\otimes} k} \underset{C}{\otimes} X_D \right\rangle$$

and

$$\Delta_{2k+1} \coloneqq \bullet \underset{C}{\otimes} X \Big|_{\mathcal{N}_{2k}} \text{ and } \Delta_{2k+2} \coloneqq \bullet \underset{D}{\otimes} \overline{X} \Big|_{\mathcal{N}_{2k+1}}$$

. Next, consider the 1-cell $\Lambda_\bullet$ in **UC** given by

$$\text{ob}(\mathcal{M}_0) \ni \mathbb{C} \overset{\Lambda_0}{\longmapsto} {}_C L^2(C)_C \in \text{ob}(\mathcal{N}_0) \text{ and } \Lambda_k \coloneqq X \underset{D}{\otimes} \bullet \Big|_{\mathcal{M}_k}$$

where the unitary connection for the squares

$$\begin{tikzcd}
\mathcal{N}_{2k-1} \arrow[r, "\bullet \underset{D}{\otimes} \overline{X}", "\Delta_{2k}"'] & \mathcal{N}_{2k} \arrow[r, "\bullet \underset{C}{\otimes} X", "\Delta_{2k+1}"'] & \mathcal{N}_{2k+1} \\
\mathcal{M}_{2k-1} \arrow[r, "\Gamma_{2k}", "\bullet \underset{D}{\otimes} \overline{X}"'] \arrow[u, "X \underset{D}{\otimes} \bullet", "\Lambda_{2k-1}"'] & \mathcal{M}_{2k} \arrow[r, "\Gamma_{2k+1}", "\bullet \underset{C}{\otimes} X"'] \arrow[u, "\Lambda_{2k}", "X \underset{D}{\otimes} \bullet"'] & \mathcal{M}_{2k+1} \arrow[u, "\Lambda_{2k+1}", "X \underset{D}{\otimes} \bullet"']
\end{tikzcd}$$

is induced by the associativity constraint of the bimodules.

In order to turn the **UC**-0-cells $\Gamma_\bullet$ and $\Delta_\bullet$ into $\mathbf{UC}^{\mathrm{tr}}$ ones, we work with the statistical dimension (same as the square root of the index) of an extremal bimodule ${}_A Y_B$, denoted by $d(Y)$. Set $\delta := d(X)$. Let $V_{\mathcal{M}_k}$ and $V_{\mathcal{N}_k}$ denote maximal sets of mutually non-isomorphic family of irreducible bimodules in $\mathcal{M}_k$ and $\mathcal{N}_k$ respectively.

Now we define the weight functions $\underline{\mu}^k$ and $\underline{\nu}^k$ on $V_{\mathcal{M}_k}$ and $V_{\mathcal{N}_k}$ respectively as follows:

$$\mu^0_{\mathbb{C}} := 1,\ \mu^k_Y := \delta^{-(k-1)} d(Y),\ \nu^k_Z := \delta^{-k} d(Z) \quad \text{for } Y \in V_{\mathcal{M}_k},\ Z \in V_{\mathcal{N}_k}$$

Since the dimension function is a linear homomorphism with respect to direct sum and Connes fusion of bimodules, we get that $\underline{\mu}^k$ and $\underline{\nu}^k$ satisfy Equation (3.6). For the same reason, the boundedness condition Equation (3.7) holds with the inequalities replaced by equality where both the bounds are 1.

4.1.1 Planar algebraic view of the associated bimodule

Let $A := \mathcal{PB}(\Gamma_\bullet)$ and $B := \mathcal{PB}(\Delta_\bullet)$ be the AFD von Neumann algebras and $H := \mathcal{PB}(\Lambda_\bullet)$ be the A-B-bimodule where $\Gamma_\bullet, \Delta_\bullet$ and $\Lambda_\bullet$ and 0-cells and 1-cells in $\mathbf{UC}^{\mathrm{tr}}$ associated to the extremal bifinite bimodule ${}_C X_D$. Denote the planar algebra associated to ${}_C X_D$ by $P = \{P_{\pm k}\}_{k \geq 0}$ where the vector spaces are given by

$$P_{+k} = \mathrm{End}\left(X \underset{D}{\otimes} \overline{X} \underset{C}{\otimes} \cdots \ k \text{ tensor components} \right)$$

and

$$P_{-k} = \mathrm{End}\left(\overline{X} \underset{C}{\otimes} X \underset{D}{\otimes} \cdots \ k \text{ tensor components} \right)$$

. Immediately from the definitions, we get the following.

(a) $A_{k+1} = P_{-k}$ and $B_k = P_{+k} = H_k$.

(b) Both the inclusions $B_k \hookrightarrow B_{k+1}$ and $H_k \hookrightarrow H_{k+1}$ are the same as $P_{+k} \hookrightarrow P_{+(k+1)}$, and $A_k \hookrightarrow A_{k+1}$ is same as $P_{-(k-1)} \hookrightarrow P_{-k}$, induced by the action of inclusion tangle by a string on the right.

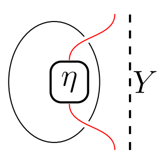
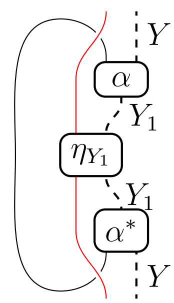
(c) Action of B_k on H_k is given by right mutiplication of P_{+k} on itself whereas that of A_k on H_k is given by the left multiplation of the left inclusion $P_{-(k-1)} \overset{\mathrm{LI}}{\hookrightarrow} P_{+k}$ induced the action of inclusion tangle by a string on the left.

(d) The trace on A_k and B_k turns out to be the normalized picture trace on $P_{-(k-1)}$ and P_{+k} respectively.

Remark 4.1.1. Let $P_{\pm\infty}$ be the union $\underset{k\geq 0}{\cup} P_{\pm k}$ of the filtered unital algebras, and $P_{\pm}$ be the von Neumann algebra generated by it acting on the GNS with respect to the canonical normalized picture trace $\mathrm{tr}_{\pm}$. Finally, the bimodule ${}_AH_B$ turns out to be the same as ${}_{P_-}L^2(P_{+\infty}, \mathrm{tr}_+)_{P_+}$ where the P_--action on left extends from treating $P_{+\infty}$ as a left-module over the subalgebra $\mathrm{LI}\,(P_{-\infty})$. As a result, the BQFS's from $\Lambda_\bullet$ to $\Lambda_\bullet$ are given by intertwiners in ${}_{P_-}\mathcal{L}_{P_+}\left(L^2\left(P_+, \mathrm{tr}_+\right)\right) = \left[\mathrm{LI}\left(P_-\right)\right]' \cap P_+$ via Theorem 3.3.9 and by Remark 3.2.4, the flat sequences correspond to elements in $\left[\mathrm{LI}\left(P_{-\infty}\right)\right]' \cap P_{+\infty}$.

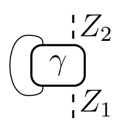
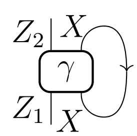
4.1.2 Loop operators and Izumi's Markov operator

We provide a description of loop operators $\{S_k : \mathrm{End}(\Lambda_k) \to \mathrm{End}(\Lambda_{k-1})\}_{k\geq 1}$ in terms of maps between intertwiner spaces. We continue to employ the graphical calculus pertaining to bimodules and intertwiners as well. Let us analyze the odd ones first. For $Y \in V_{\mathcal{M}_{2k}}$ and $\eta \in \mathrm{End}(\Lambda_{2k+1})$,

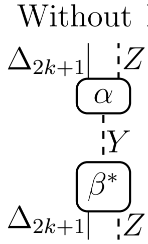
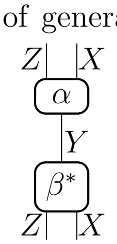
$[S_{2k+1}\eta]_Y =$  $= \displaystyle\sum_{Y_1 \in V_{\mathcal{M}_{2k+1}}} \sum_{\alpha \in \mathrm{ONB}(Y_1, \Gamma_{2k+1}Y)}$  .

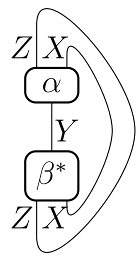
The last expression is in terms of the functors Γ_n's, Δ_n's and Λ_n's; to express it purely using bimodules and intertwiners, we prove the following relation.

Lemma 4.1.2. For $Z_1, Z_2 \in \mathrm{ob}(\mathcal{N}_{2k})$, $\gamma \in \mathcal{N}_{2k+1}(\Delta_{2k+1}Z_1, \Delta_{2k+1}Z_2)$ we have

 $= \delta^{-1}$ 

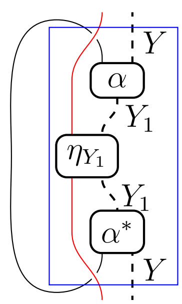
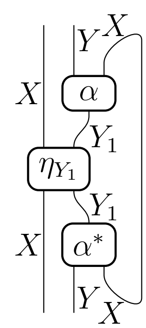
where the cap and the cup on the right (resp. left) side come from balanced spherical solution (resp. solution) to the conjugate equations for the duality of X (resp. Δ_{2k+1} commensurate with $\left(\underline{\nu}^{2k+2}, \underline{\nu}^{2k}\right)$).

Proof. Without loss of generality, we may assume $Z_1 = Z_2 = Z$ (say) is irreducible and $\gamma =$  $=$  where $Y \in V_{\mathcal{N}_{2k+1}}$. So, the right side of the equation in the statement becomes

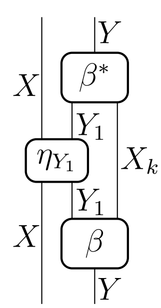
$$\delta^{-1}\,[d(Z)]^{-1}$$  $$1_Z \;=\; \frac{d(Y)}{\delta\ d(Z)}\ \langle \alpha, \beta \rangle\ 1_Z \;=\; \frac{\nu_Y^{2k}}{\nu_Z^{2k+1}}\ \langle \alpha, \beta \rangle\ 1_Z$$

where the first equality follows from traciality of spherical solutions. The last term by Equation (2.2), is same as the left side. □

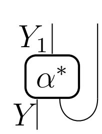
Coming back to the loop operators, we apply the lemma on the blue box below and obtain

$$[S_{2k+1}\eta]_Y \;=\; \sum_{\substack{Y_1 \in V_{\mathcal{M}_{2k+1}} \\ \alpha \in \mathrm{ONB}(Y_1, \Gamma_{2k+1} Y)}}$$  $$=\; \delta^{-1} \sum_{\substack{Y_1 \in V_{\mathcal{M}_{2k+1}} \\ \alpha \in \mathrm{ONB}\left(Y_1, Y \underset{C}{\otimes} X\right)}}$$  $$. \qquad (4.1)$$

Proposition 4.1.3. *For all $\eta \in \mathrm{End}\,(\Lambda_k)$ and $Y \in V_{\mathcal{M}_{k-1}}$, the following equation holds*

$$[S_k \eta]_Y = \sum_{\substack{Y_1 \in V_{\mathcal{M}_k} \\ \beta \in ONB(Y, Y_1 \otimes X_k)}} \frac{d(Y_1)}{\delta\ d(Y)}$$ 

where X_k is X or $\overline{X}$ according as k is even or odd.

Proof. In Equation (4.1), substituting $\beta := \left(\frac{d(Y)}{d(Y_1)}\right)^{\frac{1}{2}}$  $\overline{X}$ (which yeilds an orthonormal basis of $\mathcal{M}_{2k+1}\left(Y\,,\, Y_1 \underset{D}{\otimes} \overline{X}\right)$ as α varies over ONB $\left(Y_1, Y \underset{C}{\otimes} X\right)$), we get the desired equation for the odd case. The proof of the even case is exactly similar. □

We now recall the Markov operator (that is, a UCP map) associated to an extremal finite index subactor / bifinite bimodule defined by Izumi in [I04]. Consider the finite dimensional C*-algebra $D_k \coloneqq \operatorname{End}(\Lambda_k) \cong \underset{Y\in V_{\mathcal{M}_k}}{\oplus} {}_C\mathcal{L}_D\left(X \underset{D}{\otimes} Y\right)$ or $\underset{Y\in V_{\mathcal{M}_k}}{\oplus} {}_C\mathcal{L}_C\left(X \underset{D}{\otimes} Y\right)$ according as k is even or odd. Define the von Neumann algebra $D \coloneqq \bigoplus_{k\geq 0} D_k$. Then, Izumi's Markov operator $P : D \longrightarrow D$ is defined as

$$D \ni \underline{\eta} = \left(\eta^{(k)}\right)_{k\geq 0} \overset{P}{\longmapsto} P\underline{\eta} \coloneqq \left(S_{k+1}\eta^{(k+1)}\right)_{k\geq 0} \in D\ .$$

By [I04, Lemma 3.2], the space of P-harmonic elements $H^\infty(D, P)$ (that is, the fixed points of P) is precisely the space of bounded quasi-flat sequences corresponding to our loop operators $\{S_k\}_{k\geq 0}$.

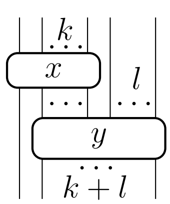
4.1.3 Temperley-Lieb - TL_δ for $\delta > 2$

Continuing with the same set up, let us further assume X is symmetrically self-dual and tensor-generates the Temperley-Lieb category for a generic modulus $\delta > 2$. This example had already been investigated extensively, in particular, by Izumi in [I04] in our context. Here, we address the question whether every UC-endormorphism of $\Lambda_\bullet$ extends to a $\mathbf{UC}^{\mathrm{tr}}$-one.

Proposition 4.1.4. *The* 1*-cell in* $\mathbf{UC}^{\mathrm{tr}}$ *corresponding to the TL-bimodule* X *possesses a BQFS to itself which is not flat.*

Proof. In [I04], Izumi showed that $H^\infty(D,P)$ (and hence $\operatorname{End}_{\mathbf{UC}^{\mathrm{tr}}}(\Lambda_\bullet)$) is 2 dimesional. So, it is enough to show that $\operatorname{End}_{\mathbf{UC}}(\Lambda_\bullet)$ is the one-dimensional space generated by the identity in it. Again, by Remark 4.1.1 and Remark 3.2.4, this boils down to showing that $[\mathrm{LI}(P_\infty)]' \cap P_\infty$ is 1-dimensional where $P = \{P_k\}_{k\geq 0}$ denotes the unshaded planar algebra associated to the symmetrically self-dual bimodule X.

Let $x \in [\mathrm{LI}(P_{-\infty})]' \cap P_{+\infty}$. Then there exists some $k \geq 0$ such that $x \in [\mathrm{LI}(P_{-\infty})]' \cap P_{+k}$, equivalently



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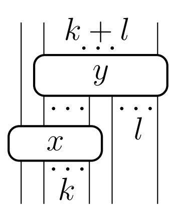
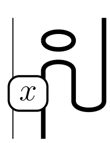
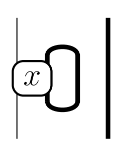


for all $y \in P_{-(k+l)}$ and $l \geq 0$. (4.2)

Using Equation (4.2) we get $x = \delta^{-k}$ 

$= \delta^{-k}$ 

$\in P_1$ where the thick line denotes k many parallel strings. Since P_1 is one-dimesnional, x must be a scalar multiple of identity. □

4.2 Directed graphs

We will discuss an example arising out of directed gaphs (where we allow multiple edges from one vertex to the other). Further, we assume the directed graphs are 'strongly connected', that is, for v, w in the vertex set, there exists a path from v to w. As a result, the corresponding adjacency matrices are irreducible and thereby, each possesses a Perron-Frobenius (PF) eigenvalue and PF eigenvectors. In terms of category and functor, it is equivalent to consider a finite semisimple category $\mathcal{M}$ and a $*$-linear functor $\Gamma \in \operatorname{End}(\mathcal{M})$ such that for simple $v, w \in \operatorname{ob}(\mathcal{M})$, there exists $k \in \mathbb{N}$ satisfying $\mathcal{M}\left(v, \Gamma^k w\right) \neq \{0\}$. From such a Γ, we build the 0-cell $\left\{\mathcal{M}_{k-1} \xrightarrow{\Gamma_k} \mathcal{M}_k\right\}_{k\geq 1}$ in $\mathbf{UC}^{\text{tr}}$ where $\mathcal{M}_k = \mathcal{M}$ and $\Gamma_k = \Gamma$ for all k, and the weight $\underline{\mu}^k$ on $\mathcal{M}_k$ is given by $d^{-k}\underline{\mu}$ where d is the PF eigenvalue of the adjacency matrix of Γ' and $\underline{\mu}$ is PF eigenvector whose sum of the coordinates is 1.

Consider the 1-cell $\Lambda_\bullet$ in $\mathbf{UC}^{\text{tr}}(\Gamma_\bullet, \Gamma_\bullet)$ by setting $\Lambda_k := \Gamma$ for $k \geq 0$, with unitary connection $W^k := 1_{\Gamma^2}$. Note that the loop operator $S_k : \operatorname{End}(\Lambda_k) \to \operatorname{End}(\Lambda_{k-1})$ is independent of k because (although the weight on $\mathcal{M}_k$ varies as k varies) our solution to conjugate equation for the duality of $\Gamma_k : \mathcal{M}_{k-1} \to \mathcal{M}_k$ is independent ; let us rename it as $S : \operatorname{End}(\Gamma) \to \operatorname{End}(\Gamma)$. More explicity, $S\eta = d^{-1}\ \boxed{\eta}\,\Big|\,\Gamma$ for all $\eta \in \operatorname{End}(\Gamma)$ where we use tracial solution to conjugate equation for Γ commensurate with $\left(\underline{\mu}, \underline{\mu}\right)$. Clearly, the range of S is contained in $\{\xi \bigodot 1_\Gamma : \xi \in \operatorname{End}(\mathrm{id}_\mathcal{M})\}$. Then an S-harmonic sequence $\{\xi_k \bigodot 1_\Gamma\}_{k\geq 0}$ is completely captured by a sequence $\{\xi_k\}_{k\geq 0}$ in the finite dimensional abelian C*-algebra $\operatorname{End}(\mathrm{id}_\mathcal{M})$ satisfying $d^{-1}\ \Gamma' \boxed{\xi_k} \Gamma = \xi_{k-1}$ for all $k \geq 1$. The operator $X := d^{-1}\ \Gamma' \boxed{} \Gamma : \operatorname{End}(\mathrm{id}_\mathcal{M}) \to \operatorname{End}(\mathrm{id}_\mathcal{M})$ is UCP and $\{\xi_k\}_k$ is X-harmonic. Using the categorical trace on natural transformations, the operator X has norm at most 1 and so is the spectral radius. Applying Proposition 3.4.5, the bounded X-harmonic sequences are linear span of elementary ones, namely, $\left\{c^{-k}\ \xi\right\}_{k\geq 0}$ where ξ is an eigenvector of X for the eigenvalue c such that $|c| = 1$. However, it is unclear whether such an elementary X-harmonic sequence contribute towards a flat sequence from $\Lambda_\bullet$ to $\Lambda_\bullet$; a necessary condition for this is $c\, 1_\Gamma \bigodot \xi = d\, \xi \bigodot 1_\Gamma$. A straight forward deduction from this condition will tell us that flat sequences are simple scalar multiples of the identity.

In the above example, if we would have started with a finite connected undirected graph Γ, then by Theorem 3.4.6, all BQFS from $\Lambda_\bullet$ to $\Lambda_\bullet$ would have been flat.

4.2.1 Vertex models

Let $\mathcal{M}$ be the category of finite dimensional Hilbert spaces, and $\Gamma := \mathrm{id}_\mathcal{M} \otimes \ell^2(X) \in \operatorname{End}(\mathcal{M})$, $\Lambda := \mathrm{id}_\mathcal{M} \otimes \ell^2(Y) \in \operatorname{End}(\mathcal{M})$ be two functors where X, Y are some nonempty finite sets. Consider the 0-cell $\Gamma_\bullet$ in $\mathbf{UC}^{\text{tr}}$ defined by $\mathcal{M}_k := \mathcal{M}$ and $\Gamma_k = \Gamma$ for all k where the weight of $\mathbb{C}$ in $\mathcal{M}_k$ is $|X|^{-k}$. For a 1-cell in $\mathbf{UC}^{\text{tr}}(\Gamma_\bullet, \Gamma_\bullet)$, we consider $\{\Lambda_k := \Lambda\}_{k\geq 0}$ with the unitary connections $W^k := \mathrm{id}_\bullet \otimes UF$ where $U : \ell^2(X) \otimes \ell^2(Y) \to \ell^2(X) \otimes \ell^2(Y)$ is a unitary and $F : \ell^2(Y) \otimes \ell^2(X) \to \ell^2(X) \otimes \ell^2(Y)$ is canonical flip map. Note that $\operatorname{End}(\Lambda) \cong M_Y(\mathbb{C})$

and the loop operators are independent of k; let us denote it by $S : M_Y(\mathbb{C}) \to M_Y(\mathbb{C})$. One can deduce the following two formula,

$$(S\eta)_{y,y'} = |X|^{-1} \sum_{\substack{x_1,x_2\in X\\ y_1,y_2\in Y}} \overline{U^{x_2y_2}_{x_1y}}\ \eta_{y_2y_1}\ U^{x_2y_1}_{x_1y'} \quad \text{and} \quad (S^*\eta)_{y,y'} = |X|^{-1} \sum_{\substack{x_1,x_2\in X\\ y_1,y_2\in Y}} U^{x_1y}_{x_2y_2}\ \eta_{y_2y_1}\ \overline{U^{x_1y'}_{x_2y_1}}$$

for all $\eta \in M_Y(\mathbb{C})$ and $y, y' \in Y$. By Theorem 3.4.6, every BQFS from $\Lambda_\bullet$ to $\Lambda_\bullet$ becomes flat.

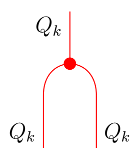
Chapter 5

Q-system completeness of UC

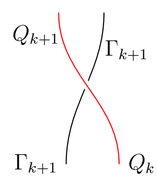
5.1 Q-system in UC

In this section, given a Q-system in **UC** for a 0-cell, we explore certain structural properties of the associated bimodules that will further enable us to construct new 0-cells and a new dualizable 1-cell in the next section, that will implement Q-system completion of **UC**.

Let $(\Gamma_\bullet, \mathcal{M}_\bullet)$ be a 0-cell in **UC** and $(Q_\bullet, W^Q_\bullet, m_\bullet, i_\bullet)$ be a Q-system in $\mathbf{UC}_1((\Gamma_\bullet, \mathcal{M}_\bullet), (\Gamma_\bullet, \mathcal{M}_\bullet))$. Graphically, each natural transformation m_k, i_k and W^Q_{k+1} will be represented by the following respective diagrams:

$m_k :=$ 

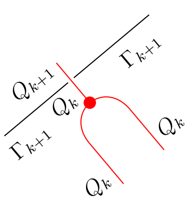
, $i_k :=$ 

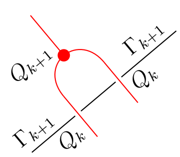
and $W^Q_{k+1} :=$ 

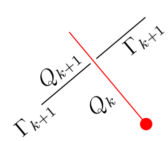
$\forall k \geq 0$

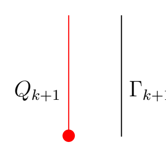
.

Pictorially, exchange relation of m_k's and i_k's with respect to $W_\bullet$ will be denoted as follows:



$=$ 

and 

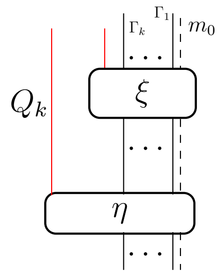
$=$ 

eventually for all k.

Remark 5.1.1. From Remark 3.1.17, we observe that for our Q-system $(Q_\bullet, m_\bullet, i_\bullet)$ in **UC** the natural transformations m_k and i_k satisfy (Q1)-(Q4) as in Definition 2.4.2 eventually for all k . For the rest of the thesis, we fix a natural number l such that m_k and i_k satisfy (Q1)-(Q4) and the exchange relations for $k \geq l$.

Consider the filtration of finite dimensional C*-algebras $\{A_k := \operatorname{End}(\Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 1}$ associated to the 0-cell $\Gamma_\bullet$ where m_0 is direct sum of a maximal set of mutually non-isomorphic simple objects in $\mathcal{M}_0$. Let $\{H_k \coloneqq \mathcal{M}_k(\Gamma_k \cdots \Gamma_1 m_0, Q_k \Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 1}$ be the right correspondence associated to $Q_\bullet$. By construction (3.1 and 3.2) , H_k is a right A_k-A_k correspondence. Thus, one may view H_k as a 1-cell in the 2-category **C*Alg** of right correspondence bimodules over pairs of C*-algebras. We will further establish that each H_k is a Q-system in $\mathbf{C^*Alg}(A_k, A_k)$ for $k \geq l$. In order to do this, we will use the following identification.

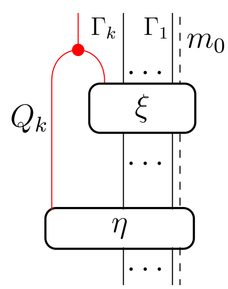
Remark 5.1.2. Let $\{Y_k \coloneqq \mathcal{M}_k(\Gamma_k \cdots \Gamma_1 m_0, Q_k^2 \Gamma_k \cdots \Gamma_1 m_0)\}_{k\geq 1}$ denote the right correspondence associated to the 1-cell $Q_\bullet \boxtimes Q_\bullet$ in **UC**. The proof of [DGGJ22, Proposition 3.12] tells us that the map $\xi \boxtimes \eta \longmapsto$



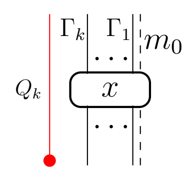
is an isomorphism between $H_k \underset{A_k}{\boxtimes} H_k$ and Y_k as right A_k-A_k correspondence.

Via the above identification, the multiplication 2-cell $m_\bullet$ and the unit 2-cell $i_\bullet$ in **UC** corresponds to the maps $\widetilde{m}_k : H_k \underset{A_k}{\boxtimes} H_k \to H_k$ and $\widetilde{i}_k : A_k \to H_k$ respectively at the level of bimodules; more explicitly

$\widetilde{m}_k(\xi \boxtimes \eta) \coloneqq$



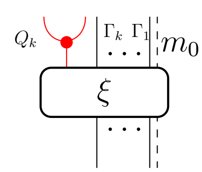
and $\widetilde{i}_k(x) \coloneqq$



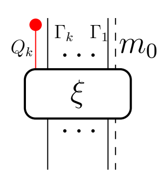
.

Proposition 5.1.3. *For each $k \geq l$, $\widetilde{m}_k$ and $\widetilde{i}_k$ are adjointable maps and hence 2-cells in* **C*Alg**. *Moreover, $(H_k, \widetilde{m}_k, \widetilde{i}_k)$ becomes a Q-system in $\mathbf{C^*Alg}(A_k, A_k)$ for each $k \geq l$.*

Proof. Using the identification in Remark 5.1.2, the adjoint of $\widetilde{m}_k$ is given by



and that of $\widetilde{i}_k$ is given by



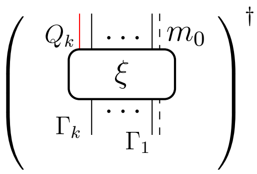
. Now using $\widetilde{m}_k$ and $\widetilde{i}_k$ and their adjoints, and the properties (Q1-Q4) of $m_\bullet$ and $i_\bullet$ mentioned at in preliminaries , associativity, unitality, frobenius property and separability of $\left(H_k, \widetilde{m}_k, \widetilde{i}_k\right)$ easily follows. □

We now explore certain structural properties of H_k.

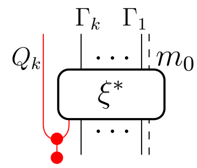
We prove the following proposition using ideas from [CPJP22].

Proposition 5.1.4. *For each $k \geq l$, the space H_k is a unital* C*-*algebra with multiplication, adjoint and unit given by*

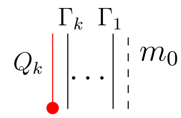
$$\xi \cdot \eta \coloneqq \widetilde{m}_k \left(\xi \boxtimes \eta \right) \quad ,$$



$=$

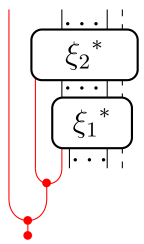
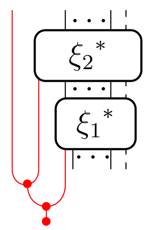
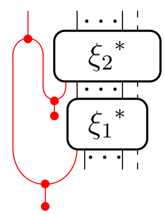


and

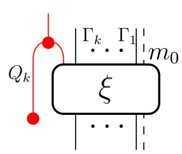
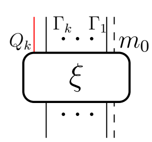
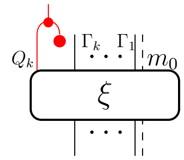


respectively for $\xi, \eta \in H_k$.

Proof. Indeed, $\xi^{\dagger\dagger} = \xi$. Again

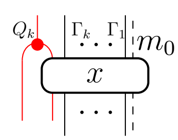
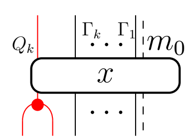
$(\xi_1 . \xi_2)^\dagger =$  $=$  $=$  $= \xi_2^\dagger . \xi_1^\dagger$

where the second equality follows from associativity and the third comes from Frobenius and unitality conditions. Also,

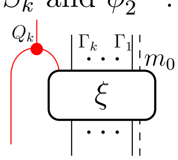
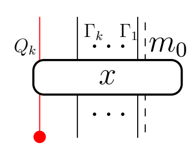
$1_{H_k} \cdot \xi =$  $=$  $=$  $= \xi \cdot 1_{H_k}.$

Hence, H_k becomes a unital *-algebra.

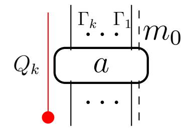
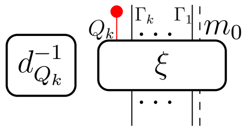
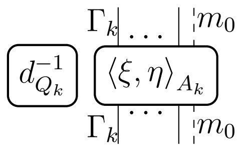
To prove that H_k is a C*-algebra, we show that it is isomorphic to a *-subalgebra of a finite dimensional C*-algebra. Define

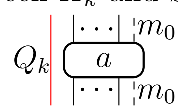
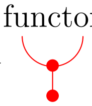
$S_k \coloneqq \Big\{ x \in \mathrm{End}(Q_k \Gamma_k \cdots \Gamma_1 m_0) \;\Big|$  $=$  $\Big\}$

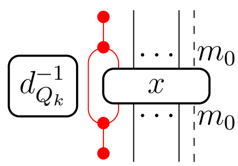
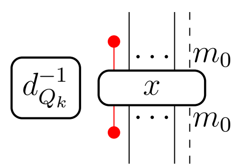
sitting inside the finite dimensional C*-algebra $\mathrm{End}(Q_k \Gamma_k \cdots \Gamma_1 m_0)$. Clearly S_k is closed under multiplication, as well as *-closed (using Frobenius property and unitality). Define $\phi_1^{(k)} : H_k \to S_k$ and $\phi_2^{(k)} : S_k \to H_k$ as follows:

$\phi_1^{(k)}(\xi) \coloneqq$  $\in S_k$ (by associativity of Q_k) and $\phi_2^{(k)}(x) \coloneqq$  .

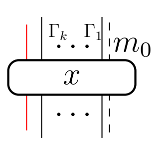
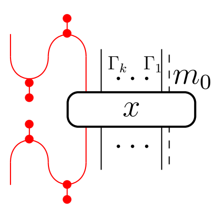
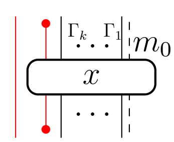
Now, it is routine to check using the axioms of Q-systems that $\phi_1^{(k)}$ and $\phi_2^{(k)}$ are unital, *-homomorphisms. Also, $\phi_1^{(k)}$ and $\phi_2^{(k)}$ are mutually inverse to each other, hence they are isomorphisms. □

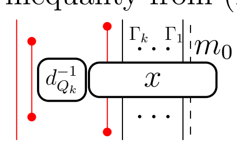
Lemma 5.1.5. *The map $\widetilde{i}_k : A_k \to H_k$ defined by $\widetilde{i}_k(a) :=$*  *is a unital inclusion of* C**-algebras. In the reverse direction, the map $E_k : H_k \to A_k$ defined by $E_k(\xi) :=$*  *(where $d_{Q_k} =$*  *) is a finite index, faithful conditional expectation satisfying $E_k\left(\eta^\dagger \cdot \xi\right) =$*  *(where $\langle\cdot,\cdot\rangle_{A_k}$ is the right A_k-valued inner product on H_k as defined in Equation (3.2)) for each $k \geq l$.*

Proof. We make use of the *-algebra isomorphisms ϕ_1, ϕ_2 between H_k and S_k and find that the map $\widetilde{i}_k : A_k \to H_k$ corresponds to $A_k \ni a \overset{\phi_1^{(k)} \circ \widetilde{i}_k}{\longmapsto} Q_k\, a =$  $\in S_k$ which is indeed an inclusion since Q_k is a bi-faithful functor. Now, Q_k is symmetrically self-dual with the solution to conjugate equation given by  . Thus, we have a conditional expectation given by

$S_k \ni x \overset{E'}{\longmapsto}$  $=$  $\in A_k$

where the equality follows from the definition of S_k and separability axiom. This conditional expectation is automatically faithful and translates into E_k (defined in the statement) via the *-isomorphism $\phi_2^{(k)}$. Now, for $x \in S_k^+$, we have

 $=$  $\leq \|d_{Q_k}\|$ 

where the first equality follows from (F1) of Fact 2.4.4 and the second inequality from (F2) of Fact 2.4.4 and the definition of S_k. We rewrite the last term as  $\leq \|d_{Q_k}\|\ \ Q_k\left(E'(x)\right)$. Hence, the conditional expectation E', and thereby E_k has finite index. □

Next, we will test the compatibility of the countable family of finite dimensional C*-algebras $\{H_k\}_{k\geq 0}$ and the inclusions $H_k \overset{I_{k+1}}{\hookrightarrow} H_{k+1}$ for $k \geq 0$ (as described in 3.3).

Lemma 5.1.6. *The inclusion* $H_k \overset{I_{k+1}}{\hookrightarrow} H_{k+1}$ *is a* $*$*-algebra homomorphism eventually for all* k*. Further, the unital filtration* $\{A_k\}_{k\geq 0}$ *of finite dimensional C*-algebras (as described* **??***) sits inside* $H_\infty = \underset{k}{\cup} H_k$ *via the inclusions* $\widetilde{i}_k : A_k \to H_k$ *eventually for all* k*. In particular, the above conditions commence when* (m_k, m_{k+1}) *and* (i_k, i_{k+1}) *start satisfying the exchange relation.*

Proof. This easily follows from the exchange relation of m_k and i_k, and the definitions of $\widetilde{m}_k$ and $\widetilde{i}_k$. ☐

Remark 5.1.7. We can obtain $\mathscr{S}_k \subset H_k$ such that $\sum\limits_{\sigma \in \mathscr{S}_k} \sigma\sigma^* = 1_{Q_k \Gamma_k \cdots \Gamma_1 m_0}$ using Lemma 3.1.3 and 3.3 .

Remark 5.1.8. We will denote the object m_0 by dashed lines and any other object by dotted lines in (ii) of the pictorial notations mentioned at the beginning of Section 2.3.2.

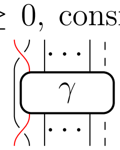
5.2 Splitting of $(Q_\bullet, m_\bullet, i_\bullet)$

In this section we will first construct a suitable 0-cell in **UC** using results from the previous section. Then move on to construct a dualizable 1-cell $X_\bullet$ from $\Gamma_\bullet$ to the newly constructed 0-cell. Subsequently we build a unitary from $\overline{X}_\bullet \boxtimes X_\bullet$ to $Q_\bullet$ which intertwine the algebra maps as well as satisfy exchange relations eventually.

Notation: Thoughout this section, given a finite dimensional C*-algebra A, we will use the notation $\mathcal{R}_A$ for the category of finite-dimensional (as a complex vector space) right A-correspondences. Note that $\mathcal{R}_A$ is a finite, semisimple C*-category.

5.2.1 New 0-cells in UC

Let $l \in \mathbb{N}$ be as in Remark 5.1.1

For each $k \geq 0$, consider the C*-algebra inclusions $A_k \overset{Q_k}{\hookrightarrow} C_k := \mathrm{End}(Q_k \Gamma_k \cdots \Gamma_1 m_0)$ and $C_k \ni \gamma \longmapsto$  $\in C_{k+1}$. Note that $Q_k(A_k) \subset S_k \subset C_k$ for all $k \geq l$. Consider the filtration of C*-algebras $\{B_k\}_{k\geq 0}$ defined as follows:

$$B_k = \begin{cases} H_k & \text{if } k \geq l \\ S_l \cap C_k & \text{if } 0 \leq k \leq l-1 \end{cases}$$

where the inclusion $B_k \hookrightarrow B_{k+1}$ is given by I_k for $k \geq l$, set inclusions for $0 \leq k \leq l-2$ and the remaining inclusion $B_{l-1} \hookrightarrow B_l$ is $\phi_2^{(l)}\Big|_{S_l \cap C_{l-1}} : S_l \cap C_{l-1} \to H_l$ (where $\phi_2^{(l)} : S_l \to H_l$ is the isomorphism defined in Proposition 5.1.4).

Define $\Delta_{k+1} := \bullet \underset{B_k}{\boxtimes} B_{k+1} : \mathcal{R}_{B_k} \to \mathcal{R}_{B_{k+1}}$ for $k \geq 0$. Each Δ_k is a bi-faithful functor (which follows from the unital inclusion $B_k \hookrightarrow B_{k+1}$ of finite dimensional C*-algebra for $k \geq 0$). Thus, we have a 0-cell $(\Delta_\bullet, \mathcal{R}_{B_\bullet}) \in \mathbf{UC}_0$.

Similarly, using the unital filtration $\{A_k\}_{k\geq 0}$ (resp., $\{C_k\}_{k\geq 0}$) of finite dimensional C*-algebras, we define another 0-cell $\Sigma_\bullet$ (resp. $\Psi_\bullet$) defined by $\Sigma_k := \bullet \underset{A_{k-1}}{\boxtimes} A_k : \mathcal{R}_{A_{k-1}} \to \mathcal{R}_{A_k}$ (resp., $\Psi_k := \bullet \underset{C_{k-1}}{\boxtimes} C_k : \mathcal{R}_{C_{k-1}} \to \mathcal{R}_{C_k}$) for $k \geq 1$.

5.2.2 Construction of dualizable 1-cell from $(\Gamma_\bullet, \mathcal{M}_\bullet)$ to $(\Delta_\bullet, \mathcal{R}_{B_\bullet})$

Our strategy is to build two dualizable 1-cells $\left(F_\bullet, W^F_\bullet\right) : \Gamma_\bullet \to \Sigma_\bullet$ and $\left(\Lambda_\bullet, W^\Lambda_\bullet\right) : \Sigma_\bullet \to \Delta_\bullet$ and define $(X_\bullet, W_\bullet)$ to be their composition in $\mathbf{UC}$ as depicted in Equation (3.5), and thereby obtaining our desired dualizable 1-cell $X_\bullet : \Gamma_\bullet \to \Delta_\bullet$ in $\mathbf{UC}$. We first prove the following easy fact.

Proposition 5.2.1. *Given a finite semisimple C*-category $\mathcal{M}$ and an object m which contains every simple object as a sub-object, the functor $F := \mathcal{M}(m, \bullet) : \mathcal{M} \to \mathcal{R}_A$ is an equivalence where $A = \mathrm{End}(m)$ and $\mathcal{R}_A$ is the category of right A-correspondences.*

Proof. For $x \in \mathrm{ob}(\mathcal{M})$, $F(x)$ becomes a right A-correspondence with the A-action and A-valued inner product defined in the following way

$$F(x) \times A \ni (u, a) \longmapsto u\, a \in F(x) \quad \text{and} \quad \langle u, v\rangle := v^* u \ .$$

For $f \in \mathcal{M}(x, y), F(f)(u) = f\, u \in F(y)$ for each $u \in F(x)$. Indeed, F-action on any morphism of $\mathcal{M}$ is adjointable ($F(f)^* = F(f^*)$) and A-linear. Clearly, each F is a faithful functor.

Let $T \in \mathcal{R}_A(F(x), F(y))$. Since every simple appears as a sub-object in m, we can find a finite subset $\mathscr{S}_x \subseteq F(x)$ such that $\displaystyle\sum_{u \in \mathscr{S}_x} u\, u^* = 1_x$. Define $f := \displaystyle\sum_{u \in \mathscr{S}_x} T(u) u^* \in \mathcal{M}(x, y)$. For $v \in F(x)$,we have,

$$\begin{aligned} T(v) &= T\left(\sum_{u \in \mathscr{S}_x} u\, u^* v\right) \\ &= \sum_{u \in \mathscr{S}_x} T(u)\, u^* v \quad \text{(since } T \text{ is right } A\text{-linear)} \\ &= F(f)(v) \quad \text{(since } F(f) = f \circ - \text{ and by definition of } f\text{)} \end{aligned}$$

Thus, F is full.

Now, we show that F is essentially surjective. Since F is fully faithful by Schur's lemma, we have, $F(x)$ is simple if x is simple. We show that for simple $H \in \mathcal{R}_A$ there is a simple x in $\mathcal{M}$ such that $F(x)_A \simeq H_A$. Choose, $\xi \in H \setminus \{0\}$ such that $\langle \xi, \xi\rangle_A \neq 0$. By spectral decomposition of $\langle \xi, \xi\rangle_A$, there is a minimal projection p in A such that $\langle \xi, \xi\rangle_A p \in \mathbb{C}p \setminus \{0\}$. Now ,since p is minimal, $\langle \xi p, \xi p\rangle_A = p\langle \xi, \xi\rangle_A p \in \mathbb{C}p \setminus \{0\}$. Without loss of generality, we assume $\xi p = \xi$. Now, H being irreducible, we have, $H = \xi A$. Now, by semi-simplicity of

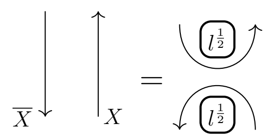
$\mathcal{M}$ there is a simple $x \in \mathcal{M}$ and an isometry $\alpha : x \to m$ such that $p = \alpha\alpha^*$. Observe that, $\alpha^* \in F(x)$ and $F(x) = \alpha^* A$. Define $T' : F(x)_A \to H_A$ as $T'(\alpha^* a) = \xi a$ for all $a \in A$. Clearly, T' is well-defined, right-A linear and onto. Thus, T' is an isomorphism. Hence, F is an equivalence. $\square$

Construction of $\left(F_\bullet, W_\bullet^F\right) \in \mathbf{UC}_1\left(\Gamma_\bullet, \Sigma_\bullet\right)$

For each $k \geq 0$, setting $m = \Gamma_k \cdots \Gamma_1 m_0$ in Proposition 5.2.1, we obtain the functor $F_k := \mathcal{M}_k(\Gamma_k \cdots \Gamma_1 m_0, \bullet) : \mathcal{M}_k \to \mathcal{R}_{A_k}$ which is an equivalence.

Proposition 5.2.2. *Suppose $\mathcal{C}$ is a C^*-2-category. Let $X \in \mathcal{C}_1(a,b)$ be dualizable with dual $\overline{X} \in \mathcal{C}_1(b,a)$ such that each component in the solution $\left(R, \overline{R}\right)$ to the conjugate equations for $\left(X, \overline{X}\right)$ are invertible. Then, there exists another solution $\left(R', \overline{R}'\right)$ to the conjugate equations for $\left(X, \overline{X}\right)$ such that R' and $\overline{R}'$ are unitaries.*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is strict. Since R and R^* are invertible, so R^*R is also invertible. Let $l := R^*R \in \operatorname{End}(1_b)$. Define $R' := R \circ l^{-\frac{1}{2}} \in \mathcal{C}_2(1_b, X \boxtimes \overline{X})$. Clearly, R' is invertible and $R'^*R' = \mathrm{id}_{1_b}$ which further implies $R'R'^* = 1_X \boxtimes 1_{\overline{X}}$. In terms of graphical calculus, the last equality can be expressed as the following identity using the conjugate equations satisfied by $(R, \overline{R})$



(5.1)

Now, define $\overline{R}' := (1_{\overline{X}} \boxtimes l^{\frac{1}{2}} \boxtimes 1_X)\overline{R}$. It is easy to verify that $\left(R', \overline{R}'\right)$ satisfy the conjugate equations for $\left(X, \overline{X}\right)$. Equation (5.1) ensures that $\overline{R}'$ is a unitary. $\square$

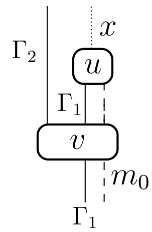
Remark 5.2.3. F_k being an equivalence is a part of an adjoint equivalence [JY21], so we may obtain an adjoint $\overline{F}_k$ of F_k, and evaluation and coevaulation implementing the duality which are both natural unitaries. Thus, for each $k \geq 0$, bi-faithfulness of F_k is immediate.

Before we describe the unitary connections for F_k's, we digress a bit to prove some results which will be useful in the construction.

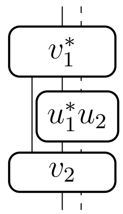
Suppose $\mathcal{N}$ is a C*-semisimple category. For $x, y \in \operatorname{Ob}(\mathcal{N})$, consider the morphism space $\mathcal{N}(x,y)$ and consider the C*-algebra $A = \operatorname{End}(x)$. Then, $\mathcal{N}(x,y)$ becomes a right-A correspondence with A-valued inner product, $\langle u, v\rangle_A = u^*v$.

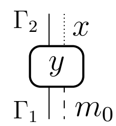
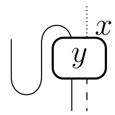
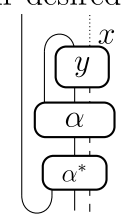
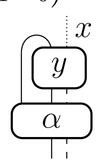
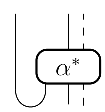
We proceed with the following lemma.

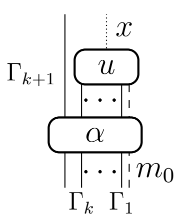
Lemma 5.2.4. *Suppose $\mathcal{M}$ and $\mathcal{N}$ are finite, C^*-semisimple categories. Let $\Gamma_1 : \mathcal{M} \to \mathcal{N}$ and $\Gamma_2 : \mathcal{N} \to \mathcal{N}$ be bi-faithful, $*$-linear functors. Then the map $T : \mathcal{N}(\Gamma_1 m_0, x) \underset{\operatorname{End}(\Gamma_1 m_0)}{\boxtimes}$*

$\mathcal{N}(\Gamma_1 m_0, \Gamma_2\Gamma_1 m_0) \to \mathcal{N}(\Gamma_1 m_0, \Gamma_2 x)$ *given by* $u \boxtimes v \xmapsto{T}$  *is a unitary as a right-*$\mathrm{End}(\Gamma_1 m_0)$*-linear map.*

Proof. Let $A = \mathrm{End}(\Gamma_1 m_0)$. Clearly, T is middle A-linear. Now,

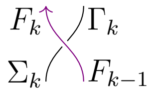
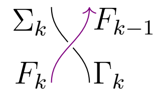
$$\langle T(u_1 \boxtimes v_1), T(u_2 \boxtimes v_2)\rangle_A =$$  $$= \langle v_1, \langle u_1, u_2\rangle_A v_2\rangle_A = \langle u_1 \boxtimes v_1, u_2 \boxtimes v_2\rangle_A .$$

Hence, T is an isometry. If we can show that T is surjective then we get our desired result from [L95]. Now, let $y \in \mathcal{N}(\Gamma_1 m_0, \Gamma_2 x)$. Then,  $=$  $= \sum_{\alpha \in \mathscr{S}}$ . Last equality follows from the fact that, we can find such a set $\mathscr{S} \subseteq \mathrm{End}\left(\overline{\Gamma}_2\Gamma_1 m_0\right)$ because of bi-faithfulness of Γ_2, Γ_1 and m_0 contains all irreducibles of $\mathcal{M}$. Now, $T\left(\sum_{\alpha\in\mathscr{S}}\right.$  $\boxtimes$  $\left.\right) =$ y. Hence, T is surjective. So, T is an unitary. $\square$

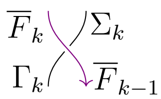
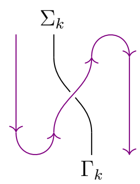
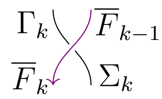
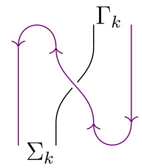
Corollary 5.2.5. *The maps* $T_x^k : F_k(x) \underset{A_k}{\boxtimes} A_{k+1\,A_{k+1}} \to \mathcal{M}_k(\Gamma_{k+1}\Gamma_k \cdots \Gamma_1 m_0, \Gamma_{k+1} x)_{A_{k+1}}$ *given by* $u \boxtimes \alpha \xmapsto{T_x^k}$  *are unitaries and they are natural in x, for each* $x \in \mathcal{M}_k$ *and* $k \geq 0$.

Proof. Clearly, T_x^k are right-A_{k+1} linear. Unitarity of T_x^k follows from Lemma 5.2.4. Naturality of T^k follows from the definition of F_k acting on morphism spaces as in Proposition 5.2.1. $\square$

We now define the unitary connections for $\{F_k\}_{k\geq 0}$ as $W_{k+1}^F := T^k : \Sigma_{k+1}F_k \to F_{k+1}\Gamma_{k+1}$ as defined in Corollary 5.2.5, for each $k \geq 0$. Pictorially we denote, for each $k \geq 0$, F_k by 

and $\overline{F}_k$ by  and for each $k \geq 1$, W_k^F by  and $\left(W_k^F\right)^*$ by  . Hence, we have a 1-cell $\left(F_\bullet, W_\bullet^F\right) \in \mathbf{UC}_1(\Gamma_\bullet, \Sigma_\bullet)$.

For each $k \geq 1$, define

$\overline{W}_k^F :=$  $:=$  and $\left(\overline{W}_k^F\right)^* :=$  $:=$ 

Since the evaluation and coevaluation are chosen (in Remark 5.2.3) to be unitaries, therefore $\overline{W}_k^F$'s are also so. We claim that $F_\bullet$ is a dualizable 1-cell in $\mathbf{UC}$ with dual $\left(\overline{F}_\bullet, \overline{W}_\bullet^F\right)$. For this, we verify that solutions to conjugate equations (as in Remark 5.2.3) satisfy exchange relations for $k \geq 0$, which is equivalent to the equations by which W_k^F's and $\overline{W}_k^F$'s become unitaries.

Remark 5.2.6. Observe that by Proposition 5.2.1, we have an adjoint equivalence $G_k : \mathcal{M}_k \to \mathcal{R}_{C_k}$ using the fact that $Q_k \Gamma_k \cdots \Gamma_1 m_0$ contains every simple of $\mathcal{M}_k$ as a subobject for each $k \geq 0$. Further, the square

$$\begin{tikzcd} \mathcal{M}_{k+1} \arrow[r, "G_{k+1}"] & \mathcal{R}_{C_{k+1}} \\ \mathcal{M}_k \arrow[u, "\Gamma_k"] \arrow[r, "G_k"'] & \mathcal{R}_{C_k} \arrow[u, "\bullet \underset{C_k}{\boxtimes} C_{k+1}"'] \end{tikzcd}$$

commutes up to a natural unitary, say W_{k+1}^G, which can be proven exactly the same was done for F_k's, and thereby yeilding a dualizable 1-cell $\left(G_\bullet, W_\bullet^G\right)$ in $\mathbf{UC}$ from $\Gamma_\bullet$ to $\Psi_\bullet$.

Picking a dual $\overline{G}_\bullet \in \mathbf{UC}_1(\Psi_\bullet, \Gamma_\bullet)$ of $G_\bullet$, we set $\left(R_\bullet, W_\bullet^R\right) := F_\bullet \boxtimes \overline{G}_\bullet \in \mathbf{UC}_1(\Psi_\bullet, \Sigma_\bullet)$. That is, $R_k = F_k \overline{G}_k : \mathcal{R}_{C_k} \to \mathcal{R}_{A_k}$ for $k \geq 0$ which along with the unitary connections are compatible with the Σ_k's and Ψ_k's.

Construction of $\left(\Lambda_\bullet, W_\bullet^\Lambda\right) \in \mathbf{UC}_1(\Sigma_\bullet, \Delta_\bullet)$ and its dual

Observe that in Section 5.2.1, for each $k \geq 0$, we have unital inclusions $A_k \hookrightarrow B_k$ of C*-algebras; in particular, for $k \geq l$, this is given in Lemma 5.1.5. As a result, the functor $\Lambda_k := \bullet \underset{A_k}{\boxtimes} B_k : \mathcal{R}_{A_k} \to \mathcal{R}_{B_k}$ turns out to bi-faithful for each $k \geq 0$. Next, we need to define the unitary connection for $\Lambda_\bullet$. We acheive this using the following easy fact.

Fact 5.2.7. *Suppose A, B, C, D are finite dimensional C*-algebras such that we have a square of unital inclusions*

$$\begin{tikzcd} C \arrow[r, hook] & D \\ A \arrow[u, hook] \arrow[r, hook] & B \arrow[u, hook] \end{tikzcd}$$

. This induces a square of categories and functors

$$\begin{tikzcd} \mathcal{R}_C \arrow[r, "\bullet \underset{C}{\boxtimes} D"] & \mathcal{R}_D \\ \mathcal{R}_A \arrow[u, "\bullet \underset{A}{\boxtimes} C"] \arrow[r, "\bullet \underset{A}{\boxtimes} B"'] & \mathcal{R}_B \arrow[u, "\bullet \underset{B}{\boxtimes} D"'] \end{tikzcd}$$

. Corresponding to this last square, there exists a unitary natural

transformation between the functors $\bullet \underset{A}{\boxtimes} B \underset{B}{\boxtimes} D$ *and* $\bullet \underset{A}{\boxtimes} C \underset{C}{\boxtimes} D$.

For $0 \leq k \leq l-1$, the unitaries W^{Λ}_{k+1} may be obtained by applying Fact 5.2.7 to the squares
$$\begin{tikzcd} \mathcal{R}_{A_{k+1}} \arrow[r, "\Lambda_{k+1}"] & \mathcal{R}_{B_{k+1}} \\ \mathcal{R}_{A_k} \arrow[u, "\Sigma_{k+1}"] \arrow[r, "\Lambda_k"'] & \mathcal{R}_{B_k} \arrow[u, "\Delta_{k+1}"'] \end{tikzcd} .$$

We now explicitly describe the unitaries $W^{\Lambda}_k : \Delta_k \Lambda_{k-1} \to \Lambda_k \Sigma_k$ for each $k \geq l+1$. For $V \in \text{Ob}\left(\mathcal{R}_{A_{k-1}}\right)$, define $\left(W^{\Lambda}_k\right)_V : V \underset{A_{k-1}}{\boxtimes} H_{k-1} \underset{H_{k-1}}{\boxtimes} H_{k H_k} \to V \underset{A_{k-1}}{\boxtimes} A_k \underset{A_k}{\boxtimes} H_{k H_k}$ as follows :

$$V \underset{A_{k-1}}{\boxtimes} H_{k-1} \underset{H_{k-1}}{\boxtimes} H_k \ni q \underset{A_{k-1}}{\boxtimes} \xi_1 \underset{H_{k-1}}{\boxtimes} \xi_2 \overset{\left(W^{\Lambda}_k\right)_V}{\longmapsto} q \underset{A_{k-1}}{\boxtimes} 1_{A_k} \underset{A_k}{\boxtimes} \xi_1 \cdot \xi_2 \;\text{ for each }\; q \in V \;.$$

It is easy to see that each $\left(W^{\Lambda}_k\right)_V$ is a unitary and natural in V, and $\left(W^{\Lambda}_k\right)^*_V$ is given as follows:

$$V \underset{A_{k-1}}{\boxtimes} A_k \underset{A_k}{\boxtimes} H_k \ni q \underset{A_{k-1}}{\boxtimes} \alpha \underset{A_k}{\boxtimes} \xi \overset{\left(W^{\Lambda}_k\right)^*_V}{\longmapsto} q \underset{A_{k-1}}{\boxtimes} 1_{H_{k-1}} \underset{A_k}{\boxtimes} \xi_1 \cdot \xi_2 \;\text{ for each }\; q \in V \;.$$

Thus, we get a 1-cell $(\Lambda_\bullet, W_\bullet) : \Sigma_\bullet \to \Delta_\bullet$ in $\mathbf{UC}$.

We now define $\left(\overline{\Lambda}_\bullet, \overline{W}^{\Lambda}_\bullet\right) \in \mathbf{UC}_1\left(\Delta_\bullet, \Sigma_\bullet\right)$ so that it becomes dual to $(\Lambda_\bullet, W_\bullet)$ in $\mathbf{UC}$. For $0 \leq k \leq l-1$, define $\overline{\Lambda}_k \coloneqq R_k \circ \left(\bullet \underset{B_k}{\boxtimes} C_k\right) : \mathcal{R}_{B_k} \to \mathcal{R}_{A_k}$ where $R_k : \mathcal{R}_{C_k} \to \mathcal{R}_{A_k}$ is the equivalence given in Remark 5.2.6.
For $k \geq l$, define $\overline{\Lambda}_k \coloneqq \bullet \underset{H_k}{\boxtimes} H_k : \mathcal{R}_{H_k} \to \mathcal{R}_{A_k}$. Here the right action of A_k on H_k is given by the inclusion $A_k \hookrightarrow H_k$ (as in Lemma 5.1.5) and the multiplication in C*-algebra H_k; however, the right A_k-valued inner product is the one defined in 3.2 (and not the one coming from conditional expectation).

Remark 5.2.8. Although the functors $\overline{\Lambda}_k$ may not be adjoint to Λ_k for $0 \leq k \leq l-1$, we will need these functors to define an adjoint of $\left(\Lambda_\bullet, W^{\Lambda}_\bullet\right)$ in $\mathbf{UC}$.

Our next job is to define the unitary connections $\left\{\overline{W}^{\Lambda}_k\right\}_{k \geq 1}$ for $\overline{\Lambda}_\bullet$. This will be divide into three different ranges for k, namely $\{1, \ldots, l-1\}$, $\{l\}$ and $\{l+1, l+2, \ldots\}$; the choice of the natural unitaries in the first two ranges could be arbitrary.

Case $0 \leq k \leq l-2$: For the unitary connection $\overline{W}^{\Lambda}_{k+1}$, we look at the following horizontally stacked squares of functors.

$$\begin{tikzcd} \mathcal{R}_{B_{k+1}} \arrow[r, "\bullet \underset{B_{k+1}}{\boxtimes} C_{k+1}"] & \mathcal{R}_{C_{k+1}} \arrow[r, "R_{k+1}"] & \mathcal{R}_{A_{k+1}} \\ \mathcal{R}_{B_k} \arrow[u, "\Delta_{k+1}"] \arrow[r, "\bullet \underset{B_k}{\boxtimes} C_k"'] & \mathcal{R}_{C_k} \arrow[u, "\bullet \underset{C_k}{\boxtimes} C_{k+1}"'] \arrow[r, "R_k"'] & \mathcal{R}_{A_k} \arrow[u, "\Sigma_{k+1}"'] \end{tikzcd} .$$

Both the squares are commutative up to natural unitaries; the left one follows from Fact 5.2.7 and the right comes from Remark 5.2.6. $\overline{W}^{\Lambda}_{k+1}$ is defined as the appropriate composition of above two natural unitaries.

Case $k = l$: To define the natural unitary $\overline{W}^{\Lambda}_{l} : \Sigma_l \; \overline{\Lambda}_{l-1} \to \overline{\Lambda}_l \; \Delta_l$, it is enough to check whether the square

$$\begin{tikzcd}
\mathcal{R}_{H_l} \arrow[rr, "\bullet \underset{H_l}{\boxtimes} H_l", "\overline{\Lambda}_l"'] & & \mathcal{R}_{A_l} \\
\mathcal{R}_{S_l \cap C_{l-1}} \arrow[u, "\bullet \underset{B_{l-1}}{\boxtimes} B_l", "\Delta_l"'] \arrow[rr, "\overline{\Lambda}_{l-1}", "R_{l-1} \circ \left(\bullet \underset{B_{l-1}}{\boxtimes} C_{l-1} \right)"'] & & \mathcal{R}_{A_{l-1}} \arrow[u, "\Sigma_l", "\bullet \underset{A_{l-1}}{\boxtimes} A_l"']
\end{tikzcd}$$

commutes up to a natural isomorphism; let us call this square $\mathbb{S}$. Consider the horizontal pair of squares,

$$\begin{tikzcd}
\mathcal{R}_{S_l} \arrow[r, "\bullet \underset{S_l}{\boxtimes} C_l"] & \mathcal{R}_{C_l} \arrow[r, "R_l"] & \mathcal{R}_{A_l} \\
\mathcal{R}_{S_l \cap C_{l-1}} \arrow[u, "\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} S_l"] \arrow[r, "\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} C_{l-1}"'] & \mathcal{R}_{C_{l-1}} \arrow[u, "\Psi_l"'] \arrow[r, "R_{l-1}"'] & \mathcal{R}_{A_{l-1}} \arrow[u, "\Sigma_l"']
\end{tikzcd}$$

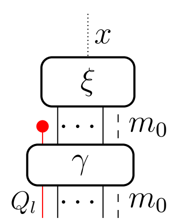
referred as $\mathbb{S}_1$; the first square of $\mathbb{S}_1$ commutes by Fact 5.2.7 and the second follows from Remark 5.2.6. Note that the bottom and the right sides of $\mathbb{S}$ matches with that of $\mathbb{S}_1$.

We next claim that the top side of $\mathbb{S}_1$ is naturally isomorphic to $\bullet \underset{S_l}{\boxtimes} S_l : \mathcal{R}_{S_l} \to \mathcal{R}_{A_l}$. To see this, consider the square

$$\begin{tikzcd}
\mathcal{R}_{C_l} \arrow[d, "\bullet \underset{C_l}{\boxtimes} C_l"'] & \mathcal{M}_l \arrow[l, "G_l"'] \arrow[d, "F_l"] \\
\mathcal{R}_{S_l} & \mathcal{R}_{A_l} \arrow[l, "\bullet \underset{A_l}{\boxtimes} S_l"]
\end{tikzcd}$$

referred as $\mathbb{S}_2$. For $x \in \mathrm{Ob}\,(\mathcal{M}_l)$, the map

$$F_l(x) \underset{A_l}{\boxtimes} S_l \ni \xi \underset{A_l}{\boxtimes} \gamma \longmapsto$$



$$\underset{C_l}{\boxtimes} 1_{C_l} \in G_l(x) \underset{C_l}{\boxtimes} C_l$$

is S_l-linear and natural in x. To show that the map is surjective, pick a basic tensor $\zeta \underset{C_l}{\boxtimes} 1_{C_l} \in G_l(x) \underset{C_l}{\boxtimes} C_l$; note that it can be expressed as the image of $\displaystyle\sum_{\sigma \in \mathscr{S}_l} \zeta \circ \sigma \underset{A_l}{\boxtimes} \phi^{(l)}_1 \left(\sigma^\dagger\right)$ where $\mathscr{S}_l$ is as in Remark 5.1.7 and $\phi^{(l)}_1 : H_l \to S_l$ is the isomorphism mentioned in Proposition 5.1.4. This concludes natural commutativity of $\mathbb{S}_2$. Now, the adjoint of the functors $\bullet \underset{C_l}{\boxtimes} C_l : \mathcal{R}_{C_l} \to \mathcal{R}_{S_l}$ and $\bullet \underset{A_l}{\boxtimes} S_l : \mathcal{R}_{A_l} \to \mathcal{R}_{S_l}$ (appearing in the square $\mathbb{S}_1$) are given by $\bullet \underset{S_l}{\boxtimes} C_l : \mathcal{R}_{S_l} \to \mathcal{R}_{C_l}$ and $\bullet \underset{S_l}{\boxtimes} A_l : \mathcal{R}_{S_l} \to \mathcal{R}_{A_l}$ respectively; this can be achieved by solving the conjugate equations using the set $\mathscr{S}_l$ again and the conditional expectations. Thus, dualizing the square $\mathbb{S}_2$, we get $\overline{F}_l \circ \left(\bullet \underset{S_l}{\boxtimes} S_{l A_l} \right) \cong \overline{G}_l \circ \left(\bullet \underset{S_l}{\boxtimes} C_{l C_l} \right)$. Now, using the fact that F_l is an adjoint equivalence and using $R_l = F_l \overline{G}_l$, we get $R_l \circ \left(\bullet \underset{S_l}{\boxtimes} C_{l C_l} \right) \cong \left(\bullet \underset{S_l}{\boxtimes} S_{l A_l} \right)$. Using this natural isomorphism and natural commutativity of the square $\mathbb{S}_1$, we obtain natural commutativity of

$$\begin{tikzcd} \mathcal{R}_{S_l} \arrow[r, "\bullet \underset{S_l}{\boxtimes} S_l"] & \mathcal{R}_{A_l} \\ \mathcal{R}_{S_l \cap C_{l-1}} \arrow[u, "\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} S_l"] \arrow[r, "R_{l-1} \circ \left(\bullet \underset{S_l \cap C_{l-1}}{\boxtimes} C_{l-1} \right)"'] & \mathcal{R}_{A_{l-1}} \arrow[u, "\Sigma_l"'] \end{tikzcd}$$

. Finally, using the isomorphism $\phi_2^{(l)} : S_l \to H_l$ (as in Proposition 5.1.4), we get our desired natural commutativity of $\mathbb{S}$. Set $\overline{W}_l^{\Lambda}$ to be a natural unitary implementing commutativity of $\mathbb{S}$.

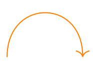
Case $k \geq l$: To define $\overline{W}_{k+1}^{\Lambda}$, we will need the solutions to conjugate equations for Λ_k and $\overline{\Lambda}_k$ for each $k \geq l$. We will use the following pictorial notations:

$$\Lambda_k := \uparrow \quad \text{and} \quad \overline{\Lambda}_k := \downarrow \quad \text{for each } \ k \geq 0$$

Definition 5.2.9.

(i)  : $\mathrm{Id}_{\mathcal{R}_{H_k}} \to \Lambda_k \overline{\Lambda}_k$ is the natural transformation defined as:

 $_V$: $V \to V \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} H_k$ is given by $q \longmapsto \sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \sigma^\dagger$ where $V \in \mathcal{R}_{H_k}$ and $\mathscr{S}_k$ is as in Remark 5.1.7.

(ii)  : $\Lambda_k \overline{\Lambda}_k \to \mathrm{Id}_{\mathcal{R}_{H_k}}$ is the natural transformation defined as :

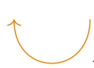
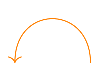
 $_V$: $V \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} H_k \to V$ is given by $q \underset{H_k}{\boxtimes} \xi_1 \underset{A_k}{\boxtimes} \xi_2 \longmapsto q.(\xi_1 \cdot \xi_2)$ where $V \in \mathcal{R}_{H_k}$.

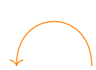
(iii)  : $\mathrm{Id}_{\mathcal{R}_{A_k}} \to \overline{\Lambda}_k \Lambda_k$ is the natural transformation defined as :

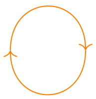
 $_V$: $V \to V \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k$ is given by $q \longmapsto q \underset{A_k}{\boxtimes} 1_{H_k} \underset{H_k}{\boxtimes} 1_{H_k}$ where $V \in \mathcal{R}_{A_k}$.

(iv)  : $\overline{\Lambda}_k \Lambda_k \to \mathrm{Id}_{\mathcal{R}_{A_k}}$ is the natural transformation defined as :

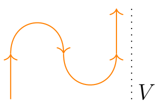
 $_V$: $V \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \to V$ is given by $q \underset{A_k}{\boxtimes} \xi_1 \underset{H_k}{\boxtimes} \xi_2 \longmapsto q.\langle \xi_2, \xi_1^\dagger \rangle_{A_k}$ where $V \in \mathcal{R}_{A_k}$.

Lemma 5.2.10. *(i)* , , ,  *satisfy conjugate equations for* $\Lambda_k, \overline{\Lambda}_k$ *for each* $k \geq l$ *.*

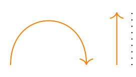
(ii) Also, $\Big($  $\Big)^* =$  *and* $\Big($  $\Big)^* =$  .

(iii)  $= \mathrm{id}_{\mathrm{Id}_{\mathcal{R}_{H_k}}}$

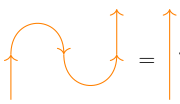
Proof. (i) We have, for every $V \in \mathcal{R}_{A_k}$, $q \in V$ and $\xi \in H_k$,

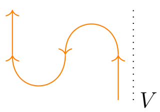


$$\left(q \boxtimes \xi\right) =$$

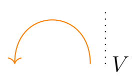


$$\left(q \boxtimes \xi \boxtimes 1_{H_k} \boxtimes 1_{H_k}\right) = q \boxtimes \xi$$

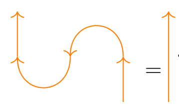
Therefore, we get . We have, for every $V \in \mathcal{R}_{A_k}$, $q \in V$ and $\xi \in H_k$,

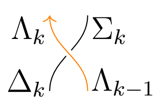


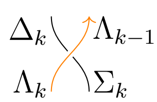
$$\left(q \underset{A_k}{\boxtimes} \xi\right) =$$

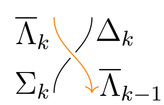


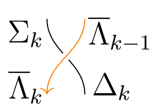
$$\left(\sum_{\sigma \in \mathscr{S}_k} q \underset{A_k}{\boxtimes} \xi \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \sigma^\dagger\right) = \sum_{\sigma \in \mathscr{S}_k} q.\langle \sigma, \xi^\dagger \rangle_{A_k} \underset{A_k}{\boxtimes} \sigma^\dagger = \sum_{\sigma \in \mathscr{S}_k} q \underset{A_k}{\boxtimes} \langle \sigma, \xi^\dagger \rangle_{A_k} \sigma^\dagger$$
$$= q \underset{A_k}{\boxtimes} \xi.$$

The last equality follows from Equation (3.2). Therefore, we get . The other equations can be proved similarly.
(ii) The proof is similar to that of (i).
(iii) It follows easily from Definition 5.2.9(i) and Definition 5.2.9(ii). □

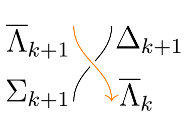
Pictorially, we denote W_k^Λ by 

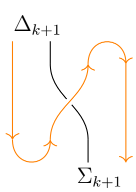
, $\left(W_k^\Lambda\right)^*$ by 

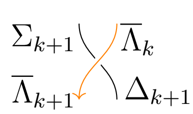
, $\overline{W}_k^\Lambda$ by 

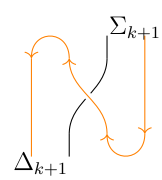
and $\left(\overline{W}_k^\Lambda\right)^*$ by 

for each $k \geq 1$. We have already defined all W_k^Λ's in page 63, and $\overline{W}_k^\Lambda$ for $1 \leq k \leq l$ in the above two cases. Now, for $k \geq l$, we define

$\overline{W}_{k+1}^\Lambda =$ 

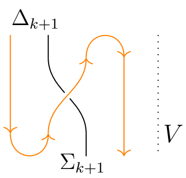
$:=$ 

and $\left(\overline{W}_{k+1}^\Lambda\right)^* =$ 

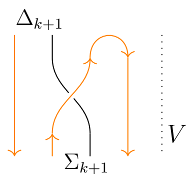
$:=$ 

which turn out to be natural unitaries by the following remark.

Remark 5.2.11. For each $k \geq l$ and $V \in \mathcal{R}_{H_k}$, $q \in V$, $\xi \in H_k$, $\alpha \in A_{k+1}$, η, $\zeta \in H_{k+1}$ the element $\left(\overline{W}^{\Lambda}_{k+1}\right)_V \left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} \alpha\right)$ can be expressed as

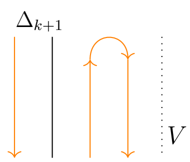


$$\left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} \alpha\right) =$$



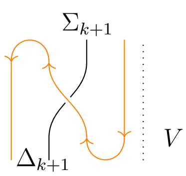
$$\left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} \alpha \underset{A_{k+1}}{\boxtimes} 1_{H_{k+1}} \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right)$$

$$=$$

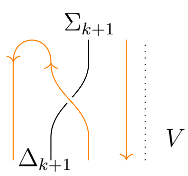


$$\left(q \underset{H_k}{\boxtimes} \xi \underset{A_k}{\boxtimes} 1_{H_k} \underset{A_{k+1}}{\boxtimes} \alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right) = q.\xi \underset{H_k}{\boxtimes} \alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}$$

and $\left(\overline{W}^{\Lambda}_{k+1}\right)^*_V \left(q \underset{H_k}{\boxtimes} \eta \underset{H_{k+1}}{\boxtimes} \zeta\right)$ can be expressed as

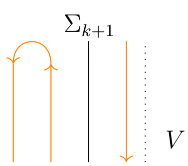


$$\left(q \underset{H_k}{\boxtimes} \eta \underset{H_{k+1}}{\boxtimes} \zeta\right) =$$



$$\left(\sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \sigma^\dagger \underset{H_k}{\boxtimes} \eta \underset{H_{k+1}}{\boxtimes} \zeta\right)$$

$$=$$

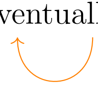


$$\left(\sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} 1_{A_{k+1}} \underset{A_{k+1}}{\boxtimes} \sigma^\dagger \cdot \eta \underset{H_{k+1}}{\boxtimes} \zeta\right)$$

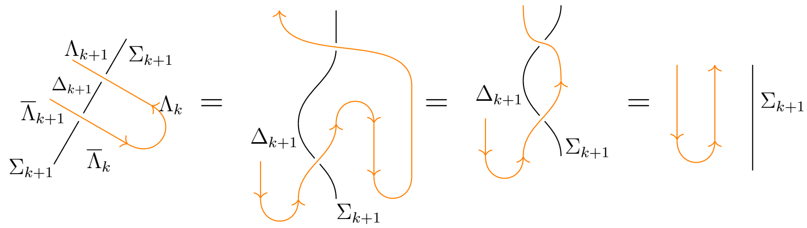
$$= \sum_{\sigma \in \mathscr{S}_k} q \underset{H_k}{\boxtimes} \sigma \underset{A_k}{\boxtimes} \langle \zeta, \eta^\dagger \cdot \sigma \rangle_{A_{k+1}} .$$

It is a straightforward verification that each $\left(\overline{W}^{\Lambda}_{k+1}\right)_V$ is a unitary and natural in V.

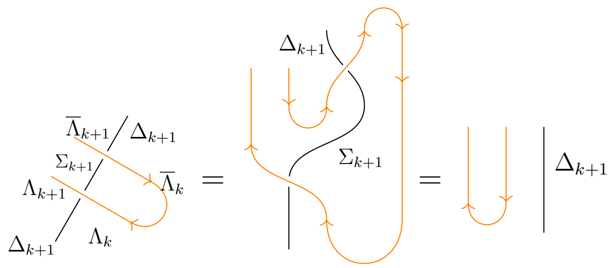
Thus, we have defined a 1-cell $\left(\overline{\Lambda}_\bullet, \overline{W}^{\Lambda}_\bullet\right)$ in **UC** from $\Delta_\bullet$ to $\Sigma_\bullet$. We need to prove that $\left(\overline{\Lambda}_\bullet, \overline{W}^{\Lambda}_\bullet\right)$ is dual to $\left(\Lambda_\bullet, W^{\Lambda}_\bullet\right)$. In order to define the solution to conjugate equation (which is in fact a pair of 2-cells in **UC**), we have the liberty to ignore finitely many terms and define them eventually (by Remark 3.1.17). By Lemma 5.2.10, we see that there are solutions to conjugate equations for Λ_k and $\overline{\Lambda}_k$ for each $k \geq l$. So, we are only left with showing exchange relations of solutions eventually.

We now verify that  and  satisfy exchange relations for $k \geq l$.

Remark 5.2.12. The solutions to conjugate equations for Λ_k and Λ_{k+1} (as in Definition 5.2.9) satisfy exchange relation eventually for all k with respect to $W^{\Lambda}_\bullet$ and $\overline{W}^{\Lambda}_\bullet$. This directly follows from the fact W^{Λ}_k's and $\overline{W}^{\Lambda}_k$'s are unitaries. Nevertheless, we still furnish a proof below. Note that



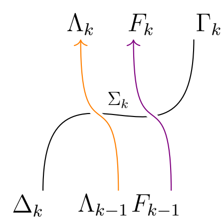
and

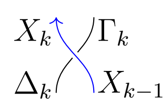
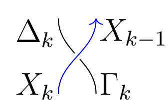


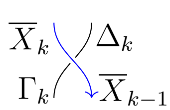
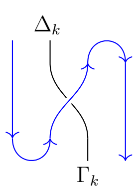
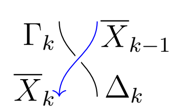
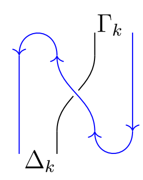
.

Hence, $\left(\Lambda_\bullet, W^\Lambda_\bullet\right) : \Sigma_\bullet \to \Delta_\bullet$ is a dualizable 1-cell in **UC** with dual $\left(\overline{\Lambda}_\bullet, \overline{W}^\Lambda_\bullet\right)$ as described above.

We are now in a position to describe our desired dualizable 1-cell $(X_\bullet, W_\bullet)$ which will split $(Q_\bullet, m_\bullet, i_\bullet)$ as Q-system.

Define $(X_\bullet, W_\bullet) := \Lambda_\bullet \boxtimes F_\bullet = \left(\{\Lambda_k F_k\}_{k\geq 0}, \left\{\right.\right.$  $\left.\left.\right\}_{k\geq 1} \right) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$.

Pictorially, we denote X_k by  , $\overline{X}_k$ by  , W_k by  and W_k^* by  .

Define $\overline{W}_k$:=  :=  and $\left(\overline{W}_k\right)^*$:=  :=  .

Thus, we arrive at our desired 1-cell $(X_\bullet, W_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$. We list some of the properties of $(X_\bullet, W_\bullet)$.

Lemma 5.2.13.

(i) $(X_\bullet, W_\bullet)$ *is a dualizable* 1*-cell in* **UC** .

(ii) $(X_\bullet, W_\bullet)$ *has a unitarily separable dual in* **UC** .

Proof. (i) $(X_\bullet, W_\bullet)$ being a composition of two dualizable 1-cells $\left(\Lambda_\bullet, W_\bullet^\Lambda\right)$ and $\left(F_\bullet, W_\bullet^F\right)$ concludes the result.

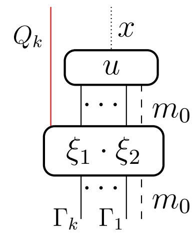
(ii) This is immediate from Remark 3.1.17 and (iii) of Lemma 5.2.10.

□

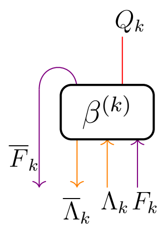
5.2.3 Isomorphism of Q-systems

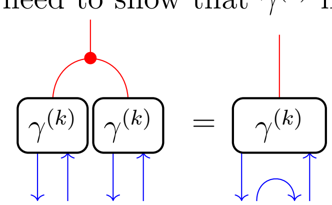
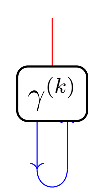
In this subsection, we build an isomorhpism between $\overline{X}_\bullet \boxtimes X_\bullet$ and $Q_\bullet$. We construct unitaries $\gamma^{(k)} : \overline{X}_k X_k \to Q_k$ for each $k \geq l$ which intertwines the mutliplication and unit maps. In the next subsection, we verify the exchange relation of $\gamma^{(k)}$ for each $k \geq l$, thus implementing isomorphism of the aforementioned Q-systems in **UC**.

For $k \geq l$ and for each $x \in \mathrm{Ob}(\mathcal{M}_k)$, define a map $\beta_x^{(k)} : \overline{\Lambda}_k \Lambda_k F_k(x) \to F_k Q_k(x)$ as follows :

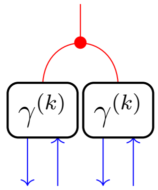
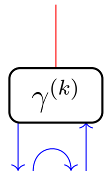
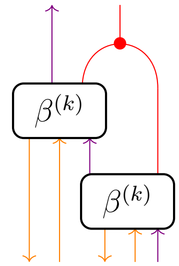
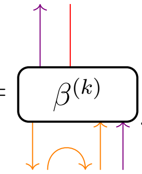
$$F_k(x) \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_{k\,A_k} \ni u \boxtimes \xi_1 \boxtimes \xi_2 \overset{\beta_x^{(k)}}{\longmapsto}$$  $$\in F_k Q_k(x)$$

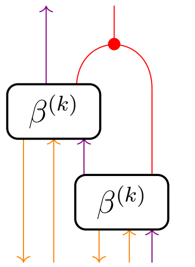
It is easy to see that, each $\beta_x^{(k)}$ is an isometry. Since, ${}_{A_k}H_k \underset{H_k}{\boxtimes} H_{k\,A_k}$ is unitarily isomorhpic to ${}_{A_k}H_{k\,A_k}$ and by application of Lemma 5.2.4 we see that $\overline{\Lambda}_k \Lambda_k F_k(x)$ and $F_k Q_k(x)$ has same dimension (as a vector space). Hence surjectiveness will follow. Thus, each $\beta_x^{(k)}$ is a unitary. Also, it easily follows that each $\beta_x^{(k)}$ is a natural in x. Thus, we get a unitary natural transformation $\beta^{(k)} : \overline{\Lambda}_k \Lambda_k F_k \to F_k Q_k$.

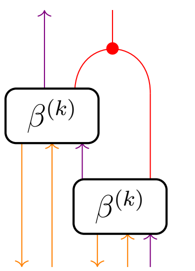
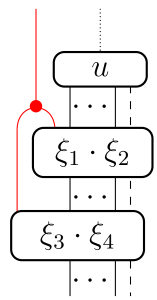
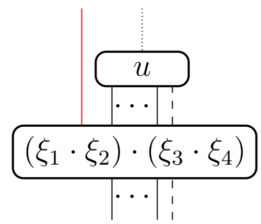
Define $\gamma^{(k)} :=$  $: \overline{X}_k X_k \to Q_k$. We show that $\gamma^{(k)}$ is an isomorphism of

Q-systems $\overline{X}_k X_k$ and Q_k for $k \geq l$. Each $\gamma^{(k)}$ is a unitary because each $\beta^{(k)}$ is so and each F_k is an adjoint equivalence (see Remark 5.2.3). We need to show that $\gamma^{(k)}$ intertwines the multiplication and unit maps. We need to show that  and  $=$  for $k \geq l$. This is what we prove next.

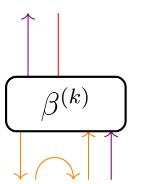
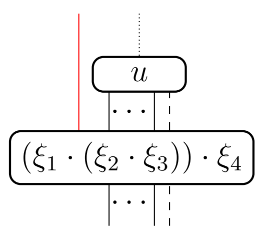
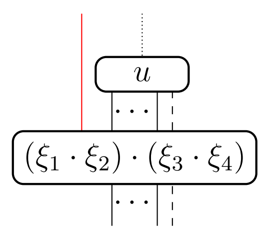
Proposition 5.2.14. *For $k \geq l$, $\gamma^{(k)} : \overline{X}_k X_k \to Q_k$ is an isomorphism of Q-systems.*

Proof. It easily follows that,  =  if and only if  = .

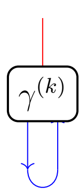
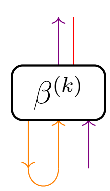
Now the map,  : $F_k(x) \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \to F_k Q_k(x)$ given as follows:

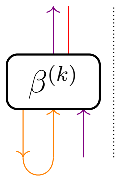
 $\left(u \boxtimes \xi_1 \boxtimes \xi_2 \boxtimes \xi_3 \boxtimes \xi_4 \right)$ =  = 

for every $u \in F_k(x)$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in H_k$. It is straightforward to show that

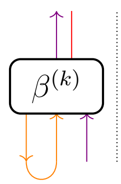
 $\left(u \boxtimes \xi_1 \boxtimes \xi_2 \boxtimes \xi_3 \boxtimes \xi_4 \right)$ =  = 

for every $u \in F_k(x)$ and $\xi_1, \xi_2, \xi_3, \xi_4 \in H_k$. The last equality follows because of associativity of H_k as shown in Proposition 5.1.4. Thus, $\gamma^{(k)}$ intertwines the multiplication maps for each $k \geq l$.

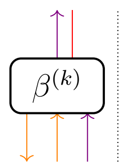
Also, it is easy to see that  =  if and only if  =  . Now, the map



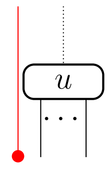
$: F_k(x) \to F_k Q_k(x)$ is given as follows :



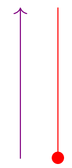
$\Big(u\Big) =$



$\Big(\sum_{\sigma \in \mathscr{S}_k} u \boxtimes \sigma \boxtimes \sigma^\dagger\Big) =$



$=$



$\Big(u\Big)$ for every $u \in F_k(x)$

where $\mathscr{S}_k \subset H_k$ is as given in Remark 5.1.7 . Thus, $\gamma^{(k)}$ intertwines the unit maps for each $k \geq l$. This concludes the proposition. □

5.2.4 Exchange relation of $\gamma^{(k)}$'s

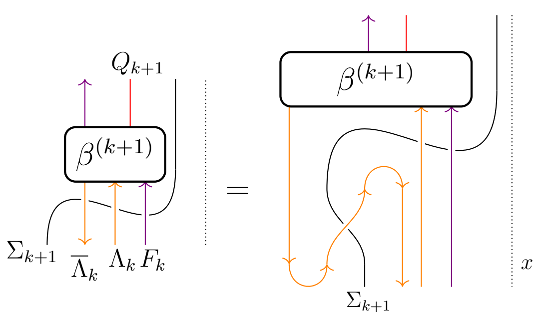
To achieve isomorphism in **UC**, we still have to show that $\gamma^{(k)}$'s satisfy exchange relation for $k \geq l$. This will establish 'splitting' of $(Q_\bullet, m_\bullet, i_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Gamma_\bullet)$ by $(X_\bullet, W_\bullet) \in \mathbf{UC}_1(\Gamma_\bullet, \Delta_\bullet)$.

Remark 5.2.15. In order to show that $\gamma^{(k)}$'s will satisfy exchange relation for $k \geq l$, it is enough to show that $\beta^{(k)}$'s also does so because solutions to conjugate equations for F_k's and $\overline{F}_k$'s satisfy exchange relations for each $k \geq l$. So instead of showing exchange relation of $\gamma^{(k)}$'s we will show that $\beta^{(k)}$'s satisfy exchange relation for $k \geq l$.

We now proceed to show that $\beta^{(k)}$'s satisfy exchange relation for $k \geq l$.

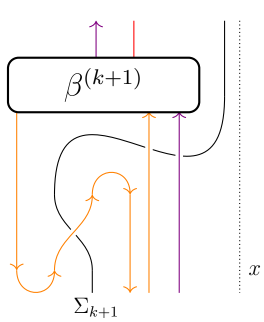
Proposition 5.2.16. *For $k \geq l$, $\beta^{(k)}$'s satisfy exchange relation.*

Proof. For $x \in \mathrm{Ob}(\mathcal{M}_k)$ the map,

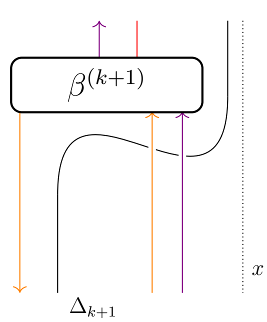


$: F_k(x) \underset{A_k}{\boxtimes} H_k \underset{H_k}{\boxtimes} H_k \underset{A_k}{\boxtimes} A_{k+1} \to F_{k+1}Q_{k+1}\Gamma_{k+1}(x)$

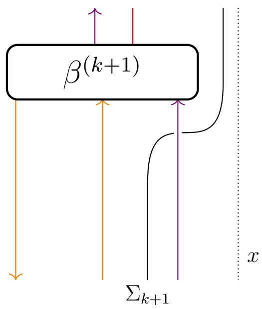
is given as follows:



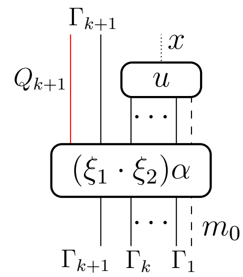
$\Big(u \underset{A_k}{\boxtimes} \xi_1 \underset{H_k}{\boxtimes} \xi_2 \underset{A_k}{\boxtimes} \alpha\Big) =$



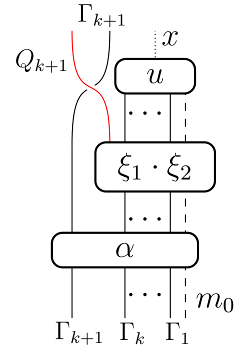
$\Big(u \underset{A_k}{\boxtimes} \xi_1 \cdot \xi_2 \underset{H_k}{\boxtimes} \alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\Big)$



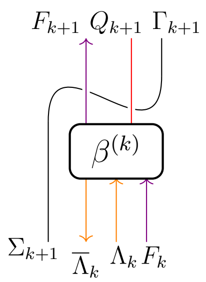
$\left(u \underset{A_k}{\boxtimes} 1_{A_{k+1}} \underset{H_k}{\boxtimes} (\xi_1 \cdot \xi_2)\alpha \underset{H_{k+1}}{\boxtimes} 1_{H_{k+1}}\right) =$



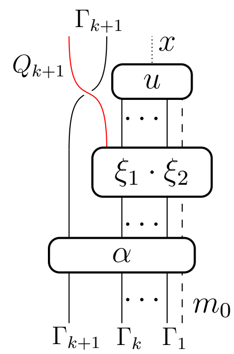
$=$



for every $u \in F_k(x), \xi_1, \xi_2 \in H_k, \alpha \in A_{k+1}$. Also, it will easily follow from the definition of $\beta^{(k)}$'s that for every $u \in F_k(x)$, ξ_1, $\xi_2 \in H_k$, $\alpha \in A_{k+1}$ we have,

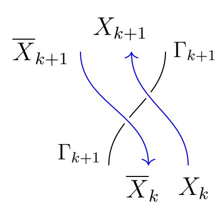


$\left(u \underset{A_k}{\boxtimes} \xi_1 \underset{H_k}{\boxtimes} \xi_2 \underset{A_k}{\boxtimes} \alpha\right) =$



Thus, $\beta^{(k)}$'s satisfy exchange relation for $k \geq l$. $\square$

From Remark 3.1.17, Proposition 5.2.14, Remark 5.2.15 and Proposition 5.2.16 we get the following theorem.

Theorem 5.2.17. *$(Q_\bullet, W^Q{}_\bullet)$ is isomorphic to* $\left(\left\{ \overline{X}_k X_k \right\}_{k \geq 0}, \left\{ \right.\right.$  $\left.\left.\right\}_{k \geq 1} \right)$ *as Q-systems in* **UC**.

Chapter 6

Concluding remarks

In this chapter, we discuss some questions arising out of this thesis for further research.

(i) Which bimodules between hyperfinite von Neumann algebras can be realized by the construction described in Section 3.2 ? This is profoundly related to the question about possible values of the index for irreducible hyperfinite subfactors and the work of S. Popa [P21].

(ii) In Section 3.2, we have considered tracial states on Bratteli digrams. A natural question is, how is the story modified if we pick arbitrary states instead of tracial ones ?

(iii) Bi-faithfulness of functors plays a major role in achieving most of our results. If we drop the bi-faithfulness condition of 0-cells and 1-cells in **UC**, then will the modified category be still Q-system complete ?

(iv) Is $\mathbf{UC}^{\text{tr}}$ Q-system complete ?

(v) Is there any relation between finite-depth Q-systems in $\mathbf{UC}^{\text{tr}}$ and 4-partite graphs and biunitary connections ?

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