Essays in the theory of allocation and voting

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To my

loving mother Shrimati Munesh

and

caring father Shri Suresh.

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Abstract

This thesis consists of two essays on allocation theory and one on voting theory. The first chapter analyses preference domains (called *priority domains*) where every strategy-proof, non-bossy and neutral allocation rule is a *priority rule*. It considers two versions of neutrality: *unanimous profile neutrality* or UPN neutrality where the neutrality axiom applies only to preference profiles where all agents have a common preference ordering and *full neutrality* or FN neutrality, where the neutrality axiom applies generally. We show that a very simple condition characterises priority domains under the UPN axiom. If these domains satisfy a mild richness condition, they must be the universal domain. The class of priority domains under the FN axiom is larger than those satisfying only UPN. We identify an FN-priority domain that is of order $\frac{1}{n}$ relative to the universal domain.

The second chapter analyses preference domains in voting environments where every strategy proof random social choice functions satisfying unanimity is a random-dictatorships. We call these *random-dictatorial* domains. Pramanik (2015) identifies a class of domains called *P-domains* which are *dictatorial* i.e. every deterministic strategy-proof social choice functions on these domains satisfying unanimity, is dictatorial. The main result of this chapter is that P-domain is random-dictatorial. A consequence of this result is that *circular domains* (Sato (2010)) are also random-dictatorial. The minimum size of a random-dictatorial domain satisfying minimal richness is shown to be twice the number of alternatives. This is the same as the corresponding lower bound for dictatorial domains. Our result stands in contrast to those in Chatterji et al. (2014) who showed that linked domains are not random-dictatorial. Linked domains were shown to be dictatorial in Aswal et al. (2003).

The third chapter attempts to provide a justification of the *non-bossiness* axiom which is pervasive in the allocation literature. It has been criticised by Thomson (2016) on the grounds that it cannot be defended by appealing to various strategic and normative criteria. We show that in some special cases, non-bossiness is a simplifying assumption that can be imposed without loss of generality by an expected welfare maximising planner in a symmetric environment. We consider the case of three objects and three agents with a planner whose goal is to maximise the expected sum of welfare with respect to a uniform prior. We show that for every strategy proof, neutral and efficient allocation

rule, there exists a strategy-proof, neutral and non-bossy allocation rule which yields the same expected welfare. We conjecture that this is true for an arbitrary number of agents. For the general case, we are able to show an equivalence in terms of expected welfare for a special class of bossy allocation rules.

Contents

Chapter 0

Introduction

This thesis considers various mechanism design in environments where monetary transfers are not permitted. It is well known that that if the set of possible preferences of agents is "rich", strategy-proofness and mild range conditions lead to impossibility results. For example, the celebrated Gibbard-Sattherwaite Theorem (Gibbard (1973), Satterthwaite (1975)) shows that strategy-proof and unanimous voting rules defined over the universal domains must be dictatorial. In the random-setting Gibbard (1977) shows that every strategy-proof and unanimous random social choice function on the universal domain must be a random dictatorship. Similarly, Svensson (1999) shows that strategy-proof allocations defined over the universal domain and satisfying neutrality and non-bossiness are priority rules (sequential dictatorships). The three essays in this thesis attempt to examine the robustness of these negative results with respect to the preference domain and to other axioms. Chapter 1 (co-authored with Arunava Sen) characterizes certain classes of priority domains. Chapter 2 demonstrates that P-domains are random-dictatorial. Chapter 3 provides a justification for the non-bossiness axiom in certain environments.

We provide a brief description of each chapter below.

Chapter 1. Priority Domains

We consider the classical problem of allocating n objects to n agents. We say that a domain of preferences is a *priority domain* if every strategy-proof, non-bossy and neutral allocation rule defined over the domain, is a priority rule. We investigate the structure of restricted domains that are priority domains. We consider two variants of the neutrality assumption. The first which we call *uniform profile neutrality* (UPN), applies only to preference profiles where all agents have the same preferences, i.e. agents are unanimous.

The UPN requires the allocation at the every unanimous profile to be a permutation of the allocation at some other unanimous profile. A stronger condition is *full neutrality* (FN) where the neutrality condition applies to all profiles, not just those where agents are unanimous. We refer to UPN and FN priority domains depending on the definition of neutrality assumed.

We identify a very simple condition called the *closure property* that is both necessary and sufficient for a domain to be a UPN priority domain. For an arbitrary profile and an arbitrary priority, the priority induces a preference ordering at the profile. The domain satisfies the closure if this artificially constructed ordering belongs to the domain. The closure property is clearly a demanding property. Every UPS-priority domain that satisfies the property that every object is first-ranked in some ordering belonging to the domain (*minimal richness*) must be the universal domain. As a consequence, restricted domains such as the domain of single-peaked preferences and the circular domain are not UPS-priority domains. However, a different class of domains which we call *range-restricted domains* are UPS-priority domains.

The situation with respect to FN-priority domains is somewhat different. The class of such domains is clearly larger than that of UPN-priority domains. In Theorem 1.2, we provide an example of an FN-priority domain that is "small" relative to the universal domain. We call such domains *lower complete Hamilton cycle* or LCHC domains. We show that the an LCHC domain is a FN-priority domain. We show that it is possible to construct LCHC domains that are approximately of size $\frac{1}{n}$ relative to the size of the universal domain. We show that unline dictatorial domains, the Priority domains have to be far richer. The only UPN-priority domain which satisfies minimal richness is the universal domain. Although we do not provide a characterization of FN-priority domains, LCHC domains are large compared to the smallest dictatorial domains.

Chapter 2. Random-dictatorship on Restricted Domains

One way to avoid the negative result of Gibbard–Satterthwaite Theorem in terms of fairness is to consider random social choice functions. Gibbard (1977) the only strategy-proof and unanimous random social choice function is a *random-dictatorship*. In this Chapter, we attempt to explore the random-dictatorial domains.

Aswal et al. (2003) introduced the concept of *dictatorial domains*. They showed that a class of domains called *linked domains* are dictatorial. Another class of dictatorial domains are circular domains Sato (2010). Pramanik (2015) introduced the β domains and γ domains which generalised linked domains and circular domains respectively. The paper showed that β domains and almost all γ domains are dictatorial.

A natural question is the connection between dictatorial domains and random-dictatorial domains. Chatterji et al. (2014) investigate the relationship between dictatorial and random-dictatorial domains. They showed that a dictatorial domain need not be random-dictatorial. In fact, they showed that stronger conditions have to be imposed on a linked domain in order for it to be random-dictatorial. Our contribution in this chapter is to show that the dictatorial γ domains (which we call P- domains) are random-dictatorial as well. This result stands in contrast to Chatterji et al. (2014) who showed that the "gap" between dictatorial and random-dictatorial domains is "large" for linked domains. We also deduce the minimum size of a random-dictatorial domain which satisfies minimal richness. The minimum size of this domain is $2m$ where m is the number of alternatives. The result in Sato (2010) showed that the minimum size of a dictatorial domain satisfying minimal richness is also $2m$.

3. Towards a defence of non-bossiness

The axiom of *non-bossiness* is widely used in the theory of allocation. But the non-bossiness axiom has been extensively criticized in Thomson (2016). According to him it cannot be justified either on strategic grounds or on normative grounds. In this chapter we aim to provide an alternative justification for non-bossiness. We argue that it is a simplifying assumption which can be made "without loss of generality" for an expected welfare maximizing planner in many situations. We provide one such context.

We consider a basic model of object allocation with equal of agents as objects. There is a planner whose objective is to maximise her expected welfare in the class of all strategy-proof and efficient allocation rules under the assumptions of private information which is independently and identically distributed and the welfare of the planner is symmetric in the identity of the agents. We simply the analysis by further assuming that the welfare of the planner is utilitarian and distribution of agents' preferences is uniform.

We present two results in this chapter. The first is for the case where the number of agents and the number of objects is equal to three. We show that for every strategy-proof, efficient and neutral allocation rule, there exists a strategy-proof, efficient and non-bossy rule which gives planner the same expected welfare. In other words, the planner does not obtain any advantage in terms of expected welfare in the case of three objects while selecting a bossy rule from the class of neutral, efficient and strategy-proof rules. We conjecture that this equivalence holds more generally. In our second result, we consider a case where the number of agents is arbitrary but equal to the agents. We consider a class of strategy-proof, efficient and bossy allocation rules which we call *lower bossy* rules. We show that every lower bossy rule is expected utility equivalent to an arbitrary priority rule. Again planner can impose the simplifying assumption of non-bossiness without loss of expected utility.

Chapter 1

Priority Domains[1]

1.1 Introduction

We consider the classical problem of allocating n objects to n agents. Each agent has a strict preference ordering over the objects, which belongs to some pre-specified preference domain. Svensson (1999) showed that the only (deterministic) allocation mechanisms defined over the universal domain and satisfying the properties of *strategy-proofness*, *non-bossiness* and *neutrality* are *priority rules*[2]. These are allocation rules where there is an exogenous ordering over all agents, say $(i_1, i_2, \ldots i_n)$. Agent i_1 moves first and picks her best object, followed by agent i_2 who picks his best object among the set of remaining objects and so on until all agents have been assigned objects. Our goal in this chapter is to examine the robustness of the Svensson (1999) characterization result with respect to the domain of preferences. We say that a domain of preferences is a *priority domain* if every strategy-proof, non-bossy and neutral allocation rule defined over the domain, is a priority rule. We investigate the structure of restricted domains that are priority domains.

Strategy-proofness and non-bossiness are standard assumptions in the mechanism design theory. An allocation rule is strategy-proof if no agent can strictly improve by misrepresenting her preferences. An allocation rule is non-bossy if no agent can change the allocations of other agents without changing her own[3]. Roughly speaking, neutrality is designed to capture the idea that the "names" of objects do not play any role in

[1] Co-authored with Arunava Sen (Indian Statistical Institute, Delhi Centre).

[2] These rules were called *serial dictatorships* in Svensson (1999). Here we follow the terminology in Moulin (2000).

[3] The non-bossiness axiom was first proposed in Satterthwaite and Sonnenschein (1981). For an extensive discussion of the interpretation and implications of the axiom, see Thomson (2016).

determining the allocation at any preference profile. If the objects are permuted at any preference profile, neutrality requires the resulting allocation to be a permutation of the original allocation, provided that the permuted profile belongs to domain of preference profiles. We consider two variants of the neutrality assumption. The first which we call *uniform profile neutrality* (UPN), applies only to preference profiles where all agents have identical preference orderings, i.e. agents are unanimous. Start with a preference profile where all agents have identical preference orderings and permute the objects in a manner such that the permuted objects generate an ordering which is also in the preference domain. The new preference profile must also be the one where all agents are unanimous. Then UPN requires the allocation at the new profile to be a permutation of the earlier allocation. A stronger condition is *full neutrality* (FN) where the neutrality condition applies to all profiles, not just those where agents are unanimous. We refer to UPN and FN priority domains depending upon the definition of neutrality assumed.

We identify a very simple condition called the *closure property* that is both necessary and sufficient for a domain to be a UPN priority domain (Theorem 1.1). Pick an arbitrary profile and an arbitrary priority. Construct the following ordering over the objects: the object assigned to the highest priority agent is first-ranked, the object assigned to the second-highest priority agent is second-ranked and so on. The domain satisfies the closure if this artificially constructed ordering belongs to the domain. The closure property is clearly a demanding property. Every UPN-priority domain that satisfies the property that every object is first-ranked in some ordering belonging to the domain satisfying (*minimal richness*) must be the universal domain (Proposition 1.2 part (ii)). As a consequence, restricted domains such as the domain of single-peaked preference orderings and the circular domain are not UPN-priority domains. However, a different class of domains which we call *range-restricted domains* are UPN-priority domains. An example of such a domain is the largest domain of preferences subject to the restriction that some object is never ranked first in any ordering in the domain.

The situation with respect to FN-priority domains is somewhat different. The class of such domains is clearly larger than that of UPN-priority domains. In Theorem 1.2, we provide an example of an FN-priority domain that is "small" relative to the universal domain. The construction of this domain uses ideas in Aswal et al. (2003). We consider domains that are symmetric in the following sense: if there is an ordering in the domain where object a_i is first-ranked and and object a_j is second-ranked, there is another preference ordering in the domain where the reverse is true, i.e. a_j is ranked first and a_i

second. Two objects are adjacent if they are ranked first and second in an ordering in the domain. A domain induces a graph in a natural way: the objects are vertices and there is an edge between any two vertices only if the corresponding objects are adjacent. We consider domains that satisfy two properties: (i) the domain's induced graph contains a Hamilton cycle and (ii) orderings of objects ranked three and lower, are unrestricted. We call such domains *lower complete Hamilton cycle* or LCHC domains. In Theorem 1.2, we show that the an LCHC domain is a FN-priority domain. We show that it is possible to construct LCHC domains that are approximately of size $\frac{1}{n}$ relative to the size of the universal domain. Our example also suggests a more general approach for constructing FN-priority domains.

An obvious way to interpret a priority rule is a sequential dictatorship. According to Svensson (1999), "our strategy-proof result is very similar to the Gibbard-Satterthwaite Theorem (Gibbard (1973); Satterthwaite (1975)). If the indivisible goods are interpreted as public goods instead of private goods as in the present chapter, the allocation mechanism is replaced by a "voting procedure" and serial dictatorship is replaced by "dictatorship"". Our results indicate that the parallel between the Gibbard-Satterthwaite Theorem and the priority result in Svensson (1999) does not extend to restricted domains. The structure of *dictatorial domains* in the voting model has been extensively studied, for instance in Aswal et al. (2003), Chatterji and Sen (2011), Pramanik (2015) and Sato (2010). The overall conclusion of these papers is that dictatorial domains can be significantly sparser than the universal domain even when they satisfy minimal richness. They can arise from restrictions on the objects that are ranked first and second (Aswal et al. (2003)). They can also be very small - for example Sato (2010) shows that the circular domain consisting of $2m$ orderings where m is the number of objects, is dictatorial. Priority domains on the other hand, have to be far richer. The only UPN-priority domain which satisfies minimal richness is the universal domain. Although we do not provide a characterization of FN-priority domains, LCHC domains are large compared to the smallest dictatorial domains.

There is an extensive literature on the allocation of discrete objects where agents have private information about their preferences. A more general class of strategy-proof allocation rules (including priority rules) defined over the universal domain is characterized in Pápai (2000) and further refined in Pycia and Ünver (2017). A related model where each agent is endowed with an object (called a house) was proposed in Shapley and Scarf (1974). A restricted domain version of the housing market model where agents have single-peaked preferences is analysed in Bade (2019). Random allocation rules over the universal domain are considered in Bogomolnaia and Moulin (2001) and Bade (2020) while Liu and Zeng (2019) and Liu (2020) look at random allocation rules over restricted domains.

This chapter is organised as follows. Section 1.2 presents the model and the axioms. Sections 1.3 and 1.4 contain results on UPN and FN-priority domains respectively. Section 1.5 concludes.

1.2 Preliminaries

Let $N = \{1, \ldots, n\}$ and $A = \{a_1, \ldots, a_n\}$ be the set of agents and the set of objects respectively. Each agent i has a strict preference ordering P_i over A where $a_j P_i a_k$ for any pair of distinct objects (a_j, a_k) implies that a_j is strictly preferred to a_k[4]. Let $\mathbb{P}$ denote the set of all strict orderings over A. The set of admissible preference orderings is called a *domain* and denoted by $\mathbb{D}$ where $\mathbb{D} \subseteq \mathbb{P}$. We shall refer to $\mathbb{P}$ as the *universal domain*. For any $P_i \in \mathbb{D}$ and $B \subset A$, we let $\max(P_i, B) = a_j$ if $a_j P_i a_k$ for all $a_k \in B \setminus \{a_j\}$; i.e. $\max(P_i, B)$ is the P_i maximal element in B. For future reference, $r_k(P_i)$ denotes the k^{th} ranked element in P_i where $k = 1, \ldots, n$. Thus $r_k(P_i) = a_j$ if $|\{a_s : a_j P_i a_s\}| = n - k$. Abusing notation slightly, we shall sometimes write $r(a_j, P_i) = k$ if $r_k(P_i) = a_j$, i.e. $r(a_j, P_i)$ is the rank of a_j in the preference ordering P_i.

A preference profile $P \in \mathbb{D}^n$ is an $n-$tuple $(P_1, \ldots, P_n)$. The i^{th} component of a profile P is the preference ordering of agent $i \in \{1, \ldots, n\}$. We shall denote profiles without subscripts, $P^1, P^2, \ldots, P^r$ etc. For any $P_i' \in \mathbb{D}$ and $P \in \mathbb{D}^n$, (P_i', P_{-i}) is the profile where P_i is replaced by P_i' in the profile P. A preference profile is *unanimous* if the preference orderings of all agents in the profile are identical. The set of all unanimous profiles will be denoted by $\mathbb{D}^n_U$.

An allocation φ is a bijection $\varphi : N \to A$. For any $i \in N$, $\varphi_i \in A$ is the object allocated to agent i. We will denote the set of allocations by Φ. An allocation rule F is a mapping $F : \mathbb{D}^n \to \Phi$; i.e. an allocation rule assigns an allocation to every preference profile. Here $F_i(P)$ is the object allocated to i at profile P according to the rule F.

A *priority* π is a bijection $\pi : N \to N$. The priority π defines a queue on the set of agents $(i_1, i_2, \ldots, i_n)$ such that $\pi(i_t) = t$ for all $t \in N$. In this queue, agent i_1 is

[4]A strict ordering is an ordering which is complete, reflexive, transitive and antisymmetric.

first, i_2 second and so on with i_n last. At every profile P, the priority π generates a *queue allocation* $\varphi^\pi(P)$ in the following way: for all $t \in N$, $\varphi^\pi_{i_t}(P) = \max(P_{i_t}, (A \setminus \{\varphi^\pi_{i_1}(P), \varphi^\pi_{i_2}(P), \ldots, \varphi^\pi_{i_{t-1}}(P)\}))$. In the queue allocation, i_1 gets the P_{i_1}-maximal element in the set A, i_2 gets the P_{i_2}-maximal element in the remainder set $A \setminus \{\varphi^\pi_{i_1}(P)\}$ and so on. An allocation rule F is a *priority rule* if there exists a priority π such that $F(P) = \varphi^\pi(P)$ for all profiles P. A priority rule with fixed priority π is denoted by F^π.

We now briefly describe some familiar requirements of allocation rules. An allocation rule F is *manipulable* if an agent has an incentive to misrepresent her preference ordering; i.e. there exists $i \in N$, $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$ such that $F_i(P_i', P_{-i}) P_i F(P_i, P_{-i})$. An allocation rule is *strategy-proof* if it is not manipulable. Priority rules are strategy-proof. This is a consequence of two features of priority rules. The first is that every agent i is presented with a set from which her P_i-maximal object is chosen. The second reason is that no agent can influence the set from which her P_i-maximal object is chosen and therefore misrepresenting her preference ordering P_i can only do her harm.

An allocation rule F is *non-bossy* if no agent can change the allocation at any profile without changing the object allocated to herself. Formally, for all $i \in N$, $P_i, P_i{}' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$, we have $F(P_i{}', P_{-i}) = F(P_i, P_{-i})$ whenever $F_i(P_i{}', P_{-i}) = F_i(P_i, P_{-i})$. The non-bossiness axiom is used widely in allocation theory - Thomson (2016) provides an extensive discussion of these issues. A priority rule satisfies non-bossiness. Pick an arbitrary agent i and suppose $F_i(P_i, P_{-i}) = F_i(P_i', P_{-i})$ for some $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{n-1}$. Then, the set of objects presented to agents following i in the priority will remain the same. Since the preference orderings of all agents other than i are unchanged, all agents following i in the priority will be presented the same sets to choose from and will choose the same objects at the profile (P_i', P_{-i}) as they did at (P_i, P_{-i}). This establishes that a priority rule satisfies non-bossiness.

An axiom related to strategy-proofness is *Maskin-Monotonicity (MM)*. For any $a_j \in A$ and $P_i \in \mathbb{D}$, we say that $P_i' \in \mathbb{D}$ is an MM transformation of P_i with respect to a_j (denoted by $P_i' \in MMT(a_j, P_i)$) if $a_j P_i x \implies a_j P_i' x$ for all $x \in A$. For any allocation φ and any profile $P \in \mathbb{D}^n$, we shall say $P' \in MMT(\varphi, P)$ if $P_i' \in MMT(\varphi_i, P_i)$ for all i. The allocation rule F satisfies MM if, for all preference profiles $P, P' \in \mathbb{D}^n$, $[P' \in MMT(F(P), P)] \implies [F(P') = F(P)]$. Let P and P' be preference profiles such that the object $F_i(P)$ "improves" in P_i' relative to P_i for each agent $i \in N$. Then MM requires the outcome of F at P' to be same as the outcome at P. The MM property is central to the Nash-implementability of social choice correspondences (see

Maskin (1999)). For one of our results, we will use the fact that an allocation function defined over an arbitrary domain of preferences that satisfies strategy-proofness and non-bossiness, also satisfies MM. This result is well-known (see Klaus and Bochet (2013)) but we state and prove it formally below.

Proposition 1.1. *Let $\mathbb{D}$ be an arbitrary domain and let F be an allocation rule $F : \mathbb{D}^n \to \Phi$. If F satisfies strategy-proofness and non-bossiness, it satisfies MM.*

Proof. Let $\mathbb{D}$ be an arbitrary domain and let $F : \mathbb{D}^n \to \Phi$ be an allocation rule satisfying strategy-proofness and non-bossiness. Let $P, P' \in \mathbb{D}^n$ be profiles such that $P'_i \in MMT(F_i(P), P_i)$ for all $i \in N$. Pick an arbitrary agent i and suppose $a_j = F_i(P'_i, P_{-i}) \neq F_i(P_i, P_{-i}) = a_k$. If $a_j P_i a_k$, then i manipulates at (P_i, P_{-i}) via P'_i contradicting the strategy-proofness of F. The remaining case is $a_k P_i a_j$. Since $P'_i \in MMT(a_k, P_i)$, $a_k P'_i a_j$ holds. Then i manipulates at (P'_i, P_{-i}) via P_i contradicting the strategy-proofness of F. Therefore $F_i(P'_i, P_{-i}) = F_i(P_i, P_{-i})$. Since F satisfies non-bossiness, $F(P_i, P_{-i}) = F(P'_i, P_{-i})$. Changing the preference orderings of agents in $N \setminus \{i\}$ successively and applying the earlier argument, we conclude $F(P) = F(P')$. $\square$

We have noted that priority rule satisfies strategy-proofness and non-bossiness. Proposition 1.1 therefore implies that priority rules satisfies MM.

An important axiom in our analysis is *neutrality.* Informally, neutrality imposes the requirement that the names of objects do not matter. Consider a preference profile and a permutation on the set of objects. The permutation induces a permuted preference profile in an obvious way. A neutral allocation rule links the outcomes at the original profile and its permuted counterpart; in particular, the permutation of the outcome at the original profile is the outcome at the permuted profile. We will consider two different notions of neutrality of an allocation rule depending on the profiles over which permutations are allowed to apply.

Let $\sigma : A \to A$ be a permutation over the set of objects. For any preference ordering $P_i \in \mathbb{D}$, the preference ordering $[\sigma \circ P_i]$ is defined as follows: for all distinct $a_j, a_k \in A$, $a_j P_i a_k \implies \sigma(a_j)\ [\sigma \circ P_i]\ \sigma(a_k)$. For any profile $P \in \mathbb{D}^n$, $[\sigma \circ P]$ is the profile obtained where the preference ordering of every agent $i \in N$ is $[\sigma \circ P_i]$. Note that an arbitrary permutation σ when applied to a profile $P \in \mathbb{D}^n$ may result in a permuted profile $[\sigma \circ P]$ that does not belong to $\mathbb{D}^n$. We would therefore like to restrict attention to *domain consistent* permutations. Let $P \in \mathbb{D}^n$. The permutation σ is P-consistent if $[\sigma \circ P] \in \mathbb{D}^n$.

We let $\Sigma(P)$ denote the set of P-consistent permutations. Finally, for every allocation φ, we let $\sigma \circ \varphi$ be the allocation defined by $\sigma \circ \varphi_i = \sigma(\varphi_i)$.

The first notion of neutrality is with respect to unanimous profiles. An allocation rule F is *Unanimous Profile Neutral* or satisfies UPN if, for all $P \in \mathbb{D}_U^n$ and $\sigma \in \Sigma(P)$, we have $F(\sigma \circ P) = \sigma \circ (F(P))$. Since P is a unanimous profile, $\sigma \circ P$ is also a unanimous profile in $\mathbb{D}^n$. Moreover, every unanimous profile can be obtained by a permutation of the profile P, that is, for every $P' \in \mathbb{D}_U^n$, there exists $\sigma \in \Sigma(P)$ such that $\sigma \circ P = P'$. Observe also that for any $P \in \mathbb{D}_U^n$, every allocation φ is a queue allocation with respect to a unique priority π. Suppose, for instance $a_{j_1} P_i a_{j_2} P_i \ldots P_i a_{j_n}$ and agent i_r is allocated the object a_{j_r}, $r = 1, \ldots, n$ in an arbitrary allocation φ. Then $\varphi = \varphi^\pi$ where π is the priority $(i_1, i_2, \ldots, i_n)$. An allocation rule F satisfies UPN if the associated priorities for the allocations $F(P)$ and $F(P')$ for any pair of profiles $P, P' \in \mathbb{D}_U^n$, are the same.

The other notion of neutrality applies to all preference profiles. Recall that $\Sigma(P)$ is the set of all permutations that are P-consistent. An allocation rule F is *Fully Neutral* or satisfies FN if, for all $P \in \mathbb{D}^n$ and $\sigma \in \Sigma(P)$, we have $F(\sigma \circ P) = \sigma \circ (F(P))$. The FN requirement is stronger than UPN because the latter applies only to a subset of all profiles. An important observation is that a priority rule satisfies FN and therefore UPN. Let the k^{th}-agent in the priority π be i_k and suppose that at the profile P, she chooses her P_{i_k}-maximal object in the set B. In the permuted profile, i_k will get her $\sigma \circ P_{i_k}$-maximal object in the permuted set $\sigma(B)$. If i_k's allocated object is a_{i_j} in profile P, it will be $\sigma(a_{i_j})$ in $\sigma \circ P$.

We introduce some specific domains which will be used in applications later in the chapter. Let $<$ be a strict ordering on the set of objects A. A (strict) preference ordering P_i is *single-peaked* (Black (1948), Arrow (1951)) if $a_k < a_j \leq r_1(P_i)$ or $r_1(P_i) \leq a_j < a_k$ imply $a_j P_i a_k$. A domain is a *single-peaked domain* (denoted by $\mathbb{D}^{SP}$) if it contains all preferences that are single-peaked (with respect to $<$). The single-peaked domain is widely used in social choice and political economy. Assume without loss of generality that $a_1 < a_2 < \cdots < a_n$. Then, the only non-trivial permutation in Σ is σ where $\sigma(a_j) = a_{n+1-j}$ for $j = 1, \ldots, n$. This is the permutation that "reverses" $<$. However, there exist profiles $P \in [\mathbb{D}^{SP}]^n$ where $\Sigma(P)$ contains other permutations. This is illustrated in Table 1.1 where P is a neutral transformation of P' with a permutation not equal to σ.

Another domain that we shall consider is the fully *circular* domain [5]. Objects are

[5]The definition of circular domain in Sato (2010) is slightly different from ours. In his definition, there are no restrictions on the ranking of objects other than those ranked either second or last for every

P_1	P_2	P_3	P_4	P'_1	P'_2	P'_3	P'_4
a_2	a_2	a_3	a_3	a_3	a_3	a_2	a_2
a_3	a_3	a_2	a_2	a_2	a_2	a_3	a_3
a_4	a_4	a_4	a_4	a_4	a_4	a_4	a_4
a_1	a_1	a_1	a_1	a_1	a_1	a_1	a_1

Table 1.1: Profiles $P = (P_1, P_2, P_3, P_4)$ and $P' = (P'_1, P'_2, P'_3, P'_4)$

arranged in a circle with a_j adjacent to a_{j+1} and a_{j-1}, $j = 1, \ldots, n$ with $a_{n+1} = a_1$ and $a_n = a_0$. The circular domain denoted by $\mathbb{D}^c$ consists of $2n$ preferences. For every object a_j, $j = \{1, \ldots, n\}$, there are exactly two preferences P_i^j and $P_i'^j$ where $a_j = r_1(P_i^j) = r_1(P_i'^j)$. Here P_i^j is the "clockwise" preference ordering $a_j P_i^j a_{j+1} P_i^j \ldots a_{j-2} P_i^j a_{j-1}$ while $P_i'^j$ is the "counter-clockwise" preference ordering $a_j P_i'^j a_{j-1} P_i'^j \ldots a_{j+2} P_i'^j a_{j+1}$. Let σ and σ' be permutations corresponding to the one-step clockwise and counter-clockwise rotations of the objects, i.e. $\sigma(a_j) = a_{j+1}$, $j = 1, \ldots n$ with $a_{n+1} = a_1$ and $\sigma'(a_j) = a_{j-1}$, $j = 1, \ldots n$. For every $P \in [\mathbb{D}^c]^n$, we have $\Sigma(P) = \{\sigma, \sigma^2, \ldots, \sigma^{n-1}, \sigma', (\sigma')^2, \ldots, (\sigma')^{n-1}\}$. Note that $\Sigma(P)$ does not depend on P.

We are now ready to define the objects that we investigate in the chapter.

Definition 1.1. The domain $\mathbb{D}$ is a *UPN-Priority domain* if every allocation rule $F : \mathbb{D}^n \to \Phi$ satisfying strategy-proofness, non-bossiness and UPN is a priority rule.

Definition 1.2. The domain $\mathbb{D}$ is a *FN-Priority domain* if every allocation rule $F : \mathbb{D}^n \to \Phi$ satisfying strategy-proofness, non-bossiness and FN is a priority rule.

In view of our earlier remarks regarding the various neutrality notions, it follows that every UPN-priority domain is an FN-priority domain. This is obvious since if a strategy-proof and non-bossy rule F defined on $\mathbb{D}$ does not satisfy UPN on then F does not satisfy FN as well.

1.3 UPN-Priority Domains

In this section we will provide a simple condition that characterizes UPN-priority domains and examine its implications.

Fix a domain $\mathbb{D}$. Let $\pi = (i_1, i_2, \ldots, i_n)$ be a priority and let $P \in \mathbb{D}^n$ be a profile. The allocation $\varphi^\pi(P)$ induces an ordering $P_i^* \in \mathbb{P}$ as follows: $r_t(P_i^*) = \varphi_{i_t}^\pi(P)$ for all

circular preference ordering.

$t = 1, \ldots, n$. The first-ranked object in P_i^* is the object given to the first agent i_1 in $\varphi^\pi(P)$, the second-ranked object in P_i^* is the object given to the second agent i_2 in $\varphi^\pi(P)$ and so on. Note that the ordering P_i^* may not be in $\mathbb{D}$. We shall denote the ordering induced by $\varphi^\pi(P)$ on P by $\lambda(\varphi^\pi(P))$.

We illustrate the construction of the ordering P_i^* in Table 1.2. Let $N = \{1, 2, 3, 4\}$, $A = \{a_1, a_2, a_3, a_4\}$ and $\pi = (1, 2, 3, 4)$. At the profile $P = (P_1, P_2, P_3, P_4)$, $\varphi^\pi(P) = (a_1, a_3, a_4, a_2)$[6]. This induces the ordering $P_i^* = \lambda(\varphi^\pi(P))$ where $a_1 P_i^* a_3 P_i^* a_4 P_i^* a_2$.

P_1	P_2	P_3	P_4	P_i^*
a_1	a_1	a_3	a_3	a_1
a_2	a_3	a_4	a_2	a_3
a_3	a_4	a_1	a_4	a_4
a_4	a_2	a_2	a_1	a_2

Table 1.2: Profile $P = (P_1, P_2, P_3, P_4)$ and P_i^*

Closure Property: The domain $\mathbb{D}$ satisfies the Closure Property if, for all profiles $P \in \mathbb{D}^n$ and for all priorities π , we have $\lambda(\varphi^\pi(P)) \in \mathbb{D}$.

Theorem 1.1. *A domain is a UPN-Priority domain if only if it satisfies the Closure Property.*

Proof. We begin by showing that a domain $\mathbb{D}$ that satisfies the Closure Property is a UPN-priority domain. Let $\mathbb{D}$ satisfy the Closure Property and let $F : \mathbb{D}^n \to \Phi$ be an arbitrary allocation rule satisfying strategy-proofness, non-bossiness and UPN. We will show that F is a priority rule. Note that F satisfies MM by virtue of Proposition 1.1.

It follows from our earlier remark regarding allocation rules satisfying UPN that there exists a priority, say $\pi = (i_1, i_2, \ldots, i_n)$ such that $F(P) = \varphi^\pi(P)$ for all $P \in \mathbb{D}_U^n$. We will complete the proof by showing that $F(P) = \varphi^\pi(P)$ for all $P \in \mathbb{D}^n$.

Let $P^* \in \mathbb{D}_U^n$ be the unanimous profile where $P_i^* = \lambda(\varphi^\pi(P))$ for all $i \in N$. Since F satisfies UPN, $F(P^*) = \varphi^\pi(P^*)$. Let $a_{i_k} = \varphi_{i_k}^\pi(P)$. By definition of the allocation $\varphi^\pi(P)$, $a_{i_k} P_{i_k} x$ for all $x \in B_k$ where $B_k = \{a_{k+1}, \ldots, a_n\}$, $k = 1, \ldots n-1$. Note that $a_{i_1} P_i^* a_{i_2} P_i^* \ldots P_i^* a_{i_n}$ for all $i \in N$. Therefore in the allocation $\varphi^\pi(P^*)$, we have $a_{i_k} P_{i_k} x$ only if $x \in B_k$. Hence $P_{i_k} \in MMT(a_{i_k}, P_{i_k}^*)$ for all $k = 1, \ldots, n$. Since F satisfies MM and $F_{i_k}(P^*) = a_{i_k}$, for all $k = 1, \ldots, n$, we have $F(P) = \varphi^\pi(P)$ as required.

[6]Here a_1, a_3, a_4 and a_2 are the objects allocated to 1,2, 3 and 4 respectively

In order to prove the second part of the Theorem, we let $\mathbb{D}$ be a domain that violates the Closure Property. Therefore, there exists a priority $\pi' = (i_1, i_2, \ldots, i_n)$ and a profile $P' \in \mathbb{D}^n$ such that $\lambda(\varphi^{\pi'}(P')) \notin \mathbb{D}$. Pick an arbitrary priority, say $\pi = (1, \ldots, n)$. Let $\alpha : N \to N$ be such that $\alpha(k) = i_k$ for all $k = 1, \ldots, n$. Let $\hat{P} \in \mathbb{D}^n$ be such that $\hat{P}_k = P'_{\alpha(k)}$ for all $k = 1, \ldots, n$. It follows immediately that $\lambda(\varphi^{\pi'}(P')) = \lambda(\varphi^{\pi}(\hat{P}))$. Hence $\lambda(\varphi^{\pi}(\hat{P})) \notin \mathbb{D}$.

We will construct an allocation rule that is not a priority rule but satisfies strategy-proofness, non-bossiness and UPN. The idea behind the proof is to partition the set $\mathbb{D}^n$ into two sets with allocations in each partition determined according to different priorities.

For notational convenience, let $\varphi^{\pi}_j(\hat{P}) = a_j$, $j = 1, \ldots, n$ so that $a_1 P^*_i a_2 P^*_i \ldots P^*_i a_n$ where $P^*_i = \lambda(\varphi^{\pi}(\hat{P}))$. By assumption $P^*_i \notin \mathbb{D}$. Let $\mathbb{D}^n_1 \subset \mathbb{D}^n$ denote the set of profiles $P \in \mathbb{D}^n$ such that (i) $a_{n-1} P_i a_n$ for $i \in \{n-1, n\}$ and (ii) $\varphi^{\pi}(P) = \varphi^{\pi}(\hat{P})$. Note that $a_{n-1}\hat{P}_{n-1}a_n$. Consider the profile $\bar{P} \in \mathbb{D}^n$ where $\bar{P}_i = \hat{P}_i$ for all $i = 1, \ldots, n-1$ and $\bar{P}_n = \bar{P}_{n-1} = \hat{P}_{n-1}$. It follows that $\bar{P} \in \mathbb{D}^n_1$ implying that $\mathbb{D}^n_1 \neq \emptyset$.

We claim that $\mathbb{D}^n_1 \cap \mathbb{D}^n_U = \emptyset$. Suppose to the contrary that a profile P exists in the intersection of the two sets. According to part (ii) of the requirement of profiles in $\mathbb{D}^n_1$, $\varphi^{\pi}(P) = \varphi^{\pi}(\hat{P})$, i.e. $\varphi^{\pi}_j(P) = a_j$ for all $j = 1, \ldots, n$. But then $P_i = P^*_i$ for all $i \in N$ which contradicts our hypothesis that $P^*_i \notin \mathbb{D}$.

Let $\bar{\pi}$ denote the priority $(1, \ldots, n-2, n, n-1)$. Define the allocation rule $F : \mathbb{D}^n \to \Phi$ as follows: for all $P \in \mathbb{D}^n$,

$$F(P) = \begin{cases} \varphi^{\bar{\pi}}(P) & \text{if } P \in \mathbb{D}^n_1 \\ \varphi^{\pi}(P) & \text{if } P \in \mathbb{D}^n \setminus \mathbb{D}^n_1 \end{cases}$$

The allocation rule F is not a priority rule. By our earlier claim all unanimous profiles belong to the set $\mathbb{D}^n \setminus \mathbb{D}^n_1$. If F were a priority rule the only candidate for the priority would be π since the allocation at unanimous profiles uniquely determines the priority. However, this is ruled out since $\varphi^{\pi}_{n-1}(\bar{P}) = a_{n-1}$ while $F_{n-1}(\bar{P}) = a_n$. We will show that F satisfies UPN, strategy-proofness and non-bossiness.

In order to see that F satisfies UPN, it suffices to observe that there are no unanimous profiles in $\mathbb{D}^n_1$.

We will show that F is strategy-proof. Note that allocation at any profile $P \in \mathbb{D}^n$

is given by some priority and the order of an agent $i \in \{1, \dots, n-2\}$ in every priority remains the same. The agent $i \in \{1, \dots, n-2\}$ can therefore never gain by reporting a different preference ordering at any profile P. The agent $j \in \{n-1, n\}$ can however improve her order in the priority by changing her preference ordering. First we consider the agent $n-1$. At any profile $P \in \mathbb{D}_1^n$, the agent $n-1$ might improve her order in the priority from n to $n-1$ by reporting an ordering P'_{n-1} such that $a_n P'_{n-1} a_{n-1}$. But $F_{n-1}(P'_{n-1}, P_{-(n-1)}) = F_{n-1}(P) = a_n$. Therefore the agent $n-1$ can never gain by reporting a different preference ordering at any profile. Now consider agent n. At any profile $P \in \mathbb{D}^n \setminus \mathbb{D}_1^n$ such that $a_{n-1} P'_{n-1} a_n$, the agent n might improve her order in the priority from n to $n-1$ by reporting an ordering P'_n such that $a_{n-1} P'_n a_n$. But $F_n(P'_n, P_{-n}) = a_{n-1} = F_n(P)$. Therefore the agent n can also never gain by reporting a different preference ordering at any profile. Hence F is strategy-proof.

We show that F satisfies non-bossiness. Pick $P \in \mathbb{D}$, $i \in \{1, \dots, n\}$ and P'_i such that $F_i(P) = F_i(P'_i, P_{-i})$. Note that the outcome of F at any profile is a priority allocation either with respect to π or $\bar{\pi}$. If the outcome at P and (P'_i, P_{-i}) is the same priority (either π or $\bar{\pi}$), $F(P) = F(P_i, P_{-i})$ follows from the fact that priority rules satisfy non-bossiness. The only cases to consider therefore are when $P \in \mathbb{D}_1^n$ and $(P'_i, P_{-i}) \in \mathbb{D}^n \setminus \mathbb{D}_1^n$ or vice-versa, i.e. agent i can effect a switch between priorities $\bar{\pi}$ and π or vice-versa by changing her preference ordering. We shall call such a switch a regime-switch.

We claim that no agent $i \in \{1, \dots, n-2\}$ can affect a regime switch. Pick $P \in \mathbb{D}_1^n$ and $i \in \{1, \dots, n-2\}$. Then $F_i(P) = F_i(P'_i, P_{-i}) = a_i$ and $(P'_i, P_{-i}) \in \mathbb{D}_1^n$. Similarly, $F_i(P) = F_i(P'_i, P_{-i}) = a_i$ implies $P \in \mathbb{D}^n \setminus \mathbb{D}_1^n \implies (P'_i, P_{-i}) \in \mathbb{D}^n \setminus \mathbb{D}_1^n$. Agent $i \in \{n-1, n\}$ can effect a regime-switch. Assume that this does occur. Observe that if $P \in \mathbb{D}^n \setminus \mathbb{D}_1^n$ and $P' \in \mathbb{D}_1^n$, then $\varphi_j^{\pi}(P) = a_j$ for all $j \in \{1, \dots, n-2\}$. Therefore it must be the case that $F_j(P) = a_j$ for all $j \in \{1, \dots, n-2\}$. Pick $i \in \{n-1, n\}$. Note that the priority of agents $j = \{1, \dots, n-2\}$ is unchanged between $\bar{\pi}$ and π. Therefore $F_j(P) = F_j(P'_i, P_{-i}) = a_j$ for all $j = \{1, \dots, n-2\}$. Since $F_i(P) = F_i(P'_i, P_{-i})$ by assumption, it must be true that $F_k(P) = F_k(P'_i, P_{-i})$ for the remaining agents $k = N \setminus \{i\}$. Therefore F satisfies non-bossiness. □

We now examine the implications of Theorem 1.1. A domain $\mathbb{D}$ satisfies the *minimal richness (MR)* property if, for all $a_j \in A$, there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) = a_j$. This axiom has been used frequently in the literature (see Aswal et al. (2003) for example). It requires that every object be ranked first in some preference ordering in the domain.

A domain $\mathbb{D}^R(B)$ is *rank-restricted with respect to* a non-empty set $B \subset A$ if there exists an integer $t \in \{1, \ldots n-1\}$ such that for all $a_j \in B$ and for all $P_i \in \mathbb{D}^R(B)$, we have $r(a_j, P_i) > t$. A domain is *rank-restricted* if there exists B with respect to which it is rank-restricted. A domain would be rank-restricted if for example, an object is never ranked first, or always ranked lower than third or always ranked last etc. Note that a rank-restricted domain must violate minimal richness. A rank-restricted domain $\mathbb{D}^R(B)$ is *maximal* if there does not exist another rank-restricted domain $\bar{\mathbb{D}}^R(B)$ such that $\mathbb{D}^R(B) \subset \bar{\mathbb{D}}^R(B)$. An example of a rank-restricted domain is the domain consisting of *all* preference orderings where the object a_j is not ranked-first.

Proposition 1.2. *The following statements hold:*

1. *(Svensson (1999)) The universal domain $\mathbb{P}$ is a UPN-priority domain.*

2. *If $\mathbb{D}$ is a UPN-priority domain and satisfies MR, then $\mathbb{D} = \mathbb{P}$.*

3. *The domains $\mathbb{D}^{SP}$ and $\mathbb{D}^c$ are not UPN-priority domains.*

4. *Every maximal rank-restricted domain $\mathbb{D}^R(B)$ is a UPN-priority domain.*

Proof. The universal domain $\mathbb{P}$ trivially satisfies the Closure Property with respect to all priorities. Part 1 of the Proposition follows immediately as an application of the first part of Theorem 1.1.

Let $\mathbb{D}$ be a UPN-priority domain. According to the second part of Theorem 1.1, $\mathbb{D}$ must satisfy the Closure Property. We show if $\mathbb{D}$ satisfies MR, it satisfies the Closure Property only if $\mathbb{D} = \mathbb{P}$. Assume to the contrary that $P_i^* \in \mathbb{P} \setminus \mathbb{D}$. Pick an arbitrary priority $\pi = (i_1, \ldots, i_n)$. Let $P \in \mathbb{D}^n$ be a profile such that $r_1(P_{i_k}) = r_k(P_i^*)$, $k = 1, \ldots, n$. The k^{th} agent in the queue π is i_k where $k = 1, \ldots, n$. The profile P is chosen such that agent i_k's first-ranked object in the preference P_{i_k}, is the k^{th}-ranked object in the ordering P_i^*. The profile $P \in \mathbb{D}^n$ by virtue of our assumption that $\mathbb{D}$ satisfies MR. By construction, $\lambda(\varphi^\pi(P)) = P_i^*$. Since $P_i^* \notin \mathbb{D}$ by assumption, the Closure Property is violated.

The single-peaked domain $\mathbb{D}^{SP}$ and the circular domain $\mathbb{D}^c$ both satisfy MR and are strict subsets of the universal domain. It follows from part (ii) above that they are not UPN-priority domains.

Let $\mathbb{D}^R(B)$ be an arbitrary maximal rank-restricted domain. Let π be the priority $(1, \ldots, n)$. Pick an arbitrary profile $P \in [\mathbb{D}^R(B)]^n$. Let $a_j \in B$. According to the definition of a rank-restricted domain, there exists $t \in \{1, \dots, n-1\}$ such that $r(a_j, P_i) > t$ for all $P_i \in \mathbb{D}^R(B)$. Therefore, $\varphi_i^\pi(P) \neq a_j$ for all $i \leq t$. In other words, none of the agents $i \in \{1, \dots t\}$ will be allocated the object a_j. Since $\mathbb{D}^R(B)$ is a maximal rank-restricted domain, it will be the case that $\lambda(\varphi^\pi(P)) \in \mathbb{D}^R(B)$. Therefore $\mathbb{D}^R(B)$ satisfies the Closure Property and Theorem 1.1 implies that it is a UPN-domain. □

1.4 FN-priority domains

In this section we show that FN-priority domains can be much smaller than UPN domains. We construct a special FN-priority domain that satisfies minimal richness and is smaller than the complete domain. In order to do so, we introduce some additional concepts.

Let $\mathbb{D}$ be an arbitrary domain. We assume that $\mathbb{D}$ satisfies the following property which we call *basic symmetry*: for all $a_i, a_j \in A$, if there exists $P_i \in \mathbb{D}$ such that $r_1(P_i) = a_i$ and $r_2(P_i) = a_j$, then there also exists $P_i' \in \mathbb{D}$ such that $r_1(P_i') = a_j$ and $r_2(P_i') = a_i$. Throughout this section, we shall confine attention to domains that satisfy basic symmetry.

Following Aswal et al. (2003), we define the graph $G(\mathbb{D})$ induced by $\mathbb{D}$ as follows: the set of vertices in the graph is the set A and (ii) vertices a_i and a_j form an edge (or are adjacent) in the graph (denoted by $a_i \sim a_j$) if there exist preference orderings P_i and P_i' such that $r_1(P_i) = a_i$, $r_2(P_i) = a_j$. In other words, a_i and a_j are connected by an edge if there exists a preference ordering in the domain where a_i and a_j are ranked first and second respectively. By basic symmetry there will also exist another preference ordering where the reverse is true. Let $A(a_i)$ denote the set of objects which are adjacent to a_i for all $a_i \in A$.

The graph $G(\mathbb{D})$ admits a *Hamilton cycle* if there exists a permutation $\lambda : \{1, \dots, n\} \to \{1, \dots, n\}$ such that $a_{\lambda(j)} \sim a_{\lambda(j+1)}$, $j = 1, \dots, n$ where $\lambda(n+1) = \lambda(1)$. A Hamilton cycle is a cycle that visits each vertex in the graph exactly once and ends at the vertex it begins. Hamilton cycles have been extensively studied in graph theory (see Wilson (1979)). The graph of the circular domain $G(\mathbb{D}^c)$ admits a Hamilton cycle. On the other hand, the graph of a single-peaked domain $G(\mathbb{D}^{SP})$ does not admit a Hamilton cycle. Suppose the underlying order $<$ on the set A is $a_1 < a_2 < \cdots < a_n$. Then $a_j \sim a_{j+1}$, $j = 1, \dots, n-1$. Since a_1 and a_n are not adjacent $G(\mathbb{D}^{SP})$ does not contain a Hamilton cycle[7].

The domain $\mathbb{D}$ is *lower complete* if there are no restrictions on preferences for ranked three or lower. Let $B \subset A$. We say preference orderings $P_i, P'_i \in \mathbb{P}$ agree on B (denoted by $P_i|_B = P'_i|_B$) if $a_j P_i a_k \Leftrightarrow a_j P'_i a_k$ for all $a_j, a_k \in B$. For any $P_i \in \mathbb{P}$, let $T(P_i) = \{r_1(P_i), r_2(P_i)\}$ and $W(P_i) = A \setminus T(P_i)$. The domain $\mathbb{D}$ is lower complete if, for all $P_i \in \mathbb{D}$ and $P'_i \in \mathbb{P}$ with $T(P_i) = T(P'_i)$, there exists $\hat{P}_i \in \mathbb{D}$ such that $\hat{P}_i|_{T(P_i)} = P_i|_{T(P_i)}$ and $\hat{P}_i|_{W(P_i)} = P'_i|_{W(P_i)}$. Suppose $P_i \in \mathbb{D}$ is such that a_i and a_j are ranked first and second respectively in P_i. In order for $\mathbb{D}$ to be lower complete, it must contain *all* preference orderings where a_i and a_j are ranked first and second respectively.

The domain $\mathbb{D}$ is a *lower complete Hamilton cycle (LCHC) domain* if it is lower complete and $G(\mathbb{D})$ admits a Hamilton cycle. The existence of Hamilton cycles is a restriction on the first and second-ranked objects of preference orderings in the domain. A LCHC domain is the "largest" domain consistent with the restrictions imposed on the first and second-ranked preference orderings. The universal domain is, of course, an LCHC domain. However, an LCHC domain can be "small" relative to the universal domain. For example, consider the lower complete domain $\bar{\mathbb{D}}$ whose graph $G(\bar{\mathbb{D}})$ consists *only* of the following edges: $a_j \sim a_{j+1}$ for all $j = 1, \dots, n$ with $a_{n+1} = a_1$. Then $|\bar{\mathbb{D}}| = 2n(n+2)!$ so that $\frac{|\bar{\mathbb{D}}|}{|\mathbb{P}|} = \frac{2}{n-1}$.

Our main result in this section is the following:

Theorem 1.2. *Every LCHC domain is an FN-priority domain.*

Proof. Let $\mathbb{D}$ be an LCHC domain and assume without loss of generality that $a_j \sim a_{j+1}$ for $j = 1, \dots, n$ where $a_{n+1} = a_1$. Let F be an arbitrary allocation rule $F : \mathbb{D}^n \to \Phi$ satisfying strategy-proofness, non-bossiness and FN. Since F satisfies UPN, there exists a priority π such that for all $P \in \mathbb{D}^n_U$, we have $F(P) = \varphi^\pi(P)$. We will complete the proof by showing that $F(P) = \varphi^\pi(P)$ for all $P \in \mathbb{D}^n$.

Let $P \in \mathbb{D}^n$ be such that $\lambda(\varphi^\pi(P)) \in \mathbb{D}$. Since F satisfies UPN, the arguments in the proof of Theorem 1.1 can be used to conclude that $F(P) = \varphi^\pi(P)$. So, pick an arbitrary $P \in \mathbb{D}^n$ such that $\lambda(\varphi^\pi(P)) \notin \mathbb{D}$. We will show that $F(P) = \varphi^\pi(P)$. We proceed as follows: we show that there exists $\bar{P}, \hat{P} \in \mathbb{D}^n$ such that:

1. $\hat{P} \in MMT(\varphi(P^u), P^u)$ for some $P^u \in \mathbb{D}^n_U$.

[7] Note that $G(\mathbb{D}^{SP})$ contains a Hamilton path.

2. $\hat{P} = \sigma \circ \bar{P}$ for some $\sigma \in \Sigma(\bar{P})$.

3. $P \in MMT(\varphi^{\pi}(\bar{P}), \bar{P})$.

Since F satisfies strategy-proofness and non-bossiness, it satisfies MM (Proposition 1.1). Since $F(P^u) = \varphi^{\pi}(P^u)$, MM implies $F(\hat{P}) = \varphi^{\pi}(\hat{P})$. i.e. the allocation at $\hat{P}$ is also according to priority π. Since F satisfies FN, $\hat{P} = \sigma \circ \bar{P}$ implies that the allocation at $\bar{P}$ is also according to priority π. Finally, MM and $P \in MMT(\varphi^{\pi}(\bar{P}), \bar{P})$ imply that the allocation at P is also according to priority π.

Assume without loss of generality that $\pi = (1, \dots n)$. We will first construct $\bar{P}$ followed by $\hat{P}$ and then P^u. Suppose $r_1(P_1) = r_1(P_2) = a_j$ (say). Then $\varphi_2(P) = r_2(P_2) = a_s$(say). By Basic Symmetry $a_j \sim a_s$. Applying the lower complete property of the LCHC domain $\mathbb{D}$, it follows that $\lambda(\varphi^{\pi}(P)) \in \mathbb{D}$ contradicting our hypothesis. Suppose $r_1(P_1) \neq r_1(P_2)$ but $r_1(P_1) \sim r_1(P_2)$, then the earlier argument can be used to deduce that $\lambda(\varphi^{\pi}(P)) \in \mathbb{D}$, a contradiction. We conclude that neither $r_1(P_1) = r_1(P_2)$ nor $r_1(P_1) \sim r_1(P_2)$ can hold.

Let $Z = \{i \in N | \varphi_i^{\pi}(P) \sim r_1(P_j) \; for \; some \; j < i\}$. We will show that $Z \neq \emptyset$. In order to verify this claim, we consider two cases.

The first case is when the first-ranked objects of all the agents are distinct. Since $G(\mathbb{D})$ admits a Hamilton cycle, the first-ranked object of the last agent in π (agent n) is adjacent to some object which must be the first-ranked object of an agent who appears earlier in the priority. Clearly $n \in Z$. The other case occurs when the first-ranked objects of some agents are repeated. Let s be the first such agent in π. Since the first-ranked object of all agents before s are distinct, these agents must be getting their first-ranked objects in $\varphi^{\pi}(P)$. Suppose $\varphi_s^{\pi}(P) = r_2(P_s)$. Since $r_1(P_s) \sim r_2(P_s)$ (using basic symmetry again) and $r_1(P_s) = r_1(P_t)$ for some $t < s$, we have $\varphi_s^{\pi}(P) \sim r_1(P_t)$ so that $s \in Z$. Suppose $\varphi_s^{\pi}(P) \neq r_2(P_s)$. Then there must exist $s' < s$ such that $r_1(P_{s'}) = r_2(P_s)$. By assumption there also exists $\hat{s} < s$ such that $r_1(P_{\hat{s}}) = r_1(P_s)$. We know that $r_1(P_s) \sim r_2(P_s)$. If $s' < \hat{s}$, then we have $\hat{s} \in Z$ and if $\hat{s} < s'$, then we have $s' \in Z$.

Since $Z \neq \emptyset$, we can pick m, the smallest integer in Z. We claim that $r_1(P_i) \neq r_1(P_j)$ for all $i, j < m$. Suppose this is false. Then there exists two agents i, j with $i < j < m$ such that $r_1(P_i) = r_1(P_j)$. Using the arguments in the earlier paragraph (the second case), we can argue that either $j \in Z$ or there exists $j' < j$ such that $j' \in Z$. In either case we contradict the assumption that m is the smallest integer in Z. Also note that $\varphi^{\pi}_m(P) \in \{r_1(P_m), r_2(P_m)\}$. If this were false, we could use earlier arguments to deduce the existence of an agent $j' < m$ such that $j' \in Z$.

We can summarize our conclusions thus far as follows:

1. There does not exist $i, j < m$ such that $r_1(P_i) = r_1(P_j)$.
2. There does not exist $i, j < m$ such that $r_1(P_i) \sim r_1(P_j)$.
3. $\varphi^{\pi}_i(P) = r_1(P_i)$ for all $i < m$ and $\varphi^{\pi}_m(P) \in \{r_1(P_m), r_2(P_m)\}$.
4. $m \geq 3$.

We now construct $\bar{P}$. We proceed in four steps. In Step I we construct $\bar{P}_m$. In Step II we construct $\bar{P}_1$ and $\bar{P}_2$. In Step III we construct $\bar{P}_3$ to $\bar{P}_{m-1}$. Finally in Step IV, we construct $\bar{P}_{m+1}$ to $\bar{P}_n$.

Step I: Construction of $\bar{P}_m$. There are two cases to consider.

Case A1: $\varphi^{\pi}_m(P) = r_2(P_m)$ i.e. $r_1(P_m) = r_1(P_i)$ for some $i < m$. Then $\bar{P}_m$ is defined as follows:

$$r_t(\bar{P}_m) = \begin{cases} r_t(P_m) & \text{for } t = 1, 2. \\ \varphi^{\pi}_t(P) & \text{for } t > m. \end{cases}$$

The top two objects in $\bar{P}_m$ are the same as in P_m and ranked in the same way. The $(m+1)^{th}$ ranked object in $\bar{P}_m$ is the object assigned to agent $(m+1)$ according to π at P, the $(m+2)^{th}$ ranked object in $\bar{P}_m$ is the object assigned to agent $(m+2)$ according to π at P, and so on. The remaining objects in $\bar{P}_m$ are ranked arbitrarily.

Case A2: $\varphi^{\pi}_m(P) \neq r_2(P_m)$. By 3 in the Conclusion above, $\varphi^{\pi}_m(P) = r_1(P_m)$. Since $m \in Z$ there exists $l < m$ such that $\varphi^{\pi}_m(P) \sim r_1(P_l)$. Then $\bar{P}_m$ is defined as follows:

$$r_t(\bar{P}_m) = \begin{cases} r_1(P_l) & \text{for } t = 1. \\ r_1(P_m) & \text{for } t = 2. \\ \varphi^{\pi}_t(P) & \text{for } t > m. \end{cases}$$

In this case, the first-ranked object in $\bar{P}_m$ is the first-ranked object of agent l where $\varphi_m^\pi(P) \sim r_1(P_l)$. In case there is more than one such agent, one of these agents is chosen arbitrarily. The second-ranked object in $\bar{P}_m$ is the first-ranked object in P_m. Note that the first-ranked and the second-ranked objects in $\bar{P}_m$ are distinct by the definition of $\sim$. The objects ranked $(m+1)$ and below are exactly the same as in Case A1. The remaining objects in $\bar{P}_m$ are ranked arbitrarily also as in Case A1.

Note that in both cases $r_2(\bar{P}_m) = \varphi_m^\pi(P)$ and $r_1(\bar{P}_m) = r_1(\bar{P}_i)$ for some $i < m$.

Step II: Construction of $\bar{P}_1$ and $\bar{P}_2$.

For notational convenience we shall denote $r_1(P_1)$ by a_j, $r_1(P_2)$ by a_k and $\varphi_m^\pi(P)$ by a_q. Note that $a_q \neq a_j \neq a_k \neq a_q$. Let $r_1(\bar{P}_m) = a_l$ where $a_l = r_1(\bar{P}_i)$ for some $i < m$ following the construction of $\bar{P}_m$ above. We will let $r_1(\bar{P}_1) = a_j$, $r_1(\bar{P}_2) = a_k$ and we choose the second-ranked objects for agents 1 and 2 carefully. All other objects are ranked arbitrarily. There are three cases to consider.

Case B1: $a_l = a_j$. Let $r_2(\bar{P}_1) = a_s$ where $a_s = r_2(\bar{P}_m) = a_q$ and let $r_2(\bar{P}_2) = a_t$ where

$$a_t \in \begin{cases} \{a_{q-1}\} & \text{if } a_k = a_{q-2}. \\ \{a_{q-2}\} & \text{if } a_k = a_{q-1}. \\ \{a_{q+2}\} & \text{if } a_k = a_{q+1}. \\ \{a_{q+1}\} & \text{if } a_k = a_{q+2}. \\ \{a_{k-1}, a_{k+1}\} \setminus \{a_{q-2}, a_q, a_{q+2}\} & \text{otherwise} \end{cases}$$

Recall that we have assumed that $a_{i-1} \sim a_i \sim a_{i+1}$ for all $i = \{1, \ldots n\}$ with $a_{n+1} = a_1$ and $a_0 = a_n$. Also a_t is well-defined because $\{a_{k-1}, a_{k+1}\}$ is a subset of $\{a_{q-2}, a_q, a_{q+2}\}$ only if $a_k \in \{a_{q-1}, a_{q+1}\}$. Note that $a_t = a_{s-2}$ only if $a_k = a_{s-1}$ and $a_t = a_{s+2}$ only if $a_k = a_{s+1}$.

Case B2: $a_l = a_k$. Let $r_2(\bar{P}_2) = a_t$ where $a_t = r_2(\bar{P}_m) = a_q$ and let $r_2(\bar{P}_1) = a_s$ where

$$a_s = \begin{cases} a_{j-1} & \text{if } a_q = a_{j+1}. \\ a_{j+1} & \text{otherwise.} \end{cases}$$

Note that $a_t \in \{a_{s-2}, a_{s+2}\}$ only if $(a_s, a_t) \in \{(a_{j-1}, a_{j+1}), (a_{j+1}, a_{j-1}), (a_{j+1}, a_{j+3})\}$.

Case B3: $a_j \neq a_l \neq a_k$. Then let $r_1(\bar{P}_1) = a_s$ where

$$a_s = \begin{cases} a_{j-1} & \text{if } a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\}. \\ a_{j+1} & \text{otherwise.} \end{cases}$$

Let $r_2(\bar{P}_2) = a_t$ where

$$a_t \in \begin{cases} \{a_{j-3}\} & \text{if } a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\} \text{ and } a_k = a_{j-2}. \\ \{a_{j-2}\} & \text{if } a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\} \text{ and } a_k = a_{j-3}. \\ \{a_{k-1}, a_{k+1}\} \setminus \{a_{j-3}, a_{j-1}, a_q\} & \text{if } a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\} \text{ and } a_k \notin \{a_{j-2}, a_{j-3}\}. \\ \{a_{j+3}\} & \text{if } a_q \notin \{a_{j+1}, a_{j+2}, a_{j+3}\} \text{ and } a_k = a_{j+2}. \\ \{a_{j+2}\} & \text{if } a_q \notin \{a_{j+1}, a_{j+2}, a_{j+3}\} \text{ and } a_k = a_{j+3}. \\ \{a_{k-1}, a_{k+1}\} \setminus \{a_{j+1}, a_q\} & \text{otherwise} \end{cases}$$

Observe that a_t is well-defined because if $a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\}$ then $\{a_{k-1}, a_{k+1}\}$ is a subset of $\{a_{j-3}, a_{j-1}, a_q\}$ only if $a_k \in \{a_{j-2}, a_j\}$. Note that $a_k = a_j$ is not possible by assumption and a_t has already been defined for the case where $a_k = a_{j-2}$. If $a_q \notin \{a_{j+1}, a_{j+2}, a_{j+3}\}$ then $\{a_{k-1}, a_{k+1}\}$ is a subset of $\{a_{j+1}, a_q\}$ only if $a_k = a_j$, which is not possible. Again note that $a_t = a_{s+2}$ either if $a_q = a_{j+3}$ and $a_k = a_{s+3}$ or if $a_q \notin \{a_{j+1}, a_{j+2}, a_{j+3}\}$ and $a_k = a_{s+1}$. Also $a_t = a_{s-2}$ either if $a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\}$ and $a_k = a_{s-1}$ or if $a_q \notin \{a_{j-1}, a_{j+1}, a_{j+2}, a_{j+3}\}$ and $a_k = a_{s-3}$.

Step III: Construction of $\bar{P}_i$ for $i \in \{3, \ldots, m-1\}$.

For all $i \in \{3, \ldots, m-1\}$, let $r_1(\bar{P}_i) = r_1(P_i)$. Denote $r_1(\bar{P}_i)$ by a_r and $r_2(\bar{P}_i)$ by a_u. The choice of a_u will depend on the choice of a_s and a_t made in Step II. As before the ranking of objects below the second rank is arbitrary.

$$a_u \in \begin{cases} \{a_{j+4}\} & \text{if } a_r = a_{j+3},\ a_s = a_{j+1} \text{ and } a_t = a_{j+2}. \\ \{a_{j+2}\} & \text{if } a_r = a_{j+3},\ a_s = a_{j+1} \text{ and } a_t \neq a_{j+2}. \\ \{a_{j+3}\} & \text{if } a_r = a_{j+2},\ a_s = a_{j+1} \text{ and } a_t \neq a_{j+2}. \\ \{a_{s-2}\} & \text{if } a_r = a_{s-1},\ a_s \in \{a_{j-1}, a_{j-2}, a_{j-3}\}. \\ \{a_{s-1}\} & \text{if } a_r = a_{s-2},\ a_s \in \{a_{j-1}, a_{j-2}, a_{j-3}\}. \\ \{a_{s+2}\} & \text{if } a_r = a_{s+1},\ a_s \notin \{a_{j+1}, a_{j-1}, a_{j-2}, a_{j-3}\}. \\ \{a_{s+1}\} & \text{if } a_r = a_{s+2},\ a_s \notin \{a_{j+1}, a_{j-1}, a_{j-2}, a_{j-3}\}. \\ A(a_r) \setminus \{a_s, a_t\} & \text{otherwise} \end{cases}$$

We claim that $r_2(\bar{P}_i)$ is well-defined because $A(a_r) \subseteq \{a_s, a_t\}$ cannot occur unless $a_r = a_{j+2}$. It can occur only either if $a_t = a_{s-2}$ and $a_r = a_{s-1}$ or if $a_t = a_{s+2}$ and $a_r = a_{s+1}$. In Case B1 of Step II we saw that $a_t = a_{s-2}$ only if $a_k = a_{s-1}$ so that $a_t = a_{s-2}$ and $a_r = a_{s-1}$ is impossible (refer to 3 of the summary of conclusions above). Similarly $a_t = a_{s+2}$ only if $a_k = a_{s+1}$. This implies that $a_t = a_{s+2}$ and $a_r = a_{s+1}$ is also ruled out. In Case B3 we saw that $a_t = a_{s+2}$ either if $a_q = a_{j+3}$ and $a_k = a_{s+3}$ or if $a_q \notin \{a_{j+1}, a_{j+2}, a_{j+3}\}$ and $a_k = a_{s+1}$. If $a_k = a_{s+1}$ then $a_r \neq a_{s+1}$. If $a_q = a_{j+3}$, then $a_s = a_{j-1}$. Thus $a_r = a_{s+1}$ would imply $a_r = a_j$ which is not possible. Also we have $a_t = a_{s-2}$ either if $a_q \in \{a_{j+1}, a_{j+2}, a_{j+3}\}$ and $a_k = a_{s-1}$ or if $a_q \notin \{a_{j-1}, a_{j+1}, a_{j+2}, a_{j+3}\}$ and $a_k = a_{s-3}$. If $a_k = a_{s-1}$ then $a_r \neq a_{s-1}$. If $a_q \notin \{a_{j-1}, a_{j+1}, a_{j+2}, a_{j+3}\}$, then $a_s = a_{j+1}$. Thus $a_r = a_{s-1}$ would imply $a_r = a_j$, which is not possible. In Case B2 we saw that $a_t \in \{a_{s-2}, a_{s+2}\}$ only if $(a_s, a_t) \in \{(a_{j-1}, a_{j+1}), (a_{j+1}, a_{j-1}), (a_{j+1}, a_{j+3})\}$. If $a_s = a_{j-1}$ then the case $a_r = a_{j+1}$ and hence $a_r = a_j$ which violates 1 in summary above. If $a_s = a_{j+1}$ and $a_t \neq a_{j+2}$ then a_t must be a_{j+3} and $a_r = a_{j+2}$ but this case is already specified in the third piece of the function a_u.

Step IV: Construction of $\bar{P}_i$ for $i > m$.

We let $\bar{P}_i$=$\bar{P}_m$ for all $i > m$. This concludes the description of the profile $\bar{P}$.

We summarize the profile $\bar{P}$ in Table 1.4 below.

$\bar{P}_1$	$\bar{P}_2$	.	$\bar{P}_i$	.	$\bar{P}_m$	.	$\bar{P}_n$
a_j	a_k	.	a_r	.	a_l	a_l	a_l
a_s	a_t	.	a_u	.	a_q	a_q	a_q
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.3: The Profile $\bar{P}$.

The profile $\bar{P}$ has been constructed so that the following properties are satisfied:

(i) $\varphi^\pi(\bar{P}) = \varphi^\pi(P)$.

(ii) The second-ranked objects in every $\bar{P}_i$ are specified so that $\bar{P}_i \in \mathbb{D}$.

(iii) $a_s \neq a_t$.

(iv) $a_s = a_q$ only if $a_l = a_j$ and $a_t = a_q$ only if $a_l = a_k$.

(v) If $a_s \in \{a_{j-1}, a_{j-2}, a_{j-3}\}$ then $a_u \in \{a_{s-1}, a_{s-2}\}$ only if $a_r \in \{a_{s-1}, a_{s-2}\}$.

(vi) If $a_s \notin \{a_{j-1}, a_{j-2}, a_{j-3}\}$ then $a_u \in \{a_{s+1}, a_{s+2}\}$ only if $a_r \in \{a_{s+1}, a_{s+2}\}$.

We now proceed to the construction of the profile $\hat{P}$. In order to do this, we construct a permutation $\sigma : A \to A$ as follows. Before that, we summarize the profile $\bar{P}$ in Table 1.4 below.

$\bar{P}_1$	$\bar{P}_2$	.	$\bar{P}_i$	.	$\bar{P}_m$	.	$\bar{P}_n$
a_j	a_k	.	a_r	.	a_l	a_l	a_l
a_s	a_t	.	a_u	.	a_q	a_q	a_q
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.4: The Profile $\bar{P}$.

In order to define permutation σ, there are five cases to consider which depend on the nature of a_s, a_k and a_t in the profile $\bar{P}$.

Case A: $a_s = a_{j-1}, a_k = a_{j-2}$ and $a_t = a_{j+1}$.

In this case, $\sigma(a_j) = a_s, \sigma(a_s) = a_j, \sigma(i) = i$ for all $i \in A \setminus \{a_j, a_s\}$. If we assume that $a_l = a_j$ then $\hat{P}$ looks as in Table 1.5.

$\hat{P}_1$	$\hat{P}_2$	.	$\hat{P}_i$	.	$\hat{P}_m$	.	$\hat{P}_n$
a_s	a_k	.	a_r	.	a_s	a_s	a_s
a_j	a_t	.	a_u	.	a_j	a_j	a_j
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.5: The Profile $\hat{P}$, Case A.

Note that we are using (iii) and (iv) of the description of $\bar{P}$.

Case B: $a_s = a_{j-1}, a_k = a_{j-3}$ and $a_t = a_{j+1}$.

In this case, $\sigma(a_j) = a_s, \sigma(a_s) = a_j, \sigma(a_k) = a_{j-2}, \sigma(a_t) = a_k, \sigma(a_{j-2}) = a_t, \sigma(i) = i$ for all $i \in A \setminus \{a_j, a_s, a_k, a_t, a_{j-2}\}$. If we further assume that $a_l = a_k$ then $\hat{P}$ looks as in Table 1.6.

Note that we are again using (iii) and (iv) of the description of $\bar{P}$.

Case C: Cases A and B do not occur and $a_s \in \{a_{j-1}, a_{j-2}, a_{j-3}\}$.

$\hat{P}_1$	$\hat{P}_2$	.	$\hat{P}_i$	.	$\hat{P}_m$	.	$\hat{P}_n$
a_s	a_{s-1}	.	a_r	.	a_{s-1}	a_{s-1}	a_{s-1}
a_t	a_k	.	a_u	.	a_k	a_k	a_k
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.6: The Profile $\hat{P}$, Case B.

In this case, $\sigma(a_j) = a_s, \sigma(a_s) = a_j, \sigma(a_k) = a_{s-1}, \sigma(a_t) = a_{s-2}, \sigma(a_{s-1}) = a_k, \sigma(a_{s-2}) = a_t, \sigma(i) = i$ for all $i \in A \setminus \{a_j, a_s, a_k, a_t, a_{s-1}, a_{s-2}\}$. If we assume that $a_j \neq a_l \neq a_k$ and $a_r = a_{s-2}$ then $\hat{P}$ looks as in Table 1.7.

$\hat{P}_1$	$\hat{P}_2$	.	$\hat{P}_i$	.	$\hat{P}_m$	.	$\hat{P}_n$
a_s	a_{s-1}	.	a_t	.	a_l	a_l	a_l
a_j	a_{s-2}	.	a_k	.	a_q	a_q	a_q
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.7: The Profile $\hat{P}$, Case C.

Note that we are using (iii), (iv) and (v) of the description of $\bar{P}$.

Case D: $a_s = a_{j+1}, a_k \neq a_{j+3}$ and $a_t = a_{j+2}$.

In this case, $\sigma(a_j) = a_s, \sigma(a_s) = a_j, \sigma(a_k) = a_t, \sigma(a_t) = a_k, \sigma(i) = i$ for all $i \in A \setminus \{a_j, a_s, a_k, a_t\}$. If we assume that $a_j \neq a_l \neq a_k$ and $a_r \notin a_{s-1}, a_{s-2}$ then $\hat{P}$ looks as in Table 1.8.

$\hat{P}_1$	$\hat{P}_2$	.	$\hat{P}_i$	.	$\hat{P}_m$	.	$\hat{P}_n$
a_s	a_t	.	a_r	.	a_l	a_l	a_l
a_j	a_k	.	a_u	.	a_q	a_q	a_q
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.8: The Profile $\hat{P}$, Case D.

Note that we are using (ii) and (iv) of the description of $\bar{P}$ and the definition of a_u in Step III of the construction of $\bar{P}$.

Case E: Case D does not occur and $a_s \notin \{a_{j-1}, a_{j-2}, a_{j-3}\}$.

In this case, $\sigma(a_j) = a_s, \sigma(a_s) = a_j, \sigma(a_k) = a_{s+1}, \sigma(a_t) = a_{s+2}, \sigma(a_{s+1}) = a_k, \sigma(a_{s+2}) = a_t, \sigma(i) = i$ for all $i \in A \setminus \{a_j, a_s, a_k, a_t, a_{s+1}, a_{s+2}\}$. If we assume that $a_j \neq a_l \neq a_k$ and

$a_r = a_{s+1}$ then $\hat{P}$ is as shown in Table 1.9.

$\hat{P}_1$	$\hat{P}_2$	.	$\hat{P}_i$	.	$\hat{P}_m$	.	$\hat{P}_n$
a_s	a_{s+1}	.	a_k	.	a_l	a_l	a_l
a_j	a_{s+2}	.	a_t	.	a_q	a_q	a_q
.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.

Table 1.9: The Profile $\hat{P}$, Case E.

Note that we are using (ii),(iv) and (vi) of the description of $\bar{P}$.

In each of the Cases A-E, we observe the following features of the profile $\hat{P}$.

(i) For each agent i, the second-ranked objects in $\hat{P}_i$ are specified in a manner such that $\hat{P}_i \in \mathbb{D}$.

(ii) $r_1(\hat{P}_1) \sim r_1(\hat{P}_2)$.

Let $P^u \in \mathbb{D}_U^n$ be the unanimous profile where $r_j(P_i^u) = \varphi_j^\pi(\hat{P})$ for all $j = \{1, \ldots, n\}$ and $i = 1, \ldots n$. Note that P^u exists by virtue of part (ii) of the description of $\hat{P}$. By assumption, $F(P^u) = \varphi^\pi(P^u)$. By construction $\hat{P} \in MMT(\varphi^\pi(P^u), P^u)$. Since F satisfies MM, we have $F(\hat{P}) = F(P^u) = \varphi^\pi(\hat{P})$. Since $\hat{P} = \sigma \circ \bar{P}$, FN implies that $F(\bar{P}) = \varphi^\pi(\bar{P})$. Finally, $P \in MMT(\varphi^\pi(\bar{P}), \bar{P})$ by construction, so that MM implies $F(P) = \varphi^\pi(P)$. This completes the proof. $\square$

The construction of the profiles $\bar{P}$ and $\hat{P}$ in the proof of Theorem 1.2 are complicated. We illustrate them in the Example below.

Example 1.1. Let $N = \{1, 2, \ldots, 6\}$ and $A = \{a_1, a_2, \ldots, a_6\}$. Suppose $a_{i-1} \sim a_i \sim a_{i+1}$ for all $i \in N$ where $a_0 = a_6$ and $a_7 = a_1$. We consider an LCHC domain. The profile P is shown in Table 1.10 below.

Assume w.l.o.g that $\pi = \{1, 2, \ldots, 6\}$. Since $a_1 \sim a_5$ does not hold, $\lambda(\varphi^\pi(P)) \notin \mathbb{D}$. Also, $m = 4$ since $\varphi_4^\pi(P) = a_6 \sim a_1 = r_1(P_1)$ and $\varphi_4^\pi(P) = a_6 \sim a_5 = r_1(P_2)$. According to our construction $\bar{P}$, $\hat{P}$ and P^u are the profiles shown in Table 1.11, 1.12 and 1.13 respectively.

The allocation in profile P^u is $\varphi^\pi(P^u) = (a_6, a_5, a_3, a_1, a_2, a_4)$. Note that $\hat{P} \in MMT(\varphi^\pi(P^u), P^u)$. Observe that $\hat{P} = \sigma \circ \bar{P}$ where σ is as follows: $\sigma(a_1) = a_6$, $\sigma(a_6) = a_1$

P_1	P_2	P_3	P_4	P_5	P_6
a_1	a_5	a_3	a_6	a_5	a_5
a_2	a_6	a_4	a_1	a_6	a_6
a_3	a_1	a_5	a_2	a_1	a_2
a_4	a_2	a_6	a_3	a_2	a_4
a_5	a_3	a_1	a_4	a_3	a_3
a_6	a_4	a_2	a_5	a_4	a_1

Table 1.10: Example 1: The Profile P.

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$	$\bar{P}_6$
a_1	a_5	a_3	a_1	a_1	a_1
a_6	a_4	a_4	a_6	a_6	a_6
a_3	a_3	a_5	a_3	a_3	a_3
a_5	a_2	a_6	a_5	a_5	a_5
a_2	a_1	a_1	a_2	a_2	a_2
a_4	a_6	a_2	a_4	a_4	a_4

Table 1.11: Example 1: The Profile $\bar{P}$.

$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$	$\hat{P}_5$	$\hat{P}_6$
a_6	a_5	a_3	a_6	a_6	a_6
a_1	a_4	a_4	a_1	a_1	a_1
a_3	a_3	a_5	a_3	a_3	a_3
a_5	a_2	a_1	a_5	a_5	a_5
a_2	a_6	a_6	a_2	a_2	a_2
a_4	a_1	a_2	a_4	a_4	a_4

Table 1.12: Example 1:The Profile $\hat{P}$.

P_1^u	P_2^u	P_3^u	P_4^u	P_5^u	P_6^u
a_6	a_6	a_6	a_6	a_6	a_6
a_5	a_5	a_5	a_5	a_5	a_5
a_3	a_3	a_3	a_3	a_3	a_3
a_1	a_1	a_1	a_1	a_1	a_1
a_2	a_2	a_2	a_2	a_2	a_2
a_4	a_4	a_4	a_4	a_4	a_4

Table 1.13: Example 1:The Profile P^u.

and $\sigma(a_j) = a_j$ for $j = \{2, 3, 4, 5\}$. By FN, $F(\bar{P}) = (a_1, a_5, a_3, a_6, a_2, a_4) = \varphi^\pi(\bar{P})$. Furthermore, $P \in MMT(\varphi^\pi(\bar{P}), \bar{P})$. Then MM implies $F(P) = (a_1, a_5, a_3, a_6, a_2, a_4) = \varphi^\pi(P)$.

The converse of Theorem 1.2 is not true. We provide an example of a priority domain over the set of four objects that neither admits a Hamiltonion cycle nor is lower complete.

Example 1.2. Assume that $A = \{a_1, a_2, a_3, a_4\}$ and $N = \{1, 2, 3, 4\}$. Suppose $\mathbb{D} = \mathbb{P} \setminus \{a_1a_2a_4a_3, a_1a_3a_2a_4, a_1a_3a_4a_2, a_1a_4a_2a_3, a_1a_4a_3a_2\}$[8]. The restricted domain $\mathbb{D}$ does not admit a Hamiltonion cycle since there exists an object $a_1 \in A$ that is connected only to a_2 in $G(\mathbb{D})$. The domain is not lower complete either since the absence of $a_1a_2a_4a_3$ in $\mathbb{D}$ restricts the preference orderings for objects ranked three or lower.

Proposition 1.3. *$\mathbb{D}$ is a FN-priority domain.*

Proof. Suppose $F : \mathbb{D}^4 \to \Phi$ is an allocation rule that satisfies strategy-proofness, non-bossiness and FN. It follows from Proposition 1.1 that F satisfies MM. By definition, F satisfies UPN since it satisfies FN. Since F satisfies UPN, there must exist a priority π such that for all $P \in \mathbb{D}^4_U$, we have $F(P) = \phi^\pi(P)$. We assume w.l.o.g. that $\pi = (1, 2, 3, 4)$.

Consider an arbitrary profile $P \in \mathbb{D}^4$ such that $\lambda(\varphi^\pi(P)) \in \mathbb{D}$. Since F satisfies UPN, we can use the arguments in the proof of Theorem 1.1 to conclude that $F(P) = \varphi^\pi(P)$. For $P \in \mathbb{D}^4$ such that $\lambda(\varphi^\pi(P)) \notin \mathbb{D}$, we consider the following cases.

1. $\lambda(\varphi^\pi(P)) = a_1a_2a_4a_3$.
   Suppose $\bar{P} \in \mathbb{D}^4$ is such that $\bar{P}_1 = \bar{P}_2 = a_2a_3a_1a_4$ and $\bar{P}_3 = \bar{P}_4 = a_3a_2a_4a_1$. Since $\lambda(\varphi^\pi(\bar{P})) \in \mathbb{D}$, we must have $F(\bar{P}) = \varphi^\pi(\bar{P}) = (a_2, a_3, a_4, a_1)$. Consider a permutation σ over A such that $\sigma(a_1) = a_3$, $\sigma(a_2) = a_1$, $\sigma(a_3) = a_2$ and $\sigma(a_4) = a_4$. Since F satisfies FN, it must be true that $F(\sigma \circ \bar{P}) = \sigma \circ F(\bar{P})$. Hence it must be the case that $F(\sigma \circ \bar{P}) = \varphi^\pi(\sigma \circ \bar{P}) = (a_1, a_2, a_4, a_3)$. By construction, it must be that $P = MMT(\varphi^\pi(\sigma \circ \bar{P}), \sigma \circ \bar{P})$. Since F satisfies MM, $F(P) = \varphi^\pi(P) = (a_1, a_2, a_4, a_3)$.

2. $\lambda(\varphi^\pi(P)) \neq a_1a_2a_4a_3$.
   We know that $r_1(\lambda(\varphi^\pi(P))) = a_1$. We denote $r_2(\lambda(\varphi^\pi(P)))$, $r_3(\lambda(\varphi^\pi(P)))$ and $r_4(\lambda(\varphi^\pi(P)))$ by a_j, a_k and a_l respectively. Note that a_j must either be a_3 or a_4. Suppose $\bar{P}$ is such that $\bar{P}_1 = a_1a_2a_3a_4$ and $\bar{P}_2 = \bar{P}_3 = \bar{P}_4 = a_ja_1a_ka_l$. Clearly $\bar{P}$ is an admissible preference profile. Consider a permutation σ over A such that $\sigma(a_1) = a_l$, $\sigma(a_l) = a_1$, $\sigma(a_j) = a_j$ and $\sigma(a_k) = a_k$. The profile $\sigma \circ \bar{P}$ is admissible since $r_1(\sigma \circ \bar{P}_1) = a_l \neq a_1$ and $r_1(\sigma \circ \bar{P}_2) = r_1(\sigma \circ \bar{P}_3) = r_1(\sigma \circ \bar{P}_4) = a_j \neq$

[8] Here we employ a new piece of notation: $a_1a_2a_4a_3$ is the preference ordering where a_1 is ranked first, a_2 second, a_4 third and a_3 last. This is to be distinguished from (a_1, a_2, a_4, a_3) which is the notation for an allocation where agent 1 receives a_1, 2 receives a_2, 3 receives a_4 and 4 receives a_3.

a_1. Moreover, $\lambda(\varphi^\pi(\sigma \circ \bar{P})) \in \mathbb{D}$ since $\lambda(\varphi^\pi(\sigma \circ \bar{P})) = a_l a_j a_k a_1$. This implies $F(\sigma \circ \bar{P}) = \varphi^\pi(\sigma \circ \bar{P})$. Since F satisfies FN, we must have $F(\bar{P}) = \sigma^{-1} \circ F(\sigma \circ \bar{P}) = (a_1, a_j, a_k, a_l)$. By construction, $P = MMT(\varphi^\pi(\bar{P}), \bar{P})$. Since F satisfies MM, $F(P) = \varphi^\pi(P) = (a_1, a_j, a_k, a_l)$.

We have established that if $F : \mathbb{D}^4 \to \Phi$ satisfies strategy-proofness, non-bossiness and FN, it must be a priority rule. This implies that $\mathbb{D}$ is a FN-priority domain. $\square$

Theorem 1.2 suggests a way to construct a more general class of FN priority domains. We can start with unanimous preference profiles and construct sequences of alternating Maskin-Monotonic and FN transformed profiles to "reach" all the profiles in $\mathbb{D}^n$. In Theorem 1.2, we only used sequences of length three: we started with a unanimous profile P^u, made a MM transformation to obtain $\hat{P}$, then a neutral transformation to obtain $\bar{P}$ and then another MM transformation to arrive at P. But these sequences could be longer in principle. We can restrict attention to alternating MM and FN transformations since successive MM and FN transformations can be obtained from a single MM and FN transformations respectively. We choose not to pursue this research direction further.

1.5 Conclusion

In this chapter we have shown that unlike the dictatorial domains in the voting setting, priority domains must be universal as long as they satisfy the minimal richness requirement. A weak notion of neutrality is required for this result. A stronger notion of neutrality, full neutrality, is compatible with priority domains that are smaller than the universal domain.

Chapter 2

Random-dictatorship on Restricted Domains

2.1 Introduction

A classic result in the theory of Mechanism Design is the Gibbard–Satterthwaite Theorem which says that every strategy-proof social choice function defined over a complete domain of preference orderings must be dictatorial, provided it satisfies the condition of unanimity. One way to avoid the consequences of this negative result is to consider random social choice functions, where the outcome at every profile of preference orderings is a lottery over the set of alternatives, thereby ameliorating the conflicts of interest between different players.

In a model where the outcomes are probabilistic, the comparison of different lotteries is a central issue. Gibbard (1977) proposed the use of the notion of stochastic dominance for this purpose. This criterion is now widely used in the literature. The paper showed that the only strategy-proof random social choice function using this notion are *unilaterals* and *duples*. In addition, if the condition of unanimity is imposed, the only strategy-proof random social choice function is a *random-dictatorship* i.e., each agent is chosen to be a dictator with a fixed probability. In this Chapter, we explore the robustness of the Gibbard's random-dictatorship result on restricted domains of preference orderings.

Aswal et al. (2003) introduced the concept of *dictatorial domains* in the following manner: a preference domain is dictatorial if every strategy-proof and unanimous social choice function defined on the domain is dictatorial. They went on to show that a class of domains called *linked domains* are dictatorial. Linked domains can be "small" in size relative to the universal domain which is, of course, also dictatorial (Gibbard (1973)). Linked domain is only one class of dictatorial domains. Sato (2010) defined *circular domains* which are not linked domains but are dictatorial as well. Pramanik (2015) introduced the class of β domains and γ domains which generalised linked domains and circular domains respectively. The paper showed that β domains and almost all γ-domains are dictatorial[1].

Random-dictatorial domains can be defined in an analogous manner: a preference domain is random-dictatorial if every strategy-proof and unanimous random social choice function defined on the domain is a random-dictatorship. A natural question is the connection between dictatorial domains and random-dictatorial domains. Chatterji et al. (2014) investigate the relationship between dictatorial and random-dictatorial domains. They showed that a dictatorial domain need not be random-dictatorial. In fact, they showed that stronger conditions need to be imposed on a linked domain in order for it to be random-dictatorial.

Our contribution in this chapter is to show that the γ-domains identified in Pramanik (2015) as dictatorial (which we call P- domains) are in fact random-dictatorial. This result stands in contrast to Chatterji et al. (2014) result on linked domains which showed that the "gap" between dictatorial and random-dictatorial domains is "large" for linked domains.

A circular domain is a P- domain; our result therefore implies that circular domains are random-dictatorial. This observation can be used to deduce the minimum size of a random-dictatorial domain which satisfies the property that every alternative is ranked first in some admissible preference orderings (This property is known as minimal richness). The minimum size of this domain is $2m$ where m is the number of alternatives. The result in Sato (2010) showed that the minimum size of a dictatorial domain satisfying minimal richness is also $2m$.

The proof of our main result is lengthy and complicated. It would have been much shorter if existing results could be used be show that P- domains satisfy the tops-only property for domains. Chatterji and Zeng (2018) provide a sufficient condition in the regard. Unfortunately, P- domains do not satisfy their conditions.

This chapter is organised as follows. Section 2.2 describes the model and basic concepts. Section 2.3 briefly reviews the existing results on dictatorial and random-

[1]These results are discussed in detail in 2.3.

dictatorial domains. Section 2.4 contains the statement and proof of the main result of the chapter. Section 2.5 is a discussion section which presents the result on the minimum size of random-dictatorial domains and the relationship of the P- domains to the conditions in Chatterji and Zeng (2018). Section 2.6 concludes.

2.2 Model and Basic Definitions

Let $A = \{a_1, \ldots, a_m\}$ be a finite set of alternatives with $m \geq 3$ and let $N = \{1, 2, \ldots, n\}$ be a finite set of agents with $n \geq 2$. Every agent i has a strict preference ordering (a complete, transitive and anti-symmetric binary relation) P_i over the set of alternatives A. For any pair of distinct alternatives (a_j, a_k), $a_j P_i a_k$ signifies that a_j is strictly preferred to a_k according to P_i. The set of strict preference orderings is denoted by $\mathbb{P}$ and referred to as the *universal domain*. The set of admissible preference orderings is denoted by $\mathbb{D} \subseteq \mathbb{P}$ and referred to as the *admissible domain*.

A profile of preference orderings $P \in \mathbb{D}^n$ is an $n-$tuple of preference orderings $(P_1, \ldots, P_n) = (P_i, P_{-i})$ where the i^{th} component of the tuple is the preference ordering of agent $i \in N$ and P_{-i} denotes the preference orderings of agents other than i. We write profiles without subscripts such as $P, P', \tilde{P}$ while the corresponding preference orderings of agent i are written with subscripts such as $P_i, P'_i, \tilde{P}_i$ respectively.

For every $P_i \in \mathbb{D}$ and $a_j \in A$, $B(a_j, P_i)$ denotes the set of alternatives in A that are *strictly better* than a_j according to the preference ordering P_i i.e., $B(a_j, P_i) = \{a_t \in A \setminus \{a_j\} \mid a_t P_i a_j\}$. We refer to $B(a_j, P_i)$ as the strict upper contour set of a_j in P_i and $B(a_j, P_i) \cup \{a_j\}$ as the upper contour set of a_j in P_i. Let $U(a_j, P_i)$ denote the upper contour set of a_j in P_i. Let $W(a_j, P_i)$ denote the set of alternative in A that are *strictly worse* than a_j according to P_i i.e., $B(a_j, P_i) = \{a_t \in A \setminus \{a_j\} \mid a_j P_i a_t\}$. Also let $M(a_j, a_k, P_i)$ denote the set of alternatives that are ranked worse than a_j but better than a_k in P_i i.e., $M(a_j, a_k, P_i) = \{a_t \in A \setminus \{a_j, a_k\} \mid a_j P_i a_t \ \& \ a_t P_i a_k\}$. Note that $M(a_j, a_k, P_i)$ may be empty.

An alternative a_j is ranked k in P_i if the cardinality of $B(a_j, P_i)$ is $k-1$. The t^{th} ranked alternative in $P_i \in \mathbb{D}$ is $r_t(P_i)$ and let $r(a_j, P_i)$ denote the rank of $a_j \in A$ in $P_i \in \mathbb{D}$. The first-ranked alternative in any preference ordering will sometimes be referred to as the *peak* of the preference ordering. For all $a_j \in A$, let $\mathbb{D}^{a_j} = \{P_i \in \mathbb{D} | r_1(P_i) = a_j\}$. For all $B \subset A$, let $\mathbb{D}^B = \{P_i \in \mathbb{D} | r_1(P_i) \in B\}$. For all $a_j, a_k \in A$ let $\mathbb{D}^{a_j, a_k} = \{P_i \in \mathbb{D} | r_1(P_i) = a_j$

and $r_2(P_i) = a_k\}$.

A (deterministic) *social choice function* (SCF) is a mapping $f : \mathbb{D}^n \to A$. The SCF f is *strategy-proof* if for all $P_i, P_i' \in \mathbb{D}$, $P_{-i} \in \mathbb{D}^{n-1}$ and $i \in N$, $f(P_i, P_{-i}) P_i f(P_i', P_{-i})$. The SCF f is *unanimous* if for all $P \in \mathbb{D}^n$ and $a_j \in A$, $f(P) = a_j$ whenever $r_1(P_i) = a_j$ for all $i \in N$. The SCF is *dictatorial* if there exist $i \in N$ such that $f(P) = r_1(P_i)$ for all $P \in \mathbb{D}^n$.

An SCF associates a deterministic outcome with every preference profile. It is strategy-proof if no agent can gain by misrepresenting his preference irrespective of the preference announcement of the other agents. An SCF is *unanimous* if the SCF respects consensus i.e., it always selects an outcome if it is ranked first by all agents. An SCF is *dictatorship* if there exists an agent called a dictator whose first ranked alternative is selected by SCF as the outcome at all preference profiles.

A random social choice function associates a probability distribution over the set A to every profile of preference orderings. Let $\mathcal{L}(A)$ denote the set of all probability distributions or lotteries over A. Formally, a *random social choice function* (RSCF) is a mapping $\varphi : \mathbb{D}^n \to \mathcal{L}(A)$. For every $a \in A$, $\varphi_a(P)$ is the probability assigned to a at profile $P \in \mathbb{D}^n$ by φ. Clearly, $\varphi_a(P) \geq 0$ and $\sum\limits_{a \in A} \varphi_a(P) = 1$. For notational convenience, we let $\varphi_j(P)$ denote the probability assigned to a_j at profile $P \in \mathbb{D}^n$ by φ. For any $B \subseteq A$, let $\varphi_B(P)$ denote the probability assigned to alternatives in B at $P \in \mathbb{D}^n$ by φ.

We follow the approach of Gibbard (1977) in order to define strategy-proofness of an RSCF. A utility function $u_i : A \to \mathbb{R}$ represents $P_i \in \mathbb{D}^n$ if for every pair of alternatives $a_j, a_k \in A$, $u_i(a_j) > u_i(a_k)$ if and only if $a_j P_i a_k$. Let $\mathbb{U}_i(P_i)$ denote the set of all utility functions that represent P_i. An RSCF φ is *strategy-proof* if for all $P_i, P_i' \in \mathbb{D}$, $P_{-i} \in \mathbb{D}^{n-1}$, $i \in N$ and $u_i \in \mathbb{U}_i(P_i)$, we have $\sum_{j=1}^m u_i(a_j)\varphi_j(P_i, P_{-i}) \geq \sum_{j=1}^m u_i(a_j)\varphi_j(P_i', P_{-i})$.

An RSCF is strategy-proof if truth-telling gives a higher expected utility than by lying for every representation of the true preference and irrespective of the announcements of other agents. This notion can also be expressed in terms of stochastic dominance as stated below.

An RSCF φ is *strategy-proof* if for all $P_i, P_i' \in \mathbb{D}$, $P_{-i} \in \mathbb{D}^{n-1}$, $i \in N$ and $k \in \{1, \ldots, m\}$, $\sum_{j=1}^k \varphi_{r_j(P_i)}(P_i, P_{-i}) \geq \sum_{j=1}^k \varphi_{r_j(P_i)}(P_i', P_{-i})$. Equivalently $\varphi_{B(a_j, P_i)}(P_i, P_{-i}) \geq \varphi_{B(a_j, P_i)}(P_i', P_{-i})$ for all $a_j \in A$, $i \in N$ and $P_i, P_i' \in \mathbb{D}$. In other words, an agent cannot increase the probability on any (strict) upper contour set by misreporting[2].

We now define counterparts of unanimity and dictatorship in the random setting. These definitions are standard in the literature, for example, see Chatterji et al. (2014). Formally, an RSCF φ is *unanimous* if for all $P \in \mathbb{D}^n$ and $a_j \in A$, $\varphi_j(P) = 1$ whenever $r_1(P_i) = a_j$ for all $i \in N$. An RSCF φ is *random-dictatorship* if there exists a probability distribution θ over N such that for all $P \in \mathbb{D}^n$ and $a_j \in A$, $\varphi_j(P) = \sum_{i \in N: r_1(P_i) = a_j} \theta(i)$ for all $a_j \in A$.

An RSCF is unanimous if it picks an alternative with probability one whenever it is ranked first by all the agents at a profile. An RSCF is a random-dictatorship if it is a fixed probability distribution over dictatorial SCFs. For instance, if $N = \{1, 2, 3, 4\}$ a random-dictatorship may select agents $1, 2, 3$ and 4 as dictators with probability $0.2, 0.5, 0.25$ and 0.05 respectively. This rule generates probability distributions over alternatives at every profile in a natural way. Consider a profile where agent 1 and 3 rank a_j first while agents 2 and 4 rank a_k and a_l first respectively. The RSCF will select a_j, a_k and a_l with probabilities 0.45, 0.5 and 0.05 respectively.

2.3 Domains and Existing Results

Our goal in this section is to review existing results on restricted domains. Our objects of interest are dictatorial and random-dictatorial domains which are defined below.

Definition 2.1. The domain $\mathbb{D} \subseteq \mathbb{P}$ is *dictatorial* if every strategy-proof and unanimous $f : \mathbb{D}^n \to A$ is a dictatorship.

Definition 2.2. The domain $\mathbb{D} \subseteq \mathbb{P}$ is *random-dictatorial* if every strategy-proof and unanimous $\varphi : \mathbb{D}^n \to \mathcal{L}(A)$ is a random-dictatorship.

A random-dictatorial domain is of course a dictatorial domain. In this chapter, we explore the reverse implication. The universal domain is both dictatorial as well as random-dictatorial. These fundamental results were established in Gibbard (1973) and Gibbard (1977) respectively.

Theorem 2.1. *A SCF $f : \mathbb{P}^n \to A$ is strategy-proof and unanimous if and only if it is dictatorial.*

[2] Since the inequality holds for all alternatives, the distinction between upper contour set or strict upper contour set is irrelevant.

Theorem 2.2. *A RSCF $\varphi : \mathbb{P}^n \to \mathcal{L}(A)$ is strategy-proof and unanimous if and only if it is random-dictatorial.*

There is a literature on restricted domains (proper subset of the universal domain) that are also dictatorial domains. The notion of connectedness of alternatives often plays a central role in the description of such domains. We briefly review some of these results.

DEFINITION 2.3. Fix a domain $\mathbb{D}$. A pair of distinct alternatives $a_j, a_k \in A$ are *connected* in $\mathbb{D}$, denoted by $a_j \sim a_k$, if there exist two preference orderings $\hat{P}_i, \bar{P}_i \in \mathbb{D}$ such that

1. $r_1(\hat{P}_i) = a_j$ and $r_2(\hat{P}_i) = a_k$
2. $r_1(\bar{P}_i) = a_k$ and $r_2(\bar{P}_i) = a_j$

DEFINITION 2.4. The domain $\mathbb{D}$ is a *linked domain* if there exists a one-to-one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that

1. $a_{\sigma(1)} \sim a_{\sigma(2)}$
2. $a_{\sigma(j)}$ is connected with at least two alternatives from the set $\{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(j-1)}\}$ for all $j \in \{3, \ldots, m\}$.

Let $G(\mathbb{D})$ be a graph with alternatives in A as vertices with an edge between any two vertices if the corresponding pair of alternatives are connected. A graph is connected if between any two distinct vertices in the graph, there is a path of distinct edges joining the two vertices. In our context, this means that for any pair of distinct alternatives $a_j, a_k \in A$, there exists a finite sequence of distinct alternatives $\{a_t\}_{t=1}^{T}$ with $T \in \{2, \ldots, m\}$, such that $a_1 = a_j$, $a_T = a_k$ and $a_t \overset{w}{\sim} a_{t+1}$ for all $t \in \{1, \ldots, T-1\}$). The length of this path is equal to the number of the edges lying on it i.e., $T - 1$.

In order for $\mathbb{D}$ to be linked, $G(\mathbb{D})$ must, of course be connected. Moreover, some additional structure on the graph is required. In particular, there must be a way of arranging the alternatives in a sequence such that the first three alternatives in the sequence are mutually connected (i.e., they form a cycle[3] in $G(\mathbb{D})$) and the subsequent alternatives are connected to at least two alternatives preceding them in the sequence.

Theorem 2.3 (Aswal et al. (2003))**.** *A linked domain is dictatorial.*

[3]Note that we have defined a path to have distinct vertices. A cycle is however defined as a path with identical beginning and the end vertex.

Another example of a dictatorial domain is the circular domain of Sato (2010).

Definition 2.5. A domain is a *circular domain* if there exists a mapping from $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ and two preference orderings $\hat{P}_i$ and $\bar{P}_i$ for all $j \in \{1, \ldots, m\}$ such that

1. $r_1(\hat{P}_i) = a_{\sigma(j)}$, $r_2(\hat{P}_i) = a_{\sigma(j+1)}$ and $r_m(\hat{P}_i) = a_{\sigma(j-1)}$.

2. $r_1(\bar{P}_i) = a_{\sigma(j)}$, $r_2(\bar{P}_i) = a_{\sigma(j-1)}$ and $r_m(\bar{P}_i) = a_{\sigma(j+1)}$.

Here we define $a_{\sigma(0)} = a_{\sigma(m)}$ and $a_{\sigma(m+1)} = a_{\sigma(1)}$.

Theorem 2.4 (Sato (2010))**.** *A circular domain is dictatorial.*

A circular domain need not be a linked domain and conversely a linked domain need not be circular. A circular domain requires the existence of preference orderings where additional restrictions are imposed on the identity of the last ranked alternatives. In contrast, linked domains place no such restrictions. Also for $m > 3$, a triple of mutually connected alternatives need not exist in the case of a circular domain but must exist in order for a domain to be linked.

Pramanik (2015) introduced the concept of β domains and γ-domains which generalize linked domains and circular domains respectively. These domains are based on notions of connectedenss that are weaker than connectedenss (Definition 2.3).

Definition 2.6 (Weakly Connected Alternatives). A pair of distinct alternatives $a_j, a_k \in A$ are *weakly connected* in a domain $\mathbb{D}$, denoted by $a_j \overset{w}{\sim} a_k$, if there exist four preference orderings $P_i^1, P_i^2, P_i^3, P_i^4 \in \mathbb{D}$ such that

1. $r_1(P_i^1) = r_1(P_i^2) = a_j$ and $r_1(P_i^3) = r_1(P_i^4) = a_k$

2. $M(a_j, a_k, P_i^1) = M(a_k, a_j, P_i^3) = \overline{M}$

3. $\overline{M} \subseteq \overline{W} = W(a_k, P_i^2)$ and $\overline{M} \subseteq \widehat{W} = W(a_j, P_i^4)$

The set of alternatives between a_j and a_k in P_i^1 and between a_k and a_j in P_i^3 is the same. Moreover this set is contained in the set of alternatives below a_k in P_i^2 and in the set of alternatives below a_j in P_i^4. For convenience we introduce some terminology. We will say that P_i^2 is an (a_j, a_k)-*weak-reversal* of P_i^1, P_i^4 is an (a_k, a_j)-*weak-reversal* of P_i^3 and P_i^3 is an (a_k, a_j)-*partner* of P_i^1. We can now restate the definition of weak connectedness

$$
\begin{array}{cccc}
P_i^1 & P_i^2 & P_i^3 & P_i^4 \\
a_j & a_j & a_k & a_k \\
\vdots & \vdots & \vdots & \vdots \\
\overline{M} & \vdots & \overline{M} & a_j \\
\vdots & a_k & \vdots & \vdots \\
a_k & \vdots & a_j & \vdots \\
\vdots & \overline{W} \supseteq \overline{M} & \vdots & \widehat{W} \supseteq \overline{M} \\
\vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots
\end{array}
$$

Figure 2.1: a_j and a_k are weakly connected

as follows: a_j and a_k are weakly connected if there exists a preference $P_i^1 \in \mathbb{D}^{a_j}$ such that $P_i^2 \in \mathbb{D}^{a_j}$ and $P_i^3 \in \mathbb{D}^{a_k}$ is an (a_j, a_k)-*weak-reversal* and $(a_k, a_j)-$*partner* of P_i^1 respectively while $P_i^4 \in \mathbb{D}^{a_k}$ is an (a_k, a_j)-*weak-reversal* of P_i^3. It is easy to see that a_j and a_k are weakly connected if they are connected. This is a special case where $\bar{M}$ is null, $P_i^1 = P_i^2$ and $P_i^3 = P_i^4$. The weak connected relationship is illustrated in Figure 2.1.

The weak connectedness relation generates a β domain in exactly the same manner as the connectedness relation generates a linked domain. Formally,

Definition 2.7. The domain $\mathbb{D}$ is a *β domain* if there exists a one-to-one function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that

1. $a_{\sigma(1)} \overset{w}{\sim} a_{\sigma(2)}$
2. $a_{\sigma(j)}$ is weakly connected to at least two alternatives from the set $\{a_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(j-1)}\}$ for all $j \in \{3, \ldots, m\}$.

A strengthening of weak connectedness leads to the notion of weak* connectedness. This notion was introduced in Pramanik (2015) where it was referred to as the SC property.

Definition 2.8. A pair of distinct alternatives $a_j, a_k \in A$ is *weak* connected* in a domain $\mathbb{D}$, denoted by $a_j \overset{*}{\sim} a_k$, if there exist four preference orderings $P_i^1, P_i^2, P_i^3, P_i^4 \in \mathbb{D}$ such that

$$
\begin{array}{cccc}
P_i^1 & P_i^2 & P_i^3 & P_i^4 \\
& & & \\
a_j & a_j & a_k & a_k \\
\left\{\begin{matrix}\vdots \\ \overline{M} \\ \vdots\end{matrix}\right. & \begin{matrix}\vdots \\ \vdots\end{matrix} & \left\{\begin{matrix}\vdots \\ \overline{M} \\ \vdots\end{matrix}\right. & \begin{matrix}\vdots \\ \vdots\end{matrix} \\
a_k & a_k & a_j & a_j \\
\begin{matrix}\vdots \\ \vdots\end{matrix} & \left\{\begin{matrix}\vdots \\ \overline{W} = \overline{M} \\ \vdots\end{matrix}\right. & \begin{matrix}\vdots \\ \vdots\end{matrix} & \left\{\begin{matrix}\vdots \\ \widehat{W} = \overline{M} \\ \vdots\end{matrix}\right.
\end{array}
$$

Figure 2.2: a_j and a_k are weak* connected

1. $r_1(P_i^1) = r_1(P_i^2) = a_j$ and $r_1(P_i^3) = r_1(P_i^4) = a_k$
2. $M(a_j, a_k, P_i^1) = M(a_k, a_j, P_i^3) = W(a_k, P_i^2) = W(a_j, P_i^4)$

The weak* connected relation is illustrated in Figure 2.2. It is different from the weak connectedness only in one respect - the set of alternatives below a_k in P_i^2 and the set of alternatives below a_j in P_i^4 is exactly $\bar{M}$.

We will use the four preference orderings in the definition of weak* connectedness repeatedly. For convenience we will say that P_i^2 is an (a_j, a_k)-*reversal* of P_i^1, P_i^4 is an (a_k, a_j)-*reversal* of P_i^1 and P_i^3 is an (a_k, a_j)-*partner* of P_i^1. Note that P_i^3 is (a_k, a_j)-*reversal* of P_i^4. We can now restate the definition of weak* connectedness as follows: a_j and a_k are weak* connected if there exists a preference $P_i^1 \in \mathbb{D}^{a_j}$ such that $P_i^2 \in \mathbb{D}^{a_j}$ is a (a_j, a_k)-*reversal*, $P_i^3 \in \mathbb{D}^{a_k}$ is (a_k, a_j)-*partner* and $P_i^4 \in \mathbb{D}^{a_k}$ is an (a_k, a_j)-*reversal* of P_i^1.

Let $\bar{G}(\mathbb{D})$ denote the graph with alternatives in A as vertices and an edge between two vertices if the corresponding pair of alternatives is weak* connected[4].

Definition 2.9. A domain $\mathbb{D}$ is a γ *domain* if $\bar{G}(\mathbb{D})$ is a connected graph.

It is important to note that the graph of a γ-domain is only required to be connected. A more demanding structure on the graph is required for both the linked domain and the β domain.

Table 3.1 illustrates a γ-domain with six alternatives where $a_1 \overset{*}{\sim} a_2 \overset{*}{\sim} a_3 \overset{*}{\sim} a_4 \overset{*}{\sim} a_5 \overset{*}{\sim} a_6$. Consider a pair of weak* connected alternatives, say (a_1, a_2). The preference

[4] Pramanik (2015) calls it the SC property.

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}	P_i^{11}	P_i^{12}
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_5	a_5	a_6	a_6
a_4	a_6	a_3	a_5	a_2	a_1	a_2	a_1	a_2	a_1	a_2	a_2
a_3	a_5	a_4	a_6	a_6	a_4	a_6	a_3	a_6	a_3	a_5	a_1
a_2	a_2	a_1	a_1	a_5	a_5	a_5	a_5	a_4	a_4	a_1	a_4
a_5	a_3	a_5	a_4	a_4	a_6	a_3	a_6	a_3	a_6	a_3	a_5
a_6	a_4	a_6	a_3	a_1	a_2	a_1	a_2	a_1	a_2	a_4	a_2

Table 2.1: A non-circular γ-domain

orderings P_i^2,P_i^3 and P_i^4 are the (a_1, a_2)-reversal, (a_2, a_1)-partner and (a_2, a_1)-reversal of P_i^1 respectively. For these quadruple of preference orderings, $\bar{M} = M(a_1, a_2, P_i^1) = \{a_3, a_4\}$. A special case of this γ-domain requiring $\bar{M} = \phi$ for every pair (a_j, a_{j+1}), $j \in \{1, \ldots, 6\}$ is a circular domain[5]. For example, a_2 will be the second and the last ranked alternative in P_i^1 and P_i^2 respectively while a_1 will be the second and the last ranked alternative in P_i^3 and P_i^4 respectively for the domain in Table 3.1 to be circular. It is clear that the class of γ-domain is significantly larger than the class of circular domains.

A graph G is a *star graph* if there exists a vertex $\bar{v}$ (called the *hub*) such that there is an edge between a pair of vertices if and only if $\bar{v}$ is one of the vertices. The maximum length of a path in a star graph with more than 2 vertices is 2.

Definition 2.10. A γ-domain is a P-domain if

1. $\bar{G}(\mathbb{D})$ is a non-star graph or
2. $\bar{G}(\mathbb{D})$ is a star graph with hub $a_j \in A$ such that there exist a pair $(a_k, a_q) \in A \backslash \{a_j\}$ such that
   (i) $a_k \sim a_q$ or
   (ii) $M(a_k, a_j, P_k^1) = M(a_q, a_j, P_q^3) = W(a_j, P_k^2) = W(a_j, P_q^4)$ for some $P_k^1, P_k^2 \in \mathbb{D}^{a_k}$ and $P_q^3, P_q^4 \in \mathbb{D}^{a_q}$.

A γ-domain with a star graph is a P-domain if there exists a pair of alternatives that are not weak* connected but are either weakly connected or satisfy reversibility of preference orderings with respect to the hub.

[5]We assume that $a_7 = a_1$. Note that a circular domain additionally requires $a_1 \overset{*}{\sim} a_6$.

Remark 2.1. In a circular domain every alternative is weak* connected to at least two other alternatives. As a result, the graph associated with a circular domain is a non-star graph. Consequently, a circular domain is a P-domain.

We end this section by stating the result of Pramanik (2015).

Theorem 2.5 (Pramanik (2015))**.** *A domain is dictatorial if it is a β domain or a P-domain.*

2.4 Main Results

In this section, we state and prove the main result of the chapter.

Theorem 2.6. *Every P-domain is random-dictatorial.*

Proof. The proof uses induction on the number of agents. In the first step we prove the result for two agents. The second step is an induction step and is completed by showing that a P-domain satisfies Condition α in Chatterji et al. (2014) which allows us to apply their *Ramification Theorem*.

In the two agent case, the proof establishes random-dictatorship on progressively larger sets of profiles. The initial step is to demonstrate random-dictatorship on profiles where the peaks of the two agents are weak* connected. We then "spread" random-dictatorship to arbitrary profiles.

Step 1: Let $N = \{1, 2\}$ and let $\varphi : \mathbb{D}^2 \to \mathcal{L}(A)$ be a strategy-proof and unanimous RSCF.

Throughout this proof, we shall use the following convention; we shall always write a profile as an ordered pair of preference orderings where the first element in the pair refers to the preference ordering of agent 1 and the second element to the preference ordering of agent 2. Thus $(\bar{P}_i, P'_i)$ is the preference profile where agent 1 has preference ordering $\bar{P}_i$ and agent 2 has preference ordering P'_i.

In Lemmas 2.1 - 2.3 below, we assume $a_j \sim a_k$. We let $P^1_j, P^2_j \in \mathbb{D}^{a_j}$, $P^3_k, P^4_k \in \mathbb{D}^{a_k}$ be such that P^2_j is an (a_j, a_k)-weak reversal and P^3_k is a (a_k, a_j)-partner of P^1_j and P^4_k is an (a_k, a_j)-weak reversal of P^3_k as specified in the definition of weak connectedness.

Lemma **2.1.** $\varphi_j(P^1_j, P^4_k) + \varphi_k(P^1_j, P^4_k) = \varphi_j(P^2_j, P^3_k) + \varphi_k(P^2_j, P^3_k) = 1.$

Proof. Suppose $\varphi_l(P_j^1, P_k^4) > 0$ where $a_j P_k^4 a_l$. Then agent 2 can manipulate by announcing a_j as peak and obtaining a_j as an outcome with probability one. Similarly, $\varphi_l(P_j^1, P_k^4) = 0$ for all a_l such that $a_k P_j^1 a_l$. However, $W(a_k, P_j^1) \cup W(a_j, P_k^4) = W(a_k, P_j^2) \cup W(a_j, P_k^3) = A \setminus \{a_j, a_k\}$ thus proving that $\varphi_j(P_j^1, P_k^4) + \varphi_k(P_j^1, P_k^4) = 1$. The other equality follows from an analogous argument. □

Let $\varphi_j(P_j^1, P_k^4) = \beta$ and $\varphi_j(P_j^2, P_k^3) = \alpha$. From Lemma 2.1, $\varphi_k(P_j^1, P_k^4) = 1 - \beta$ and $\varphi_k(P_j^2, P_k^3) = 1 - \alpha$.

Lemma **2.2.** *$\beta = \alpha$.*

Proof. Since the peaks of P_j^4 and P_j^3 are the same, the strategy-proofness of φ implies that $\varphi_k(P_j^1, P_k^3) = 1 - \beta$. Similarly, since the peaks of P_j^1 and P_j^2 are the same, strategy-proofness implies that $\varphi_j(P_j^1, P_k^3) = 1 - \alpha$. Since $\varphi_j(P_j^1, P_k^3) + \varphi_k(P_j^1, P_k^3) \leq 1$, we get $\alpha + 1 - \beta \leq 1$ i.e., $\alpha \leq \beta$.

Again since the peaks of P_k^4 and P_k^3 are the same, strategy-proofness of φ implies that $\varphi_k(P_j^2, P_k^4) = 1 - \alpha$. Similarly, since the peaks of P_j^1 and P_j^2 are the same, strategy-proofness implies that $\varphi_j(P_j^2, P_k^4) = \beta$. Since $\varphi_j(P_j^2, P_k^4) + \varphi_k(P_j^2, P_k^4) \leq 1$, we get $\beta + 1 - \alpha \leq 1$ i.e., $\beta \leq \alpha$.

Hence $\beta = \alpha$. □

We can summarise the two lemmas above as follows. Suppose a_j and a_k are weakly connected. For profiles where the preference orderings of the agents are as specified in the definition of weak connectedness and the peak of agent 1 is a_j while that of 2 is a_k then φ is a random-dictatorship where the peak of agent 1 is chosen with probability β and the peak of agent 2 is chosen with probability $1 - \beta$. When the peaks of agent 1 and 2 are a_k and a_j respectively, then the probabilities of choosing the peaks of agents 1 and 2 are β' and $1 - \beta'$ respectively. However we cannot assert $\beta = \beta'$ at the moment.

In the next lemma, we extend the random-dictatorship property to arbitrary profiles where the peaks of agents 1 and 2 are a_j and a_k respectively.

Lemma **2.3.** *Let $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$. Then $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_j, P_k) = 1 - \beta$.*

Proof. The first step of the argument is to show that $\varphi_j(P_j^1, P_k) = \beta$ and $\varphi_k(P_j^1, P_k) = 1 - \beta$. We know from Lemma 2.2 that $\varphi_j(P_j^1, P_k^3) = \beta$ and $\varphi_k(P_j^1, P_k^3) = 1 - \beta$. Since the peaks of P_k and P_k^3 are both a_k, $\varphi_k(P_j^1, P_k) = 1 - \beta$ i.e., $\varphi_j(P_j^1, P_k) \leq \beta$. Suppose there exists a_l such that $\varphi_l(P_j^1, P_k) = \delta > 0$. Observe that $a_l \in M(a_j, a_k, P_j^1)$; otherwise agent 1 will manipulate by announcing a_k as the peak. Since P_k^3 is a (a_k, a_j) -partner of P_j^1, $a_l \in M(a_k, a_j, P_k^3)$. The total probability of the upper contour set of a_l in P_k^3 at profile (P_j^1, P_k^3) is $1 - \beta$ (by Lemma 2.2) but is at least $1 - \beta + \delta$ in (P_j^1, P_k). This contradicts the strategy-proofness of φ. By an analogous argument, $\varphi_j(P_j, P_k^3) = \beta$ and $\varphi_k(P_j, P_k^3) = 1 - \beta$.

The strategy-proofness of φ implies that $\varphi_j(P_j, P_k) = \varphi_j(P_j^1, P_k) = \beta$ and $\varphi_k(P_j, P_k) = \varphi_k(P_j, P_k^3) = 1 - \beta$. □

Note that if a_j and a_k are weak* connected then they are weakly connected as well. Hence Lemma 2.3 can be rewritten for the special case when $a_j \overset{*}{\sim} a_k$.

Corollary 2.1. *Suppose $a_j \overset{*}{\sim} a_k$. Let $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$. Then $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_j, P_k) = 1 - \beta$.*

Henceforth we shall employ an important piece of notation. For the fixed a_j and a_k that we consider (where $a_j \overset{*}{\sim} a_k$), we denote $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_j, P_k) = 1-\beta$ where $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$. In the following Lemmas, we let $P_j^1, P_j^2 \in \mathbb{D}^{a_j}$, $P_k^3, P_k^4 \in \mathbb{D}^{a_k}$ be such that P_j^2 is an (a_j, a_k)-reversal, P_k^3 is a (a_k, a_j)-partner and P_k^4 is an (a_k, a_j)-reversal of P_j^1 as specified in the definition of weak* connectedness. The next lemma considers the case where a_j is weak* connected to a_k and a_k is weak* connected to a_q.

LEMMA **2.4.** *Suppose $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q$. Let $P_q \in \mathbb{D}^{a_q}$ and $P_k \in \mathbb{D}^{a_k}$. Then $\varphi_q(P_q, P_k) = \beta$ and $\varphi_k(P_q, P_k) = 1 - \beta$.*

Proof. Since P_j^2 is an (a_j, a_k)-reversal of P_j^1, $M(a_j, a_k, P_j^1) = W(a_k, P_j^2)$. Therefore it must be that either $a_q P_j^1 a_k$ or $a_q P_j^2 a_k$ holds. We assume hereafter that $a_q P_j^1 a_k$ without loss of generality. From hereon, we let $P_k^5, P_k^6 \in \mathbb{D}^{a_k}$ and $P_q^7, P_q^8 \in \mathbb{D}^{a_q}$ be such that P_k^6 is an (a_k, a_q)-reversal, P_q^7 is a (a_q, a_k)-partner and P_q^8 is an (a_q, a_k)-reversal of P_k^5. The existence of such orderings are guaranteed by the fact that a_q and a_k are weak* connected. Suppose it is the case that $a_q P_k^5 a_j$ without loss of generality. Since P_q^8 is an (a_q, a_k)-reversal of P_k^5, we have $a_j P_q^8 a_k$.

It follows from Corollary 2.1 that $\varphi_j(P_j^1, P_k) = \beta$ and $\varphi_q(P_q^8, P_k) = \alpha$ for some $\alpha \in [0, 1]$. We claim that $\beta = \alpha$.

If $\alpha > \beta$, the probability weight of $B(a_k, P_j^1)$ in profile (P_j^1, P_k) is β while it is α in profile (P_q^8, P_k), contradicting the strategy-proofness of φ. Similarly, if $\alpha < \beta$ then the agent 1 will manipulate at (P_q^8, P_k) by reporting P_j^1. Therefore $\alpha = \beta$. $\square$

In Lemma 2.4, we begin with a situation where the peaks of agents are weak* connected. If an agent changes her preference ordering such that her peak remains weak* connected to the unchanged peak of the other agent then probabilities of their peaks remain unchanged. The following lemma assumes a cycle of weak* connections involving three alternatives and demonstrates that interchanging the peaks of the two agents will not change the probability of the peak of any agent.

Lemma **2.5.** *Suppose $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q \overset{*}{\sim} a_j$. Then for $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$, $\varphi_k(P_k, P_j) = \varphi_j(P_j, P_k)$.*

Proof. From Corollary 2.1, we know that $\varphi_k(P_k, P_j) = \alpha$ for some $\alpha \in [0, 1]$. We claim that $\alpha = \beta$.

Since $a_q \overset{*}{\sim} a_j \overset{*}{\sim} a_k$, it follows from Lemma 2.4 that for all $P_j \in \mathbb{D}^{a_j}$, $P_k \in \mathbb{D}^{a_k}$ and $P_q \in \mathbb{D}^{a_q}$, $\varphi_q(P_j, P_q) = \varphi_j(P_j, P_k) = \beta$. Since $a_k \overset{*}{\sim} a_q \overset{*}{\sim} a_j$, it follows from the same argument that $\varphi_k(P_k, P_q) = \varphi_q(P_j, P_q) = \beta$. Again since $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q$, it must be that $\varphi_j(P_k, P_j) = \varphi_k(P_k, P_q) = \beta$. This completes the proof. $\square$

Lemma 2.5 can be summarized as follows. Suppose there are three alternatives (which comprise the set, say B) that are mutually weak* connected. Then, the probability of the peak of any agent in profiles with distinct peaks belonging to B, is the same, irrespective of the identity of the peak. The next lemma deals with the case when a_j is weak* connected to a_k and a_k is weak* connected to a_q but there is no cycle.

Lemma **2.6.** *Suppose $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q$ but $a_q \overset{*}{\sim} a_j$ does not hold. Then for $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$, $\varphi_k(P_k, P_j) \neq \varphi_j(P_j, P_k)$ only if $M(a_j, a_q, P_j^1) \cap M(a_q, a_k, P_q^7) \neq \phi$ or $M(a_j, a_k, P_j^2) \cap M(a_q, a_j, P_q^8) \neq \phi$.*

Proof. It follows from Corollary 2.1 that $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_k, P_j) = \alpha$ for some $\alpha \in [0, 1]$. Our claim is that $\alpha \geq \beta$.

It follows from Corollary 2.1 that the probability weight of $B(a_j, P_k^3)$ according to φ at (P_j^1, P_k^3) is $1 - \beta$. Since φ is strategy-proof, the probability weight of the same set of alternatives at (P_j^1, P_q^8) is at most $1 - \beta$. We have assumed that P_k^3 is a (a_k, a_j)-partner of P_j^1 i.e., $W(a_j, P_k^3) = W(a_k, P_j^1)$. Since $a_q P_j^1 a_k$ by assumption (see the proof of Lemma 2.4), $W(a_k, P_j^1) \subseteq W(a_q, P_j^1)$. But we know from Lemma 2.1 that the probability weight of $W(a_q, P_j^1)$ at (P_j^1, P_q^8) must be 0. Therefore, the probability weight of $W(a_j, P_k^3)$ at (P_j^1, P_q^8) is 0 as well. Since $B(a_j, P_k^3) \cup W(a_j, P_k^3) = A \setminus \{a_j\}$, we must have $\varphi_j(P_j^1, P_q^8) \geq \beta$.

The probability weight of $B(a_q, P_k^6)$ at (P_k^6, P_q^8) is α according to Lemma 2.4. Again since φ is strategy-proof, the probability weight of the same set of alternatives at (P_j^1, P_q^8) is at most α. We have assumed that $W(a_q, P_k^6) = W(a_k, P_q^8) \subseteq W(a_j, P_q^8)$ and therefore the probability weight of $W(a_q, P_k^6)$ at (P_j^1, P_q^8) is 0 from Lemma 2.1, implying that $\varphi_q(P_j^1, P_q^8) \geq 1 - \alpha$.

The arguments in the two earlier paragraphs in conjunction with the fact that $\varphi_j(P_j^1, P_q^8) + \varphi_q(P_j^1, P_q^8) \leq 1$ implies $\alpha \geq \beta$.

The next step in the proof is to prove the contra-positive. Suppose $M(a_j, a_q, P_j^1) \cap M(a_q, a_k, P_q^7) = M(a_j, a_k, P_j^2) \cap M(a_q, a_j, P_q^8) = \phi$. Since $M(a_j, a_k, P_j^2) \cap M(a_q, a_j, P_q^8) = \phi$, the only alternatives which can be given a positive probability at (P_j^2, P_q^8) are a_j, a_q and a_l such that $a_l P_q^8 a_j$ and $a_k P_j^2 a_l P_j^2 a_q$ [6]. We claim that $\varphi_l(P_j^2, P_k^8) = 0$. Suppose not i.e., $\varphi_l(P_j^2, P_k^8) = \delta > 0$. Since $\varphi_k(P_k^4, P_q^8) = \alpha$, we have $\varphi_j(P_j^2, P_k^8) = \alpha$. This follows from the fact that φ is strategy-proof and P_j^2 is an (a_j, a_k)-partner of P_k^4 implying that the upper contour set of a_k in P_j^2 is the same as the upper contour set of a_j in P_k^4. Since P_q^8 is a (a_q, a_k)-partner of P_k^6 and since $a_l P_q^8 a_j P_q^8 a_k$, it must be the case that $a_l P_k^6 a_q$. The probability weight of $B(a_q, P_k^6)$ at (P_k^2, P_q^8) therefore is $\alpha + \delta$ while it is α at (P_k^6, P_q^8) thereby contradicting the strategy-proofness of φ. Thus $\varphi_q(P_j^2, P_k^8) = 1 - \alpha$.

Using similar arguments and the fact that P_k^3 is an (a_k, a_j)-partner of P_j^1 while P_q^7 is an (a_q, a_k)-partner of P_k^5, we can infer that $\varphi_j(P_j^1, P_q^7) = \beta$ and $\varphi_q(P_j^1, P_q^7) = 1-\beta$. Since φ is strategy-proof, $\varphi_j(P_j^1, P_q^8) = \varphi_j(P_j^2, P_q^8) = \alpha$ and $\varphi_q(P_j^1, P_q^8) = \varphi_j(P_j^1, P_q^7) = 1 - \beta$. Since $\varphi_j(P_j^1, P_q^8) + \varphi_q(P_j^1, P_q^8) \leq 1$, we get $\alpha \leq \beta$.

Hence we have proved that if $M(a_j, a_q, P_j^1) \cap M(a_q, a_k, P_q^7) =$
$M(a_j, a_k, P_j^2) \cap M(a_q, a_j, P_q^8) = \phi$, then $\alpha = \beta$. □

Remark 2.2. The strategy-proofness of φ further implies that $\varphi_j(P_j^2, P_q^7) = \varphi_j(P_j^1, P_q^7) = \beta$ and $\varphi_q(P_j^2, P_q^7) = \varphi_j(P_j^2, P_q^8) = 1 - \alpha$. Since P_j^2 is an (a_j, a_k)-partner of P_k^4, the probability weight of the upper contour set of a_k in P_j^2 at profile (P_j^2, P_q^7) is α as argued above. Likewise, since P_q^7 is an (a_q, a_k)-partner of P_k^5, the probability weight of the upper contour set of a_k in P_q^7 at profile (P_j^2, P_q^7) is $1 - \beta$. Since $\alpha \geq \beta$ as concluded above,

[6] Any alternative in $M(a_j, a_k, P_j^2)$ will lie below a_j in P_k^8 and therefore cannot get a positive probability from Lemma 2.1.

$\alpha + 1 - \beta \geq 1$ implying that only alternatives in $M(a_j, a_k, P_j^2) \cup M(a_q, a_k, P_q^7)$ can get a positive probability at (P_j^2, P_q^7). We claim that $M(a_j, a_k, P_j^2) \cap M(a_q, a_k, P_q^7) = \phi$ implies that $\alpha = \beta$. If not then since $\alpha \geq \beta$, it must be that $\alpha > \beta$. the sum of $\varphi_j(P_j^2, P_q^7)$ and $\varphi_q(P_j^2, P_q^7)$ must be strictly less than 1. Hence there must be a set of alternatives $C \subset M(a_j, a_k, P_j^2)$ which has a probability weight of $\alpha - \beta$. And there also must a set of alternatives B disjoint with C which carries an equal probability weight. This implies that the total probability weight of all alternatives in A is $\beta + 1 - \alpha + (\alpha - \beta) + (\alpha - \beta) > 1$. Hence $M(a_j, a_k, P_j^2) \cup M(a_q, a_k, P_q^7) = \phi$ implies that $\alpha = \beta$.

Remark 2.3. Using arguments symmetrical to those in Lemma 2.6 and Remark 2.2, we can deduce that $\varphi_j(P_q^8, P_1^1) = \varphi_j(P_q^8, P_1^2) = 1 - \beta$, $\varphi_q(P_q^8, P_1^1) = \varphi_q(P_q^7, P_1^1) = \alpha$, $\varphi_j(P_q^7, P_1^2) = \varphi_j(P_q^7, P_1^1) = 1 - \alpha$ and $\varphi_q(P_q^7, P_1^2) = \varphi_q(P_q^8, P_1^2) = \beta$. If $\alpha = \beta$ then it follows from Lemma 2.6 and Remark 2.2 that $\varphi_j(P_j, P_q) = \varphi_q(P_q, P_j) = \beta$ and $\varphi_q(P_j, P_q) = \varphi_j(P_q, P_j) = 1 - \beta$ for $P_j \in \{P_j^1, P_j^2\}$ and $P_q \in \{P_q^7, P_q^8\}$.

We now consider a special case of the Lemma above in which $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q \sim a_j$.

Lemma **2.7.** *Suppose $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q \sim a_j$. Then for $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$, $\varphi_k(P_k, P_j) = \varphi_j(P_j, P_k)$.*

Proof. From Corollary 2.1 and 2.6, it follows that $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_k, P_j) = \alpha$ where $\alpha \geq \beta$. Suppose $\alpha > \beta$. It follows from Remark 2.2 that $\varphi_j(P_j^2, P_q^7) + \varphi_q(P_j^2, P_q^7) = (1 - \alpha + \beta) < 1$. But we know from Lemma 2.3 that $\varphi_j(P_j^2, P_q^7) + \varphi_q(P_j^2, P_q^7) = 1$. Hence $\alpha = \beta$. □

In the following lemma we generalise the result in Remark 2.3 to any profile where the two agents have distinct peaks from the set $\{a_j, a_q\}$.

Lemma **2.8.** *Suppose $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q$. If $\varphi_j(P_j, P_k) = \varphi_k(P_k, P_j)$ then for all $P_j \in \mathbb{D}^{a_j}$, $P_k \in \mathbb{D}^{a_k}$ and $P_q \in \mathbb{D}^{a_q}$, it must be that $\varphi_j(P_j, P_q) = \varphi_q(P_q, P_j) = \beta$ and $\varphi_q(P_j, P_q) = \varphi_j(P_q, P_j) = 1 - \beta$.*

Proof. The idea of this proof is similar to the proof of Corollary 2.1 but we describe it in detail anyway. The first step is to show that $\varphi_j(P_j^1, P_q) = \beta$ and $\varphi_q(P_j^1, P_q) = 1 - \beta$. We know from Remark 2.3 that $\varphi_q(P_j^1, P_q^8) = 1 - \beta$. Since the peaks of P_q and P_q^8 are both a_q, $\varphi_q(P_j^1, P_q) = 1 - \beta$ i.e., $\varphi_j(P_j^1, P_q) \leq \beta$. Suppose there exists $a_l \in A \setminus \{a_j, a_q\}$ such that $\varphi_l(P_j^1, P_q) = \delta > 0$. It must be the case that $a_l \in M(a_j, a_q, P_j^1)$, otherwise agent 1 will manipulate by announcing a_q as the peak. We have assumed that $a_q P_j^1 a_k$. Since

P_k^3 is (a_k, a_j)-partner of P_j^1, it follows that $a_l P_k^3 a_j$ and $a_q P_k^3 a_j$. The total probability of the strict upper contour set of a_j in P_k^3 at profile (P_j^1, P_k^3) is $1-\beta$ (by Lemma 2.1) but is at least $1-\beta+\delta$ at (P_j^1, P_q). This contradicts the strategy-proofness of φ. Hence $\varphi_j(P_j^1, P_q) = \beta$.

The next step is to show that $\varphi_j(P_j, P_q^8) = \beta$ and $\varphi_q(P_j, P_q^8) = 1-\beta$. Since $\varphi_j(P_j^2, P_q^8) = \beta$ from Remark 2.3, it follows that $\varphi_j(P_j, P_q^8) = \beta$ and $\varphi_q(P_j, P_q^8) \leq 1-\beta$. We need to show that $\varphi_q(P_j, P_q^8)$ is indeed $1-\beta$. If $\varphi_l(P_j, P_q^8) = \delta > 0$ for some $a_l \in A \setminus \{a_j, a_q\}$ then $a_l \in M(a_q, a_j, P_q^8)$, otherwise agent 2 will manipulate by announcing a_j as the peak. Recall that $a_j P_q^8 a_k$. Since P_k^6 is (a_k, a_q)-partner of P_q^8, it follows that $a_l P_k^6 a_q$ and $a_j P_k^6 a_q$. The total probability of the strict upper contour set of a_q in P_k^6 at profile (P_k^6, P_q^8) is β (by Lemma 2.1) but is at least $\beta+\delta$ in (P_j, P_q^8). This contradicts the strategy-proofness of φ. Hence $\varphi_q(P_j, P_q^8) = 1-\beta$.

The strategy-proofness of φ implies that $\varphi_j(P_j, P_q) = \varphi_j(P_j^1, P_q) = \beta$ and $\varphi_q(P_j, P_k) = \varphi_q(P_j, P_q^8) = 1-\beta$. Using a symmetrical argument, we can conclude that $\varphi_q(P_q, P_j) = \beta$ and $\varphi_j(P_q, P_j) = 1-\beta$. $\square$

LEMMA **2.9.** *Suppose $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q \overset{*}{\sim} a_w$. Then $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$, $\varphi_k(P_k, P_j) = \varphi_j(P_j, P_k)$.*

Proof. It follows from Corollary 2.1 that $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_k, P_j) = \alpha$ for some $\alpha \in [0, 1]$. We have already established in Lemma 2.6 that $\alpha \geq \beta$. What remains to be shown is $\alpha \leq \beta$.

In order to do this we need to introduce a few more preference orderings. The existence of these preference orderings is again guaranteed by the fact that $a_q \overset{*}{\sim} a_w$. Let $P_q^9, P_q^{10} \in \mathbb{D}^{a_q}$ and $P_w^{11}, P_w^{12} \in \mathbb{D}^{a_w}$ be such that P_q^{10} is an (a_q, a_w)-reversal, P_w^{11} is a (a_w, a_q)-partner and P_w^{12} is an (a_w, a_q)-reversal of P_q^9. Let $a_w P_q^9 a_k$ without loss of generality. Since P_q^{12} is an (a_w, a_q)-reversal of P_q^9 we have $a_k P_q^{12} a_q$.

Replacing P_j^1, P_k^3, P_k^6 and P_q^8 with P_k^5, P_q^7, P_q^{10} and P_w^{12} respectively and replicating the arguments used in the first step of Lemma 2.6, we can deduce that $\alpha \leq \beta$. This completes the proof of the Lemma. $\square$

Remark 2.4. Lemmas 2.8, 2.8 and 2.9 together imply that if $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q \overset{*}{\sim} a_w$ then $\varphi_k(P_k, P_w) = \varphi_w(P_w, P_k) = \beta$ and $\varphi_w(P_k, P_w) = \varphi_k(P_w, P_k) = 1-\beta$ where $P_k \in \mathbb{D}^{a_k}$ and $P_q \in \mathbb{D}^{a_q}$.

Let $B \subseteq A$. The RSCF φ is a random-dictatorship over B with weights $(\beta, 1-\beta)$ for some $\beta \in [0,1]$, if for all $a_j, a_k \in B$, $P_j \in \mathbb{D}^{a_j}$ and $P_k \in \mathbb{D}^{a_k}$, we have $\varphi_j(P_j, P_k) = \beta$ and $\varphi_k(P_j, P_k) = 1-\beta$.

Lemma **2.10.** *Suppose $B = \{a_1, a_2, a_3, a_4\}$ with $a_1 \overset{*}{\sim} a_2 \overset{*}{\sim} a_3 \overset{*}{\sim} a_4$. If $\varphi_1(P_1', P_2') = \beta$ for some $P_1' \in \mathbb{D}^{a_1}$ and $P_2' \in \mathbb{D}^{a_2}$ then φ is a random-dictatorship over B with weights $\beta, 1-\beta$.*

Proof. We know from lemmas 2.1 to 2.9 above that for $s \in \{1,2\}$ and $t \in \{1,2\}$, $\varphi_s(P_s, P_{s+t}) = \varphi_{s+t}(P_{s+t}, P_s) = \beta$, $\varphi_{s+t}(P_s, P_{s+t}) = \varphi_s(P_{s+t}, P_s) = 1-\beta$.

We need to establish that $\varphi_1(P_1, P_4) = \varphi_4(P_4, P_1) = \beta$ and $\varphi_4(P_1, P_4) = \varphi_1(P_4, P_1) = 1-\beta$.

Let $\bar{P}_1, \hat{P}_1 \in \mathbb{D}^{a_1}$, $\bar{P}_2, \hat{P}_2 \in \mathbb{D}^{a_2}$, $\bar{P}_3, \hat{P}_3 \in \mathbb{D}^{a_3}$ and $\bar{P}_4, \hat{P}_4 \in \mathbb{D}^{a_4}$ such that

1. $\hat{P}_1$ is an (a_1, a_2)-reversal, $\bar{P}_2$ is a (a_2, a_1)-partner and $\hat{P}_2$ is an (a_2, a_1)-reversal of $\bar{P}_1$.

2. $\hat{P}_3$ is an (a_3, a_4)-reversal, $\bar{P}_4$ is a (a_4, a_3)-partner and $\hat{P}_4$ is an (a_4, a_3)-reversal of $\bar{P}_3$.

3. $a_4 \bar{P}_1 a_2$ and $a_3 \bar{P}_4 a_1$.

As usual, the existence of such preference orderings is guaranteed by the fact that $a_1 \overset{*}{\sim} a_2$ and $a_3 \overset{*}{\sim} a_4$ as well as the fact that if for some alternatives (a_j, a_k), $\hat{P}_j$ is an (a_j, a_k)-reversal of $\bar{P}_j$ then $a_l \bar{P}_j a_k$ or $a_l \hat{P}_j a_k$ for all $a_l \in A \setminus \{a_j, a_k\}$.

Let $B(a_4, \hat{P}_1) \cup W(a_4, \hat{P}_1) = B(a_4, \hat{P}_3) \cup W(a_4, \hat{P}_3) = A \setminus \{a_4\} = Y$. The probability weight of $B(a_4, \hat{P}_3)$ at $(\hat{P}_3, \hat{P}_4)$ is β which is at least as large as the probability weight of $B(a_4, \hat{P}_3)$ at $(\hat{P}_1, \hat{P}_4)$. Since $\hat{P}_3$ is a (a_3, a_4)-partner of $\hat{P}_4$, the probability weight of $W(a_4, \hat{P}_3) = W(a_3, \hat{P}_4)$ is 0 at $(\hat{P}_1, \hat{P}_4)$ according to Lemma 2.1. Therefore, the probability weight of Y at $(\hat{P}_1, \hat{P}_4)$ is at most β. We also know that the probability weight of Y at $(\hat{P}_1, \hat{P}_4)$ is at least as large as the probability weight of $B(a_4, \hat{P}_1)$ at $(\hat{P}_1, \hat{P}_4)$ which in turn is at least as large as the probability weight of $B(a_4, \hat{P}_1)$ at $(\hat{P}_2, \hat{P}_4)$ which is β. Hence the probability weight of Y at $(\hat{P}_1, \hat{P}_4)$ is β implying that $\varphi_4(\hat{P}_1, \hat{P}_4) = 1-\beta$.

Let $B(a_1, \bar{P}_4) \cup W(a_1, \bar{P}_4) = B(a_1, \bar{P}_2) \cup W(a_1, \bar{P}_2) = A \setminus \{a_1\} = Z$. The probability weight of $B(a_1, \bar{P}_2)$ at $(\bar{P}_1, \bar{P}_2)$ is $1-\beta$ which is at least as large as the probability weight of $B(a_1, \bar{P}_2)$ at $(\bar{P}_1, \bar{P}_4)$. Since $\bar{P}_2$ is a (a_2, a_1)-partner of $\bar{P}_1$ and since the probability weight of $W(a_1, \bar{P}_2) = W(a_2, \bar{P}_1)$ at both $(\bar{P}_1, \bar{P}_4)$ is 0 according to Lemma 2.1[7], the probability weight of Z at $(\bar{P}_1, \bar{P}_4)$ is at most $1-\beta$. The probability weight of Z at $(\bar{P}_1, \bar{P}_4)$ is at least as large as the probability weight of $B(a_1, \bar{P}_4)$ at $(\bar{P}_1, \bar{P}_4)$ which in turn is at least as large as the probability weight of $B(a_1, \bar{P}_4)$ at $(\bar{P}_1, \bar{P}_3)$[8] which is $1-\beta$. Hence the probability weight of Z at $(\bar{P}_1, \bar{P}_4)$ is $1-\beta$, implying that $\varphi_1(\bar{P}_1, \bar{P}_4) = \varphi_1(\hat{P}_1, \bar{P}_4) = \beta$.

The probability weight of $B(a_4, \hat{P}_3)$ at $(\hat{P}_3, \hat{P}_4)$ is at least as large as the probability weight of $B(a_4, \hat{P}_3)$ at $(\bar{P}_1, \hat{P}_4)$. We know that the former is equal to β. Since $\hat{P}_3$ is a (a_3, a_4)-partner of $\hat{P}_4$, the probability weight of $W(a_4, \hat{P}_3) = W(a_3, \hat{P}_4)$ at $(\bar{P}_1, \hat{P}_4)$ is 0. This implies that the probability weight of Y at $(\bar{P}_1, \hat{P}_4)$ is at most β and $\varphi_4(\bar{P}_1, \hat{P}_4) \geq 1-\beta$. Likewise, the probability weight of $B(a_1, \bar{P}_2)$ at $(\bar{P}_1, \bar{P}_2)$ is at least as large as the probability weight of $B(a_1, \bar{P}_2)$ at $(\bar{P}_1, \hat{P}_4)$. We know that the latter is equal to $1-\beta$. Since $\bar{P}_2$ is a (a_2, a_1)-partner of $\bar{P}_1$, the probability weight of $W(a_2, \bar{P}_1) = W(a_1, \bar{P}_2)$ at $(\bar{P}_1, \hat{P}_4)$ is 0. This implies that the probability weight of Z at $(\bar{P}_1, \hat{P}_4)$ is at most $1-\beta$ and $\varphi_1(\bar{P}_1, \hat{P}_4) \geq \beta$. The last two inequalities together with the fact that φ is strategy-proof imply that $\varphi_1(\bar{P}_1, \hat{P}_4) = \varphi_1(\hat{P}_1, \hat{P}_4) = \beta$ and $\varphi_4(\bar{P}_1, \hat{P}_4) = \varphi_4(\bar{P}_1, \bar{P}_4) = 1-\beta$.

Since φ is strategy-proof, $\varphi_1(\hat{P}_1, \bar{P}_4) = \varphi_1(\bar{P}_1, \bar{P}_4) = \beta$ and $\varphi_4(\hat{P}_1, \bar{P}_4) = \varphi_4(\hat{P}_1, \hat{P}_4) = 1-\beta$. Using arguments symmetrical to those used in paragraphs above, we can conclude that $\varphi_1(P_1, P_4) = \varphi_4(P_4, P_1) = \beta$ and $\varphi_4(P_1, P_4) = \varphi_1(P_4, P_1) = 1-\beta$ for $P_1 \in \{\bar{P}_1, \hat{P}_1\}$ and $P_4 \in \{\bar{P}_4, \hat{P}_4\}$.

We now generalise this result to cover all the preference orderings in $\mathbb{D}^{a_1}$ and $\mathbb{D}^{a_4}$. We will use the same argument as in Lemma 2.8 for the proof. Our first step is to show that $\varphi_1(\bar{P}_1, P_4) = \beta$ and $\varphi_4(\bar{P}_1, P_4) = 1-\beta$. We know that $\varphi_4(\bar{P}_1, \hat{P}_4) = 1-\beta$. Since the peaks of P_4 and $\hat{P}_4$ are identical, $\varphi_4(\bar{P}_1, P_4) = 1-\beta$ i.e., $\varphi_1(\bar{P}_1, P_4) \leq \beta \leq \beta$. Suppose there exists $a_l \in A \setminus \{a_1, a_4\}$ such that $\varphi_l(\bar{P}_1, P_4) = \delta > 0$. It must be the case that $a_l \in M(a_1, a_4, \bar{P}_1)$, otherwise agent 1 will manipulate by announcing a_4 as the peak. We have assumed that $a_4 \bar{P}_1 a_2$. Since $\bar{P}_2$ is (a_2, a_1)-partner of $\bar{P}_1$, it follows that $a_l \bar{P}_2 a_1$ and $a_4 \bar{P}_2 a_1$. The total probability of the strict upper contour set of a_1 in $\bar{P}_2$ at profile $(\bar{P}_1, \bar{P}_2)$ is $1-\beta$ but is at least $1-\beta+\delta$ at $(\bar{P}_1, P_4)$. This contradicts the strategy-proofness of φ. Hence $\varphi_1(\bar{P}_1, P_4) = \beta$.

The next step is to show that $\varphi_1(P_1, \hat{P}_4) = \beta$ and $\varphi_4(P_1, \hat{P}_4) = 1-\beta$. Since $\varphi_1(\hat{P}_1, \hat{P}_4) = \beta$, it follows that $\varphi_1(P_1, \hat{P}_4) = \beta$ and $\varphi_4(P_1, \hat{P}_4) \leq 1-\beta$. If $\varphi_l(P_1, \hat{P}_4) = \delta > 0$ for some $a_l \in A \setminus \{a_1, a_4\}$ then $a_l \in M(a_4, a_1, \hat{P}_4)$, otherwise agent 2 will manipulate

[7] We have assumed that $a_4 \bar{P}_1 a_2$

[8] Note that $a_3 \bar{P}_4 a_1$.

by announcing a_1 as the peak. Recall that $a_1 \hat{P}_4 a_3$. Since $\hat{P}_3$ is (a_3, a_4)-partner of $\hat{P}_4$, it follows that $a_l \hat{P}_3 a_4$ and $a_1 \hat{P}_3 a_4$. The total probability of the strict upper contour set of a_4 in $\hat{P}_3$ at profile $(\hat{P}_3, \hat{P}_4)$ is β (by Lemma 2.1) but is at least $\beta + \delta$ in $(P_1, \hat{P}_4)$. This contradicts the strategy-proofness of φ. Hence $\varphi_4(P_1, \hat{P}_4) = 1 - \beta$.

Strategy-proofness of φ implies that $\varphi_1(P_1, P_4) = \varphi_1(\bar{P}_1, P_4) = \beta$ and $\varphi_4(P_1, P_4) = \varphi_4(P_1, \hat{P}_4) = 1 - \beta$. Using a symmetrical argument, we can conclude that $\varphi_4(P_4, P_1) = \beta$ and $\varphi_1(P_4, P_1) = 1 - \beta$. □

Lemma **2.11.** *Let $\beta \in [0, 1]$ and $4 \leq k \leq m$. Suppose for all injections $\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, m\}$, φ is a random-dictatorship with weights $(\beta, 1-\beta)$ over $B = \{a_{\sigma(1)}, \dots, a_{\sigma(k)}\} \subset A$ such that $a_{\sigma(1)} \overset{*}{\sim} a_{\sigma(2)} \overset{*}{\sim} \dots \overset{*}{\sim} a_{\sigma(k)}$. If $a_{k+1} \in A \setminus B$ is such that $a_{k+1} \overset{*}{\sim} a_{\sigma(k)}$ then φ is a random-dictatorship over $B' = B \cup \{a_{k+1}\}$ with weights $(\beta, 1 - \beta)$.*

Proof. To prove this Lemma, we will make use of the notation similar to what we adopted in Lemma 2.10. Let $\bar{P}_{\sigma(1)}, \hat{P}_{\sigma(1)} \in \mathbb{D}^{a_{\sigma(1)}}$, $\bar{P}_{\sigma(2)}, \hat{P}_{\sigma(2)} \in \mathbb{D}^{a_{\sigma(2)}}$, $\bar{P}_{\sigma(k)}, \hat{P}_{\sigma(k)} \in \mathbb{D}^{a_{\sigma(k)}}$ and $\bar{P}_{k+1}, \hat{P}_{k+1} \in \mathbb{D}^{a_{k+1}}$ be such that

1. $\hat{P}_{\sigma(1)}$ is an $(a_{\sigma(1)}, a_{\sigma(2)})$-reversal, $\bar{P}_{\sigma(2)}$ is a $(a_{\sigma(2)}, a_{\sigma(1)})$-partner and $\hat{P}_{\sigma(2)}$ is an $(a_{\sigma(2)}, a_{\sigma(1)})$-reversal of $\bar{P}_{\sigma(1)}$.
2. $\hat{P}_{\sigma(k)}$ is an $(a_{\sigma(k)}, a_{k+1})$-reversal, $\bar{P}_{k+1}$ is a $(a_{k+1}, a_{\sigma(k)})$-partner and $\hat{P}_{k+1}$ is an $(a_{k+1}, a_{\sigma(k)})$-reversal of $\bar{P}_{\sigma(k)}$.
3. $a_{k+1} \bar{P}_{\sigma(1)} a_{\sigma(2)}$ and $a_{\sigma(k)} \bar{P}_{k+1} a_{\sigma(1)}$.

From the statement of the theorem, we know that for all $j \in \{1, \dots, k\}$, φ is a random-dictatorship over the set $B' \setminus \{a_{\sigma(j)}\}$. We only need to show that for $P_{\sigma(1)} \in \mathbb{D}^{a_{\sigma(1)}}$ and $P_{k+1} \in \mathbb{D}^{a_{k+1}}$, $\varphi_{\sigma(1)}(P_{\sigma(1)}, P_{k+1}) = \varphi_{k+1}(P_{k+1}, P_{\sigma(1)}) = \beta$ and $\varphi_{k+1}(P_{\sigma(1)}, P_{k+1}) = \varphi_{\sigma(1)}(P_{k+1}, P_{\sigma(1)}) = 1 - \beta$.

Following the same method of proof as in Lemma 2.10, we first show that random-dictatorship spreads to $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ and $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$. The next step is to demonstrate the spread to arbitrary preference orderings.

Let $B(a_{k+1}, \hat{P}_{\sigma(1)}) \cup W(a_{k+1}, \hat{P}_{\sigma(1)}) = B(a_{k+1}, \hat{P}_{\sigma(k)}) \cup W(a_{k+1}, \hat{P}_{\sigma(k)}) = A \setminus \{a_{k+1}\} = Y$. Using the fact that the probability weight of $W(a_{k+1}, \hat{P}_{\sigma(k)})$ is 0 at $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ and the fact that the probability weight of $B(a_{k+1}, \hat{P}_{\sigma(k)})$ at $(\hat{P}_{\sigma(k)}, \hat{P}_{k+1})$ is at least as large as the probability weight of $B(a_{k+1}, \hat{P}_{\sigma(k)})$ at $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ which is $1 - \beta$, we get that the

the probability weight of Y at $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ is at most $1-\beta$. However, the probability weight of Y at $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ is at least as large as the probability weight of $B(a_{k+1}, \hat{P}_{\sigma(1)})$ at $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ which is in turn as least as large as β that is the probability weight of $Y = B(a_{k+1}, \hat{P}_{\sigma(1)})$ at $(\hat{P}_{\sigma(2)}, \hat{P}_{k+1})$. Hence the probability weight of $Y = A\{a_{k+1}\}$ at $(\hat{P}_{\sigma(1)}, \hat{P}_{k+1})$ is $1-\beta$ implying that $\varphi_{k+1}(\hat{P}_{\sigma(1)}, \hat{P}_{k+1}) = 1-\beta$.

Let $B(a_1, \bar{P}_{k+1}) \cup W(a_1, \bar{P}_{k+1}) = B(a_1, \bar{P}_{\sigma(2)}) \cup W(a_1, \bar{P}_{\sigma(2)}) = A \setminus \{a_1\} = Z$. The probability weight of $B(a_1, \bar{P}_{\sigma(2)})$ at $(\bar{P}_{\sigma(1)}, \bar{P}_{\sigma(2)})$ is $1-\beta$ which is at least as large as the probability weight of $B(a_1, \bar{P}_{\sigma(2)})$ at $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$. Since the probability weight of $W(a_1, \bar{P}_{\sigma(2)}) = W(a_2, \bar{P}_{\sigma(1)})$ is 0 at $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$, the probability weight of Z at $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$ is at most $1-\beta$. However, the probability weight of Z at $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$ is at least as large as the probability weight of $B(a_1, \bar{P}_{k+1})$ at $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$ which in turn is as large as the probability weight of $B(a_1, \bar{P}_{k+1})$ at $(\bar{P}_{\sigma(1)}, \bar{P}_{\sigma(k)})$ that is $1-\beta$. Hence the probability weight of $Z = A \setminus \{a_1\}$ at $(\bar{P}_{\sigma(1)}, \bar{P}_{k+1})$ is $1-\beta$ which implies that $\varphi_1(\bar{P}_{\sigma(1)}, \bar{P}_{k+1}) = \beta$.

The next step is to show that $\varphi_1(\bar{P}_{\sigma(1)}, \hat{P}_{k+1}) = \beta$ and $\varphi_4(\bar{P}_{\sigma(1)}, \hat{P}_{k+1}) = 1-\beta$. The probability weight of $B(a_{k+1}, \hat{P}_{\sigma(k)})$ at $(\bar{P}_{\sigma(1)}, \hat{P}_{k+1})$ is at most β which is the probability weight of $B(a_{k+1}, \hat{P}_{\sigma(k)})$ at $(\hat{P}_{\sigma(k)}, \hat{P}_{k+1})$. Since the probability weight of $W(a_{k+1}, \hat{P}_{\sigma(k)}) = W(a_{k+1}, \hat{P}_{k+1})$ at $(\bar{P}_{\sigma(1)}, \hat{P}_{k+1})$ is 0, the probability weight of $Y = A \setminus \{a_{k+1}\}$ at $(\bar{P}_{\sigma(1)}, \hat{P}_{k+1})$ is at most β and $\varphi_4(\bar{P}_{\sigma(1)}, \hat{P}_{k+1}) \geq 1-\beta$. Similarly the probability weight of $B(a_1, \bar{P}_{\sigma(2)})$ at $(\bar{P}_{\sigma(1)}, \bar{P}_2)$ is at least as large as $1-\beta$ which is the probability weight of $B(a_1, \bar{P}_{\sigma(2)})$ at $(\bar{P}_{\sigma(1)}, \hat{P}_{k+1})$. Again since the probability weight of $W(a_1, \bar{P}_{\sigma(2)}) = W(a_2, \bar{P}_{\sigma(1)})$ at $(\bar{P}_{\sigma(1)}, \hat{P}_{k+1})$ is 0, the probability weight of $Z = A \setminus \{a_1\}$ at $(\bar{P}_{\sigma(1)}, \hat{P}_{k+1})$ is at most $1-\beta$ and $\varphi_1(\bar{P}_{\sigma(1)}, \hat{P}_{k+1}) \geq \beta$. The last two inequalities imply that $\varphi_1(\bar{P}_{\sigma(1)}, \hat{P}_{k+1}) = \varphi_1(\hat{P}_{\sigma(1)}, \hat{P}_{k+1}) = \beta$ and $\varphi_4(\bar{P}_{\sigma(1)}, \hat{P}_{k+1}) = \varphi_4(\bar{P}_{\sigma(1)}, \bar{P}_{k+1}) = 1-\beta$.

Strategy-proofness of φ implies that $\varphi_1(\hat{P}_{\sigma(1)}, \bar{P}_{k+1}) = \varphi_1(\bar{P}_{\sigma(1)}, \bar{P}_{k+1}) = \beta$ and $\varphi_4(\hat{P}_{\sigma(1)}, \bar{P}_{k+1}) = \varphi_4(\hat{P}_{\sigma(1)}, \hat{P}_{k+1}) = 1-\beta$. Using symmetric arguments, we conclude that $\varphi_1(P_{\sigma(1)}, P_{k+1}) = \varphi_{k+1}(P_{k+1}, P_{\sigma(1)}) = \beta$ and $\varphi_{k+1}(P_{\sigma(1)}, P_{k+1}) = \varphi_1(P_{k+1}, P_{\sigma(1)}) = 1-\beta$.

The next step is generalise this result to all preference orderings in $\mathbb{D}^{a_1}$ and $\mathbb{D}^{a_{k+1}}$. Using arguments similar to the one used in Lemma 2.10 and just replacing a_4 by a_{k+1} and a_3 by a_k, we can conclude that $\varphi_1(\bar{P}_1, P_{k+1}) = \beta$ and $\varphi_{k+1}(P_1, \hat{P}_{k+1}) = 1-\beta$. It follows from the strategy-proofness of φ that $\varphi_1(P_1, P_{k+1}) = \varphi_1(\bar{P}_1, P_{k+1}) = \beta$ and $\varphi_{k+1}(P_1, P_{k+1}) = \varphi_{k+1}(P_1, \hat{P}_{k+1}) = 1-\beta$. Using a symmetrical argument, we can conclude that $\varphi_{k+1}(P_{k+1}, P_1) = \beta$ and $\varphi_1(P_{k+1}, P_1) = 1-\beta$. $\square$

Lemma **2.12.** *A P-domain with a non-star graph is random-dictatorial.*

Proof. We make a couple of observations here. The first is that a non-star connected graph with 3 vertices is cyclic. The second is that the maximum length of a path between two vertices in a connected non-star graph G, with more than 3 vertices, is greater than 2.

Suppose $m = 3$. Then $\bar{G}(\mathbb{D})$ must be cyclic i.e., if $A = \{a_1, a_2, a_3\}$ then $a_1 \overset{*}{\sim} a_2 \overset{*}{\sim} a_3 \overset{*}{\sim} a_1$. It follows from Lemmas 2.4 and 2.7 that the random-dictatorship spreads to the entire domain which in this case happens to be the universal domain.

Suppose $m > 3$. According to Proposition 2.4 (see Appendix), the maximum length of a path in $\bar{G}(\mathbb{D})$ is greater than 2. Let $\{a_t\}_{t=1}^{T}$ with $4 \le T \le m$ be a path in $\bar{G}(\mathbb{D})$. Lemma 2.11 demonstrates that random-dictatorship spreads to this entire path with weights, say $(\beta, 1-\beta)$. Let $a_k \in A \setminus \{a_1, \ldots, a_T\}$. Since the graph is connected, there must be a path π between a_k and some $a_j \in \{a_1, \ldots, a_T\}$ such that none of the alternatives in $\{a_1, \ldots, a_T\} \setminus \{a_j\}$ lie in π. The maximum of lengths of the sub-paths $\pi_1^1 = \{a_1, \ldots, a_j\}$ and $\pi_2^1 = \{a_j, \ldots, a_T\}$ of $\{a_t\}_{t=1}^{T}$ is at least $\frac{T}{2}$ and $\frac{T-1}{2}$ for T even and odd respectively. Since $T \ge 4$, $\frac{T}{2} \ge 2$ and $\frac{T-1}{2} \ge 2$ for T even and odd respectively. Let π^2 contain π and π_1^1 as sub-paths and let π^3 contains π and π_2^1 as sub-paths. Since the minimum length of π is 1, the maximum of the lengths of π^2 and π^3 is greater than 2 and both π^2 and π^3 include at least one edge from $\{a_t\}_{t=1}^{T}$. Suppose the length of π^2 is greater than that of π^3 without loss of generality. It follows from Lemma 2.11, that the random-dictatorship spreads to π^2 with weights $(\beta, 1-\beta)$. If the length of π^3 is 2 then from Lemma 2.7 random-dictatorship spreads to π^3 as well. $\square$

Lemma **2.13.** *A P-domain with a star graph is random-dictatorial.*

Proof. Let a_j be the hub of the P-domain with a star graph. By definition of the P-domain, there must exist $a_k, a_q \in A \setminus \{a_j\}$ such that $a_k \sim a_q$ or $M(a_k, a_j, P_k^1) = M(a_q, a_j, P_q^3) = W(a_j, P_k^2) = W(a_j, P_q^4)$ for some $P_k^1, P_k^2 \in \mathbb{D}^{a_k}$ and $P_q^3, P_q^4 \in \mathbb{D}^{a_q}$.

Case 1: Suppose $a_k \sim a_q$. Since $a_k \overset{*}{\sim} a_j \overset{*}{\sim} a_q$, it follows from Lemma 2.7 that $\varphi_k(P_k, P_j) = \beta$. Therefore, for all $a_x \in A \setminus \{a_j, a_k\}$, we have $\varphi_x(P_x, P_k) = \varphi_k(P_k, P_x) = \varphi_x(P_x, P_j) = \varphi_k(P_k, P_j) = \beta$ and $\varphi_k(P_x, P_k) = \varphi_x(P_k, P_x) = \varphi_j(P_k, P_j) = 1 - \beta$.

Case 2: Suppose $M(a_k, a_j, P_k^1) = M(a_q, a_j, P_q^3) = W(a_j, P_k^2) = W(a_j, P_q^4)$ for some $P_k^1, P_k^2 \in \mathbb{D}^{a_k}$ and $P_q^3, P_q^4 \in \mathbb{D}^{a_q}$. Suppose $\varphi_k(P_k, P_j) = \beta$ and $\varphi_j(P_j, P_k) = \alpha$ for all

$P_k \in \mathbb{D}^{a_k}$ and $P_j \in \mathbb{D}^{a_j}$. It follows from Corollary 2.1 and Lemma 2.4 that $\varphi_q(P_q, P_j) = \beta$ and $\varphi_j(P_j, P_q) = \alpha$. Since $a_k \overset{*}{\sim} a_j \overset{*}{\sim} a_q$, it follows from Lemma 2.6 that $\alpha \geq \beta$.

Since $M(a_k, a_j, P_k^1) = M(a_q, a_j, P_q^3)$, it must be that $a_j P_k^1 a_q$ and $a_j P_q^3 a_k$[9]. Since $M(a_k, a_j, P_k^1) = W(a_j, P_q^4)$, it follows that $M(a_k, a_j, P_k^1) \subset W(a_k, P_q^4)$. If any alternative in $W(a_k, P_q^4)$ is chosen with a positive probability at the profile (P_k^1, P_q^4) then agent 2 will manipulate by reporting a_k as her peak. The probability weight of the upper contour set of a_j in P_k^1 at profile (P_k^1, P_q^4) must be at least as large as the probability weight of the same set at (P_j, P_q^4) for all $P_j \in \mathbb{D}^{a_j}$. Hence $\varphi_k(P_k^1, P_q^4) \geq \alpha$. Using similar arguments, we can infer that $\varphi_q(P_k^2, P_q^3) \geq 1 - \beta$. Since φ is strategy-proof, it follows that $\varphi_k(P_k^2, P_q^4) \geq \alpha$ and $\varphi_q(P_k^2, P_q^4) \geq 1 - \beta$. Hence $(\alpha + 1 - \beta) \leq 1$ implying that $\alpha \leq \beta$.

Hence $\alpha = \beta$. □

This completes the proof of Step 1.

Step 2: In this step, we extend the result to $n > 2$ by showing that P- domains satisfy the conditions of Ramification theorem in Chatterji et al. (2014).

Definition 2.11. A domain $\mathbb{D}$ satisfies *minimally richness* if for all $a_j \in A$, $\mathbb{D}^{a_j} \neq \phi$.

Definition 2.12. A domain $\mathbb{D}$ satisfies Condition α if there exist three distinct alternatives $a_j, a_k, a_q \in A$ and three preference orderings $P_i^j \in \mathbb{D}^{a_j}$, $P_i^k \in \mathbb{D}^{a_k}$, $P_i^q \in \mathbb{D}^{a_q}$ such that

1. $a_k P_i^j a_q$, $a_q P_i^k a_j$ and $a_j P_i^q a_k$
2. $W(a_k, P_i^j) \cup W(a_q, P_i^k) \cup W(a_j, P_i^q) = A$

Theorem 2.7 (Chatterji et al. (2014) Ramification Theorem)**.** *Let $\mathbb{D}$ satisfies minimal richness and Condition α. Then the following statements are equivalent:*

1. *$\varphi : \mathbb{D}^2 \to \mathcal{L}(A)$ is unanimous and strategy-proof $\Rightarrow$ φ is a random-dictatorship.*
2. *$\varphi : \mathbb{D}^n \to \mathcal{L}(A)$ is unanimous and strategy-proof $\Rightarrow$ φ is a random-dictatorship for $n \geq 2$.*

Proposition 2.1. *A P- domain satisfies Condition α as well as minimal richness.*

[9] If $a_q P_k^1 a_j$ then $M(a_k, a_j, P_k^1) \neq M(a_q, a_j, P_q^3)$ since $a_q \in M(a_k, a_j, P_k^1)$

We show that the result holds for a γ-domain which implies that it holds for a P-domain as well. For a γ-domain $\mathbb{D}$, $|\mathbb{D}^{a_j}| \geq 2$ for all $a_j \in A$ since a_j is weak* connected to some other alternative in A. Moreover, since $m \geq 3$, there must be distinct $a_j, a_k, a_q \in A$ such that $a_j \overset{*}{\sim} a_k \overset{*}{\sim} a_q$. Therefore, there must be $P_j^1, P_j^2 \in \mathbb{D}^{a_j}$, $P_k^3, P_k^4, P_k^5, P_k^6 \in \mathbb{D}^{a_k}$ and $P_q^7, P_q^8 \in \mathbb{D}^{a_q}$ such that $M(a_j, a_k, P_j^1) = M(a_k, a_j, P_k^3) = W(a_k, P_j^2) = W(a_j, P_k^4)$ and $M(a_k, a_q, P_k^5) = M(a_q, a_k, P_q^7) = W(a_q, P_k^6) = W(a_k, P_q^8)$.

Suppose $a_q P_j^1 a_k$ and $a_j P_j^8 a_k$ without loss of generality. The existence of such preference orderings is guaranteed by the definition of the γ-domain. Then P_j^2, P_k^3 and P_q^8 are three preference orderings such that

1. $a_k P_j^2 a_q$, $a_q P_k^3 a_j$ and $a_j P_q^8 a_k$
2. $W(a_k, P_j^2) \cup W(a_q, P_k^3) \cup W(a_j, P_q^8) = A$ [10].

Since a P-domain satisfies condition α, it is random-dictatorial for all $n \geq 2$. This completes the proof of Step 2 and Theorem 2.6. □

We have demonstrated in the Section 2.3 that a circular domain is a P-domain. The following corollary follow from Theorem 2.6.

Corollary 2.2. *A Circular domain is random-dictatorial.*

2.5 Discussion

In this section, we present a result on the minimum size of a random-dictatorial domain and discuss some features of P- domains.

2.5.1 Minimal size of random-dictatorial domains

The question regarding the minimum size of random-dictatorial domain is trivial without assumptions of the richness of the domain. For instance, the domain consisting of a single preference ordering is trivially a random-dictatorial domain. For that reason, we impose the condition of minimal richness defined in section 2.4.

Aswal et al. (2003) have shown in the following result that a domain of size less than $2m$ cannot be dictatorial and hence random-dictatorial.

[10]Note that since P_k^3 is (a_k, a_j)-reversal of P_j^2, $W(a_k, P_j^2) \cup W(a_q, P_k^3) = A \setminus \{a_k\}$.

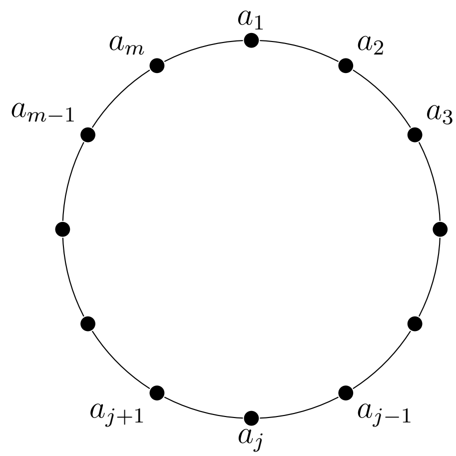

Figure 2.3: A Circular domain $\mathbb{D}^c$

Theorem 2.8. *If a domain satisfies the unique seconds property*[11]*, then it is non-dictatorial.*

If a minimally rich domain $\mathbb{D}$ has a size less than $2m$ then there must be some alternative $a_j \in A$ such that $|\mathbb{D}^{a_j}| = 1$. As a result, $\mathbb{D}$ must satisfy the unique second property. Consequently it is not dictatorial and therefore not random-dictatorial[12].

Proposition 2.2. *Let $\mathbb{D}$ be a random-dictatorial domain satisfying minimal richness. Then $|\mathbb{D}| \geq 2m$. Moreover, this bound is tight.*

Proof. In view of Theorem 2.8, we only need to show the tightness of the bound. Consider a specific circular domain $\mathbb{D}^c$ which includes exactly two preference orderings $\bar{P}_i^j$ and $\hat{P}_i^j$ for every alternative $a_j \in A$ such that the following holds:

- $a_j \bar{P}_i^j a_{j+1} \bar{P}_i^j a_{j+2} \ldots a_{j-2} \bar{P}_i^j a_{j-1}$
- $a_j \hat{P}_i^j a_{j-1} \hat{P}_i^j a_{j-2} \ldots a_{j+2} \hat{P}_i^j a_{j+1}$

Preference orderings $\bar{P}_i^j$ and $\hat{P}_i^j$ are the ordering obtained by ranking the alternatives in the circle shown in Figure 2.3 in the clockwise and the counter-clockwise directions starting from a_j. Clearly $\mathbb{D}^c$ has $2m$ preference orderings. It is also a circular domain so that Corollary 2.2 applies. $\square$

[11] A domain $\mathbb{D}$ satisfies the unique seconds property if there is a pair of alternatives $a_j, a_k \in A$ such that $r_1(P_i) = a_j \Rightarrow r_2(P_i) = a_k$ for all $P_i \in \mathbb{D}^n$.

[12] Note that a random-dictatorial domain is always dictatorial while the converse is not true.

Chatterji et al. (2014) have shown that more structure on the graph (i.e., a higher number of connections) is required for a linked domain to be random-dictatorial. It therefore seems intuitive that the minimal size of a random-dictatorial domain might be larger the minimal size of a dictatorial domain. But since we know from 2.2 that the minimal circular domain with $2m$ preference orderings is random-dictatorial, the minimum size of a random-dictatorial domain turns out to be equal to the minimum size of a dictatorial domain[13]. The next corollary is related to the minimal size of a random-dictatorial domain.

Corollary 2.3. *The minimum size of a random-dictatorial domain is* $2m$*.*

2.5.2 Tops-onlyness

It is worthwhile noting that neither γ domains nor the β domains satisfy the sufficient conditions, formulated in Chatterji and Zeng (2018), for *tops-onlyness* of a random social choice functions.

Definition 2.13. A RSCF $\varphi : \mathbb{D}^n \to \mathcal{L}(A)$ satisfies the *tops-only property* if for all $P, P' \in \mathbb{D}^n$ such that $r_1(P_i) = r_1(P'_i)$ for all $i \in N$, $\varphi_j(P) = \varphi_j(P')$ for all $j \in 1, \dots, m$. A domain $\mathbb{D}$ is *tops-only* if every strategy-proof and unanimous $\varphi : \mathbb{D}^n \to \mathcal{L}(A)$ satisfies tops-only property.

Chatterji and Zeng (2018) proved that a minimal-rich domain is tops-only whenever it satisfies the *interior* and the *exterior* properties[14]. We show below that these conditions are not satisfied by circular domains and hence by P- domains.

Proposition 2.3. *Assume* $m > 3$*. Every minimal circular domain violates the interior property. There exist minimal circular domains that violate the exterior property.*

Proof. Let $\mathbb{D}$ be a minimal circular domain. Pick a_1, a_2 such that $a_1 \overset{*}{\sim} a_2$. According to the definition of a circular domain, there exists two preference orderings, say P_i^1 and P_i^2 such that $r_1(P_i^1) = r_1(P_i^2) = a_1$, $r_2(P_i^1) = a_2$ and $r_m(P_i^2) = a_2$. Since $\mathbb{D}$ is minimal, these two are the only two preference orderings which belong to $\mathbb{D}^{a_1}$. Since $m > 3$, there does not exist a sequence of preference orderings in $\mathbb{D}^{a_1}$, originating in P_i^1 and terminating at

[13] Sato (2010) have shown circular domain to be dictatorial

[14] Appropriate definitions can be found in Chatterji and Zeng (2018).

P_i^1	P_i^2	P_i^3	P_i^4	P_i^5	P_i^6	P_i^7	P_i^8	P_i^9	P_i^{10}	P_i^{11}	P_i^{12}
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4	a_5	a_5	a_6	a_6
a_2	a_6	a_3	a_1	a_4	a_2	a_5	a_3	a_6	a_4	a_1	a_5
a_3	a_5	a_4	a_6	a_5	a_1	a_6	a_1	a_1	a_3	a_2	a_4
a_4	a_4	a_5	a_5	a_6	a_6	a_1	a_6	a_2	a_2	a_3	a_3
a_5	a_3	a_6	a_4	a_1	a_5	a_2	a_2	a_3	a_1	a_4	a_2
a_6	a_2	a_1	a_3	a_2	a_4	a_3	a_5	a_4	a_6	a_5	a_1

Table 2.2: Circular domain

P_i^2 such that successive preference orderings in the sequence differ only by a switch of adjacent alternatives.

For the second part of the Proposition, refer to the domain in Table 2.2. We claim that the domain doesn't satisfy the exterior property. For a pair of preference orderings P_i, P_i' in the domain with different peaks and for any pair of alternatives (a_j, a_k) ranked similarly in the two preference orderings (say $a_j P_i a_k$ and $a_j P_i' a_k$), the exterior property requires the existence of a sequence of preference orderings in the domain, originating in P_i and terminating at P_i', such that the pair (a_j, a_k) is isolated[15] in every successive pair of preference orderings in the sequence. Consider the pair of preference orderings (P_i^8,P_i^1) and the pair of alternatives (a_1, a_6). It can be verified that there does not exist any preference P_i in the domain such that the pair (a_1, a_6) is isolated in the pair (P_i^8, P_i). □

2.6 Conclusion

In this chapter, we have demonstrated that P-domains are random-dictatorial and derived the lower bound of the size of random-dictatorial domains satisfying minimal-richness.

2.7 Appendix

In this section, we prove a result which we require in the proof of Theorem 2.6.

[15] A pair of alternatives (a_j, a_k) is isolated in a pair of preference orderings $(\bar{P}_i, \hat{P}_i)$ if there exist $k \in \{1, \ldots, (m-1)\}$ such that the upper contour set of $r_k(\bar{P}_i)$ in $\bar{P}_i$ is identical to the upper contour set of $r_k(\hat{P}_i)$ in $\hat{P}_i$ and a_j belongs to these sets if and only if a_k does not. Details can be found in Chatterji and Zeng (2018).

Proposition 2.4. *For any graph with more than three vertices, the maximum length of a path in a connected graph is more than 2 if and only if it is non-star graph.*

Proof. Throughout this proof, we shall denote a graph G by a pair (V, E) where V and E are the sets of vertices and edges in the graph respectively. For a pair of vertices $v_i, v_j \in V$, $(v_i, v_j) \in E$ if v_i and v_j are connected by an edge in the graph.

If G is a star graph then the maximum length of a path in G is 2 by definition. To complete the proof, we show that if G is non-star graph then there is a path of length 3 . We will prove this by induction on $|V|$.

Suppose $V = \{v_1, v_2, v_3, v_4\}$. Since G is connected there must be a vertex, say v_1 which forms edges with two other vertices, say v_2 and v_3. We consider two cases, $(v_4, v_1) \in E$ and $(v_4, v_1) \notin E$. Suppose $(v_4, v_1) \in E$. Since G is a non-star graph, at least one of the edges (v_4, v_2), (v_4, v_3) or (v_2, v_3) must belong to E. If $(v_4, v_2) \in E$ then there is a path of length 3 between v_4 and v_3. If $(v_4, v_3) \in E$ then there is a path of length 3 between v_4 and v_2. If however, $(v_2, v_3) \in E$ then there is a path of length 3 between v_3 and v_4. Suppose $(v_4, v_1) \notin E$. Since G is connected, then either (v_4, v_2) or (v_4, v_3) belongs to E. In either case, as argued earlier, we get a path of length 3 in G. We have shown that the result is true for $|V| = 4$.

We assume that the result holds for all connected non-star graphs with maximum $k-1$ vertices where $k > 4$. Let $V = \{v_1, \ldots, v_k\}$. Suppose $G = (V, E)$ is a connected non-star graph. We need to show that G contains a path of length 3. Suppose we remove a vertex (alternative) say v_j from G. Let E_j be set of edges in G such that v_j is one of the vertices in every edge in E_j. There are three possibilities for the resulting graph $(V \setminus \{v_j\}, E \setminus E_j)$. It is (i) a non-star graph (ii) a disconnected graph or (iii) a star graph. If (i) holds then by our assumption it contains a path of length 3. Adding v_j and E_j back to the $(V \setminus \{v_j\}, E \setminus E_j)$ will not remove this path from G. If (ii) holds then v_j has an edge with at least one vertex from every (disconnected) component of $(V \setminus \{v_j\}, E \setminus E_j)$. If every component of $(V \setminus \{v_j\}, E \setminus E_j)$ has exactly one vertex then v_j must be the hub of (V, E) implying that (V, E) is a star graph which is a contradiction. This implies that at least one component of $(V \setminus \{v_j\}, E \setminus E_j)$, say G' contains more than one vertex. Suppose $(v_j, v_r) \in E_j$ without loss of generality where $v_r \neq v_s \in G'$. Note that there must be a path between v_r and v_s by definition of a component. Since v_j is connected to at least one vertex from every other component, there must be a path $(v_j, v_t) \in E_j$ such v_t is not a vertex in G'. This implies that there is a path $(v_s, \ldots, v_r, v_j, v_t)$ in E

with length greater or equal to 3. Suppose $(V \setminus \{v_j\}, E \setminus E_j)$ is a star graph with hub say v_q. If $E_j = \{(v_j, v_q\}$ then (V, E) is a star graph with hub v_q. Therefore E_j is not a singleton set implying that $(v_j, v_r) \in E_j$ where $v_r \neq v_s \in V \setminus \{v_j, v_q\})$. Hence there is a path $(v_j, v_r, v_q, v_s) \in G$ which has a length of 3. This completes the proof. $\square$

Chapter 3

Towards a defence of non-bossiness

3.1 Introduction

The axiom of *non-bossiness* is pervasive in the theory of allocation. The notion was first introduced by Satterthwaite and Sonnenschein (1981) and since then has been widely used in the literature, for example in Svensson (1999), Pápai (2001), Kojima (2010), Kongo (2013) and Afacan (2012)[1].

The non-bossiness axiom has been extensively criticized in Thomson (2016). He argued that it cannot be justified either on strategic grounds or on normative grounds such as fairness, arbitrariness, consistency, consistency and welfare dominance. Our objective is in this chapter is to provide an alternative justification for non-bossiness. We argue that it is a simplifying assumption which can be made "without loss of generality" for an expected welfare maximizing planner in many situations. We provide one such context.

We consider a basic model of object allocation with an equal number of agents and objects. There is a planner whose objective is to maximise her expected welfare in the class of all strategy-proof and efficient allocation rules. The preference ordering or type of an agent is her private information. We make two critical assumptions. The first is that the welfare of the planner is symmetric in the identity of the agents and the second is that preference orderings or types of agents (ex-ante beliefs of the planner) are distributed independently and identically. For the ease of exposition, we make two further simplifying assumptions. We assume that the welfare of the planner is additive

[1]Some authors have assumed non-bossiness implicitly as a consequence of the group-strategy-proofness, for example Pycia and Ünver (2017). Group-strategy-proofness however can be criticised on the ground that it is not compatible with private information.

(utilitarian) in the utilities of the agents. The second is that the preference orderings of agents are distributed uniformly. The results can be generalised to other anonymous welfare functions and distributions.

In this chapter, we present two results. The first concerns the case where the number of agents and the number of objects is equal to three. We show that for every strategy-proof, efficient and neutral allocation rule, there exists a strategy-proof, efficient and non-bossy rule which gives planner the same expected welfare. From Svensson (1999) we know that a neutral, strategy-proof and non-bossy allocation rule must be a *priority rule*. In other words, the planner does not obtain any advantage in terms of expected welfare in the case of three objects while selecting a bossy rule from the class of neutral, efficient and strategy-proof rules. It is an open question that this equivalence holds more generally.

In our second result, we consider a case where the number of agents is arbitrary but equal to the agents. We consider a class of bossy allocation rules which we call *lower bossy* rules. In these rules, a particular agent always goes first in the queue. The identity of the second agent in the queue depends upon the preference orderings of the first while the identity of the third agent depends upon the preference orderings of the first two and so on. This rule is strategy-proof and efficient. We show that every lower bossy rule is EUE to an arbitrary priority rule. Again planner can impose the simplifying assumption of non-bossiness without loss of expected utility.

This chapter is organised as follows. The section 2 introduces the notations and definitions. Section 3 deals with the case of the three agents and three objects while section 4 discusses lower-bossy rules for arbitrary n. Section 5 concludes.

3.2 Model and Basic Definitions

Let $N = \{1, 2, \ldots, n\}$ and $A = \{a_1, \ldots, a_n\}$ denote the set of agents and objects respectively. Agent $i \in N$ has a strict preference ordering P_i over A, where $a_j P_i a_k$ for any pair of distinct objects (a_j, a_k) signifies that a_j is strictly preferred to a_k according to P_i[2]. Let $\mathbb{P}$ denote the set of all strict orderings over A. We shall refer to $\mathbb{P}$ as the *universal domain*. Clearly $P_i \in \mathbb{P}$ for all $i \in N$. A preference profile $P = (P_1, \ldots, P_n) \in \mathbb{P}^n$. The i^{th} component of a profile P is the preference ordering P_i of agent $i \in N$. We shall denote

[2]A strict ordering is an ordering which is complete, reflexive, transitive and antisymmetric.

profiles without subscripts, $P^1, P^2, \ldots, P^r$ etc. For any $P_i' \in \mathbb{P}$ and $P \in \mathbb{P}^n$, (P_i', P_{-i}) is the profile where P_i is replaced by P_i' at the profile P.

For any $P_i \in \mathbb{P}$ and $B \subset A$, we let $\max(P_i, B) = a_j$ if $a_j P_i a_k$ for all $a_k \in B \setminus \{a_j\}$; i.e. $\max(P_i, B)$ is the P_i maximal element in B. For future reference, $r_k(P_i)$ denotes the k^{th} ranked element in P_i where $k = 1, \ldots, n$. Thus $r_k(P_i) = a_j$ if $|\{a_s : a_j P_i a_s\}| = n - k$. Abusing notation slightly, we shall sometimes write $r(a_j, P_i) = k$ if $r_k(P_i) = a_j$, i.e. $r(a_j, P_i)$ is the rank of a_j in the preference ordering P_i.

An allocation φ is a bijection $\varphi : N \to A$. For any $i \in N$, $\varphi_i \in A$ is the object allocated to agent i. We will denote the set of allocations by Φ. An allocation rule F is a mapping $F : \mathbb{P}^n \to \Phi$; i.e. an allocation rule assigns an allocation to every preference profile. Here $F_i(P)$ is the object allocated to i at profile P according to the rule F.

A *priority* π is a bijection $\pi : N \to N$. The priority π defines a queue on the set of agents $(i_1, i_2, \ldots, i_n)$ such that $\pi(i_t) = t$ for all $t \in N$. In this queue, agent i_1 is first, i_2 second and so on with i_n last. At every profile P, the priority π generates a *queue allocation* $\varphi^\pi(P)$ in the following way: for all $t \in N$, $\varphi^\pi_{i_t}(P) = \max(P_{i_t}, (A \setminus \{\varphi^\pi_{i_1}(P), \varphi^\pi_{i_2}(P), \ldots, \varphi^\pi_{i_{t-1}}(P)\}))$. In the queue allocation, i_1 gets the P_{i_1}-maximal element in the set A, i_2 gets the P_{i_2}-maximal element in the remainder set $A \setminus \{\varphi^\pi_{i_1}(P)\}$ and so on. An allocation rule F is a *priority rule* if there exists a priority π such that $F(P) = \varphi^\pi(P)$ for all profiles P. For convenience we let F^π denote the priority rule generated by the priority π.

We now briefly describe some familiar requirements for the allocation rules.

An allocation rule F is *efficient* if for all $P \in \mathbb{P}^n$, $F(P)$ is Pareto-efficient, i.e., there doesn't exist an allocation $\varphi \in \Phi$ such that $\varphi_i P_i F_i(P)$ or $\varphi_i = F_i(P)$ for all $i \in N$ and $\varphi_i P_i F_i(P)$ for some $i \in N$. It is well known that $F(P)$ is Pareto efficient if there exists a priority π at P such that $F(P) = F^\pi(P)$. Note that the π could depend on P.

An allocation rule F is *manipulable* if an agent has an incentive to misreport her preference ordering; i.e. there exists $i \in N$, $P_i, P_i' \in \mathbb{P}$ and $P_{-i} \in \mathbb{P}^{n-1}$ such that $F_i(P_i', P_{-i}) P_i F(P_i, P_{-i})$. An allocation is *strategy-proof* if it is not manipulable. Priority rules are strategy-proof. This is a consequence of two features of priority rules. The first is that every agent i is presented with a set from her P_i-maximal object is chosen. Misreporting her preference ordering P_i can only harm her. The second reason is that no agent can influence the set from which her P_i-maximal object is chosen.

Let $\sigma : A \to A$ be a permutation over the set of objects. We denote the set of all permutations by Σ. Consider a preference profile and a permutation of the set of objects. The permutation induces a permuted preference profile in an obvious way. For any preference ordering $P_i \in \mathbb{P}$, the preference ordering $[\sigma \circ P_i]$ is defined as follows: for all distinct $a_j, a_k \in A$, $a_j P_i a_k \implies \sigma(a_j)\ [\sigma \circ P_i]\ \sigma(a_k)$. For any profile $P \in \mathbb{P}^n$, $[\sigma \circ P]$ is the profile obtained where the preference ordering of every agent $i \in N$ is $[\sigma \circ P_i]$. For every allocation φ, we let $\sigma \circ \varphi$ be the allocation defined by $(\sigma \circ \varphi)_i = \sigma(\varphi_i)$. An allocation rule F is *Neutral* or satisfies *neutrality* if, for all $P \in \mathbb{P}^n$ and $\sigma \in \Sigma$, we have $F(\sigma \circ P) = \sigma \circ (F(P))$. Neutrality imposes the requirement that the names of objects do not matter. A neutral allocation rule links the outcomes at the original profile and its permuted counterpart; in particular, the permutation of the outcome at the original profile is the outcome of the permuted profile.

An important axiom in our analysis is *non-bossiness*. An allocation rule F is *non-bossy* if no agent can change the allocation at any profile without changing the object allocated to herself. Formally, for all $i \in N$, $P_i, P_i' \in \mathbb{P}$ and $P_{-i} \in \mathbb{P}^{n-1}$, we have $F(P_i', P_{-i}) = F(P_i, P_{-i})$ whenever $F_i(P_i', P_{-i}) = F_i(P_i, P_{-i})$. An allocation rule F is *bossy* if it not non-bossy. The non-bossiness axiom is used widely in allocation theory - Thomson (2016) provides an extensive discussion of these issues. A priority rule satisfies non-bossiness. Pick an arbitrary agent i and suppose $F_i(P_i, P_{-i}) = F_i(P_i', P_{-i})$ for some $P_i, P_i' \in \mathbb{P}$ and $P_{-i} \in \mathbb{P}^{n-1}$. Then, the set of objects presented to the agent following i in the priority will remain the same. Since the preference orderings of all agents other than i are unchanged, all agents following i in the priority will be presented the same sets to choose from and will choose the same objects at the profile (P_i', P_{-i}) as they did at (P_i, P_{-i}).

We assume the existence of a social planner whose objective is to choose an allocation rule which maximises expected welfare. Let P_i be a preference ordering. The utility of i^{th} agent from object a_j at P_i is $u_i(a_j, P_i)$ where $u_i(a_j, P_i) = (n+1) - r(a_j, P_i)$. Fix an allocation rule F. The utility of agent i at profile P upon receiving the allocation $F_i(P)$ is $u_i(F(P)) = u_i(F_i(P), P_i)$. The welfare of the planner at profile P under F is given by

$$W^F(P) = \sum_{i=1}^{n} u_i(F(P))$$

The expected welfare of the planner for the allocation rule F is given by

$$E\left(W^F\right) = \sum_{P \in P^N} W^F(P)$$

The expression for expected welfare assumes a uniform distribution over preference profiles. The more accurate expression should include a multiplicative constant which we omit since we are using expected welfare only for comparison between allocation rules. The allocation rules F and G are *expected utility equivalent* (or EUE) if $E(W^F) = E(W^G)$.

3.3 The case of three objects

In this section, we consider the case where there are three agents $N = \{1, 2, 3\}$ and three objects $A = \{a_1, a_2, a_3\}$. We define an allocation rule, called the *modified priority rule* below.

Let $P \in \mathbb{P}$ be a profile. We say that P is a 0-profile for a pair of objects (a_s, a_t) if all agents agree on the ranking of a_s and a_t. It is a 1-profile otherwise. We define an indicator function $\mathbb{1}_{\{a_s,a_t\}} : \mathbb{P}^n \to \{0, 1\}$ such that

$$\mathbb{1}_{\{a_s,a_t\}}(P) = \begin{cases} 0 & \text{if P is a 0-profile for the pair } (a_s, a_t) \\ 1 & \text{otherwise} \end{cases}$$

Definition 3.1. A modified priority rule is specified by a pair $\langle i_1, \lambda\rangle$ where $i_1 \in N$ and λ is a mapping $\lambda : \{0, 1\} \to N \setminus \{i_1\}$. The modified priority rule specified by $\langle i_1, \lambda\rangle$ is denoted by $F^{\langle i_1,\lambda\rangle}$. For every $P \in \mathbb{P}$, $F^{\langle i_1,\lambda\rangle}$ generates a priority $\pi(P) = \{i_1, i_2, i_3\}$, where $i_2 = \lambda(\mathbb{1}_{A\setminus\{\max\{A,P_{i_1}\}\}}(P))\}$ and $i_3 = N \setminus \{i_1, i_2\}$.

For every profile $P \in \mathbb{P}^3$, $F^{\langle i_1,\lambda\rangle}$ allocates the agent i_1 her peak object, say a_1 without loss of generality. For the pair of remaining objects (a_2, a_3), the rule decides the second and the third agent in the priority $\pi(P)$ at P, on the basis of whether a_2 and a_3 are ranked similarly in P_i for all $i \in N$ or not. For example, consider P^1 and P^2 are shown in Table 3.1. The allocation rule $F_1^{\langle 1,\lambda\rangle}$ with $\lambda(0) = 2$ and $\lambda(1) = 3$, allocates objects as encircled in Table 3.1.

A priority rule is a special case of a modified priority rule where $\lambda(0) = \lambda(1)$.

Proposition 3.1. *Every efficient, neutral and strategy-proof allocation rule is a modified priority rule.*

Proof. We begin the proof by showing that a modified priority rule is efficient, neutral

P_1^1	P_2^1	P_3^1	P_1^2	P_2^2	P_3^2
a_1	a_2	a_1	a_1	a_2	a_2
a_2	a_3	a_2	a_3	a_1	a_3
a_3	a_1	a_3	a_2	a_3	a_1

Table 3.1: Profiles P^1 and P^2

and strategy-proof.

Let $F^{\langle i_1,\lambda\rangle}$ be a modified priority rule. Since the rule allocates at P according to priority $\{i_1, i_2, i_3\}$, where $i_2 = \lambda(\mathbb{1}_{A\setminus\{\max\{A,P_{i_1}\}\}}(P))\}$ and $i_3 = N \setminus \{i_1, i_2\}$, $F^{\langle i_1,\lambda\rangle}(P)$ is Pareto efficient for all $P \in \mathbb{P}^3$. This implies that $F^{\langle i_1,\lambda\rangle}$ is efficient.

Let $\sigma : A \to A$ be an arbitrary permutation over the set of objects. For any two distinct objects $a_j, a_k \in A$, we have $a_j P_i a_k$ if and only if $\sigma(a_j)(\sigma \circ P_i)\sigma(a_k)$ for all $i \in N$ and $P_i \in \mathbb{P}$. An arbitrary profile P is therefore a 0-profile or a 1-profile for objects in the set $A \setminus r_1(P_{i_1})$ if and only if $\sigma \circ P$ is a 0-profile or a 1-profile respectively for objects in the set $A \setminus r_1(\sigma \circ P_{i_1})$. It follows from definition of $F^{\langle i_1,\lambda\rangle}$ that the priority according to $F^{\langle i_1,\lambda\rangle}$ at P and $\sigma \circ P$ must be identical. Hence $F^{\langle i_1,\lambda\rangle}(\sigma \circ P) = \sigma \circ F^{\langle i_1,\lambda\rangle}(P)$ implying that $F^{\langle i_1,\lambda\rangle}$ is neutral.

The first agent in the priority at any profile according to $F^{\langle i_1,\lambda\rangle}$ will always be i_1. It is obvious that only the agent last in the priority at a profile will intend to manipulate. We already know that the priority rules are strategy-proof. It is interesting therefore to consider the case in which the range of λ is not a singleton set. Let $i_1 = 1$, $\lambda(0) = 2$ and $\lambda(1) = 3$ without loss of generality. Suppose P is a 0-profile for the objects in the set $A \setminus r_1(P_1)$. This implies that $r_2(P_1)P_3r_3(P_1)$ and therefore $F_3^{\langle 1,\lambda\rangle}(P) = r_3(P_1)$. Suppose the agent 3 reports P_3' as her preference ordering for which $r_3(P_1)P_3'r_2(P_1)$. The profile (P_{-3}, P_3') is clearly a 1-profile[3]. It follows from our assumption on λ that $F_3^{\langle 1,\lambda\rangle}(P_{-3}, P_3') = r_3(P_1)$ which implies that agent 3 does not gain from misreporting. Suppose P is a 1-profile for the objects in the set $A\setminus r_1(P_1)$. In case either $r_3(P_1)P_2r_2(P_1)$ and $r_2(P_1)P_3r_3(P_1)$ or $r_2(P_1)P_2r_3(P_1)$ and $r_3(P_1)P_3r_2(P_1)$, agent $i \in \{2, 3\}$ is allocated $\max(P_i, A \setminus \{r_1(P_1)\})$. Hence, no agent can gain by misreporting. In case $r_3(P_1)P_2r_2(P_1)$ and $r_3(P_1)P_3r_2(P_1)$, it follows that $F_2^{\langle 1,\lambda\rangle}(P) = r_2(P_1) = \min(P_2, A \setminus \{r_1(P_1)\})$. Suppose agent 2 misreports her preference ordering to P_2' for which
$r_2(P_1)P_2'r_3(P_1)$. Since (P_2', P_{-2}) is still a 1-profile, we have $F_2^{\langle 1,\lambda\rangle}(P_2', P_{-2}) = r_2(P_1)$

[3]From now on, we may call a profile 0-profile or 1-profile without specifying the pair of objects in the set $A \setminus r_1(P_{i_1})$

implying that agent 2 does not gain by misreporting. Hence, $F^{\langle i_1, \lambda \rangle}$ is strategy-proof.

The next step is to show that if F is efficient, neutral and strategy-proof allocation rule then there must exist an agent $i_1 \in N$ such that $F_{i_1}(P) = \max(P_{i_1}, A)$ for all $P \in \mathbb{P}^3$. We introduce some terms for future reference in the proof. A profile is *unanimous* if the preference orderings of all the agents are identical at that profile. A profile is *top-unanimous* if the peaks of preference orderings of all the agents are identical at that profile. A profile is a *zero-conflict* profile if every agent's preference ordering has a distinct peak. Since F is neutral, the priority at every unanimous profile according to F will be identical by definition. Suppose the priority according to F at unanimous profiles is $\{1, 2, 3\}$ without loss of generality. Let P be an arbitrary profile such that $r_1(P_i) = a_1$ for all $i \in N$. Suppose $F_1(P) \neq a_1$. We assume that $F_2(P) = a_1$ without loss of generality. Let P' and P'' be such that $P'_i = P_i$ for $i \in \{1, 3\}$, $P''_i = P'_i$ for $i \in \{2, 3\}$ and $P''_1 = P'_2 = P_3$. Since F is strategy-proof, we have $F_2(P') = a_1$ and $F_1(P'') \neq a_1$. But this is in contradiction to our assumption since P'' is a unanimous profile. This implies that $F_1(P) = a_1$. Since F is neutral, agent 1 gets her peak for every top-unanimous profile. Agent 1 must also get her peak at profiles with distinct peaks for every agent, otherwise the allocation will not be Pareto inefficient implying that F is not efficient.

We need to show that agent 1 gets her peak a_1 if peak of exactly one agent is different from that of hers. Let that agent be 3 without loss of generality. Let P be an arbitrary profile such that $r_1(P_1) = r_1(P_2) = a_1$ and $r_1(P_3) \neq a_1$. Since F is efficient, $F_3(P) \neq a_1$. Suppose that $F_2(P) = a_1$. Strategy-proofness of F implies that if $F_2(P) = a_1$ then $F_2(P') = a_1$ for any profile P' such that $r_1(P'_1) = r_1(P'_2) = a_1$ and $P'_3 = P_3$. We introduce a few notations for convenience in order to prove that agent 1 always gets her peak. For all $i \in N$ and $j \in \{1, 2, 3\}$, let $\bar{P}^j_i$ and $\hat{P}^j_i$ denote preference orderings such that $r_1(\bar{P}^j_i) = r_1(\hat{P}^j_i) = a_j$, $a_{j+1}\bar{P}^j_i a_{j+2}$ and $a_{j+2}\hat{P}^j_i a_{j+1}$ where we define $a_4 = a_1$ and $a_5 = a_2$.

Consider the top-unanimous profile $(\hat{P}^1_1, \bar{P}^1_2, \bar{P}^1_3)$. We have shown that $F_1(\hat{P}^1_1, \bar{P}^1_2, \bar{P}^1_3) = a_1$. Suppose it is the case that $F_2(\hat{P}^1_1, \bar{P}^1_2, \bar{P}^1_3) = a_2$. Since F is neutral, it follows that $F_3(\bar{P}^1_1, \hat{P}^1_2, \hat{P}^1_3) = a_2$. Suppose $r_1(P_3) = a_2$. Since F is strategy-proof, $F_3(\hat{P}^1_1, \bar{P}^1_2, P_3) = a_3$. By assumption, $F_2(\hat{P}^1_1, \bar{P}^1_2, P_3) = a_1$. Hence $F_1(\hat{P}^1_1, \bar{P}^1_2, P_3) = a_2$. Since $r_3(\hat{P}^1_i) = a_2$, strategy-proofness of F implies that $F_1(\bar{P}^3_1, \bar{P}^1_2, P_3) = a_2$. Since $(\bar{P}^3_1, \bar{P}^1_2, P_3)$ is zero-conflict profile, $F_1(\hat{P}^3_1, \bar{P}^1_2, P_3) = a_3$ which is a contradiction. Therefore, if $r_1(P_3) = a_2$ then $F_2(\hat{P}^1_1, \bar{P}^1_2, \bar{P}^1_3) \neq a_2$. Suppose $r_1(P_3) = a_3$. Since F is strategy-proof, $F_3(\bar{P}^1_1, \hat{P}^1_2, P_3) = a_2$. Since $F_2(\bar{P}^1_1, \hat{P}^1_2, P_3) = a_1$, we have $F_1(\bar{P}^1_1, \hat{P}^1_2, P_3) = a_3$. Since $r_3(\bar{P}^1_i) = a_3$, strategy-proofness implies that $F_1(\hat{P}_1^2, \hat{P}_2^1, P_3) = a_3$. But this is in contradiction to the fact that $(\hat{P}_1^2, \hat{P}_2^1, P_3)$ is a zero-conflict profile. Hence, if $r_1(P_3) = a_3$ then we have $F_2(\hat{P}_1^1, \bar{P}_2^1, \bar{P}_3^1) \neq a_2$.

We have established thus far that if $F_2(P) = a_1$ then $F_2(\hat{P}_1^1, \bar{P}_2^1, \bar{P}_3^1) = a_3$ and $F_2(\bar{P}_1^1, \hat{P}_2^1, \hat{P}_3^1) = a_2$. Since F is efficient and since the allocation to agent 1 is her peak at every top-unanimous profile, we have $F_3(\bar{P}_1^1, \hat{P}_2^1, \bar{P}_3^1) = a_2$ and $F_3(\hat{P}_1^1, \bar{P}_2^1, \hat{P}_3^1) = a_3$. It is the case that either $r_2(P_3) = a_1$ or not. Suppose $r_2(P_3) = a_1$. Again, there are two sub-cases to consider. Suppose it is the case that $r_1(P_3) = a_2$. Since $F_3(\bar{P}_1^1, \hat{P}_2^1, \bar{P}_3^1) = a_2$, we have $F_3(\bar{P}_1^1, \hat{P}_2^1, P_3) = a_2$. This implies that $F_1(\bar{P}_1^1, \hat{P}_2^1, P_3) = a_3$. Since $r_3(\bar{P}_i^1) = a_3$, it follows from strategy-proofness that $F_1(\bar{P}_1^2, \hat{P}_2^1, P_3) = a_3$. Since F is neutral and $F_2(\hat{P}_1^1, \bar{P}_2^1, \bar{P}_3^1) = a_3$, we have $F_2(\bar{P}_1^2, \hat{P}_2^2, \hat{P}_3^2) = a_3$. The strategy-proofness of F implies that $F_2(\bar{P}_1^2, \hat{P}_2^1, P_3) = a_3$ which is a contradiction. Suppose it is the case that $r_1(P_3) = a_3$. Since $F_3(\hat{P}_1^1, \bar{P}_2^1, \bar{P}_3^1) = a_3$, we have $F_3(\hat{P}_1^1, \bar{P}_2^1, P_3) = a_3$. This implies that $F_1(\hat{P}_1^1, \bar{P}_2^1, P_3) = a_2$. But since $r_3(\hat{P}_i^1) = a_2$ and since F is strategy-proof, it follows that $F_1(\hat{P}_1^3, \bar{P}_2^1, P_3) = a_2$. The neutrality of F and $F_2(\bar{P}_1^1, \hat{P}_2^1, \hat{P}_3^1) = a_2$ imply that $F_2(\hat{P}_1^1, \bar{P}_2^1, \bar{P}_3^1) = a_2$ which is a contradiction. Hence we have shown that $r_2(P_3) = a_1$ and $F_2(P) = a_1$ cannot both be true. Therefore, if $r_2(P_3) = a_1$ then we have $F_1(P) = a_1$. Since $(\hat{P}_1^2, \hat{P}_2^2, \hat{P}_3^2)$ is a unanimous profile, we have $F_2(\hat{P}_1^2, \hat{P}_2^2, \hat{P}_3^2) = a_1$. The strategy-proofness of F implies that $F_2(\hat{P}_1^2, \hat{P}_2^1, \hat{P}_3^2) = a_1$. If $F_1(\hat{P}_1^2, \hat{P}_2^1, \hat{P}_3^2)$
$\neq a_2$ then $F_1(\hat{P}_1^2, \hat{P}_2^1, \hat{P}_3^2) = a_3$. This implies that $F_1(\bar{P}_1^2, \hat{P}_2^1, \hat{P}_3^2) = a_3$ which contradicts the result that $r_2(P_3) = a_1$ implies $F_1(P) = a_1$. Hence we have $F_1(\hat{P}_1^2, \hat{P}_2^1, \hat{P}_3^2) = a_2$. Similarly, we have $F_2(\bar{P}_1^3, \bar{P}_2^3, \bar{P}_3^3) = a_1$. Since F is strategy-proof, it follows that $F_2(\bar{P}_1^3, \bar{P}_2^1, \bar{P}_3^3) = a_1$. It cannot be that $F_1(\bar{P}_1^3, \bar{P}_2^1, \bar{P}_3^3) = a_2$ because otherwise $F_1(\bar{P}_1^1, \bar{P}_2^1, \bar{P}_3^3) = a_2$ which again contradicts the result that $r_2(P_3) = a_1$ implies $F_1(P) = a_1$. Hence we have $F_1(\bar{P}_1^3, \bar{P}_2^1, \bar{P}_3^3) = a_3$.

Suppose $r_2(P_3) \neq a_1$. Again we have two sub-cases to consider. Suppose it is the case that $r_1(P_3) = a_2$. Since $F_3(\bar{P}_1^1, \hat{P}_2^1, \bar{P}_3^1) = a_2$, it follows from strategy-proofness of F that $F_3(\bar{P}_1^1, \hat{P}_2^1, \bar{P}_3^2) = a_2$. Since $F_2(P) = a_1$, we have $F_1(\bar{P}_1^1, \hat{P}_2^1, \bar{P}_3^2) = a_3$. The strategy-proofness of F implies that $F_1(\hat{P}_1^2, \hat{P}_2^1, \bar{P}_3^2) = a_3$. Since F is neutral, we have $F_2(\hat{P}_1^2, \bar{P}_2^2, \bar{P}_3^2) = a_1$. This implies that $F_2(\hat{P}_1^2, \hat{P}_2^1, \bar{P}_3^2) = a_1$ and $F_3(\hat{P}_1^2, \hat{P}_2^1, \bar{P}_3^2) = a_2$. Since F is strategy-proof, the latter is in contradiction to the fact $F_3(\hat{P}_1^2, \hat{P}_2^1, \hat{P}_3^2) \neq a_2$. Suppose it is the case that $r_1(P_3) = a_3$. Since $F_3(\hat{P}_1^1, \bar{P}_2^1, \hat{P}_3^1) = a_3$, it follows from strategy-proofness of F that $F_3(\hat{P}_1^1, \bar{P}_2^1, \hat{P}_3^3) = a_3$. Therefore, we have $F_1(\hat{P}_1^1, \bar{P}_2^1, \hat{P}_3^3) = a_2$. Since F is strategy-proof, we get $F_1(\bar{P}_1^3, \bar{P}_2^1, \hat{P}_3^3) = a_2$. It follows from neutrality that

$F_2(\bar{P}_1^3, \hat{P}_2^3, \hat{P}_3^3) = a_1$. This implies that $F_2(\bar{P}_1^3, \bar{P}_2^1, \hat{P}_3^3) = a_1$ and $F_3(\bar{P}_1^3, \bar{P}_2^1, \hat{P}_3^3) = a_3$. This contradicts the fact that F is strategy-proof since $F_3(\bar{P}_1^3, \bar{P}_2^1, \bar{P}_3^3) \neq a_3$. Therefore, $r_2(P_3) \neq a_1$ also implies that $F_1(P) = a_1$.

We have shown thus far that there exists an agent $i_1 \in N$ who gets his peak at every profile according to F. Let $i_1 = 1$, without loss of generality. Next we need to show that there cannot exist a pair of 0-profiles with different priorities according to F. We will prove this by contradiction. Suppose there exist two 0-profiles P^1 and P^2 such that the priorities according to F are $\pi^1 = \{1, 2, 3\}$ and $\pi^2 = \{1, 3, 2\}$ respectively. Let $P^3 = \sigma \circ P^2$ such that $\sigma(P_3^2) = P_3^1$. It is obvious that $P_3^1 = P_3^3$. Since F is neutral, P^3 is a 0-profile for the objects in the set $A \setminus \{r_1(P_1^3)\}$ and the priority at P^3 is π^2. We assume without loss of generality that $P_3^1 = \bar{P}_3^1$ where $\bar{P}_i^1$ is as defined above. We assume that $r_1(P_1^1) = a_1$ without loss of generality. Since $P_3^1 = P_3^3$ and since P^1 and P^3 are 0-profiles, it must be that $P_1^1 = P_1^3$. This implies that P^1 and P^3 can possibly differ only in the preference ordering of agent 2. If $P_2^1 = P_2^3$ then $P_1 = P_3$ with different priorities on the same profile. Therefore, we must have $P_2^1 \neq P_2^3$. We will consider two sub-cases depending upon the identity of $r_1(P_1^3)$. Suppose that $r_1(P_1^3) = a_1$. Since P^1 and P^3 are both 0-profiles and since $P_3^1 = P_3^3$, it must be that $P_1^1 = P_1^3$. Since priorities are different at P^1 and P^3, we get $F_1(P^1) = F_1(P^3) = a_1$, $F_2(P^1) = a_2$ and $F_2(P^3) = a_3$. Agent 2 will then manipulate at P^3 by misreporting P_2^1.

Suppose that $r_1(P_1^3) = a_j$ where $j \in \{2, 3\}$. We define $k \in \{2, 3\} \setminus \{j\}$. We have assumed that $a_1 P_3^3 a_k$. Since agent 1 agrees with 2 and 3 at P^3, we have $a_j P_1^3 a_1 P_1^3 a_k$ and $a_1 P_2^3 a_k$. Let P^4 be a neutral transformation of P^3 such that $P_1^4 = P_1^1$. Since P^4 is a neutral transformation of P^3, it is a 0-profile in which $a_2 P_i^4 a_3$ for $i \in \{2, 3\}$ and the allocation at P^4 is according to π^2. We construct a profile $P^5 = (P_1^1, P_2^1, P_3^4)$ such that P^5 differs from both P^1 and P^4 in the preference ordering of exactly one agent. Since the allocation at P^1 is according to π^1, we have $F_3(P^1) = a_3$ where $a_3 = r_3(P_3^1)$. Since the allocation at P^4 is according to π^2, we have $F_2(P^4) = a_3$. Since P_4 and P_5 differ only in the preference ordering of agent 2 and since $F_1(P^5) = a_1$, it follows from strategy-proofness that $F_2(P^5) = a_3$, otherwise agent 2 will manipulate at P^4 by reporting P_2^1. Similarly, since P_1 and P_5 differ only in the preference ordering of agent 3, strategy-proofness of F implies that $F_3(P^5) = a_3$, which is a contradiction. Using analogous arguments for the other cases on $r_1(P_1^1)$, we can show that no two distinct 0-profiles can have different priorities according to F. The next step is to show that there cannot exist a pair of 1-profiles with different priorities according to F such that the outcomes on both profiles change if they exchange priorities. The proof is very similar to the proof for the case of pair of 0-profiles but we still describe it in detail for the sake of completeness.

Suppose there exist two 1-profiles P^1 and P^2 such that the priorities according to F are $\pi^1 = \{1,2,3\}$ and $\pi^2 = \{1,3,2\}$ respectively. Moreover, we need to have $F^{\pi^1}(P^1) \neq F^{\pi^2}(P^1)$ and $F^{\pi^1}(P^2) \neq F^{\pi^2}(P^2)$. Since the priorities $\{1,2,3\}$ and $\{1,3,2\}$ lead to same allocations for 1-profiles where agents 2 and 3 disagree over the set of objects in $A \setminus \{r_1(P_1)\}$, it must be that agents 2 and 3 agree with each other but disagree with 1, at P^1 as well as P^2. Let P^3 be a neutral transformation of P^2 i.e., $P^3 = \sigma \circ P^2$ such that $\sigma(P^2_3) = P^1_3$. This implies that $P^3_3 = P^1_3$. Since F is neutral, P^3 is a 1-profile as agents 2 and 3 will agree but disagree with 1 at P^3. The case that $P^3 = P^1$ is trivial. We assume without loss of generality that $P^1_3 = \bar{P}^1_3$ where $\bar{P}^1_i$ is as defined above. We further assume that $r_1(P^1_1) = a_1$ without loss of generality. Since agent 1 has to disagree with 3 over the pair (a_2, a_3) at P^1, we have $P^1_1 = \hat{P}^1_1$. Since 2 has to agree with 3 over (a_2, a_3), we will have $a_2 P^1_2 a_3$. We will consider two sub-cases depending upon the identity of $r_1(P^3_1)$. Suppose that $r_1(P^3_1) = a_1$. Since 1 has to disagree with 3 over (a_2, a_3) at P^3, we have $P^3_1 = \hat{P}^1_1$. Moreover, it must be that $a_2 P^3_2 a_3$. Since the priority at P^1 and P^3 are π^1 and π^2 respectively, we have $F_2(P^1) = a_2$ and $F_2(P^3) = a_3$. Agent 2 will manipulate at P^3 by reporting P^1_2. Suppose that $r_1(P^3_1) = a_j$ where $j \in \{2,3\}$. We define $k \in \{2,3\} \setminus \{j\}$. We have assumed that $a_1 P^3_3 a_k$. Since agent 1 does not agree with 2 and 3 at P^3, we have $a_j P^3_1 a_k P^3_1 a_1$ and $a_1 P^3_2 a_k$. Let P^4 be a neutral transformation of P^3 such that $P^4_1 = P^1_1$. Since P^4 is a neutral transformation of P^3, it is a 1-profile in which $a_2 P^4_i a_3$ for $i \in \{2,3\}$ and the allocation at P^4 is according to π^2. We construct a profile $P^5 = (P^1_1, P^1_2, P^4_3)$ such that P^5 differs from both P^1 and P^4 in the preference ordering of exactly one agent. Since the allocation at P^1 is according to π^1, we have $F_3(P^1) = a_3$ where $a_3 = r_3(P^1_3)$. Since the allocation at P^4 is according to π^2, we have $F_2(P^4) = a_3$. Since P_4 and P_5 differ only in the preference ordering of agent 2 and since $F_1(P^5) = a_1$, it follows from strategy-proofness that $F_2(P^5) = a_3$ otherwise agent 2 will manipulate at P^4 by reporting P^1_2. Similarly, since P_1 and P_5 differ only in the preference ordering of agent 3, strategy-proofness of F implies that $F_3(P^5) = a_3$ which is a contradiction. Using analogous arguments for the other cases on $r_1(P^1_1)$, we can show that no two distinct 1-profiles can have different priorities according to F.

We have proved that in the case of 3 agents and 3 objects, every neutral, efficient and strategy-proof allocation rule F allocates at every 0-profile according to some priority π^1 beginning with i_1 and allocates at every 1-profile according to some (not necessarily different) priority π^2 beginning with i_1. We assume without loss of generality that $\pi^1 = \{i_1, i_2, i_3\}$ and $\pi^1 = \{i_1, i_2', i_3'\}$ where $i_2 \neq i_3 \in N \setminus \{i_1\}$ and $i_2' \neq i_3' \in N \setminus \{i_1\}$. Suppose $F^{\langle i_1, \lambda \rangle}$ be a modified allocation rule such that $\lambda(0) = i_2$ and $\lambda(1) = i_2'$. By definition, $F^{\langle i_1, \lambda \rangle}(P)$ is according to π^1 if P is a 0-profile and according to π^2 if P is a 1-profile. Therefore, $F(P) = F^{\langle i_1, \lambda \rangle}(P)$ for all $P \in \mathbb{P}^3$. This completes the proof of the Proposition. □

Proposition 3.1 will now be used to prove the main result of this section.

Theorem 3.1. *For every efficient, neutral and strategy-proof allocation rule, there exists an EUE priority rule.*

Proof. We know from Proposition 3.1 that F is a modified priority rule for some $i_1 \in N$ and a mapping $\lambda : \{0, 1\} \to N \setminus \{i_1\}$. If the range of λ is a singleton, then F is a priority rule and there is nothing to prove. Assume therefore that $\lambda(0) \neq \lambda(1)$. Assume further without loss of generality that $i_1 = 1$ and the priority at 0-profiles according to F is $\pi = \{1, 2, 3\}$. We say that agent 2 and 3 are remaining agents.

We claim that $W^F(P) - W^{F^\pi}(P) = W^{F^\pi}(P') - W^F(P')$ for all P and P' such that $P_1' = P_1$, $P_2' = P_3$ and $P_3' = P_2$.

For a 0-profile P, $W^F(P) - W^{F^\pi}(P) = 0$. Suppose P be a 1-profile such that one of the three cases holds. The remaining agents disagree over objects in $A \setminus \{r_1(P_1)\}$ or the preference orderings of remaining agents are identical or the first ranked object in preference ordering of one of the remaining agent is ranked last in the preference ordering of the other. Then $W^F(P) - W^{F^\pi}(P) = 0$ as well. $W^F(P) - W^{F^\pi}(P) \neq 0$ only for a 1-profile such that $r_1(P_1)$ is ranked second only by agent 3. Note that the priority at P must be $\{1, 3, 2\}$. Suppose $W^F(P) - W^{F^\pi}(P) > 0$ for some P. Since $r_1(P_1)$ is ranked second by agent 3, $F_3(P) = r_1(P_3)$ while $F_2(P)$ is ranked just below $F_3(P)$ by agent 2. If the priority at P were π then agent's 3 allocation would change from his best to worst object while agent 2 will just move one rank up. Let P' be such that the preference orderings of the remaining agents are interchanged. Since agent 2 now has $r_1(P_1)$ as her second ranked object, he gets his worst object while agent 3 gets her best object in $A \setminus \{r_1(P_1)\}$ which is just above $F_2(P')$. If the priority at P' was π then agent 2 will move from his worst to best while 3 will move down by just one rank. Therefore, it must be that $W^F(P) - W^{F^\pi}(P) = W^{F^\pi}(P') - W^F(P')$. This completes the proof. □

According to Theorem 3.1 there is no advantage in selecting a bossy rule for an expected utility maximising planner who is constrained to select an allocation rule from the class of neutral, efficient and strategy-proof rules. An open question is whether expected utility equivalence holds for an arbitrary n where n is the number of agents and objects.

3.4 Lower Bossy Rules

In this section, we discuss a special class of bossy allocation rules which we call lower bossy[4]. We will show these rules to be EUE to a priority rule for any n.

Definition 3.2. F is *lower bossy* if there exist $i_1 \in N$ and $\sigma^j : (P_{i_1}, \ldots, P_{i_{j-1}}) \to N \setminus \{i_1, \ldots, i_{j-1}\}$ for all $j \in \{2, \ldots, n\}$ and $P \in \mathbb{P}^n$ such that $F_{i_j}(P) = \max(A \setminus \{F_{i_1}(P), \ldots, F_{i_{j-1}}(P)\}, P_{i_j})$ for all $j \in \{1, \ldots, n\}$ and $P \in \mathbb{P}^n$. Here $i_j = \sigma^j(P_{i_1}, \ldots, P_{i_{j-1}})$ for all $j \in \{2, \ldots, n\}$

As an example of a lower bossy rule, consider $N = \{1, 2, 3\}$, $A = \{a, b, c\}$, $i_1 = 1$ and for all P, $\sigma^2(P)$ is defined as follows:

$$\sigma^2(P) = \begin{cases} 2 & \text{if } aP_1b \\ 3 & \text{otherwise} \end{cases}$$

For the unanimous profiles P and P' such that for all i, aP_ibP_ic and $bP'_iaP'_ic$, the priorities are $\{1, 2, 3\}$ and $\{1, 3, 2\}$ respectively. The lower-bossy rule in this example is clearly not neutral.

Abusing the notation slightly, for all $j \in \{2, \ldots, (n-1)\}$, we write i_j as a function of P. A lower bossy allocation rule F is efficient by definition since F allocates at every P according to a priority $(i_1, i_2(P), \ldots, i_n(P))$. Since agent i_j for $j \in \{2, \ldots, (n-1)\}$ cannot change the preference orderings of agents $(i_1, i_2, \ldots, i_{j-1})$ at any profile, her position in the priority as well as the set of objects available to her remains unchanged. This implies that a lower bossy rule is strategy-proof. We call the rule lower bossy because given a profile, any agent $i_j \in N$ for $j \in \{2, \ldots, (n-1)\}$ can influence the allocation of only the agents down in the queue at that profile.

[4]For another class of bossy rules, see Raghavan (2020)

The following theorem establishes an equivalence between a lower bossy rule and a priority rule.

Theorem 3.2. *A lower bossy allocation rule is EUE to an arbitrary priority rule F^π.*

Proof. For this proof we will use induction on the position j of the bossy agent in the priority at a profile. The proof involves decreasing the degree of bossiness of the allocation rule iteratively.

Let F be a lower bossy allocation rule. Consider an arbitrary profile $\bar{P}$. There exist $i_1(\bar{P}), \ldots, i_n(\bar{P})$ by definition[5]. Since F is lower bossy, $i_j(\bar{P})$ continues to be at j^{th} position in the priority at profiles P such that $P_{i_t} = \bar{P}_{i_t}$ for all $t \in \{1, \ldots, j-1\}$ and $j \in \{2, \ldots, n\}$. Before proceeding, we add a notation for convenience. Let $i_j^F(P)$ denote the agent at j^{th} position in the queue at P according to F.

We decrease the degree of bossiness of the allocation rule F by defining F^j for all $j \in \{2, \ldots, n-1\}$ to be such that for all $k \in N$ and $\bar{P}, P \in \mathbb{P}^n$ such that $P_{i_s(\bar{P})} = \bar{P}_{i_s(\bar{P})}$ for all $s \in \{1, \ldots, n-j-1\}$, we have $i_k^{F^j}(P) = i_k^F(\bar{P})$. The positions of $i_{n-j}^F(\bar{P}), \ldots, i_n^F(\bar{P})$ according to F^j at at such profiles, remains identical to their respective positions at $\bar{P}$ according to F. We claim that F^2 is EUE to a priority rule.

Claim 3.1. $E(W^{F^2}) = E(W^F)$

Proof. By definition, F^2 does not influence the position of the first $(n-2)$ agents in the priority at any profile according to F. Hence $u_{i_j^F(P)}(F(P)) = u_{i_j^F(P)}(F^2(P))$ for all $P \in \mathbb{P}^n$ and $j \in \{1, \ldots, (n-2)\}$ implying that

$$\sum_{P \in \mathbb{P}^n} \sum_{k=i_1^F(P)}^{i_{n-2}^F(P)} u_k(F(P))) = \sum_{P \in \mathbb{P}^n} \sum_{k=i_1^F(P)}^{i_{n-2}^F(P)} u_k(F^2(P)) \tag{3.1}$$

Fix a $\bar{P}$. Suppose for some $\hat{P}$ such that $\hat{P}_{N \setminus \{i_{n-2}^F(\bar{P}),\, i_{n-1}^F(\bar{P}),\, i_n^F(\bar{P})\}} = \bar{P}_{N \setminus \{i_{n-2}^F(\bar{P}),\, i_{n-1}^F(\bar{P}),\, i_n^F(\bar{P})\}}$ and $\hat{P}_{i_{n-2}^F(\bar{P})} \neq \bar{P}_{i_{n-2}^F(\bar{P})}$, we have

$$u_{i_{n-1}^F(\bar{P})}(F(\hat{P})) + u_{i_n^F(\bar{P})}(F(\hat{P})) > (<) u_{i_{n-1}^F(\bar{P})}(F^2(\hat{P})) + u_{i_n^F(\bar{P})}(F^2(\hat{P})) \tag{3.2}$$

[5]Suppose there is a set of agents who can mutually exchange positions at $\bar{P}$ without a change in the allocation to any of them. We can break the ties and deduce their order uniquely at $\bar{P}$ by making their preference orderings identical.

For P' such that $P'_{N\setminus\{i^F_{n-1}(\bar{P}),\, i^F_n(\bar{P})\}} = \hat{P}_{N\setminus\{i^F_{n-1}(\bar{P}),\, i^F_n(\bar{P})\}}$, $P'_{i^F_{n-1}(\bar{P})} = \hat{P}_{i^F_n(\bar{P})}$ and $P'_{i^F_n(\bar{P})} = \hat{P}_{i^F_{n-1}(\bar{P})}$, the following relations must hold:

$$u_{i^F_{n-1}(\bar{P})}(F(P')) + u_{i^F_n(\bar{P})}(F(P')) < (>)\ u_{i^F_{n-1}(\bar{P})}(F^2(P')) + u_{i^F_n(\bar{P})}(F^2(P')) \tag{3.3}$$

$$\sum_{P\in\{\hat{P},P'\}} \sum_{k\in\{i^F_{n-1}(\bar{P}),\, i^F_n(\bar{P})\}} u_k(F(P)) = \sum_{P\in\{\hat{P},P'\}} \sum_{k\in\{i^F_{n-1}(\bar{P}),\, i^F_n(\bar{P})\}} u_k(F^2(P)) \tag{3.4}$$

Together 1, 2, 3 and 4 imply that $W^{F^2}(\bar{P}) = W^F(\bar{P})$, $W^{F^2}(\hat{P}) > (<) W^F(\hat{P})$, $W^{F^2}(P') < (>) W^F(P')$ and $\sum\limits_{P\in\{\bar{P},\hat{P},P'\}} W^{F^2}(P) = \sum\limits_{P\in\{\bar{P},\hat{P},P'\}} W^F(P)$. This implies that $E(W^{F^2}) = E(W^F)$. □

We have shown that if we remove the influence of $i^F_{n-2}(\bar{P})$ on the agents down in the order by assuming a fixed priority over them at profiles such that for all $i \in \{i^F_1(\bar{P}), \ldots, i^F_{n-3}(\bar{P})\}$, the preference orderings of the agent i is unchanged then the expected welfare remains unchanged for the new allocation rule. The choice of the fixed priority over the agents $\{i^F_{n-1}(\bar{P}), i^F_n(\bar{P})\}$ does not matter due to the symmetric argument we made in the proof of Claim 3.1.

Induction step: We assume that F^t and F are EUE for some $t \in \{2, \ldots, n-2\}$. We need to prove that F^{t+1} and F are EUE.

Using arguments similar to the ones made in the proof for Claim 3.1, we can infer that

$$\sum_{P\in\mathbb{P}^n} \sum_{k=i_1(P)}^{i_{n-t-1}(P)} u_k(F^t(P))) = \sum_{P\in\mathbb{P}^n} \sum_{k=i_1(P)}^{i_{n-t-1}(P)} u_k(F^{t+1}(P)) \tag{3.5}$$

Consider an arbitrary $\bar{P}$. Suppose there exists a $\hat{P}$ such that $\hat{P}_{i^F_s(\bar{P})} = \bar{P}_{i^F_s(\bar{P})}$ for all $s \in \{1, \ldots, n-t-2\}$, $\hat{P}_{i^F_{n-t-1}(\bar{P})} \neq \bar{P}_{i^F_{n-t-1}(\bar{P})}$ and the following inequality holds:

$$\sum_{k=i^F_{n-t}(\bar{P})}^{i^F_n(\bar{P})} u_k\left(F^t\left(\hat{P}\right)\right) > (<) \sum_{k=i^F_{n-t}(\bar{P})}^{i^F_n(\bar{P})} u_k\left(F^{t+1}\left(\hat{P}\right)\right) \tag{3.6}$$

Let $\theta : \{(n-t), \ldots, n\} \to \{(n-t), \ldots, n\}$ be a permutation such that for all $j \in \{(n-t), \ldots, n\}$ and for all we have $i^{F^t}_j(\hat{P}) = i^{F^{t+1}}_{\theta(j)}(\bar{P})$. It must be the case that the following relations hold for P' such that $P'_{i^F_s(\bar{P})} = \hat{P}_{i^F_s(\bar{P})}$ for all $s \in \{1, \ldots, n-t-1\}$, $P'_{i^F_{\theta(s)}(\bar{P})} = \hat{P}_{i^F_s(\bar{P})}$ for all $s \in \{n-t, \ldots, n\}$:

$$\sum_{k=i_{n-t}^F\left(\bar{P}\right)}^{i_n^F\left(\bar{P}\right)} u_k\left(F^t\left(P'\right)\right) < (>) \sum_{k=i_{n-t}^F\left(\bar{P}\right)}^{i_n^F\left(\bar{P}\right)} u_k\left(F^t\left(P'\right)\right) \tag{3.7}$$

$$\sum_{P\in\left\{P',\hat{P}\right\}} \sum_{k=i_{n-t}^F\left(\bar{P}\right)}^{i_n^F\left(\bar{P}\right)} u_k\left(F^t\left(\hat{P}\right)\right) = \sum_{P\in\left\{P',\hat{P}\right\}} \sum_{k=i_{n-t}^F\left(\bar{P}\right)}^{i_n^F\left(\bar{P}\right)} u_k\left(F^{t+1}\left(\hat{P}\right)\right) \tag{3.8}$$

Together 5, 6, 7 and 8 imply that $W^{F^{t+1}}(\bar{P}) = W^{F^t}(\bar{P})$, $\quad W^{F^{t+1}}(\hat{P}) > (<) W^{F^t}(\hat{P})$, $W^{F^{t+1}}(P') < (>) W^{F^t}(P')$ and $\sum_{P\in\left\{\bar{P},\hat{P},P'\right\}} W^{F^{t+1}}(P) = \sum_{P\in\left\{\bar{P},\hat{P},P'\right\}} W^{F^t}(P)$. This implies that $E(W^{F^{t+1}}) = E(W^{F^t}) = E(W^F)$. This completes the proof. $\square$

3.5 Conclusion

In this chapter, we have attempted to provide a justification for the axiom of non-bossiness. In particular we have shown that it can be imposed without loss of generality by an expected utility maximising planner in certain symmetric settings.

Bibliography

Afacan, M. (2012): "On The "Group Non-bossiness" Property," *Economics Bulletin*, 32, 1571–1575.

Arrow, K. J. (1951): *Social Choice and Individual Values*, Wiley, New York.

Aswal, N., S. Chatterji, and A. Sen (2003): "Dictatorial domains," *Economic Theory*, 22, 45–62.

Bade, S. (2019): "Matching with single-peaked preferences," *Journal of Economic Theory*, 180, 81–99.

——— (2020): "Random serial dictatorship: the one and only," *Mathematics of Operations Research*, 45, 353–368.

Black, D. (1948): "On the rationale of group decision-making," *The Journal of Political Economy*, 23–34.

Bogomolnaia, A. and H. Moulin (2001): "A new solution to the random assignment problem," *Journal of Economic theory*, 100, 295–328.

Chatterji, S. and A. Sen (2011): "Tops-only domains," *Economic Theory*, 46, 255–282.

Chatterji, S., A. Sen, and H. Zeng (2014): "Random dictatorship domains," *Games and Economic Behavior*, 86, 212–236.

Chatterji, S. and H. Zeng (2018): "On random social choice functions with the tops-only property," *Games and Economic Behavior*, 109, 413–435.

Gibbard, A. (1973): "Manipulation of voting schemes: a general result," *Econometrica*, 587–601.

——— (1977): "Manipulation of schemes that mix voting with chance," *Econometrica*, 45, 665–681.

Klaus, B. and O. Bochet (2013): "The relation between monotonicity and strategy-proofness," *Social Choice and Welfare*, 40, 41–63.

Kojima, F. (2010): "Impossibility of stable and nonbossy matching mechanisms," *Economics Letters*, 107, 69–70.

Kongo, T. (2013): "On non-bossy matching rules in two-sided matching problems," *International Journal of Economic Theory*, 9, 303–311.

Liu, P. (2020): "Random assignments on sequentially dichotomous domains," *Games and Economic Behavior*, 121, 565–584.

Liu, P. and H. Zeng (2019): "Random assignments on preference domains with a tier structure," *Journal of Mathematical Economics*, 84, 176–194.

Maskin, E. (1999): "Nash Equilibrium and Welfare Optimality," *Review of Economic Studies*, 66, 39–56.

Moulin, H. (2000): "Priority Rules and other Asymmetric Rationing Methods," *Econometrica*, 68, 643–684.

Pápai, S. (2000): "Strategyproof assignment by hierarchical exchange," *Econometrica*, 68, 1403–1433.

Pramanik, A. (2015): "Further results on dictatorial domains," *Social Choice and Welfare*, 45, 379–398.

Pycia, M. and M. U. Ünver (2017): "Incentive compatible allocation and exchange of discrete resources," *Theoretical Economics*, 12, 287–329.

Pápai, S. (2001): "Strategyproof and Nonbossy Multiple Assignments," *Journal of Public Economic Theory*, 3, 257–271.

Raghavan, M. (2020): "Influence in private-goods allocation," *Journal of Mathematical Economics*, 89, 14–28.

Sato, S. (2010): "Circular domains," *Review of Economic Design*, 14, 331–342.

Satterthwaite, M. A. (1975): "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of economic theory*, 10, 187–217.

Satterthwaite, M. A. and H. Sonnenschein (1981): "Strategy-Proof Allocation Mechanisms at Differentiable Points," *Review of Economic Studies*, 48, 587–597.

Shapley, L. and H. Scarf (1974): "On cores and indivisibility," *Journal of mathematical economics*, 1, 23–37.

Svensson, L.-G. (1999): "Strategy-proof allocation of indivisible goods," *Social Choice and Welfare*, 16, 557–567.

Thomson, W. (2016): "Non-bossiness," *Social Choice and Welfare*, 47, 665–696.

Wilson, R. J. (1979): *Introduction to graph theory*, Pearson Education India.