Indian Statistical Institute
Semester 2 (2002-2003)
B. Stat 1st Year
Semestral Exam
Probability Theory 2

Date and Time: 9.5.03, 10:30-1:30 Total Points 75

The maximum you can score is 70. Answers must be justified with clear and precise arguments. Each question must be answered on a separate page and more than one answers to a question will not be accepted. If there are more than one answers to a question, only the first answer will be examined.

1. (X, Y) is a continuous bivariate random variable with density

$$f(x,y) = \begin{cases} \frac{1}{4}[1 + xy(x^2 - y^2)] & \text{if } |x| \le 1, |y| \le 1, \\ 0 & \text{otherwise} \end{cases}$$

$5 + 5 = 10$ pts.

(a) Find the marginals of X and Y.

(b) What is their covariance?

2. (a) On a straight line of length a two points are taken independently following uniform distribution. Find the probability that the distance between the points is greater than b $(b < a)$.

(b) If X and Y are independent with densities $\alpha e^{-\alpha x}$ and $\beta e^{-\beta x}$, $(x > 0)$, respectively find the density of $X + Y$ in closed form. $10 + 5 = 15$ pts.

3. If X and Y are independent with $X \sim N(0, 1)$ and $Y \sim \chi^2_n$ find the density of

$$\frac{X}{\sqrt{Y/n}}$$

in closed form. 10 pts.

4. (a) If X, Y are i.i.d. unif(0, 1) find the probability density function of the random variable

$$\frac{X}{X+Y}.$$

(b) Find the expectation of $\frac{X}{X+Y}$. $10 + 5 = 15$ pts.

5. Recall that a bivariate normal density with zero means, unit variances, and correlation coefficient ρ is

$$\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}\{x^2 - 2\rho xy + y^2\}}.$$

Let ϕ_1 and ϕ_2 be two bivariate normal densities with zero means, unit variances but different correlations coefficients ρ_1 and ρ_2. Consider the bivariate density $f = \frac{1}{2}(\phi_1 + \phi_2)$. $10 + 5 = 15$ pts.

P.T.O

(a) Find the marginals of f. Find the correlation coefficient corresponding to the density f.

(b) Show that f is not a bivariate normal density.

6. (a) If X_1 and X_2 are identically distributed then show that for any t

$$P(|X_1 - X_2| > t) \leq 2P(|X_1| > t/2).$$

(b) Let X be a random variable and denote $EX^i = \mu_i, i = 1, 2, 3, 4$, all finite. Show that the determinant

$$\begin{vmatrix} 1 & \mu_1 & \mu_2 \\ \mu_1 & \mu_2 & \mu_3 \\ \mu_2 & \mu_3 & \mu_4 \end{vmatrix} \geq 0.$$

(Hint: Consider the quadratic form in a, b, c, obtained from $E(a + bX + cX^2)^2$.)

5 + 5 = 10 pts.

INDIAN STSTISTICAL INSTITUTE

SEMESTRAL-II EXAMINATION

B.STAT(2002-03) I Year

Vectors & Matrices II

Date: 06.05.2003 Full Marks: 60 Time: 3 hrs

Note:There are 7 questions carrying 80 marks.
Maximum you can score is 60.

1.(a) Let V be a finite-dimensional vector space over the field $\mathbb{C}$ and A is a linear operator on V into V.

(i) Show that A has an *eigen value*.

(ii) Assume that x_i is an *eigen vector* of A belonging to the eigen value λ_i and $\lambda_i \neq \lambda_j$ for $i \neq j$, $1 \leq i, j \leq r$. Prove that $\{x_i, 1 \leq i \leq r\}$ is a linearly independent set.

(b) Consider the vector space $\mathbb{R}^2$ over the field $\mathbb{R}$. Construct an operator $A : \mathbb{R}^2 \to \mathbb{R}^2$ such that A does not possess any *characteristic root*.

[3+6+6=15]

2. Let x, y be vectors in a finite-dimentional complex inner-product space V. Prove :

(i) x and y are orthogonal if and only if $\|\alpha x + \beta y\|^2 = \|\alpha x\|^2 + \|\beta y\|^2$ for all pairs of scalars α and β.

(ii) $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$.

[5+2=7]

3. Apply $Gram - Schmidt$ $orthogonalization$ process to obtain an $orthonormal$ basis from the basis $\{(1,0,1), (1,-1,0), (1,1,1)\}$ for $\mathbb{R}^3$.

[10]

4. Find the $eigen$ $values$ and corresponding $eigen$ $vectors$ of the matrix

$$\begin{pmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$$

[15]

P.T.O

5. Let V be a finite-dimensional complex inner-product space. Let $J : V \to V$ be an onto(not assumed to be linear) map that satisfies $J^2 = I$ and $\langle Jx, Jy \rangle = \langle y, x \rangle$ for all $x, y \in V$.

(a) Give an example of such a map on the *unitary space* $\mathbb{C}^2$.

(b) Prove that (i) $\langle Jx, y \rangle = \langle Jy, x \rangle$,

(ii) $J(x+y) = Jx + Jy$ and ; $x, y \in V$

(iii) $J(\alpha x) = \bar{\alpha} Jx$ $\alpha \in \mathbb{C}$.

[$2 + 5 = 7$]

6.(a) Let $A = ((a_{ij}))$ be a matrix of size n such that

$$0 \leq a_{ij} \leq 1 \text{ for } 1 \leq i, j \leq n \quad \text{and} \quad \sum_{j=1}^{n} a_{ij} = 1 \text{ for } 1 \leq i \leq n.$$

Show that (i) 1 is an *eigen value* of A and (ii) $|\lambda| \leq 1$ for each eigen value λ of A.

(b) Let $T : \mathbb{R}^5 \to \mathbb{R}^5$ be a linear transformation. Prove that there exists a subspace M of $\mathbb{R}^5$ such that $dim(M) = 1$ and M is invariant under T.(Hint: 5 is an odd number).

[(2+4)+4=10]

7.(a) Using only the *elementary row operations* find A^{-1} and solve $Ax = b$ where A is the non-singular matrix $\begin{pmatrix} 0 & 1 & 1 & 2 \\ 1 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$ and $b^T = (2\ 1\ 0\ 4)$.

(b) Find the value of the *determinant* of the matrix

$$\begin{pmatrix} 1 + a_1 & a_2 & \cdots & a_n \\ a_1 & 1 + a_2 & \cdots & a_n \\ . & . & \cdots & . \\ a_1 & a_2 & \cdots & 1 + a_n \end{pmatrix}$$

[$12 + 4 = 16$]

STOP

INDIAN STATISTICAL INSTITUTE

B. Stat. I Year (2002-2003), Analysis - II

Semestral Examination

Time: 3 hrs: Max. Marks 60: Date: 02-05-2003.

Note: Answer ANY FOUR questions. Each question carries 15 marks.

1. (a) If $a_n > 0$ for $n = 1, 2, \cdots$, show that the infinite product $\prod_{n=1}^{\infty}(1 + a_n)$ converges if, and only if $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\{g_n(x)\}$ is a sequence of real-valued functions on a set X such that $\sum_{n=1}^{\infty} |g_n(x)|$ converges uniformly on X, then show that the infinite product $\prod_{n=1}^{\infty}(1 + g_n(x))$ converges uniformly on X.

(c) Show that the infinite product $\prod_{n=1}^{\infty}(1 - \frac{x^2}{n^2})$ converges uniformly on any compact subset of $\mathbb{R}$.

[6+6+3=15]

2. (a) Show that the series $\sum_{n=0}^{\infty} x^n(1-x)$ converges pointwise but not uniformly on $[0, 1]$, whereas the series $\sum_{n=0}^{\infty}(-1)^n x^n (1-x)$ converges uniformly on $[0, 1]$.

(b) Show that $\int_0^{\infty} e^{-x^2} \cos(2xy)\, dx$ is uniformly convergent on $(-\infty, \infty)$. If $F(y) = \int_0^{\infty} e^{-x^2} \cos(2xy)\, dx$, show that $F(y)$ is differentiable on $(-\infty, \infty)$ and that $F'(y) = -2yF(y)$. Hence, find the value of $F(y)$. (You may use the result $\int_0^{\infty} e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$).

[8+7=15]

3. (a) If $a > 0$, prove that

$$\lim_{h \to 0} \int_{-a}^{a} \frac{h}{h^2 + x^2}\, dx = \pi.$$

P.T.O.

(b) If $f(x)$ is continuous on $[-1,1]$, prove that
$$\lim_{h\to 0}\int_{-1}^{1}\frac{h}{h^2+x^2}f(x)\,dx=\pi\cdot f(0).$$

(c) If $f(x)$ is Riemann integrable on $[0,A]$ for every $A>0$ and if $f(x)\to 1$ as $x\to\infty$, show that
$$\lim_{t\to 0+} t\int_{0}^{\infty}e^{-tx}f(x)\,dx=1.$$

[5+5+5=15]

4. (a) Assuming that $\frac{\pi^2}{3}+4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\theta=\theta^2$ for $-\pi\le\theta\le\pi$, find a function whose Fourier Series is $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3}\sin n\theta$.

(b) Use (a) to evaluate $\sum_{n=1}^{\infty}\frac{1}{n^6}$.

(c) Find the Fourier Series of the function $f(\theta)$ defined on $(-1,1]$ by:
$$\begin{aligned} f(\theta) &= \cos\theta, \quad 0\le\theta\le 1\\ &= -\cos\theta, \quad -1<\theta<0\end{aligned}$$
and extended to the whole of $\mathbb{R}$ as a periodic function of period 2.

[6+4+5=15]

5. (a) Let $f(x)$ be Riemann integrable on $[a,b]$. Then, show that for each β
$$\lim_{\alpha\to\infty}\int_{a}^{b}f(t)\sin(\alpha t+\beta)\,dt=0.$$

(b) Suppose that $f(x)$ is Riemann integrable on $[a+\epsilon,b]$ for each $\epsilon>0$, and that the improper integral $\int_{a+}^{b}|f(x)|dx$ is convergent. Show that, for each β
$$\lim_{\alpha\to\infty}\int_{a+}^{b}f(t)\sin(\alpha t+\beta)\,dt=0.$$

(c) Use (a) to show that
$$\int_{0}^{\infty}\frac{\sin x}{x}\,dx=\frac{\pi}{2}.$$

[6+4+5=15]

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination : 2002-03
B.Stat.(Hons.) I Year
Statistical Methods II

Date : 25.04.2003 **Maximum Marks : 100** **Duration : 3 Hours**

Answer Question No.5 and ANY THREE questions from the rest . Marks allotted to each question are given within the parentheses . Standard notations and symbols are used .

1. (a) Express a partial correlation coefficient of a lower order in terms of partial correlation coefficients of higher order .
(b) Show that
$$\frac{b_{12.34\ldots p}}{b_{21.34\ldots p}}=\frac{s_{1.23\ldots p}}{s_{2.13\ldots p}}\cdot\frac{s_{1.34\ldots p}}{s_{2.34\ldots p}}\;.$$

(15 +10) = [25]

2. (a) Show that the multiple correlation coefficient $r_{1.23\ldots p}$ is the maximum total correlation that may hold between x_1 and a linear function of x_2 , x_3 ,…, x_p .
(b) If r_{1i} = r (i = 2,3,…,p) and $r_{ij}=r'$ (i,j = 2,3,…,p , i≠j) , then what is $r_{1.23\ldots p}$?

(15 + 10) =[25]

3. (a) Starting from the Pearsonian differential equation derive the p.d.f. of the Pearsonian Type I curve . Discuss in details how you would estimate the unknown parameters of the distribution based on a given frequency distribution for which the Pearsonian Type I curve is appropriate . Also indicate how you would fit such a probability distribution to the given data .

(b) Show that , for the Pearsonian family of distributions ,
$$\frac{mean-mode}{s.d.}=\frac{\sqrt{\beta_1}(\beta_2+3)}{2(5\beta_2-6\beta_1-9)}\;.$$

(15 + 10) = [25]

4. Give an algorithm to simulate observations from each of the following probability distributions having the pd.f.'s :

(a) f(x) = Constant . $\left(1+\frac{x}{a}\right)^p \exp[-px/a]$, $-a\le x<\infty$, p > 0

(b) f(x) = Constant . $\left(1-\frac{x^2}{a^2}\right)^m$, $-a\le x\le a$, m > 0 .

P.T.O. (15 + 10) = [25]

5. The following is the frequency distribution of right-hand grip for 345 European males :

Right-hand grip (in lb.)	Frequency
29.5 - 39.5	1
39.5 - 49.5	2
49.5 - 59.5	12
59.5 - 69.5	52
69.5 - 79.5	99
79.5 - 89.5	101
89.5 - 99.5	55
99.5 - 109.5	17
109.5 - 119.5	5
119.5 - 129.5	1
Total	345

Fit a normal probability curve to the above data and test for goodness of fit .

(20 + 5) =[25]

Indian Statistical Institute
Semester 2 (2002-2003)
B. Stat 1st Year
Mid-semestral Exam
Probability Theory 2

Friday 28.2.2003, 10:30-1:30 Total Points $5 \times 6 = 30$

Answers must be justified with clear and precise arguments.

1. (a) Suppose X is a nonnegative continuous random variable with finite first moment. Show that $\lim_{x\to\infty} x(1-F(x)) = 0$.

(b) If X is a nonnegative continuous random variable such that both EX and $E(1/X)$ exist then show that $E(1/X) \geq 1/(EX)$.

2. Two random variables X and Y are said to be independent if for any two subsets of the real line A and B we have $P(X \in A \text{ and } Y \in B) = P(X \in A)P(Y \in B)$.

(a) Suppose X and Y are independent and both $U(0,1)$. Show that $E|X - Y| \leq 1/2$. (Hint: use an inequality involving $|X-Y|, |X-1/2|, |Y-1/2|$.)

(b) Suppose X is $N(0,1)$. Show that sign(X) and $|X|$ are independent.

3. (a) A nonnegative random variable Y is said to have a lognormal distribution if $\log Y$ has a $N(\mu, \sigma^2)$ distribution. Find the density of such a Y.

(b) A random number Θ is chosen from the $U(-\pi/2, \pi/2)$ distribution. What is the density of the following random variable

$$X = \tan\Theta?$$

4. If $X \sim N(\mu, 1)$ show that

$$Y = \frac{1-\Phi(X)}{\phi(X)}$$

has expection $1/\mu$.

5. A random variable Y has the following distribution function:

$$F(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & 0 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

(a) Describe how you can get such a random variable with the above df using an exponential random variable X with parameter 1.

(b) Find the expectation of Y.

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : (2002-2003)
B-Stat(Hons) I year, Second Semester
Computational Techniques and Programming - II

Date : 26.02.2003   Maximum marks : 50   Duration : 2 hours

Note: *Answer all the questions.*

1. (a) Define the operators $E,\ \delta,\ \mu,\ \nabla,\ \mathbf{1}$ and $\mathbf{0}$.

   (b) Prove the following equations.

   i. $(E^{1/2} - \frac{1}{2}\delta)^2 - \frac{1}{4}\delta^2 = 1$.

   ii. $E^{1/2} - \frac{1}{2}\delta - \mu = \mathbf{0}$.

   (c) For a polynomial f, show that $Ef = \left(\sum_{k=0}^{\infty}\nabla^k\right) f$ [3+4+5=12]

2. Let f(-2) = -50, f(-1) = 6, f(0) = 10, f(1) = 10, f(2) = 30, f(3) = 190. Find approximate values of f'(-2) and f'(3) by stating clearly the formulas you are going to use. [4+4=8]

3. Write the flow chart for finding the determinant of any $n \times n$ real matrix. [10]

4. Describe the difference between a macro and a function. Implement the swapping of two values in a C program using a macro and a function. [2+2+2=6]

5. (a) Write a C code to find the size(in bits) of an unsigned integer without using the *sizeof* operator.

   (b) Explain recursion in C programming language using a program for calculating **n**!. [3+3=6]

6. Choose the correct answer. [8]

   (i) If the type of the function is omitted then the default type of the function is ....

   (a) void (b) int (c) char (d) none

   (ii) Let i = 5 and s[i++] = 'A'.

   (a) s[5] is A (b) s[6] is A (c) s[7] is A (d) None of the above

(P.T.O)

(iii) The communication between the functions takes place by the passed arguments and values returned by the functions, and through external variables.
(a) True (b) False

(iv) The source code of a function can be split into multiple files.
(a) True (b) False

(v) {int x, *ip; x = 5; ip = &x; *ip++;} then the value of x is:
(a) 6 (b) 5 (c) Syntax error (d) None of the above

(vi) #define p(x) (x * x)
main(){int y = 4, z; z = p(y+1); }
The value of z is:
(a) 25 (b) 9 (c) 16 (d) None of the above

(vii) {int a = 2, b = 4, c = 6; if(a = 3) b = 5; if(a = 0) c = 1; printf("%d,%d\n",b,c);}
The output is:
(a) 4,6 (b) 5,1 (c) 4,1 (d) 5,6

(viii) printf(" %d\n",printf("\nHello, World"));
The output is:
(a) Hello, World (b) Syntax Error (c) Hello, World 13
(d) Hello, World 14

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INDIAN STATISTICAL INSTITUTE
Mid-Semestral Examination – 2002-2003
B. Stat I Year
Vectors and Matrices II

Date : 24.2.03 Maximum Marks : 80 Duration : 2½ hours

Answer all questions. The maximum you can score is 80.

1. Let $C = \begin{pmatrix} A & O \\ O & I_s \end{pmatrix}$ be a real matrix of order n with A, a square matrix of order r, I_s, the identity matrix of order $s(= n-r)$ and O standing for submatrices with all entries zero.

(i) Consider $f : M(r, \mathbb{R}) \to \mathbb{R}$ defined by $f(A) = \det(C)$ where C is as above. Show that f satisfies all the three defining properties of the determinant and, hence, deduce that $\det(C) = \det(A)$.

(ii) Similarly derive that $\det(D) = \det(B)$ where $D = \begin{pmatrix} I_r & O \\ O & B \end{pmatrix}$ is a real matrix of order n with I_r, the identity matrix of order r and B a submatrix of order $s(= n-r)$.

(iii) Show that $\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \begin{pmatrix} A & O \\ O & I_s \end{pmatrix}\begin{pmatrix} I_r & O \\ O & B \end{pmatrix}$ where A is a square matrix of order r, B is a square matrix of order s. Deduce that

$$\det\begin{pmatrix} A & O \\ O & B \end{pmatrix} = \det(A)\det(B).$$

[12+8+10=30]

2. Find the determinant of the matrix of order n

$$A = \begin{pmatrix} a & b & b & \cdots\cdots & b \\ b & a & b & \cdots\cdots & b \\ b & b & a & \cdots\cdots & b \\ . & . & . & \cdots\cdots & . \\ . & . & . & \cdots\cdots & . \\ b & b & b & \cdots\cdots & a \end{pmatrix} \text{ with } a \neq b$$

from the first principles.

[20]

Contd....(2)

3. Compute the determinant of

$$\begin{pmatrix} 1-x & 1 & 1 \\ 1 & 1-x & 1 \\ 1 & 1 & 1-x \end{pmatrix}$$

Can you genralize this result ?

[8+12=20]

4. Solve the system of linear equations

$$\begin{aligned} 2x + y \quad &= 0 \\ 3y + z &= 1 \\ x \quad + 4z &= 2 \end{aligned}$$

by applying the elementary row operations.

[30]

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INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : 2002-2003
B.Stat. (Hons.) I Year
Statistical Methods II

Date :20. 02.2003 **Maximum Marks : 100** **Duration : 3 Hours**

Answer all questions . Marks allotted to each question are given within the parentheses . Standard notations and symbols are used .

1. (a) Two variables ξ and η are made up of the sum of a number of terms as follows :

$$\xi = \sum_{i=1}^{n} x_i + \sum_{i=1}^{a} u_i$$
$$\eta = \sum_{i=1}^{n} x_i + \sum_{j=1}^{b} v_j$$

where the variables x , u and v are all referred to their means , have unit standard deviations and are uncorrelated .

Show that

$$r_{\xi,\eta} = \frac{n}{\sqrt{(n+a)(n+b)}} .$$

(b) Using these results , show that if

$$X = \sum_{i=1}^{6} x_i \text{ , } A = x_1 \text{ , } B = x_2 + x_3 \text{ , } C = x_4 + x_5 + x_6$$

and the x's are subject to the above conditions , then the multiple regression equation of X on A , B and C is given by

$$\frac{\hat{X}}{\sqrt{6}} = 0.4082.A + 0.5774.\frac{B}{\sqrt{2}} + 0.7071.\frac{C}{\sqrt{3}} .$$

(c) Calculate $r_{X.ABC}$ and $r_{XC.AB}$ and interpret the results .

(10 + 10 + 5) = [25]

2. (a) Show that for the trivariate distribution ,

$$1- r_{1.23}^2 = (1-r_{12}^2)(1-r_{13.2}^2)$$

P.T.O.

(b) Deduce that (i) $r_{1.23} \geq r_{12}$, (ii) $r_{1.23}^2 = r_{12}^2 + r_{13}^2$ if $r_{23} = 0$,

(iii) $1 - r_{1.23}^2 = \frac{(1-\rho)(1+2\rho)}{(1+\rho)}$ provided all the coefficients of zero order are equal to ρ .

(iv) If $r_{1.23} = 0$, show that x_1 is uncorrelated with any of the other variables .

(5 + 5x4) = [25]

3. In the course of an experiment , 15 mosquitoes were put in each of 120 jars and were next subjected to a dose of D.D.T. . After 4 hours the number alive in each jar was counted and the following frequency distribution was obtained :

No. of mosquitoes alive	Frequency (No. of jars)
0	2
1	12
2	14
3	22
4	28
5	17
6	13
7	10
8	2

Fit a suitable probability distribution to the above data assuming that each mosquito has a common probability of survival . Also test for goodness of fit .

(15 + 10) = [25]

4. (a) Simulate 20 observations from an exponential distribution having p.d.f.

$$f(x, \theta) = \theta \exp(-x\theta) \text{ for } 0<x<\infty .$$

You may take $\theta = 0.2$.

(b) Using the above observations , simulate 20 observations from the following linear regression model

$$y_i = \beta x_i + \varepsilon_i$$

where given x_i's , ε_i's are assumed to be independently normally distributed as $N(0, \sigma^2 x_i^g)$. You may take $\beta = 3$, $\sigma = 5$ and g = 2 .

(c) Use your sample observations to get an unbiased estimate of β .

(10 + 10 + 5) = [25]

INDIAN STATISTICAL INSTITUTE

B. Stat. I Year (2002-2003), Analysis - II

Mid-Semestral Examination

Time: 3 hrs: Max. Marks 30: Date: 17-02-2003.

Note: Each question carries 8 marks. You may answer all the questions. But the maximum you can score is 30.

1. (a) Let $f(x)$ be a real-valued continuous function defined on $[1, \infty)$. When do you say that $\int_1^\infty f(x)dx$ is convergent?

(b) Let $f(x)$ be a real-valued continuous function defined on $[1, \infty)$. For each $n = 1, 2, \cdots$, let $a_n = \int_1^n f(x)dx$. If $\int_1^\infty f(x)dx$ is convergent, then show that $\lim_{n\to\infty} a_n$ exists and is equal to $\int_1^\infty f(x)dx$.

(c) Give an example to show that the converse of (b) need not be true.

(d) If $f(x)$ is a continuous function on $[1, \infty)$ which is decreasing to 0, then show that $\int_1^\infty f(x)dx$ converges if, and only if the sequence $\{a_n\}$ (defined in (b)) converges.

(e) For what values of s, does $\int_2^\infty \frac{dx}{x(\log x)^s}$ converge?. Justify your answer.

2. (a) Let $\gamma(t) = (\gamma_1(t), \gamma_2(t)), t \in [0, 1]$ be a curve in $\mathbb{R}^2$ defined on $[0, 1]$. If $\gamma_1'(t)$ and $\gamma_2'(t)$ are continuous on $[0, 1]$, show that γ is rectifiable and its length $\Lambda(\gamma) = \int_0^1 \sqrt{(\gamma_1'(t))^2 + (\gamma_2'(t))^2}dt$.

(b) Prove that the curve

$$y = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

is rectifiable, whereas the curve

$$y = \begin{cases} x \sin(\frac{1}{x}) & \text{if } 0 < x \leq 1 \\ 0 & \text{if } x = 0. \end{cases}$$

is not rectifiable.

P.T.O

Indian Statistical Institute
Semester 1 (2002-2003)
B. Stat 1st Year
Semestral Exam
Probability Theory 1

Thursday 19.12.2002, 10:30-1:30 Total Points $7 \times 11 = 77$

The maximum you can score is 70. Answers must be justified with clear and precise arguments.

1. Suppose $N, n, N > n$, are positive integers. Construct a random experiment and use 'total probability is 1' to prove the formula

$$1 + \frac{N-n}{N-1} + \frac{(N-n)(N-n-1)}{(N-1)(N-2)} + \cdots + \frac{(N-n)(N-n-1)\cdots 1}{(N-1)(N-2)\cdots(n+1)n} = \frac{N}{n}.$$

2. Suppose n is a positive integer and $p_1, \cdots, p_k$ are all the distinct prime divisors of n. Using the principle of inclusion and exclusion prove that the number of integers less than n and prime to it is

$$n\Pi_{i=1}^{k}(1 - \frac{1}{p_i}).$$

3. (a) Seven balls are distributed randomly into seven cells. Given that exactly two cells are empty find the (conditional) probability of a triple occupancy of some cells.
(b) A bag contains n balls, of which an unknown number K are white and $n - K$ are black and $P(K = i) = 1/(n+1), i = 0(1)n$. Two balls are drawn with replacement and found to be white. Find the probability that the next ball drawn will be black given the above information. (An expression in closed form is required.)

4. A coin is tossed $(m + n)$ times, $m > n$. Find the probability of at least m consecutive heads in closed form.

5. A player tosses a coin and is to score 1 for every head and 2 for every tail that turns up. He is to play on until his score reaches or passes n. If p_n is the chance of attaining exactly n find p_n. (An expression in closed form is required.)

6. The product of two independent one digit numbers ξ and η uniformly distributed on $\{0, 1, \cdots, 9\}$ can be written as $\xi \times \eta = 10\zeta_2 + \zeta_1, 0 \le \zeta_i \le 9$. Find the probability distribution of ζ_1 and ζ_2. Are they independent?

P.T.O

3. Define a sequence of functions $\{f_n\}$ on $[0, 1]$ inductively as follows.

$$f_0(x) = 1 \quad \text{for all} \quad x \in [0,1]$$
$$f_n(x) = \sqrt{x f_{n-1}(x)} \quad \text{for} \quad x \in [0,1], \quad n = 1, 2, \cdots$$

(a) Show that for each x, $\{f_n(x)\}$ is a decreasing sequence. What is $\lim_{n\to\infty} f_n(x)$ for each $x \in [0, 1]$.
(b) Show that the sequence of functions $\{f_n(x)\}$ converges uniformly on $[0, 1]$.

4. (a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges uniformly on $[1 + \epsilon, \infty)$, for any $\epsilon > 0$.
(b) Show that the series $\sum_{n=2}^{\infty} \frac{\log n}{n^x}$ converges uniformly on $[1 + \epsilon, \infty)$, for any $\epsilon > 0$.
(c) If $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$, $x \in (1, \infty)$ show that f is differentiable on $(1, \infty)$. What is $f'(x)$?

5. (a) Suppose that the series $\sum_{n=0}^{\infty} a_n$ converges. For $|x| < 1$, let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Show that the limit $\lim_{x\to 1-} f(x)$ exists and is equal to $\sum_{n=0}^{\infty} a_n$.
(b) If each $a_n \ge 0$, if $\sum_{n=0}^{\infty} a_n$ diverges, and if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ exists for $|x| < 1$, then show that $\lim_{x\to 1-} f(x) = \infty$.
(c) If each $a_n \ge 0$, if $f(x) = \sum_{n=0}^{\infty} a_n x^n$ exists for each x, $|x| < 1$, and if $\lim_{x\to 1-} f(x)$ exists and is equal to A, then show that $\sum_{n=0}^{\infty} a_n$ converges to A.

7. n distinguishable particles are arranged randomly into N distinguishable cells. Unaffected by the other particles any particle can fall in any fixed cell with probability $1/N$. Let X denote the number of empty cells. Find $E(X), V(X)$ and find the limits of these as $n \to \infty, (n/N) \to \alpha$.

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : (2002-2003)

B-Stat(Hons) I year

Computational Techniques and Programming - I

Date : 16.12.02 Maximum marks : 100 Duration : 195 minutes

Note: This paper carries 112 marks. Answer as much as you can.

1.(a) State the conditions for the method of iteration to be applicable for solving an equation $x = f(x)$.
(b) Apply the method of iteration to solve for x, correct up to 3 places after the decimal, in the following equation
$x^3 = 2x^2 + x + 2$, where x is known to be positive, by justifying your choice of the starting point of the method. [3+15=18]

2. (a) Define the k^{th} divided difference of a function f with respect to $x_0, x_1, ..., x_k$.
(b) Show that, for any permutation of $x_0, x_1, ..., x_k$, the k^{th} divided difference of f remains unchanged. [2+8]

3. Write short notes on any two errors occurring in numerical techniques. [10]

4. Apply Newton's forward interpolation formula to approximate the value of f(-1.7) if f(-2)=66, f(-1)=25, f(0)=20, f(1)=21 and f(2)=46. [8]

5. Show that the central difference operator δ satisfies the following property
$\delta^{2m+1} f(x_k + \frac{h}{2}) = h^{2m+1}(2m+1)! f[x_{k-m}, x_{k-m+1}, ..., x_k, ..., x_{k+m}, x_{k+m+1}]$,
where $x_i \in I\!R\ \forall i$, $x_i - x_{i-1} = h > 0\ \forall i$, f is a continuous function and f$[y_0, y_1, ..., y_r]$ denote the r^{th} divided difference of f relative to $y_0, y_1, ..., y_r$. [10]

6. Derive the 3^{rd} order Newton Cotes formula of numerical integration. [10]

7. Determine the values of the indicated substrings given below. Assume that ALPHA has the value 'ABCDEFGHIJ1234567890' with length 20.
(a) ALPHA(1:1) (b) ALPHA(3:5)
(c) ALPHA(11:) (d) ALPHA(:) [4]

(P.T.O)

8. Write FORTRAN programs to solve the following problems.

(a) Read any two matrices A and B. Compute C as the product of A and B Print the results using proper format. Please make the necessary assumptions

(b) Write a function subprogram to compute n!, where n is a positive integer. Then use it to compute the sum given below

$$\sum_{n=1}^{n=k}(-1)^{n+1}\frac{1}{n!},$$

where the value of k is to be read in the main program.

(c) Compute the roots of the equation, $ax^2 + bx + c = 0$, for any given real values of a, b and c.

(d) Read an input string and convert every numeric character in it to a blank.

[5+5+8+5=23]

9. Convert the following decimal numbers to the corresponding binary and hexadecimal numbers.

(a) 123 (b) 0.325 (c) 12.25 [6]

10. Perform the following arithmetic operation in binary, using signed 2's complement method.

- 8.75 + 2.25 + 6.50 [8]

11. Prove the Boolean relation given below with the help of truth table.

A.(B + C) = A.B + A.C [5]

——X——

INDIAN STATISTICAL INSTITUT

First Semestral Examination:(2002-2003)

B.Stat(Hons)-I Year

Vector & Matrices I

Date: 13.12.2002 **Maximum Marks: 100** **Duration: $3\frac{1}{2}$hrs.**

Note:There are 5 questions carrying 120 marks.
You may answer any.Maximum you can score is 100.

1.(a) Let V and W be vector spaces, each of dimension n, over the field $\mathbb{C}$ and let $T : V \to W$ be a linear transformation.Prove that T is an isomorphism if and only if $\{Tv_i, 1 \le i \le n\}$ is a <u>*basis*</u> for W for any <u>*basis*</u> $\{v_i, 1 \le i \le n\}$ of V.

(b) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by $T(x, y) = (ax + by, cx + dy)$ where a, b, c, d are fixed real numbers. Show that T *is a linear isomorphism* iff $ad - bc \neq 0$. Write explicitly T^{-1} when it exists. What are the matrices of T and T^{-1} (whenever the latter exists) with respect to the standard basis for $\mathbb{R}^2$ over $\mathbb{R}$?

(c) Let $M(n, \mathbb{R})$ be the vector space over $\mathbb{R}$ of all real matrices of *order* $n(n \ge 2)$.Let $W = \{A \in M(n, \mathbb{R}) : trace(A) = 0\}$.Find $dim(W)$ without finding a basis for W. State precisely the result on *linear functionals* you would like to employ to obtain the above result.

[7+16+7=30]

2.(a) Consider the subspace $M = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 + x_3 + x_4 = 0\}$ of $\mathbb{R}^4$.Can you find a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^4$ such that $R(T) = M$?

(b) Let $T : V \to V$ be a linear transformation such that $R(T) = ker(T)$, *the kernel of* T. What can you say about $T^2 (= T o T)$? Construct such a map $T : \mathbb{R}^2 \to \mathbb{R}^2$.

[6+10=16]

3.(a) Suppose that V is a finite-dimensional vector space over $\mathbb{R}$ with basis $\{x_1, \cdots, x_n\}$. Suppose that $\alpha_1, \alpha_2, \cdots, \alpha_n$ are pairwise distinct reals. If A is a linear transformation on V into V such that $Ax_j = \alpha_j x_j, j = 1, 2, \cdots, n$, and if B is a linear transformation on V into V such that B *commutes with* A(i.e., $BA = AB$), then show that there exist reals $\beta_1, \beta_2, \cdots, \beta_n$ such that $Bx_j = \beta_j x_j, j = 1, 2, \cdots, n$.

(b) Prove that if B is a linear transformation on a finite-dimensional vector space V over $\mathbb{R}$ into itself, and if B *commutes* with *every* linear transformation on V into V, then there exists a real number β such that $Bx = \beta x$ for all x in V.

(c) For which values of α is the following matrix invertible ?

$$\begin{pmatrix} 1 & \alpha & 0 \\ \alpha & 1 & \alpha \\ 0 & \alpha & 1 \end{pmatrix}$$

(d)Let V be a vector space over a field $\mathcal{F}$. Suppose U and W are subspaces of V such that each x in V has a unique representation $x = u + w$ where $u \in U$ and $w \in$ W. Show that U and W are complements to each other.

[8+8+6+4=26]

(P.T.O)

4.(a) Show that the vectors $x_1 = (1,1,1)$, $x_2 = (1,1,-1)$ and $x_3 = (1,-1,1)$ form a *basis* for $\mathbb{R}^3$ over the field $\mathbb{R}$. If $\{f_1, f_2, f_3\}$ is the *dual basis*, and if $x = (0,1,0)$, find $< x, f_1 >$, $< x, f_2 >$ and $< x, f_3 >$.

(b) Suppose that $1 \leq m < n$ and that $f_1, f_2, \cdots, f_m$ are linear functionals on an $n - \textit{dimensional}$ vector space V over $\mathbb{R}$. Under what conditions on the scalars $\alpha_1, \alpha_2, \cdots, \alpha_m$ is it true that there exists a vector x in V such that $< x, f_j > = \alpha_j$ for $j = 1, 2. \cdots, m$?

(c) Let $\mathcal{P}$ stand for the vector space over $\mathbb{R}$ of all polynomials with real coefficients. If $\{\alpha_0, \alpha_1, \alpha_2, \cdots\}$ is an arbitrary sequence of real numbers, and if $p(t) = \sum_{i=0}^{n} a_i t^i$ is an element of $\mathcal{P}$,define $\lambda(p) = \sum_{i=0}^{n} a_i \alpha_i$. Prove that λ is an element of the dual space $\mathcal{P}'$ and that every element of $\mathcal{P}'$ can be obtained in this manner by a suitable choice of the α's .

[6+6+10=22]

5.(a) Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation defined by:

$$T_\theta(x,y) = (x\cos\theta - y\sin\theta,\ x\sin\theta + y\cos\theta),\ 0 \leq \theta < 2\pi.$$

Let $X = \{(1,0),(0,1)\}$, $Y = \{(1,1),(1,-2)\}$ be two bases of $\mathbb{R}^2$ over $\mathbb{R}$. Find the *matrix representstions* $[T_\theta; X]$, $[T_\theta; Y]$ and $[T_\theta; X, Y]$.

(b) Let A be the linear transformation on $\mathcal{P}_n$ defined by $(Ap)(t) = p(t+1)$, and let $\{p_0, p_1, \cdots p_{n-1}\}$ be the basis of $\mathcal{P}_n$ defined by $p_j(t) = t^j$, $j = 0, 1, \cdots, n-1$. Find the matrix of A with respect to this basis. [Here $\mathcal{P}_n$ denotes the set of all real polynomials of degree at most $n-1$].

[(6+6+6)+8=26]

STATISTICAL METHODS I

B-Stat I, First Semester 2002-2003

Semestral Examination

Date: 09.12.2002 Maximum marks: 100 Duration: 3 hours.

1. (a) Prove the following results:

(i) The measure of kurtosis β_2 is always greater than 1 for any variable.

(ii) The absolute difference between the mean and the median of any variable never exceeds the standard deviation of the variable.

(iii) Gini's mean difference cannot be greater than $\sqrt{2}$ times the standard deviation of the variable.

(b) Construct any artificial data set involving 2×2 tables which demonstrates Simpson's paradox. How can this apparent paradox be explained? [(4+6+4)+6=20 points]

2. Consider the joint distribution of several attributes, each at 2 levels (present and absent). Let the capital letters A, B, C, ..., denote the presence of the attributes, and the Greek letters α, β, γ, ... denote their absence.

(a) Show that for n such attributes, the total number of class frequencies (including those of order $0, 1, \ldots, n$) is 3^n. [Note: The class frequency of a particular class is the number of items assigned to that class, and except for the total frequency (denoted by n), the other class frequencies are denoted by putting the corresponding symboles within "( )" brackets. Thus $(AB\gamma)$ represents the frequency of the class where A and B are present, but C is absent. n represents the class frequency of order 0; (A), (B), ..., (α) (β), ..., represents class frequencies of order 1, (AB), ... represents class frequencies of order 2, etc.]

(b) For $n = 3$, show that

$$(AB) - (ABC) \leq (A) - (AC).$$

(c) 100 children took three examinations. 40 passed the first, 39 passed the second, and 48 passed the third. 10 passed all three, 21 failed all three. 9 passed the first two and failed the third, 19 failed the first two and passed the third.
Find how many passed at least two examinations. Show that for the question asked certain of the given frequencies are not necessary. Which are they? [6+6+8 = 20 points]

3. Let $(x_1, y_1), \ldots, (x_n, y_n)$ represent n bivariate observations. We are interested in the regression of y on x.

(a) Derive the least squares estimates of a and b, the intercept and the slope parameter of the regression line of y on x.

(b) Show that $Var(Y) = r^2 s_y^2$, where r is the correlation coefficient between x and y, $s_y^2 = Var(y)$, and Y is the predicted value of y based on the fitted regression line in (a).

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(c) Show that $Cov(Y, e) = 0$, where e is the residual (error) in prediction.

(d) Using part (c) or otherwise, show that $Var(e) = r^2(1 - s_y^2)$.

(e) Show that $Cov(y, Y) = |r|$. [6+4+3+3+4 = 20 points

4. (a) Let there be k groups of data on x and y, with means $\bar{x}_i$ and $\bar{y}_i$, variances s_{xi}^2 and s_{yi}^2, and correlations r_i in the i-th group ($i = 1, 2, \ldots, k$). Show that the correlation for the combined data is

$$r = \frac{\sum_{i=1}^k n_i r_i s_{xi} s_{yi} + \sum_{i=1}^k n_i (\bar{x}_i - \bar{x})(\bar{y}_i - \bar{y})}{\left[\sum_{i=1}^k n_i s_{xi}^2 + \sum_{i=1}^k n_i (\bar{x}_i - \bar{x})^2\right]^{1/2} \left[\sum_{i=1}^k n_i s_{yi}^2 + \sum_{i=1}^k n_i (\bar{y}_i - \bar{y})^2\right]^{1/2}}$$

where $\bar{x}$ and $\bar{y}$ are the grand means of x and y and n_i is the number of pairs in the i-th group.

(b) Two judges rank 10 competitors in a certain art competition. Following are the ranks given by them. Measure the association between the judges using Spearman's r_R and

	Competitor									
	1	2	3	4	5	6	7	8	9	10
Ranks given by Judge A	5	1	4	2	7	3	6	8	10	9
Ranks given by Judge B	10	5	1	2	3	4	7	6	8	9

Kendall's τ, and comment on the strength of the association. [10+10=20 points

5. A market research company is comparing two different TV programs, A and B. Let p be the proportion of viewers who favour program A over program B. The company is interested in testing the hypothesis,

$$H_0 : p = 0.5, \text{ against}$$

$$H_1 : p \neq 5.$$

A random sample of 25 viewers are selected. Let X be the number among them who favour program A. Let x be the observed value of X.

(a) Which of the following rejection regions is most appropriate in this context and why?

$$R_1 = [x : x \leq 7 \text{ or } x \geq 18], \quad R_2 = [x : x \leq 8], \quad R_3 = [x : x \geq 17].$$

(b) What is the probability distribution of X when H_0 is true? Using this find the level of the test for the rejection region you chose in part (a).

(c) Find the power of the test at $p = 0.3$ for the rejection region you chose in part (a).

(d) Using your selected region what would your decision be if 6 of the 25 people queried favoured program A. [5+5+5+5=20 points]

INDIAN STATISTICAL INSTITUTE

B. Stat. I Year (2002-2003), Analysis - I

First Semestral Examination

Time: 3 hrs: Max. Marks 60: Date: 5-12-2002.

Note: This paper carries 70 marks. You may answer all the questions. But the maximum you can score is 60.

1. (a) Let $\{a_n\}$ be a sequence of real numbers such that $a_n > 0$ for all n and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = l$ exists. Show that $\lim_{n\to\infty} \sqrt[n]{a_n}$ exists and is equal to l.

(b) Use (a) to show that
$$\lim_{n\to\infty} \frac{n}{(n!)^{\frac{1}{n}}} = e.$$

[6+4=10]

2. Let f be a real valued function defined on $I\!R$. Suppose there exists $\alpha, 0 < \alpha < 1$, such that $|f(x) - f(y)| < \alpha|x - y|$ for all $x, y \in I\!R$. Let $x_0 \in I\!R$ and define the sequence $\{x_n\}$ by:

$$x_1 = f(x_0)$$

$$x_{n+1} = f(x_n), n = 1, 2, 3, \cdots.$$

Show that $\{x_n\}$ is a Cauchy sequence in $I\!R$. If $y = \lim_{n\to\infty} x_n$, show that $f(y) = y$. Show also that there is only one point $y \in I\!R$ such that $f(y) = y$.

[10].

3. (a) A real-valued function f defined on an interval (a, b) is said to be a convex function if

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y)$$

whenever $x, y \in (a, b)$ and $0 < \lambda < 1$. If f is convex on (a, b) and if $a < x < y < z < b$, show that

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x} \leq \frac{f(z) - f(y)}{z - y}.$$

(b) If f is convex on (a, b), show that f is continuous on (a, b). Show also that if ϕ is an increasing convex function defined on the range of f, then $\phi(f(x))$ is also convex on (a, b).

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(c) If f is continuous on (a, b) such that

$$f(\frac{x+y}{2}) \leq \frac{f(x)+f(y)}{2}$$

for all $x, y \in (a, b)$, prove that f is convex on (a, b).

(d) Suppose f is differentiable on (a, b). Prove that f is convex on (a, b) if, and only if f' is monotonically increasing on (a, b). Now suppose f is twice differentiable on (a, b). Prove that f is convex on (a, b) if, and only if $f'' \geq 0$ for all $x \in (a, b)$.

[4+7+7+7=25]

4. (a) Suppose f has a continuous derivative f' on an interval I. Suppose that the compact interval $[a, b]$ is contained in I. Let $\epsilon > 0$. Show that there is $\delta > 0$ such that whenever $x, y \in [a, b], 0 < |x - y| < \delta$, then we have

$$|\frac{f(y) - f(x)}{y - x} - f'(x)| < \epsilon.$$

(b) Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b). Show that there is $\xi \in (a, b)$ such that

$$(f(b) - f(a))g'(\xi) = (g(b) - g(a))f'(\xi).$$

(c) Assume that the functions $f(x)$ and $g(x)$ are continuous on $[0, 1]$, and differentiable on $(0, 1)$. Also assume that $f(0) = 0, g(0) = 0$ and $f'(x)$ and $g'(x)$ are positive for $x \in (0, 1)$. Prove that if $\frac{f'(x)}{g'(x)}$ is an increasing function of x, then $\frac{f(x)}{g(x)}$ is also an increasing function of x.

[7+4+6=17]

5. (a) If f is a bounded real valued function which is Riemann integrable on the compact interval $[a, b]$, then show that the function $|f|$ is also Riemann integrable on $[a, b]$.

(b) Give a function f which is not Riemann integrable on $[0, 1]$, but $|f|$ is Riemann integrable.

[5+3=8]

INDIAN STATISTICAL INSTITUTE

First Semestral Examination : (2002-2003)

B.Stat. - 1st Year.
Remedial English

Date : 2.12.02 **Maximum Marks : 100** **Duration : 3 Hrs.**

Section A – 60 Marks.
Writing Skill

Q.1. Write a Composition on any one of the following topics. (30)

a) A friendship in your life you Cherish and your reason for doing so.
b) Is God dead ?
c) Make an impartial assessment of the statement :
" Money is Happiness "

Q.2. Write a letter on any one of the following topics : (15)

a) Imagine you want to register your claim for a Scooter. Write a letter to the l agent asking for information about the procedure for registration.

b) Write a letter to the editor of a well-known newspaper about the nuisance of hawkers in crowded streets.

Q.3. Write a paragraph on any one of the following topics : (15)

a) "The surest way to remain poor is to be an honest man".
b) "Man is the marker of his own destiny".

Contd.pg.no.2...

Section B – 20 Marks.
Grammar

Q.4. State the different meanings in each of the following pair of words : 5 x 1 (5)

a) Beneficial ; Beneficient.
b) Artist ; Artisan
c) Antipathy ; Apathy.
d) Elemental ; Elementary.
e) Errant ; Arrant.

Q.5. Idiomatic Comparisons : 5 x 1 (5)

a) Robin's old servant is as blind as a ______________
b) Kiran is as cheerful as a ______________
c) The new class teacher's voice is as loud as ______________
d) He left the office with a mind as heavy as ______________
e) Raju's father's decision is as firm as ______________

Q.6. One word Substitution : 10 x 1/2 (5)

a) Not definitely or clearly expressed
b) Incapable of being reached.
c) An assembly of worshippers
d) Give tit for tat
e) Incapable of being overcome
f) Incapable of being put into practice.
g) A word or a practice no longer in use.
h) The science of Reasoning.
i) An absolute government
j) Belonging to all parts of the world.

Q.7. Fill in the blanks with suitable collective nouns : 5 x 1 (5)

a) A ___________ of Mountains
b) A ___________ of Wolves
c) A ___________ of Goods.
d) A ___________ of Geese.
e) A ___________ of Musicians.

Contd.pg.no.3.....

Section - C - 20 Marks.
Literature

Q.8. Attempt any One : 1 x 10 = (10)

a) How does advertising damage Nature, Art, Language and Youth ?
b) Discuss after C.E.M. Joad the merits and defects of our Civilization.
c) Explain the Significance of the title of Tagore's Poem "Where the Mind is Without Fear ".

Q.9. Attempt any one of the following : 5 x 1 = (5)

a) How has Shakespeare described the fourth, fifth and sixth stages of a man's life in The Seven Ages of Man ?

b) What problem did Shamnath face with his mother ? How was the problem solved ?

Q.10. Attempt any five of the following : 5 x 1 = (5)

a) Explain the term "the crescendo of success".
b) What are the 'narrow domestic walls' that Tagore speaks of in his poem ?
c) What is a Cockney dialect ?
d) What does Shaw mean by 'Good English" ?
e) Why was the league of Nations formed ?
f) What does Shakespeare mean by "Seeking the bubble reputation/even in the cannon's mouth" ?
g) Why does the mother not wear her jewels in the story ' The Boss Came to dinner' ?

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Indian Statistical Institute
Semester 1 (2002-2003)
B. Stat 1st Year
Mid-semestral Exam
Probability Theory 1

Friday 4.10.2002, 10:30-1:30 Total Points $6 \times 10 = 60$

Answers must be justified with clear and precise arguments.

1. (a) A box contains a white balls and b black balls. If the balls are drawn one by one at random without replacement find the probability that the last ball drawn is white.

(b) In the same set up, for a fixed k, find the probability that the k-th ball drawn is white.

2. If a six digit number is selected by sampling with replacement from the numbers $0, 1, \cdots, 9$ find the probability that the sum of the first three digits will be equal to the sum of the last three digits.

3. Ten manuscripts are arranged in 30 files (3 files for one manuscript). If 6 files are selected at random find the probability that there won't be an entire manuscript among them.

4. A cell contains N chromosomes, between any two of which an interchange of parts may occur. If r interchanges occur (which can happen in $\binom{N}{2}^r$ distinct ways), show that the probability that exactly m chromosomes will be involved is

$$\binom{N}{2}^{-r} \binom{N}{m} \sum_{k=2}^{m} (-1)^{m-k} \binom{m}{k} \binom{k}{2}^r.$$

5. (a) Prove that $P(S_1 \geq 0, \cdots, S_{2n-1} \geq 0, S_{2n} = 0) = 2f_{2n+2}$.

(b) Prove that the probability that before epoch $2n$ there occur exactly r returns to the origin equals the probability that a return takes place at epoch $2n$ and is preceded by at least r returns. (You can use $P(S_1 \neq 0, \cdots, S_{2n} \neq 0) = P(S_{2n} = 0)$)

6. The price of a ticket is Rs. 5. In a line for ticket there are $2n$ persons of whom n have only Rs. 5 bills and the rest n have only Rs. 10 bills. The ticket seller has no change to begin with. What is the probability that not a single customer will be waiting for change?

INDIAN STATISTICAL INSTITUTE
B. Stat. I Year (2002-2003), First Semester
Analysis - I: Mid-Semestral Examination
October 1, 2002. Time: 3 hrs Max. Marks 30.

1. Let S be a non-empty subset of $I\!R$, and let $x_0 \notin S$. Show that x_0 is a limit point of S if, and only if, there is a sequence $\{x_n\}_{n=1}^{\infty}$ such that $x_n \in S$ for each $n = 1, 2, \cdots$ and $\lim_{n\to\infty} x_n = x_0$. [5]

2. Let a and b be two positive real numbers such that $a < b$. Define two sequences $\{a_n\}$ and $\{b_n\}$ by:

$$a_1 = a, \quad b_1 = b,$$
$$a_2 = \sqrt{a_1 b_1}, \quad b_2 = \frac{a_1 + b_1}{2},$$
$$a_3 = \sqrt{a_2 b_2}, \quad b_3 = \frac{a_2 + b_2}{2},$$

and in general,

$$a_n = \sqrt{a_{n-1} b_{n-1}}, \quad b_n = \frac{a_{n-1} + b_{n-1}}{2},$$

for $n = 2, 3, \cdots$. Show:

(a) the sequence $\{a_n\}$ converges.
(b) the sequence $\{b_n\}$ converges.
(c) the two sequences converge to the same real number.

[5]

3. (a) Let $\{a_n\}$ and $\{b_n\}$ be two bounded sequences of positive real numbers. Show that

$$\limsup_{n\to\infty}(a_n b_n) \leq (\limsup_{n\to\infty} a_n).(\limsup_{n\to\infty} b_n).$$

(b) By giving an example, show that the assertion in (a) need not be true if we do not assume $\{a_n\}$ and $\{b_n\}$ to be positive.

[5]

P.T.O

4. (a) Show that the series $\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$ converges, and find the sum of the series.

(b) Check the convergence of the following series:

i. $\sum_{n=1}^{\infty} (\frac{1}{n} - \frac{1}{2^n})$.

ii. $\sum_{n=1}^{\infty} \frac{1}{n^{1+\frac{1}{n}}}$.

(c) Find all the values of a, b with $a > b > 0$ and for which the series

$$\sum_{n=1}^{\infty} \frac{1}{a^n - b^n}$$

converges.

[12]

5. Let $\sum a_n$ and $\sum b_n$ be two series with $a_n > 0$ and $b_n > 0$ for each n. Suppose there is a positive integer N such that

$$\frac{a_{n+1}}{a_n} \leq \frac{b_{n+1}}{b_n} \text{ for } n \geq N,$$

and suppose that $\sum b_n$ converges. Show that $\sum a_n$ also converges.

[3]

INDIAN STATISTICAL INSTITUTE
Mid-semester Examination : (2002-2003)
B-Stat(Hons) I year
Computational Techniques and Programming - I

Date : 27.9.02 Maximum marks : 100 Duration : 195 minutes

Note: *Answer all the questions.*

1. (a) State the conditions under which the method of iteration converges to the solution of an equation.
(b) Prove that the method of iteration indeed converges to the solution of an equation under the stated conditions.
(c) Find an upper bound for the error for the method of iteration.
(d) Find the value of x (correct up to 4 decimal places) using the method of iteration if $x^4 = x + 8$. [3+15+6+8=32]

2. (a)Formulate the problem of finding 5^{th} root of 20 using Newton Raphson Method.
(b) Show that the Newton Raphson method for finding the 5^{th} root of 20 indeed converges to $20^{\frac{1}{5}}$ if the starting point is 40.
(c) Find the value of $20^{\frac{1}{5}}$ correct upto four decimal places using Newton Raphson method with the starting number being 40. [5+5+8=18]

3. (a) Describe the iteration sequence for the method of Regula Falsi for finding root of an equation.
(b) Use Regula Falsi to find the value of $20^{\frac{1}{5}}$ by taking the first two iterates to be the same as the first two iterates of the question 2.(c). Run the algorithm for a maximum of 10 iterations. Stop the algorithm if the difference between two consecutive iterates is less than 10^{-4}. [2+8=10]

4. Convert the following mathematical expressions into valid Fortran codes without using library functions:
(i) $\frac{a}{b} - \frac{c}{d}$ (ii) $\frac{a}{b-c}$ (iii) $\frac{m}{[1-(v/c)^2]^{1/2}}$
(iv) $\frac{a}{bc}$ (v) $\sqrt{\sqrt{\frac{a-b}{c+4d}}}$ [1+1+2+1+2=7]

5. Convert the following mathematical expressions into valid Fortran codes:
(i) $e^b \tan^3(a)$ (ii) $\frac{a^{c-1}}{\ln(b)}$ (iii) $\tan^{-1}\sqrt{\sin^2|a|}$ [2+2+2=6]

(P.T.O)

6. Predict the output of the following program segments.

(a)

```
      INTEGER P
      P = 0
      DO 10 I = 1,2
      DO 10 J = 1,2*I
      DO 10 K = I,J
      P = P+1
      PRINT *,I,J,P
10    CONTINUE
      STOP
      END
```

(b)

```
      REAL F
      Y = 2
      PRINT *,F(F(F(Y)))
      STOP
      END
      REAL FUNCTION F(X)
      F = X**2
      RETURN
      END
```

[4+3=7]

7. Write FORTRAN programs to solve the following problems.

(a) Merge two one-dimensional arrays A and B into a new one dimensional array C, such that even subscripts of C correspond to successive elements of A and the odd subscripts correspond to successive elements of B. [For example, if A = (7,3,9,13) and B = (5,8,10,2), then C = (5,7,8,3,10,9,2,13).] You can make the necessary assumptions for writing the program.

(b) Assign the values of a real 10×10 array A so that each element A(i,j)(where i,j=1,....10) has a value equal to the sum of its indices (e.g. A(6,4) = 10).

(c) Approximate the value of e by the infinite series $e = \sum_{n=0}^{n=\infty} \left(\frac{1}{n!}\right)$ correct up to six decimal places.

(d) Write a program which reads in a list of names(last name, first name) and phone number as a single string, and returns the list in alphabetical order.

[5+3+5+7=20]

X

INDIAN STATISTICAL INSTITUTE

Midsemester Examination : (2002-2003)

B. Stat 1st Year

Statistical Methods -I

Date: 25. 09. 2002 Maximum marks: 100 Duration: 3 hours.

1. (a) Explain the meaning of the following terms, clearly pointing out the differences between them: (i) controlled study, (ii) randomized study, (iii) observational study. Why can causal inference not be drawn from observational studies?

(b) The following are zinc concentrations (in mg/ml) in the blood for two groups of rats. Group A received a dietary supplement of calcium and group B did not. Researchers are interested in variations in zinc level as a side effect of dietary supplementation of calcium.

Group A: 1.31, 1.45, 1.12, 1.16, 1.30, 1.50, 1.20, 1.22, 1.42, 1.14, 1.23, 1.59, 1.10, 1.53, 1.52, 1.17, 1.49, 1.62, 1.29.

Group B: 1.13, 1.71, 1.39, 1.15, 1.33, 1.00, 1.03, 1.68, 1.76, 1.55, 1.34, 1.47, 1.74, 1.74, 1.19, 1.15, 1.20, 1.59, 1.47.

Draw back to back stem and leaf plots, and side by side box plots for the above data, pointing out the mean, median and the other quartiles. Comment on the comparative features of the two distributions. [10 + 15 points]

2. (a) For a 2×2 contingency table with cell frequencies a, b, c and d respectively, find the coefficient of contingency C and Tschuprow's coefficient T in terms of the cell frequencies.

(b) Show that if δ is the deviation between the observed and expected (under independence) frequencies of the AB cell in a 2×2 table, then, with usual notations,

P. T. O

$$(AB)^2 + (\alpha\beta)^2 - (\alpha B)^2 - (A\beta)^2 = \{(A) - (\alpha)\}\{(B) - \beta)\} + 2N\delta.$$

[10 + 5 points]

3. The first four moments of a distribution about the value 4 (i.e. $\frac{1}{n}\sum_{i=1}^{n}(X_i - 4)^r, r = 1, \ldots, 4$) are -1.5, 17, -30 and 108, respectively. Find the first four raw and central moments. [10 + 10 points]

4. In a frequency distribution the classes are of equal width w, the i-th class having frequency f_i and mid point x_i ($i = 1, 2, \ldots, k$). Let

$$f_i' = \sum_{j=1}^{i} f_j, \quad f_i'' = \sum_{j=1}^{i} f_j', \quad F_1 = \sum_{i=1}^{k} f_i', \quad F_2 = \sum_{i=1}^{k} f_i''.$$

Determine the standard deviation s of the variable in terms of w, F_1 and F_2. [15 points]

5. To test the ability of auto mechanics to identify simple engine problems an automobile with a single such problem was taken to 40 different car repair facilities. Only 4 out of the 40 mechanics correctly identified the problem.

   (a) Construct a test of hypothesis to test whether the true proportion of auto mechanics who can determine the problem correctly is less than 0.2. Clearly define your parameter of interest, the null and the alternative hypothesis, and the rejection region.

   (b) Find the type I error of the test you have constructed in part (a).

   (c) Find the power of the test in part (a) when the true proportion of mechanics being able to identify the problem is 0.1.

   (d) On the basis of the above data (4 correct detections in a total of 40), will your test in part (a) reject the null hypothesis or fail to do so. [10+5+5+5 points]

INDIAN STATISTICAL INSTITUTE

PERIODICAL EXAMINATION(2002-03)

B.Stat(Hons) I Year(2002-03)

Vectors & Matrices I

Date: 23.09.2002 **Full Marks: 100** **Time: $2\frac{1}{2}$ hrs**

Note: There are 6 questions carrying 112 marks.
Maximum one can score is 100.

1. (a) Can you find a vector space over $\mathbb{Q}$, the set of rational numbers, which is not a vector space over $\mathbb{R}$?
   (b) Let W_1 and W_2 be subspaces of a vector space V over the field $\mathcal{F}$. When is $W_1 \cup W_2$ a subspace of V ?

[4+8=12]

2. (a) Show that the vectors (a, b) and (c, d) in $\mathbb{R}^2$ are linearly independent iff $ad - bc$ is not equal to 0.

   (b) Let $\mathcal{P}_n$ denote the vector space of all polynomials with real coefficients of degree $\leq n$ ($n \geq 1$). Show that $\{x, 3x^2, 5 + x\}$ is a basis of $\mathcal{P}_2$. What about $\{2x, x^2 - 3x, 2x^2\}$ in $\mathcal{P}_2$? Also prove that $\{1, x - c, (x - c)^2, (x - c)^3\}$, where $c \in \mathbb{R}$. is a basis for $\mathcal{P}_3$.

[6+(5+5+8)=24]

3. Let V be a finite-dimensional vector space over a field. Suppose W_1 and W_2 are subspaces of V with $dim(W_1) + dim(W_2) > dim(V)$, then show that $W_1 \cap W_2 \neq \{\theta\}$. What can you say if we have $dim(W_1) + dim(W_2) = dim(V)$ for any choice of W_1 & W_2 ?

[6+8=14]

4. (a) Let X be a non-empty set and $x_0 \in X$. Let $V = F(X, \mathbb{R})$ be the vector space of all real-valued functions on X. Let $W = \{f \in V : f(x_0) = 0\}$. Show that W is a subspace of V. Does there exist a subspace W' of V which is a complement of W in V?

   (b) Let C be the vector space of all convergent real sequences and let W be the subspace of constant sequences and C_0 the subspace of all sequences convergent to 0. Show that C_0 and W are each other's complements.

[(4+10)+8=22]

P.T.O

5. Let $V = M(n, \mathbb{R})$, the vector space of all $n \times n$ real matrices. If $A = ((a_{ij})) \in V$, then define $trace\,(A) = \sum_{i=1}^{n} a_{ii}$.

Let $W = \{A \in V : trace(A) = 0\}$. Show that W is a subspace of V. Find a basis for W and $dim(W)$.

[5+15=20]

6. Let f be a non-zero linear functional on a vector space V over the scalar field $\mathcal{F}$, and α be an arbitrary scalar . Does there necessarily exist a vector x in V such that $f(x) = \alpha$? If $K(f) = \{x \in V : f(x) = 0\}$, show that $K(f)$ is a subspace of V and has a complement W such that $dim(W) = 1$.

$[5 + 15 = 20]$

STOP