4

INDIAN STATISTICAL INSTITUTE

B. Stat. II Year: 2006-2007 (First Semester)

Mid-Semestral Examination: Analysis III

Date: 04 September 2006; Time: 3 hours; Max. Marks: 100.

Note: You may answer all the questions. But the maximum you can score is 100.

1. (a) Let

$$f(x,y) = \frac{xy^2}{x^2+y^4} \quad \text{if} \quad (x,y) \neq (0,0)$$
$$f(0,0) = 0.$$

Show that the partial derivatives of f exist everywhere on $\mathbb{R}^2$. Show also that f is not continuous at $(0,0)$.

(b) Let f be a real-valued function defined on an open set U in $\mathbb{R}^n$. If the partial derivatives $\frac{\partial f}{\partial x_i}(i = 1,\cdots,n)$ exist on U and if there is $M > 0$ such that $|\frac{\partial f}{\partial x_i}(\boldsymbol{x})| \leq M$ for every $\boldsymbol{x} \in U$ and for every $i = 1,\cdots,n$, then show that f is continuous on U.

[10+15=25]

2. Let L be a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$ and let $f : \mathbb{R}^n - \{\mathbf{0}\} \mapsto \mathbb{R}$ be defined by

$$f(\boldsymbol{x}) = \frac{< L\boldsymbol{x}, \boldsymbol{x} >}{\|\boldsymbol{x}\|^2}.$$

($< \boldsymbol{x}, \boldsymbol{y} >$ is the inner product of $\boldsymbol{x}$ and $\boldsymbol{y}$). Let $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \neq \mathbf{0}$. For any unit vector $\boldsymbol{v} \in \mathbb{R}^n$, find the directional derivative of f at $\boldsymbol{x}$ in the direction $\boldsymbol{v}$. Hence find the derivative of f at $\boldsymbol{x}$.

[15]

3. Let U be an open set in the complex plane $\mathbb{C}$, and let $z_0 = x_0 + iy_0 \in U$. Let $f : U \mapsto \mathbb{C}$, $f(z) = f(x+iy) = u(x,y) + iv(x,y)$ be (complex) differentiable at $z_0 = x_0 + iy_0$. Show that the functions $u(x,y), v(x,y) : \mathbb{R}^2 \mapsto \mathbb{R}$ are differentiable at (x_0, y_0).

[15]

4. Let U be an open set in $\mathbb{R}^3$ and let $f : U \mapsto \mathbb{R}$ have continuous partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ on U. Let $(x_0, y_0, z_0) \in U$ and suppose $f(x_0, y_0, z_0) = 0$ and $\frac{\partial f}{\partial z}(x_0, y_0, z_0) \neq 0$. Show that there is $\epsilon > 0$ and a unique real-valued function $g(x,y)$ defined on $V = (x_0 - \epsilon, x_0 + \epsilon) \times (y_0 - \epsilon, y_0 + \epsilon)$ such that

P.T.O.

(a) $g(x_0, y_0) = z_0$.

(b) $(x, y, g(x, y)) \in U$ for all $(x, y) \in V$.

(c) $f(x, y, g(x, y)) = 0 \ \forall (x, y) \in V$.

(d) g is continuously differentiable on V.

[25]

5. Find, by the method of Lagrange multipliers, the shortest distance between the ellipse $x^2 + 9y^2 = 9$ and the line $x + 5y = 10$.

[10]

6. (a) Let U be a convex open set in $\mathbb{R}^n$ and let $\boldsymbol{f} = (f_1, \cdots, f_n)$ be a continuously differentiable function from U to $\mathbb{R}^n$ satisfying $\frac{\partial f_i}{\partial x_j}(\boldsymbol{x}) = \frac{\partial f_j}{\partial x_i}(\boldsymbol{x})$ for all $\boldsymbol{x} \in U$ and for $i, j = 1, \cdots, n$. Construct a continuously differentiable function $\phi : U \mapsto \mathbb{R}$ such that $\nabla\phi = \boldsymbol{f}$ on U.

(b) Let $\boldsymbol{f} = (f_1, f_2)$ be defined on $\mathbb{R}^2 - \{(0,0)\}$ by:

$$f_1(x, y) = \frac{-y}{x^2 + y^2} \quad f_2(x, y) = \frac{x}{x^2 + y^2}.$$

Show that there does not exist a continuously differentiable function $\phi : \mathbb{R}^2 - \{(0,0)\} \mapsto \mathbb{R}$ such that $\nabla\phi = \boldsymbol{f}$.

[10+10=20]

INDIAN STATISTICAL INSTITUTE

Mid-semestral examination, B.Stat II year

PROBABILITY THEORY- III

06-09-06 Time: Two hours

Maximum you can score is 40 marks. Justify your steps by quoting theorems you are using.

1. Let X_1, X_2, X_3 be i.i.d. Uniform (0,1) random variables. Find the density of their geometric mean, namely, $Z = (X_1X_2X_3)^{1/3}$.

[5]

2. $Y_1 < Y_2 < \cdots < Y_5$ be an order statistic of size five from Uniform (0,1). Put $Z_1 = Y_1, Z_2 = Y_2 - Y_1, \cdots, Z_5 = Y_5 - Y_4$. Find the joint density of $Z_1, \cdots, Z_5$.

Find the density of $Z_2 + Z_4$.

[5+3]

3. The random variables X, Y, Z have joint density

$$f(x, y, z) = Ce^{-\frac{3}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{2}z^2 + xy + xz} \quad \text{for} \quad (x, y, z) \in R^3$$

Find the constant C. Calculate the variance-covariance matrix. Find the conditional density of $Y - X$ given $Z - X$.

[5+3+4]

4. (a) For each n, let X_n be a random variable taking values $n^n, n^{-n}, -n^n$ $-n^{-n}$ each with probability $\frac{1}{4n^2}$ and zero with probability $1 - \frac{1}{n^2}$. Show that $\frac{X_1 + \cdots + X_n}{n} \longrightarrow 0$ a.e.

(b) Let $X_1, X_2, \cdots$ be a sequence of independent random variables. X_1 is Uniform (0,1). For $n \geq 2$, X_n is Uniform $(-\sqrt{3}, +\sqrt{3})$. Show that

$$\frac{X_1 + X_2 + \cdots + X_n}{\sqrt{n-1}} \Rightarrow N(0,1)$$

[4+6]

5. (a) Let $\varphi(t)$ be the characterstic function of a random variable X and suppose that $\varphi(1) = 1$. Show that X is a discrete random variable.

(b) Let $Z = X_1 + iX_2$ where X_1 and X_2 are independent standard normal. Calculate $E(Z^2)$.

[6+4]

INDIAN STATISTICAL INSTITUTE

Mid-Semester Examination : (2006 – 2007)

B. Stat. II Yr.

C & Data Structures

Date: 8th September 2006   Max. Marks: 100   Duration: 120 min.

Answer all the questions.

1. Write a **C** function that will reverse a linked list while traversing it only once. At the conclusion, each node should point to the node that was previously its predecessor; the *head* should point to the node that was formerly at the end, and the node that was formerly first should have a **NULL** link.

   Is the header of a linked list usually a static variable or a dynamic variable? 10 + 2

2. What is the major difference between a linked list and a contiguous list? What are the operations that can be done on a *Stack*?

   A program requires two stacks of same type of data. If two stacks are used separately, one might overflow while the other has sufficient unused space. A neat way to avoid this problem is to put all the space in one array and let one stack grow from one end and the other at the other end in opposite directions. In this way if one stack turns out to be large and the other small, they will still fit, and there will be no overflow until all the space is used. Declare a new data type *DoubleStack* and write algorithms for four operations **PushA**, **PushB**, **PopA**, and **PopB** to handle the two stacks within one *DoubleStack*, the above situation. 2 + 2 + 20

3. Define the term *Queue*. What operations can be done on a *Queue*?

   Write an algorithm for a function which takes a line of text from the terminal and produces a single character output according the following rule. The input text is supposed to consist of two parts separated by a colon ':'. You must use a *Queue* to keep track of the input.

N	No colon in the line.
L	The left part is longer.
R	The right part is longer.
D	The left and right parts have same length but are different
S	The left and right parts are exactly the same.

Examples:

Input	*Output*
Sample Sample	N
Short:Long	L
Sample:Sample	S

2 + 10

4. A *Scroll* is a data structure intermediate to a *Deque* and a *Queue*. In a *Scroll* all additions to the list are at its one end, but deletions can be made at either ends. Now, suppose that data items numbered 1, 2, 3, 4, 5, 6 come in the input stream in this order. By using a *Scroll*, which of the following rearrangements can be obtained in the output order?

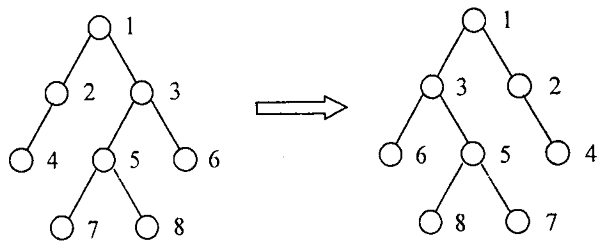
   (a) 1 2 3 4 5 6   **(c)** 1 5 2 4 3 6   **(e)** 1 2 6 4 5 3
   (b) 2 4 3 6 5 1   **(d)** 4 2 1 3 5 6   **(f)** 5 2 6 3 4 1   6 × 2

Turn Over

5. Define the term *Binary Tree*. For a non-empty *Binary Tree* T, if n_0 is the number of leaf nodes and n_2 is the number of nodes with degree 2, prove that $n_0 = n_2 + 1$.

   Write algorithms for deletion and insertion operations on a *Binary Search Tree*. 2 + 3 + 15

6. Write an algorithm that will interchange all left and right subtrees in a *Binary Tree*. (See the example in the following figure.)



   Suppose that you are given two sequences ABCDEFGHI and BCAEDGHFI that correspond to the **prorder** and **inorder** traversals of a *Binary Tree*. Reconstruct the *Binary Tree*. 5 + 10

7. Write an algorithm for finding the height of a *Binary Tree*. 5

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2006-2007

B.Stat. (Hons.) 2nd Year. 1st Semester

Statistical Methods III

Date: September 12, 2006 **Maximum Marks: 60** **Duration: 3 and 1/2 hours**

- This question paper carries 75 points. Answer as much as you can. However, the maximum you can score is 60.
- You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. (Minimal sufficiency in multiple linear regression) Suppose $Y_1, \ldots, Y_n$ are independent with

$$Y_i \sim \mathrm{N}(\beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_p x_{i,p}, \sigma^2),\ i = 1, \ldots, n,$$
$$\boldsymbol{\theta} \stackrel{def}{=} (\beta_0, \beta_1, \beta_2, \ldots, \beta_p, \sigma),\ \beta_0, \beta_1, \beta_2, \ldots, \beta_p \in \mathbb{R},\ \sigma > 0.$$

The matrix $\mathbf{X}$ defined by

$$\mathbf{X} = \begin{pmatrix} 1 & x_{1,1} & x_{1,2} & \cdots & x_{1,p} \\ 1 & x_{2,1} & x_{2,2} & \cdots & x_{2,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{i,1} & x_{i,2} & \cdots & x_{i,p} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_{n,1} & x_{n,2} & \cdots & x_{n,p} \end{pmatrix}_{n\times(p+1)}$$

has rank $p+1$. The entries of the matrix $\mathbf{X}$ are fixed (non-random). Find a minimal sufficient statistic for $\boldsymbol{\theta}$. [7]

2. (A general problem of estimation of which a special case is one of estimating the probability that a particular telephone operating center, out of n such, remains idle throughout the day) Suppose $X_1, \ldots, X_n \stackrel{\text{i.i.d.}}{\sim}$ Poisson(θ), $\theta \in \Theta = (0, \infty)$. Consider the problem of estimating $\psi(\theta) = \exp(-a_0\theta)$, where a_0 is a known non-zero constant.

(a) Assuming $S \stackrel{def}{=} X_1 + \cdots + X_n$ to be sufficient for θ and complete, find the UMVUE of $\psi(\theta)$.

[P.T.O.]

(b) Denote the UMVUE of $\psi(\theta)$ by $T_{0,n}$. Is $T_{0,n}$ consistent for $\psi(\theta)$? Give reasons.

(c) Find the MLE of $\psi(\theta)$. Denote it be $T_{1,n}$. Find the expression for the MSE of $T_{1,n}$.

(d) Show that $T_{1,n}$ is A.N. $(\exp(-a_0\theta), \exp(-2a_0\theta)a_0^2\theta/n)$. [5+3+(1+4)+4 = 17]

3. (Method of moments estimation of gamma shape parameter) Suppose $X_1, \ldots, X_n$ are i.i.d. having gamma distribution with scale parameter θ_1 and shape parameter θ_2. Write $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\theta_1, \theta_2 > 0$. The parameter space is $\Theta = (0,\infty) \times (0,\infty)$. Both θ_1 and θ_2 are unknown.

(a) Obtain a method of moments estimate of θ_2.

(b) Denote the estimate you obtain in (a) by T_n. Decide, with adequate reasons, if T_n is consistent. [5+4 = 9]

4. (Maximum likelihood estimation of reliability in the case of normal distribution with known standard deviation) Suppose $X_1, \ldots, X_n \overset{\text{i.i.d.}}{\sim} N(\theta, \sigma_0^2)$, $\theta \in \Theta = \mathbb{R}$, and σ_0 is a known positive number. Consider the problem of estimating $\psi(\theta) = P_\theta(X_1 > u_0)$, where u_0 is a known real number.

(a) Find the MLE of $\psi(\theta)$.

(b) Denote the MLE of $\psi(\theta)$ by T_n. Show that the MSE of T_n is given by

$$\Phi_2\left(\delta\sqrt{\frac{n}{n+1}}, \delta\sqrt{\frac{n}{n+1}}; \frac{1}{n+1}\right) - 2\Phi(\delta)\left[\Phi\left(\delta\sqrt{\frac{n}{n+1}}\right) - \frac{1}{2}\Phi(\delta)\right], \quad \delta \overset{def}{=} \frac{u_0 - \theta}{\sigma_0},$$

where

$$\Phi(u) \overset{def}{=} P(Z \le u), \quad Z \sim N(0,1),$$
$$\Phi_2(u_1, u_2; \rho) \overset{def}{=} P(Z_1 \le u_1, Z_2 \le u_2), \quad (Z_1, Z_2) \sim N_2(0,0,1,1,\rho).$$

[2+8 = 10]

5. (Maximum likelihood estimation of Cauchy location parameter when the scale parameter is known) Let $X_1, \ldots, X_n$ denote a random sample of size 50 from a Cauchy distribution with unknown location parameter θ and known scale parameter equal to 1.0. The parameter space is $\Theta = \mathbb{R}$. Suppose we want to find the the MLE of θ.

(a) Derive the likelihood equation.

(b) Find the expressions for successive iterations if we want to solve the likelihood equation by Newton-Raphson method. What is an initial choice for θ and why?

(c) Find the expressions for successive iterations if we want to solve the likelihood equation by the method of scoring. [Note. You may assume the expression for the Fisher information $I(\theta)$.]

(d) Show that the observations $X_1, \ldots, X_n$ can also be considered to have been generated first by generating n i.i.d. observations, denoted $\tau_1, \ldots, \tau_n$, from Gamma(1/2, 1/2) distribution and then generating n independent observations, one each from $N(\theta, 1/\tau_i)$, $i = 1, \ldots, n$.

(e) Use (d) above to explain how the EM algorithm can be used to find the MLE of θ. [Note. Your answer should contain clear descriptions of the E-step and the M-step.] [3+(3+2)+2+6+7 = 23]

6. State the *invariance principle*. Illustrate its use through an example where the problem is one of estimating a parameter. [2+7 = 9]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE
MID-SEMESTER EXAMINATION: 2006-07
B. STAT. II: 2006-07
ECONOMICS-I

Date: 15/09/06 Maximum Marks: 40 Duration: 3 Hours

Answer Each Group in a Separate Script.

Group A

This part of the question carries 25 marks. Answer as much as you can. The maximum you can score is 20.

1. (a) Define price elasticity of demand. What are perfectly inelastic, inelastic, unit elastic and perfectly elastic items? Illustrate graphically.
   (b) What are 'necessary', 'inferior' and 'luxury' items? Give examples.
   [5+6=11]
2. (a) Plot the indifference maps for each of the following cases, indicating the direction of preferences in each case.
   (i) Two commodities purchased by the consumer are 'bads'.
   (ii) A consumer is saturated with some positive amount of commodity 1, but for commodity 2, which is a 'good', saturation point does not exist.
   (iii) Commodity 1 is 'good' and commodity 2 is 'garbage'.

   (b) Write down formally the consumer's utility maximisation problem. Show that the utility maximising bundle must lie on the budget line. Under what conditions is the utility maximising bundle unique? Justify your answer.
   [6+8=14]

Group B

Answer as many questions as you can. Each question carries 4 marks. However, you cannot score more than 20 marks.

P. T. O

1. A firm has the production function: $f(x_1,x_2,x_3,x_4)=\min\{x_1,x_2\}+\min\{x_3,x_4\}$ where x_i is the amount of the i-th factor. The factor price vector is (1,2,3,4). What is the vector of conditional factor demands to produce 1 unit of output?

2. Consider the production function $f(x_1,x_2)=\min\{ax_1,bx_2\}$. Find the cost function associated with this production function. What will be the cost function if the production function is $f(x_1,x_2)=ax_1+bx_2$.

3. Consider a profit-maximizing firm which takes input and output prices as given. Show that its profit function (i.e., profit as a function of input and output prices in equilibrium) is homogeneous of degree one.

4. Show that if the production function is homogeneous, the expansion path will be a straight line passing through the origin.

5. Production of a commodity involves two inputs whose marginal productivities are, respectively, $MP_1=x_2-ax_1$ and $MP_2=x_1-bx_2$; $a,b>0$. Derive the expression of the production function.

6. Given the production function,

$$f(x_1,x_2)=A[ax_1^{-\beta}+(1-a)x_2^{-\beta}]^{-\frac{1}{\beta}} \quad ,A>0,\ 0<a<1,\ \beta>-1,\ \beta\neq 0$$

show that $f(x_1,x_2)\to\min\{x_1,x_2\}$ as $\beta\to\infty$.

7. Prove that if $f(x)$ is a homothetic technology and x and x' produce the same level of output, then tx and tx' must also produce the same level of output where $t\geq 0$ and x and x' are two vectors of inputs.

INDIAN STATISTICAL INSTITUTE

Mid-semestral Examination 2006-07

B.Stat II Year

Physics I

Maximum Marks 100 — Date: 15.9.06 — Duration 3 hours

Note: Use different Answer Sheets for different groups

Group A

Attempt Question No. 1 and any five from the rest

1a) Show that the vectors
A=3i+ 2j+k,
B=i- 3j+ 5k,
C=2i+ j- 4k
form a right angled triangle.

2

1b) Interpret the symbol ∇.**A**.
Explain its difference with **A**.∇.

2

1c) Write the Gauss's law in its integral form. Explain the physical significance of the law. Then derive its differential form.

3

1d) Explain why a conductor is an equipotential and also say why the charge density is zero inside it.

3

2a) Write all the conditions followed from the definition of irrotational fields.

2

2b) A particle moves so that its position vector is given by **r**=cosω t**i**+ sinω t**j** where ω is a constant. Show that
(i) the velocity **v** is perpendicular to **r**.

P.T.O

(ii) the acceleration **a** is directed toward the origin and has magnitude proportional to the distance from the origin.
(iii) **r**× **v** is a constant vector.
Can you make a comment on the motion of this particle and what is this acceleration called?

6

3a) Show that if ϕ is any solution to Laplace's equation then $\nabla\phi$ is a vector which is both solenoidal and irrotational.

3

3b)Show that the vector $\mathbf{F} = yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}$ can be written both as the gradient of a scalar and the curl of a vector. Find scalar and vector potentials for this function.

5

4. You have a vacuum diode. The anode is held at a positive potential V_0. Assuming the one dimensional (x-directional)motion of electrons find out the relation of charge density ρ and v, the speed of the electrons. Hence deduce the current-potential relationship in the steady state and discuss about its nature.

8

5. The electric potential of some configuration is given by

$$\phi(\mathbf{r}) = A\frac{e^{-\lambda r}}{r}$$

where A and λ are constants. Find the electric field $E(\mathbf{r})$. Then show that

$$\oint_S \mathbf{E}.d\mathbf{a} + \frac{1}{\lambda^2}\int_V \phi d\tau = \frac{q}{\epsilon_0}$$

where S is the surface, V the volume, of any sphere centered at q.

8

6. A point charge Q is situated a distance D ($D \geq R$) from the centre of a grounded conducting sphere of radius R. Find the potential outside the sphere.

8

7a). Find the electric field outside a uniformly charged solid sphere of radius R and total charge q.

4

7b). Find the electric field a distance z above one end of a straight line segment of length L, which carries a uniform line charge λ.

4

8a) State and explain Stoke's theorem.

2

8b) Evaluate $\oint_C (y - sinx)dx + cosxdy)$, where C is the path OABO with the vertices $O(0,0), A(\pi/2, 0)$ and $B(\pi/2, 1)$ directly.

3

8c) Find the electric potential at points far from a dipole. Give your result in terms of the dipole moment.

3

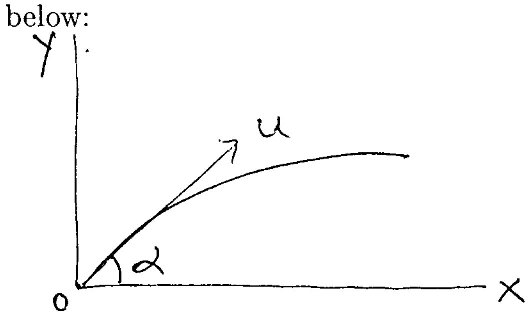
Group B

Answer all questions :

1. An inextensible string of negligible mass hanging over a smooth peg at the point A connects the mass m_1 on a frictionless inclined plane of inclination θ to the horizontal, to another mass m_2 which is hanging vertically from the peg at A. Use D'Alembert's principle to show that the masses will be in equilibrium if $m_2 = m_1 \sin\theta$. (10)

2. A particle of mass M moves on a plane in the field of force given by (in polar coordinates) $\mathbf{F} = -\hat{\mathbf{n}}_\mathbf{r}\, k\, r\cos\theta$, where k is a constant and $\hat{\mathbf{n}}_\mathbf{r}$ is a unit vector in the radial direction. Write the Lagrangian and Lagrange's equation of motion. (10)

3. A particle of mass m is projected with initial velocity u at an angle α with the horizontal as shown below:



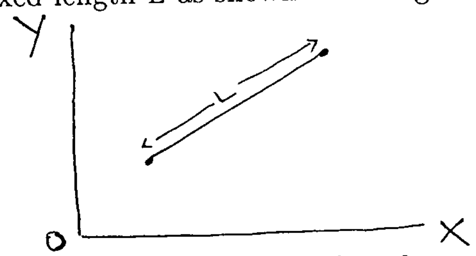
Using Lagrange's equations, show that the path of the projectile is

$$y = xtan\alpha - \frac{g}{2}\frac{x^2}{u^2cos^2\alpha}$$

g being the acceleration due to gravity. (10)

P.T.O

4. Consider a system of two particles (confined to the horizontal plane XY) joined by a massless rod of fixed length L as shown in the figure.



Suppose that the system is so constrained that the centre of the rod cannot have a velocity component perpendicular to the rod. Write down the constraint equations.

(5)

5. Show that the total kinetic energy T of a system of particles is given by $T = \frac{1}{2}MV^2 + \frac{1}{2}\sum_i m_i \left(\frac{d\mathbf{r}'_i}{dt}\right)^2$, where

M is the total mass of the system of particles.
$\mathbf{V}$ is the velocity of the centre of mass relative to the origin O
$\mathbf{r}'_i$ is the radius vector from the center of mass to the ith particle. (5)

6. Two masses m_1 and m_2 are interacting by a potential energy $V(\mathbf{r})$.

   (i) Write down the Lagrangian in terms of the centre of mass $\mathbf{R}$ and relative position $\mathbf{r}$.

   (ii) Using $\mathbf{R}$ and $\mathbf{r}$ as generalized coordinates obtain Lagrange's equations.

   (iii) Calculate the total angular momentum in the centre of mass reference frame and give the physical significance. (1+2+2)

7. Consider a particle of mass m and charge q moving with uniform velocity in electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$ respectively. You are given that $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$ and $\mathbf{B} = \nabla \mathrm{x} \mathbf{A}$, $\mathbf{A}(\mathbf{r}, t)$ and $V(\mathbf{r}, t)$ being electromagnetic scalar and vector potentials respectively.
Write down the Lagrangian for the system.
Hence show that $m\ddot{x} = q(E_x + \dot{y}B_z - \dot{z}B_y)$ (1+4)

Mid-Semestral Examination (B. Stat-II, Biology I, 2006), Date: Sept 15, 2006

Answer any five; All questions carry equal marks; Full marks = 40; Time = 2.5 hours

1. a). One theory suggests that mitochondria arose during evolution from the invasion of small oxygen-using cells into large non-oxygen using cells. What evidence supports this theory? [4]

   b). What kind of endoplasmic reticulum would you expect to predominate in pancreatic cells that secrete digestive enzymes? [4]

2. a). The liver detoxifies many ingested substances by oxidizing them with molecular oxygen in the presence of an enzyme catalyst. One possible product of these reactions is hydrogen peroxide. How does the organism protect itself from the toxic effects of hydrogen peroxide produced? [4]

   b). If pancreatic cells synthesizing digestive enzymes for export are supplied with radioactive amino acids, the pathway of proteins from synthesis to export can be followed. In what order does the radioactivity appear in the organelles involved in this pathway? [4]

3. a). What are the major differences between plant and animal cells with respect to structure and functions of various organelles? [4]

   b). What is turgor pressure and what advantage do plant cells get out of it? [4]

4. a). What are the three elements of the cytoskeleton? How do they differ from each other (a) in diameter and (b) in composition? [4]

   b). What are the functions of each type of cytoskeletal element? What is a reason for the lack of a cytoskeleton in prokaryotic cells? [4]

5. a). Mention differences between aerobic and anaerobic oxidation of glucose in humans. [3]

   b). Which of the two processes is efficient in synthesizing energy-rich molecules and why? Discuss with examples. [3]

   c). Write down two reactions in TCA cycle where NADH and $FADH_2$ molecules are produced. [2]

P.T.O

6. a). How do cells in our body get amino acids? Describe elaborately mentioning the different steps. [3]

b). What are the functions of amino acids in our body under normal and starving conditions? Discuss with examples. [3]

c). "Some plant proteins are inferior to animal proteins". Explain why? And how they could be made equivalent to animal proteins. [2]

INDIAN STATISTICAL INSTITUTE
B. Stat. II Year: 2006-2007 (First Semester)
Semestral Examination: Analysis III

Date: 24/11/2006; Time: 3 hours; Max. Marks: 100.

Note: This paper carries 110 marks. You may answer all the questions. But the maximum you can score is 100.

1. Evaluate the line integral

$$\int_C \frac{(x+y)dx-(x-y)dy}{x^2+y^2}.$$

where C is the circle $x^2+y^2=a^2$ traversed once in the anticlockwise direction.

[10]

2. (a) Let S be a bounded subset of $\mathbb{R}^n$ such that the Jordan content $c(S)=0$. Let $f : S \mapsto \mathbb{R}^n$ be a function such that $\|f(\boldsymbol{x})-f(\boldsymbol{y})\| \leq M\|\boldsymbol{x}-\boldsymbol{y}\|$ (for some $M>0$), for all $\boldsymbol{x},\boldsymbol{y} \in S$. Show that $c(f(S))=0$.

   (b) Let $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a linear, one-one map. Show that for any bounded Jordan measurable subset S of $\mathbb{R}^n$, $T(S)$ is Jordan measurable.

[10+15=25]

3. Let H be the parallelogram in $\mathbb{R}^2$ whose vertices are $(1,1),(3,2),(4,5)$, and $(2,4)$. Find the affine map T which sends $(0,0)\mapsto(1,1),(1,0)\mapsto(3,2),(0,1)\mapsto(2,4)$. Use T to convert the integral $\alpha=\iint_H e^{x-y}\,dx\,dy$ to an integral over $[0,1]\times[0,1]$ and thus compute α.

[10]

4. By transforming to polar coordinates, find the value of the improper integral

$$\int_0^{\alpha\sin\beta}\{\int_{y\cot\beta}^{\sqrt{\alpha^2-y^2}}\log(x^2+y^2)dx\}dy$$

$(0<\beta<\pi/2)$.

[10]

P.T.O

5. Suppose $u(x,y)$ and $v(x,y)$ are continuously differentiable real-valued functions defined on an open set containing $D = \{(x,y) : x^2+y^2 \le 1\}$. Let $F(x,y) = (v(x,y), u(x,y))$ and $G(x,y) = (\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x} - \frac{\partial v}{\partial y})$. Find

$$\iint_D F \cdot G \; dxdy$$

if it is given that $u(x,y) = 1$ and $v(x,y) = y$ on $\{(x,y) : \; x^2+y^2 = 1\}$.

[15]

6. Compute the area of the torus obtained by rotating the circle

$$x = 0, \quad y - a = r\cos\theta, \quad z = r\sin\theta \quad (0 \le \theta < 2\pi)$$

about the z-axis. (Here $r < |a|$.)

[15]

7. Use Stokes' theorem to evaluate the line integral

$$\int_C (y\; dx + z\; dy + x\; dz)$$

where C is the curve of intersection of the sphere $x^2+y^2+z^2 = a^2$ and the plane $x+y+z=0$.

[10]

8. Let ϕ be a real-valued function, which is twice continuously differentiable on an open set containing the ball $\{(x,y,z) : \; x^2+y^2+z^2 \le 1\}$. Assume that ϕ is never zero and satisfies,

$$\|\nabla\phi\|^2 = 4\phi, \quad \text{and} \quad \operatorname{div}(\phi\nabla\phi) = 10\phi.$$

Evaluate the surface integral

$$\iint_S \frac{\partial\phi}{\partial n}\, dS,$$

where S is the surface of the unit ball with centre at the origin, and $\frac{\partial\phi}{\partial n}$ is the directional derivative of ϕ in the direction of the unit outer normal to S.

[15]

INDIAN STATISTICAL INSTITUTE
First Semester Examination: 2006-07
B. Stat. II Year
Probability Theory III

Date: 28.11.06 **Maximum Marks: 60** **Duration: 3 Hours**

This paper is set for 66 Marks
Maximum you can score is 60 marks.
Conclusions without proper justifications will be treated as guess work.

1. X, Y are independent standard normal. Show that the expected value of their maximum is $1/\sqrt{\pi}$.

[6]

2. Suppose that $Y_1 < Y_2 < \ldots < Y_n$ is order statistic of size n from uniform (0,1). Put $Z_i = (Y_i/Y_{i+1})^i$ for $1 \le i \le n-1$. Show that $Z_i, \ldots, Z_{n-1}$ is a sample of size n-1 from uniform (0,1).

[6]

3. $(X_i)_{i\ge 1}$ are i.i.d random variables. $P(X_i = 1) = p = 1 - P(X_i = 0)$. Put $Y_i = X_i X_{i+1}$.

(a) Calculate the mean and variance of Y_i.

(b) Calculate var $\left(\sum_1^n Y_i\right)$.

(c) Show that $\frac{1}{n}\sum_1^n Y_i \to EY_1$ in probability.

[3+5+4]

4. (a) If X_n is a Poisson random variable with parameter n, show that $\frac{X_n - n}{\sqrt{n}} \Rightarrow N(0,1)$.

(b) Show that $e^{-n}\sum_{k=0}^{n}\frac{n^k}{k!} \to \frac{1}{2}$ as $n \to \infty$.

(c) Let f be a bounded continuous function on $[0,\infty)$. Let α_n be the n fold integral,

$$\alpha_n = \int_0^\infty \!\!\ldots\! \int_0^\infty f\left(\frac{x_1 + \ldots + x_n}{n}\right) e^{-(x_1+\ldots+x_n)}\, dx_1\, dx_2 \ldots dx_n .$$

Show that $\alpha_n \to f(1)$.

[3+4+5]

P.T.O.

5. Suppose that $\varphi_1, \ldots, \varphi_5$ are characterstic functions and $\lambda_1, \ldots, \lambda_5$ are strictly positive numbers. Show that $\Psi(t) = e^{\sum_{k=1}^{5} \lambda_k [\varphi_k(t) - 1]}$ is a characterstic function. [6]

6. Let $Y_1, \ldots, Y_n$ be i.i.d N (0,1) and $Y = (Y_1, \ldots, Y_n)'$. Let A be an $n \times n$ symmetric matrix. Show that $Y'AY$ is a χ^2 random variable iff A is idempotent, that is, $A^2 = A$. When it is χ^2, what is its degrees of freedom. [6]

7. Consider a branching process starting with one individual (generation 0). The number of offspring of an individual is either 0 or 1 or 2 or 3 with equal probabilities. What is the probability that the process survives for ever? what is the expected number of individuals in the third generation? [6]

8. Let $(X_n)_{n \geq 1}$ be i.i.d uniform (0,1) and $(S_n)_{n \geq 0}$ be their partial sum sequence, that is, $S_0 = 0$ and for $n \geq 1$, $S_n = \sum_{1}^{n} X_i$.

a) Show that $S_n \to \infty$ almost surely.

b) Calculate $P(S_k \leq 1)$.

c) Define an integer valued random variable N by: N is the first time the partial sum exceeds one. That is, $N = k$ iff $S_{k-1} \leq 1$ & $S_k > 1$. Show $EN = e$.

[3+4+5]

□□□□□

INDIAN STATISTICAL INSTITUTE

First Semestral Examination: 2006–2007

B.Stat. (Hons.) 2nd Year. 1st Semester

Statistical Methods III

Date: December 05, 2006 **Maximum Marks: 100** **Duration: 4 hours**

- This question paper carries 113 points. Answer as much as you can, but you must answer part 2 of question # 5. The maximum you can score in all questions excluding part 2 of question # 5 is 84.
- You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

---

1. Suppose $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, $\boldsymbol{\theta} \equiv (\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$. Let the observations be grouped into disjoint sets (classes) $\mathcal{X}_1, \ldots, \mathcal{X}_k$ such that $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset \, (i \neq j)$ and that $\cup_i \mathcal{X}_i = \mathbb{R}$. The precise form of each $\mathcal{X}_j$ is given by

$$\mathcal{X}_j = \begin{cases} (-\infty, u_1) & \text{if } j = 1, \\ [u_{j-1}, u_j) & \text{if } j = 2, \ldots, k-1, \\ [u_{k-1}, \infty) & \text{if } j = k, \end{cases}$$

where the quantities $u_1 < u_2 < \cdots < u_{k-1}$ are known to us. Denote by N_j, the random variable giving the number of observations falling in the class $\mathcal{X}_j$, $j = 1, \ldots, k$. The data consist of N_j, $j = 1, \ldots, k$. Notice that $\sum_{j=1}^{k} N_j = n$. Describe how you will estimate the MLE of $\boldsymbol{\theta}$, based on these data and by employing the EM algorithm. [11]

[**Hint.** You can assume that the conditional pdf of X_1 given $N_1 = n_1, \ldots, N_k = n_k$ is given by

$$f(x) = \frac{n_j}{n} \frac{\phi(\frac{x-\mu}{\sigma})}{\sigma \cdot \left[\Phi(\frac{u_j - \mu}{\sigma}) - \Phi(\frac{u_{j-1} - \mu}{\sigma})\right]}, \quad x \in \mathcal{X}_j, \; j = 1, \ldots, k,$$

where $\phi(\cdot), \Phi(\cdot)$ are the pdf and cdf of the standard normal distribution, respectively.]

2. Let $X_1, \ldots, X_n$ be independent and each distributed uniformly over the set of integers $\{1, 2, \ldots, \theta\}$. Find a UMP test for testing $H_0 : \theta = \theta_0$, at level $1/\theta_0^n$ against the alternative $\theta \neq \theta_0$. [9]

3. Suppose $X_{i,1}, \ldots, X_{i,n}$ are i.i.d. Poisson(θ_i), $\theta_i \in (0, \infty)$, $i = 1, 2$. Also, suppose that all the $X_{i,j}$'s are independent. Develop a suitable test for $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 > \theta_2$ based on $S_i \stackrel{def}{=} X_{i,1} + \cdots + X_{i,n}$, $i = 1, 2$. [9]

P.T.O.

4. Suppose $(X_{i,1}, X_{i,2})$, $i = 1, \ldots, n$, are i.i.d. $N_2(\theta_1, \theta_2, \sigma_1^2, \sigma_2^2, \rho)$, $\theta_1, \theta_2 \in \mathbb{R}, \sigma_1, \sigma_2 > 0, \rho \in (-1, 1)$. Define $T_i \stackrel{def}{=} \sum_{l=1}^n X_{i,l}$, $S_{ij} \stackrel{def}{=} \sum_{l=1}^n (X_{i,l} - T_i/n)(X_{j,l} - T_j/n)$, $i, j = 1, 2$. Based on $T_1, T_2, S_{11}, S_{22}, S_{12}$, develop a suitable test for $H_0 : \sigma_1 = \sigma_2$ against $H_1 : \sigma_1 \neq \sigma_2$. Obtain an expression for the P-value. [11+3 = 14]

5. Suppose $Y_1, \ldots, Y_n$ are independent with $Y_i \sim N(\beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_p x_{i,p}, \sigma^2)$, $i = 1, \ldots, n$, $\beta_0, \beta_1, \beta_2, \ldots, \beta_p \in \mathbb{R}, \sigma > 0$. Denote by **A**, the $n \times (p+1)$ matrix $((a_{ij}))$ defined as follows : $a_{ij} = 1 \; \forall \; i$ if $j = 1$, and for $j > 1$, $a_{ij} = x_{i,j-1}, i = 1, \ldots, n$. Assume that **A** has rank $p + 1$ $(n > p + 1)$. The entries of the matrix **A** are fixed (non-random).

(1) Suppose we are interested in testing the following hypothesis about the parameter $\beta \stackrel{def}{=} (\beta_0, \beta_1, \ldots, \beta_p)^T$:

$$H_0 : \beta_1 = \cdots = \beta_p = 0 \text{ against } H_1 : H_0 \text{ is false.}$$

(a) Denote by SS_{Reg}, SS_{Res}, the *regression/model sum of squares*, and *residual/error sum of squares*, respectively. Argue that a suitable test statistic for testing H_0 against H_1 is the following:

$$F \stackrel{def}{=} \frac{SS_{Reg}/p}{SS_{Res}/(n-p-1)},$$

large values of F being significant.

(b) Show that under H_0, $F \sim F_{p,n-p-1}$.

(c) Argue that the critical region for testing H_0 against H_1 is given by the following:

we reject H_0 if $F > F_{\alpha;p,n-p-1}$,
where $F_{p;d_1,d_2} \stackrel{def}{=} (1-p)$-th quantile of F distribution with d_1, d_2 df's.

(d) Find the non-null distribution of F.

(2) Suppose now that we are interested in inferring about the parametric function $\psi \equiv \psi(\beta_0, \ldots \beta_p) \stackrel{def}{=} \sum_{i=0}^p c_i \beta_i$, where $c_0, \ldots, c_p$ are known constants.

(a) Find, with adequate justification, an appropriate estimate of ψ.

(b) Obtain an expression for the estimated mean square error of your estimate in (a).

(c) Discuss how you will test the hypothesis $H_0 : \psi = 0$ against $H_1 : \psi \neq 0$.

[(8+8+2+10)+(4+5+7) = 44]

6. (1) What do you mean by a *variance stabilizing transformation*?

(2) Suppose $X_1, X_2, \ldots \stackrel{i.i.d.}{\sim}$ Poisson(λ), $\lambda > 0$. Define $T_n = \sum_{i=1}^n X_i/n$. Find a variance stabilizing transformation $h(T_n)$ of T_n. [3+7 = 10]

7. (1) A time series with a periodic component can be constructed from

$$x_t = u_1 \sin(2\pi\nu_0 t) + u_2 \cos(2\pi\nu_0 t),$$

where u_1, u_2 are independent random variables with zero means and $E(u_1^2) = E(u_2^2) = \sigma^2$. The constant ν_0 determines the period or time taken by the process to make one complete cycle. Show that this series is weakly stationary. Find its autocovariance function.

(2) Describe briefly how you can estimate the trend of a time series by method of moving average.

(3) Describe briefly how you can interpret the correlogram of a time series.

[(4+4)+4+4 = 16]

****** *Best of Luck!* ******

Indian Statistical Institute

First Semestral Examination (B. Stat-II, Biology I, 2006)

Date: 8.12.06

Answer any five; All questions carry equal marks; Full marks = 50; Time = 2.5 hours

1. (a) Distinguish protein from DNA chemically and functionally. (5)

   (b) State the functions of the three different types of RNA used in protein synthesis. (5)

2. What are recessive allele and disease? If both husband and wife are carriers of a recessive allele for albinism, determine the chances of the occurrence of the following, in a family of four children:

   (a) all four normal

   (b) three normal and one albino

   (c) two normal and two albino and

   (d) the first child be a normal girl (2X5)

3. (i) Nucleotide compositions of four different viruses were found to be: (a) 35% A, 35% T, 15% G and 15%C; (b) 35% A, 15% T, 20% G and 30% C; (c) 35% A, 30% U, 30% G and 5% C. Mention the physical nature of the nucleic acids present in these viruses. (3)

   (ii) What will be the expected genotypes and phenotypes of the children from the couples having the genotypes, (a) A/A and B/B, (b) A/B and O/O, (c) A/O and B/O and (d) A/O and O/O? (A, B and O are alleles for blood group) (7)

4. (a) Of father and son, if both have defective color vision, then which of the parents do you think could have transmitted the defective allele to the son? Show the genotypes with pedigree. (5)

   (b) A normal woman, whose father had hemophilia, marries a normal man. What is the chance that their first child will have hemophilia? (5)

P. T.O

(Both defective color vision and hemophilia are controlled by X-linked recessive alleles.)

5. Mention the functions of the codons in protein synthesis. What do you mean by initiation and termination codons and how many of them should be present in an active gene and why? How many different DNA sequences are possible with 6 codons: ATG, TAA, AGG, GGG, CGT, TAC such that the sequences start with the codon ATG and end with the codon TAA. (3+3+4)

6. (a) Explain mitosis and meiosis with examples. In a cross of "Aa Bb X Aa Bb", what fraction of the progeny will have recessive genotype for at least one gene?

(b) Explain why two brothers from the same parents are not genetically identical (twin should be excluded). (3+4+3)

7. a) The function of circulating white blood cells is to engulf invading bacteria in a process known as phagocytosis. Suggest what happens to the bacteria after they are engulfed. (5)

b) What are enzymes? Discuss the special features of enzyme-catalyzed reactions. (5)

INDIAN STATISTICAL INSTITUTE

First Semestral Examination (2006-07)
B. Stat II year
Physics I

Date: 8.12.06

Maximum Marks 100 Duration 3 hours

Note: Use different Answer Sheets for different groups

Group A

Answer any five

1. A metal sphere of radius r_1 carries a charge q. It is surrounded, out to radius r_2, by linear dielectric of permittivity ϵ.

(a) Find the potential at the centre (relative to infinity).

(b) Compute the surface(outer and inner) bound charges. [6+4]

2. A point charge q is brought a distance d away from an infinite, conducting, grounded plane.

(a) Find the surface charge density induced on the plane .

(b) Compute the total induced charge on the plane. [6+4]

3 (a) Find the magnetic field a distance h from a long straight wire carrying a steady current I (use Ampere's law).

(b) Find the force per unit length between two long, parallel wires a distance d apart, carrying currents I_1 and I_2 when i) I_1 and I_2 are in the same direction, ii) I_1 and I_2 are in the opposite direction.

(c) Show that $\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r})$ is a possible vector potential for a uniform magnetic field $\mathbf{B}$ where $\mathbf{r}$ denotes the position of the field point. [3+4+3]

4 (a) What is the universal flux rule?

(b) Explain how Maxwell fixed the correction to Ampere's law.

(c) Starting from Maxwell's equation derive

(i) Coulomb's law and (ii) Continuity equation [1+3+(3+3)]

P.T.O

5 (a) You have two loops of wire at rest. A steady current I is passed through the loop 1. Show that the the mutual inductance of these two loops is a geometrical quantity. How the mutual inductance will be affected if the same current is passed through the loop 2 instead of loop 1.

(b) If the current I in a loop is changed with time what will be the e.m.f. induced in that loop?

(c) What is the unit of inductance?

(d) Find the self inductance of a toroidal coil with rectangular cross section (inner radius a, outer radius b and height h) which carries a total of N turns.
(Given, Flux through a single turn is $\frac{\mu_0 N I h}{2\pi} ln(\frac{a}{b})$.)

[(4+2)+1+1+2]

6 (a) Show that the magnetic vector potential due to magnetic dipoles is proportional to $1/r^2$

(b) Using Maxwell's equation (in source free space with $\epsilon_0 = \mu_0 = 1$) obtain

$$\frac{\partial}{\partial t}\left(\frac{E^2+B^2}{2}\right) + \nabla \cdot (\mathbf{E} \times \mathbf{B}) = 0$$

[5+5]

7 (a) Current I is flowing through a wire of length L and V is the potential difference between the ends of the wire. Using Poynting's theorem find out the energy per unit time passing in through the surface of the wire and show that it gives the Joule heating law.

(b) Find the ratio of the displacement current and conduction current when an alternating current $\mathbf{E} = E_0 cos\omega t$ is applied to a conductor.

[5+5]

Group B

Answer all questions :

1(a). Find the equation of motion for the total angular momentum of a conservative system of particles in which a cyclic coordinate q_j is such that dq_j corresponds to a rotation of the system of particles around some axis. Hence show that the conservation of the conjugate momentum corresponding to the cyclic coordinate corresponds to conservation of angular momentum. Give relevant diagram. [7]

1(b). A particle moves in a conservative field of force produced by a homogeneous mass distribution uniformly distributed in the half plane $x = 0$, $y > 0$. The force generated by a volume element of the distribution is derived from a potential that is proportional to the mass of the volume element and is a function only of the scalar distance from the volume element. State the conserved quantities in the motion of the particle. [3]

2(a). Two spaceships A and B are moving in opposite directions. An observer on Earth measures the speed of A to be v_1c and the speed of B to be v_2c where v_1,v_2 are constants such that $v_1 < 1$, $v_2 < 1$. Find the velocity of B with respect A. [5]

2(b). An astronaut takes a trip to a planet located at 8 lightyears from earth. The astronaut measures the time of the one-way journey to be 6 years. If the spaceship moved at a constant speed of 0.8c, how can 8 lightyear distance be reconciled with the 6years duration measured by the astronaut? [5]

3(a). A particle of unit mass moves in a uniform gravitational field. Assume the time dependence of the position of the particle as $x(t) = At^2 + Bt$ where A, and B are constants. Is the action really the smallest for the actual path given above? [5]

3(b). Suppose we have a thread of fixed length l, the endpoints of which lie along the x-axis . We wish to find the shape of the curve that this thread should assume so that it encloses the maximum area by variational method under constraint.Write the relevant Euler-Lagrange equations. [2.5]

3(c). Write the expression for the vector quantity which is conserved besides the angular momentum in Kepler's problem.Hence show how it leads to the orbit equation for the Kepler's problem. [2.5]

4. Two equal masses are connected by springs having equal spring constants so that the masses are free to slide on a frictionless table. The points to which the ends of the

P.T.O

springs are attached are fixed. Obtain the Hamiltonian and Hamilton's equations of motion. [6+4]

5(a). If the transformation equations between two sets of cordinates are

$$p = 2(1 + q^{\frac{1}{2}} cosp) q^{\frac{1}{2}} sinp \quad ; \quad Q = log(1 + q^{\frac{1}{2}} cosp)$$

Then show that
(i) the transformation is canonical. [2]
and (ii) the generating function of this transformation is $F_3 = -(e^Q - 1)^2 tanp$ [4]

5(b). Making use of the Poisson brackets, show that for a one dimensional system with the Hamiltonian $H = \frac{p^2}{2} - \frac{1}{2q^2}$
there is a constant of the motion $D = \frac{pq}{2} - Ht$ [4]

Indian Statistical Institute

First Semestral Examination: (2006–2007)
B.Stat.(Hons.) – II year
Economics I

Date: 08/12/2006 *Maximum Marks –**60*** *Duration: **3 hours***

Answer each Group on a separate answer script.

Group A

Answer any **three** questions. The maximum you can score in this part is **30**.

1. (a) 'For any given set of values of prices and income it cannot be the case that for a consumer all goods are inferior'. Is this statement true? Justify your answer.

(b) Define 'Giffen goods' and 'Inferior goods'. Show that all Giffen goods are inferior, but the converse is not true.

(c) Can two indifference curves cross? Justify your answer.

(d) Suppose the consumer's utility function for a commodity x is given by u=$logx$. Is this a valid utility function? Will you call the commodity a "bad"? Explain your answers.

[2+4+2+3=11]

2. (a) Consider the indirect utility function

$$v(p_1, p_2, m) = \frac{m}{p_1 + p_2}$$

(i) Derive the Marshallian demand functions
(ii) What is the expenditure function?
(iii) Comment on the relationship between the two goods and draw the indifference curves.

(b) Suppose that a consumer's utility function is given by $u(x_1, x_2) = x_1 x_2$. Determine the utility maximizing choice of goods 1 and 2 in terms of prices p_1, p_2 and income m. Verify that both are normal goods.

[(3+1+3)+4=11]
P.T.O

3. (a) Explain the idea of Revealed Preference (RP).

(b) Check whether the Weak Axiom of RP is satisfied in each of the following cases. Give reasons for your answer.

(i) Income m=20, $(p_1, p_2) = (1, 1)$; the choice is (5, 15)
Income m=20, $(p_1, p_2) = (2, 0.50)$; the choice is (8, 8)

(ii) Income m=5, $(p_1, p_2) = (1, 2)$; the choice is (1, 2)
Income m=5, $(p_1, p_2) = (2, 1)$; the choice is (2, 1)

(iii) Income m=4, $(p_1, p_2) = (2, 1)$; the choice is (1, 2)
Income m=4, $(p_1, p_2) = (1, 2)$; the choice is (2, 1)

[2+3×3=11]

4. (a) In the context of choice under uncertainty, what is the 'Expected Utility Property'? Under what type of transformation is this property preserved?

(b) Let C_1 and C_2 represent consumption in states 1 and 2, respectively and let $U(C_i) = C_i$, i=1, 2. Suppose p_i is the probability that state i will occur. Does the form $p_1\sqrt{C_1} + p_2 C_2$ have the expected utility property? Explain.

(c) An individual's utility function over money payoff W is of the form $U(W) = A\sqrt{W}$, where A>0. He faces a situation where he receives $Rs.$784 with probability $\frac{1}{3}$ and $Rs.$100 with probability $\frac{2}{3}$. Find the 'certainty equivalent' payoff.

(d) Suppose an individual's utility from wealth (W) is given by $U(W) = a + bW + cW^2$, b>0, c<0. Show that the Arrow-Pratt absolute risk aversion measure is an increasing function of W.

[3+1+4+3=11]
P.T.O

Group B

Answer any **three** questions. The maximum you can score in this part is **30**.

1. Consider the market for a particular good. There are two types of customers: those of type 1 are the low demand customers, each with a demand function of the form $p = 10 - q$, and those of type 2, who are the high demand customers, each has a demand function of the form $p = 2(10 - q)$. There are 30 customers of type 1 and 20 of type 2. The firm producing the product is a monopolist in this market and has a cost function $C(q) = 4q^2$.

(a) Suppose the firm is unable to prevent the customers from selling the good to one another, so that the monopolist cannot charge different customers different prices. What prices per unit will the monopolist charge to maximize its total profit and what will be the equilibrium quantities to be supplied to the two groups in equilibrium?

(b) Now suppose that the firm realizes that by asking for IDs it can identify the types of the customers (for instance, type 1's are students who can be identified using their student IDs). It can thus charge different per unit prices to the two groups, if it is optimal to do so. Find the profit maximizing prices to be charged to the two groups.

[6+5=11]

2. Consider a monopolist whose cost function is $C(q)$ with $C'(q) > 0$ and the demand function $p(q)$ with $p'(q) < 0$.

(a) Given an initial equilibrium of the monopolist, suppose that a tax, $t > 0$ per unit of output is imposed so that when the consumers pay a price p_d, the monopolist receives a price $p_s = p_d - t$. What will be the effect of such a tax on monopoly equilibrium output?

(b) Now suppose that this unit tax is replaced by a value tax $\tau > 0$ (therefore $p_d = (1 + \tau)p_s$) in such a way that the final price facing the consumers is the same under either scheme. What will be its implication for tax revenue?

[5+6=11]
P.T.O

3. On a tropical island there are 100 boat builders, numbered 1 through 100. Each builder can build up to 12 boats a year and each builder maximizes profit given the market price. Let y denote the number of boats built per year by a particular builder, and for each i, from 1 to 100, boat builder has a cost function $C(y) = 11 + iy$. Assume that in the cost function the fixed cost, 11, is a quasi-fixed cost, that is, it is only paid if the firm produces a positive level of output. If the price of a boat is 25, how many builders will choose to produce a positive amount of output and how many boats will be built per year in total?

[11]

4. (a) What do you mean by a 'dominated strategy'? Given the following payoff matrix, solve for the equilibrium strategies of the players by the method of iterated elimination of dominated strategies? What kind of logic is involved in getting the solution?

		Player B		
Player A	Strategy	L	C	R
	T	(1, 0)	(1, 3)	(3, 0)
	M	(0, 2)	(0, 1)	(3, 0)
	B	(0, 2)	(2, 4)	(5, 0)

(b) What do you mean by 'Nash equilibrium'? Consider a simultaneous move game of two players, each having three strategies. Construct a 3×3 payoff matrix such that this game has a unique Nash equilibrium but it cannot be solved by the method of iterated elimination of dominated strategies.

[6+5=11]

INDIAN STATISTICAL INSTITUTE

Backpaper Examination: 2006–2007
B.Stat. (Hons.) 2nd Year. 1st Semester
Statistical Methods III

Date: 6.2.07 **Maximum Marks: 100** **Duration: 4 hours**

- This question paper carries 104 points. Answer as much as you can. However, the maximum you can score is 100.
- You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

---

1. Suppose $X_1, \ldots, X_n$ are i.i.d. Gamma(α, β), $\boldsymbol{\theta} \equiv (\alpha, \beta)$, $\alpha, \beta \in (0, \infty)$, with pdf given by

$$f(x; \alpha, \beta) = \frac{\alpha^\beta \exp(-\alpha x) x^{\beta-1}}{\Gamma(\beta)}, \quad x > 0.$$

Find a minimal sufficient statistic for $\boldsymbol{\theta}$. [7]

2. Suppose $Y_1, \ldots, Y_n$ are independent with $Y_i \sim \text{N}(\beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_p x_{i,p}, \sigma^2)$, $i = 1, \ldots, n$, $\boldsymbol{\theta} \equiv (\beta_0, \beta_1, \beta_2, \ldots, \beta_p, \sigma)$, $\beta_0, \beta_1, \beta_2, \ldots, \beta_p \in \mathbb{R}, \sigma > 0$. Denote by $\mathbf{A}$, the $n \times (p+1)$ matrix $((a_{ij}))$ defined as follows : $a_{ij} = 1 \; \forall \; i$ if $j = 1$, and for $j > 1$, $a_{ij} = x_{i,j-1}, i = 1, \ldots, n$. Assume that $\mathbf{A}$ has rank $p+1$ $(n > p+1)$. The entries of the matrix $\mathbf{A}$ are fixed (non-random). Find the maximum likelihood estimate of $\boldsymbol{\theta}$. [9]

3. Suppose $X_1, \ldots, X_n$ are i.i.d. having gamma distribution with scale parameter θ_1 and shape parameter θ_2. Write $\boldsymbol{\theta} = (\theta_1, \theta_2)$, $\theta_1, \theta_2 > 0$. The parameter space is $\Theta = (0, \infty) \times (0, \infty)$. Both θ_1 and θ_2 are unknown.

(1) Obtain method of moments estimate of θ_2.

(2) Denote the estimate you obtain in (1) by T_n. Decide, with adequate reasons, if T_n is consistent. [5+4 = 9]

4. Let $(X_0, \ldots, X_s)$ have multinomial distribution with pmf given by

$$P(X_0 = x_0, \ldots, X_s = x_s) = \frac{n!}{x_0! \cdots x_s!} p_0^{x_0} \cdots p_s^{x_s}, \quad x_j \geq 0 \text{ for each } j, \; \sum_{j=0}^{s} x_j = n,$$

$$p_j \geq 0 \text{ for each } j, \; \sum_{j=0}^{s} p_j = 1.$$

Show that the uniformly minimum variance unbiased estimate of $p_i p_j$ is $X_i X_j / [n(n-1)]$. [10]

P.T.O.

5. Suppose $X_1, \ldots, X_n$ are i.i.d. $N(\mu, \sigma^2)$, $\boldsymbol{\theta} \equiv (\mu, \sigma)$, $\mu \in \mathbb{R}$, $\sigma > 0$. Let the observations be grouped into disjoint sets (classes) $\mathcal{X}_1, \ldots, \mathcal{X}_k$ such that $\mathcal{X}_i \cap \mathcal{X}_j = \emptyset\ (i \neq j)$ and that $\cup_i \mathcal{X}_i = \mathbb{R}$. The precise form of each $\mathcal{X}_j$ is given by

$$\mathcal{X}_j = \begin{cases} (-\infty, u_1) & \text{if } j = 1, \\ [u_{j-1}, u_j) & \text{if } j = 2, \ldots, k-1, \\ [u_{k-1}, \infty) & \text{if } j = k, \end{cases}$$

where the quantities $u_1 < u_2 < \cdots < u_{k-1}$ are known to us. Denote by n_j, the frequency of the j-th class $\mathcal{X}_j$, $j = 1, \ldots, k$. The data consist of n_j, $j = 1, \ldots, k$. Notice that $\sum_{j=1}^{k} n_j = n$. Describe how you will estimate the MLE of $\boldsymbol{\theta}$, based on these data and by employing the EM algorithm. [11]

6. Let $X_1, \ldots, X_n$ be independent and each distributed uniformly over the set of integers $\{1, 2, \ldots, \theta\}$. Find a UMP test for testing $H_0 : \theta = \theta_0$, at level $1/\theta_0^n$ against the alternative $\theta \neq \theta_0$. [8]

7. Suppose $X_{i,1}, \ldots, X_{i,n_i}$ are i.i.d. Bernoulli$(1, \theta_i)$, $\theta_i \in (0,1)$, $i = 1, 2$. Also, suppose that all the $X_{i,j}$'s are independent. Develop a suitable large-sample test for $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 \neq \theta_2$ based on $S_i \stackrel{def}{=} X_{i,1} + \cdots + X_{i,n_i}$, $i = 1, 2$. [8]

8. Suppose $(X_{i,1}, X_{i,2})$, $i = 1, \ldots, n$, are i.i.d. $N_2(\theta_1, \theta_2, \sigma_1^2, \sigma_2^2, \rho)$, $\theta_1, \theta_2 \in \mathbb{R}, \sigma_1, \sigma_2 > 0, \rho \in (-1, 1)$. Define $T_i \stackrel{def}{=} \sum_{l=1}^{n} X_{i,l}$, $S_{ij} \stackrel{def}{=} \sum_{l=1}^{n} (X_{i,l} - T_i/n)(X_{j,l} - T_j/n)$, $i, j = 1, 2$. Based on $T_1, T_2, S_{11}, S_{22}, S_{12}$, develop a suitable test for $H_0 : \theta_1 = \theta_2$ against $H_1 : \theta_1 > \theta_2$. [9]

9. Suppose $Y_1, \ldots, Y_n$ are independent with $Y_i \sim N(\beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_p x_{i,p}, \sigma^2)$, $i = 1, \ldots, n$, $\beta_0, \beta_1, \beta_2, \ldots, \beta_p \in \mathbb{R}, \sigma > 0$. Denote by $\mathbf{A}$, the $n \times (p+1)$ matrix $((a_{ij}))$ defined as follows : $a_{ij} = 1\ \forall\ i$ if $j = 1$, and for $j > 1$, $a_{ij} = x_{i,j-1}, i = 1, \ldots, n$. Assume that $\mathbf{A}$ has rank $p+1$ $(n > p+1)$. The entries of the matrix $\mathbf{A}$ are fixed (non-random). Develop a suitable test for $H_0 : \sigma = \sigma_0$ against $H_1 : \sigma > \sigma_0$. [9]

10. Suppose $X_{i,1}, \ldots, X_{i,n_i}$ are i.i.d. $N(\theta_i, \sigma^2)$, $i = 1, 2$. Also, suppose that all the $X_{i,j}$'s are independent. Define $\bar{X}_i \stackrel{def}{=} \sum_{j=1}^{n_i} X_{i,j}/n_i$, $S_i^2 \stackrel{def}{=} \sum_{j=1}^{n_i} (X_{i,j} - \bar{X}_i)^2/(n_i - 1)$, $i = 1, 2$. Define

$$T \stackrel{def}{=} \frac{\sqrt{\dfrac{n_1 n_2}{n_1 + n_2}}(\bar{X}_1 - \bar{X}_2)}{\sqrt{\dfrac{(n_1-1)S_1^2 + (n_2-1)S_2^2}{n_1 + n_2 - 2}}}.$$

Find the distribution of T. [8]

11. **(1)** A time series with a periodic component can be constructed from

$$x_t = u_1 \sin(2\pi\nu_0 t) + u_2 \cos(2\pi\nu_0 t),$$

where u_1, u_2 are independent random variables with zero means and $E(u_1^2) = E(u_2^2) = \sigma^2$. The constant ν_0 determines the period or time taken by the process to make one complete cycle. Show that this series is weakly stationary. Find its autocovariance function.

(2) Describe briefly how you can estimate the trend of a time series by method of moving average.

(3) Describe briefly how you can interpret the correlogram of a time series.

[(4+4)+4+4 = 16]

****** *Best of Luck!* ******

7

INDIAN STATISTICAL INSTITUTE

Mid-Semestral Examination

Second Semester 2006–2007

B. Stat (Second year)

Elements of Algebraic Structures

Date: 19 February, 2007 | Maximum Marks: 40

Duration: 2 hours 30 minutes

Answer all questions.

We shall adopt the following notation:

$$\begin{aligned} \mathbb{Z} &= \text{additive group of integers} \\ \mathbb{Q} &= \text{additive group of rationals} \\ \mathbb{Z}_n &= \text{the additive group of integers modulo } n \\ S_n &= \text{permutation group of } n \text{ elements} \\ GL_n(\mathbb{R}) &= \text{multiplicative group of invertible } n \times n \text{ matrices over } \mathbb{R} \end{aligned}$$

(1) Find all homomorphisms from $\mathbb{Q}$ to $\mathbb{Z}$. 5

(2) Prove that every group G can be realised as a subgroup of $GL_n(\mathbb{R})$ for some n. 8

(3) Let G be a group of order $2p$, where p is an odd prime. Show that G is either cyclic or isomorphic to the dihedral group. 7

(4) Find all conjugacy classes in S_5. Find the number of elements in each conjugacy class. Prove that the alternating group A_5 is the only non-trivial normal subgroup in S_5. 8

(5) Let G be a group of order 105. Show that G has a normal subgroup of order 35. 8

(6) Find the order of the automorphism group $\mathbb{Z}_{p^2}$, where p is a prime. Prove that the automorphism group of $\mathbb{Z}_{p^2}$ is cyclic. 9

INDIAN STATISTICAL INSTITUTE

B. Stat II: 2006 – 2007

Mid-semester Examination

Economic Statistics and Official Statistics

Duration: 3 hours | Maximum marks: 100 | Date: 21.02.2007

(Answer Part A and Part B in separate answer sheets.)

Part A

Economic Statistics

*(Answer question no. 1 and any **one** from question nos. 2 and 3.)*

1. Suppose the distribution of income of persons in a community follows Pareto distribution. Find the ratio of shares of income of the top 25 percent people to that of bottom 25 percent people. Also derive Elteto-Frigyes measures. How will these measures change if the distribution is truncated from below at Q_1, the first quartile? [12+12+2=26]

2. What are the different types of qualitative data usually observed in economic analysis? Write down, with illustrations, some ways to treat these data to arrive at meaningful conclusions? What are the precautions to be taken in analyzing such data and why? [2+10+2=14]

3. Write short notes on any two of the following:
   (i) Graphical test of Pareto distribution.
   (ii) Problems with data.
   (iii) Properties of Lorenz Curve. [7+7=14]

4. Assignments [10]

Part B

Official Statistics

(Answer all questions)

1. Describe the functions of CSO. [5]
2. What are the divisions of NSSO? Describe the activities of any one Division of NSSO. [4+6]
3. Write a short note on Economic Census. [5]
4. (a) What is Annual Survey of Industries? Describe its scope and coverage. [10]
   Or
   (b) Write a note on sampling scheme of ASI. [10]
5. Differentiate between GDP and National income. [7]
6. What are the production boundaries in calculation of national income? [7]
7. How important is Environmental accounting? Give your arguments. [6]

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: (2006-2007)

Course Name: **B. Stat.-2nd Year**

Subject: **Biology-II**

Date.23.2.07..Maximum Marks...**30**........Duration: ....**1 hour 30 minutes**....

Please suggest answer(s) if you think the question is incorrectly framed or simply wrong in your opinion; Numbers in the parentheses denote marks. 20 marks from Group-A and Group-B (10 marks) should be answered.

GROUP-A (20 marks)

1. What are the broad aims and objectives of plant improvement by traditional methods? *(4)*
2. Please define following with examples: Normally and often self pollinated crops; Normally and often cross pollinated crops. *(4)*
3. Define biodiversity? What do you mean by α-, β- and γ– diversity? *(4)*
4. What do you mean by in situ and ex-situ conservation of biodiversity? Give examples. *(4)*
5. What are the different methods of selection? Describe the salient features of pureline and mass selection? *(4)*
6. What are the differences between the different selection methods adopted for improving self- and cross-pollinated crops? *(4)*
7. What are the major causes for loss of bio-diversity? Write salient points. *(4)*

GROUP-B (10 marks)

Question

1. (a) Determine the values of a and b so that (0, 0) is a sink of

$$\frac{dx}{dt} = ax - by$$

$$\frac{dy}{dt} = bx + 2y$$

(b) Write down the Lotka – Volterra model for predator-prey interaction. Find the dynamic behavior of the model around the biologically feasible equilibria.

3+2+5=10

Indian Statistical Institute

Mid-semester Examination (2006-2007)
B.Stat. II Year
Economics II

23.2.07 **Maximum Marks 40** **Duration 150 minutes**

1. a. Explain the value added and the income method of calculating GDP.

b. Consider the following information regarding a firm in the domestic economy in a given period . (All figures are in crores of rupees)

Revenue earned by the firm	~~19~~ 80,000
Unsold output of the firm	2,000
Raw materials purchased from other firms	30,500
Unused portion of the raw materials purchased	2,500
(Wages and salaries paid to households of which 1000 is paid to foreign technical experts for their advice)	22,000
Rent paid to another firm for hiring the office premises	500
Dividend paid to households of which 1/5th goes to foreigners	10,000
Interest paid to commercial banks on loans taken	12,000
Business transfers	0
Net indirect taxes paid	1,500
Depreciation	3,300

Derive the firm's contribution to GDP and NI using the value added method. Find out the firm's contribution to different components of NI. Does the firm make any contribution to the final expenditure? **8+(6+6+2)**

2. a. Explain the concepts of current account balance, capital account balance and official reserve settlement balance. Show that the sum of the three is identically equal to zero.

b. Suppose that in an economy in a given year the stock of foreign exchange of the central bank declined by Rs.15,000 crore to accommodate the balance of payments deficit at the prevailing exchange rate. The net inflow of foreign loan to the domestic sector is Rs.25,000 crore. In addition foreigners purchased shares of domestic companies worth Rs.10,000 crore. Domestic economic agents did not buy any foreign financial assets. Net investment in the domestic sector is Rs.80,000 crore. Difference between NNP and private disposable income is Rs.50,000 crore. Business transfers and net foreign transfers are zero, G = Rs.40,000 crore, C = Rs.90,000 crore, net indirect tax = Rs.20,000. Find out, current account balance, capital account balance, official settlement balance and national income.

8+(1+1+1+11)

INDIAN STATISTICAL INSTITUTE

Mid-Semester Exam: 2006-2007

B. Stat. (Hons.) – II Year

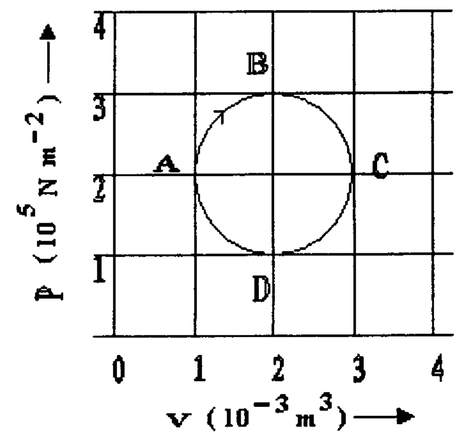
Physics - II

Group – A (Thermodynamics and Stat. Mech.)

Date : 23.2.07 Maximum Marks : 15 Duration : 1 Hour

(Note: all questions carry equal marks)

1. The molar specific heat at constant volume of an ideal gas is 3R/2. If 1 mole of the gas undergoes a quasi-static cycle which appears as a circle in the accompanying PV diagram
   a) Show that the net work done by the gas in one cycle is 314 J.
   b) Find the difference in internal energy between states C and A.
   c) Calculate the amount of heat absorbed by the gas in the path ABC.



2. Consider a gas obeying the equation of state $(P + a/V^2)(V - b) = RT$.
   a) Calculate its internal energy U(T,V).
   b) Find changes in C_p and C_v due to variations in P and V respectively, at fixed T.
   c) State how these results differ from those of an ideal gas.

3. a) Two identical bodies of constant heat capacity C_p have the same initial temperature T_i. If a refrigerator working between the two bodies cools down one of them to temperature T_c, find the minimum work required to do so.

   b) One mole of an ideal gas (in thermally insulating container) is allowed to expand freely so that its volume doubles. Calculate the entropy change for this irreversible process.

P.T.O

Group B

Maximum Marks 15 Duration 1 hour

Attempt any two from the three questions

1(a). What was the new idea for radiation of energy that was introduced by Planck Explain how this idea helped to explain the photo-electric effect.

1(b). What is Compton effect ? Find the expression of the change of wavelength of th photon when it is scattered by an angle θ by an electron.

3+4

2. Consider the 2×2 matrices σ_x, σ_y and σ_z.
(i) Find the eigen values and eigen functions of the following operator

$$\sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z$$

(ii)Show that

$$\frac{1}{2}[I + \sin\theta\cos\phi\sigma_x + \sin\theta\sin\phi\sigma_y + \cos\theta\sigma_z]$$

is a projection operator, I being the identity matrix.
(iii) Show that each spin component σ_i $(i = x, y, z)$ commutes with $\sigma^2 (= \sigma_x^2 + \sigma_y^2 + \sigma_z^2)$.

$3 + 2 + 2$

3(a). If A is self adjoint operator, then show that e^{iA} is a Unitary operator.

3(b). Show that the following two states $|0>$ and $\frac{1}{\sqrt{2}}[|0> + |1>]$, $|0>$ and $|1>$ being eigen states of σ_z, can not be cloned

3(c). A spin-1/2 particle has been prepared in the following state

$$|\psi> = \sqrt{\frac{2}{5}}|0> + \sqrt{\frac{3}{5}}|1>$$

If spin is measured along x-axis, what is the probability of getting the result to b up?

$2 + 3\frac{1}{2} +$

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: 2006-2007
B.Stat. (Hons.) 2nd Year. 2nd Semester
Statistical Methods IV

Date: February 26, 2007 **Maximum Marks: 60** **Duration: 3 hours**

- Answer all the questions.
- You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Suppose $\boldsymbol{X} \sim N_p(\boldsymbol{0}, \boldsymbol{\Sigma})$, where $\boldsymbol{\Sigma} = \sigma^2(1-\rho)\boldsymbol{I}_p + \sigma^2\rho\boldsymbol{J}_p$, with $\boldsymbol{J}_p$ being the $p \times p$ matrix having all entries equal to 1, and $\sigma > 0, -1/(p-1) < \rho < 1$. Write $\boldsymbol{X} = (X_1, X_2, \ldots, X_p)^T$. Define $\bar{X} = (X_1 + X_2 + \cdots + X_p)/p$, $S^2 = \sum_{i=1}^{p}(X_i - \bar{X})^2$. Find Var$(S^2)$. [9]

2. Suppose $(X_{i,1}, X_{i,2})^T$, $i = 1, \ldots, n$, are i.i.d. $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^2$ and $\boldsymbol{\Sigma} = ((\sigma_{ij}))$ is positive definite. Let $\rho \stackrel{def}{=} \sigma_{12}/\sqrt{\sigma_{11}\sigma_{22}}$. Define $T_k = \sum_{i=1}^{n} X_{i,k}$, $k = 1, 2$, $S_{12} = \sum_{i=1}^{n}(X_{i,1} - T_1/n)(X_{i,2} - T_2/n)$. Also, define $r = S_{12}/\sqrt{S_{11}S_{22}}$. We have seen earlier that a test-statistic for testing $H_0 : \rho = 0$ against $H_1 : \rho \neq 0$ is given by $T \stackrel{def}{=} \sqrt{n-2}\, r/\sqrt{1-r^2}$. Use appropriate results about Wishart distribution to find the null distribution of T. [14]

3. Suppose $\mathbf{X}_i$ is an $n_i \times p$ $(n_i \geq p+1)$ data matrix from $N_p(\boldsymbol{\mu_i}, \boldsymbol{\Sigma})$, $i = 1, 2$. The parameters $\boldsymbol{\mu}_1$, $\boldsymbol{\mu}_2 \in \mathbb{R}^p$, and $\boldsymbol{\Sigma}$ is a positive definite matrix. Let $n \stackrel{def}{=} n_1 + n_2$.

(a) Define Mahalanobis D^2 statistic based on $\mathbf{X}_1$ and $\mathbf{X}_2$.

(b) Define $F = [n_1 n_2(n-p-1)D^2]/[n(n-2)p]$. Show that F has a non-central F distribution with parameters to be obtained by you. [3+10 = 13]

4. Suppose $\mathbf{X}$ is an $n \times p$ $(n \geq p+1)$ data matrix from $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. The parameters $\boldsymbol{\mu}$, $\boldsymbol{\Sigma}$ are unknown, but it is known that $\boldsymbol{\Sigma}$ is a positive definite matrix. Write $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_p)^T$.

(a) Derive explicitly the algebraic form of the maximum likelihood estimate of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ subject to the restriction $\mu_1 = \cdots = \mu_{p_1} = 0$, where $1 \leq p_1 < p$ is fixed and known.

(b) Let $1 \leq p_1 < p$. Derive explicitly the algebraic form of the likelihood ratio test for $H_0 : \mu_1 = \cdots = \mu_{p_1} = 0$ versus $H_1 : H_0$ *is false*. Find a suitable equivalent of the test statistic which has a standard distribution, to be obtained by you, as the null distribution. [9+(6+9) = 24]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE

Mid-semester Examination: (2006 - 2007)

Course Name : B. Stat II

Subject Name : SQC & OR

Date: 02 March 2007 Maximum Marks 100 Duration 3 hours

Note: This paper carries 110 marks. You may answer as many questions as you like, but the maximum you can score is 100.

1. A certain farming organization operates three farms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. The data for the upcoming season is as shown below:

Farm	Usable Acreage	Water Availability (in cubic feet)
1	400	1,500
2	600	2,000
3	300	900

The organization is considering planting crops which differ primarily in their expected profit per acre and in their consumption of water. Further more, the total acreage that can be devoted to each of the crops is limited by the amount of appropriate harvesting equipment available.

Crop	Maximum Acreage	Water Consumption (in cubic feet per acre)	Expected profit per Acre (Rs.)
A	700	5	4,000
B	800	4	3,000
C	300	3	1,000

In order to maintain a uniform workload among the farms, it is the policy of the organization that the percentage of the usable acreage planted be the same at each farm. However, any combination of the cops may be grown at any of the farms. The organization wishes to know how much of each crop should be planted at the respective farms in order to maximize expected profit.

Formulate this problem as an LP model in order to maximize expected total profit. [15]

2. Consider the following problem:-

$$P: \text{Maximize } z = x_1 + x_2$$

$$\text{subject to} \quad -x_1 + x_2 + x_3 \leq 2$$

$$-2x_1 + x_2 - x_3 \leq 1$$

$$x_1, x_2, x_3 \geq 0$$

a. Write down the dual D of the problem P.

b. Solve the dual D

c. Determine the primal optimal solution using the complementary slackness conditions.

[5+5+10=20]

3. i. An electronics company manufactures a special type of cathode ray tubes on a mass production basis. The production rate is 500 tubes per hour. A random sample of size 50 units is taken every hour. A control chart for fraction non-conforming (p) indicates that the current process average in $\bar{p} = 0.20$.

a. Find the 3σ, control limits for the control chart.

b. When the process is in-control, how often would false alarms be generated?

c. If the process average deteriorates to $\bar{p} = 0.30$, after how many samples will the control chart be able to detect this shift?

d. Suggest two methods of reducing the out-of-control ARL.

ii Find the probability of detecting a shift of 2σ in process mean on the first sample following the shift in an $\bar{X}$ - chart with the usual 3-sigma limits and sub group size $n = 5$

[(3+5+5+5)+7=25]

4. a. Consider a double sampling plan with the following parameters:

$n_1 = 50$ $c_1 = 1$ $r_1 = 5$ $n_2 = 100$ $c_2 = 7$

Find the probability that a second sample with be taken if the lot has fraction defective $p = 0.06$

b. Show that AOQ = p (1 – AFI)

c. Suppose that a single sampling acceptance rectification plan $n = 150$, $c = 1$ is being used for receiving inspection where the vendor ships the product in lots of size $N = 3000$. Find the AOQL for this plan.

[10+5+10=25]

5. a. Define quality costs. Give their broad categories for a manufacturing organization.

b. When should we be interested in obtaining the optimal solution of the primal by solving the dual?

c. Distinguish between Type A and Type B OC-curves.

d. When one should apply $\bar{X} - s$ chart instand of $\bar{X} - R$ chart and why?

[5+2+5+3 = 15]

6. Choose the best answer:

i. A feasible solution to an LP problem :

a. must satisfy all of the problem's constraints simultaneously.
b. need not satisfy all of the constraints, only some of them.
c. must be a corner point of the feasible region.
d. must optimize the value of the objective function.

ii. Which of the following statements is true with respect to the optimal solution of an LP problem:

a. every LP problem has an optimal solution.
b. Optimal solution of an LP problem always occurs at an extreme point.
c. at optimal solution all resources are used completely.
d. if an optimal solution exists, there will always be at least one at a corner point.

iii. If two constraints do not intersect in the first quadrant of the graph, then :

a. the problem is in feasible
b. the solution is unbounded
c. one of the constraint is redundant
d. none of the above.

iv. To formulate a problem for solution by the simplex method, we must add artificial variable to :

a. only equality constraints
b. only 'greater than' constraints
c. both (a) and (b)
d. none of the above.

v. If nothing is known concerning the pattern of variation of a set of numbers, we can calculate the standard deviation of a set of numbers and state that $\overline{X} + 3\sigma$ will include at least :

a. 89% of all the numbers
b. 95% of all the numbers
c. 99.7% of all the numbers
d. None of the above.

vi. If the distribution of defectives among various lots is found to follow the law of chance, we can conclude that :

a. the manufacturing process was in control.
b. the product was well mixed before dividing into lots
c. either (a) or (b) is true
d. all lots should be accepted.

vii. Consumer's risk of 10% means that :

a. the probability that a sampling plan will reject "good" material is 10%.
b. the probability that a sampling plan will accept "poor" material is 10%.
c. the acceptable quality level of the lot is 10%.
d. the unacceptable quality level of the lot is 10%.

viii. When measurements show a lack of statistical control, the standard error of the average:

a. is related to confidence limits.
b. is a measure of process variability.
c. is simple to compute
d. has no meaning.

ix. AOQL means :

a. average outgoing quality level
b. average outgoing quality limit
c. average outside quality limit
d. anticipated optimum quality level.

x. The PDCA wheel is attributed to :

a. Shewhart
b. Juran
c. Dodge and Romig
d. Deming

[1 x 10 = 10]

20

INDIAN STATISTICAL INSTITUTE

Semestral Examination

Second Semester 2006–2007

B. Stat (Second year)

Elements of Algebraic Structures

Date: May 4, 2007 Maximum Marks: 60

Duration: 3 hours

Answer all questions.

(1) Show that $\mathbb{Z} \times \mathbb{Z}$ is not an integral domain. Find all idempotent elements of this ring. Determine all ring homomorphisms from $\mathbb{Z} \times \mathbb{Z}$ to $\mathbb{Z}$. 2+2+6

(2) Prove that the principal ideal (x) of the polynomial ring $\mathbb{Z}[x]$ is a prime ideal but not a maximal ideal. Find a maximal ideal of $\mathbb{Z}[x]$ containing (x). 5+5

(3) Let $\mathbb{Z}[i] = \{m + ni \in \mathbb{C} : m, n \in \mathbb{Z}\}$ be the Gaussian ring of integers. What is the smallest subfield of $\mathbb{C}$ containing $\mathbb{Z}[i]$? Justify your answer. 6

(4) Let E be an algebraic extension field of F. Show that every subring S of E containing F is a subfield of E. Show that it is not necessarily true if E is not algebraic. 5+5

(5) (a) Let n be any positive integer and let $f(x)$ be an irreducible polynomial of degree n over a field F. If d is the dimension of the splitting field of $f(x)$ over F then show that $n \leq d \leq n!$.

(b) Find the splitting field of $x^{p^3} - 1$ over $\mathbb{Z}_p$.

5+3

(6) Prove or disprove: A finite field of 81 elements has a subfield of 9 elements. 6

(7) (a) Find all quadratic polynomials over $\mathbb{Z}_2$. Which of these polynomials are irreducible?

(b) Use the information obtained in (a) to prove that there exists a unique field (upto isomorphism) containing 4 elements. 2+2+4

(8) Let $p_1, p_2, \ldots, p_n$ be n distinct primes. Set $\alpha_i = \sqrt[p_i]{2}$. Show that

$$[\mathbb{Q}(\alpha_1, \alpha_2, \ldots, \alpha_n) : \mathbb{Q}] = p_1 p_2 \cdots p_n.$$

8

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination : (2006-2007)

B. Stat. II Year

Physics II

Group A

Date: 8.5.07 Maximum Marks 30 Duration $1\frac{1}{2}$ hour

Attempt any four questions

1. An ideal gas occupies a volume of 10^{-3} m^3 at temperature 3 K and pressure 10^3 Pa. The internal energy of the gas at this initial point is taken to be zero. Now the gas undergoes the following cycle:
The temperature is raised to 300 K at constant volume; the gas is then expanded adiabatically till the temperature is 3 K; followed by an isothermal compression to the original volume.

(a) Plot the cycle on a $P-V$ diagram.

(b) Calculate the work done and the heat transferred in each process and the internal energy at the end of each process.

(c) How efficient is the cycle? [Given: $C_v = \frac{3R}{2}$, $\gamma = \frac{5}{3}$]

$1 + 5\frac{1}{2} + 1$

2(a) Determine (asymptotically) the number of accessible microstates (cells) in the two dimensional phase space for a one dimensional harmonic oscillator, having energy between 0 and E.

(b) Two systems separately in equilibrium, are brought in thermal contact with each other such that only exchange of energy takes place. Under what condition would the composite system be in equilibrium? Which macrostate is identified with the equilibrium state?

$3\frac{1}{2}+4$

3(a) Show that in the canonical ensemble, the entropy of a system can be written as $S = -k\sum_r P_r \ln P_r$, where P_r is the probability of the system to be in the r-state.

P.T.O

(b) From the above, deduce $S = k \ln \Omega$ for a microcanonical ensemble.

(c) What is the value of S if the system is in the ground state, which is either unique or degenerate?

$3\frac{1}{2}$+2+2

4. An ideal gas of N identical, monoatomic particles is confined in volume V at temperature T. The single particle partition function is given by $Z_1(V,T) = \frac{V}{h^3}(2\pi mkT)^{3/2}$. Assuming the particles to be indistinguishable,

(a) Obtain the Helmholtz free energy F for the N particle system.

(b) Evaluate $-\frac{\partial F}{\partial T}$ at constant V and N. Explain the significance of the result?

$3 + 4\frac{1}{2}$

5 (a) Find the relative root-mean-square fluctuation in the number of particles N. When is the fluctuation negligible?

(b) Consider a classical system of N, one dimensional harmonic oscillators which are mutually distinguishable. Evaluate the energy U of the system using the grand partition function Λ.

4+$3\frac{1}{2}$

Group B

Date: Maximum Marks 30 Duration $1\frac{1}{2}$ hour

Attempt any four questions

1(a) When photons of energy 4.25 eV strikes the surface of metal A, the released photoelectrons have maximum kinetic energy T_A eV and the corresponding de-Broglie wavelength, λ_A. The maximum kinetic energy of photoelectrons liberated from a metal B by photons of energy 4.7 eV is $T_B(= T_A - 1.5)$ eV. If the de-Broglie wavelength of these photoelectrons is $\lambda_B(= 2\lambda_A)$ then calculate the work-functions of metals A and B.

(b) The wavefunction of a particle of mass m confined in a 1-dimensional region; $0 \le x \le L$ is given by:

$$\psi(x;t) = \sqrt{\frac{2}{L}}\sin(\frac{\pi x}{L})e^{-i\omega t}.$$

(where ω is a constant, independent of both x and t). Find the Potential which confines it.

$3\frac{1}{2}$+4

2(a) A particle of mass m is moving under the influence of the following potential:

$$V(x) = \frac{1}{2}m\omega^2x^2; \quad x > 0$$
$$= \infty; \quad x \le 0$$

What are the possible values of energy if a measurement for energy of the particle is performed.

(b) A simple harmonic oscillator is in it's ground state. Calculate the probability of finding it in classically forbidden region.

4+$3\frac{1}{2}$

3. A particle of mass m moves freely inside a unidimensional infinite potential well of length L with walls at $x = 0$ and $x = L$.
(a) Find the values of energy that one can get in a measurement of energy for the particle.

P.T.O

(b)Write down the energy eigenfunctions associated with the particle .
(c) Find the probability of finding the particle at time t within a region $\frac{L}{3} \leq x \leq \frac{2L}{3}$ if initially it was in the ground state.

$2\frac{1}{2}+2\frac{1}{2}+2\frac{1}{2}$

4. The z-component of the spin of an electron is measured and found to be $+1$ (in units of $\frac{\hbar}{2}$)
If a subsequent measurement is made for the x-component of the spin; what are
(a) the possible results?
(b) What are the probabilities of finding these results?
(c)What would be the corresponding probabilities if the spin is measured along y-direction rather than in x-direction ?
(d) Show that operators (represented by 2×2 Pauli spin matrices) corresponding to spin along x and z directions do not commute.

$1 + 1\frac{1}{2} + 2\frac{1}{2} + 2\frac{1}{2}$

5(a) Show that the following four states of two qubits A and B form an orthogonal basis:

$$|B_1\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B]$$

$$|B_2\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|0\rangle_B - |1\rangle_A|1\rangle_B]$$

$$|B_3\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|1\rangle_B + |1\rangle_A|0\rangle_B]$$

$$|B_4\rangle = \frac{1}{\sqrt{2}}[|0\rangle_A|1\rangle_B - |1\rangle_A|0\rangle_B]$$

where $|0\rangle$ and $|1\rangle$ are eigen states of σ_z.

(b) Take any one of the above states and show that it cannot be written as a product state.

(c) If the composite system of two particles is prepared in the state $|B_4\rangle$(given above) and measurement of z-component of spin on one of the particles yields up $(+1)$, then what is the probability that a measurement of the x-component of the spin results in up $(+1)$ for the second particle ?

$2\frac{1}{2} + 2\frac{1}{2} + 2\frac{1}{2}$

INDIAN STATISTICAL INSTITUTE
Second Semestral Examination: (2006 – 2007)
B. Stat II Year
Biology II

Date: 08.05.07 Maximum Marks 50 Duration Three hours .
Answer any five questions
(Number of copies of the question paper required 20)

1. Define Moisture Availability Index? Draw a suitable rice calendar with the following data 2+8

Week No.	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Rainfall (mm) at 0.5 Prob.	0	23	5	0	22	27	38	25	48	60	68	72	90	80	25	12	5	0	0
PET (mm)	35	32	33	27	23	22	20	24	20	19	18	18	16	20	22	27	29	30	34

2. Briefly explain the criteria for determining the essentiality of plant nutrients. Calculate the quantity of Vermicompost (VC), Urea, Single super phosphate and Muriate of potash required for 1 ha of rice crop to meet the nutrient requirement of 100 kg N, 60 kg P_2O_5 and 60 kg K_2O per hectare. 25% of the required N is to be given through VC. 2+8

3. Classify rice and rice culture depending on eco-geographical situation. Write down the suitable agrotechniques for optimization of rice production. Critically highlight the variation in yield of winter (Aman) rice and summer (Boro) rice. 2+5+3

4. Define drought. Write about different types of drought. Write in detail about the contingent cropping system for early season drought. 2+3+5

5. Write short notes on any five of the following: 2 x 5
a) Onset of Monsoon
b) Rainfall quantum and distribution
c) Rice nursery
d) Phytoclimate
e) Field capacity
f) Available moisture holding capacity of soil
g) Intercropping and Mixed cropping
h) Plant protection chemicals

6. Write five major differences between plant and animal growth. Briefly describe about three major signaling events in mammalian development. What are the defining properties of stem cell? What do you mean by morphogen gradient and half mouse embryo? Describe briefly how teratoma develops? 2.5+1.5+2+2+2

B-Stat II Year 2007
End Semester

Course: Economics II

Date- 8.5.07

Subject:Macroeconomics

Time 3 Hours Maximum Marks 60

All questions carry equal marks

Answer any three questions

1. (i) Suppose that the income earners in an economy plan to save more at every given level of income. However, following this, aggregate saving in the new equilibrium in the economy is found to be less. Explain this in the framework of the simple Keynesian model. **(ii)** Do you observe the same kind of phenomenon in an IS-LM model, where investment is a function of rate of interest alone? **(iii)** Consider a simple Keynesian model where planned consumption is a proportional function of income. Now the saving function shifts downward by 4 units. Following this, level of saving in the new equilibrium is found to rise by 12 units. Marginal propensity to consume with respect to GDP (≡Y) is 0.3. Derive the new consumption function.

[6+4+10+2]

2. **(i)**Explain how a tax function where tax collection increases with income acts as a built-in stabilizer in the simple Keynesian model.**(ii)** Consider a simple Keynesian model where $C = 100 + 0.9YD$, . YD (disposable income) $= Y - T$, $T = 40 + .5Y$, $T_{max} = 1040$, $I = 90 + .45Y$ and $G = 76$,

P.T.O

(where $T_{\max}$ is the cap on total tax collection and other notations have usual connotations).

Write down the aggregate demand function. Show it in a diagram. Comment on the equilibrium in the model. How does your answer change when $T_{\max}$ is raised to 1240? [4+16+2]

3(i). Following a shift in the money demand function, the equilibrium in the IS-LM model

is found to change from $(Y = 3000, r = 4\%)$ to $(Y = 2875, r = 5\%)$. It is given that the marginal propensity to spend with respect to Y(GDP) = (3/5). Find out how much planned investment is crowded out in the process.(Planned investment is a function of rate of interest alone.All relations in the model are linear).

(ii). Suppose that the government takes an additional loan of Rs.100 from the central bank. Suppose that CRR=10%. There is no currency holding. At the time of government borrowing there was an excess demand of Rs.120 for commercial bank credit. How much additional broad money will be created? Explain the process.

[12+8+2]

4.Discuss the Mundell Flemming model to show that in a situation of perfect capital mobility under fixed exchange rate regime monetary policy is totally ineffective while fiscal policy has full multiplier effect as in the simple Kenesian model.

[22]

INDIAN STATISTICAL INSTITUTE

B. Stat II: 2006 – 2007

Second Semester Examination

Economic Statistics and Official Statistics

Duration: 3 hours Maximum marks: 100 Date: 11.05.2007

[Answer Group A and Group B in separate answer scripts. Answer question no. 1 and any two of the rest of the questions of Group A and all questions of Group B. Allotted marks are given in brackets[] at the end of each question.]

Group A: Economic Statistics

1. The following data show the percentage distribution of different religions in a State in India among all households and non-poor households. The overall Head Count Ratio of the State is 30%.

Races	Percentage distribution among	
	All households	Non-poor households
Hindu	75.4	79.12
Muslim	19.0	14.24
Christian	4.5	5.40
Other	1.1	1.24
Total	100.0	100.00

Compute the Head Count ratio of each of four religions in the State. Also calculate the percentage distribution over different regions of the poor households in the State. [25]

2. Write a brief account on the different types of errors in index number formulae used in practice and the choice of appropriate index numbers in the light of these errors. [20]

3. Define and prove moment distribution property of a two-parameter lognormal distribution. Hence derive the concentration curve and the concentration ratio of a commodity assuming a constant elasticity form of the Engel curve. Also describe the estimation procedures proposed by Iyengar. How are these procedures superior to the usual Least Squares estimation procedures of the implied double-log form? [20]

4. Write short notes on any two of the following:
   (i) Forms of Engel curves
   (ii) Pareto form vs. log-normal form of distribution of income.
   (iii) Treatment of qualitative data in economic analysis.
   (iv) Measures of Concentration in business and industry. [10×2=20]

Group B: Official Statistics

1. (a) Describe the importance of Foreign Trade Statistics. Describe the process of preparing official statistics on Import [10]

   OR

   (b) What are the types of Agricultural Statistics available in India. Write about nine fold classification of Land for this purpose. [10]

2. Define Index of Industrial Production. Highlight its shortcomings. [10]
3. Examine whether Wholesale Price Index No. can measure inflation properly. [8]
4. Define (a) Ex-factory Price (b) Current Price vs. Constant Price [7]

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: 2006–2007

B.Stat. (Hons.) 2nd Year. 2nd Semester

Statistical Methods IV

Date: May 15, 2007 **Maximum Marks: 100** **Duration: 4 hours**

- This question paper carries 102 points. Answer as much as you can. However, the maximum you can score is 100.
- You should provide as much details as possible while answering a question. You must state clearly any result stated (and proved) in class you may need in order to answer a particular question.

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1. Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d. $\mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}$ is a $p \times p$ positive definite matrix. Let $\boldsymbol{\mu}$ be unknown.

(a) Obtain the likelihood ratio test statistic λ for testing $\mathrm{H}_0 : \boldsymbol{\Sigma} = \sigma^2 \boldsymbol{I}_p$ *for an unknown positive* σ against $\mathrm{H}_1 : \mathrm{H}_0$ *is false* in terms of the eigenvalues of the sample dispersion matrix $\boldsymbol{S}$.

(b) What can you say about the null distribution of λ? Give reasons. [9+6 = 15]

2. Suppose $\boldsymbol{X} \stackrel{def}{=} (X_1, \ldots, X_p)^{\mathrm{T}}$ is a random vector with a positive definite dispersion matrix $\boldsymbol{\Sigma}$.

(a) Denote the multiple correlation coefficient between X_1 and $(X_2, \ldots, X_p)^{\mathrm{T}}$ by $\bar{R}$. Derive an expression for $\bar{R}$ in terms of the entries of $\boldsymbol{\Sigma}$.

(b) Suppose now that $\boldsymbol{X} \sim \mathrm{N}_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where $\boldsymbol{\mu} \in \mathbb{R}^p$ and $\boldsymbol{\Sigma}$ are both unknown. Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ be i.i.d. observations on $\boldsymbol{X}$. Suppose, moreover, that we are interested in testing $\mathrm{H}_0 : \bar{R} = 0$ against $\mathrm{H}_1 : \bar{R} > 0$.

(1) Obtain the likelihood ratio test statistic λ for testing H_0 against H_1 in terms of the sample multiple correlation coefficient R.

(2) Obtain the null distribution of a suitable equivalent of λ, to be obtained by you, which involves the sample multiple correlation coefficient R. [8+(9+8) = 25]

3. Suppose $Y_{i1}, \ldots, Y_{in_i}$ are i.i.d. $\mathrm{N}(\theta_i, \sigma^2)$, $i = 1, 2, 3$, $n_i \geq 2$ for every i; where $\theta_1, \theta_2, \theta_3 \in \mathbb{R}$ and $\sigma > 0$ are all unknown. Suppose, moreover, that all the Y_{ij}'s are independent. Suppose we are interested in testing $\mathrm{H}_0 : \theta_1 = \theta_2 = \theta_3$ against $\mathrm{H}_1 : \mathrm{H}_0$ *is false*.

(a) Obtain the likelihood ratio test statistic λ for this testing problem.

[P.T.O.]

(b) Find a suitable equivalent of λ which has a standard distribution, to be obtained by you, as the non-null-distribution. [8+11 = 19]

4. Suppose $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ are i.i.d. $N_2(\theta_1, \theta_2, \sigma_1^2, \sigma_2^2, \rho)$, where $\theta_1, \theta_2 \in \mathbb{R}, \sigma_1, \sigma_2 > 0, -1 < \rho < 1$. Denote by r_n, the sample correlation coefficient, based on $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$. Show that the limiting distribution of $\sqrt{n}(r_n - \rho)/(1-\rho^2)$, as $n \to \infty$, is $N(0,1)$. [13]

5. Consider n i.i.d random variables, denoted $X_1, \ldots, X_n$, having an exponential distribution with location parameter θ and scale parameter σ, $\theta \in \mathbb{R}, \sigma > 0$. Denote the order statistics by $X_{1:n} \le \cdots \le X_{n:n}$. Suppose that we have a Type II censored sample where only the first r order statistics $X_{1:n} \le \cdots \le X_{r:n}$ are observed.

(a) Find the MLE of (θ, σ), denoted by $(\hat{\theta}, \hat{\sigma})$.

(b) Show that $T_1 \stackrel{def}{=} n(\hat{\theta} - \theta)/\sigma$ is a standard exponential variable and that $T_2 \stackrel{def}{=} r\hat{\sigma}/\sigma$ is a gamma variable with parameters to be obtained by you. [8+11 = 19]

6. Suppose $X_1, X_2, \ldots$ are i.i.d. $N(\mu, \sigma^2)$, where $\theta \in \mathbb{R}$, and $\sigma > 0$ are unknown. Denote by $X_{1:n} \le \cdots \le X_{n:n}$, the order statistics corresponding to $X_1, \ldots, X_n$. Let $r_n \stackrel{def}{=} [n/4] + 1$, $s_n \stackrel{def}{=} [3n/4] + 1$. Also, let $T_n \stackrel{def}{=} X_{s_n:n} - X_{r_n:n}$ be the interquartile range. Obtain, with reasons, a consistent estimate of σ which is a multiple of T_n. You have to establish consistency of your estimate. [11]

****** *Best of Luck!* ******

INDIAN STATISTICAL INSTITUTE

Second Semestral Examination: (2006 – 2007)

B. STAT, II YEAR

Demography

Date: 18 May 2007 Maximum Marks: 50 Duration: 3 hrs

Answer any five questions

(Symbols and notations have their usual meaning)

1. a) Show that $t.d_{x+1} = d_{x+t} - (1-t)d_x$. .[Note: Use uniform distribution of deaths and l_{x+t} is linear for $0 \le t \le 1$]

b) The following values of q_x have been derived from a mortality experience of an insect. Construct d_x and T_x columns of the corresponding life table by assuming uniform distribution of deaths.

x	q_x
0	0.15
1	0.10
2	0.25
3	0.50

[5, 5]

2. a) List three basic axioms of the *pure birth* process?

b) In a *birth-death* process it is given that $\lambda_n \Delta t$ is the probability that a birth occurs in a small interval of length Δt and $\mu_n \Delta t$ is the probability that a death occurs in a small interval of length Δt, when the population size is n. If basic axioms of such a process hold, then find an expression for P_n, assuming steady-state solution exists.

[2, 8]

3. a) Explain how do you obtain the relation ${}_nK_{x+t}^{(t)} = {}_nK_x^{(0)} \frac{{}_nL_{x+t}}{{}_nL_x}$ for the method of population projection using life table survival ratios.

(Note: Here ${}_nK_{x+t}^{(t)}$ is the population at time t whose ages are between x and $x+n$)

b) Write linear difference equations involved in the method in (a).

[6, 4]

[P. T. O.]

4. Answer any four of the following

a) Using the Reed and Merrell method, obtain the relation between ${}_nq_x$ and ${}_nm_x$ (use either linear or non-linear form for l_x)

b) Brief description of Stable population.

c) Describe how to compute Net Reproduction Rate.

d) Prove ${}_tq_x = t.q_x$ for $t \in [0,1]$, assuming uniform distribution of deaths.

e) Write any two major limitations of *crude death rate*.

[2.5 × 4=10]

5. a) Given below information on standard and non-standard populations. Find the standardized death rates by direct and indirect methods.

Age groups	Standard Population		Non-Standard Population	
	Pop^n.	Death Rate	Pop^n.	Death Rate
< 10	15,600	25	14,600	18
10 – 25	17,000	10	16,000	12
25 – 50	40,000	5	30,000	4
50 – 60	15,000	8	12,000	10
60 – 75	10,000	12	11,000	15
75 +	8,000	30	6,000	25

[10]

6. a) Prove that ${}_nq_x = \int_0^n {}_tp_x\mu_{x+t}dt$.

b) Using the Makeham's law of mortality $\mu_x = A + Bc^x$, derive an expression for the number of survivors at age x in a life table.

[5, 5]

INDIAN STATISTICAL INSTITUTE

Back-paper Examination

Second Semester 2006–2007

B. Stat (Second year)

Elements of Algebraic Structures

Date: 31.7.2007

Maximum Marks: 100

Duration: 3 hours

Answer all questions.

(1) (a) Let G be a finite abelian group with elements $a_1, a_2, \ldots, a_n$. If G has exactly one element y of order 2 then show that $a_1a_2\ldots a_n = y$.

(b) Show that $\mathbb{Z}_p \setminus \{0\}$ has exactly one element of order 2.

(c) Hence show that $(p-1)! \equiv -1 \bmod p$. 4+7+5

(2) Prove that every group of order 15 is cyclic. 12

(3) (a) Let S_n denote the symmetric group of n elements. What is a k cycle for $k > 1$? Find the number of k cycles in S_n.

(b) Let σ_1 and σ_2 be two disjoint cycles of lengths k_1 and k_2 respectively. What is the order of the product $\sigma_1\sigma_2$? Justify your answer.

5+6

(4) Consider any transcendental number $\alpha \in \mathbb{C}$ over $\mathbb{Q}$. What is the dimension of the field $\mathbb{Q}(\alpha)$ over $\mathbb{Q}$. Justify your answer. 6

(5) Prove or disprove:

(a) The polynomial $f(x) = x^4 + 2x + 2$ is irreducible over $\mathbb{Q}$.

(b) The polynomial $g(x) = x^3 - 4x^2 - 8$ is irreducible over $\mathbb{Z}_5$. 4+6

(6) Let F be a finite field.

(a) Define a polynomial over F which vanishes at every point of F.

(b) Prove that F can not be algebraically closed. 3+6

P.T.O

(7) Find an ideal in the ring $\mathbb{Z} \times \mathbb{Z}$ which is not a prime ideal. Find a prime ideal which is not maximal. Justify your answer in each case. 7+7

(8) Let $C^0[0,1]$ be the set of all continuous real valued functions defined on the interval $[0,1]$. It is a ring under the pointwise addition and multiplication. Show that for each $a \in [0,1]$ $M_a = \{f \in C^0[0,1] : f(a) = 0\}$ is a maximal ideal of this ring. 7

(9) Let E be an extension field of a field F. Show that all elements of E which are algebraic over F form a subfield of E containing F.

Prove that the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$ is an infinite extension of $\mathbb{Q}$. 6+9