Variants of vertex and edge colorings of graphs

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by

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"Color is my daylong obsession, joy, and torment." — Claude Monet

To my mother (Latika Pattanayak) and my son Hasu (Aarki Pattanayak)

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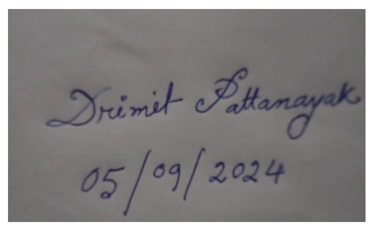

Abstract

A *k-linear coloring* of a graph G is an edge coloring of G with k colors so that each color class forms a *linear forest*—a forest whose each connected component is a path. The *linear arboricity* $\chi'_l(G)$ of G is the minimum integer k such that there exists a k-linear coloring of G. Akiyama, Exoo and Harary conjectured in 1980 that for every graph G, $\chi'_l(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ where $\Delta(G)$ is the maximum degree of G. First, we prove the conjecture for 3-degenerate graphs. This establishes the conjecture for graphs of treewidth at most 3 and provides an alternative proof for the conjecture in some classes of graphs like cubic graphs and triangle-free planar graphs for which the conjecture was already known to be true. Next, we prove that for every 2-degenerate graph G, $\chi'_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$ if $\Delta(G) \geq 5$. We conjecture that this equality holds also when $\Delta(G) \in \{3, 4\}$ and show that this is the case for some well-known subclasses of 2-degenerate graphs. All the above proofs can be converted into linear time algorithms that produce linear colorings of input 3-degenerate and 2-degenerate graphs using a number of colors matching the upper bounds on linear arboricity proven for these classes of graphs. Motivated by this, we then show that for every 3-degenerate graph, $\chi'_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$ if $\Delta(G) \geq 9$. Further, we show that this line of reasoning can be extended to obtain a different proof for the linear arboricity conjecture for all 3-degenerate graphs. This proof has the advantage that it gives rise to a simpler linear time algorithm for obtaining a linear coloring of an input 3-degenerate graph G using at most one more color than the linear arboricity of G.

A *p-centered coloring* of a graph G, where p is a positive integer, is a coloring of the vertices of G in such a way that every connected subgraph of G either contains a vertex with a unique color or contains more than p different colors. As p increases, we get a hierarchy of more and more restricted colorings, starting from proper vertex colorings, which are exactly the 1-centered colorings. Debski, Felsner, Micek and Schroder proved

that bounded degree graphs have p-centered colorings using $O(p)$ colors. But since their method is based on the technique of entropy compression, it cannot be used to obtain a description of an explicit coloring even for relatively simple graphs. In fact, they ask if an explicit p-centered coloring using $O(p)$ colors can be constructed for the planar grid. We answer their question by demonstrating a construction for obtaining such a coloring for the planar grid.

List of publications/communications

- Manu Basavaraju, Arijit Bishnu, Mathew C. Francis, and Drimit Pattanayak. The linear arboricity conjecture for 3-degenerate graphs. Graph-Theoretic Concepts in Computer Science - 46th International Workshop, *WG 2020*, Leeds, UK, June 24–26, 2020, Revised Selected Papers, volume 12301 of *Lecture Notes in Computer Science*, pages 376—387. Springer, 2020.

- Mathew C. Francis and Drimit Pattanayak. A p-centered coloring for the grid using $O(p)$ colors. Published in *Discrete Mathematics*, volume 347, number 1, pages 113670, year 2024.

- Manu Basavaraju, Arijit Bishnu, Mathew C. Francis, and Drimit Pattanayak. The linear arboricity conjecture for graphs of low degeneracy. arXiv:2007.06066 [math.CO]. https://doi.org/10.48550/arXiv.2007.06066.

List of symbols

- $V(G)$: Set of vertices of graph G.
- $E(G)$: Set of edges of graph G.
- $\bar{N}_G(v)$: Set of incident edges at $v \in V(G)$.
- $N_G(v)$: Set of neighboring vertices at $v \in V(G)$.
- $d_G(v)$: Degree of $v \in V(G)$.
- $\Delta(G)$: Maximum degree of G.
- P_G: Set of pivots in G.
- $\bar{P}_G(v)$: Set of pivot edges at $v \in P_G$.
- $\mathcal{L}_G$: Collection of linear colorings of G.
- ${}^k\mathcal{L}_G$: Collection of k-linear colorings of G.
- $Part(X)$: Collection of partitions of the set X.
- $[x]_\alpha$: The unique part in partition α that contains x.

List of Figures

Contents

Chapter 1

Introduction

All graphs that we consider in this thesis will be simple and undirected. Thus, a graph G consists of a set of "vertices", denoted by $V(G)$, and a set of "edges" $E(G)$, where each edge is a subset of $V(G)$ of cardinality 2. For two vertices $u, v \in V(G)$, we shall denote the edge $\{u, v\}$ as simply uv. If $uv \in E(G)$, then we say that the vertices u and v are "adjacent" in G, or that there is an "edge between" u and v. We also say that the edge uv is "incident on" the vertex u (as well as on the vertex v). The "degree" of a vertex is the number of edges that are incident on it. A "subgraph" of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$; in other words, it is a graph that can be obtained by removing some vertices and edges from G (note that the removal of a vertex results in the removal of all edges incident on that vertex). An "induced subgraph" of G is a graph H such that $V(H) \subseteq V(G)$ and $E(H) = \{uv \in E(G) \colon u, v \in V(H)\}$. Thus, an induced subgraph of G is a graph that can be obtained by G using only the removal of some vertices. To denote the fact a graph H having $V(H) = S$ is an induced subgraph of G, we sometimes say that "H is the subgraph induced by S in G". For an edge $uv \in E(G)$, the graph G' obtained by the "contraction" of the edge uv is the graph obtained from G by removing u and v and then adding a new vertex that is adjacent to the vertices that are neighbours of either u or v in G. A *minor* of a graph G is a graph which can be obtained by repeatedly applying the operations of edge removal, vertex removal, or edge contraction.

1.1 Graph coloring

We shall mainly be concerned with the topic of "graph coloring". Graph coloring is a subject with a diversity of research directions. Typically, a graph coloring problem deals with the assignment of labels, which shall be called "colors", satisfying certain properties to specific kinds of structures in the graph. For example, the problem of assigning colors to the vertices of a graph satisfying certain properties is generally called a "vertex coloring" problem. The problems where edges are to be assigned colors are called "edge coloring" problems. In most coloring problems, we are interested in minimizing or maximizing the total number of different colors used. In this thesis, we shall be mainly concerned with some special vertex and edge coloring problems.

We start by giving the formal definition of a vertex/edge coloring of a graph. A *vertex coloring* of a graph is an assignment of colors to the vertices of a graph. Analogously, an *edge coloring* of a graph is an assignment of colors to the edges of a graph. As mentioned before, we usually want the colors assigned to the vertices or edges to satisfy some property; for example, in a *proper vertex coloring*, we want that any two vertices that have an edge between them receive different colors (see Figure 1.1 for an example). In the case of proper vertex coloring, the relevant question is to find such a coloring using the minimum possible number of colors; in fact, finding a proper vertex coloring using the maximum possible number of colors is trivial since one could simply assign each vertex a different color.

More formally, let G be a graph and K be some set of colors. A *vertex coloring* is a mapping $c : V(G) \to K$. Similarly, an *edge coloring* of G is a mapping $c : E(G) \to K$. For any $i \in K$, the set $c^{-1}(i)$ is called the *color class* corresponding to the color i; i.e.

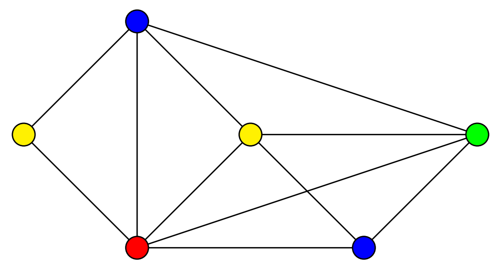

Figure 1.1: An example of a proper vertex coloring

when c is a vertex (respectively, edge) coloring, a color class of c is a set of vertices (respectively, edges) all of which receive the same color in the coloring c. Clearly, the color classes of c partition $V(G)$ (respectively, $E(G)$) when c is a vertex (respectively, edge) coloring. Thus a vertex or edge coloring of a graph can also be seen as a partition of $V(G)$ or $E(G)$ respectively. Thus, a vertex or edge coloring problem is essentially the problem of finding a partition of the vertex set or edge set satisfying certain properties. For example, the problem of finding a proper vertex coloring of a graph using the minimum number of colors is just another way of stating the problem of partitioning the vertex set of a graph into the minimum number of parts in such a way that each part is an "independent set", which is a set of vertices such that no two of them have an edge between them.

Just like in the case of vertex colorings, an edge coloring is said to be *proper* when at each vertex of the graph, no two edges incident to the vertex get the same color (see Figure 1.2 for an example). One can see that in a proper edge coloring, each color class forms a "matching", which is a set of edges such that no two of them have a vertex in common. As before, since finding a proper edge coloring of a graph using the maximum number of colors is trivial (each edge can be assigned a different color), the relevant question here is to find a proper edge coloring of the graph using the minimum possible number of colors.

The notion of proper vertex and edge colorings of graphs is almost as old as graph

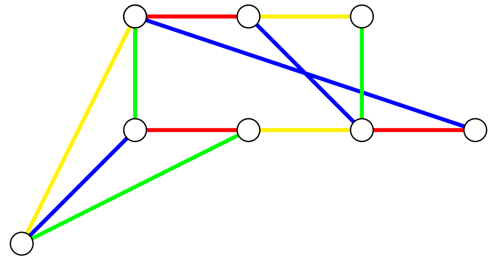

Figure 1.2: An example of a proper edge coloring

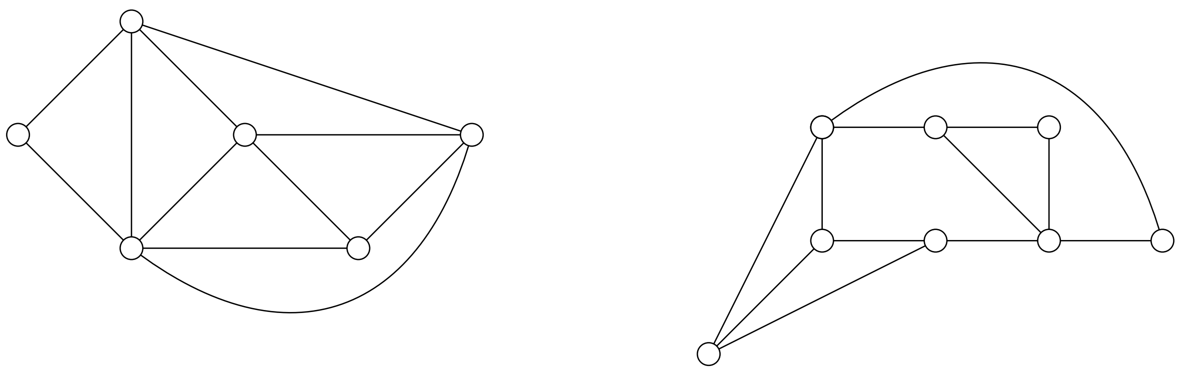

Figure 1.3: Both the graphs in Figures 1.1 and 1.2 are planar as each of them has a planar drawing as shown above.

theory itself. In fact, one of the foundational problems of the field of graph theory was the famous "four color problem", now known as the Four Color Theorem, since its resolution in 1975 by Appel, Haken, and Koch [9, 10, 11]. The *chromatic number* of a graph G, denoted as $\chi(G)$, is the minimum number of colors required in any proper vertex coloring of the graph. A *planar graph* is a graph that can be drawn on the plane in such a way that two edges intersect only at one of their common endpoints, i.e. no edges "cross". Such a drawing of a graph is called a "planar drawing" of the graph (see Figure 1.3). The four color problem was originally stated as the question of whether the regions of any map can be colored with four colors in such a way that two regions that share a boundary always get different colors. This question turns out to be just the question of whether $\chi(G) \leq 4$ for every planar graph G. The Four Color Theorem showed that this is indeed the case: every planar graph has a proper vertex coloring using at most 4 colors. There are special kinds of planar graphs that have been studied in the literature whose chromatic number is smaller than 4. For example, Grötzsch's Theorem [36] states that every triangle-free planar graph (a planar graph that does not contain three mutually adjacent vertices) has chromatic number at most 3. Likewise, the chromatic number of an outerplanar graph (a graph that has a planar drawing in which all vertices lie on the boundary of the outermost region), and more generally any partial 2-tree, is at most 3; in fact, this follows from the fact that the chromatic number of any *2-degenerate graph* is at most 3 (see Section 1.2 for the definition of 2-degenerate graphs). In fact, the question of what is a necessary and sufficient condition for a planar graph to have a chromatic number of at most 3 is an interesting open question that has received considerable attention in the literature (see [66, 18]).

Note that the chromatic number of a graph can be as large as the number of vertices in the graph, but not any larger since giving a different color to each vertex of the graph yields a proper vertex coloring. In fact, there are graphs for which these many colors will be required: a complete graph on n vertices, which we shall denote as K_n, requires n colors in any proper vertex coloring, since there is an edge between each pair of vertices in a complete graph. Thus $\chi(K_n) = n$. It is easy to see that a properly vertex colored graph remains properly vertex colored when edges and/or vertices are removed from it. This implies that $\chi(G) \geq \chi(H)$ for any subgraph H of G. It then follows that for any graph G, $\chi(G) \geq \omega(G)$, where $\omega(G)$ is the size of the largest "clique" (a clique is a set of mutually adjacent vertices) in G. Upper bounds better than n for $\chi(G)$ can be derived by relating it with other graph invariants like the maximum degree. For example, it can be shown that $\chi(G) \leq \Delta(G) + 1$ for every graph G, where $\Delta(G)$ is the maximum degree of a vertex in G. Brooks' Theorem [19, 53] states that any connected graph G has $\chi(G) \leq \Delta(G)$ whenever G is not a complete graph or an odd cycle (see next chapter for the definition of the cycle graph). Note that we shall abbreviate $\Delta(G)$ to just Δ when the graph being referred to is clear from the context.

As for proper edge coloring, the minimum number of colors required in any proper edge coloring of a graph G is called the *chromatic index* of G and is denoted as $\chi'(G)$. It is quite easy to see that in any proper edge coloring, all the edges that are incident with any one vertex have to be given different colors. This means that for any graph G, $\chi'(G) \geq \Delta(G)$. Vizing [70] and Gupta [40] independently proved that any graph G can be properly edge colored with at most $\Delta(G) + 1$ colors. The graphs which can be properly edge colored with $\Delta(G)$ colors are called *class 1 graphs* and those which are not class 1 are called *class 2 graphs*. Many well known classes of graphs are class 1; for example, bipartite graphs, planar graphs having maximum degree at least 7 [63].

There are many active research directions in the study of proper vertex and edge colorings of graphs. There are several unanswered questions about the chromatic number of graphs that remain open despite considerable effort from researchers over many decades. The famous Hadwiger Conjecture aims to generalise Four Color Theorem to much wider classes of graphs. It states that any graph that does not contain K_t as a minor has chromatic number $\chi(G) < t$ (see the next chapter for definition of minor of a graph). Since planar graphs do not contain K_5 as a minor, the case $t = 5$ of Hadwiger's Conjecture implies the Four Color Theorem. This conjecture is known to be true for $t \in \{1, 2, \ldots, 6\}$ [62]. Erdős et al. [16] state that Hadwiger Conjecture is one of the "deepest unsolved problems in graph theory". Kostochka [47] and Thomason [68] independently showed that any graph without a K_t minor can be colored with $O(t\sqrt{\log t})$ colors. Hadwiger Conjecture was shown to be true for line graphs by Reed and Seymour in 2004 [60]. Another famous unsolved problem about the chromatic number of a graph is Reed's Conjecture [2], which states $\chi(G) \leq \left\lceil \frac{\Delta(G)+\omega(G)+1}{2} \right\rceil$. The Gyárfás-Sumner Conjecture, independently stated by Gyárfás [41] and Sumner [67], asserts that for each tree T there exists a function $f_T : \mathbb{N} \to \mathbb{N}$ such that every graph G satisfying $\chi(G) \geq f_T(\omega(G))$ contains an induced subgraph isomorphic to T. As of now, this conjecture has been shown to be true only for very few kinds of trees (see [64]).

As for proper edge coloring, an as yet unresolved conjecture of Vizing [71] states that every planar graph having maximum degree 6 is class 1. Note that there are class 2 planar graphs having maximum degree Δ for each $\Delta \in \{2, 3, 4, 5\}$ [71].

Many variants of vertex and edge colorings have been studied in the literature (refer to [44] for a detailed overview of many types of graph colorings). We give a brief overview of a few of them.

Star coloring of a graph is a proper vertex coloring in which any path on 4 vertices uses at least 3 different colors [31]. It is not difficult to see that these are exactly the proper vertex colorings in which the union of any two color classes induces a subgraph that is isomorphic to a disjoint union of "stars". (A star is the graph $K_{1,t}$, for some $t \in \mathbb{N}$. See Chapter 2 for the definition of the notation $K_{n,m}$). The minimum number of colors required in any star coloring of a graph G is called the *star chromatic number* of G, and is denoted by $\chi_s(G)$. The best known bound on $\chi_s(G)$ in terms of $\Delta(G)$ is due to Esperet and Parreau [29], who showed that for any graph G, $\chi_s(G) \leq 2\sqrt{2}\Delta^{3/2} + \Delta$.

Acyclic vertex coloring is a proper vertex coloring in which each cycle contains vertices of at least 3 different colors [37]; or in other words, there are no "bi-colored" cycles in the graph. In this case, it can be seen that these are exactly the proper vertex colorings that satisfy the property that the union of any two color classes induces a subgraph isomorphic to a forest. Thus, every star coloring is also an acyclic vertex coloring, which implies that the minimum number of colors required in any acyclic vertex coloring of a graph G, denoted by $\chi_a(G)$, is at most $\chi_s(G)$. Alon et al. [6] showed that $\chi_a(G) \leq \lceil 50\Delta^{4/3} \rceil$ and Kostochka and Stocker [48] showed that $\chi_a(G) \leq \left\lfloor \frac{(\Delta+1)^2}{4} \right\rfloor + 1$. The best known bounds on $\chi_a(G)$ for general graphs in terms of the maximum degree are $\chi_a(G) \leq 2.835\Delta^{4/3}+\Delta$ for all values of Δ by Sereni and Volec [65], and $\chi_a(G) \leq \frac{3}{2}\Delta^{4/3} + 5\Delta$, when $\Delta \geq 24$ by Gonçalves et al. [35].

Similarly, an *acyclic edge coloring* is a proper edge coloring in which each cycle has at least 3 colors [8]. Thus it is an edge coloring in which every vertex as at most one edge of each color incident on it, and also the union of any two color classes forms a subgraph that is isomorphic to a forest. The minimum number of colors required in any acyclic edge coloring of a graph G is known as its *acyclic chromatic index*, denoted by $\chi'_a(G)$. The Acyclic Edge Coloring Conjecture, stated independently by Fiamčík [32] and Alon et al. [8], asserts that every graph G has an acyclic edge coloring using at most $\Delta(G) + 2$ colors; i.e. for every graph G, $\chi'_a(G) \leq \Delta(G) + 2$. Despite much work on this conjecture, the best bound on $\chi'_a(G)$ in terms of $\Delta(G)$ that is known is $\chi'_a(G) \leq 4\Delta - 4$ [29].

A *total coloring* of a graph is a coloring of vertices and edges so that no pair of adjacent vertices, no pair of adjacent edges, and no edge and one of its end vertices get the same color. The total chromatic index $\chi''(G)$ of a graph G is the minimum number of colors required in any total coloring of the graph. The Total Coloring Conjecture, independently stated by Behzad [13] and Vizing [69], states that $\chi''(G) \leq \Delta(G) + 2$. The conjecture remains open for general graphs. Bollobás and Harris [17] showed that $\chi''(G) \leq \frac{11}{6}\Delta(G)$, when $\Delta(G)$ is large enough, and Kostochka [46] showed that $\chi''(G) \leq 3\Delta(G)$, when $\Delta(G) \geq 6$. Molloy and Reed [54] showed that there is a constant c such that for every graph G, $\chi''(G) \leq \Delta(G) + c$.

In a *strong edge coloring* of a graph G, the edges in each color class has to be a matching and the edges should also form an induced subgraph of G. The minimum number of colors required in any strong edge coloring of a graph G is denoted by $\chi'_s(G)$, and is called the *strong chromatic index* of the graph. This notion was introduced by Erdős and Nešetřil (see [42]), who also conjectured that for any graph G, $\chi'_s(G) \leq$

$\frac{5}{4}\Delta(G)^2$ when $\Delta(G)$ is even and $\chi'_s(G) \leq \frac{5\Delta(G)^2-2\Delta(G)+1}{4}$ when $\Delta(G)$ is odd. This conjecture also remains open despite much research on the topic (see [76] for a survey).

There also variants of vertex and edge colorings in which the colorings need not be proper. Note that if we do not require a vertex (resp. edge) coloring to be proper, then we are relaxing our condition that each color class has to be an independent set (resp. matching). Thus many problems that are stated in terms of partitioning the vertex set or edge set of a graph can be reformulated as coloring problems. For example, a conjecture of Erdős and Gallai [28] states that the edge set of every graph on n vertices can be partitioned into at most $\lceil \frac{n}{2} \rceil$ paths. This can be reformulated as follows: the edges of every graph on n vertices can be colored with at most $\lceil \frac{n}{2} \rceil$ colors in such a way that the edges in each color class form a path. Another example is the notion of *conflict-free colorings on open neighbourhoods* of graphs. These are vertex colorings of graphs in which the neighbourhood of every vertex x contains a uniquely colored vertex; i.e. a vertex whose color is different from that of every other neighbour of x (see [1, 14]).

Graph coloring problems have found applications in a variety of different problems in computer science; for example, clustering algorithms [52] and register allocation [20]. In such contexts, the algorithmic complexity of computing a coloring of the required type becomes important. Unfortunately, for most coloring problems, it turns out that computing a coloring using the optimum number of colors is NP-complete. The problem of determining if an input graph has a proper vertex coloring using 3 colors was one of the earliest problems that was shown to be NP-complete [34]. Even more interestingly, even though it follows from Vizing's Theorem that every graph G of maximum degree Δ has chromatic index either Δ or $\Delta + 1$, it is NP-hard to determine for an input graph which of these is the case; i.e. determining if a graph is class 1 or class 2 is NP-hard [51]. In a similar vein, even though every planar graph is known to have a proper vertex coloring using at most 4 colors, it is NP-hard to determine if the chromatic number given planar graph is at least 4 [34].

Even then, the proofs for many known upper bounds on coloring problems are constructive and they can be used to construct efficient algorithms that compute a coloring of the required type using at most the number of colors given by the upper bound. For example, the proof of the Four Color Theorem yields an $O(n^2)$ time algorithm that produces a coloring of any input planar graph on n vertices using at most 4 colors [61].

In this thesis, we shall derive upper bounds for two coloring problems: one a variant of edge coloring and the other a vertex coloring.

The first problem that we study is a kind of edge coloring known as *linear coloring*. A graph whose each connected component is a path is called a *linear forest*. Linear coloring of a graph is an edge coloring of the graph in which each color class is a linear forest. We prove that a well-known conjecture called the "Linear Arboricity Conjecture" is true for a certain class of graphs called "3-degenerate graphs" (see Section 1.2 for details). We also construct efficient algorithms to compute linear colorings using optimal or close to optimum number of colors for 2-degenerate and 3-degenerate graphs.

We then study a form of vertex coloring known as *p-centered coloring* (here, p is a natural number). A vertex coloring of a graph G is said to be a p-centered coloring if in any connected subgraph H of G, either there is a vertex whose color is different from that of every other vertex of H, or H contains more than p different colors. It turns out that 1-centered colorings are the same as proper vertex colorings and 2-centered colorings are the same as star colorings. Thus as p increases, we get proper colorings with additional special properties. We answer a question asked in the literature regarding the existence of p-centered colorings using $O(p)$ colors for the infinite grid. The next two sections explain these results in more detail. We refer [24] for any terms not defined here.

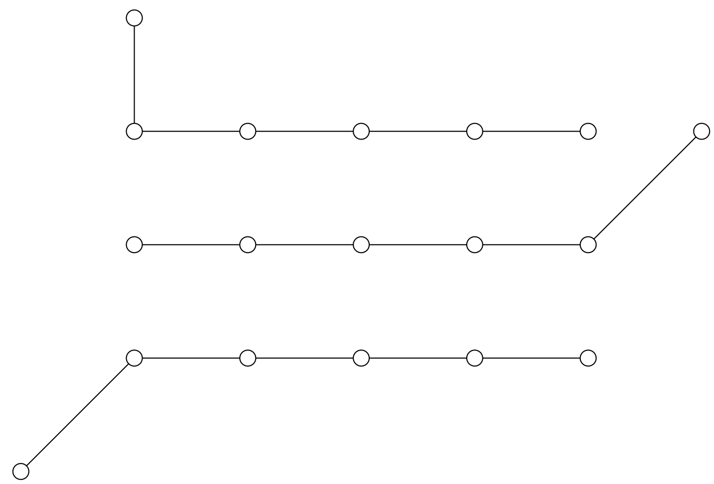

Figure 1.4: An example of a linear forest.

1.2 Linear colorings

As defined above, a linear forest is a graph whose each component is a path. Note that linear forests are exactly the acyclic graphs having maximum degree at most 2. A linear coloring of a graph G is an edge coloring α such that for every color $i \in \alpha(E(G))$, the subgraph of G formed by the edges $\alpha^{-1}(i)$ is a linear forest. The class of all linear colorings of G is denoted by $\mathcal{L}_G$. Note that a proper edge coloring of a graph G is also a linear coloring of G. For any graph G, the linear chromatic index $\chi'_l(G) = \min\{|\rho(E(G))| : \rho \in \mathcal{L}_G\}$, i.e. it is the minimum number of colors required in any linear coloring of G.

A graph G is d-degenerate if any subgraph H of G has a vertex u satisfying $d_H(u) \leq d$. Alternatively, it is a graph from which one can repeatedly remove a vertex of degree at most d until no vertices remain.

Note that for any graph G, $\chi'_l(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil$, since in any linear coloring of G, there can be at most 2 edges of the same color incident with any vertex. In fact, as noted by Harary [43], if G is a Δ-regular graph, then $\chi'_l(G) \geq \lceil \frac{\Delta+1}{2} \rceil$. The Linear Arboricity Conjecture suggests that this lower bound for regular graphs is also an upper bound

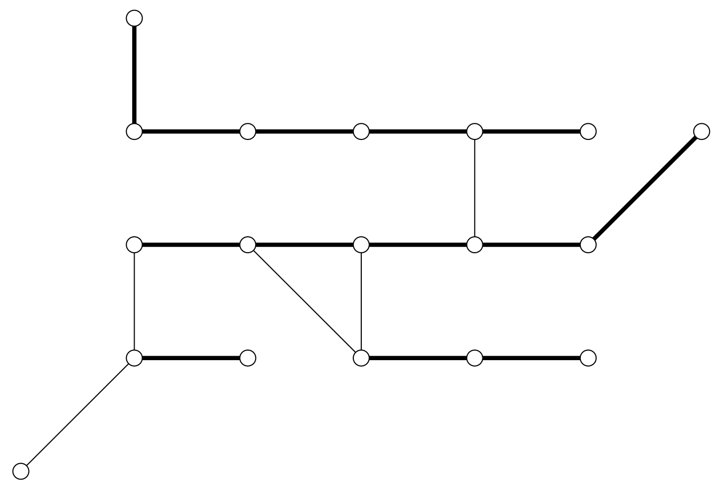

Figure 1.5: An example of a linear coloring: thin and thick edges form two linear forests

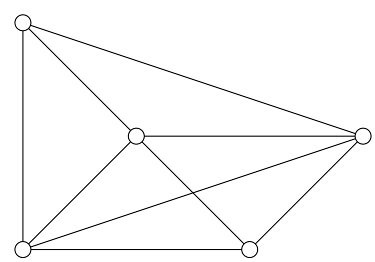

Figure 1.6: An example of a 3-degenerate graph

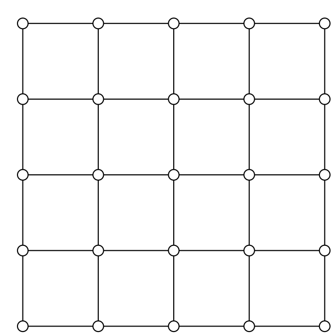

Figure 1.7: An example of a 2-degenerate graph

for the linear arboricity of general graphs.

We state the conjecture below.

Conjecture 1 (Akiyama, Exoo, Harary (1981))**.** *For any graph G, $\chi'_l(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

The conjecture has been proven for all graphs G such that $\Delta(G) \in \{3, 4, 5, 6, 8, 10\}$ [3, 4, 27, 38] and was shown to be true for planar graphs by Wu and Wu [73, 74]. Cygan et al. [22] proved that the linear arboricity of planar graphs which have $\Delta \geq 10$ is $\lceil \frac{\Delta}{2} \rceil$. Yang et al. [75] have recently verified the conjecture for graphs that do not contain K_5 as a minor.

Works of Alon [5], Alon and Spencer [7], and Ferber et al. [30] show that the conjecture holds asymptotically; in particular, for any $\epsilon > 0$ there exists a Δ_0 such that $\chi'_l(G) \leq (\frac{1}{2} + \epsilon)\Delta(G)$ whenever $\Delta(G) \geq \Delta_0$. The best bound in this direction currently is the result of Lang and Postle [50], who showed in 2023 that for every graph G having a large enough maximum degree $\Delta(G)$, $\chi'_l(G) \leq \frac{\Delta}{2} + 3\sqrt{\Delta} \log^4 \Delta$.

From Vizing's Theorem [70], which says that any graph can be properly edge colored with $\Delta + 1$ colors, we get that $\chi'_l(G) \leq \Delta(G) + 1$ for any graph G. The best known general bound for linear arboricity is $\lceil \frac{3\Delta}{5} \rceil$ when Δ is even and $\lceil \frac{3\Delta+2}{5} \rceil$ for Δ odd, obtained by Guldan [38, 39].

Kainen [45] showed that $\chi'_l(G) \leq \left\lceil \frac{\Delta(G)+k-1}{2} \right\rceil$ for every k-degenerate graph G. This result implies that the linear arboricity conjecture is true for 2-degenerate graphs. In fact, a stronger statement is known to be true for 2-degenerate graphs: the acyclic chromatic index of 2-degenerate graphs is at most $\Delta + 1$ [12]. Thus the edges of any 2-degenerate graph G can be properly colored using at most $\Delta(G) + 1$ colors such that the union of any two color classes is a forest. Since the union of any two color classes in such a coloring will always be a linear forest, we get that $\chi'_l(G) \leq \left\lceil \frac{\chi'_a(G)}{2} \right\rceil$, and so the linear arboricity conjecture for 2-degenerate graphs follows from the fact that for every 2-degenerate graph G, $\chi'_a(G) \leq \Delta(G) + 1$.

Although arboricity (minimum number of colors required to color the edges of a graph so that all the color classes are forests) can be computed in polynomial time [33], computing linear arboricity is NP-hard [58]. As $\chi'_l(G) \geq \lceil \frac{\Delta}{2} \rceil$ for any graph G, a 2-factor approximation algorithm for computing linear arboricity can be obtained using Vizing's Theorem. Cygan et al. [22] showed an $O(n \log n)$ algorithm that produces a linear coloring of every planar graph on n vertices with the optimum number of colors when $\Delta(G) \geq 9$. Duncan, Eppstein and Kobourov [26] gave an $O(n)$ algorithm to construct a 2-linear coloring of a graph of maximum degree 3 using depth-first search.

Our results

In Chapter 3, we prove the Linear Arboricity Conjecture for 3-degenerate graphs (Section 3.1). A consequence of this result is that the conjecture holds for the class of graphs having treewidth at most 3 (also called partial 3-trees; refer Chapter 2 for definition of treewidth). Our method can also serve as an alternative proof for the validity of the Linear Arboricity Conjecture on triangle-free planar graphs, Halin graphs, and cubic graphs, for which the conjecture is already known to be true [4]. Note that 3-degenerate graphs generalize cubic graphs, and that they can be non-planar and can have arbitrarily large treewidth. We can take a graph G of arbitrarily large treewidth and construct a graph G' by subdividing each edge (for every edge, we delete it and introduce a new vertex whose only neighbours are the end points of the deleted edge). It is easy to see that G' is 2-degenerate, and hence also 3-degenerate. Since G' contains G as a minor, it follows that the treewidth of G' is at least as large as that of G.

Cygan et al. [22] showed that for every planar graph G, $\chi'_l(G) \leq \left\lceil \frac{\Delta(G)}{2} \right\rceil$, if $\Delta(G) \geq 10$. This upper bound does not hold for all planar graphs with $\Delta \leq 5$ and for the range $6 \leq \Delta \leq 9$, it is open whether every planar graph has a $\lceil \frac{\Delta}{2} \rceil$-linear coloring.

We show in Section 3.2 that every 2-degenerate graph G has $\chi'_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$ if

Graph class	$\Delta \leq 2$	$3 \leq \Delta \leq 4$	$5 \leq \Delta \leq 8$	$\Delta \geq 9$
2-degenerate	$\lceil \frac{\Delta+1}{2} \rceil$	$\lceil \frac{\Delta+1}{2} \rceil^\star$	$\lceil \frac{\Delta}{2} \rceil$	$\lceil \frac{\Delta}{2} \rceil$
3-degenerate	$\lceil \frac{\Delta+1}{2} \rceil$	$\lceil \frac{\Delta+1}{2} \rceil^\star$	$\lceil \frac{\Delta+1}{2} \rceil^\star$	$\lceil \frac{\Delta}{2} \rceil$

Table 1.1: Upper bounds on linear arboricity: The improved bounds due to results obtained in this thesis are shown in blue. The bounds marked by $\star$ are not known to be tight.

$\Delta(G) \geq 5$. The statement is not true when $\Delta \leq 2$ (any cycle is a counterexample), but we conjecture that $\chi'_l(G) = 2$ for every 2-degenerate graph G when $\Delta(G) \in \{3, 4\}$. As evidence towards this conjecture, we prove in Section 3.2.2 the existence of 2-linear colorings in various subclasses of 2-degenerate graphs having $\Delta \leq 4$. An implication of one of these results is that every 2-degenerate graph on n vertices with maximum degree 4 and having $2n - 3$ edges (the maximum possible number of edges) contains a Hamiltonian path. Outerplanar graphs are 2-degenerate graphs; so our results on 2-degenerate graphs also hold for outerplanar graphs.

In Chapter 4, we show that for every 3-degenerate graph G, $\chi'_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$ if $\Delta(G) \geq 9$. This proof, which is based on an approach different from that of Chapter 3, also yields a different proof for the Linear Arboricity Conjecture on 3-degenerate graphs. This approach has the benefit that converting this proof into a linear-time algorithm is more straightforward. Moreover, the linear colorings constructed using this approach have some special properties.

We show in Chapter 5 how all our proofs can be converted into linear time algorithms that generate linear colorings of input graphs using a number of colors that matches the upper bounds obtained.

Table 1.1 summarizes the upper bounds on linear arboricity that were obtained.

1.3 Centered colorings

A *centered coloring* is a vertex coloring such that every connected subgraph contains a vertex having a unique color. Nešetřil and Ossona de Mendez [55] introduced the *treedepth* of a graph and showed that it is exactly the minimum number of colors needed in a centered coloring of the graph. The treedepth of a graph G is defined as follows. A *rooted forest* is a graph whose each connected component is a rooted tree. The "height" of a rooted forest is the maximum length of a path from a root to a leaf in the rooted forest. Let G be a graph and let F_G be a rooted forest of minimum height having the property that any two adjacent vertices in G have an ancestor-descendant relationship in the forest F_G; i.e. if $uv \in E(G)$, then both u and v lie on some path from a root to a leaf in F_G. The height of F_G is called the treedepth of G. The minimum number of colors required in a centered coloring of a graph G is the same as the treedepth of G. The treedepth of a graph also turns out to be equal to some other graph parameters that were known in the literature (we refer the reader to [55, 49] for more details).

In Figure 1.8, a graph G is shown with a centered coloring using 4 colors. The corresponding tree F_G of height 4 is also shown in the figure. It is not difficult to see that there is no centered coloring for this graph using 3 colors. If at all it were possible, then there must exist a rooted forest F_G having height at most 3. Since G is connected F_G must be a rooted tree, and since the graph contains triangles, the height of F_G must be exactly equal to 3. Then as xpq is a triangle, one of x, p or q is the root of T. But since ysr is also a triangle, one of y, s, r also has to be the root of F_G; this is a contradiction.

Centered colorings have connections to notions such as "conflict-free coloring" of hypergraphs. A hypergraph is a pair $H = (V, E)$ where $V = V(H)$ (vertex set) is any set and $E = E(H)$ (edge set) is a collection of subsets of $V(H)$. A vertex coloring of a hypergraph is a mapping $f : V \rightarrow C$ where C is a set of colors. A vertex coloring

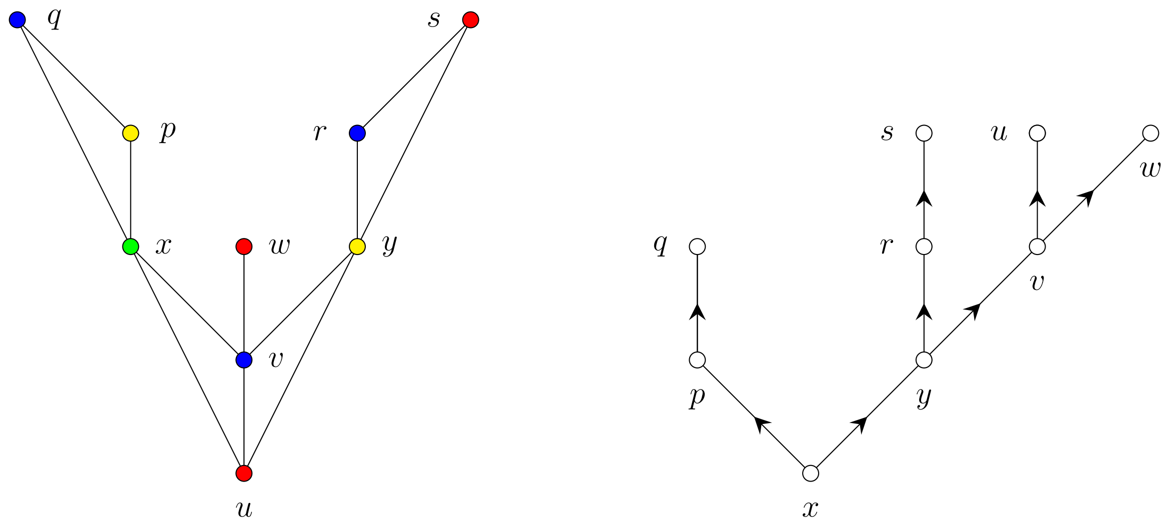

Figure 1.8: An example of a graph G with a centered coloring using 4 colors, and a rooted forest F_G of height 4 rooted at x

g of a hypergraph H is conflict-free if for any $e \in E(H)$ there is a vertex $v \in e$ such that $g(v) \notin g(e \setminus \{v\})$. Conflict-free coloring was introduced in [57]. From a graph G we can define a hypergraph H_G where we take $V(H_G) = V(G)$ and we define edge set $E(H_G) = \{V(A) : A \text{ is a connected subgraph of } G\}$. Notice that a centered coloring of G is nothing but a conflict-free coloring of H_G.

A vertex coloring of a graph G is *p-centered* if in any connected subgraph of G, either there is a uniquely colored vertex or there are more than p colors, where $p \in \mathbb{N}$. It is not difficult to see that a p-centered coloring of a graph G is a coloring of $V(G)$ such that for every $i \leq p$, the union of i color classes induces a subgraph of G whose each connected component has a centered coloring using at most i colors. Thus, a p-centered coloring can be said to be a "low treedepth coloring".

It can be seen that a 1-centered coloring is nothing but a proper vertex coloring as follows. In a vertex coloring, if any two adjacent vertices receive the same color, then the connected subgraph containing those two vertices and the edge between them contains no unique color and also does not contain more than one color. This means that 1-

centered colorings are all proper vertex colorings. On the other hand, any connected subgraph containing at least two vertices in a properly vertex colored graph contains at least one edge, and therefore contains more than one color. Thus proper vertex colorings are 1-centered colorings.

Now consider a 2-centered coloring of a graph. As a 2-centered coloring is also a 1-centered coloring, we know that it is a proper vertex coloring too. If any path on four vertices is colored using just 2 colors, then the path is a connected subgraph that neither contains a unique color, nor contains more than 2 colors, which contradicts the fact that the coloring is a 2-centered coloring. Thus, in a 2-centered coloring of a graph, any path of four vertices must contain at least three colors. This shows that 2-centered colorings are star colorings. Now consider a star coloring of a graph G. Suppose that it is not a 2-centered coloring of G. Then it contains a connected subgraph H that contains at most 2 colors, say "red" and "blue", and contains no uniquely colored vertex. Consider any vertex v in H. If in H, there is no vertex that is at a distance 2 from v, then every vertex in $V(H) \setminus \{v\}$ is a neighbour of v, which implies that v is a uniquely colored vertex in H, a contradiction. Thus we can assume that every vertex has some vertex at distance 2 from it. Now let v be any vertex in H. Assume without loss of generality that v is colored "red". Let $u \in V(H)$ be a vertex that is at distance 2 from v in H. Let $x \in V(H)$ be the middle vertex on a path of length 2 between u and v; i.e. x is a common neighbour of u and v. Clearly, x is colored "blue" and u is colored "red". Now by our earlier observation, we know that there is a vertex y at distance 2 from x. Let z be the middle vertex on a path of length 2 between x and y. As before, we have that z is colored "red" and that y is colored "blue". Clearly, either $z \neq u$ or $z \neq v$. Let us assume without loss of generality that $z \neq u$. Then $uxzy$ is a path on four vertices in H, and therefore it is a path on four vertices in G containing just 2 colors. This contradicts the fact that the coloring of G that we have is a star coloring. Thus, 2-centered colorings are exactly the star colorings. For larger values of p, p-centered colorings become more and more restricted.

The notion of p-centered colorings of graphs was introduced by Nešetřil and Ossona de Mendez [55] in the course of their development of the theory of "structurally sparse" graph classes. In 2004, DeVos et al. [23] showed that every proper minor closed class of graphs has a *low treewidth coloring*: there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that the vertex set of any graph in the class can be partitioned into $f(p)$ parts in such a way that for every $i \leq p$, the union of any i parts induces a subgraph of treewidth at most $i-1$. Since the treedepth of a graph is always at least one more than its treewidth [15] (in fact, one more than its pathwidth) and the treewidth of any graph is equal to the maximum of the treewidth of its connected components, low treedepth colorings are a generalization of low treewidth colorings. The following generalization of the result of DeVos et al. was shown in [55]: for every proper minor closed class of graphs, there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that for every integer $p \geq 1$, every graph in the class has a p-centered coloring using at most $f(p)$ colors. In a landmark paper, Nešetřil and Ossona de Mendez [56] improved this further and showed that the classes of graphs having low treedepth colorings, the classes of graphs having low treewidth colorings, the classes of graphs for which there exists a function $f : \mathbb{N} \to \mathbb{N}$ such that every graph in the class has a p-centered coloring using at most $f(p)$ colors for every $p \geq 1$, are all the same and characterized these classes as the classes of graphs with *bounded expansion* (see [56] for details).

It was first shown by Pilipczuk and Siebertz [59] that for every proper minor closed family of graphs, there exists a *polynomial* function $f : \mathbb{N} \to \mathbb{N}$ such that for every positive integer p, every graph in the class has a p-centered coloring using at most $f(p)$ colors; or in other words, proper minor closed families of graphs "admit polynomial centered colorings". Dębski, Felsner, Micek and Schröder [25] combined different techniques to obtain several path-breaking results on p-centered colorings, including substantial improvements to and tightening of some bounds in [59]. Perhaps the most surprising result in their work is their proof using entropy compression that graphs of bounded degree have p-centered colorings using just $O(p)$ colors: they showed that for any positive integer p, every graph G having maximum degree Δ has a p-centered coloring using $O(\Delta^{2-\frac{1}{p}}p)$ colors.

Our result

The technique of Dębski, Felsner, Micek and Schröder does not provide an explicit construction of a p-centered coloring using $O(p)$ colors for graphs of bounded maximum degree, and they remark that an explicit construction of such a coloring is not known even for the planar grid. In Chapter 6, we give a simple and direct construction of a p-centered coloring of the planar grid, also known as the two dimensional grid, using $O(p)$ colors.

Chapter 2

Preliminaries

We give some common graph theoretic definitions here. For any graph G, the notation $V(G)$ and $E(G)$ shall denote its vertex set and edge set respectively. When $u, v \in V(G)$, and $uv \in E(G)$, we say that u and v are adjacent in G, or that u and v are neigbhours of each other. All graphs that we consider will be loopless and simple; i.e. no vertex is adjacent to itself, and there is at most one edge between any pair of vertices. For a vertex $u \in V(G)$, the "neighbourhood of u in G", denoted as $N_G(u)$ is the set $\{v \in V(G) \colon uv \in E(G)\}$. The set of edges "incident on u", denoted by $\bar{N}_G(u)$ is the set $\{xy \in E(G) \colon x = u\}$. The "degree" of a vertex $u \in V(G)$ is $d_G(u) = |N_G(u)| = |\bar{N}_G(u)|$. The "maximum degree" of G, denoted by $\Delta(G) = \max\{u \in V(G) \colon d(u)\}$. Whenever G is clear from the context, we write $N(u)$ for $N_G(u)$ and $d(u)$ for $d_G(u)$, for $u \in V(G)$, and Δ instead of $\Delta(G)$. A graph H is a subgraph of a graph G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If H is a subgraph of G, we write $H \subseteq G$. For any $U \subseteq V(G)$, let $G[U]$ denote the induced subgraph of G on U. That is $V(G[U]) = U$ and $E(G[U]) = \{uv \in E(G) : u, v \in U\}$. A "connected component", or simply "component", of a graph G is a maximal connected subgraph of the graph G. If $U \subseteq E(G)$, then $G - U$ is a graph satisfying $V(G-U) = V(G)$ and $E(G-U) = E(G) \setminus U$. For $e \in E(G)$, we abbreviate $G - \{e\}$ to just $G - e$. If $U \subseteq V(G)$, then $G - U$ is a graph with $V(G-U) = V(G) \setminus U$ and

$E(G-U) = E(G) \setminus \bigcup_{u \in U} \bar{N}_G(u)$. For $v \in V(G)$, we abbreviate $G - \{v\}$ to just $G - v$. Planar graphs are those graphs which can be drawn on the Euclidean plane such that no two edges cross (such a drawing of a planar graph is called a planar embedding of the graph). Trees and forests, Halin graphs are examples of planar graphs. Outerplanar graphs are planar graphs which have a planar embedding in which every vertex lies on the boundary of the unbounded face.

We let K_n denote the complete graph of n vertices and C_n the cycle on n vertices. We denote by $K_{m,n}$ the complete bipartite graph whose two partite sets are of size m and n.

Let G be a graph. A *tree decomposition* of G is a pair $(T, \{X_t\}_{t \in V(T)})$ where T is a tree and for each $t \in V(T)$, $X_t \subseteq V(G)$ such that:

1. $\bigcup_{t \in V(T)} X_t = V(G)$. That is for any $v \in V(G)$ there is a $t \in V(T)$ such that $v \in X_t$.

2. If $uv \in E(G)$, then there is $t \in V(T)$ such that $u, v \in X_t$.

3. For any $v \in V(G)$, if $K = \{t \in V(T) : v \in X_t\}$, then the induced subgraph $T[K]$ of T is connected.

If $s = \min\{|X_t| : t \in V(T)\}$, then the *width* of $(T, \{X_t\}_{t \in V(T)})$ is defined as $s - 1$. The minimum width among all tree decompositions of G is the *treewidth* of G, denoted by $tw(G)$.

The treewidth of K_n is $n - 1$. The treewidth of any tree is 1. It is easy to see that treewidth is a "minor monotone" property: if H is a minor of G, then $tw(H) \leq tw(G)$. Every graph G is $tw(G)$-degenerate. A *k-tree* is a graph G which is either a K_{k+1} or is the graph obtained by adding a vertex of degree k to a k-tree such that neighbors of the newly added vertex form a clique. It is folklore that an edge maximal graph of treewidth k is a k-tree. A subgraph of a k-tree is a *partial k-tree*. Note that partial k-

trees are exactly the graph of treewidth at most k. Outerplanar graphs have treewidth at most 2; i.e. they are partial 2-trees. Partial 2-trees form a subclass of planar graphs and are also known as "series-parallel graphs".

Planar graphs are the class of graphs that contain no K_5 or $K_{3,3}$ as a minor (Kuratowski's Theorem, see [24]). Partial 2-trees are exactly the graphs that do not contain a K_4 as a minor. Outerplanar graphs are exactly graphs that do not contain a K_4 or a $K_{2,3}$ as a minor. Being partial 2-trees, outerplanar graphs are 2-degenerate. Even though outerplanar graphs have treewidth at most 2, planar graphs can have arbitrarily large treewidth. A common example is the $n \times n$ grid (a 5×5 is shown in Figure 1.7), which has treewidth $\Omega(n)$. Nevertheless, every planar graph is 5-degenerate.

Chapter 3

Linear arboricity of 3-degenerate and 2-degenerate graphs

Let c be an edge coloring of a graph G. Observe that $\bigcup_{i \in c(E(G))} \{c^{-1}(i)\}$ is a partition of the edge set $E(G)$ and that $c = \bigcup_{e \in E(G)} \{(e, c(e))\} \subseteq E(G) \times K$. Recall that the class of all linear colorings of G is denoted by $\mathcal{L}_G$. We also define ${}^k\mathcal{L}_G = \{\alpha \in \mathcal{L}_G : |\alpha(E(G))| = k\}$. Any coloring $\alpha \in {}^k\mathcal{L}_G$ is called a k-linear coloring of G. Let G be a graph and $\mathcal{C}_G = \bigcup_{H \subseteq G} \mathcal{L}_H$. We define a binary operation $\oplus$ on $\mathcal{C}_G$. Let $f \in \mathcal{L}_A$ and $g \in \mathcal{L}_B$ where $A, B \subseteq G$. We define $f \oplus g = (f \cup g) \setminus \bigcup_{e \in E(A) \cap E(B)} \{(e, f(e))\}$. If f is a linear coloring of the graph A and g is a linear coloring of the graph B then $f \oplus g$ is an edge coloring of the graph $A \cup B$ (where $V(A \cup B) = V(A) \cup V(B)$ and $E(A \cup B) = E(A) \cup E(B)$). In $E(A) \setminus E(B)$ the coloring takes color from f and $E(B)$ takes the colors from g. For any graph G, the linear chromatic index $\chi'_l(G) = \min\{|\rho(E(G))| : \rho \in \mathcal{L}_G\}$ is the minimum number of colors used in any linear coloring of G.

Recall that a graph G is d-degenerate if any $H \subseteq G$ has a vertex $u \in V(H)$ satisfying $d_H(u) \leq d$; this is equivalent to saying that one can repeatedly remove a vertex of degree at most d from a d-degenerate graph until no vertices remain. We define a set P_G for any d-degenerate graph G. When $\Delta(G) > d$, let $U = \{v \in V(G) : d_G(v) \leq d\}$. For this case $P_G = \{u \in V(G) \setminus U : d_{G-U}(u) \leq d\}$. If $\Delta(G) \leq d$, then $P_G = V(G)$. The set P_G is called the set of *pivots* of G. For any $v \in P_G$ we define $\bar{P}_G(v) = \{uv \in \bar{N}_G(v) : d_G(u) \leq d\}$. The set $\bar{P}_G(v)$ is called the set of *pivot edges* of G at $v \in P_G$. Note that $P_G \subseteq V(G)$ and for $v \in P_G$, $\bar{P}_G(v) \subseteq \bar{N}_G(v) \subseteq E(G)$. Note that every vertex in P_G has at most d neighbors having degree more than d.

Definition 1. *Let G be a graph and $u, v \in V(G)$ satisfying $uv \notin E(G)$ and $N_G(u) \cap N_G(v) = \emptyset$. We define $G/(u,v)$ to be a graph with $V(G/(u,v)) = V(G) \setminus \{v\}$ and $E(G/(u,v)) = (E(G) \setminus \bar{N}_G(v)) \cup \{uy : y \in N_G(v)\}$. We say that $G/(u,v)$ is a graph obtained from G by* identifying *v with u.*

For any graph G, a vertex $v \in V(G)$ and linear coloring $\alpha : E(G) \to K$ we define three sets $Once_\alpha(v) = \{s \in \alpha(E(G)) : |\alpha^{-1}(s) \cap \bar{N}_G(v)| = 1\}$, $Twice_\alpha(v) = \{s \in \alpha(E(G)) : |\bar{N}_G(v) \cap \alpha^{-1}(s)| = 2\}$ and $Missing_\alpha(v) = K \setminus (Once_\alpha(v) \cup Twice_\alpha(v))$. Also we define $Colors_\alpha(v) = Once_\alpha(v) \cup Twice_\alpha(v)$. Whenever the coloring is understood, we simply write $Missing(v)$, $Once(v)$, $Twice(v)$ and $Colors(v)$.

The first result that we prove in this chapter is that the Linear Arboricity Conjecture holds for 3-degenerate graphs. Our theorem that leads to this result is stated below.

Theorem 1. *For any 3-degenerate graph G, if $\Delta(G) \leq 2k-1$, where $k \in \mathbb{N}$, then $\chi'_l(G) \leq k$.*

This is an extension of the result for subcubic graphs (subcubic graphs are graphs having maximum degree at most 3; it is easy to see that they form a subclass of 3-degenerate graphs) by Akiyama, Exoo and Harary [3].

3.1 Proof Theorem 1

Note that if G contains isolated vertices, then we can simply remove them and compute a k-linear coloring of the resulting graph; this will be a k-linear coloring of G. So we assume that G contains no isolated vertices. For any 3-degenerate graph G we consider the set P_G and for $v \in P_G$ we have $\bar{P}_G(v) \neq \emptyset$. Let $uv \in \bar{P}_G(v)$. The proof is by induction on $|E(G)|$. If $E(G) = \emptyset$ the theorem is true. Let us assume that $|E(G)| > 0$. Assume that the theorem is true for all 3-degenerate graphs having less than $|E(G)|$ edges and $\Delta(G) \leq 2k - 1$. We show that $\chi'_l(G) \leq k$. Clearly, $d_G(u) \leq 3$.

Lemma 1. *If $d_G(v) < 2k - 1$, then $\chi'_l(G) \leq k$.*

Proof. Let $H = G - uv$. Hence $|E(H)| < |E(G)|$. From the induction hypothesis, there is an $\alpha \in {}^k\mathcal{L}_H$.

Observe that $|Once_\alpha(v)| + 2|Twice_\alpha(v)| = d_H(v) \leq 2k - 3$ and $|Once_\alpha(v)| + |Twice_\alpha(v)| + |Missing_\alpha(v)| = k$. Simplifying, we get $2|Missing_\alpha(v)| + |Once_\alpha(v)| \geq 3$. Hence we get the following two possibilities.

1. $|Once_\alpha(v)| \geq 3$ when $Missing_\alpha(v) = \emptyset$.

2. $|Once_\alpha(v) \cup Missing_\alpha(v)| \geq 2$ when $|Missing_\alpha(v)| \geq 1$.

For the first case we select a color $i \in Once_\alpha(v) \backslash Colors_\alpha(u)$ and $\beta = \alpha \oplus \{(uv, i)\} \in {}^k\mathcal{L}_G$. For the second case let $i \in Missing_\alpha(v)$. If $Colors_\alpha(u) \neq \{i\}$, then $\alpha \oplus \{(uv, i)\} \in {}^k\mathcal{L}_G$. Otherwise, let $j \in (Once_\alpha(v) \cup Missing_\alpha(v)) \setminus \{i\}$ and $\alpha \oplus \{(uv, j)\} \in {}^k\mathcal{L}_G$. This completes the lemma.

□

Here onwards we shall assume that $d_G(v) = 2k - 1$. Also we assume that $k \geq 2$ as the case when $k = 1$ is easy. Observe that v has at most 3 neighbors of degree more than 3. Therefore, v has at least $2k - 4$ neighbors of degree at most 3. If $k = 2$ then $\Delta(G) \leq 3$. This implies that every neighbor of v has degree at most 3. If $k \geq 3$, then also there exists 2 neighbors of v of degree at most 3. Thus, there always exists $w \in N_G(v) \setminus \{u\}$ such that $d_G(w) \leq 3$.

Lemma 2. *If $uw \in E(G)$, then $\chi'_l(G) \leq k$.*

Proof. Let $x \in N_G(u) \setminus \{v, w\}$ and $y \in N_G(w) \setminus \{v, u\}$. If x does not exist then add a new vertex x with $d_G(x) = 1$ adjacent to u. Similarly, if y does not exist add a new vertex y with $d_G(y) = 1$ adjacent to w. Let us assume that $H = G - \{u, w\}$. At first assume that $x \neq y$. Since $|E(H)| < |E(G)|$, by the induction hypothesis, there exists an $\alpha \in {}^k\mathcal{L}_H$. Observe that $d_H(x) \leq 2k - 2$. Therefore, there exists a color $i \in Once_\alpha(x) \cup Missing_\alpha(x)$. Let $\beta = \alpha \oplus \{(ux, i)\}$. Now, $d_H(v) = 2k - 3$. We know that either of the following is true:

1. $|Once_\alpha(v)| = 3$.

2. $|Missing_\alpha(v)| = 1$ and $|Once_\alpha(v)| = 1$.

For the first case, let $j, t \in Once_\alpha(v) \setminus \{i\}$ and $Once_\alpha(v) \setminus \{j, t\} = \{l\}$. Let $\beta_1 = \beta \oplus \{(uv, j), (uw, t), (vw, l)\}$. Observe that $d_H(y) \leq 2k - 2$. Thus we have either $Missing_\alpha(y) \geq 1$ or $Once_\alpha(y) \geq 2$. Let $m \in Missing_\alpha(y) \cup (Once_\alpha(y) \setminus \{l\})$. Then we have $\beta_1 \oplus \{(yw, m)\} \in {}^k\mathcal{L}_G$.

Now, we consider the second case. Let $Missing_\alpha(v) = \{t\}$ and $Once_\alpha(v) = \{j\}$. Let $\beta_1 = \beta \oplus \{(uv, t), (uw, j), (vw, t)\}$. Notice that $d_H(y) \leq 2k - 2$. Hence, either $|Missing_\alpha(y)| \geq 1$ or $|Once_\alpha(y)| \geq 2$. Let us assume that $l \in Missing_\alpha(y) \cup (Once_\alpha(y) \setminus \{i\})$. Then, we have $\beta_1 \oplus (yw, l) \in {}^k\mathcal{L}_G$.

Now we consider the case when $x = y$. Clearly in H, $d_H(x) \leq 2k - 3$. There are the following possibilities for x.

1. $|Once_\alpha(x)| \geq 3$.

2. $|Once_\alpha(x) \cup Missing_\alpha(x)| \geq 2$ and $|Missing_\alpha(x)| \geq 1$.

At first assume that $\{i, j, k\} \subseteq Once_\alpha(x)$ and $Once_\alpha(v) = \{p, q, r\}$. Let $\beta = \alpha \oplus \{(uv, p), (vw, q), (uw, r)\}$. Now let $s \in Once_\alpha(x) \setminus \{p, r\}$, and $t \in Once_\alpha(x) \setminus \{s, q\}$. Then $\beta \oplus \{(xu, s), (xw, t)\} \in {}^k\mathcal{L}_G$.

Now let us assume that $Once_\alpha(v) = \{p, q, r\}$ and $i \in Missing_\alpha(x)$ and $j \in (Missing_\alpha(x) \cup Once_\alpha(x)) \setminus \{i\}$. Let $\beta = \alpha \oplus \{(xu, i), (uw, i), (xw, j)\}$. Let $s \in Once_\alpha(v) \setminus \{i\}$ and $t \in Once_\alpha(v) \setminus \{s, j\}$. Then $\beta \oplus \{(uv, s), (vw, t)\} \in {}^k\mathcal{L}_G$. The case when $|Once(x)| \geq 3$, $|Missing(v)| = 1$ and $|Once(v)| = 1$ can be handled similarly.

Now let us assume that $p \in Missing_\alpha(v)$ and $q \in (Missing_\alpha(v) \cup Once_\alpha) \setminus \{p\}$ and $i \in Missing_\alpha(x)$ and $j \in (Missing_\alpha(x) \cup Once_\alpha(x)) \setminus \{i\}$. Let $\beta = \alpha \oplus \{(uv, p), (uw, q), (vw, p)\}$. Let $\beta_1 = \beta \oplus \{(xu, i), (xw, j)\}$. Then $\beta_1 \in {}^k\mathcal{L}_G$.

□

Lemma 3. *If* $(N_G(u) \cap N_G(w)) \setminus \{v\} \neq \emptyset$*, then* $\chi'_l(G) \leq k$*.*

Proof. Let $z \in (N_G(u) \cap N_G(w)) \setminus \{v\}$. We assume that $u \notin N_G(w)$. From Lemma 2 we assume that $x \in N_G(u) \setminus \{v, z\}$ and $y \in N_G(w) \setminus \{v, z\}$. Let us define $H = G - \{u, w\}$. Since $|E(H)| < |E(G)|$ there is an $\alpha \in {}^k\mathcal{L}_H$. Also $d_H(v) = 2k - 3$ and also $d_H(z) \leq 2k - 3$. There are two possibilities for v and z. Hence if $b \in \{v, z\}$, the following holds.

1. $|Once_\alpha(b)| \geq 3$.

2. $|Missing_\alpha(b)| \geq 1$ and $|Missing_\alpha(b) \cup Once_\alpha(b)| \geq 2$.

Since, $d_H(x)$, $d_H(y) \leq 2k - 2$ if $b \in \{x, y\}$ there are the following two possibilities if we assume that $x \neq y$.

1. $|Once_\alpha(b)| \geq 2$.

2. $Missing_\alpha(b) \neq \emptyset$.

We treat v and z symmetrically. First let us assume that $x \neq y$. Suppose that $|Once_\alpha(z)| \geq 3$, $|Missing_\alpha(v) \cup Once_\alpha(v)| \geq 2$ and $|Missing_\alpha(v)| \geq 1$. Let $i \in Missing_\alpha(v)$, $j \in (Missing_\alpha(v) \cup Once_\alpha(v)) \setminus \{i\}$ and $\{p, q, r\} \subseteq Once_\alpha(z)$ so that p, q, r are all distinct from each other. Also let $t \in Missing_\alpha(y) \cup (Once_\alpha(y) \setminus \{j\})$,

$s \in Missing_\alpha(x) \cup Once_\alpha(x)$, $l \in \{p,q,r\} \setminus \{t,j\}$ and $m \in \{p,q,r\} \setminus \{s,l\}$. Let $\beta = \alpha \oplus \{(uv,i),(vw,j),(wy,t),(ux,s),(wz,l),(uz,m)\}$. Clearly, $\beta \in {}^k\mathcal{L}_G$.

Now let $|Once_\alpha(z)| \geq 3$ and $|Once_\alpha(v)| \geq 3$. Suppose that $\{i,j,l\} \subseteq Once_\alpha(v)$ and $\{p,q,r\} \subseteq Once_\alpha(z)$. Let $t \in Missing_\alpha(y) \cup (Once_\alpha(y) \setminus \{j\})$ and $h \in \{i,l\} \setminus \{t\}$. Assume that $\beta = \alpha \oplus \{(vw,j),(wy,t),(uv,h)\}$. Let $s \in Missing_\alpha(x) \cup (Once_\alpha(x) \setminus \{h\})$. Let $\beta_1 = \beta \oplus \{(ux,s)\}$. If $h \in \{p,q,r\}$, then let $f = h$ otherwise let $f \in \{p,q,r\} \setminus \{t,j\}$. Let $\beta_2 = \beta_1 \oplus \{(zw,f)\}$. Let $m \in \{p,q,r\} \setminus \{f,s\}$. Thus $\beta_2 \oplus \{(uz,m)\} \in {}^k\mathcal{L}_G$.

Now let us consider the case $|Missing_\alpha(v)|$,$|Missing_\alpha(z)| \geq 1$ and $|Missing_\alpha(v) \cup Once_\alpha(v)| \geq 2$ and $|Missing_\alpha(z) \cup Once_\alpha(z)| \geq 2$. Let $i \in Missing_\alpha(v)$, $j \in (Missing_\alpha(v) \cup Once_\alpha(v)) \setminus \{i\}$, $p \in Missing_\alpha(z)$ and $q \in (Missing_\alpha(z) \cup Once_\alpha(z)) \setminus \{p\}$. If $i \neq p$, then let $\beta = \alpha \oplus \{(uv,i), (wv,i),(zu,p),(zw,p)\}$ else let $\beta = \alpha \oplus \{(uv,i),(zw,i),(vw,j),(zu,q)\}$. Let $s \in Missing_\alpha(x) \cup (Once_\alpha(x) \setminus \{\beta(uz)\})$ and $t \in Missing_\alpha(y) \cup (Once_\alpha(y) \setminus \beta(wv))$. Then $\beta \oplus \{(ux,s),(wy,t)\} \in {}^k\mathcal{L}_G$.

Now we consider the case $x = y$. The possibilities for x are the same that for v and z since $d_H(x) \leq 2k-3$. We treat v, x and z symmetrically. Then there are four possibilities.

First let us assume that $|Once_\alpha(x)| \geq 3$, $|Once_\alpha(v)| \geq 3$ and $|Once_\alpha(z)| \geq 3$. Let $\{i,j\} \subseteq Once_\alpha(v)$, $p \in Once_\alpha(x) \setminus \{j\}$, and $q \in Once_\alpha(x) \setminus \{p,i\}$. If $i \in Once_\alpha(z)$, we define $s = i$, else we let $s \in Once_\alpha(z) \setminus \{p,j\}$. Finally, let $t \in Once_\alpha(z) \setminus \{s,q,i\}$. Let $\beta = \alpha \oplus \{(uv,i),(vw,j),(xw,p),(xu,q),(zw,s),(uz,t)\}$. Then $\beta \in {}^k\mathcal{L}_G$.

For the second case let us assume that $i \in Missing_\alpha(x)$, $j \in (Once_\alpha(x) \cup Missing_\alpha(x)) \setminus \{i\}$, $|Once_\alpha(v)| \geq 3$, and $|Once_\alpha(z)| \geq 3$. Let $\{p,q\} \subseteq Once_\alpha(v)$. We assume without loss of generality that $j \neq q$. Let $s \in Once_\alpha(z) \setminus \{j,q\}$ and $t \in Once_\alpha(z) \setminus \{s,p\}$. Then $\alpha \oplus \{(uv,p),(vw,q),(xw,j),(ux,i),(zw,s),(zu,t)\} \in {}^k\mathcal{L}_G$.

For the third case let us assume that $i \in Missing_\alpha(x)$, $j \in (Once_\alpha \cup Missing_\alpha) \setminus \{i\}$. Let $s \in Missing_\alpha(v)$ and $t \in (Missing_\alpha(v) \cup Once_\alpha(v)) \setminus \{s\}$. Let $\beta = \alpha \oplus$

Figure 3.1: Deleting edges uv, vw, wx to obtain the graph H (left), and identifying w with u to get H' (right).

$\{(ux, i), (xw, j), (vw, s), (uw, t)\}$. Let $q \in Once_\alpha(z) \backslash \{p, t\}$. Then $\beta \oplus \{(zw, p), (zu, q)\} \in {}^k\mathcal{L}_G$.

For the fourth case let us assume that $i \in Missing_\alpha(x)$, $j \in (Once_\alpha \cup Missing_\alpha) \setminus \{i\}$. Let $s \in Missing_\alpha(v)$ and $t \in (Missing_\alpha(v) \cup Once_\alpha(v) \setminus \{s\})$. Also assume that $p \in Missing_\alpha(z)$ and $q \in Once_\alpha(z) \cup Missing_\alpha(z)$. Let $\beta = \alpha \oplus \{(ux, i), (wx, j), (uv, t), (wv, s)\}$. Let $q \in Once_\alpha(z) \backslash \{j\}$ and $p \in Once_\alpha(z) \backslash \{q, t\}$. Then $\beta \oplus \{(zu, p), (zw, q)\} \in {}^k\mathcal{L}_G$. □

Now we are ready to prove the theorem. Let $x, y \in N_G(w) \setminus \{v\}$ (we can always add degree 1 vertices x, y when they do not exist). By Lemma 2 we assume that $uw \notin E(G)$. Also by Lemma 3 we assume that $N_G(u) \cap N_G(w) = \{v\}$. Let $H = G - \{uv, vw, wx\}$ and $H' = H/(u, w)$ (see Figure 3.1). Observe that H' is a 3-degenerate graph. As $|E(H')| < |E(G)|$ there is a linear coloring $\alpha' \in {}^k\mathcal{L}_{H'}$. Let $t = \alpha'(uy)$. Let $\alpha = (\alpha' \setminus \{(uy, t)\}) \cup \{(wy, t)\}$. Observe that $d_H(x) \leq 2k - 2$. Therefore, either $|Missing_\alpha(x)| \geq 1$ or $|Once_\alpha(x)| \geq 2$. Let $i \in Missing_\alpha(x) \cup (Once_\alpha(x) \setminus \{\alpha(wy)\})$. Assume that $\beta = \alpha \oplus \{(wx, i)\}$. As $d_H(v) = 2k - 3$ we have the following possibilities.

1. $|Missing_\alpha(v)| = 1$ and $|Once_\alpha(v)| = 1$.

2. $|Once_\alpha(v)| = 3$.

For the first case let $Missing_\alpha(v) = \{i'\}$ and $Once_\alpha(v) = \{j\}$. First let us assume that either $i' \in Twice_\beta(u)$ or $i' \in Twice_\beta(w)$. From the symmetry of u and

w let us assume that $i' \in Twice_\beta(u)$. Let $\beta_1 = \beta \oplus \{(uv, j), (vw, i')\}$. Notice, that $i' \in Twice_\beta(w)$ will imply that $\alpha' \notin {}^k\mathcal{L}_{H'}$. Thus $\beta_1 \in {}^k\mathcal{L}_G$. So let us assume that $i' \notin Twice_\beta(u) \cup Twice_\beta(w)$. If $j \in Twice_\beta(u) \cup Twice_\beta(w)$ then let $\beta_1 = \beta \oplus \{(uv, i'), (vw, i')\}$. Observe that β_1 is a linear coloring. Therefore, let us assume that $i', j \notin Twice_\beta(u) \cup Twice_\beta(w)$. If there is a path between u and v of color j in β, then let $\beta_1 = \beta \oplus \{(uv, i'), (vw, j)\}$. Otherwise, let $\beta_1 = \beta \oplus \{(uv, j), (vw, i')\}$. In both the cases we have $\beta_1 \in {}^k\mathcal{L}_G$.

For the second case let us assume that $Once_\beta(v) = \{i', j, l\}$. Let $L \subseteq Once_\beta(v)$ such that if $b \in L$ then there is a b colored path having end vertices u and v in β. Notice that $L \subseteq Once_\beta(u) \cap Once_\beta(v)$. As $d_H(u) \leq 2$, we have $0 \leq |L| \leq 2$. Also observe that if $Twice_\beta(u) \neq \emptyset$ then $L = \emptyset$. Let $|L| = 2$ and $s \in L \setminus Twice_\beta(w)$ and $t \in \{i', j, l\} \setminus L$. Let us assume that $\beta_1 = \beta \oplus \{(vw, s), (uv, t)\}$. Otherwise, let $r \in \{i', j, l\} \setminus Colors_\beta(w)$ and $m \in \{i', j, l\} \setminus \{r\} \cup Twice_\beta(u) \cup L$. Define $\beta_1 = \beta \oplus \{(uv, m), (vw, r)\}$. Then, $\beta_1 \in {}^k\mathcal{L}_G$.

This proves Theorem 1.

As a corollary, we get that the Linear Arboricity Conjecture is true for 3-degenerate graphs.

Corollary 1. *For every 3-degenerate graph G, $\chi'_l(G) \leq \left\lceil \frac{\Delta(G)+1}{2} \right\rceil$.*

We present an alternate proof of this result in Chapter 4.

3.2 2-degenerate graphs

We prove some results on 2-degenerate graphs in the section. These are continuations of our results on 3-degenerate graphs. For any 2-degenerate graph G, the definition of P_G and $\bar{P}_G(v)$ are same as before.

3.2.1 2-degenerate graphs with $\Delta > 4$

Theorem 2. *Let G be any 2-degenerate graph with $\Delta(G) \leq 2k$, where $k \in \mathbb{N} \setminus \{0, 1, 2\}$. Then $\chi'_l(G) \leq k$.*

Proof. Again, note that we can assume that G contains no isolated vertices. The proof is by induction on $|E(G)|$. If $E(G) = \emptyset$ then the theorem is true. Let us assume that $|E(G)| > 0$. Observe that there exists $v \in P_G$ and $uv \in \bar{P}_G(v)$. Notice that $d_G(u) \leq 2$. Let $H = G - uv$. First assume that $d_G(v) < 2k$. From the induction hypothesis, we assume that there is a linear coloring $\alpha \in {}^k\mathcal{L}_H$. Since $d_H(v) = |Once_\alpha(v)| + 2|Twice_\alpha(v)| \leq 2k - 2$ and $|Missing_\alpha(v)| + |Once_\alpha(v)| + |Twice_\alpha(v)| = k$, it is true that $2|Missing_\alpha(v)| + |Once_\alpha(v)| \geq 2$. Hence there are the following two possibilities.

1. $|Missing_\alpha(v)| = 0$ and $|Once_\alpha(v)| \geq 2$.

2. $|Missing_\alpha(v)| \geq 1$.

Note that $d_H(u) \leq 1$. When $|Missing_\alpha(v)| = 0$ let $i \in Once_\alpha(v) \setminus Colors_\alpha(u)$ else let $i \in Missing_\alpha(v)$. The coloring $\beta = \alpha \oplus \{(uv, i)\} \in {}^k\mathcal{L}_G$. Therefore, we assume that $d_G(v) = 2k$ and therefore $d_H(v) = 2k - 1$. Notice that since $k \geq 3$, we have $2k \geq 6$. Therefore, $|\bar{P}_G(v)| \geq 4$. Let $\{vx, vw, vu\} \subset \bar{P}_G(v)$. Let $H' = H - \{vw, vx\}$. From the induction hypothesis there exists a linear coloring $\alpha' \in {}^k\mathcal{L}_{H'}$. Notice that $|Once_{\alpha'}(v)| + 2|Twice_{\alpha'}(v)| = d_{H'}(v) = 2k - 3$ and $|Missing_{\alpha'}(v)| + |Once_{\alpha'}(v)| + |Twice_{\alpha'}(v)| = k$. Combining, we get $2|Missing_{\alpha'}(v)| + |Once_{\alpha'}(v)| = 3$. One of the following is true.

1. $|Once_{\alpha'}(v)| = 3$ when $|Missing_{\alpha'}(v)| = 0$.

2. $|Once_{\alpha'(v)}| = 1$ when $|Missing_{\alpha'}(v)| = 1$.

At first let us consider that $|Once_{\alpha'}(v)| = 3$ and $Once_{\alpha'}(v) = \{c_0, c_1, c_2\}$. Notice that $d_{H'}(u) = d_{H'}(w) = d_{H'}(x) = 1$. For all $i \in \{0, 1, 2\}$, let $f(c_i)$ denote the other end vertex of the path of color c_i starting at v. Let us assume that $A = \{f(c_0), f(c_1), f(c_2)\} \cap$

$\{u, w, x\}$. If $f(c_i) \in A$ then $c_{(i+1) \bmod 3}$ is chosen for the edge $vf(c_i)$. The remaining colors are given arbitrarily to remaining edges. This gives a k-linear coloring of G. Now let us consider the case when $|Missing_{\alpha'}(v)| = 1$ and $|Once_{\alpha'}(v)| = 1$. Assume that $Missing_{\alpha'}(v) = \{a\}$ and $Once_{\alpha'}(v) = \{b\}$. Without loss of generality, let us assume that if at all there is any a colored path (in the coloring α' of H') between any two vertices in $\{u, w, x\}$ it is between w, x. Also without loss of generality let us assume that the b colored path starting from v does not end at x. Then the coloring $\alpha' \oplus \{(uv, a), (wv, a), (xv, b)\} \in {}^k\mathcal{L}_G$.

□

Corollary 2. *For any 2-degenerate graph G, $\chi'_l(G) = \left\lceil \frac{\Delta(G)}{2} \right\rceil$, whenever $\Delta(G) \geq 5$.*

3.2.2 2-degenerate graphs with $\Delta \leq 4$

We conjecture that any 2-degenerate graph of maximum degree 4 has a linear coloring with 2 colors.

Conjecture 2. *For any 2-degenerate graph G, with $\Delta(G) \leq 4$, it is true that $\chi'_l(G) \leq 2$.*

Graphs with large number of edges

We prove the Conjecture 2 when $|E(G)| \geq 2|V(G)| - 5$ (note that a 2-degenerate graph G of more than 1 vertex can have at most $2(|V(G)| - 3) + 3 = 2|V(G)| - 3$ edges). We define a class $\mathcal{F}_k$ (where $k \in \mathbb{N}$) of 2-degenerate graphs of maximum degree at most 4. The class $\mathcal{F}_k$ contains those 2-degenerate graphs G satisfying $\Delta(G) \leq 4$ and $|E(G)| = 2|V(G)| - k$. Observe that $\{\mathcal{F}_k : k \geq 2\}$ is a partition of the class of 2-degenerate graphs of maximum degree at most 4. It is to be noted that $\mathcal{F}_2 = \{K_1\}$. Observe that for all $G \in \mathcal{F}_k$ and any $v \in V(G)$ with $d_G(v) \leq 2$, we have $G - v \in \mathcal{F}_l$ where $l \leq k$.

Lemma 4. *If $G \in \cup_{k=2}^{5} \mathcal{F}_k$ and disconnected, then G has at most two components and at least one component of G is a K_1.*

Proof. Let $G_1, G_2, \ldots, G_r$ be the components of G. It can be seen that $|E(G)| = 2(|V(G_1)|+|V(G_2)|+\ldots+|V(G_r)|)-(l_1+l_2+\ldots+l_r)$ where $G_i \in \mathcal{F}_{l_i}$ and $i \in \{1, 2, \ldots, r\}$. Notice that $2 \leq l_i \leq 5$ for $i \in \{1, 2, \ldots, r\}$. If $r > 2$, then $|E(G)| \leq 2|V(G)| - 6 < 2|V(G)| - k$ such that $k \in \{2, 3, 4, 5\}$. Which is a contradiction. Therefore we assume that $r \leq 2$. When $r = 2$, it is a contradiction when $l_1, l_2 \geq 3$. Therefore at least one of l_1,l_2 is 2. That is there is a component that is a K_1. This completes the proof. □

Definition 2. *Let $\alpha \in \mathcal{L}_G$. We define a set $M_{\alpha,G} = \{v \in V(G) : d_G(v) = 2$ and $|Colors_\alpha(v)| = 1\}$. We call the set $M_{\alpha,G}$ the set of* monochromatic vertices *of G with respect to the linear coloring α.*

Theorem 3. *Every 2-degenerate graph G having $\Delta(G) \leq 4$ and $|E(G)| \geq 2|V(G)| - 5$ has a 2-linear coloring α satisfying $|M_{\alpha,G}| \leq 1$.*

Proof. Let $k \in \{2, 3, 4, 5\}$. Let $G \in \mathcal{F}_k$. The proof is by induction on $|E(G)|$. Notice that when $\Delta(G) \leq 2$, G is a collection of cycles or paths and the graph G has a 2-linear coloring with at most one monochromatic vertex (color the edges alternatively). By Lemma 4, there can be at most one component that is not K_1 of G. So G has a linear coloring with 2 colors such that $|M_{\alpha,G}| \leq 1$ (it is a linear coloring in which only odd cycles have a monochromatic vertex). In this case the theorem is true. The base case is when $|E(G)| = 0$. The base case is trivially true. Now we let that $|E(G)| > 1$ and $\Delta(G) > 2$. Since $\Delta(G) > 2$ there is a $v \in P_G$ such that $d_G(v) > 2$ and $uv \in \bar{P}_G(v)$ (where $u \in N_G(v)$). Clearly, $d_G(u) \leq 2$.

If $d_G(u) = 1$, then let $H = G - u$. Observe that $H \in \mathcal{F}_l$ where $2 \leq l < 5$. As $|E(H)| < |E(G)|$ there is a linear coloring α of H such that $|M_{\alpha,H}| \leq 1$. Notice that $Once_\alpha(v) \cup Missing_\alpha(v) \neq \emptyset$. Let $i \in Once_\alpha(v) \cup Missing_\alpha(v)$. The coloring $\alpha \oplus \{(uv, i)\}$ is a linear coloring of G with 2 colors.

Now assume that $d_G(u) = 2$. Let $d_G(v) = 3$. Let $H = G - u$. Notice that $H \in \mathcal{F}_k$. Therefore there is a linear coloring α of H with colors i, j such that $|M_{\alpha,H}| \leq 1$. As $d_H(v) = 2$, without loss of generality we assume one of the following is true.

1. $Missing_\alpha(v) = \{i\}$.

2. $Once_\alpha(v) = \{i, j\}$.

Let $N_G(u) = \{v, w\}$. Observe that since $d_H(w) < 4$ there exists $t \in Missing_\alpha(w) \cup Once_\alpha(w)$. Let $\alpha' = \alpha \oplus \{(uw, t)\}$.

Let $Colors_{\alpha'}(u) = \{i\}$. Then for the first case let $\beta = \alpha' \oplus \{(uv, i)\}$. For the second case let $\beta = \alpha' \oplus \{(uv, j)\}$. Now let $Colors_{\alpha'}(u) = \{j\}$. Then as the first case we assume that $\beta = \alpha' \oplus \{(uv, i)\}$. For the second case we assume that $\beta = \alpha' \oplus \{(uv, i)\}$. It can be seen that β is a linear coloring of G with two colors satisfying $|M_{\alpha',G}| \leq 1$.

Now let us consider that $d_G(v) = 4$. Clearly, there is $wv \in \bar{P}_G(v) \setminus \{uv\}$. We assume that $d_G(w) = 2$. Otherwise, we are in the case when $d(u) = 1$, which was already handled earlier. Let us assume the case when $N_G(u) = N_G(w) = \{v, x\}$. Note that $x \in P_G$. Therefore, we assume that $d_G(x) = 4$. Let $H = G - \{u, w\}$. Notice that $H \in \mathcal{F}_k$. Therefore, we can assume that there is a linear coloring α of H with colors i, j such that $|M_{\alpha,H}| \leq 1$. Without loss of generality, we assume that either $M_{\alpha,H} \cap \{x, v\} = \{v\}$ or $M_{\alpha,H} \cap \{x, v\} = \emptyset$. For the former case, we assume that $Missing_\alpha(v) = \{i\}$ and let $\beta = \alpha \oplus \{(uv, i), (wv, i), (xu, i), (xw, j)\}$. For the latter case let $\beta = \alpha \oplus \{(uv, j), (wv, i), (xu, i), (xw, j)\}$. Therefore in both the cases $\beta \in {}^2\mathcal{L}_G$ satisfying $|M_{\beta,G}| \leq 1$.

Now let us consider that $N_G(u) = \{v, x_u\}$ and $N_G(w) = \{v, x_w\}$ and $x_u \neq x_w$. Let $H = G - \{uv, wv\}$ and $H' = H/(u, w)$. As $H' \in \mathcal{F}_k$ we assume that there is a linear coloring α' of H' with colors i, j satisfying $|M_{\alpha',H'}| \leq 1$. Let $\alpha = (\alpha' \setminus \{(ux_w, \alpha'(ux_w))\}) \cup \{(wx_w, \alpha'(ux_w))\}$. Let $Colors_\alpha(v) = \{i, j\}$. Also assume that $Colors_\alpha(u) = \{i\}$. Suppose that there is a path of color i having end vertices v and u in the coloring *alpha* of

H'. Then we define $\beta = \alpha \oplus \{(uv, j), (vw, i)\}$. Since u and w are symmetric, we can now assume that there is no monochromatic path to u or w from v. Then we again declare β in the same way. Now assume that $Colors_\alpha(v) = \{i\}$. Now let us assume without loss of generality that $Colors_\alpha(u) = \{i\}$, $Colors_\alpha(w) = \{j\}$. Let $\beta = \alpha \oplus \{(uv, j), (wv, j)\}$. In all the cases β is a linear coloring with 2 colors of G satisfying $|M_{\alpha,G}| \leq 1$.

□

We say that a 2-degenerate graph G is "maximal" if $|E(G)| \geq 2|V(G)| - 3$.

Corollary 3. *Any maximal 2-degenerate graph G of maximum degree at most 4 contains a Hamiltonian path.*

Proof. Let $G = \mathcal{F}_2 \cup \mathcal{F}_3$. If $G = K_1$, this is trivially true. Therefore, we assume that $G \in \mathcal{F}_3$. By Theorem 3, G has a linear coloring with 2 colors. Since the color classes are linear forests, each of these color classes can contain at most $|V(G)| - 1$ edges. As $|E(G)| = |V(G)| - 3$ we must have a color class with $|V(G)| - 1$ edges. Clearly, the edges in this color class form a Hamiltonian path of G. □

Neither of the two conditions in the above corollary can be relaxed: the graph $K_{2,4}$ and the 2-tree on 6 vertices shown in Figure 3.2 have no Hamiltonian paths. Note that the former is a 2-degenerate graph having maximum degree 4, but is not maximal, whereas the latter is a maximal 2-degenerate graph, but has maximum degree more than 4.

Bipartite graphs

Theorem 4. *Any bipartite 2-degenerate graph G of maximum degree at most 4 has a linear coloring α with 2 colors where $|M_{\alpha,G}| = 0$.*

Proof. Let G be a bipartite 2-degenerate graph with $\Delta(G) \leq 4$. We do an induction on both $|V(G)|$ and $|E(G)|$. That is the theorem is true for graphs having either smaller

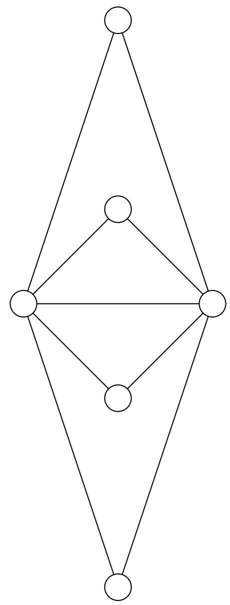

Figure 3.2: A maximal 2-degenerate graph with $\Delta = 5$ without any Hamiltonian path.

number of vertices or smaller number of edges than G. The base case is when $|V(G)| = 1$ or G is a disjoint union of paths or cycles (even). In both the cases, the theorem is true since any proper edge coloring of G is a 2-linear coloring having no monochromatic vertices. Therefore, we assume that $\Delta(G) > 2$. Thus there exist $v \in P_G$ and $uv \in \bar{P}_G(v)$ such that $d_G(v) > 2$.

Let us first assume that $d_G(u) = 1$. Let $H = G - uv$. Since $|E(H)| < |E(G)|$ there exists a linear coloring α of H with 2 colors satisfying $M_{\alpha,H} = \emptyset$. As $d_H(v) \in \{2, 3\}$, $Missing_\alpha(v) \cup Once_\alpha(v) \neq \emptyset$. Let $i \in Missing_\alpha(v) \cup Once_\alpha(v)$. Now, $\alpha \oplus (uv, i)$ is a linear coloring of G with 2 colors satisfying $|M_{\alpha,G}| = 0$.

Thus we assume that $d_G(u) = 2$. First we assume that $d_G(v) = 3$. Let $H = G - uv$. Observe that $|E(H)| < |E(G)|$. Thus there exists a linear coloring α of H with colors i, j satisfying $|M_{\alpha,H}| = 0$. Notice that $d_H(v) = 2$. Clearly, $Colors_\alpha(v) = \{i, j\}$. Without loss of generality, let $Colors_\alpha(u) = \{i\}$. The coloring $\alpha \oplus \{(uv, j)\}$ is our desired coloring.

Now we shall assume that $d_G(v) = 4$. Note that there exists $w \in N_G(v) \setminus \{u\}$. If w is of degree 1 in G, we follow the previous argument. Therefore we assume that $d_G(w) = 2$. First assume that $N_G(u) \cap N_G(w) = \{v, x\}$. Notice that $x \in P_G$. If $d_G(x) = 3$, we follow the previous argument. Therefore, we assume that $d_G(x) = 4$. Let us assume that $H = G - \{u, w\}$. As $|V(H)| < |V(G)|$ there exists a linear coloring α of H with colors i, j satisfying $|M_{\alpha,H}| = 0$. Observe that $Colors_\alpha(x) = Colors_\alpha(v) = \{i, j\}$. The coloring $\alpha \oplus \{(uv, i), (xu, j), (wv, j), (xw, i)\}$ is a linear coloring with 2 colors of G satisfying $|M_{\alpha,G}| = 0$. Now let us assume that $N_G(u) \setminus \{v\} = \{x_u\}$, $N_G(w) \setminus \{v\} = \{x_w\}$ and $x_u \neq x_w$. Let $H = G - \{uv, wv\}$ and $H' = H/(u, w)$. As $|E(H')| < |E(G)|$ there exists a linear coloring α' of H' with colors i, j such that $|M_{\alpha',H'}| = 0$. Let $\alpha = (\alpha' \setminus \{(ux_w, \alpha'(ux_w))\}) \cup \{(wx_w, \alpha'(ux_w))\}$. Clearly, $\alpha \in {}^2\mathcal{L}_H$. Note that $Colors_\alpha(u) \neq Colors_\alpha(w)$ since otherwise u would have been monochromatic in α. So without loss of generality let us assume that $Colors_\alpha(u) = \{i\}$, $Colors_\alpha(w) = \{j\}$ and $Colors_\alpha(v) = \{i, j\}$. We see that $\beta = \alpha \oplus \{(uv, j), (vw, i)\}$ is a valid linear coloring for the graph G using 2 colors and having $M_{\beta,G} = \emptyset$. □

Partial 2-trees

A partial 2-tree is a graph G which has a tree decomposition of width at most 2. Partial 2-trees are 2-degenerate graphs and are closed under taking minor. We state the following folklore result as an observation.

Observation 1. *Every partial 2-tree G contains one of the following configurations.*

1. *A vertex of degree at most one,*

2. *Two adjacent vertices each of degree two,*

3. *Two non-adjacent vertices of degree two that have the same neighborhood,*

4. *A triangle containing a vertex of degree 2, a vertex of degree 3, and a vertex of degree at least 3,*

5. *Two triangles having no common edges, but having a common vertex of degree 4, and in each triangle, apart from the common vertex, there is a vertex of degree 2 and a vertex of degree at least 4.*

Definition 3. *Let G be a graph and $V' = \{v \in V(G) : d_G(v) = 2\}$. We let $\mathcal{P}(V')$ denote the power set of V'. Define $T(G) = \{\mathcal{S} \subseteq \mathcal{P}(V')\colon$ for every $S \in \mathcal{S}$, $|S| = 2$ and for distinct $S, S' \in \mathcal{S}$, we have $S \cap S' = \emptyset\}$. For some $\mathcal{S} \in T(G)$, the linear coloring $f \in {}^2\mathcal{L}_G$ is said to "satisfy" $\mathcal{S}$ if $|M_{f,G} \cap S| \leq 1$ for every $S \in \mathcal{S}$.*

Theorem 5. *Let G be a partial 2-tree having maximum degree at most 4. If $\mathcal{S} \in T(G)$, then there is an $f \in {}^2\mathcal{L}_G$ such that f satisfies $\mathcal{S}$.*

Proof. We shall prove by induction on $|V(G)|$. The theorem is true when $|V(G)| = 1$. We prove when $|V(G)| > 1$. By Observation 1, G contains one of the configurations from $(1) - (5)$. In each case we show how to find a linear coloring satisfying $\mathcal{S}$.

(1) Let $u \in V(G)$ such that $d_G(u) = 1$ and $N_G(u) = \{v\}$. Let $H = G - u$. Using the induction hypothesis, assume that there exists $f \in {}^2\mathcal{L}_H$ such that $f(E(H)) = \{1, 2\}$ satisfying $\mathcal{S} \setminus \{S : v \in S \in \mathcal{S}\}$. Let $i \in Missing_f(v)$ if $d_G(v) = 2$ else let $i \in \{1, 2\} \setminus Twice_f(v)$. Clearly, $g = f \oplus \{(uv, i)\} \in {}^2\mathcal{L}_G$. Also observe that g satisfies $\mathcal{S}$.

(2) Suppose that G contains two adjacent vertices u, v such that $d_G(u) = d_G(v) = 2$. Consider the partial 2-tree $H = G - uv$. If there exist $u', v' \in V(G)$ such that $\{\{u, u'\}, \{v, v'\}\} \subseteq \mathcal{S}$, let $\mathcal{S}' = (\mathcal{S} \cup \{\{u', v'\}\}) \setminus \{\{u, u'\}, \{v, v'\}\}$, otherwise let $\mathcal{S}' = \mathcal{S} \setminus \{S\colon u \in S \text{ or } v \in S\}$. By the induction hypothesis, there exists $f \in {}^2\mathcal{L}_H$ satisfying $\mathcal{S}'$ (since H satisfies (1),H has a 2-linear coloring that satisfies $\mathcal{S}'$). For any $x \in V(G)$ we denote by x' a vertex such that $\{x, x'\} \in \mathcal{S}$ if such x' exists. If u' does not exist, $u' = v$

(which means that $\{u', v'\} = \{u, v\} = \{u, u'\} = \{v, v'\}$), or if u' is not monochromatic, then color uv with a color so that v is not monochromatic (since we can afford to let u be monochromatic). Otherwise, color uv with a color so that u is not monochromatic. It is easy to see that we have a 2-linear coloring of G. Clearly, if one of u', v' does not exist, or if $\{u', v'\} = \{u, v\}$, then the 2-linear coloring that we have constructed satisfies $\mathcal{S}$. Otherwise, $\{u', v'\} \in \mathcal{S}'$, and so only at most one of u', v' can be monochromatic in the coloring f of H given by the induction hypothesis. So in this case too, we have a 2-linear coloring of G that satisfies $\mathcal{S}$.

(3) Suppose that G contains two non-adjacent degree 2 vertices u, v such that $N_G(u) = N_G(v) = \{x, y\}$. Let $H = G - \{u, v\}$. Clearly, H is a partial 2-tree. If there exist $u', v' \in V(G)$ such that $\{u, u'\}, \{v, v'\} \in \mathcal{S}$, let $\mathcal{S}' = (\mathcal{S} \cup \{\{u', v'\}\}) \setminus \{\{u, u'\}, \{v, v'\}\}$, otherwise let $\mathcal{S}' = \mathcal{S} \setminus \{S \colon u \in S \text{ or } v \in S\}$. If $d_H(x) = d_H(y) = 2$, then define $\mathcal{S}'' = \mathcal{S}' \cup \{\{x, y\}\}$, otherwise define $\mathcal{S}'' = \mathcal{S}'$. By the induction hypothesis, there exists a 2-linear coloring f of H that satisfies $\mathcal{S}''$. We construct a 2-linear coloring of G as follows. Assign every edge $e \in E(G - \{u, v\})$ the color $f(e)$. Note that as $\{x, y\} \in \mathcal{S}''$, at least one of x, y is non-monochromatic in the coloring f of H. This means that in order to construct a 2-linear coloring of G, we can color the edges ux, vx, uy, vy in such a way that u is not monochromatic and also in such a way that v is not monochromatic. As before, if u' does not exist, $u' = v$ or if u' is not monochromatic, then we color the edges ux, vx, uy, vy such that v is not monochromatic, otherwise we color those edges so that u is not monochromatic. It is easy to verify that the 2-linear coloring constructed in this manner satisfies $\mathcal{S}$.

(4) Suppose that the vertices u, v, w form a triangle in G and $d_G(u) = 2$ and $d_G(v) \geq 3$, $d_G(w) \geq 3$. Consider the partial 2-tree $H = G - \{u\}$. If there is $u' \in V(G)$ such that $\{u, u'\} \in \mathcal{S}$ then we let $\mathcal{S}' = (\mathcal{S} \setminus \{u, u'\}) \cup \{\{v, u'\}\}$ otherwise we let $\mathcal{S}' = \mathcal{S}$. By the induction hypothesis there is a 2-linear coloring f of H using the colors $\{1, 2\}$

that satisfies $\mathcal{S}'$. We will now construct a 2-linear coloring of G by assigning colors to the edges uv and uw. First we color uw with an arbitrary color from $Missing_f(w) \cup Once_f(w)$ (this set is nonempty as $d_H(w) \leq 3$). Note that $d_H(v) = 2$ and if u' exists, only one of u' or v can be monochromatic in the coloring f of H. If v is monochromatic in H, then we color uv with the color in $Missing_f(v)$, and otherwise we color uv with a color in $\{1, 2\}$ that is different from the color of uw. It is easy to verify that we now have a 2-linear coloring of G that satisfies $\mathcal{S}$.

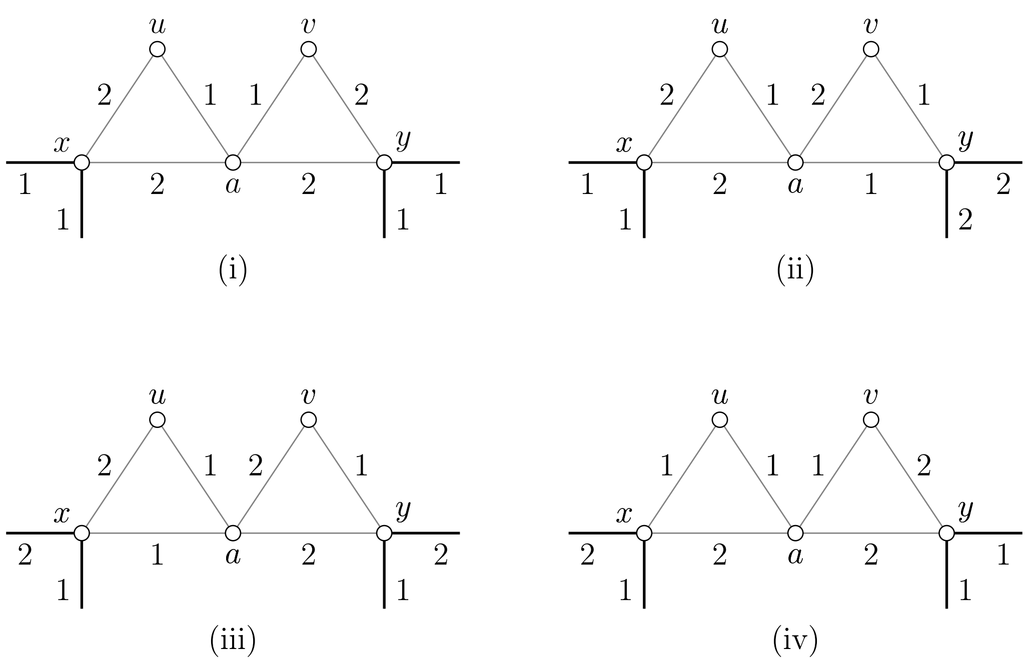

Figure 3.3: Constructing a 2-linear coloring of G from a 2-linear coloring of H in case (5).

(5) Suppose that both $\{a, u, x\}$ and $\{a, v, y\}$ induce triangles in G, where $d_G(u) = d_G(v) = 2$ and $d_G(x) = d_G(y) = 4$. We consider the partial 2-tree $G = H - \{u, v, a\}$. Note that $d_H(x) = d_H(y) = 2$. If there exists $u' \neq v$ such that $\{u, u'\} \in \mathcal{S}$, then define $\mathcal{A} = (\mathcal{S} \setminus \{\{u, u'\}\}) \cup \{\{y, u'\}\}$, otherwise let $\mathcal{A} = \mathcal{S}$. Now if there exists $v' \neq u$ such that $\{v, v'\} \in \mathcal{A}$, then we let $\mathcal{A}' = (\mathcal{A} \setminus \{\{v, v'\}\}) \cup \{\{x, v'\}\}$ otherwise let $\mathcal{A}' = \mathcal{A}$.

Finally, we define $\mathcal{S}' = \mathcal{A}' \setminus \{\{u, v\}\}$. Let f be a 2-linear coloring of H using the colors $\{1, 2\}$ that satisfies $\mathcal{S}'$ that exists by the induction hypothesis. We shall assign colors to the edges ux, ua, va, vy, ax, ay to construct a 2-linear coloring of G. If both x and y are monochromatic or if both x and y are non-monochromatic, then we can color these edges in such a way that both u and v are non-monochromatic as shown in Figures 3.3(i)–3.3(iii). If y is monochromatic and x is non-monochromatic, then we can color these edges as shown in Figure 3.3(iv) so that v is non-monochromatic and u is monochromatic. Symmetrically, if x is monochromatic and y is non-monochromatic, then we can color these edges in such a way that u is non-monochromatic and v is monochromatic. It is easy to verify that in all cases, we have a 2-linear coloring of G that satisfies $\mathcal{S}$.

□

Chapter 4

Optimal linear coloring of 3-degenerate graphs

Recall that given a graph with a linear coloring, a vertex in it is called a monochromatic vertex if it has degree 2 and both edges incident to it have the same color. In this chapter, we first prove that every 3-degenerate graph G has a $\left\lceil\frac{\Delta(G)}{2}\right\rceil$-linear coloring containing no monochromatic vertices if $\Delta(G) \geq 9$. We then extend the techniques used to show that every connected 3-degenerate graph G has a $\left\lceil\frac{\Delta(G)+1}{2}\right\rceil$-linear coloring that contains no monochromatic vertices unless G is an odd cycle (in which case it has a 2-linear coloring containing exactly one monochromatic vertex).

4.1 Some preliminary definitions

Let G be a 3-degenerate graph having $\Delta(G) \geq 3$. If $\Delta(G) = 3$, then every vertex is a pivot and every edge is a pivot edge; in this case, we denote by v some vertex having $d_G(v) = 3$. On the other hand, if $\Delta(G) > 3$, we denote by v a pivot having $d_G(v) \geq 4$. Let F be the set of pivot edges incident on v ($F = \bar{P}_G(v)$). It can be seen that $F \neq \emptyset$. Define $X = \{x \in V(G) : xv \in F\}$. Clearly, for each $x \in X$, we have $d_G(x) \leq 3$. Let

$I = \{xx' \in E(G)\colon x, x' \in X \text{ and } d_G(x) = d_G(x') = 2\}$. Let $H = G - (F \cup I)$. Let $W = \{x \in X\colon d_H(x) = 1\}$. It is not difficult to see that W is an independent set in H as well as in G. For each vertex $w \in W$, let $\overline{w}$ denote its unique neighbor in H. In H, we now choose a maximal set of pairs of vertices from W such that the vertices in each pair can be identified with each other without introducing multiple edges. For this purpose, we define an auxiliary graph A with $V(A) = W$ and $E(A) = \{xy\colon \overline{x} \neq \overline{y}\}$. Let M be a maximal matching of A. Let W' denote the vertices of $V(A) = W$ that remain unmatched by M. Due to the maximality of M, for any two vertices $a, b \in W'$, we have $\overline{a} = \overline{b}$. Thus if $W' \neq \emptyset$, there exists a vertex $\widetilde{w} \in V(H) \setminus (W \cup \{v\})$ such that $\widetilde{w} = \overline{w}$ for every $w \in W'$. We can now identify the vertices in W which are matched to each other by M to construct a graph H' from H.

It is easy to see that the graph H' is a 3-degenerate graph having $\Delta(H') \leq \Delta(G)$. Note that any linear coloring $c_{H'}$ of H' that contains no monochromatic vertices can be converted into a linear coloring c_H of H containing no monochromatic vertices and using the same number of colors by just splitting back the vertices that were identified during the construction of H' from H. We say that c_H is the linear coloring of H "corresponding to" the linear coloring $c_{H'}$ of H'.

For any set $S \subseteq W$, we say that "S satisfies property $\mathcal{P}$" if $|S| \geq 3$ and for each $x \in S$, if there exists $x' \in W$ such that $xx' \in M$, then we also have $x' \in S$.

Claim 1. *Let $c_{H'}$ be a linear coloring of H' containing no monochromatic vertices, and let c_H be the linear coloring of H corresponding to $c_{H'}$. Let $S \subseteq W$. If S satisfies property $\mathcal{P}$, then there exist $x, y \in S$ such that the colors of the edges $x\overline{x}$, $y\overline{y}$ are different in c_H.*

Proof of claim. Suppose S satisfies property $\mathcal{P}$. Assume for the sake of contradiction that for each $x \in S$, $c_H(x\overline{x}) = 1$ (say). Suppose that there exist $u, w \in S$ such that $uw \in M$. Then the edges $u\overline{u}$ and $w\overline{w}$ got their colors in c_H from the colors of two edges incident on a degree 2 vertex in $c_{H'}$. Then this vertex has two edges of color 1 incident on it in $c_{H'}$, which contradicts the fact that there are no monochromatic vertices in $c_{H'}$. As S satisfies property $\mathcal{P}$, we can now conclude that $S \subseteq W'$. As $|S| \geq 3$, we have three distinct vertices $p, q, r \in S$ such that $c_H(p\overline{p}) = c_H(q\overline{q}) = c_H(r\overline{r}) = 1$. Since $S \subseteq W'$, we have that $\overline{p} = \overline{q} = \overline{r} = \widetilde{w}$. Thus $c_H(p\widetilde{w}) = c_H(q\widetilde{w}) = c_H(r\widetilde{w}) = 1$. Then three edges incident on the vertex $\widetilde{w}$ have the same color in c_H, contradicting the fact that c_H is a linear coloring of H. This proves the claim.

4.2 Maximum degree at least 9

Theorem 6. *Every 3-degenerate graph G having maximum degree $\Delta(G) \leq k$, where $k \geq 9$, has a $\lceil \frac{k}{2} \rceil$-linear coloring in which there are no monochromatic vertices.*

Proof. We prove this by induction on $|E(G)|$. If $\Delta(G) \leq 2$, then G is a disjoint union of cycles and paths; in this case, G clearly has a 3-linear coloring in which there is no monochromatic vertex, and we are done. So we can assume that $\Delta(G) \geq 3$.

Let the sets F, I, X, W, and the graphs H, H' be as defined in Section 4.1. Since $|E(H')| < E(G)|$, and $\Delta(H') \leq \Delta(G) \leq k$, we have by the induction hypothesis that there is a $\lceil \frac{k}{2} \rceil$-linear coloring $c_{H'}$ of H' in which there are no monochromatic vertices. Let c_H be the linear coloring of H corresponding to $c_{H'}$. We show how to construct a $\lceil \frac{k}{2} \rceil$-linear coloring of G that does not contain any monochromatic vertices, starting from the coloring c_H. We first construct a $\lceil \frac{k}{2} \rceil$-linear coloring c of $G - I$ that does not contain any monochromatic vertices by extending c_H. Once this is done, the linear coloring c can be easily further extended to the required $\lceil \frac{k}{2} \rceil$-linear coloring of G by assigning to each edge $xx' \in I$ a color that is different from both $c(vx)$ and $c(vx')$ (this can be done since $\lceil \frac{k}{2} \rceil \geq 5$). We describe below how the $\lceil \frac{k}{2} \rceil$-linear coloring c of $G - I$ is constructed. Note that W is exactly the set of vertices in X that have degree 2 in

$G - I$. In the following, we denote by $d(u)$ the degree of a vertex u in the graph $G - I$.

Let $\mathcal{C}$ denote the set of edge colorings (that are not necessarily linear colorings) of $G - I$ using colors in $\{1, 2, \ldots, \lceil \frac{k}{2} \rceil\}$ that can be obtained from c_H by coloring the edges in F using the colors from $Missing(v) \cup Once(v)$ such that every color in $Missing(v)$ is given to at most two edges of F and every color in $Once(v)$ is given to at most one edge of F (here, the sets $Missing(v)$ and $Once(v)$ are with respect to the coloring c_H). In other words, $\mathcal{C}$ is the set of edge colorings of $G - I$ that are extensions of c_H in which at most two edges of the same color are incident on v. Notice that $\mathcal{C} \neq \emptyset$, since we can always generate a coloring that belongs to $\mathcal{C}$ by the following procedure: each color in $Missing(v)$ is assigned to some two edges of F, and each color in $Once(v)$ is assigned to some edge in F, so that every edge in F gets colored. This can be done because $d(v) \leq \Delta(G) \leq k$. We make the following observation.

Since c_H did not contain any monochromatic vertex, in every coloring in $\mathcal{C}$, no vertex has more than two edges of the same color incident on it.

Thus, for any coloring in $\mathcal{C}$, the subgraph formed by the edges having the same color is a disjoint union of cycles and paths. Note that given any coloring in $\mathcal{C}$, permuting the colors on the edges in F always gives another coloring in $\mathcal{C}$.

Among the colorings in $\mathcal{C}$, let c denote a coloring that contains the smallest possible number of monochromatic vertices, and subject to that, contains the smallest possible number of monochromatic cycles. We claim that c is a linear coloring of $G - I$ that does not contain any monochromatic vertices. From here onward, the sets $Missing(v)$ and $Once(v)$ shall be with respect to the coloring c.

Suppose that there exists a monochromatic vertex u in c. Since there are no monochromatic vertices in c_H and $d(v) \geq 3$, we can conclude that $u \in W$. Suppose first that there exists a color $i \in Missing(v) \cup (Once(v) \setminus \{c(uv)\})$. Then we can change the color of uv to i so that u is no longer a monochromatic vertex. It is easy to see that we have not introduced any new monochromatic vertex, and hence we have a coloring in $\mathcal{C}$ that has fewer monochromatic vertices than c, which contradicts our choice of c. So we can assume that $Missing(v) = \emptyset$ and that $Once(v) \subseteq \{c(uv)\}$, which implies that $|Missing(v) \cup Once(v)| \leq 1$. This means that $d(v) \geq k-1$ (note that this immediately implies that $d(v) \geq 8$). As v is a pivot, it follows that $|F| = |X| \geq k-4$.

Next, suppose that there exists $x \in X \setminus W$. Then $d(x) \in \{1, 3\}$. If $c(uv) \neq c(xv)$, then we can interchange the colors of uv and xv so as to obtain a coloring in $\mathcal{C}$ having a smaller number of monochromatic vertices than c, which contradicts our choice of c. So we can assume that for every $y \in X \setminus W$, $c(uv) = c(yv)$. Since at most one edge in F other than uv can have the color $c(uv)$, we can now assume that $X \setminus W = \{x\}$, and also that $c(xv) = c(uv)$. Note that this implies that for every $y \in W \setminus \{u\}$, $c(yv) \neq c(uv)$. Since $|X| \geq k-4$, we now have $|W| \geq k-5$. Let $S = W$. As $k \geq 9$, we then have $|S| \geq 4$. Clearly, S satisfies property $\mathcal{P}$. By Claim 1, we know that there exists $y \in S$ such that $c(y\overline{y}) \neq c(u\overline{u}) = c(uv)$ (note that $y \neq u$). We can then interchange the colors of yv and uv to obtain a coloring in $\mathcal{C}$ that has fewer monochromatic vertices than c, which contradicts the choice of c. So we can assume that $X \setminus W = \emptyset$, or in other words $X = W$. If there exists $w \in W \setminus \{u\}$ such that $c(vw) = c(uv)$, then let $S = W \setminus (\{w\} \cup \{w' \in W \colon ww' \in M\})$; otherwise, let $S = W$. It is easy to see that $|S| \geq |W| - 2 = |X| - 2 \geq k - 6 \geq 3$. Thus S satisfies property $\mathcal{P}$. By Claim 1, it follows that there exists $y \in S$ such that $c(y\overline{y}) \neq c(u\overline{u}) = c(uv)$. Now, by interchanging the colors of the edges uv and yv, we can obtain a coloring in $\mathcal{C}$ that has fewer monochromatic vertices than c, which contradicts the choice of c. We can thus conclude that there are no monochromatic vertices in c.

Next let us suppose that c contains a monochromatic cycle. Let each edge of this cycle have color 1 (say). As there are no monochromatic cycles in c_H, we can infer that an edge $uv \in F$ is contained in this monochromatic cycle. Let $u'v$ be the other edge incident to v that is contained in the monochromatic cycle. Clearly, $c(uv) = c(u'v) = 1$. As there are no monochromatic vertices in c, we have $d(u) \geq 3$ and $d(u') \geq 3$, which implies that $u, u' \notin W$. As $uv \in F$, we further have that $d(u) = 3$. Let 2 be the color of the edge incident on u that is not part of the monochromatic cycle. We denote by P the monochromatic path from v to u containing only edges colored 1, and not containing the edge uv; i.e. it is the path obtained by removing the edge uv from the monochromatic cycle. Since there is only one edge colored 2 incident on u, we know that there is a maximal monochromatic path, all of whose edges are colored 2, having one endpoint u. Let us denote this path by Q.

Suppose that there exists $i \in Missing(v) \cup (Once(v) \setminus \{2\})$. Then we can change the color of uv to i so as to obtain a coloring in $\mathcal{C}$ that does not contain any monochromatic vertices and contains fewer monochromatic cycles than c. As this is a contradiction to the choice of c, we can assume that $Missing(v) = \emptyset$ and $Once(v) \subseteq \{2\}$. This implies that $d(v) \geq k - |Once(v)|$. As before, since v is a pivot, we have $|F| = |X| \geq k - |Once(v)| - 3$. Note that since $|Once(v)| \leq 1$, this means that $|F| = |X| \geq k - 4$.

Claim 2. *Let $xv \in F$ such that $x \notin \{u, u'\}$. Let c' be the coloring in $\mathcal{C}$ obtained from c by exchanging the colors of the edges uv and xv. If c' does not have fewer monochromatic cycles than c, then $c(xv) = c'(uv) = 2$ and c' contains a monochromatic cycle colored 2 containing the edge uv.*

Proof of claim. It is clear that the monochromatic cycle colored 1 in c is no longer a monochromatic cycle in c', since $c'(uv) = c(xv) \neq 1$. If there is a new monochromatic cycle in c', then clearly, it has to contain either the edge xv or the edge uv. In the former case, i.e. there is a monochromatic cycle colored 1 in c' containing the edge xv, since P is a path colored 1 from v to u in c' as well, we have that u also belongs to this cycle. But this contradicts the fact that there is only one edge colored 1 incident on u in c'. Thus, if at all a new monochromatic cycle arises in c', it has to be one containing the edge uv. Since the only color other than 1 that appeared on the edges incident on u in c was 2, it follows that this new monochromatic cycle is colored 2, which implies that $c'(uv) = c(xv) = 2$. This proves the claim.

First, suppose that v is not contained in Q. If there exists $x \in X \setminus (W \cup \{u, u'\})$, then we exchange the colors of the pivot edges xv and uv to obtain a new coloring c' in $\mathcal{C}$. Clearly, the coloring c' does not contain any monochromatic vertices. By Claim 2 and our choice of c, we have that c' contains a monochromatic cycle colored 2 containing the edge uv, which we shall denote by C. Then $C - uv$ is a path in c', all of whose edges are colored 2, from u to v. The edge xv is not on this path since $c'(xv) = 1$, and therefore $C - uv$ is a path colored 2 in c too, implying that v is contained in the path Q in c, contradicting our assumption that v is not contained in Q. So we can assume that $X \setminus W \subseteq \{u, u'\}$. As $|X| \geq k - 4$, we now have $|W| \geq k - 6 \geq 3$. Let $S = W$. It is easy to see that S satisfies property $\mathcal{P}$, and therefore by Claim 1, we have that there exists $y \in S$ such that $c(y\overline{y}) \neq 1$. We now exchange the colors of the edges uv and yv to obtain a new coloring c' in $\mathcal{C}$. By our choice of y, it follows that there are no monochromatic vertices in c'. Then by our choice of c, we have that c' does not have fewer monochromatic cycles than c, which implies by Claim 2 that there is a monochromatic cycle colored 2 containing the edge uv in c'. As before, this implies that there is a monochromatic path colored 2 between u and v in c, which contradicts our assumption that v does not lie on Q.

So we can assume that v is contained in Q. Let zv be the first edge on Q that is incident on v (when traversing the path Q starting from u), and let Q_z denote the subpath of Q between u and z. Since there are no monochromatic vertices in c, we know that $d(z) \geq 3$. Suppose that $zv \in F$, i.e. $z \in X$. Then we exchange the colors of zv and uv to obtain a new coloring c' in $\mathcal{C}$. It is easy to see that this does not create any monochromatic vertices. By our choice of c, we then have that c' does not contain fewer monochromatic cycles than c. Then by Claim 2, we know that c' has a monochromatic cycle colored 2 containing the edge uv, which we shall denote by C. Note that even in the coloring c', the path Q_z is a monochromatic path colored 2 between u and z. Thus, the fact that u is contained in C implies that z is contained in C, which contradicts the fact that exactly one edge colored 2 is incident on z in c'. We can thus conclude $zv \notin F$, or in other words, $z \notin X$.

If there exists $x \in X \setminus W$ such that $c(xv) \notin \{1, 2\}$, then we can exchange the colors of the edges xv and uv to obtain a new coloring c' in $\mathcal{C}$. Clearly, c' does not contain any monochromatic vertices, and therefore by our choice of c, it should contain at least as many monochromatic cycles as c. Then by Claim 2, we have that $c(xv) = 2$, which contradicts the fact that $c(xv) \notin \{1, 2\}$. We can thus assume that for every $x \in X \setminus W$, $c(xv) \in \{1, 2\}$. Since there can be at most two edges of each color incident on v, and $zv \notin F$ is an edge colored 2 incident on v, we have that $|X \setminus W| \leq 3 - |Once(v)|$. Thus, $|W| \geq |X| - 3 + |Once(v)|$. Recalling that $|X| \geq k - |Once(v)| - 3$, we now have that $|W| \geq k - 6 \geq 3$. Note that for every $y \in W$, we have $c(yv) \neq 1$, since $u, u' \notin W$. Suppose that there exists $y \in W$ such that $c(y\overline{y}) \neq 1$ and $c(yv) \neq 2$, then we exchange the colors of yv and uv to obtain a new coloring c' in $\mathcal{C}$. Clearly, c' does not contain any monochromatic vertices, and therefore by the choice of c and Claim 2, it follows that $c'(uv) = 2$, which contradicts the fact that $c(yv) \neq 2$. So we can assume that for every $y \in W$ such that $c(yv) \neq 2$, we have $c(y\overline{y}) = 1$.

Let $S = W$. Since S satisfies property $\mathcal{P}$, we have by Claim 1 that there exists $y \in S$ such that $c(y\overline{y}) \neq 1$. Then by the above observation, we have that $c(yv) = 2$. Since zv and yv are two edges colored 2 incident on v, we now have that for every $p \in X \setminus \{y\}$, $c(pv) \neq 2$, and also that $|Once(v)| = 0$. Then using our previous observation that for every $x \in X \setminus W$, $c(xv) \in \{1, 2\}$, we can conclude that for every $x \in X \setminus W$, $c(xv) = 1$. This implies that $|X \setminus W| \leq 2$. This gives $|W| \geq |X| - 2 \geq k - |Once(v)| - 5 = k - 5 \geq 4$.

Thus there exists $w \in W \setminus \{y\}$ such that $c(wv) \neq c(y\overline{y})$. Notice that $c(wv) \neq 2$, which implies by our observation from the previous paragraph that $c(w\overline{w}) = 1$. We now construct a new coloring c' in $\mathcal{C}$ by setting $c'(uv) = c(wv)$, $c'(yv) = c(uv) = 1$, $c'(wv) = c(yv) = 2$, and by giving every other edge the same color as it has in c. Then c' is a coloring in $\mathcal{C}$ containing no monochromatic vertices. Since $c(wv) \notin \{1, 2\}$, we have that there is no monochromatic cycle containing uv in c'. Since y and w are not monochromatic vertices in c', it is clear that neither yv nor wv are contained in monochromatic cycles in c'. This implies that c' contains fewer monochromatic cycles than c, which contradicts our choice of c. □

Corollary 4. *Every 3-degenerate graph G having $\Delta(G) \geq 9$ has a $\left\lceil \frac{\Delta(G)}{2} \right\rceil$-linear coloring.*

4.3 Maximum degree less than 9

Theorem 7. *Any 3-degenerate graph G having $\Delta(G) \leq 7$ has a 4-linear coloring containing no monochromatic vertices.*

Proof. We prove this by induction on $|E(G)|$. Clearly, if $\Delta(G) \leq 2$, then G has a 4-linear coloring containing no monochromatic vertices. So we assume that $\Delta(G) \geq 3$.

Let the sets F, I, X, W, and the graphs H, H' be as defined in Section 4.1. As $|E(H')| < |E(H)|$, by the induction hypothesis, there is a 4-linear coloring $c_{H'}$ of H' that contains no monochromatic vertices. Let c_H be the 4-linear coloring of H containing no monochromatic vertices corresponding to $c_{H'}$. We assume that the colors used in $c_{H'}$ and c_H are from $\{1, 2, 3, 4\}$. As in the earlier proof, let $\mathcal{C}$ denote the edge colorings using colors $\{1, 2, 3, 4\}$ of $G - I$ obtained by extending c_H by coloring the edges in F in such a way that no color occurs more than twice on the edges incident on v. Again, it is easy to see that any coloring in $\mathcal{C}$ has the property that each vertex has at most two edges of the same color incident on it. Let c be a coloring in $\mathcal{C}$ containing the least number of monochromatic vertices, and subject to that, the least number of monochromatic cycles. In the following, the sets $Missing(v)$, $Once(v)$, and $Twice(v)$ are with respect to the coloring c, and we denote by $d(u)$ the degree of a vertex u in $G - I$.

Claim 3. *There are no monochromatic vertices in c.*

Proof of claim. For the sake of contradiction, let us assume that u is a monochromatic vertex in c. Clearly, $u \in W$. If there exists any color $i \in (Missing(v) \cup Once(v)) \setminus \{c(uv)\}$, we can recolor uv with i to obtain a coloring in $\mathcal{C}$ having lesser number of monochromatic vertices than c, contradicting our choice of c. Therefore, we assume that $Missing(v) = \emptyset$ and $Once(v) \subseteq \{c(uv)\}$. This means that $|Missing(v) \cup Once(v)| \leq 1$, which implies that $d(v) \geq 7$. Since $\Delta(G) \leq 7$, we have $d(v) = 7$ and therefore $Once(v) = \{c(uv)\}$, and also that $|X| \geq 4$. If there is a vertex $w \in X \setminus W$, then we can exchange the colors of wv and uv to obtain a coloring in $\mathcal{C}$ that contains fewer monochromatic vertices than c, which again contradicts our choice of c. So we assume that $X = W$, which implies that $|W| \geq 4$. The set W satisfies property $\mathcal{P}$. Then by Claim 1, there exists $z \in W$ such that $c(z\bar{z}) \neq c(u\bar{u})$. Now exchanging the colors on the edges uv and zv gives a coloring in $\mathcal{C}$ with fewer monochromatic vertices than c, again leading to the same contradiction (notice that $c(zv) \neq c(uv)$ since $Once(v) = \{c(uv)\}$). This proves the claim.

Claim 4. *There are no monochromatic cycles in c.*

Proof of claim. Let C be a monochromatic cycle in c. Then as c_H did not contain any monochromatic cycles, there exists $wv \in E(C) \cap F$. Using Claim 3, we infer that $d(w) = 3$, which means that $w \in X \setminus W$. Let us denote by 1 the color of the edges in C, and by 2 the color of the edge incident on w that is not in $E(C)$. Let P be the maximal path formed by edges of color 2 starting from w. Let x denote the end vertex of P other than w. Notice that since $d(v) \leq 7$, there exists $i \in Once(v) \cup Missing(v)$. Clearly, $i \neq 1$. If $x \neq v$, then we can recolor wv with the color i to obtain a coloring in $\mathcal{C}$ with no monochromatic vertices and fewer monochromatic cycles than c, which contradicts our choice of c. So we assume that $x = v$. If there is any $i \in (Once(v) \cup Missing(v)) \setminus \{2\}$, then we can recolor wv with i to again obtain a coloring in $\mathcal{C}$ that contains no monochromatic vertices and fewer monochromatic cycles than c, which contradicts our choice of c. So we infer that $d(v) = 7$ and $Once(v) = \{2\}$. If there exists $u \in (X \setminus \{w\}) \cap V(C)$, then recoloring uv with 2 gives a coloring in $\mathcal{C}$ that contradicts the choice of c. Therefore, we can assume that $(X \setminus \{w\}) \cap V(C) = \emptyset$. Then if there is any $u \in X \setminus \{w\}$, then $c(uv) \neq 1$, and exchanging the colors on the edges wv and uv gives a coloring in $\mathcal{C}$ that contradicts our choice of c. Hence we can assume that $X \setminus W = \{w\}$ and therefore $|W| \geq 3$. Then W satisfies property $\mathcal{P}$, and by Claim 1, there exists $u \in W$ such that $c(u\bar{u}) \neq 1$. Now we can exchange the colors on the edges uv and wv to again get a coloring in $\mathcal{C}$ with no monochromatic vertices and fewer monochromatic cycles than c, which contradicts our choice of c. This proves the claim.

From Claims 3 and 4, it follows that c is a 4-linear coloring of $G - I$ containing no monochromatic vertices. We can now extend c to a 4-linear coloring of G that does not contain any monochromatic vertices by coloring each edge $xx' \in I$ using a color that is different from $c(vx)$ and $c(vx')$. This completes the proof. □

Theorem 8. *A 3-degenerate graph G having $\Delta(G) \leq 5$ has 3-linear coloring.*

Proof. As before, we prove this by induction on $|E(G)|$. Since G has a 3-linear coloring containing no monochromatic vertices if $\Delta(G) \leq 2$, we assume that $\Delta(G) \geq 3$.

Let the sets F, I, X, W, and the graphs H, H' be as defined in Section 4.1. As $|E(H')| < |E(H)|$, by the induction hypothesis, there is a 3-linear coloring $c_{H'}$ of H' that contains no monochromatic vertices. We now construct a new 3-linear coloring $c'_{H'}$

of H', also containing no monochromatic vertices, by modifying $c_{H'}$ a little if required. Let $W = \{u, w\}$ and $\bar{u} = \bar{w} = x$ (say). Suppose that $c_{H'}(ux) = c_{H'}(wx)$. Since x has degree at most 5 in H', there is a color i of $c_{H'}$ that does not occur twice on the edges incident on x. We now recolor one of the edges ux or wx in $c_{H'}$ with the color i to obtain a new edge coloring $c'_{H'}$ of H'. It is easy to see that $c'_{H'}$ is also a 3-linear coloring of H' having no monochromatic vertices. If $|W| \neq 2$ or if $W = \{u, w\}$, but $\bar{u} \neq \bar{w}$ or $c_{H'}(u\bar{u}) \neq c_{H'}(w\bar{w})$, then we let $c'_{H'} = c_{H'}$. Thus, the coloring $c'_{H'}$ of H' has the special property that if $W = \{u, w\}$ and $\bar{u} = \bar{w}$, then $c'_{H'}(u\bar{u}) \neq c'_{H'}(w\bar{w})$.

Let c_H be the 3-linear coloring of H containing no monochromatic vertices corresponding to $c'_{H'}$. We assume that the colors used by $c'_{H'}$ and c_H are from $\{1, 2, 3\}$. We again let $\mathcal{C}$ denote the edge colorings using colors in $\{1, 2, 3\}$ of $G - I$ obtained by extending c_H by coloring the edges in F in such a way that no color occurs more than twice on the edges incident on v. As before, every coloring in $\mathcal{C}$ has the property that each vertex has at most two edges of the same color incident on it. Let c be a coloring in $\mathcal{C}$ containing the least number of monochromatic vertices, and subject to that, the least number of monochromatic cycles. In the following, the sets $Missing(v)$, $Once(v)$, and $Twice(v)$ are with respect to the coloring c, and we denote by $d(u)$ the degree of a vertex u in $G - I$.

Claim 5. *There are no monochromatic vertices in c.*

Proof of claim. Let u be a monochromatic vertex in c. Let $i \in (Once(v) \cup Missing(v)) \setminus \{c(uv)\}$. Then we can recolor the edge uv with the color i to obtain a coloring in $\mathcal{C}$ that contains fewer monochromatic vertices than c, which contradicts our choice of c. So $Missing(v) = \emptyset$ and $Once(v) \subseteq \{c(uv)\}$. This implies that $d(v) \geq 5$. Since $\Delta(G) \leq 5$, we have $d(v) = 5$, which implies that $Once(v) = \{c(uv)\}$ and $|X| \geq 2$. If $w \in X \setminus W$, then we can exchange the colors of uv and wv to get a coloring in $\mathcal{C}$ with fewer monochromatic vertices, again contradicting our choice of c. So $|W| = |X| \geq 2$.

If $c(uv) = c(u\bar{u}) \neq c(w\bar{w})$ for some $w \in W$, then we can exchange the colors on the edges uv and wv to obtain a coloring in $\mathcal{C}$ which will lead to the usual contradiction to our choice of c. Hence it must be the case that $c(u\bar{u}) = c(w\bar{w})$ for all $w \in W$. From Claim 1, it follows that $|W| = 2$. Let $W = \{u, w\}$. Since $c(u\bar{u}) = c(w\bar{w})$, and the fact that there are no monochromatic vertices in $c'_{H'}$, we get that $\bar{u} = \bar{w} = x$ (say). Note that we have $c'_{H'}(u\bar{u}) = c(u\bar{u}) = c(w\bar{w}) = c'_{H'}(w\bar{w})$. We now have a contradiction to the special property of $c'_{H'}$ that was observed above. This proves the claim.

Claim 6. *There are no monochromatic cycles in c.*

Proof of claim. Suppose that there is a monochromatic cycle C in the coloring c of $G - I$. Since there are no monochromatic cycles in c_H, we have that there exists $uv \in E(C) \cap F$. Let 1 denote the color of the edges of C. From Claim 5 it must be that $d(u) = 3$. Let 2 denote the color of the edge incident on u that does not belong to C. Let P be the maximal path whose edges are colored 2 that starts at u, and let x denote its end vertex other than u. Suppose first that $x \neq v$. Clearly, since $d(v) \leq 5$, there exists $i \in \mathit{Once}(v) \cup \mathit{Missing}(v)$. We can recolor the edge uv with the color i to obtain a coloring in $\mathcal{C}$ having no monochromatic vertices and having fewer monochromatic cycles than c, which contradicts our choice of c. So let us assume that $x = v$. Now if there exists $i \in (\mathit{Once}(v) \cup \mathit{Missing}(v)) \setminus \{2\}$, then we can recolor the edge uv with i to obtain a coloring in $\mathcal{C}$, which will again lead to the same contradiction to the choice of c. So we can assume that $\mathit{Missing}(v) = \emptyset$ and $\mathit{Once}(v) = \{2\}$. Note that this implies that $d(v) = 5$. Suppose there exists $w \in W$. Notice that since there are no monochromatic vertices in c, we have $c(wv) \notin \{1, 2\}$. If $c(w\bar{w}) \neq 1$, then we can exchange the colors on the edges uv and wv to obtain a coloring in $\mathcal{C}$ which will again lead to the same contradiction. On the other hand, if $c(w\bar{w}) = 1$, then we can color uv with $c(wv)$ and wv with 2 to get another coloring in $\mathcal{C}$ which will also lead to the same contradiction. So we can assume that $W = \emptyset$. Since $d(v) = 5$, this implies that $|X \setminus W| \geq 2$. Thus there exists $w \in (X \setminus W) \setminus \{u\}$. If $wv \in E(C)$, we can recolor the edge wv with 2 to get a coloring in $\mathcal{C}$ which again will contradict our choice of c as before. So we can assume that $wv \notin E(C)$, which means that $c(wv) \neq 1$. Then we can exchange the colors of the edges uv and wv to again get a coloring in $\mathcal{C}$ with no monochromatic vertices and fewer monochromatic cycles, contradicting our choice of c. This proves the claim.

From Claims 5 and 6, it follows that there is a 3-linear coloring of $G - I$ containing no monochromatic vertices. Now as before, we can color every edge $xx' \in I$ with a color not in $\{c(vx), c(vx')\}$ to obtain a 3-linear coloring of G containing no monochromatic vertices. □

Theorem 9. *Any connected graph of maximum degree at most 3 is either an odd cycle or has a 2-linear coloring without any monochromatic vertex.*

Proof. We prove this by induction on $|E(G)|$. Let G be a connected graph of maximum degree 3. Observe that if G is an odd cycle, then for any vertex $u \in V(G)$, there is a 2-linear coloring of G in which the only monochromatic vertex is u. If G is an even cycle or a path, then any proper edge coloring of G using 2 colors is a 2-linear coloring of G having no monochromatic vertex. Suppose that G contains a vertex u such that $d_G(u) = 1$. Let $N_G(u) = \{v\}$. Let $H = G - u$. Clearly, H is a connected graph. Notice that $|E(H)| < |E(G)|$. If H is an odd cycle, then we have by the induction hypothesis that H has a 2-linear coloring c in which v is the only monochromatic vertex. Otherwise, we have by the induction hypothesis that H has a 2-linear coloring c having no monochromatic vertices. In either case, we extend c to a 2-linear coloring of G containing no monochromatic vertices by coloring the edge uv with a color in $Once(v) \cup Missing(v)$.

So we assume that G contains no vertex of degree 1. If there is no vertex of degree more than 2, then G is a cycle, in which case we are already done as noted above. So there exists $v \in V(G)$ such that $d_G(v) = 3$. Let $u \in N_G(v)$. Let $H = G - uv$.

First, suppose that H is disconnected. Clearly, H has two connected components, say C_u and C_v, containing u and v respectively. By the induction hypothesis, we can assume that C_u (resp. C_v) has a 2-linear coloring c_u (resp. c_v) using the colors $\{1, 2\}$, such that if C_u (resp. C_v) is an odd cycle, then the only monochromatic vertex in c_u (resp. c_v) is u (resp. v) and both edges incident on u (resp. v) are colored 1 in c_u (resp. c_v); otherwise, c_u (resp. c_v) contains no monochromatic vertices. Moreover, we assume that if u is a degree 1 vertex in C_u, then the only edge incident on u is colored 1 in c_u. We construct a 2-linear coloring of G as follows. First, color the edges of C_u with the colors they have in c_u and the edges of C_v with the colors they have in c_v. Now, we color uv with 2 to obtain the required 2-linear coloring of G.

Next, suppose that H is connected. If H is an odd cycle, then by the induction hypothesis, there is a 2-linear coloring c of H in which v is the only monochromatic vertex. Now coloring uv with the color in $Missing(v)$ gives a 2-linear coloring of G with no monochromatic vertices. On the other hand, if H is not an odd cycle, then by the induction hypothesis, there exists a 2-linear coloring c of H using the colors $\{1, 2\}$ containing no monochromatic vertices. If there exists $i \in Missing(u) \cup Missing(v)$, then we can color uv with i to obtain a 2-linear coloring of H containing no monochromatic vertices. So we assume that $Missing(u) = Missing(v) = \emptyset$. If for some color $i \in \{1, 2\}$, there is no path of color i between u and v in the coloring c of H, then we color uv with i to obtain the required 2-linear coloring of G. So we can assume that there is both a path of color 1 and a path of color 2 between u and v in the coloring c of H. Let $w \in N_G(v) \setminus \{u\}$. Let us assume without loss of generality that $c(vw) = 1$. Now changing the color of vw to 2 and then coloring uv with 1 gives a 2-linear coloring of H with no monochromatic vertices. □

Chapter 5

Linear time algorithms

We now show that the proofs of all the upper bounds we have derived for linear arboricity can be converted into linear-time algorithms that produce a linear coloring of an input 3-degenerate or 2-degenerate graph using at most the number of colors given by the corresponding upper bound. We describe the algorithmic framework in detail for the case of 3-degenerate graphs, as that is more technically involved than the algorithms for 2-degenerate graphs.

We first present an algorithm that computes a k-linear coloring of a 3-degenerate graph G having $\Delta(G) \leq 2k - 1$. Our algorithm will be a linear time algorithm; i.e. having a running time of $O(n + m)$, where n and m are the number of vertices and edges in G respectively. Since G is 3-degenerate, we have $m \leq 3n - 6$, and therefore our algorithm will also be an $O(n)$-time algorithm. We assume that the input graph G is available in the form of an adjacency list representation.

Our general strategy will be to convert the inductive proof of Theorem 1 into a recursive algorithm, but there are some important differences, the main one being that the algorithm computes a more general kind of edge coloring using k colors, which may not always be a k-linear coloring. The algorithm follows the proof of Theorem 1 and removes some edges and if needed identifies two vertices to obtain a smaller graph G'

for which an edge coloring of the desired kind is found by recursing on it. The graph G' is changed back into G by splitting back any identified vertices and adding the removed edges. The newly added edges are then colored to obtain an edge coloring of the desired kind for G. During this process, we never change the color of an edge that is already colored. We shall first discuss why our algorithm needs to compute a generalized version of k-linear coloring.

If the algorithm were to construct a k-linear coloring of G from a k-linear coloring of G' according to the proof of Theorem 1, and still have overall linear runtime, we would like to be able to decide the right color to be given to an uncolored edge uv in $O(1)$ time. This means that we need data structures that allow us to determine in $O(1)$ time a color i for uv such that:

(i) $i \notin Twice(u) \cup Twice(v)$, and

(ii) if $i \in Once(u) \cap Once(v)$, there is no path colored i having endvertices u and v.

The requirement (i) can be met by storing the sets $Once(u)$ and $Missing(u)$ for every vertex u as described in Section 5.1.2.

For (ii), we could store a collection of "path objects" representing the monochromatic paths in the current coloring in such a way that by examining these objects, we can determine in $O(1)$ time whether there is a monochromatic path of color i having endvertices u and v. In particular, for a monochromatic path P having endvertices u and v, we could store the pointer to the path object representing P on the vertices u and v or on the first and last edges of P. In this way, given a vertex u and a color $i \in Once(u)$, we can determine in $O(1)$ time the other endvertex of the path colored i starting at u. Note that as an edge uv gets colored with color i, a path of color i can get extended (if $i \in Once(u) \cap Missing(v)$ or $i \in Missing(u) \cap Once(v)$) or two paths of color i can get fused into one path of color i (if $i \in Once(u) \cap Once(v)$). If we store the pointer to a path object in each vertex (or each edge) of the path, then it becomes difficult to fuse two paths in $O(1)$ time as we cannot afford to visit every vertex (or edge) of the path to change the pointer stored on that vertex (or edge). It is sufficient if we store the pointer to the path object only on the endvertices of the path as mentioned above, since we never need to know what the internal vertices of a path are. This also allows us to fuse two paths in $O(1)$ time. Since no edge that already has a color is ever recolored, a monochromatic path never gets split into two paths or gets shortened when an edge is colored. But a monochromatic path might need to get split into two monochromatic paths when a vertex is split into two. Since the internal vertices of a path do not store the pointer to the corresponding path object, the vertex to be split does not provide us with a pointer to the path object corresponding to the monochromatic path that needs to be split. In short, we cannot update the path objects so as to split this monochromatic path into two paths. We solve this problem by making sure that two paths that meet at a point that will be split later are never fused together into one path. This is explained in more detail below.

We say that a path having an endvertex u and containing the edge uv is "ending at u through uv". Suppose that a vertex w is identified with a vertex u when G' is constructed from G. It is clear from the proof of Theorem 1 that in G', the vertex u has degree at most 3, and there is possibly an edge uy that corresponds to an original edge wy in G. Before recursing to find the coloring for G', we mark the vertex-edge pair (u, uy) as "special" (we call this a "special vertex-edge incidence"; more details given in Section 5.1). This mark, which can be stored inside the adjacency list of u, indicates that while computing the coloring for G', a monochromatic path ending at u through uy should not be fused with another monochromatic path ending at u, even if they have the same color. Thus, while splitting the vertex u back into u and w, no path needs to be split. Note that this means that after the coloring for G' is computed, we might have a path object for a path P colored i ending at u through uy and another path object for a path Q also colored i and ending at u, but through a different edge (as these paths will not be fused). If the other endvertices of P and Q are x and x' respectively, then we can no longer detect that in this coloring of G', there is a monochromatic path colored i starting at x and ending at x', as there is no path object having endvertices x and x'. This means that the edge xx', if it exists, could get colored i, creating a monochromatic cycle colored i. We will allow this to happen, since this monochromatic cycle will anyway get destroyed when the vertex u is split into u and w while recovering G back from G'. Thus, at any stage of the recursion, we compute a coloring for a graph in which certain vertex-edge pairs have been marked as special, and this coloring is not a k-linear coloring any more as it could contain monochromatic cycles. We call this kind of coloring a "pseudo-k-linear coloring". Since the path objects that we store do not correspond to maximal monochromatic paths anymore, we call them "segments" instead of paths. We now define these notions more rigorously.

5.1 Pseudo-k-linear colorings and segments

We define a *vertex-edge incidence* of a graph G to be a pair consisting of a vertex and an edge incident with it; i.e. it is a pair of the form (u, uv) where $u, v \in V(G)$ and $uv \in E(G)$. A subgraph H of G is said to *contain in its interior* a vertex-edge incidence (u, uv) if it contains the edge uv and $d_H(u) \geq 2$.

Given a graph G and a set S of vertex-edge incidences in it, a mapping $c : E(G) \to \{1, 2, \ldots, k\}$ is said to be a *pseudo-k-linear coloring* of (G, S) if each color class is a disjoint union of paths and cycles such that every such cycle contains in its interior at least one vertex-edge incidence in S .

Note that a pseudo-k-linear coloring of $(G, \emptyset)$ is a k-linear coloring of G and also that a k-linear coloring of G is a pseudo-k-linear coloring of (G, S) for any set S of vertex-edge incidences of G. Our algorithm computes a pseudo-k-linear coloring for an input

(G, S), where G is a graph with $\Delta(G) \leq 2k-1$ and S is a set of vertex-edge incidences of G. Observe that a path that has u as an endvertex does not contain in its interior the vertex-edge incidence (u, uv) even if uv belongs to the path. A *segment* in a graph G is a list $(e_1, e_2, \dots, e_k)$ of edges of G, where $k \geq 1$, such that for each $i \in \{1, 2, \dots, k-1\}$, e_i and e_{i+1} are both incident with a common vertex, and no vertex is incident with more than two edges in $\{e_1, e_2, \dots, e_k\}$. Clearly, the union of the edges in a segment gives a path or a cycle in the graph G. Given a segment $(e_1, e_2, \dots, e_k)$, we define the *terminal vertices* of this segment as follows. If $k = 1$, then the two endpoints of e_1 are the terminal vertices of the segment. If $k > 1$, then the endpoint of e_1 that is not incident with e_2 and the endpoint of e_k that is not incident with e_{k-1} are the terminal vertices of this segment. Further, if u is a terminal vertex incident with the edge e_1 and v is a terminal vertex incident with the edge e_k, then (u, e_1) and (v, e_k) are said to be the *terminal vertex-edge incidences* of the segment. In this case, we also sometimes say that this segment is "ending at u through e_1" and "ending at v through e_k". Note that every segment has exactly two terminal vertex-edge incidences. A segment is said to *contain in its interior* a vertex-edge incidence (u, uv) if the segment contains uv but u is not a terminal vertex.

Let c be a pseudo-k-linear coloring of (G, S) and let H be a monochromatic path or cycle of color i in a color class of c. A maximal segment of H that does not contain in its interior any vertex-edge incidences from S is called a *monochromatic segment of* (G, S) having color i (the coloring c is assumed to be clear from the context). Observe that H decomposes into a collection of pairwise edge-disjoint monochromatic segments in a unique way. A monochromatic cycle that contains in its interior a single vertex-edge incidence $(u, uv) \in S$ thus decomposes into a single monochromatic segment whose first and last edges are the two edges of the cycle incident on u. We say that a monochromatic segment of (G, S) is *clean* if neither of its terminal vertex-edge incidences are in S. At a given point of time, we maintain a set of segment objects, one corresponding to each monochromatic segment of (G, S) under the current pseudo-k-linear coloring. The segment object corresponding to a monochromatic segment stores just the terminal vertex-edge incidences of the segment.

Observation 2. *We can extend a pseudo-k-linear coloring c of* $(G - uv, S)$ *to a pseudo-k-linear coloring of* (G, S)*, for some* $uv \in E(G)$*, by giving* uv *a color* i *if:*

(i) $i \notin Twice(u) \cup Twice(v)$*, and*

(ii) there is no clean monochromatic segment of color i *having* u *and* v *as the terminal vertices.*

5.1.1 Encoding the graph

We assume that the input graph G is available in the following representation (if not, this representation can be easily computed in linear time in the initialization phase). We maintain a list `Edges` of the edges of the graph. For each vertex u, we maintain a list `Adj`(u) of the edges incident with u. The node in `Adj`(u) corresponding to an edge e incident with u stores the pointer to the node for e in the list `Edges`. For an edge uv, let N_u and N_v be the nodes corresponding to uv in the lists `Adj`(u) and `Adj`(v) respectively. The node for uv in the list `Edges` stores (u, N_u, v, N_v). For every vertex u, we store its degree $d_G(u)$ in the current graph G. The degree $d(u)$ of a vertex u in the original input graph is assumed to be known at all times. Thus if we have the pointer to the node for an edge uv in the list `Edges`, the list `Adj`(u), or the list `Adj`(v), then the edge can be removed from the graph in $O(1)$ time. It is easy to see that adding an edge, identifying a vertex of degree at most 1 with another of degree at most 2, and splitting a vertex of degree at most 3 into two vertices can all be done in $O(1)$ time in this representation. If $(u, uv) \in S$, then this fact is stored by setting a binary flag in the node corresponding to the edge uv in $\texttt{Adj}(u)$ to true. Note that using our representation, given just the pointer to a node in the $\texttt{Adj}(u)$, we can find u in $O(1)$ time using the `Edges` list.

5.1.2 Encoding the coloring

We color the edges of the graph using the integers $\{1, 2, \ldots, k\}$. Every node in the list `Edges` also contains a field in which the color assigned to the corresponding edge is stored. Every vertex u maintains two lists of colors $\texttt{Onc}(u)$ and $\texttt{Miss}(u)$, to store the sets $Once(u)$ and $Missing(u)$ respectively. We simply let $\texttt{Onc}(u)$ be a list that contains one node for each color in $Once(u)$. The node corresponding to a color i in $\texttt{Onc}(u)$ also stores the pointer to the node in $\texttt{Adj}(u)$ corresponding to the edge colored i incident with u. But we cannot store all the colors in $Missing(u)$ in the list $\texttt{Miss}(u)$, because if we do, then initializing these lists for all the vertices will take too long ($\Omega(nk)$ time). To overcome this, we will use a trick from [21]: we store in $\texttt{Miss}(u)$ only the colors in $Missing(u) \cap \{1, 2, \ldots, \min\{d(u)+2, k\}\}$, where $d(u)$ is the degree of u in the graph given as input to the algorithm. Note that at any stage of the algorithm, if G is the graph being colored by the recursive coloring procedure, for any vertex $u \in V(G)$, $d_G(u) \leq d(u)$. This way of storing the list $\texttt{Miss}(u)$ ensures that the total size of these lists $\sum_{u \in V(\hat{G})} |\texttt{Miss}(u)| \leq \sum_{u \in V(\hat{G})} (d(u)+2)$ which is $O(n+m) = O(n)$ (as 3-degenerate graphs have at most $3n-6$ edges, for $n \geq 3$), where $\hat{G}$ is the initial input graph. Thus we can initialize all these lists in linear time. This trick works even though $\texttt{Miss}(u)$ may not contain all the colors in $Missing(u)$ because we never need to check if some particular color is in $Missing(u)$ unless $d_G(u) \leq 3$, in which case we can do it in $O(1)$ time by just checking the colors of all the edges incident with u. For every vertex u such that $d_G(u) > 3$, we will only need to find some two colors in $Missing(u)$. The following observation shows we will never go wrong when trying to do this.

Observation 3. *If* $\texttt{Miss}(u) \neq Missing(u)$*, then* $|\texttt{Miss}(u)| \geq 2$*.*

Proof. If $d(u) + 2 > k$, we store all the colors in $Missing(u)$ in $\texttt{Miss}(u)$, so we have nothing to prove. So let us assume that $d(u) + 2 \leq k$. Then both the sets $\{1, 2, \ldots, d(u) + 2\}$ and $Missing(u)$ are subsets of $\{1, 2, \ldots, k\}$. This implies that $|\texttt{Miss}(u)| = |Missing(u) \cap \{1, 2, \ldots, d(u) + 2\}| \geq |Missing(u)| + d(u) + 2 - k$. Since $|Missing(u)| \geq k - d_G(u)$ and $d_G(u) \leq d(u)$, we get $|\texttt{Miss}(u)| \geq 2$. □ □

The list $\texttt{Miss}(u)$ needs to be updated when an edge incident with u gets colored, say with a color i, that is in $\texttt{Miss}(u)$. Now that we have colored uv with i, we need to remove i from the list $\texttt{Miss}(u)$. We may not have the pointer to the node corresponding to i in $\texttt{Miss}(u)$ since we may have determined that $i \in Missing(u)$ by checking the adjacency list of u (as $d_G(u) \leq 3$). We cannot afford to search the list $\texttt{Miss}(u)$ to find and remove the node corresponding to the color i (note that the length of $\texttt{Miss}(u)$ could be $d(u) + 2$ which can much larger than $d_G(u)$). For this purpose, we follow [21] and maintain an array $missingindex(u)$ of size $\min\{d(u) + 2, k\}$ that contains pointers to the nodes in the list $\texttt{Miss}(u)$. For each $i \in \texttt{Miss}(u)$, the i-th element of the array $\texttt{Ptrs}(u)$ will store the pointer to the node in $\texttt{Miss}(u)$ corresponding to the color i. Thus, given a color i in the list $\texttt{Miss}(u)$, we can use the array $\texttt{Ptrs}(u)$ to remove the node corresponding to i from the list $\texttt{Miss}(u)$ in $O(1)$ time. It is clear that the array $\texttt{Ptrs}(u)$ can also be initialized in linear time during the initialization phase and updated in $O(1)$ time whenever a node is removed from the list $\texttt{Miss}(u)$.

Let (G, S) be the graph and set of vertex-edge incidences at any stage of the algorithm. We maintain a "segment object" corresponding to each monochromatic segment of (G, S) under the current pseudo-k-linear coloring. For a monochromatic segment having terminal vertex-edge incidences (u, uv) and (x, xy), we store the pointer to its segment object in the node for uv in $\texttt{Adj}(u)$ and the node for xy in $\texttt{Adj}(x)$. The segment object will store the pointers to these two nodes as well. Note that by our definition of monochromatic segments, in a pseudo-k-linear coloring of (G, S), if monochromatic segments σ and σ' (possibly $\sigma = \sigma'$) having the same color end at a vertex u through distinct edges e and e', then at least one of (u, e) or (u, e') is in S. Every time we extend a pseudo-k-linear coloring by coloring an uncolored edge, we have to update the collection of segment objects so that they correspond exactly to the monochromatic segments under the new coloring. Our strategy for doing this will be as follows. When an edge uv gets colored i, we first create a new segment object corresponding to the segment containing just uv. At this point, we may have two segment objects corresponding to two segments having u or v as a common terminal vertex, and so the segment objects may not necessarily correspond to monochromatic segments under the new coloring. In order to correct this, whenever some segment objects represent segments whose union gives a monochromatic segment under the new coloring, we "fuse" them into a single segment object that represents the new monochromatic segment.

Observation 4. *A new segment object containing a single edge can be created in $O(1)$ time. Two segment objects can be fused into a single segment object in $O(1)$ time.*

Proof. If we want to create a segment object containing a single edge uv, the pointer to whose node in the list `Edges` is known, we first use this node to find the nodes N_u and N_v in the lists $\texttt{Adj}(u)$ and $\texttt{Adj}(v)$ respectively corresponding to uv. We create a segment object σ and store in it the pointers to N_u and N_v and we store the pointer to σ in N_u and N_v. It is clear that this process takes just $O(1)$ time.

Suppose that we have two segment objects σ_1 and σ_2 corresponding to segments ending at a vertex u through distinct edges e_1 and e_2 incident with u respectively. Let (u'_1, e'_1) be the terminal vertex-edge incidence of σ_1 other than (u, e_1) and (u'_2, e'_2) be the terminal vertex-edge incidence of σ_2 other than (u, e_2). From the segment objects σ_1 and σ_2, we can find the nodes N_1 in $\texttt{Adj}(u'_1)$ and N_2 $\texttt{Adj}(u'_2)$ corresponding to e'_1 and e'_2 respectively. We now create a new segment object σ containing the pointers to N_1 and

N_2—i.e. it represents a segment having terminal vertex-edge incidences (u_1', e_1') and (u_2', e_2'). We replace the pointers to σ_1 and σ_2 in N_1 and N_2 respectively with pointers to σ. We can now destroy the objects σ_1 and σ_2 and remove their pointers from the nodes corresponding to e_1 and e_2 in $\texttt{Adj}(u)$. It is easy to see that all this can be done in $O(1)$ time. □ □

The following observation is easy to see.

Observation 5. *For a vertex $u \in V(G)$ such that $d_G(u) \leq 3$, we can compute $Once(u)$, $Twice(u)$ and $Colors(u)$ in $O(1)$ time.*

Lemma 5. *Let c be a pseudo-k-linear coloring of $(G - uv, S)$. Then c can be extended in $O(1)$ time to a pseudo-k-linear coloring of $(G, S \cup S')$, where $S' \subseteq \{(u, uv), (v, uv)\}$, by coloring the edge uv with a color i provided that $d_G(u) \leq 3$, $i \notin Twice(u)$, the pointer to a node containing color i in either $\texttt{Miss}(v)$ or $\texttt{Onc}(v)$ is known, and there is no clean monochromatic segment of color i having terminal vertices u and v.*

Proof. It is clear that by coloring the edge uv with the color i, we obtain a pseudo-k-linear coloring of $(G, S \cup S')$. We only need to show that we can update our data structures encoding the coloring in $O(1)$ time.

We assume that we have the pointer to the node N_{uv} for uv in the list $\texttt{Edges}$ and also the pointer to a node N_v containing the color i in one of the lists $\texttt{Miss}(v)$ or $\texttt{Onc}(v)$. We first set the color field in N_{uv} to i and create a new segment object for a segment σ colored i with terminal vertex-edge incidences (u, uv) and (v, uv).

Since $d_G(u) \leq 3$, we can use Observation 5 to check if $i \in Once(u)$. If $i \notin Once(u)$, then we know that $i \notin Colors(u)$. In this case, we use the array $missingindex(u)$ to find the node for i in $\texttt{Miss}(u)$ and remove it in $O(1)$ time if $i \leq d(u) + 2$, and we do nothing if $i > d(u) + 2$. Add to $\texttt{Onc}(u)$ a node containing the color i and the pointer to the node corresponding to the edge uv in $\texttt{Adj}(u)$. Let us now suppose that $i \in Once(u)$;

i.e. there is an edge e_u colored i incident with u. We can traverse $\texttt{Onc}(u)$ to find the node containing i (and e_u) and remove it in $O(1)$ time as $|\texttt{Onc}(u)| \leq d_G(u) - 1 \leq 2$. Furthermore, from the node for e_u in $\texttt{Adj}(u)$, we find the pointer to the segment object representing the monochromatic segment σ_u colored i ending at u through e_u. If neither (u, e_u) nor (u, uv) are in $S \cup S'$, then we fuse the segments σ_u and σ into a single segment. By Observation 4, this can be done in $O(1)$ time, and since σ_u cannot be a clean segment having u and v as terminal vertices (by the assumption in the lemma) the segment represented by the new segment object cannot be a monochromatic cycle that contains no vertex-edge incidence from $S \cup S'$ in its interior.

Now if N_v is a node from the list $\texttt{Miss}(v)$, we can easily remove it from the list $\texttt{Miss}(v)$ in $O(1)$ time. From the node N_{uv} in the list $\texttt{Edges}$, we can find a pointer to the node for uv in the list $\texttt{Adj}(v)$. We add to the list $\texttt{Onc}(v)$ a node containing i and this pointer, and we are done. So let us suppose that N_v is a node from the list $\texttt{Onc}(v)$. From N_v, we can find the node in the list $\texttt{Adj}(v)$ corresponding to the edge e_v incident with v colored i. From this node, we can find the pointer to the segment object that represents the monochromatic segment σ_v colored i ending at v through e_v. In the current collection of segment objects, let σ' be the segment containing the edge uv. Now if neither (v, uv) nor (v, e_v) are in $S \cup S'$ (this implies that $\sigma' \neq \sigma_v$ as there was no clean monochromatic segment with terminal vertices u and v), fuse σ' and σ_v into a single monochromatic segment. Again this can be done in $O(1)$ time by Observation 4. □ □

5.2 Maintaining the *Eligible* list

The *Eligible* list stores the pivots that have at least one neighbor of degree at most 3. Each vertex stores a pointer to its node in the list, in case it is in the list. Thus, we can check in $O(1)$ time if a particular vertex is in the list *Eligible* and if needed, also remove it from the list. Recall that for each vertex u, we maintain a variable $d_G(u)$ that stores its current degree. In addition, we also store another variable $d'_G(u)$ that maintains the number of neighbors of u that have degree at most 3. Note that a vertex u is a pivot when $d_G(u) - d'_G(u) \leq 3$. We make sure that the list *Eligible* always contains exactly those vertices u for which $d_G(u) - d'_G(u) \leq 3$ and $d'_G(u) \geq 1$. Whenever our algorithm removes some edges or identifies two vertices in order to create a smaller graph, we update the *Eligible* list accordingly. Note that for both these operations, we can update $d_G(u)$ and $d'_G(u)$, for any vertex u for which these parameters change, in $O(1)$ time. For example, if an edge uv is removed, we update the *Eligible* list as follows. As will be seen in Section 5.3, our algorithm always removes an edge uv such that $d_G(u) \leq 3$. When this happens, we decrease $d_G(u)$ and $d_G(v)$ by one, and since u had degree at most 3 before, we decrease $d'_G(v)$ by one. If v also had degree at most 3 before, we decrease $d'_G(u)$ also by one. If now $d'_G(u) = 0$ or $d'_G(v) = 0$, we remove that vertex from the list *Eligible* if it is present in the list. Note that when the edge uv is removed, u or v will not become eligible to be in the *Eligible* list if it was not already in the list. If v has now become a vertex of degree exactly 3, for each neighbor w of v (there are 3 such neighbors), we increment $d'_G(w)$ by 1, and if for any of them this results in $d_G(w) - d'_G(w) \leq 3$ (clearly, $d'_G(w) \geq 1$), we add w to the *Eligible* list if it is not already present in the list. Observe that all of these operations associated with the removal of an edge take only $O(1)$ time in total. Similarly, we can update the *Eligible* list in $O(1)$ time after an identification operation too as follows. Note that we always identify a vertex w of degree at most 1 with a vertex u of degree at most 2. We update $d_G(u)$ and $d'_G(u)$ and remove w from the list *Eligible* if it is present in the list. Further, if now $d'_G(u) \geq 1$, we add u to the list *Eligible* if it is not already in it.

5.3 The algorithm

We now describe the algorithm. The proof of correctness of the algorithm and the fact that it runs in $O(1)$ time shall be clear from the description that we provide.

The initialization phase of the algorithm consists of doing some preprocessing in order to compute the degrees of each vertex, determine the pivots, and to construct the list *Eligible* of pivots that have at least one pivot edge incident with them. It is easy to see that this stage takes linear time. The data structures required to encode the graph and the k-linear coloring (see Sections 5.1.1 and 5.1.2) are also initialized during this phase. Note that after the initialization, no edges are colored, so for every vertex u, we will have $\texttt{Onc}(u) = \emptyset$ and $\texttt{Miss}(u)$ will contain all the colors in $\{1, 2, \ldots, \min\{d(u) + 2, k\}\}$. Also, there will be no segment objects. Then we invoke a recursive procedure $\textsc{Color}(G, S)$, setting $S = \emptyset$, which works as follows.

The procedure $\textsc{Color}(G, S)$

The procedure does not take G and S as parameters, but rather expects that they are encoded in the data structures: i.e. it assumes that the graph G is encoded in the lists `Edges` and the lists $\texttt{Adj}(u)$, for each vertex $u \in V(G)$, and that the set S of vertex-edge incidences is available in the form of binary-valued flags inside nodes in the `Adj` lists of the vertices (i.e. the flag is set to true for the node corresponding to an edge uv in $\texttt{Adj}(u)$ if and only if $(u, uv) \in S$). The procedure returns a pseudo-k-linear coloring of (G, S) by filling data in the lists `Miss` and `Onc` of each vertex and also by constructing a collection of segment objects representing the monochromatic segments of (G, S) under the coloring, as explained in Section 5.1.2. We now describe the procedure in detail. Note that whenever we say "color e with i", where e is an edge and i a color, we mean that we use Lemma 5 to assign the color i to the edge e in $O(1)$ time.

If the list *Eligible* is empty, then the graph G is empty, and therefore the procedure simply returns without constructing any coloring. Suppose that the list *Eligible* is not empty. Then let v be the first vertex in *Eligible*. Determine a neighbor u of v such that $d_G(u) \leq 3$. This will take only $O(1)$ time as v, being a pivot, has at most three neighbors having degree more than 3 and being in the list *Eligible*, v has at least one neighbor of degree at most 3.

If $d_G(v) < 2k-1$, then modify G to $G' = G - uv$. As mentioned in Sections 5.1.1 and 5.2, we can do this in $O(1)$ time. Note that even though we remove the nodes N_{uv}, N_u, N_v corresponding to uv from `Edges`, `Adj`(u), `Adj`(v) respectively, we retain them for later use, as the nodes N_u and N_v contain information about S in their binary flags. Observe that the set S has now become $S' = S \setminus \{(u, uv), (v, uv)\}$. Construct a pseudo-$k$-linear coloring of (G', S') by invoking $\textsc{Color}(G', S')$. Modify G' back to G by adding the edge uv. While doing this, we add back the stored nodes N_{uv}, N_u and N_v to the lists `Edges`, `Adj`(u) and `Adj`(v) respectively. Note that this restores S from S'. If there is a color i in `Onc`(v) such that $i \notin Colors(u)$ (this check can be performed in $O(1)$ time by Observation 5), then color uv with i. Note that since $|Colors(u)| \leq 2$, we will never need to traverse beyond the third element in `Onc`(v) to find the color i and hence this check can be performed in $O(1)$ time. Otherwise, it must be the case that $|$`Onc`$(v)| \leq 2$, which implies that $Missing(v) \neq \emptyset$ and that $|Missing(v) \cup Once(v)| \geq 2$. By Observation 3, we have that `Miss`$(v) \neq \emptyset$. Let i be the first color in the list `Miss`(v). If $i \notin Twice(u)$, we color the edge uv with i. Otherwise, if $|Missing(v)| \geq 2$, then we have from Obesrvation 3 that $|$`Miss`$(v)| \geq 2$, so we take the second color j in the missing list and give that color to uv. Lastly, if $|Missing(v)| = 1$, then $|Once(v)| \geq 1$, and therefore we take the first color j in `Onc`(v) and color uv with j. We thus obtain a pseudo-k-linear coloring of (G, S) in $O(1)$ time.

So let us now assume that $d_G(v) = 2k-1$. As explained in the proof of Theorem 1, then there exists a vertex $w \in N_G(v) \setminus \{u\}$ such that $d_G(w) \leq 3$. Clearly, this vertex can be found in $O(1)$ time as v is a pivot. We first check if $uw \in E(G)$. Note that this check can be done in $O(1)$ time as $d_G(u), d_G(w) \leq 3$. If $uw \in E(G)$, then we can follow the steps in the proof of Lemma 2 to compute a pseudo-k-linear coloring for (G, S) as we did before. Similarly, if there exists a vertex $z \in (N(u) \cap N(w)) \setminus \{v\}$ (again this can be checked in $O(1)$ time as $d_G(u), d_G(w) \leq 3$), we follow the steps in the proof of Lemma 3 to compute a pseudo-k-linear coloring for (G, S). In the both the above cases, it can be easily seen that the steps outside the recursive call to the procedure COLOR can be done in $O(1)$ time.

Now suppose that $uw \notin E(G)$ and that $N(u) \cap N(w) = \{v\}$. The algorithm in this case too follows the proof of Theorem 1 closely. We shall only describe the algorithm for the case when $d_G(w) = 3$, as the procedure for other cases can be easily deduced from this procedure (or we could add dummy vertices of degree 1 as neighbors of w to make $d_G(w) = 3$). Let $N(w) = \{v, x, y\}$. Remove the edges uv, vw, wx from G (the pointers to the nodes for these edges in `Edges` can be obtained in $O(1)$ time as $d_G(u), d_G(w) \leq 3$, and hence they can be removed in $O(1)$ time). Observe that as the nodes corresponding to these edges have disappeared from `Adj`(u), `Adj`(v), `Adj`(w) and `Adj`(x), the set of special vertex-edge incidences S has now changed to $S_1 = S \setminus \{(u, uv), (v, uv), (v, vw), (w, vw), (w, wx), (x, wx)\}$. We now identify the vertex w with u—in other words, the edge wy has to change one of its endpoints from w to u. We do this by modifying the node N corresponding to wy in the list `Edges`. Further, we add a node N_u containing a pointer to N to `Adj`(u) and remove the node N_w containing the pointer to N from `Adj`(w) (also decrease the degree of w to zero and update the *Eligible* list as mentioned in Section 5.2). Let us call the graph so obtained G'. Note that the set of special vertex-edge incidences S_1 has now changed to $S_2 = S_1 \setminus \{(w, wy)\}$ if $(y, wy) \notin S_1$ or to $S_2 = (S_1 \setminus \{(w, wy), (y, wy)\}) \cup \{(y, uy)\}$ if $(y, wy) \in S_1$.

We now modify the set S_2 to $S' = S_2 \cup \{(u, uy)\}$ by essentially setting the requisite binary flag to true in the node N_u in the list $\texttt{Adj}(u)$. The procedure $\textsc{Color}(G', S')$ is invoked to construct a pseudo-k-linear coloring of (G', S'). In order to modify this into a pseudo-k-linear coloring of (G, S), we first split the vertex u back into u and w. For this, we just change the endpoint u to w in the node N, remove the node N_u from $\texttt{Adj}(u)$ and add N_w back to $\texttt{Adj}(w)$, while increasing the degree of w to 1. Note that this step automatically recovers S_1 from S'. Add the edges uv, vw, wx. While doing this, we restore the nodes that we removed from $\texttt{Adj}(u)$, $\texttt{Adj}(v)$, $\texttt{Adj}(w)$ and $\texttt{Adj}(x)$, so that the set S_1 changes back to S. We now color these edges as explained in the proof of Theorem 1, with the only difference being that instead of checking whether there is a path of some color i having u and v as endvertices, we check whether there is a clean monochromatic segment of color i having u and v as terminal vertices. We explain in detail below.

Observation 6. *Given vertices u, v and an edge e incident with v, we can check in $O(1)$ time if there exists a clean monochromatic segment having (v, e) as a terminal vertex-edge incidence and u as a terminal vertex.*

Proof. We assume that we have the node N for e in $\texttt{Adj}(v)$. We check if the pointer to some segment object is stored in N. If not, then we can immediately conclude that the clean monochromatic segment that we seek does not exist. So let us suppose that N contains the pointer to a segment object σ. Let N' be the node other than N whose pointer is stored σ. If $N' \in \texttt{Adj}(u)$ (this can be checked in $O(1)$ time as mentioned in Section 5.1.1) and the binary flags in N and N' indicating membership in S are both set to false, we conclude that σ represents a clean monochromatic segment having (v, e) as a terminal vertex-edge incidence and u as a terminal vertex. Clearly, all this takes only $O(1)$ time. □ □

As noted in the proof of Theorem 1, we first color wx. If $\texttt{Miss}(x) \neq \emptyset$, then we color wx with a color in $\texttt{Miss}(x)$. Otherwise, by Observation 3, we have that $Missing(x) = \emptyset$, which implies that $|\texttt{Onc}(x)| \geq 2$. By checking at most the first two nodes of $\texttt{Onc}(x)$, we find a color i in $\texttt{Onc}(x)$ that is different from the color of wy. Color wx with i. We shall now color uv and vw, again following the proof of Theorem 1.

We have two cases: either $|Missing(v)| = 1$ and $|Once(v)| = 1$, or $|Once(v)| = 3$. In the former case, we have $|\texttt{Onc}(v)| = 1$ and by Observation 3, we also have $|\texttt{Miss}(v)| = 1$. Let i be the color in $\texttt{Miss}(v)$ and j the color in $\texttt{Onc}(v)$. If $i \in Twice(u)$, then color uv with j and vw with i. Otherwise, if $i \in Twice(w)$, then color vw with j and uv with i. So let us consider the case when $i \notin Twice(u) \cup Twice(w)$. Again following the proof, if $j \in Twice(u) \cup Twice(w)$, we color both uv and vw with i. So we assume that $j \notin Twice(u) \cup Twice(w)$. From the node in $\texttt{Onc}(v)$ containing the color j, we can find the node in $\texttt{Adj}(v)$ containing an edge e_j colored j incident with v. Using this node, we determine if there is a clean monochromatic segment having (v, e_j) as a terminal vertex-edge incidence and u as a terminal vertex as described in Observation 6. If there is, we color uv with i and vw with j, and otherwise, we color uv with j and vw with i. This works even though the coloring we have now is a pseudo-k-linear coloring (which may contain monochromatic cycles) .

Observation 7. *Let G be a graph that is the disjoint union of paths and cycles. If P_1 and P_2 are two distinct non-zero-length paths in G such that $V(P_1) \cap V(P_2) \neq \emptyset$, then one of the endvertices of P_1 or one of the endvertices of P_2 must be a vertex of degree two in G.*

Thus since $j \in Once(v)$ and $j \notin Twice(u) \cup Twice(w)$, if there is a path colored j from v to u, then by Observation 7, there can be no path colored j from v to w. Since clean monochromatic segments colored j are also paths colored j, there cannot be two clean monochromatic segments colored j, one having terminal vertices u, v and another having terminal vertices v, w.

We shall now consider the case when $|Once(v)| = 3$. For this case, our algorithm and its proof of correctness differs slightly from the proof of Theorem 1. We traverse the nodes of $\texttt{Onc}(v)$ to find the three colors i, j, ℓ in $Once(v)$. From these nodes, we can find the nodes in $\texttt{Adj}(v)$ corresponding to the edges e_i, e_j, e_ℓ incident with v having colors i, j, ℓ respectively. Using Observation 6, we determine the set $L = \{p \in \{i, j, \ell\}:$ there exists a clean monochromatic segment having (v, e_p) as a terminal vertex-edge incidence and u as a terminal vertex$\}$. It is clear that $|L| \leq 2$ as u has at most two colored edges incident with it.

Claim 7. *If $Twice(u) \neq \emptyset$, then $L \subseteq Twice(u)$ and $|L| \leq 1$.*

If there exists $p \in Twice(u)$, then both the colored edges incident with u have color p, i.e. $Colors(u) = Twice(u) = \{p\}$. Then any monochromatic segment having u as a terminal vertex must have color p, which implies that $L \subseteq \{p\} = Twice(u)$. Clearly, this also means that $|L| \leq 1$. This proves the claim.

Suppose that $|L| = 2$. Then by the above claim, $Twice(u) = \emptyset$. Let $p \in L \setminus Twice(w)$ (p exists as $|Twice(w)| \leq 1$). Clearly, there is at most one edge colored p incident with each of u, v, w. Thus, by Observation 7, there is no path having color p between v and w, and therefore there is no monochromatic segment having color p and terminal vertices v, w. Therefore, in this case, we color vw with p and uv with the color in $\{i, j, \ell\} \setminus L$. If $|L| \leq 1$, color vw with a color $r \in \{i, j, \ell\} \setminus Colors(w)$ and uv with a color in $\{i, j, \ell\} \setminus (\{r\} \cup Twice(u) \cup L)$ (note that this set is nonempty by our claim above). This completes the description of the algorithm.

5.4 The 2-degenerate cases

We now outline how the proofs of the upper bounds obtained for the linear arboricity of 2-degenerate graphs can also be converted into linear-time algorithms that compute linear colorings that use the same number of colors as the corresponding upper bounds.

The proof of Theorem 2 can be converted into a linear time algorithm that outputs a $\left\lceil \frac{\Delta(G)}{2} \right\rceil$-linear coloring of an input 2-degenerate graph G having $\Delta(G) \geq 5$ by utilizing the framework described above (note that the algorithm can just use k-linear colorings instead of pseudo-k-linear colorings as we never identify vertices and split them later, so no monochromatic paths ever get split into two paths). Note that this means that the problem of computing linear arboricity is linear time solvable in 2-degenerate graphs having maximum degree at least 5.

It can be seen that using the techniques and data structures described in this section, the proof of Theorem 3 can be converted into a linear time algorithm that given an input 2-degenerate graph G having $\Delta(G) \leq 4$ and $|E(G)| \geq 2|V(G)| - 5$ constructs a 2-linear coloring of G containing at most one monochromatic vertex. Again, it is not hard to see that the proof of Theorem 4 can be converted into a linear time algorithm that computes a 2-linear coloring with no monochromatic vertex for any input bipartite 2-degenerate graph having maximum degree at most 4.

The proof of Theorem 5 can also be converted into a linear time algorithm that computes a 2-linear coloring of any input partial 2-tree having maximum degree at most 4 as explained below. Note that since we do not identify a pair of vertices and then split them back during the induction step, no monochromatic paths need to be split during the algorithm, which means that it can work with just 2-linear colorings instead of pseudo-2-linear colorings. For the algorithm to work in linear time, one of the five configurations listed in Observation 1 has to be found in $O(1)$ time during the induction step. This can be accomplished, for example, as follows. For any vertex u, we can determine whether it is part of a configuration of one of the types mentioned in Observation 1 by doing a local search starting from u. Since every vertex has at most 4 neighbors, this can be done in $O(1)$ time. At every stage of the algorithm, we maintain a list of all the configurations in the current graph. At the start, we compute the list of all configurations in the original graph G in linear time by performing the local search starting from all vertices of G. With each vertex, we also store a list of pointers to the configurations that it is a part of. (Note that a vertex cannot be part of more than $O(1)$ configurations. Further, note that a naive approach may lead to the same configuration being detected more than once—the local search from each of its vertices may detect the configuration. Therefore the list of all configurations may contain duplicates, but this does not affect the runtime complexity of the algorithm as a configuration is duplicated only as many times as the number of vertices in it; i.e. $O(1)$ times.) During the induction step, we remove vertices or edges to obtain a smaller graph, and in this process the degrees of at most five vertices in the resultant graph will be different from their degrees in the original graph. We remove all configurations that these vertices were a part of, and then determine all configurations that they are a part of in the new graph by again doing a local search starting from them. This takes only $O(1)$ time. The only remaining detail is regarding the implementation of the list $\mathcal{S}$ of pairs of degree two vertices. We do not explicitly store the list $\mathcal{S}$, but instead, each pair $\{u, v\}$ of this list can be encoded by storing a pointer to v on u and a pointer to u on v. It is easy to see that these pointers can be easily manipulated for achieving the modifications to the list $\mathcal{S}$ that we need.

5.5 A simpler algorithm for 3-degenerate graphs

The proof of Theorem 6 can also be converted into a linear-time algorithm that produces a $\left\lceil \frac{\Delta(G)}{2} \right\rceil$-linear coloring of an input 3-degenerate graph G having $\Delta(G) \geq 9$ as follows. The input to the algorithm is a 3-degenerate graph having maximum degree at most k, where $k \geq 9$, and the algorithm generates a $\lceil \frac{k}{2} \rceil$-coloring of the graph (which can be retrieved from the `Edges` list, as in previous algorithms) that does not contain any monochromatic vertices. At every step, the algorithm modifies the current graph to obtain a smaller graph on which it recurses to generate a $\lceil \frac{k}{2} \rceil$-coloring that does not contain monochromatic vertices, which is then extended to a $\lceil \frac{k}{2} \rceil$-coloring of the current graph that also does not contain any monochromatic vertices. Following the proof of Theorem 6, the algorithm converts the graph G at any particular stage into the smaller graph H' by picking a pivot v, removing all the pivot edges incident on v as well as any edges between two neighbours of v each having degree two, and then finally pairing up and identifying as many degree one vertices as possible that were earlier neighbours of v—in the language of the proof of Theorem 6, we remove all edges in $F \cup I$ and then pair up the vertices in W and identify them to construct H'. It is not difficult to see that if the number of pivot edges incident on v in G is $t = |F|$, then G can be modified into the graph H' in $O(t)$ time. In fact, it is straightforward to see that the graph G can be modified into the graph $H = G - (F \cup I)$ in $O(t)$ time. For converting H to H', we initialize a list L of "sets" (which are again implemented as lists). Initially, $L = \{\{u\} : u \in W\}$. We now arbitrarily choose two vertices u, u' from two different sets in L and check if their single neighbour in H is the same vertex. If yes, we merge the two sets from which u and u' were chosen. Otherwise, we identify u with u' and remove both u and u' from their respective sets in L. If some set in L becomes empty in the process, we remove it from L. In $O(t)$ time, the list L will either become empty or will contain only a single set. If L is nonempty, then the single set that it contains is the set W' from the proof of Theorem 6. The algorithm now recurses on H' to produce a $\lceil \frac{k}{2} \rceil$-coloring of H' that contains no monochromatic vertices. Once this is done, we split back the vertices that were identified to obtain the graph H back from H', but retaining the colors on the edges. Note that while doing this, no monochromatic path will get split, since the vertices in H' that get split have degree two and are not monochromatic in the coloring of H'. This means that even though we identify vertices and split them back, we can work with just $\lceil \frac{k}{2} \rceil$-linear colorings and do not need pseudo $\lceil \frac{k}{2} \rceil$-linear colorings. Now it only remains to be shown that the edges in $F \cup I$ can be added back to H and colored in just $O(t)$ time in order to obtain a $\lceil \frac{k}{2} \rceil$-linear coloring of G. Since $d_G(v) \leq t+3$, and coloring the edges of I once we have the required $\lceil \frac{k}{2} \rceil$-linear coloring of $G-I$ is easy, we only need to show that the $\lceil \frac{k}{2} \rceil$-linear coloring of H can be extended to a $\lceil \frac{k}{2} \rceil$-linear coloring of $G-I$ containing no monochromatic vertices in $O(d_G(v))$ time. We start by iterating through the available colors $1, 2, \ldots, \lceil \frac{k}{2} \rceil$ in ascending order. For each color $i \in \{1, 2, \ldots, \lceil \frac{k}{2} \rceil\}$, we assign i to two as yet uncolored edges in F if i does not appear on the (at most three) edges incident on v in H, and we assign i to one as yet uncolored edge in F if i appears on exactly one of the edges incident on v in H (if i occurs twice on the edges incident on v, we do not assign i to any edges in F). Each time an edge uv in F gets colored with a color i in this way, we update the color of that edge in the `Edges` list, and also modify the lists `Miss`(v), `Onc`(v), `Miss`(u), and `Onc`(u) as usual. But we do not modify any of the "path objects"—i.e., we do not alter our record of the monochromatic paths in the coloring at this stage; in fact, the edge coloring of $G - I$ that we have at this stage may contain monochromatic cycles and/or monochromatic vertices, both of which we want to avoid. Nevertheless, it is clear that it takes only $O(d_G(v))$ time to color all the edges of F in this way. We now follow the proof of Theorem 6 in order to permute the colors on the edges in F so as to get the required $\lceil \frac{k}{2} \rceil$-linear coloring of $G - I$. Note that the path objects can be used for detecting the presence of monochromatic cycles and also for eliminating them in the way described in the proof of Theorem 6. It is not difficult to see that the same kind of data structures can be used to encode the partial linear colorings generated during the various stages of the algorithm implying that the linear arboricity of 3-degenerate graphs having maximum degree at least 9 is linear time computable.

The same ideas can be adopted for converting the proofs of Theorem 7 and Theorem 8 into linear-time algorithms that generate linear colorings for 3-degenerate graphs of maximum degree 7 and 5 using at most 4 and at most 3 colors respectively. It is also straightforward to convert the proof of Theorem 9 into a linear-time algorithm that constructs a 2-linear coloring for 3-degenerate graphs of maximum degree 3.

Chapter 6

p-centered colorings of grids

In this chapter, we prove that the "two dimensional grid graph" (see Definition 4) (shown in Figure 6.1) has a p-centered coloring using $O(p)$ colors, for every positive integer p.

We require the following definitions for the purpose of proof.

For any set X, a partition α of X is a collection of nonempty subsets of X such that for any $x \in X$, $|\{A \in \alpha : x \in A\}| = 1$. Sets in α are called parts. For any $x \in X$, $[x]_\alpha$ is the unique part in α containing x. We denote by $Part(X)$ the collection of all partitions of X. For any set X, a partition α of X, and any nonempty set Y, a partition of $Y \cap X$ induced by α is $\alpha_Y = \{A \cap Y : A \in \alpha\} \setminus \{\emptyset\}$. A vertex coloring of a graph G is an element $\alpha \in Part(V(G))$. If $\alpha \in Part(V(G))$ and $v \in V(G)$ we declare that the

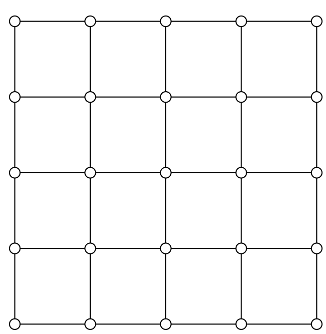

Figure 6.1: The 5×5 grid

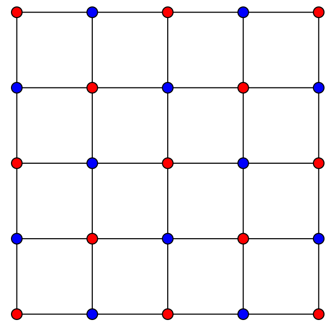

Figure 6.2: A 1-centered coloring of the 5×5 grid

color of v is $[v]_\alpha$. Note that for any graph G and $p \in \mathbb{N} \setminus \{0\}$, a p-centered coloring of G is a vertex coloring ρ of G such that for any connected subgraph H of G, if $|\rho_{V(H)}| \leq p$, then there is a part $A \in \rho_{V(H)}$ such that $|A| = 1$.

Definition 4. *The two dimensional grid is a graph G with $V(G) = \mathbb{N}^2$ and $E(G) = \{(a,b)(c,d) : (a,b),(c,d) \in \mathbb{N}^2$ and $|a-c| + |b-d| = 1\}$.*

We now describe a p-centered coloring of the two dimensional grid and prove that the coloring is indeed p-centered. Here onwards, we denote by G the two dimensional grid. We state our result below.

Theorem 10. *For the two dimensional grid G, for every $p \in \mathbb{N}$ there is a p-centered coloring ρ such that $|\rho| \leq f(p)$ where $f : \mathbb{N} \to \mathbb{N}$ such that $f(p) \in O(p)$.*

Note that the two dimensional grid G is a bipartite graph. The two partite sets are $A = \{(x,y) \in V(G) : x + y \text{ is even}\}$ and $V(G) \setminus A$. Since 1-centered coloring is just a proper vertex coloring, there is a 1-centered coloring with 2 colors for the two dimensional grid. From here onwards, we assume that $p > 1$.

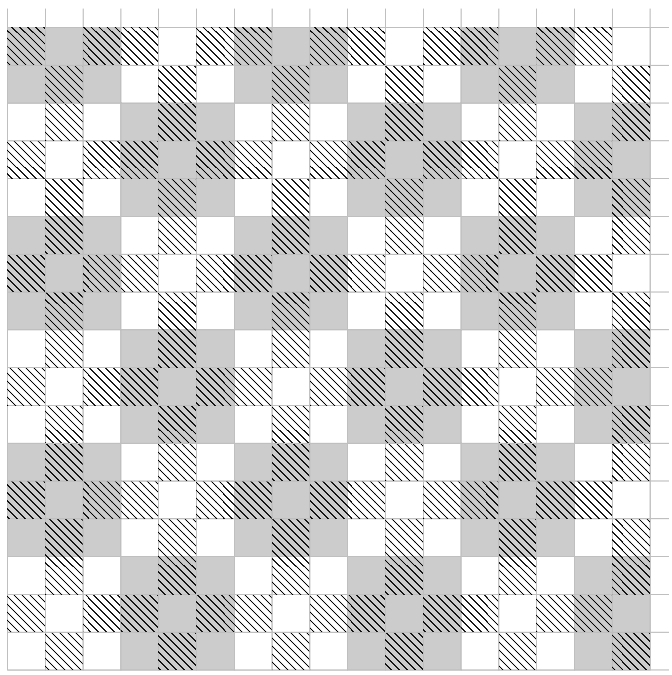

Figure 6.3: The figure shows the two partitions of $V(G)$. The cell at position (i, j) is: (a) white if $(i, j) \in R$ and gray if $(i, j) \in C$, (b) hatched if $(i, j) \in B$ and not hatched if $(i, j) \in A$.

6.1 The coloring

Definition 5. *For two natural numbers $a, b \in \mathbb{N}$, we define the interval $[a, b] = \{n \in \mathbb{N} : a \leq n \leq b\}$. Notice that when $a > b$, $[a, b] = \emptyset$.*

Definition 6. *For $(a, b) \in \mathbb{N}^2$, we define $\pi_x : \mathbb{N}^2 \to \mathbb{N}, (a, b) \mapsto a$ and $\pi_y : \mathbb{N}^2 \to \mathbb{N}, (a, b) \mapsto b$.*

For any $a, b \in \mathbb{Z}$ we define $mod(a, b) = a - b \left\lfloor \frac{a}{b} \right\rfloor$. For $a \in \mathbb{N}$ and $A \subseteq \mathbb{N}$ we write $aA = \{a \cdot b : b \in A\}$. We define lg for log with base 2. First partition the set $\mathbb{N}^2$ into $\{R, C\}$. Define $R = \{(x, y) \in \mathbb{N}^2 : mod(\left\lfloor \frac{x}{3} \right\rfloor + \left\lfloor \frac{y}{3} \right\rfloor, 2) = 0\}$ and $C = \mathbb{N}^2 \setminus R$. Also, we partition $V(G)$ into $\{A, B\}$, such that $A = \{(a, b) \in V(G) : 2 \mid (mod(a, 3) + mod(b, 3))\}$(for $a, b \in \mathbb{Z}$ write $b \mid a$ when $mod(a, b) = 0$ and $b \nmid a$ when $mod(a, b) \neq 0$) and $B = V(G) \setminus A$. We define a partition $\rho = \bigcup_{i=0}^{6p+5} \{\{(a, b) \in R \cap A : mod(a, 6p + 6) = i\}\}$ of $R \cap A$. Also we define a partition $\lambda = \bigcup_{i=0}^{6p+5} \{\{(a, b) \in C \cap A : mod(b, 6p + 6) = i\}\}$ of $C \cap A$. Notice that $|\rho| = |\lambda| = 6p + 6$. Refer Figure 6.3 for an illustration of the partition of $\mathbb{N}^2$ into $\{R, C\}$ and also into $\{A, B\}$.

Now let us partition $\mathbb{N}^2$ in another way. Let p be a power of 2. We define $P_i = \pi_x^{-1}(4p/2^i\mathbb{N}) \cup \pi_y^{-1}(4p/2^i\mathbb{N})$ where $i \in [0, lg(4p)]$. Let us define $Q_i = P_i \setminus \bigcup_{j \in [0, i-1]} P_j$ for each $i \in [0, lg(4p)]$. Now for each Q_i where $i \in [0, lg(4p)]$ we define a partition α_i. We declare that for any $(a, b), (x, y) \in Q_i$ and for any $i \in [0, lg(4p)]$, $[(a, b)]_{\alpha_i} = [(x, y)]_{\alpha_i}$ whenever $8p/2^i \mid (x - a)$ and $8p/2^i \mid (b - y)\}$. We define $\alpha = \bigcup_{i \in [0, lg(4p)]} \alpha_i$. Observe that $|\alpha| \leq 2 \left(\sum_{i \in [0, lg(4p)]} \frac{8p}{2^i} \right) \leq 32p$.

We define a new partition β on B. We declare that for any $(a, b), (x, y) \in B$, $[(a, b)]_\beta = [(x, y)]_\beta$ whenever $mod(a, 3) = mod(x, 3)$ and $mod(b, 3) = mod(y, 3)$ and $[(\left\lfloor \frac{a}{3} \right\rfloor, \left\lfloor \frac{b}{3} \right\rfloor)]_\alpha = [(\left\lfloor \frac{x}{3} \right\rfloor, \left\lfloor \frac{y}{3} \right\rfloor)]_\alpha$.

We claim that $\beta \cup \rho \cup \lambda$ is our desired coloring. Notice that $\beta \cup \rho \cup \lambda \leq 4(32p) + 2(6p + 6) = 140p + 12$.

Lemma 6. *For any subgraph G' of G, if $\pi_x(V(G'))$ and $\pi_y(V(G'))$ are some intervals such that $|\pi_x(V(G'))| \leq 4p$ and $|\pi_y(V(G'))| \leq 4p$, then there exists $q \in \alpha_{V(G')}$ such that $|q| = 1$.*

Proof. Let $i = min\{j \in [0, lg(4p)] : Q_j \cap V(G')\}$. If $i = 0$, then the lemma is true as $|\pi_x(V(G'))| \leq 4p < 8p$ and $|\pi_y(V(G'))| \leq 4p < 8p$. Let $(x, y) \in Q_i \cap V(G')$. We claim that $[(x, y)]_{\alpha_{V(G')}}| = 1$. So we assume that $i > 0$. Suppose that there exists $(x', y') \in [(x, y)]_{\alpha_{i_{V(G')}}} \setminus \{(x, y)\}$. As $(x, y), (x', y') \in Q_i$, we infer that $(x, y), (x', y') \in P_i$ and $(x, y), (x', y') \notin P_{i-1}$. We see that $8p/2^i \mid (x' - x)$ and $8p/2^i \mid (y' - y)$. If $x \neq x'$, without loss of generality, let us assume that $x < x'$. Notice that $4p/2^{i-1} \nmid x$ and $4p/2^{i-1} \nmid x'$ ($4p/2^{i-1} = 8p/2^i$). Therefore, there is always a $c \in [x, x'] \setminus \{x, x'\} \subset \pi_x(V(G'))$ such that $8p/2^i \mid c$. As $\pi_x(V(G'))$ is an interval, there exists $(c, d) \in V(G')$. Now it is clear that $(c, d) \in P_{i-1} \cap V(G')$. Hence, we can conclude that $(c, d) \in Q_j$ for some $j < i$. This is a contradiction. Thus, we have that $x = x'$. Similarly, we conclude that $y = y'$. This completes the proof. □

Lemma 7. *For any connected subgraph G' of G if $|\pi_x(V(G'))| \geq 6p+9$ then $|\rho_{V(G')}| > p$.*

Proof. As G' is connected there is a subpath P of G' such that $|\pi_x(V(P))| \geq 6p + 9$. Let us partition $\mathbb{N}$ into the collection $\kappa = \{[t, t+5] : t \in \mathbb{N} \text{ and } 6 \mid t\}$. Let $S = \{a \in \kappa : |a \cap \pi_x(V(P))| = 6\}$. Notice that it is not possible that $|S| < p$ and there is a $c \in \kappa$ such that $|c \cap \pi_x(V(P))| = 5$ when $|S| = p$. Let $S' \subseteq S$ such that $|S'| = p + 1$ and $\bigcup_{a \in S'} a$ is also an interval when $|S| > p$ or $S' = S \cup \{c \cap \pi_x(V(P))\}$ when $|S| = p$. Therefore $|S'| = p+1$. Note that if $a \in S'$, a is an interval. Let $a \in S'$. Observe that, there is $t \in a$ such that $3 \mid t$ and also $[t-2, t+1] \subset a$ as $|a| \geq 5$. Since P is connected there is an edge $(t-1, y)(t, y) \in E(P)$ such that $y \in \mathbb{N}$. Observe that either $(t-1, y) \in R$ or $(t, y) \in R$. First let us assume that $(t, y) \in R$. If $(t, y) \notin A$ then $2 \nmid mod(t, 3) + mod(y, 3)$ which implies that $mod(y, 3) = 1$. Since $|\pi_x(V(P)) \cap a| \geq 5$ there is $(x', y') \in N(t, y) \cap V(P) \cap R$ such that $2 \mid \mod(x', 3) + \mod(y', 3)$ which implies that $(x', y') \in A$ (recall that for any vertex (x,y) in G, $N(x,y) = \{(x',y') \in \mathbb{N}^2 : |x-x'|+|y-y'| = 1\})$. Now let $(t-1,y) \in R$. If $(t-1,y) \notin A$, then $2 \nmid mod(t-1,3)+mod(y,3)$ which implies that $mod(y,3) = 1$. Thus there is a $(x',y') \in N(t-1,y) \cap V(P) \cap R$ such that $2 \mid mod(x',3)+mod(y',3)$ which implies that $(x',y') \in A$. Therefore, $\pi_x^{-1}(a) \cap V(P) \cap R \cap A \neq \emptyset$ for any $a \in S'$. Note that $|S'| = p+1$ and for each $a \in S'$ we get an unique element $f(a) \in \pi_x^{-1}(a) \cap V(P) \cap R \cap A($ for each a we choose a vertex u in $\pi_x^{-1}(a) \cap V(P) \cap R \cap A$ as $f(a)$ such that $\pi_x(u)$ is the minimum). Note that for two distinct $a,b \in S'$, $\pi_x(f(a)) \neq \pi_x(f(b))$ and $|\pi_x(f(a)) - \pi_x(f(b))| \leq 6p+5$ which imply that $[f(a)]_\rho \neq [f(b)]_\rho$. Therefore $|\rho_{V(G')}| > p$ as $|S'| > p$. □

Lemma 8. *For any connected subgraph G' of G if $|\pi_y(V(G'))| \geq 6p+9$, then $|\lambda_{V(G')}| > p$.*

Proof. The proof is similar to that of Lemma 7. □

Now we are ready to prove the theorem. We define a map $I : \mathbb{N}^2 \to \mathbb{N}^2, (x,y) \mapsto (\lfloor \frac{x}{3} \rfloor, \lfloor \frac{y}{3} \rfloor)$.

6.2 Proof of Theorem 10

Let H be a connected subgraph of G. If either $|\pi_x(V(H))| \geq 6p+9$ or $|\pi_y(V(H))| \geq 6p+9$ then from Lemma 7 and Lemma 8 either $|\rho_{V(H)}| > p$ or $|\lambda_{V(H)}| > p$, respectively. Therefore, let us assume that $|\pi_x(V(H))| < 6p+9$ and $|\pi_y(V(H))| < 6p+9$. We claim that the components of the induced subgraph of G on A are C_4 or K_1. Let $(a,b) \in A$. If $mod(a,3) = mod(b,3) = 1$, then $N((a,b)) \setminus \{(a,b)\} \subset B$. Thus for this case the connected component of $G[A]$ containing (a,b) is K_1. The other case is when $(a,b) \in A \setminus \{x \in \mathbb{N} : mod(x,3) = 1\}^2$. Let $(a,b) \in R$. Without loss of generality let us assume that $mod(a,3) = 0$ and $mod(b,3) = 2$. Let $(x,y) \in N((a,b)) \cap A$. Then only possibility is that $(x,y) \in C$. Similarly, one can show that if $(a,b) \in C$ and

$(x, y) \in N((a, b)) \cap A$ then $(x, y) \in R$. Also observe that $|N((a, b) \cap A| = 2$ and if $(x, y), (x', y') \in N((a, b)) \cap A$ ((x, y) and (x', y') are distinct), then $|\frac{y-y'}{x-x'}| = 1$. Observe that if $(a, b), (c, d) \in R \cap (A \setminus \{x \in \mathbb{N} : mod(x, 3) = 1\}^2)$ and $|a - b| + |c - d| = 2$, then $N((a, b)) = N((c, d))$. Therefore any component in the induced subgraph on $V(G) \setminus \{x \in \mathbb{N} : mod(x, 3) = 1\}^2$ is a C_4. Hence if $V(H) \cap B = \emptyset$, then H must be a subgraph of either C_4 or K_1. In both the cases, from the definition of the partitions λ and ρ there is a $A \in \lambda_{V(H)} \cup \rho_{V(H)}$ such that $|A| = 1$. Now assume that $V(H) \cap B \neq \emptyset$. We claim that $\pi_x(I(V(H) \cap B))$ and $\pi_y(I(V(H) \cap B))$ are intervals. We only show that $\pi_x(I(V(H) \cap B))$ is an interval since proof of the other claim is similar. Let us suppose that $x, x' \in \pi_x(I(V(H) \cap B))$. Let $b \in ([x, x'] \setminus \{x, x'\})$. Clearly, there exists $(b_1, b_2) \in [3x, 3x + 2] \times [3x', 3x' + 2] \cap \pi_x(V(H))^2$. Since $3b + 1 \in [b_1, b_2] \setminus \{b_1, b_2\}$ and $\pi_x(V(H))$ is an interval we have $3b+1 \in \pi_x(V(H))$. Thus there exists $a \in \mathbb{N}$ such that $(3b + 1, a) \in V(H)$. If $(3b + 1, a) \notin B$, then $mod(a, 3) = 1$. Since $(3b + 1, a) \in V(H)$ and H is connected and $|V(H)| > 1$, there exists some neighbour u of $(3b + 1, a)$ such that $u \in V(H) \cap B$. Note that $\pi_x(I(u)) = b$. Thus, $\pi_x(I(V(H) \cap B))$ is an interval. Symmetrically, $\pi_x(I(V(H) \cap B))$ is an interval. As $|\pi_x(V(H))| \leq 6p + 8$ and $|\pi_y(V(H))| \leq 6p+8$, $|\pi_x(I(V(H) \cap B))| \leq 2p+4$ and $|\pi_y(V(H) \cap B)| \leq 2p+4$. As $p > 1$ we have $2p+4 \leq 4p$. Therefore applying Lemma 6 we get that there exists $q \in \alpha_{I(V(H))}$ such that $|q| = 1$. Let $u \in I^{-1}(q) \cap V(H)$ then $[u]_{\beta_{V(H)}} = \{u\}$. This completes the proof.

6.3 Generalizing to all values of p

The above theorem is applicable to all values of $p \in \mathbb{N}$. Let us assume that $p = 1$. A two dimensional grid is a bipartite graph. So there is a proper coloring with only 2 colors. Any proper coloring is an 1-centered coloring. Suppose that $p > 1$. Let p' be the smallest power of 2 that is greater than p. Observe that $p' \geq 2$ and $p' \geq 2p$. By the above construction we get a p'-centered coloring with at most $140p' + 12$ colors. Notice that any p'-centered coloring of the two dimensional grid is also a p-centered coloring of the two dimensional grid as $p \leq p'$. Thus for any $p > 2$ we get a p-centered coloring with at most $280p+12$ colors. Thus for any value of $p \in \mathbb{N}$ there is a p-centered coloring of the two dimensional grid using $O(p)$ colors.

Chapter 7

Conclusion

In this work we studied two graph coloring problems. We give below some concluding remarks and thoughts on future research directions on these two problems.

Linear colorings

The work done on the first problem, namely the Linear Arboricity Conjecture, has established that this long standing conjecture is true for the class of 3-degenerate graphs, which has many interesting and well-studied subclasses. A natural way to build further upon this work would be to try to prove the conjecture for 4-degenerate graphs. It should become harder since if one could prove the conjecture for k-degenerate graphs for every $k \in \mathbb{N}$, then the Linear Arboricity Conjecture itself would be proved. Our approach for proving the conjecture on 3-degenerate graphs crucially hinges on the operation of identifying a pair of nonadjacent vertices. In both proofs that we have presented, we identify a pair of nonadjacent vertices u and v such that $d(u) + d(v) \leq 3$, so that the new vertex created by the identification is a vertex of degree at most 3. This guarantees that the new graph after the identification operation is also a 3-degenerate graph, albeit with a lesser number of vertices, and we can argue inductively on this

Figure 7.1: The two cases for 4-degenerate graphs.

graph. If one were to repeat this approach for 4-degenerate graphs, in the final step of the proof of Theorem 1, we will have two ways to construct the graph H, as shown in Figure 7.1. In both cases, we construct H' from H by identifying the vertex w with the vertex u. Now we inductively construct a linear coloring of H' using at most k colors (recall that $\Delta(G) \leq 2k-1$), and obtain a partial linear coloring c of H by splitting back the identified vertex into u and w. We now have to find colors that can be given to edges that were removed to obtain H from G. For the case that is shown on the left of Figure 7.1, we need to color the edges wx and wz. But it is possible that in the partial coloring c of H, we have $Missing(x) = Missing(z) = \{i\}$ and $Once(x) = Once(z) = \emptyset$. This forces us to extend c into a linear coloring of H by assigning the color i to both wx and wz. But it is possible that $c(wy) = i$, and therefore c cannot be extended into a linear coloring of H. Similarly, for the case shown on the right of Figure 7.1, it could happen that $c(wy) = c(wz) = i$, $Missing(x) = \{i\}$, and $Once(x) = \emptyset$. Once again, we cannot extend c into a linear coloring of H. These kinds of problems also come up when trying to adapt the proof of the Linear Arboricity Conjecture for 3-degenerate graphs given in Chapter 4 to 4-degenerate graphs.

p-centered colorings

An end goal of our research on p-centered colorings is to produce a constructive proof for Dębski, Felsner, Micek and Schröder's result that every graph G having maximum degree Δ has a p-centered coloring using $O(\Delta^{2-\frac{1}{p}}p)$ colors. We sought to understand how such a proof would look like by trying to construct p-centered colorings using $O(p)$ colors for specific graphs with bounded degree. A natural next step would be to try to extend our proof to bounded degree graphs similar to the two dimensional grid, like the three dimensional grid, or the two dimensional grid with diagonals added in each cycle of length four. But adapting the technique that we used for the two dimensional grid to these graphs doesn't look straightforward.

In our approach to color the two dimensional grid, we implicitly use a centered coloring with $O(p)$ colors for the $p \times p$ grid. This was possible since the treedepth of a $p \times p$ grid is $O(p)$. The treewidth of a $p \times p \times p$ grid is $\Omega(p^2)$ [72]. Since the treewidth of any graph is at most its treedepth [15], this means that any centered coloring of a $p \times p \times p$ grid has to use $\Omega(p^2)$ colors. This leads us to believe that a different approach will be required to construct a p-centered coloring using $O(p)$ colors for the three dimensional grid.

As for the two dimensional grid with diagonals added in each cycle of length four, which we shall call the "extended grid", our approach fails for a different reason. One of the crucial ingredients in our proof for the grid graph was a coloring that contained no "large violators"; i.e. a coloring such that every connected subgraph that spans more than some $c \cdot p$ rows or columns, for some constant c, contains more than p colors. It seems difficult to construct such a coloring for the extended grid. For the normal grid graph, we could divide the vertices into two classes R and C, so that we could color each vertex in the class R with its row number modulo $c \cdot p$, for some constant c, and each vertex in the class C with its column number modulo $c \cdot p$. Then any connected subgraph H has the property that if it intersects vertices in two adjacent columns, then it has to intersect a vertex of class C from at least one of those columns. Thus, any connected subgraph that spans $c \cdot p$ columns contains vertices of type C from at least $\frac{c \cdot p}{2}$ of those columns. The reason why this happens is that removal of vertices of type C from any two adjacent columns disconnects the graph. Moreover, since induced subgraph of formed by the vertices that we removed consists of only isolated vertices, we could use one color, or a constant number of colors, to color all of those vertices. On the other hand, if we attempt to partition the vertices of the extended grid into classes such that removal of some number of vertices from one of the classes disconnects the graph into two components, neither of which are "small", then the removed vertices would induce a "large" connected subgraph in the extended grid, and therefore if we use just a constant number of colors to color all of them, then this connected subgraph would be "large violator". Therefore, we feel that as a first step towards finding a p-centered coloring using $O(p)$ colors for the extended grid, one first needs to find a coloring for the extended grid that does not contain any "large violators".

Open problems

We would like to present some open problems that we tried but failed. These are the problems which we believe need to be studied if one is to expand upon the work done in this thesis.

1. Is there a linear coloring with $\left\lceil \frac{\Delta(G)+1}{2} \right\rceil$ colors for 4-degenerate graphs?

2. Is there a 2-linear coloring for any 2-degenerate graph with maximum degree 4?

3. It seems natural to ask the following question: Is the problem of computing linear arboricity NP-hard when restricted to 2-degenerate graphs of maximum degree at most 4? We do not know the answer to this question, but would like to remark here that this problem seems similar to Conjecture 4 in [22].

4. Show an explicit construction for a p-centered coloring using $O(p)$ colors for the 3-dimensional grid.

5. Show an explicit construction for a p-centered coloring using $O(p)$ colors for the planar grid with both diagonals added inside each cycle of length four.

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