Essays on Evaluation Aggregation, Strategy-proof Social Choice, and Myopic-Farsighted Stable Matching

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Contents

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To Family and Friends.

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1 Introduction

The thesis comprises of six chapters on evaluation aggregation, social choice and matching. A brief introduction to each of the six chapters is provided below.

In Chapter 2, we consider collective evaluation problems, where individual grades given to candidates are combined to obtain a collective grade for each of these candidates. In this paper, we prove the following two results: (i) a collective evaluation rule is update monotone and continuous if and only if it is a min-max rule, and (ii) a collective evaluation rule is update monotone and consistent if and only if it is an extreme min-max rule.

Chapters 3,4 and 5 deals with strategic social choice problems where a social planner needs to decide an outcome for a society from a finite set of feasible outcomes based on the preferences of the agents in the society. Agents preferences are their private information and agents are strategic meaning that they manipulate the outcome by misreporting their preferences whenever that is beneficial for them. The objective of the social planner is to design a rule that no agent can manipulate.

In Chapter 3, we consider domains that satisfy pervasiveness and top-connectedness, and we provide a necessary and sufficient condition for the existence of non-dictatorial, Pareto optimal, and group strategy-proof choice rules on those domains.

In Chapter 4, we consider choice functions that are unanimous, anonymous, symmetric, and group strategy-proof and consider domains that are single-peaked on some tree. We prove the following three results in this setting. First, there exists a unanimous, anonymous, symmetric, and group strategy-proof choice function on a path-connected domain if and only if the domain is single-peaked on a tree and the number of agents is odd. Second, a choice function is unanimous, anonymous, symmetric, and group strategy-proof on a single-peaked domain on a tree if and only if it is the pairwise majority rule (also known as the tree-median rule) and the number of agents is odd. Third, there exists a unanimous, anonymous, symmetric, and strategy-proof choice function on a strongly path-connected domain if and only if the domain is single-peaked on a tree and the number of agents is odd. As a corollary of these results, we obtain that there exists no unanimous, anonymous, symmetric, and group strategy-proof choice function on a path-connected domain if the number of agents is even.

In Chapter 5, we consider weak domains, that is, set of preferences that may admit indifference. We show that every unanimous and strategy-proof random social choice function on any weak domain containing all strict preferences is weak random dictatorial. On weak single-peaked domains, we show that a random social choice function is Pareto optimal and strategy-proof if and only if it is an extreme probabilistic fixed ballot rule. Next, we consider single-plateaued domains and provide the structure of unanimous and strategy-proof random social choice functions on these domains.

Chapter 6 considers the problem of designing strategy-proof social choice rules in an incomplete information framework. More formally, agents have beliefs about the preferences of the other agents and they tend to manipulate whenever that improves the expected outcome according to their belief. We explore the structure of locally ordinal Bayesian incentive compatible (LOBIC) random Bayesian rules (RBRs). We show that under lower contour monotonicity, for almost all prior profiles (with full Lebesgue measure), a LOBIC RBR is locally dominant strategy incentive compatible (LDSIC). We further show that for almost all prior profiles, a unanimous and LOBIC RBR on the unrestricted domain is random dictatorial, and thereby extend the result in [40] for Bayesian rules. Next, we provide a sufficient condition on a domain so that for almost all prior profiles, unanimous RBRs on it are tops-only. Finally, we provide a wide range of applications of our results on single-peaked (on arbitrary graphs), hybrid, multiple single-peaked, single-dipped, single-crossing, multi-dimensional separable domains, and domains under partitioning. Since OBIC implies LOBIC by definition, all our results hold for OBIC RBRs.

Chapter 7 considers the many-to-one two-sided matching problem. Agents are assumed to be heterogeneous with respect to their ability to foresee the consequences of a block, and thereby categorized as myopic and farsighted. We study the structure of stable matchings and stable sets in this setting.

2

On update monotone, continuous, and consistent collective evaluation rules

2.1 Introduction

We consider (collective) evaluation problems where agents grade candidates based on their level of excellence, and where these individual judgments are to be aggregated to obtain a collective grade for each of these candidates. In education environments, this is daily practice. Here students are graded for different subjects and a final overall grade determines the performance of the students compared to each other. Other examples of these are, for instance, the so-called majority aggregation rules as proposed by [5] and [6] or the linguistic decision rules described in [38]. The fundamental problem is to find rules with desirable properties that take all the individual grades as inputs and produce a collective evaluation as output.

On the one hand, grades may express an evaluation result on a more or less absolute scale. Examples of such problems include those where agents have to evaluate competitors based on their performances using a predefined grade scale, like for instance in music contests, teaching environments, or certain sport disciplines, such as e.g. gymnastics, or figure skating. On the other hand, these grades can be interpreted as individual quality assessments enabling agents to order a relatively large group of candidates. At job vacancies, grades or quality expressions may support committee members in ordering larger amounts of applicants. Decision-making based on qualitative information has a wide range of practical applications like online auctions, personnel evaluation, and supply chain management. We use linguistic/qualitative grading scales to allow explicitly for individual interpretations of these. Strictly speaking a grade $A+$ in history given by teacher i is not comparable to a grade $B-$ in geography given by teacher j. Linguistic qualifications like 'good' or 'perfect' leave this individual interpretation more open than numerical ones as the latter automatically refer to an absolute scale.

The literature on collective evaluation rules, also known as linguistic decision rules, is very extensive (see [90], [91] , [92], [88] and [86]). [89] proposes linguistic max and min operators. The max (min) operator chooses the maximum (minimum) grade given by the agents to a candidate. Later, [90] generalizes these operators as linguistic max-min weighted averaging operators. In a similar spirit, [91] and [92] propose linguistic median and weighted median operators which choose the median and weighted median grade for a candidate, respectively.

[45] present a linguistic ordered weighted averaging operator which is based upon convex combinations of grades ([30]). [12] consider a multi-person multi-criteria decision problem for group decision making in a linguistic context and provide a human-consistent definition of consensus and a procedure for its computation.

The use of the Borda count in linguistic decision making problems is introduced by [39]. They provide two ways of extending the Borda rule to an evaluative framework either by taking into account all agent's opinions or by only considering the favorable ones for each candidate when compared with each other. A comprehensive survey of the literature on linguistic decision rules can be found in [87].

[4] introduce the notion of order functions. They show that an evaluation rule is unanimous, anonymous, monotonic, and strategy-proof if and only if it is an order function.

Judgment aggregation considers situations where there is a collection of propositions and a set of judges each having a binary opinion (accept/reject) for each proposition. A judgment aggregation rule chooses a set of propositions based on the approval/disapproval of the judges. Regularly in judgment aggregation, propositions may be interrelated, whereas in our model, no such interconnection is considered amongst the grades assigned by the agents over the candidates. There is, however, a small branch of judgment aggregation literature, dealing with non-binary opinion, to which our model relates (see [66], [34], [33] and [32]).

Therefore, in literature several collective evaluation rules are proposed and discussed with respect to their advantages and disadvantages. Here, we intend to look at the converse of this, that is, we identify three properties of collective evaluation rules and characterize all rules satisfying those properties. These three properties are *update monotonicity, continuity*, and *consistency*.

The implication of update monotonicity is as follows. Suppose a grade, say 'very good', is collectively decided for a candidate. Suppose further that an agent, who previously graded this candidate as 'average', now changes his evaluation to 'good' while leaving all grades of all other candidates and all other agents unchanged. In a sense, the evaluation of this agent moves in favor of the outcome 'very good'. Update monotonicity says that the collective evaluation of the candidate also in the new situation is 'very good'. Thus, update monotonicity ensures that the outcome does not change when agents change their evaluations towards the outcome.

Continuity is a well-known property of an aggregation rule when candidates are elements of a Euclidean space. Here, we have adopted this idea for the case of finitely many candidates. Continuity ensures that small changes in grades lead to small changes in the collective evaluation. In other words, it ensures that the collective evaluation rule is not too sensitive to some grade changes of some agents.

The implication of consistency is as follows. Let 'very good', 'good', and 'average' be three consecutive grades. Consider a situation where agents are divided into two groups such that all members of one group grade a certain candidate as 'average' while all other agents grade him/her as 'good'. Suppose, further, that all the agents agree on the grade of every other candidate. Now, consider another situation which differs from the previous case in the following way: the group of agents, who graded the candidate as 'average' in the previous case, now all grade him/her as 'good', while the agents in the other group now grade him/her as 'very good'. In a sense, the judgment of each agent has shifted uniformly in some particular direction for the candidate. The consistency property says that the collective decision for the candidate in the latter case shifts equally in the same direction, that is, if it was 'good' ('average') in the former case, then it will be 'very good' ('good') now. Thus, this property ensures some type of consistency in the evaluation rule.

We provide two different characterization results in this paper. In the first one, we show that a collective evaluation rule is update monotone and continuous if and only if it is a min-max rule. Such a rule is based on a unique collection of parameters indicating for each candidate, say a, and each subset of agents, say S, the lowest possible collective grade candidate a can get, when all agents that are not in S give candidate a the highest possible grade. Let this lowest grade be denoted by β_a^S. Given the grades assigned by the agents, the outcome for candidate c is now determined as follows. First, for every subset S of agents the maximum among the grades given to c by the agents in S and this parameter β_a^S is determined. This yields for each such subsets S a highest grade for a. The min-max rule chooses the minimum of all these highest grades. In our second result, we show that a collective evaluation rule satisfies update monotonicity and consistency if and only if it is an extreme min-max or extreme max-min rule. Here the parameters β_a^S are either the highest or lowest possible grade. Well-known collective evaluative operators in the literature such as min, max, median, etc. are special cases of extreme min-max or extreme max-min operators.

Min-max and max-min rules (for one dimension) arise in the context of strategic social choice on a single-peaked domain. [63] shows that every unanimous and strategy-proof social choice function on a one dimensional single-peaked domain is a min-max or max-min rule. Our model captures the single-peaked property by means of the fact that grades are ordered. It is worth noting that our model is a multi-dimensional setting where each candidate comprises a dimension. However, as it follows from our results, the combination of update monotonicity and continuity, and the combination of update monotonicity and consistency, each implies the candidate-wise property. That is, the collective evaluation of a candidate depends only on the grades assigned to this candidate by the agents. Another important thing to note is the following. Since every min-max rule is strategy-proof and every collective evaluation rule satisfying any of the above mentioned combinations of properties is some type of a min-max rule, it follows that strategy-proofness can be obtained as a by-product of these properties. Thus, our paper builds up a connection between the theories of collective evaluation problems and social choice.

2.2 The model

Let $A = \{a_1, \ldots, a_m\}$ denote the finite set of *m candidates* and let $N = \{1, \ldots, n\}$ denote the finite set of *n agents*. The set of *k grades* is denoted by *G*, where these grades are indicated by the numbers 1 up to *k*. We emphasize that a higher number corresponds to a higher grade, that is, only ordinal information is needed here.[1]

An *evaluation u* is a function from *A* to *G*. It assigns to every candidate $a \in A$ a grade $u(a)$. The interpretation is related to that of ordinal utility functions: the higher the grade of a candidate, the better is the evaluation of the candidate. We denote by *U* the set of all evaluations. A *profile* u_N is an element of U^n, i.e., an n-tuple of evaluations, where agent *i*'s evaluation is denoted by u_i. For a profile $u_N \in U^n$, u_i denotes the evaluation of agent *i* and $u_i(a)$ denotes the grade agent *i* gives to candidate *a*.

A *collective evaluation rule* assigns to each profile a (collective) evaluation. More formally, a collective evaluation rule is a function $\phi : U^n \to U$. The grade of a candidate $a \in A$ at evaluation $\phi(u_N)$ is denoted by $\phi_a(u_N)$. As we have discussed in Section 2.1, some commonly known collective evaluation rules in the

[1] Denoting grades by numbers is no more than a convention. While the ordinal structure of numbers represents betterness comparisons between grades, the arithmetic structure of numbers (including ratios or differences) carries no information, i.e., is insignificant. For instance, the comparison $4 > 2$ means that grade 4 is higher than grade 2, but the identity $4/2 = 2$ does not mean that grade 4 is twice as high as grade 2, and the difference comparison $4 - 2 > 2 - 1$ does not mean that grade 4 is more distant to grade 2 than grade 2 to grade 1. Although we do not formally represent non-ordinal information, non-ordinal propositions about grades need not be regarded as meaningless. Ordinalists would regard non-ordinal propositions such as 'these grades are more distant from one another than those' as meaningless. Others would regard them as meaningful properties which are not captured arithmetically. Ordinalists would regard two evaluations as equivalent if they induce the same weak order over candidates; for instance, evaluating a certain candidate as 'good' and all others as 'bad' would be equivalent to evaluating him as 'excellent' and all others as 'good'. The ability to encompass ordinalist and non-ordinalist notions of value counts as an advantage of our formal framework over the standard relational framework of Arrovian preference aggregation.

literature are min, max and median rules.

In what follows, we discuss some properties of collective evaluation rules.

Definition 2.2.1 *A collective evaluation rule ϕ is called* anonymous *if for all permutations π of $\{1, \ldots, n\}$ and all profiles $u_N, v_N \in U^n$ such that $u_i = v_{\pi(i)}$ for all $i \in \{1, \ldots, n\}$, we have $\phi(u_N) = \phi(v_N)$.*

In words, a collective evaluation rule is anonymous if the identities of the agents do not play any role in the aggregation procedure.

Definition 2.2.2 *A collective evaluation rule ϕ is* update monotone *if for all profiles $v_N, w_N \in U^n$ such that for all candidates $a \in A$ and all agents $i \in N$, either $v_i(a) \leq w_i(a) \leq \phi_a(v_N)$ or $\phi_a(v_N) \leq w_i(a) \leq v_i(a)$, we have $\phi(v_N) = \phi(w_N)$.*

A collective evaluation rule is update monotone if outcomes do not change whenever agents change their individual grades (judgments) towards the outcomes.[2]

Definition 2.2.3 *A collective evaluation rule ϕ is* uncompromising *if for all profiles $v_N, w_N \in U^n$ with the property that for all candidates $a \in A$ and all agents $i \in N$, $v_i(a) < \phi_a(v_N)$ implies $w_i(a) \leq \phi_a(v_N)$ and $\phi_a(v_N) < v_i(a)$ implies $\phi_a(v_N) \leq w_i(a)$, we have $\phi(v_N) = \phi(w_N)$.*

A collective evaluation rule is uncompromising if the aggregated grade does not change whenever agents do not change their side with respect to the outcome. For instance, suppose that the aggregated grade of a candidate at a profile is 'good' and the judgment of an agent about that candidate at that profile is 'bad'. Then, uncompromisingness says that if that agent changes his/her judgment to 'very bad' or to 'average' or to 'good', then the aggregated judgment of the candidate will not change.

Remark 2.2.4 *Uncompromisingness implies update monotonicity.*[3]

Definition 2.2.5 *A collective evaluation rule ϕ is* candidate-wise *if for all candidates a and all profiles $v_N, w_N \in U^n$ such that $v_i(a) = w_i(a)$ for all agents $i \in N$, we have $\phi_a(v_N) = \phi_a(w_N)$.*

The candidate-wise property says that the outcome grade of a candidate depends *only* on the individual grades of the agents about that candidate.

[2] [16] introduce the notion of update monotonicity in the context of preference aggregation.

[3] Note that uncompromisingness implies the following weaker property for a collective evaluation rule ϕ: for all profiles $v_N, w_N \in U^n$, if for all $i \in N$ and all $a \in A$, we have $w_i(a) < v_i(a) < \phi_a(v_N)$ or $w_i(a) > v_i(a) > \phi_a(v_N)$, then $\phi(v_N) = \phi(w_N)$. This implication is a kind of "inverse update monotonicity": if individuals move away from the outcome, then the outcome stays the same. In fact, this is the reason why the name "uncompromising" is given to this axiom.

2.3 A characterization of the update monotone and continuous collective evaluation rules

In this section, we provide a characterization of the update monotone and continuous collective evaluation rules. First, we introduce the notion of continuity.

2.3.1 Continuity

Continuity ensures that a 'small' change in the individual grades can only lead to a 'small' change in the outcome. Here, by small change we mean that exactly one agent changes his/her (individual) grade for a particular candidate to exactly one grade above or below.

Definition 2.3.1 *The distance between two evaluations u and v is defined as $d(u,v) = \sum_a |u(a) - v(a)|$. For two profiles $u_N, v_N \in U^n$, we define the distance between them as $d(u_N, v_N) = \sum_{i \in N} d(u_i, v_i)$.*

Definition 2.3.2 *A collective evaluation rule $\phi : U^n \to U$ is said to be continuous if for all profiles $u_N, v_N \in U^n$, $d(u_N, v_N) = 1$ implies $d(\phi(u_N), \phi(v_N)) \leq 1$.*

In Appendix .3, we show that our notion of continuity actually boils down to that in standard mathematics by considering natural topologies on U^n and U.

2.3.2 Min-max rules

Now, we introduce a class of collective evaluation rules called min-max rules. Such a collective evaluation rule can be described by certain minimax (or equivalently maximin) mechanisms. These rules are well known in the social choice literature for single-peaked preferences ([63]). Here the structure of single-peakedness is induced by the underlying natural ordering of the grades. It is well known that the class ranges from simple rules, like dictatorship or constant rules, to rules which treat all the agents equally, such as choosing the median grade ([4]).

Definition 2.3.3 *For all candidates $a \in A$, let $\beta_a = (\beta_a^S)_{S \subseteq N}$ be a list of 2^N parameters satisfying: (i) $\beta_a^S \in G$ for all coalitions $S \subseteq N$, (ii) $\beta_a^{\emptyset} = k$, $\beta_a^N = 1$, and (iii) for any coalition $S \subseteq T$, $\beta_a^T \leq \beta_a^S$. Then, a collective evaluation rule $\phi : U^n \to U$ is called a min-max rule with respect to $(\beta_a)_{a \in A}$ if*

$$\phi_a(u_N) = \min_{S \subseteq N} \{\max(\{\beta_a^S\} \cup \{u_i(a) : i \in S\})\}.$$

Clearly, every min-max rule is candidate-wise.

Definition 2.3.4 *A min-max rule with respect to parameters $\beta = (\beta_a^S)_{a \in A, S \subseteq N}$ is called a generalized median rule if for all candidates $a \in A$, $\beta_a^S = \beta_a^T$ for all $S, T \subseteq N$ with $|S| = |T|$.*

Note that a min-max rule is anonymous if and only if it is a generalized median rule.

2.3.3 Results

We now present the main result of this section that characterizes all update monotone and continuous collective evaluation rules.

Theorem 2.3.5 *A collective evaluation rule $\phi : U^n \to U$ is update monotone and continuous if and only if it is a min-max rule.*

The proof of this theorem is relegated to Appendix .1.

The following corollary is immediate from Theorem 2.3.5.

Corollary 2.3.1 *A collective evaluation rule $\phi : U^n \to U$ is update monotone, continuous, and anonymous if and only if it is a generalized median rule.*

In what follows, we show that update monotonicity and continuity are independent of each other.

Remark 2.3.6 (Update monotonicity is independent from continuity) *Let $N = \{1, 2\}$ and let $A = \{a, b\}$. Suppose $G = \{1, 2, 3\}$ where $1 \equiv$ bad, $2 \equiv$ average and $3 \equiv$ good. Consider the collective evaluation rule, say ϕ, given in Table 2.3.1. The rule ϕ assigns the outcome $(1, 2)$ at every profile except the ones where agent 1 announces $(1, 1)$, in which case it assigns the outcome $(1, 1)$.*

It can be verified that ϕ is continuous. We show that it violates update monotonicity. Consider the profiles $v_N = ((2, 1), (1, 1))$ and $w_N = ((1, 1), (1, 1))$. Note that $\phi(v_N) = (1, 2)$ and $\phi_a(v_N) = w_1(a) < v_1(a)$, $v_1(b) = w_1(b) < \phi_b(v_N)$, $v_2(a) = w_2(a) = \phi_a(v_N)$, and $v_2(b) = w_2(b) < \phi_b(v_N)$. Further, note that $\phi(w_N) = (1, 1) \neq \phi(v_N)$, which is a contradiction to update monotonicity.

Remark 2.3.7 (Continuity is independent from update monotonicity) *Let $N = \{1, 2\}$ and let $A = \{a, b\}$. Suppose $G = \{1, 2, 3\}$ where $1 \equiv$ bad, $2 \equiv$ average and $3 \equiv$ good. Consider the collective evaluation rule, say ϕ, given in Table 2.3.2. According to this rule, agent 1 is the dictator except when he/she announces $(2, 1)$, in which case the outcome is $(2, 2)$. It is straightforward that ϕ is update monotone. We show that it violates continuity. Consider the profiles $v_N = ((1, 1), (1, 1))$ and $w_N = ((2, 1), (1, 1))$. Note that $d(v_N, w_N) = 1$, and $\phi(v_N) = (1, 1)$, $\phi(w_N) = (2, 2)$. However, since $d(\phi(v_N), \phi(w_N)) = 2$, this violates continuity.*

1 \ 2	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,3)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(2,1)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(2,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(2,3)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(3,1)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(3,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(3,3)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)

Table 2.3.1

2.4 A characterization of the update monotone and consistent collective evaluation rules

In this section, we introduce the notion of *consistency* and provide a characterization of the collective evaluation rules that are update monotone and consistent.

2.4.1 Consistency

First, we introduce the notion of minimal conflict profiles.

Definition 2.4.1 *For a candidate $a \in A$, a grade l with $l < k$, and a coalition $S \subseteq N$, a profile $v_N \in U^n$ is a* minimal conflict profile *at (a, l, S) if $v_i(a) = l$ for all agents $i \in S$, $v_j(a) = l + 1$ for all agents $j \in N - S$, and $v_s(x) = v_t(x)$ for all agents $s, t \in N$ and all candidates $x \in A - \{a\}$.*

In words, for a minimal conflict profile at (a, l, S), the candidate $a \in A$ is assigned the grade l by all the agents in S and the grade $l + 1$ by all the agents in $N - S$. Further, all the agents agree on the grade of every other candidate.

Now, we are ready to define the notion of consistency. It ensures that the individual grades are aggregated over minimal conflict profiles in a consistent manner. Note that it does not impose any condition on the outcome at other profiles. Thus, it is a weak consistency requirement.

Definition 2.4.2 *A collective evaluation rule ϕ is* consistent *if for all candidates $a \in A$, all coalitions $S \subseteq N$, all grades l, l' with $l, l' < k$, and all profiles $v_N, w_N \in U^n$ such that v_N is a minimal conflict profile at (a, l, S) and w_N is a minimal conflict profile at (a, l', S), we have*

1 \ 2	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)
(2,1)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)
(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)
(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)
(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)

Table 2.3.2

$$\phi_a(v_N) = l \textit{ if and only if } \phi_a(w_N) = l'.$$

The implication of consistency is as follows. Consider a candidate $a \in A$ and a coalition $S \subseteq N$. Consider all the profiles that are minimal conflict at a between S and $N - S$ (but at different grades). Then, consistency says that if a collective evaluation rule selects the lower (higher) grade at any of these profiles, then it must select the lower (higher) grade at every profile.

2.4.2 Extreme min-max rules

In this section, we introduce a particular type of collective evaluation rules, called the extreme min-max rules, which we will use in our characterization of the collective evaluation rules that are update monotone and consistent. Verbally, an extreme min-max rule is a min-max rule where the parameters always take extreme values, that is, either the lowest possible grade or the highest possible grade. These rules are introduced in [18] in the context of strategic social choice, where it is shown that every efficient and strategy-proof social choice rule on a single-peaked domain with outside option is an extreme min-max rule.

Definition 2.4.3 *A min-max rule with respect to parameters $\beta = (\beta_a^S)_{a \in A, S \subseteq N}$ is called an extreme min-max rule if $\beta_a^S \in \{1, k\}$ for all candidates $a \in A$ and all coalitions $S \subseteq N$.*

2.4.3 Ordered rules

In this subsection, we introduce one more class of collective evaluation rules called ordered rules.

The notion of ordered rules is introduced in [4]. However, in their paper, they call these rules order functions.

Let $a \in A$ be a candidate and $k \leq n$, and let $u_N \in U^n$ be a profile. Call l the k-th ordered grade of a at u_N, if l is such that $|\{i \in N \mid u_i(a) \leq l\}| \geq k$ and $|\{i \in N \mid u_i(a) < l\}| < k$. Now, we are ready to define ordered rules.

Definition 2.4.4 *Let $\underset{\sim}{k} = (k_1, \ldots, k_m)$ be such that $k_i \leq n$ for all $i = 1, \ldots, m$. Then,* $\underset{\sim}{k}$-th ordered rule *$\phi^{\underset{\sim}{k}} : U^n \to U$ is defined as follows: for all candidates $a_i \in A$ and all profiles $u_N \in U^n$, $\phi_{a_i}^{\underset{\sim}{k}}(u_N)$ is the k_i-th ordered grade of a_i at u_N.*

Consider a $\underset{\sim}{k}$-th ordered rule. For all $i = 1, \ldots, m$, if

- $k_i = 1$, then it is known as the min operator,
- $k_i = n$, then it is known as the max operator,
- $k_i = n/2 + 1$ if n is even and $k_i = (n+1)/2$ if n is odd, then it is known as the median operator.

It is straightforward to see that every ordered rule is an extreme min-max rule.

It can be verified that every extreme min-max rule (and hence every ordered rule) is update monotone, consistent and uncompromising. Additionally, ordered rules are anonymous.

2.4.4 Results

In this section, we provide a characterization of all collective evaluation rules that are update monotone and consistent.

Theorem 2.4.5 *A collective evaluation rule $\phi : U^n \to U$ is update monotone and consistent if and only if it is an extreme min-max rule.*

The proof of this theorem is relegated to Appendix .2.

The following corollary is immediate from Theorem 2.4.5.

Corollary 2.4.1 *A collective evaluation rule is $\phi : U^n \to U$ is update monotone, consistent, and anonymous if and only if it is some $\underset{\sim}{k}$-th ordered rule.*

In what follows, we show that update monotonicity and consistency are independent of each other.

Remark 2.4.6 (Update monotonicity is independent from consistency) *Let $N = \{1, 2\}$ and let $A = \{a, b\}$. Suppose $G = \{1, 2, 3\}$ where $1 \equiv$ bad, $2 \equiv$ average and $3 \equiv$ good. Consider the collective evaluation rule, say ϕ, given in Table 2.4.1. Note that the collective grade for candidate a is dictatorially* *determined by agent 1 and that for candidate b is almost also solely determined by agent 1 except in six profiles. Those profiles have the property that agents 1 and 2 maximally differ on the grade for candidate b, while agreeing on the grade for candidate a. At these profiles, compromising grade 2 is chosen as the collective grade for b.*

It can be verified that ϕ is consistent. We show that it violates update monotonicity. Consider the profiles $v_N = ((1,1),(1,3))$ and $w_N = ((1,1),(1,2))$. Note that $\phi(v_N) = (1,2)$ and $v_1(a) = w_1(a) = \phi_a(v_N)$, $v_1(b) = w_1(b) = \phi_b(v_N)$, $v_2(a) = w_2(a) = \phi_a(v_N)$, and $\phi_b(v_N) = w_2(b) < v_2(b)$. Further, note that $\phi(w_N) = (1,1) \neq \phi(v_N)$, which is a contradiction to the update monotonicity.

1 \ 2	(1,1)	(1,2)	(1,3)	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)
(1,1)	(1,1)	(1,1)	(1,2)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)	(1,1)
(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)	(1,2)
(1,3)	(1,2)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)	(1,3)
(2,1)	(2,1)	(2,1)	(2,1)	(2,1)	(2,1)	(2,2)	(2,1)	(2,1)	(2,1)
(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)	(2,2)
(2,3)	(2,3)	(2,3)	(2,3)	(2,2)	(2,3)	(2,3)	(2,3)	(2,3)	(2,3)
(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,1)	(3,2)
(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)	(3,2)
(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,3)	(3,2)	(3,3)	(3,3)

Table 2.4.1

Remark 2.4.7 (Consistency is independent from update monotonicity) *Let $N = \{1,2\}$ and let $A = \{a,b\}$. Suppose $G = \{1,2,3\}$. Consider the collective evaluation rule, say ϕ, such that $\phi((1,1),(1,1)) = (1,1)$ and $\phi(u_N) = (2,2)$ for all profiles $u_N \neq ((1,1),(1,1))$. It is straightforward that ϕ is update monotone. We show that it violates consistency. Consider the profiles $v_N = ((1,2),(1,1))$ and $w_N = ((1,3),(1,2))$. Note that v_N is a minimal conflict profile at $(b,1,\{2\})$ and w_N is a minimal conflict profile at $(b,2,\{2\})$. Further, note that $\phi_b(v_N) = 2$ and $\phi_b(w_N) = 2$, which violates the consistency.*

2.5 Conclusion

In this paper, we have characterized the collective evaluation rules based on some properties of those. Theorem 2.3.5 shows that a collective evaluation rule is update monotone and continuous if and only if it is a min-max rule ([63]), and Theorem 2.4.5 shows that a collective evaluation rule is update monotone and consistent if and only if it is an extreme min-max rule. In the literature of strategic social choice theory, min-max rules appear as the unanimous (tops-only) and strategy-proof social choice functions on the single-peaked domains, and extreme min-max rules appear as the efficient and strategy-proof social choice functions on the single-peaked domains with outside option. In our model, the single-peakedness is inherited by the ordering over the grades. It is worth noting that unanimity (efficiency) and strategy-proofness can be guaranteed by the combination of update monotonicity and continuity or by the combination of update monotonicity and consistency. Thus, this paper has established a relation between the traditional social choice theory and the aggregations of evaluations.

Appendix

.1 Proof of Theorem 2.3.5

Proof: (**If part**) Let ϕ be a min-max rule. It follows from [2] that ϕ is uncompromising. By Remark 2.2.4, ϕ is update monotone. We proceed to show ϕ is continuous. Let $u_N, v_N \in U^n$ be two profiles such that $v_i(a) = u_i(a) + 1$ and $u_j(x) = v_j(x)$ for all $(j, x) \in N \times A$ such that $(j, x) \neq (i, a)$. It is sufficient to show that $|\phi(u_N) - \phi(v_N)| \leq 1$. First, note that by uncompromisingness, $\phi_x(u_N) = \phi_x(v_N)$ for all $x \neq a$. If $\phi_a(u_N) \neq u_i(a)$, then again by uncompromisingness, $\phi_a(u_N) = \phi_a(v_N)$. So, suppose $\phi_a(u_N) = u_i(a)$. Assume for contradiction $|\phi(u_N) - \phi(v_N)| \geq 2$. Assume without loss of generality, $\phi_a(v_N) = \phi_a(u_N) + 2$. However, by using uncompromisingness at v_N, this means $\phi_a(v_N) = \phi_a(u_N)$, a contradiction. ■

(**Only-if part**) The proof of the only-if part follows from the following lemmas.

Lemma .1.1 *Every update monotone and continuous collective evaluation rule $\phi : U^n \to U$ is candidate-wise.*

Proof: Let $\phi : U^n \to U$ be an update monotone and continuous collective evaluation rule. Assume for contradiction that ϕ is not candidate-wise, that is, there exist profiles $u_N, v_N \in U^n$ and a candidate $a \in A$ such that $u_i(a) = v_i(a)$ for all agents $i \in N$ and $\phi_a(u_N) \neq \phi_a(v_N)$. Let
$\mathcal{X} = \{(i, x) \in N \times A \mid u_i(x) \neq v_i(x)\}$. If $\mathcal{X}$ is empty, then $u_N = v_N$, and hence $\phi_a(u_N) = \phi_a(v_N)$. Suppose that $\mathcal{X}$ is not empty. Consider $(i, b) \in \mathcal{X}$. Without loss of generality assume that $u_i(b) = l < l' = v_i(b)$. Consider the profile $u'_N \in U^n$ where only agent i changes his/her grade for only candidate b from l to $l + 1$, and everything else remain same as in u_N. More formally, u'_N is such that $u'_i(b) = l + 1$ and $u'_j(x) = u_j(x)$ for all $(j, x) \in N \times A$ such that $(j, x) \neq (i, b)$.

Claim 1. $\phi_a(u_N) = \phi_a(u'_N)$.

Proof of the claim. We distinguish the following two cases.

Case 1: Suppose $\phi_b(u_N) \neq \phi_b(u'_N)$.

Note that since from u_N to u'_N, only agent i changes his/her grade for only b from l to $l+1$, we have $d(u_N, u'_N) = 1$. Therefore, by continuity, $\phi_b(u_N) \neq \phi_b(u'_N)$ implies $\phi_x(u_N) = \phi_x(u'_N)$ for all $x \neq b$. This, in particular, implies $\phi_a(u_N) = \phi_a(u'_N)$.

Case 2: Suppose $\phi_b(u_N) = \phi_b(u'_N)$. We distinguish two further sub-cases.

Case 2.1: Suppose $\phi_b(u_N) = \phi_b(u'_N) \geq l+1$.

Note that by the construction of u'_N, this means from u_N to u'_N, only agent i moves his/her grade for the candidate b towards $\phi_b(u_N)$ and nothing else changes, that is, $u_j(x) = u'_j(x)$ for all $(j, x) \in N \times A$ such that $(j, x) \neq (i, b)$, and $u_i(b) \leq u'_i(b) \leq \phi_b(u_N)$. Therefore, we have by update monotonicity, $\phi(u_N) = \phi(u'_N)$, which in particular means $\phi_a(u_N) = \phi_a(u'_N)$.

Case 2.2: Suppose $\phi_b(u_N) = \phi_b(u'_N) \leq l$.

Since $u_j(x) = u'_j(x)$ for all $(j, x) \in N \times A$ such that $(j, x) \neq (i, b)$ and $\phi_b(u'_N) \leq u_i(b) \leq u'_i(b)$, we have by update monotonicity, $\phi(u_N) = \phi(u'_N)$, which in particular means $\phi_a(u_N) = \phi_a(u'_N)$. This completes the proof of the claim. □

If $u'_i(b) \neq v_i(b)$, then consider the profile $u''_N \in U^n$ such that $u''_i(b) = l+2$ and $u''_j(x) = u'_j(x)$ for all $(j, x) \in N \times A$ such that $(j, x) \neq (i, b)$. By using similar logic as for Claim 1, we can show that $\phi_a(u''_N) = \phi_a(u'_N)$. Continuing in this manner, we construct a profile $\bar{u}_N \in U^n$ such that $\phi_a(\bar{u}_n) = \phi_a(u_N)$ and $\bar{u}_j(x) \neq v_j(x)$ if and only if $(j, x) \in \mathcal{X} - \{(i, b)\}$. Next, we take $(i', b') \in \mathcal{X} - \{(i, b)\}$, and use similar logic to construct a profile $\hat{u}_N \in U^n$ such that $\phi_a(\hat{u}_n) = \phi_a(u_N)$ and $\hat{u}_j(x) \neq v_j(x)$ if and only if $(j, x) \in \mathcal{X} - \{(i, b), (i', b')\}$. Continuing in this manner, we arrive at the profile v_N and deduce $\phi_a(v_N) = \phi_a(u_N)$, which contradicts our initial assumption that $\phi_a(v_N) \neq \phi_a(u_N)$. This completes the proof of Lemma .1.1. ■

Lemma .1.2 *Every update monotone and continuous collective evaluation rule $\phi : U^n \to U$ is uncompromising.*

Proof: Let $\phi : U^n \to U$ be an update monotone and continuous collective evaluation rule. By Lemma .1.1, ϕ is candidate-wise. Therefore, assume that $A = \{a\}$. Let a profile $u_N \in U^n$ be such that $u_i(a) < \phi_a(u_N)$. Take a profile $v_N \in U^n$ such that $v_i(a) \leq \phi_a(u_N)$. It is enough to show $\phi_a(v_N) = \phi_a(u_N)$ where $u_j(a) = v_j(a)$ for all $j \neq i$. If $u_i(a) \leq v_i(a) \leq \phi_a(u_N)$, then we are done by update monotonicity of ϕ. Suppose $v_i(a) < u_i(a) < \phi_a(u_N)$. Suppose further that $v_i(a) = u_i(a) - 1$. Since $v_i(a) < u_i(a) < \phi_a(u_N)$, by continuity, we have $v_i(a) < u_i(a) \leq \phi_a(v_N)$. By update monotonicity, this means $\phi_a(v_N) = \phi_a(u_N)$. The proof for arbitrary $v_i(a) < u_i(a)$ follows by repeated application of this argument. ■

Lemma .1.3 *Every uncompromising and candidate-wise collective evaluation rule $\phi : U^n \to U$ is a min-max rule.*

The proof of this lemma follows from [2].

.2 Proof of Theorem 2.4.5

Proof: (**If part**) Let $\phi : U^n \to U$ be an extreme min-max rule. Since ϕ is a min-max rule, by the proof of the if part of Theorem 2.3.5, it follows that ϕ is update monotone. We show that it is consistent. Consider a candidate $a \in A$ and a coaliton $S \subseteq N$. For all grades $l = 1, \ldots, k-1$, consider a minimal conflict profile $u^l_N \in U^n$ such that $u^l_i(a) = l$ for all agents $i \in S$, $u^l_i(a) = l+1$ for all agents $l \in N - S$, and $u^l_i(x) = u^l_j(x)$ for all agents $i, j \in N$ and all candidates $x \in A - \{a\}$. Consider the profile $v_N \in U^n$ where $v_i(a) = 1$ for all agents $i \in S$ and $v_i(a) = k$ for all agents $i \in N - S$. By the definition of ϕ, $\phi_a(v_N) = \beta^S_a \in \{1, k\}$. Without loss of generality, let $\phi_a(v_N) = 1$. We show that $\phi_a(u^l_N) = l$ for all $l = 1, \ldots, k-1$. Take $l \in \{1, \ldots, k-1\}$. Consider the profile $\bar{v}_N \in U^n$ such that $\bar{v}_i(a) = l$ for all agents $i \in S$ and $\bar{v}_j(b) = v_j(b)$ in all other cases, that is, if $j \notin S$ or if $b \neq a$. Suppose $\phi_a(\bar{v}_N) > l$. By uncompromisingness, this means $\phi_a(v_N) > l$, a contradiction. So, $\phi_a(\bar{v}_N) \leq l$. Suppose $\phi_a(\bar{v}_N) < l$. By uncompromisingness, this means $\phi_a(\tilde{v}_N) < l$, where $\tilde{v}_i(a) = k$ for all agents $i \in N$. However, by the definition of min-max rule, we have $\phi(\tilde{v}_N) = k$, a contradiction. So, $\phi(\bar{v}_N) = l$. Now, consider the profile $\hat{v}_N \in U^n$ such that $\hat{v}_i(a) = l+1$ for all agents $i \in N - S$ and $\hat{v}_j(b) = \bar{v}_j(b)$ for all other cases, that is, if $j \in S$ or if $b \neq a$. By uncompromisingness, $\phi_a(\hat{v}_N) = l$. Note that by the construction of $\hat{v}_N$, $\hat{v}_i(a) = u^l_i(a)$ for all agents $i \in N$. Therefore by the candidate-wise property of ϕ, it follows that $\phi_a(u^l_N) = l$. This completes the proof of the if part of the theorem.

(**Only-if part**) The proof of the only-if part follows from the following lemmas. Our first lemma shows that every update monotone and consistent collective evaluation rule is candidate-wise.

Lemma .2.1 *Every update monotone and consistent collective evaluation rule $\phi : U^n \to U$ is candidate-wise.*

Proof: Let $\phi : U^n \to U$ be an update monotone and consistent collective evaluation rule. Consider a candidate $a \in A$ and profiles $v_N, w_N \in U^n$ such that $v_i(a) = w_i(a)$ for all agents $i \in N$. It is sufficient to prove that $\phi_a(v_N) = \phi_a(w_N)$. Without loss of generality, assume for contradiction that $\phi_a(v_N) < \phi_a(w_N)$. Because ϕ is update monotone, we can assume that $v_i(x) = \phi_x(v_N)$ and $w_i(x) = \phi_x(w_N)$ for all agents $i \in N$ and all candidates $x \in A - \{a\}$. Let $S = \{i \in N \mid v_i(a) \leq \phi_a(v_N)\}$. Let $\widehat{v}_N, \widehat{w}_N \in U^n$ be two profiles such that

(i) $\widehat{v}_N$ is a minimal conflict profile at $(a, \phi_a(v_N), S)$,
(ii) $\widehat{w}_N$ is a minimal conflict profile at $(a, \phi_a(w_N) - 1, S)$, and
(iii) $\widehat{v}_i(x) = v_i(x) = \phi_x(v_N)$ and $\widehat{w}_i(x) = w_i(x) = \phi_x(w_N)$ for all agents $i \in N$ and all candidates $x \in A - \{a\}$.

Note that $\widehat{v}_i(a) = \phi_a(v_N)$ and $\widehat{w}_i(a) = \phi_a(w_N) - 1$ for all agents $i \in S$ and $\widehat{v}_i(a) = \phi_a(v_N) + 1$ and $\widehat{w}_i(a) = \phi_a(w_N)$ for all agents $i \in N - S$. By update monotonicity, $\phi_a(\widehat{v}_N) = \phi_a(v_N)$ and

$\phi_a(\widehat{w}_N) = \phi_a(w_N)$. This means a low grade for a is chosen at $\widehat{v}_N$ whereas a high grade for a is chosen at $\widehat{w}_N$ contradicting that ϕ is consistent. ■

Our next Lemma shows that every update monotone and consistent collective evaluation rule is uncompromising.

Lemma .2.2 *Every update monotone and consistent collective evaluation rule $\phi : U^n \to U$ is uncompromising.*

Proof: Let $\phi : U^n \to U$ be an update monotone and consistent collective evaluation rule. Consider a candidate $a \in A$. As ϕ is candidate-wise by Lemma .2.1, it is sufficient to prove that $\phi_a(v_N) = \phi_a(w_N)$ for all profiles $v_N, w_N \in U^n$ such that for all agents $i \in N$, $v_i(a) < \phi_a(v_N)$ implies $w_i(a) \leq \phi_a(v_N)$, $\phi_a(v_N) < v_i(a)$ implies $\phi_a(v_N) \leq w_i(a)$, and $v_i(x) = w_i(x)$ for all candidates $x \in A - \{a\}$. To the contrary, suppose $\phi_a(v_N) \neq \phi_a(w_N)$. Without loss of generality, assume $\phi_a(w_N) < \phi_a(v_N)$. By means of update monotonicity, we assume that $v_i(x) = w_i(x) = \phi_x(v_N)$ for all agents $i \in N$ and all candidates $x \in A - \{a\}$. Let $S = \{i \in N \mid v_i(a) < \phi_a(v_N)\}$ and $T = \{i \in N \mid v_i(a) > \phi_a(v_N)\}$. Consider the following two profiles $\widehat{v}_N, \widehat{w}_N \in U^n$ such that

(i) for all agents $i \in N$ and all candidates $x \in A - \{a\}$, $\widehat{v}_i(x) = v_i(x) = \phi_x(v_N) = \widehat{w}_i(x)$,

(ii) for all agents $i \in S$, $\widehat{w}_i(a) = \phi_a(w_N)$ and $\widehat{v}_i(a) = \phi_a(v_N) - 1$, and

(iii) for all agents $i \in N - S$, $\widehat{w}_i(a) = \phi_a(w_N) + 1$ and $\widehat{v}_i(a) = \phi_a(v_N)$.

Note that $\widehat{v}_N$ is a minimal conflict profile at $(a, \phi_a(v_N) - 1, S)$ and $\widehat{w}_N$ is a minimal conflict profile at $(a, \phi_a(w_N), S)$. Update monotonicity implies $\phi_a(\widehat{w}_N) = \phi_a(w_N)$ and $\phi_a(\widehat{v}_N) = \phi_a(v_N)$. However, as $\widehat{v}_N$ and $\widehat{w}_N$ are minimal conflict profiles, this contradicts the consistency of ϕ. ■

The following Lemma shows that for every update monotone and consistent collective evaluation rule, the outcome grade of a candidate at a profile is either one of the labels announced by the agents for that candidate at that profile or one of the highest or the lowest labels for that candidate.

Lemma .2.3 *Suppose $\phi : U^n \to U$ is an update monotone and consistent collective evaluation rule. Let $u_N \in U^n$ be a profile and let $a \in A$ be a candidate. Then, $\phi_a(u_N) \in \{u_1(a), \ldots, u_n(a)\} \cup \{1, k\}$.*

Proof: Let $\phi : U^n \to U$ be an update monotone and consistent collective evaluation rule. Let $u_N \in U^n$ be a profile and let $a \in A$ be a candidate. We show that $\phi_a(u_N) \in \{u_1(a), \ldots, u_n(a)\} \cup \{1, k\}$. To the contrary, suppose $\phi_a(u_N) \notin \{u_1(a), \ldots, u_n(a)\} \cup \{1, k\}$. Let $S = \{i \in N \mid u_i(a) < \phi_a(u_N)\}$. Because $\phi_a(u_N) \notin \{u_1(a), \ldots, u_n(a)\}$, we have $N - S = \{i \in N \mid u_i(a) > \phi_a(u_N)\}$. Also, as $\phi_a(u_N) \notin \{1, k\}$, we have $\phi_a(u_N) + 1 \in G$ and $\phi_a(u_N) - 1 \in G$. Let $\widehat{v}_N \in U^n$ be a minimal conflict profile at $(a, \phi_a(u_N) - 1, S)$ and $\widehat{w}_N$ be a minimal conflict profile at $(a, \phi_a(u_N), S)$. By Lemma .2.1, $\phi_a(\widehat{v}_N)$ and $\phi_a(\widehat{w}_N)$ are independent of $\widehat{v}_i(x)$ and $\widehat{w}_i(x)$ for all agents $i \in N$ and all candidates $x \in A - \{a\}$. Because ϕ is update monotone, it follows that $\phi_a(\widehat{v}_N) = \phi_a(u_N) = \phi_a(\widehat{w}_N)$. However, this contradicts that ϕ is consistent. ■

Now, we are ready to complete the proof of Theorem 2.4.5. Let $\phi : U^n \to U$ be an update monotone and consistent collective evaluation rule. By Lemma .2.2 and the if-part of Theorem 2.3.5, this means ϕ is continuous. Now, by Theorem 2.3.5, we obtain that ϕ is a min-max rule. Suppose $(\beta_a^S)_{a \in A, S \subseteq N}$ are the parameters of ϕ. It remains to show that $\beta_a^S \in \{1, k\}$ for all candidates $a \in A$ and all $S \subseteq N$. Consider a candidate $a \in A$ and a coalition $S \subseteq N$. Note that by the definition of min-max rule, β_a^S is the outcome of ϕ at a profile where $u_i(a) = 1$ for all $i \in S$ and $u_i(a) = k$ for all agents $i \in N - S$. By Lemma .2.3, the outcome of ϕ at such a profile is either 1 or k, which completes the proof of the only-if part of the theorem.
■

.3 Topological justification of continuity

A function F from a set $\mathcal{A}$ to another set $\mathcal{B}$ is continuous if the inverse image $F^{-1}(B)$ of an open set B in $\mathcal{B}$ is open in $\mathcal{A}$. Thus, the definition of continuity requires the notion of open sets, that is, the notion of topologies on both the domain $\mathcal{A}$ and the range $\mathcal{B}$ of F. Such a topology is standard when $\mathcal{A}$ and $\mathcal{B}$ are Euclidean spaces. However, in case of finite sets, there is no unified notion of topologies. In what follows, we define a natural topology on graphs and show that the notion of continuity that arises from this topology coincides with that in Definition 2.3.2.

Let $\mathcal{G}_\mathcal{A} = (\mathcal{A}, \mathcal{E}_\mathcal{A})$ and $\mathcal{G}_\mathcal{B} = (\mathcal{B}, \mathcal{E}_\mathcal{B})$ be (undirected) graphs with vertex sets $\mathcal{A}$ and $\mathcal{B}$, respectively. In our case, these graphs are defined as follows.

The set $\mathcal{A}$ is the set of profiles $\mathcal{U}^n$ and $\{u_N, v_N\} \in \mathcal{E}_\mathcal{A}$ if and only if there are $a \in A$ and $i \in N$ such that

(i) $u_j(x) = v_j(x)$ for all $j \in N$ and all $x \in A$ such that either $j \neq i$ or $a \neq x$, that is, for all agent-candidate pairs (j, x) such that $(j, x) \neq (i, a)$, and

(ii) $u_i(a) = v_i(a) + 1$.

Thus, two profiles form an edge if and only if they differ minimally, that is, only one agent differs from one of them to the other and that too by exactly one grade for one particular candidate.

The set $\mathcal{B}$ is the set of evaluations $\mathcal{U}$ and $\{u, v\} \in \mathcal{E}_\mathcal{B}$ if and only if there is $a \in A$ such that

(i) $u(x) = v(x)$ for all $x \in A$ with $a \neq x$, and

(ii) $u(a) = v(a) + 1$.

Thus, two evaluations form an edge if and only if they differ in a minimal way, that is, by exactly one grade of one particular candidate.

Define the topology $\tau_{\mathcal{A}}$ as follows: a subset O of $\mathcal{A} \cup \mathcal{E}_{\mathcal{A}}$ is open if $x \in O \cap \mathcal{A}$ and $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$ for some $y \in \mathcal{A}$ imply $\{x, y\} \in O$. This, in particular, means, for instance, that every collection of edges is open, and every set O with the following property is open: if a vertex is in O, then all its adjacent edges are also in O. Similarly, define the topology $\tau_{\mathcal{B}}$ as the collection of subsets O of $\mathcal{B} \cup \mathcal{E}_{\mathcal{B}}$ such that if $x \in O \cap \mathcal{B}$ and $\{x, y\} \in \mathcal{E}_{\mathcal{B}}$ for some $y \in \mathcal{B}$, then $\{x, y\} \in O$.

For a vertex z in $\mathcal{A}$, define its neighborhood $\mathcal{N}_{\mathcal{G}_{\mathcal{A}}}(z) = \{z\} \cup \{\{z, v\} : \{z, v\} \in \mathcal{E}_{\mathcal{A}}\}$. So, the neighborhood of z in $\mathcal{G}_{\mathcal{A}}$ consists of the vertex z and all the edges that are adjacent to z. Similarly, we define the neighborhood $\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(z)$ for any $z \in \mathcal{B}$.

Call a function $\widetilde{F}$ from $\mathcal{A} \cup \mathcal{E}_{\mathcal{A}}$ to $\mathcal{B} \cup \mathcal{E}_{\mathcal{B}}$ an *extension* of a function F from $\mathcal{A}$ to $\mathcal{B}$ if $F(x) = \widetilde{F}(x)$ for all $x \in \mathcal{A}$. We say that $\widetilde{F}$ is *continuous* if $\widetilde{F}^{-1}(O) \in \tau_{\mathcal{A}}$ for all $O \in \tau_{\mathcal{B}}$.

Lemma .3.1 *Let $\widetilde{F}$ from $\mathcal{A} \cup \mathcal{E}_{\mathcal{A}}$ to $\mathcal{B} \cup \mathcal{E}_{\mathcal{B}}$ be a continuous extension of a function F from $\mathcal{A}$ to $\mathcal{B}$. Suppose $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$. Then $\widetilde{F}(\{x, y\}) \in \mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x))$.*

Proof: Note that $\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x)) \in \tau_{\mathcal{B}}$. So, $\widetilde{F}^{-1}(\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x))) \in \tau_{\mathcal{A}}$. But by definition $x \in \widetilde{F}^{-1}(\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x)))$ and therewith $\mathcal{N}_{\mathcal{G}_{\mathcal{A}}}(x) \subseteq \widetilde{F}^{-1}(\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x)))$. As $\{x, y\} \in \mathcal{N}_{\mathcal{G}_{\mathcal{A}}}(x)$ it follows that $\widetilde{F}(\{x, y\}) \in \mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x))$. ■

Note that an open set O in $\mathcal{G}_{\mathcal{A}}$, i.e., an element of $\tau_{\mathcal{A}}$, consists of a collection of edges in $\mathcal{E}_{\mathcal{A}}$ and some neighborhoods $\mathcal{N}_{\mathcal{G}_{\mathcal{A}}}(x)$ of the vertices $x \in O$.

Lemma .3.2 *Let $\widetilde{F}$ from $\mathcal{A} \cup \mathcal{E}_{\mathcal{A}}$ to $\mathcal{B} \cup \mathcal{E}_{\mathcal{B}}$ be an extension of a function F from $\mathcal{A}$ to $\mathcal{B}$. Further, let $\widetilde{F}(\{x, y\}) \in \mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x))$ for all $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$. Suppose $z \in \mathcal{B}$. Then, $\widetilde{F}^{-1}(\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(z))$ is in $\tau_{\mathcal{A}}$.*

Proof: If there is no x in $\mathcal{A}$ such that $F(x) = z$, then $\widetilde{F}^{-1}(\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(z))$ is either empty or consist of (only) some edges in $\mathcal{E}_{\mathcal{A}}$, and hence is open. So, suppose x in $\mathcal{A}$ is such that $F(x) = z$. It is sufficient to prove that $\mathcal{N}_{\mathcal{G}_{\mathcal{A}}}(x) \subseteq \widetilde{F}^{-1}(\mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(z))$. However, this follows by the assumption that $\widetilde{F}(\{x, y\}) \in \mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x))$ for all $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$. ■

The above implies the following corollary.

Corollary .3.1 *Let $\widetilde{F}$ from $\mathcal{A} \cup \mathcal{E}_{\mathcal{A}}$ to $\mathcal{B} \cup \mathcal{E}_{\mathcal{B}}$ be an extension of a function F from $\mathcal{A}$ to $\mathcal{B}$. Then,*
(i) $\widetilde{F}$ is continuous if and only if $\widetilde{F}(\{x, y\}) \in \mathcal{N}_{\mathcal{G}_{\mathcal{B}}}(F(x))$ for all $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$, and
(ii) if $F(x) \neq F(y)$ and $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$, then $\widetilde{F}(\{x, y\}) = \{F(x), F(y)\} \in \mathcal{E}_{\mathcal{B}}$.

Proof: The proof of (i) follows from Lemmas .3.1 and .3.2. For the proof of (ii), let $F(x) \neq F(y)$ and $\{x, y\} \in \mathcal{E}_{\mathcal{A}}$. It is sufficient to prove that $\widetilde{F}(\{x, y\}) = \{F(x), F(y)\}$. By Lemma .3.1, $\widetilde{F}(\{x, y\})$ is in the intersection of $\mathcal{N}_{\mathcal{G}_\mathcal{B}}(F(x))$ and $\mathcal{N}_{\mathcal{G}_\mathcal{B}}(F(y))$. Now, the proof follows from the fact that these two neighborhoods have only $\{x, y\}$ in common. ■

Note that for two distinct profiles u_N and v_N, we have $\{u_N, v_N\} \in \mathcal{E}_\mathcal{A}$ if and only if $d(u_N, v_N) = 1$, and for two distinct evaluations u and v, we have $\{u, v\} \in \mathcal{E}_\mathcal{B}$ if and only if $d(u, v) = 1$. Therefore, it follows from Corollary .3.1 that a function $F : \mathcal{A} \to \mathcal{B}$ is continuous with respect to the topology defined above if and only if $d(u_N, v_N) = 1$ implies $d(F(u_N), F(v_N)) \leq 1$ as required in Definition 2.3.2.

3

A characterization of possibility domains under Pareto optimality and group strategy-proofness

3.1 Introduction

We consider domains satisfying two mild conditions, namely pervasiveness and top-connectedness. Almost all well-known domains in both one-dimension and multiple-dimensions satisfy these conditions. We provide a necessary and sufficient condition on these domains for the existence of non-dictatorial, Pareto optimal, and group strategy-proof choice rules. By applying our result, we show that a domain on a graph admits such rules if and only if the graph has a terminal node. We further show that generalized circular domains, partially single-peaked domains, a large class of separable domains and lexicographic domains do not admit non-dictatorial, Pareto optimal, and group strategy-proof choice rules.

A closely related paper is [75] where the same analysis is done for unanimous and strategy-proof choice rules. However, they impose some technical condition on the domains (apart from pervasiveness and top-connectedness). [79] shows that circular domains and [1] and [72] show that a particular type of partially single-peaked domains do not admit non-dictatorial, unanimous, and strategy-proof choice rules. Their notions of these domains are quite restrictive. We show that in the presence of unanimity, if one strengthens strategy-proofness by group strategy-proofness, then he/she cannot obtain a non-dictatorial rule on a much generalized class of such domains.

Most striking consequence of our result is that separable or lexicographic multi-dimensional domains do not admit non-dictatorial, Pareto optimal, and group strategy-proof choice rules–even if the marginals are restricted, for instance, single-peaked. This is in sharp contrast with the result in [14] where it is shown that component-wise dictatorial rules (they are *not* dictatorial) are unanimous and strategy-proof on these domains.

3.2 Model

For any set B, we denote by $\mathbb{L}(B)$ the set of all linear orders (transitive, antisymmetric, and complete binary relations) on B. An element of $\mathbb{L}(B)$ is called a *preference* over B.

Let $N = \{1, \ldots, n\}$ be a set of *agents* and let A be a set of *alternatives*, where $|A| \geq 3$ and $n \geq 2$. A *domain* (of admissible preferences) $\mathbb{D}$ is a subset of $\mathbb{L}(A)$. We denote by $\tau(R)$ the best (top-ranked) alternative in $R \in \mathbb{D}$. For all the domains we consider in this paper, it is assumed that for all $x \in A$ there is $R \in \mathbb{D}$ with $\tau(R) = x$.

A *profile* p is an N-tuple of individual preferences in $\mathbb{D}^N$. A profile p is unanimous if $p(i) = p(j)$ for all $i, j \in N$. A subset S of N is called a *coalition*. The restriction of a profile p to a coalition S is denoted by $p|_S$.

For a preference R and two alternatives x and y, we write $R \equiv xy\cdots$ to mean x is the best and y is the second-best alternative in R. To save parentheses we write $xy \in R$ instead of $(x, y) \in R$, which has the usual interpretation that x is (weakly) preferred to y at R.

The notion of inseparable pairs is introduced in [49]. An ordered pair of alternatives (x, y) is called an *inseparable top-pair* if for all $R \in \mathbb{D}$ with $\tau(R) = x$, $yz \in R$ for all $z \in A \setminus \{x, y\}$.

A domain $\mathbb{D}$ is called *pervasive* if for all $x, y \in A$, $R \equiv xy\cdots$ is in $\mathbb{D}$ implies there is $R' \equiv yx\cdots$ in $\mathbb{D}$. Two alternatives x and y in A are called *directly top-connected*, denoted by $x \leftrightsquigarrow y$, if there are $R, R' \in \mathbb{D}$ such that $R \equiv xy\cdots$ and $R' \equiv yx\cdots$. A domain $\mathbb{D}$ is called *top-connected* if for every two alternatives x, y there is a sequence $x_1 = x, \ldots, x_k = y$ such that x_l and x_{l+1} are directly top-connected for all $l = 1, \ldots, k-1$. The notion of top-connectedness is introduced in [3].

In this paper, we restrict our attention to the domains that are both pervasive and top-connected.

We now introduce choice rules and a few properties of those. A *choice rule* is a function $\phi : \mathbb{D}^N \to A$. A choice rule ϕ is said to be *unanimous* if for all unanimous profiles the rule ϕ selects the (common) best alternative of the agents at the profile. A choice rule ϕ is *Pareto optimal* if for all profiles p there is no alternative x different from $\phi(p)$ such that $x\phi(p) \in p(i)$ for all agents $i \in N$. A choice rule ϕ is called *dictatorial* with an agent i as the *dictator*, if for all profiles p, $\phi(p) = \tau(p(i))$. A choice rule ϕ is *strategy-proof* if for all agents $i \in N$ and all profiles p and q with $p|_{N\setminus\{i\}} = q|_{N\setminus\{i\}}$, we have either

$\phi(p) = \phi(q)$ or $\phi(p)\phi(q) \in p(i)$. A choice rule ϕ is *group strategy-proof* if for all coalitions S and all profiles p and q with $p|_{N\setminus S} = q|_{N\setminus S}$, we have either $\phi(p) = \phi(q)$ or $\phi(p)\phi(q) \in p(i)$ for some $i \in S$. Note that group strategy-proofness and unanimity imply Pareto optimality.

3.3 Results

We now present the main result of this paper and discuss a few applications of it.

Theorem 3.3.1 *Let $\mathbb{D}$ be a pervasive and top-connected domain. Then there exist non-dictatorial, Pareto optimal, and group strategy-proof choice rules on $\mathbb{D}$ if and only if $\mathbb{D}$ has an inseparable top-pair.*

3.3.1 Domains on graphs

Let $G = (A, \mathcal{E})$ be an undirected connected graph over A. A preference R respects G if for all $x, y \in A \setminus \tau(R)$, x lies in *every* path from $\tau(R)$ to y implies $xy \in R$.[1] A domain $\mathbb{D}$ respects a graph G if it contains *all* preferences that respect G.

Note that by definition, every domain on a graph is pervasive and top-connected. Furthermore, a domain on a graph has an inseparable top-pair if and only if the graph has a terminal node.[2] Therefore, we have the following corollary of Theorem 3.3.1.

Corollary 3.3.1 *Let G be an arbitrary connected graph and let $\mathbb{D}$ be the domain with respect to G. Then, there exists a non-dictatorial, Pareto optimal, and group strategy-proof choice rule on $\mathbb{D}$ if and only if G has a terminal node.*

3.3.2 Generalized circular domains

Generalized circular domains are generalization of the circular domains analyzed in [79]. Suppose that $A = \{x_1, \ldots, x_m\}$. A domain $\mathcal{C}$ is called generalized circular if for all $x_k \in A$ there exist $R, R' \in \mathcal{C}$ such that $R \equiv x_k x_{k+1} \cdots$ and $R' \equiv x_k x_{k-1} \cdots$, where $x_{m+1} = x_1$ and $x_0 = x_m$. Note that the domain with respect to the graph $G = (A, \mathcal{E})$, where $\mathcal{E} = \{\{x_1, x_2\}, \ldots, \{x_{k-1}, x_k\}, \{x_k, x_1\}\}$, is the unrestricted domain. Thus, generalized circular domains are *not* special cases of domains on graphs. Further note that by definition, a generalized circular domain $\mathcal{C}$ is pervasive and top-connected. Clearly, such a domain does not have an inseparable top-pair. Thus, we have the following corollary of Theorem 3.3.1.

Corollary 3.3.2 *Every Pareto optimal and group strategy-proof choice rule on a generalized circular domain is dictatorial.*

[1] A path in a graph is a sequence of nodes such that every two consecutive nodes share an edge.
[2] A node is called terminal if its degree is one.

It is worth mentioning that the result in [79] does not apply to generalized circular domains. In fact, there are non-dictatorial, unanimous, and strategy-proof choice rules on these domains.

3.3.3 Partially single-peaked domains

[72] and [1] consider multiple single-peaked domains and partially single-peaked domains, respectively, and provide a characterization of the unanimous and strategy-proof choice rules on such domains. Here, we generalize these domains and provide a necessary and sufficient condition for the existence of non-dictatorial, Pareto optimal, and group strategy-proof choice rules.

For ease of presentation, we assume that the set of alternatives is the integers $\{1, \ldots, m\}$. A preference R is single-peaked over an interval $[\underline{x}, \overline{x}]$ if for all $x, y \in A$ such that at least one of x and y is in $[\underline{x}, \overline{x}]$, we have $\tau(R) \leq x < y$ or $y < x \leq \tau(R)$ implies $xy \in R$.

A domain is partially single-peaked over a collection of intervals $[\underline{x}^1, \overline{x}^1], \ldots, [\underline{x}^k, \overline{x}^k]$ if each preference in it is single-peaked over all those intervals. A domain $\mathbb{D}$ is single-peaked if it is partially single-peaked over the interval $[1, m]$. A domain is tail-single-peaked if it is partially single-peaked over the intervals of the form $[1, \underline{x}]$ and $[\overline{x}, m]$, where $2 < \underline{x} < \overline{x} < m - 1$.

Corollary 3.3.3 *A pervasive and top-connected partially single-peaked domain $\mathbb{D}$ admits a non-dictatorial, Pareto optimal, and group strategy-proof choice rule if and only if it is tail-single-peaked.*

Corollary 3.3.3 also holds for a class of single-crossing domains ([78]) since those domains are pervasive and top-connected.

3.3.4 Multi-dimensional domains

Let $A = \prod_{l \in M} A^l$ where $M = \{1, \ldots, k\}$ is a finite set of components, and for each component $l \in M$, the component set A^l contains finitely many elements with $|A^l| \geq 2$.

A preference R is *lexicographic* if there exists a (unique) component order $R^\circ \in \mathbb{L}(M)$ and a (unique) marginal preference $R^l \in \mathbb{L}(A^l)$ for each $l \in M$ such that for all $a, b \in A$, we have $\left[a^l b^l \in R^l \text{ for some } l \in M \text{ and } a^{l'} = b^{l'} \text{ for all } l'l \in R^\circ\right] \Rightarrow [ab \in R]$.

A preference R is *separable* if there exists a (unique) marginal preference R^l for each $l \in M$ such that for all $a, b \in A$, we have $[a^l b^l \in R^l \text{ and } a^{-l} = b^{-l} \text{ for some } l \in M] \Rightarrow [ab \in R]$.

For a collection of domains $\mathbb{D}^\circ \subseteq \mathbb{L}(M), \mathbb{D}^1 \subseteq \mathbb{L}(A^1), \ldots, \mathbb{D}^k \subseteq \mathbb{L}(A^k)$, we denote by $\mathcal{S}(\mathbb{D}^1, \ldots, \mathbb{D}^k)$ the set of all separable preferences with marginal preferences in $\mathbb{D}^1 \times \ldots \times \mathbb{D}^k$, and by $\mathcal{L}(\mathbb{D}^\circ, \mathbb{D}^1, \ldots, \mathbb{D}^k)$ the set of all lexicographically separable preferences with component orders in $\mathbb{D}^\circ$ and marginal preferences in $\mathbb{D}^1 \times \ldots \times \mathbb{D}^k$.

We say $\mathbb{D}^\circ \subseteq \mathbb{L}(M)$ satisfies anti-regularity if for all $l \in M$, there is $R^\circ \in \mathbb{D}^\circ$ such that l is the bottom-ranked component in R°.

Note that if $\mathbb{D}^l$ is pervasive and top-connected for all $l = 1, \ldots, k$, then $\mathcal{S}(\mathbb{D}^1, \ldots, \mathbb{D}^k)$ is also pervasive and top-connected. Further, if $\mathbb{D}^0$ is anti-regular, then $\mathcal{L}(\mathbb{D}^0, \mathbb{D}^1, \ldots, \mathbb{D}^k)$ too is pervasive and top-connected. Verification of these facts is left to the reader.

Corollary 3.3.4 *Let $\mathbb{D}^l$ be pervasive and top-connected for all $l = 1, \ldots, k$. Then, every Pareto optimal and group strategy-proof choice rule on the separable domain $\mathcal{S}(\mathbb{D}^1, \ldots, \mathbb{D}^k)$ is dictatorial. Furthermore, if $\mathbb{D}^0$ is anti-regular, then every Pareto optimal and group strategy-proof choice rule on the lexicographic domain $\mathcal{L}(\mathbb{D}^0, \mathbb{D}^1, \ldots, \mathbb{D}^k)$ is dictatorial.*

Remark 3.3.2 *It is worth noting that Corollary 3.3.4 does not hold if we replace Pareto optimality and group strategy-proofness by unanimity and strategy-proofness. In fact, it is shown in [14] that every unanimous and strategy-proof rule on the maximal lexicographic or the maximal separable domain is component wise dictatorial. To the contrary, with Pareto optimality and group strategy-proofness, we get dictatorship even when marginals are restricted, for instance, are single-peaked.*

Appendix

.1 Proof of Theorem 3.3.1

If part of the proof follows from Example 3.1 of [75]. The proof of the only-if part is also similar to the proof of the only-if part of Theorem 1 in [75] with the only difference that Lemmas 3 and 5 are to be modified for our case. We present below these modified lemmas.

The following notations and notions are used in the proofs of Lemmas 1, 2, and 4 in [75] and also in the proofs below. For a coalition S, we denote by $((R)^S, (R')^{N\setminus S})$ a profile p where $p(i) = R$ for all $i \in S$ and $p(i) = R'$ for all $i \in N \setminus S$. We call such a profile $(S, N \setminus S)$-*unanimous*. Additionally, if $\tau(R) = x$ and $\tau(R') = y$, then such a profile is said to be xy-$(S, N \setminus S)$-*unanimous*. We say a coalition S is decisive on a pair of alternatives (x, y) for a choice rule ϕ if $\phi(p) = x$ for all $xy - (S, N \setminus S)$-unanimous profiles p, and we say that a coalition S is *decisive* for ϕ if for all $R \in \mathbb{D}$ with $\tau(R) = x$ and all $p|_{N\setminus S} \in \mathbb{D}^{N\setminus S}$, we have $\phi(R^S, p|_{N\setminus S}) = x$. We say that a choice rule ϕ is *alternative decisive* if for all coalitions S either S is decisive or $N \setminus S$ is decisive.

Lemma .1.1 (Modified Lemma 3) *Let $\phi : \mathbb{D}^N \to A$ be a Pareto optimal and strategy-proof choice rule. Let $a \leftrightsquigarrow b$ and $b \leftrightsquigarrow c$, where a, b and c are three different alternatives. Let $S \subseteq N$ be decisive on all ab-$(S, N \setminus S)$-unanimous profiles. Then S is decisive on all bc-$(S, N \setminus S)$-unanimous profiles.*

Proof: Let $\phi : \mathbb{D}^N \to A$ be a Pareto optimal and strategy-proof choice rule. Let $a \leftrightsquigarrow b$ and $b \leftrightsquigarrow c$, where a, b and c are three different alternatives. Let $S \subseteq N$ be decisive on all ab-$(S, N \setminus S)$-unanimous profiles.

We show that S is decisive on all bc-$(S, N \setminus S)$-unanimous profiles. Assume for contradiction that S is not decisive on all bc-$(S, N \setminus S)$-unanimous profiles. Then Lemma 2 of [75] implies $N \setminus S$ is decisive on all bc-$(S, N \setminus S)$-unanimous profiles. Consider R^{ab} in $\mathbb{D}^a$ and R^{cb} in $\mathbb{D}^c$ such that $R^{ab} \equiv ab \cdots$ and $R^{cb} \equiv cb \cdots$. Take a profile p such that $p(i) = R^{ab}$ for all $i \in S$ and $p(i) = R^{cb}$ for all $i \in N \setminus S$. As ϕ is Pareto optimal, this means $\phi(p) \in \{a, b, c\}$. Consider a profile q where $q(i) = R^{ba}$, with $R^{ba} \equiv ba \cdots$, for all $i \in S$ and $q(i) = p(i)$ for all $i \in N \setminus S$. As $N \setminus S$ is decisive on all bc-$(S, N \setminus S)$-unanimous profiles, $\phi(q) = c$. This means $\phi(p) \notin \{a, b\}$ as otherwise strategy-proofness will be violated at q via p. Similarly we have $\phi(p) \notin \{b, c\}$ as S is decisive on all ab-$(S, N \setminus S)$-unanimous profiles. So, $\phi(p) \notin \{a, b, c\}$, a clear contradiction. So, S is decisive on all bc-$(S, N \setminus S)$-unanimous profiles. ■

Lemma .1.2 (Modified Lemma 5) *Let for all finite and non-empty subsets N' of N every Pareto optimal and group strategy-proof choice rule $\phi' : \mathbb{D}^{N'} \to A$ be alternative decisive. Let $\phi : \mathbb{D}^N \to A$ be a Pareto optimal and group strategy-proof choice rule. Then ϕ is dictatorial.*

Proof: Let $N = \{1, \ldots, n\}$ and consider the following sequence of choice functions ϕ^1 up to ϕ^{n-1} from $\mathbb{D}^{N^1}$ up to $\mathbb{D}^{N^{n-1}}$ to A, respectively, where $N^k = \{1, \ldots, k\}$ and ϕ^k is defined as follows. For all k fix the same linear order R. For $p \in \mathbb{D}^{N^k}$ define $\phi^k(p) = \phi(p, R^{N \setminus N^k})$. Because ϕ is strategy-proof it follows that ϕ^k is strategy-proof for all $k \in \{1, \ldots, n-1\}$. Note that if ϕ^k is dictatorial with dictator i_k in $\{1, \ldots, k\}$, then $\{i_k\}$ is decisive at ϕ at every profile profile q where $q(j) = R$ for all $j \in N \setminus \{i_k\}$. Because ϕ is alternative decisive this yields that ϕ is dictatorial with dictator i_k. So, dictatorship of ϕ^k implies dictatorship of ϕ.

Now either N^{n-1} is decisive at ϕ or $\{n\}$ is decisive. In the latter case we have the desired result that ϕ is dictatorial. In the former case we have ϕ^{n-1} is unanimous by decisiveness of N^{n-1} at ϕ. Also ϕ is group strategy-proof. As group strategy-proofness and unanimity imply Pareto optimality, ϕ is Pareto optimal. Therefore ϕ^{n-1} is alternative decisive. So, either N^{n-2} or $\{n-1\}$ is decisive at ϕ^{n-1}. In case $\{n-1\}$ is decisive at ϕ^{n-1} we have that ϕ^{n-1} is dictatorial. Hence, ϕ is dictatorial, the desired result. In case N^{n-2} is decisive at ϕ^{n-1} we may proceed to ϕ^{n-2} and prove similarly that either ϕ is dictatorial with dictator $n-2$ or ϕ^{n-3} is a unanimous and strategy-proof choice rule. This process stops certainly at ϕ^1 as obviously $\{1\}$ is decisive at ϕ^1 and therefore ϕ is dictatorial with agent 1 as the dictator. ■

4

Necessary and sufficient conditions for pairwise majority decisions on path-connected domains

4.1 Introduction

We consider standard social choice problems where a group of agents have to collectively decide an alternative from a set of feasible alternatives. A choice function selects an alternative for every collection of individual preferences.

We impose desirable conditions on choice functions such as unanimity, anonymity, symmetry, and group strategy-proofness. A choice function is unanimous if, whenever all the individuals have the same preference, their common top-ranked alternative is chosen. It is called anonymous if it treats all the individuals equally. Symmetry ensures that if the role of two alternatives (at the top of preferences) are interchanged at certain type of profiles, the outcome is also interchanged accordingly. A choice function is called group strategy-proof if no group of agents can be strictly better off by misrepresenting their preferences, and is called strategy-proof if no individual can be better off by misrepresenting his/her preference.

A preference is called single-peaked on a tree if the alternatives can be arranged on a tree[1] so that preference declines as one moves away from the top-ranked alternative. Such preferences are well-known in the literature for their usefulness in modelling public good location problems.

We assume a mild structure called path-connectedness (see, [3]) on the domains we consider in this paper. Theorem 4.4.1 shows that a path-connected domain admits unanimous, anonymous, symmetric, and group strategy-proof choice functions if and only if it is single-peaked on a tree and the number of agents is odd. It follows as a corollary of this result that there exists no unanimous, anonymous, symmetric, and group strategy-proof choice function on a path-connected domain if the number of agents is even. When the number of agents is odd, Theorem 4.4.2 characterizes all unanimous, anonymous, symmetric, and group strategy-proof choice functions on single-peaked domains on trees as the tree-median rule. Finally, we investigate what happens if we replace group strategy-proofness by strategy-proofness. Theorem 4.5.2 says that if we strengthen the notion of path-connectedness in a suitable manner, then the conclusion of Theorem 4.4.1 can be achieved with strategy-proofness, that is, a strongly path-connected domain admits unanimous, anonymous, symmetric, and strategy-proof choice functions if and only if it is a single-peaked domain on a tree.

An alternative is called the pairwise majority winner at a profile if it beats every other alternative according to pairwise majority comparison and a choice function is called the pairwise majority rule if it selects the pairwise majority winner at every preference profile. [27] argued that if such a majority winner exists at a profile, we should choose it on the basis of "straightforward reasoning". The analysis of the pairwise majority rule dates back to [10], [27], and [56]. [9] shows that the pairwise majority rule exists on domains that are single-peaked on a line. Later, [31] generalizes this result by showing that the pairwise majority rule exists on a domain even if the domain is single-peaked on a tree. [41] consider the problem of locating a public facility and show that the outcome of the pairwise majority rule on a single-peaked domain on a tree minimizes the total distance traversed by the users to go to the facility. They further prove that this property holds for a single-peaked domain only when the underlying graph is a tree. [63] characterizes the pairwise majority rule on domains that are single-peaked on a line. [28] shows that strategy-proof and tops-only SCFs on a single-peaked domain on a tree can be recursively decomposed into medians of constant and dictatorial rules.

[81] consider single-peaked domains on tree when preferences are Euclidean with respect to the graph distance and show that an SCF on such a domain is strategy-proof and unanimous if and only if it is an extended generalized median voter scheme. [65] introduce a class of generalized single-peaked domains based on an abstract betweenness property and show that an SCF is strategy-proof on a sufficiently rich domain of generalized single-peaked preferences if and only if it takes the form of voting by issues. They

[1] A connected graph is called a tree if it has no cycle.

also provide a characterization of such domains that admit SCFs satisfying strategy-proofness, unanimity, neutrality, and non-dictatorship/anonymity. We provide a detailed discussion on the connection of our paper with these papers in Section 4.6.

[60] considers the problem of preference aggregation with exactly two alternatives and characterizes the pairwise majority aggregation rule in this setting by means of always decisiveness, equality, symmetry, and positive responsiveness. Later, [48] and [83] provide necessary and sufficient conditions on a domain so that the pairwise majority aggregation rule is transitive.

It is worth mentioning that the tree-median rule coincides with the pairwise majority rule on domains that are single-peaked on a tree.[2] Thus, the main contributions of our paper can be considered as (i) a characterization of domains that are single-peaked on trees by means of choice functions satisfying natural conditions such as unanimity, anonymity, symmetry, and group strategy-proofness/strategy-proofness, and (ii) a characterization of the pairwise majority rule on these domains as the only choice function satisfying the above mentioned properties. Thus, in addition to the existing results where single-peakedness on trees is proved to be sufficient for the existence of the pairwise majority rule, we show that under some natural conditions, it is also necessary for the same.

Characterizing domains by means of the choice functions that they admit is considered as an important problem in the literature. [24] characterize single-peaked domains on arbitrary trees by means of strategy-proof, unanimous, tops-only random social choice functions satisfying a compromise property and [71] shows that every minimally rich and connected Condorcet domain which contains at least one pair of completely reversed orders must be single-peaked.

The rest of the paper is organized as follows. Section 4.2 presents the notion of single-peaked domains on trees and Section 4.3 introduces the notion of the tree-median rule. Main results of the paper are presented in Section 4.4. Section 4.5 shows how group strategy-proofness can be replaced with strategy-proofness in our main result. All the proofs, as well as the independence of the axioms used in our main result, are collected in the Appendix.

4.2 Domains and their properties

Let A denote the set of *alternatives* and let $N = \{1, \ldots, n\}$ denote the set of *n agents*, where n is at least 2. We denote by $\mathbb{L}(A)$ the set of all linear orders (reflexive, transitive, antisymmetric, and complete binary relations) on A. An element of $\mathbb{L}(A)$ is called a *preference*. Note that preferences are strict by definition. An admissible set of agents' preferences (or a domain) $\mathbb{D}$ is a subset of $\mathbb{L}(A)$. A *profile* is a collection of preferences, one for each agent. More formally, a profile p is an element of $\mathbb{D}^n$.

[2]Despite the fact that the tree-median rule is nothing but the pairwise majority rule, we use the former term as for the special case when the tree is a line, this rule is called the median rule in the literature.

For ease of presentation, we do not use braces for singleton sets and use the following notations throughout the paper. Let R be a preference and let a and b be two alternatives (not necessarily distinct) in A. To save parentheses, we write $ab \in R$ instead of $(a, b) \in R$, which has the usual interpretation that a is (weakly) preferred to b at R. When a and b are distinct, we write $R \equiv \cdots ab \cdots$ to mean that a is ranked just above b at R. In line with this, we write $R \equiv ab \cdots$ to mean that a is the top-ranked and b is the second-ranked alternative at R. Notations like $R \equiv \cdots a \cdots b \cdots$, $R \equiv a \cdots$, and $R \equiv \cdots a$ have self-explanatory interpretations.

The top-ranked alternative at a preference R is denoted by $\tau(R)$. The set of the top-ranked alternatives of the preferences in a domain $\mathbb{D}$ is denoted by $\tau(\mathbb{D})$, that is,
$\tau(\mathbb{D}) = \{a \in A : \tau(R) = a \text{ for some } R \in \mathbb{D}\}$. We assume that $\tau(\mathbb{D})$ is a finite set of m alternatives.

Next, we introduce the notion of graphs. An (undirected) graph $G = (V(G), E(G))$ is a tuple where $V(G)$ is the set of vertices and $E(G) \subseteq \{\{a, b\} : a, b \in V(G)\}$ is the set of edges. A sequence of vertices $x^1, \ldots, x^k$ is called a path in G if $\{x^l, x^{l+1}\} \in E$ for all $1 \leq l < k$. A path $x^0, x^1, \ldots, x^k$ in G is called a *cycle* if $k \geq 3$, $x^0 = x^k$, and $x^s \neq x^t$ for all $0 \leq s < t \leq k$. A graph is called a *tree* if it has no cycles. For a tree and two vertices a and b, we denote by $\pi(a, b)$ (whenever the tree is clear from the context) the unique path between a and b.

Two alternatives a and b in A are called *top-connected* (in $\mathbb{D}$) if there are $R, R' \in \mathbb{D}$ such that $R \equiv ab \cdots$ and $R' \equiv ba \cdots$. We use the notation $a \leftrightsquigarrow b$ to mean that a and b are top-connected. The induced graph of a domain $\mathbb{D}$ is defined as the undirected graph $\mathcal{G}(\mathbb{D}) = (\tau(\mathbb{D}), E)$, where E is the set of edges consisting of all pairs of top-connected alternatives, that is, $E = \{\{a, b\} \subseteq \tau(\mathbb{D}) : a \leftrightsquigarrow b\}$. Two alternatives a and b are called *path-connected* if there is a path from a to b in $\mathcal{G}(\mathbb{D})$. A domain $\mathbb{D}$ is called path-connected if every two alternatives in $\tau(\mathbb{D})$ are path-connected (see, [3]).

A subset S of N is called a coalition. The restriction of a profile p to a coalition S is denoted by $p|_S$. For a coalition S and preferences R and R' in $\mathbb{D}$, the N-tuple $((R)^S, (R')^{N \setminus S})$ denotes the profile p where $p(i) = R$ for all agents i in S and $p(i) = R'$ for all agents i in $N \setminus S$.

We introduce the notion of single-peaked domains on trees. A preference is single-peaked on a tree if it has the property that as one goes far away along any path from its top-ranked alternative, preference decreases.

Definition 4.2.1 *Let T be a tree with $V(T) \subseteq A$. A domain $\mathbb{D}$ is called single-peaked on T if $\tau(\mathbb{D}) = V(T)$ and for all $R \in \mathbb{D}$ and all $a, b \in \tau(\mathbb{D})$, $a \in \pi(\tau(p(i)), b)$ implies $ab \in R$.*

Note that for a domain $\mathbb{D}$ that is single-peaked on a tree, there is no restriction on the ordering of the alternatives outside $\tau(\mathbb{D})$. We present an example of a single-peaked domain on a tree.

Example 4.2.2 *Let the set of alternatives be* $A = \{a_1, a_2, a_3, a_4, a_5, a_6, a_7\}$*. Consider the tree* T *in Figure 4.2.1 with* $V(T) = \{a_1, a_2, a_3, a_4, a_5\}$*. In Table 4.2.1, we present a single-peaked domain on this tree.*

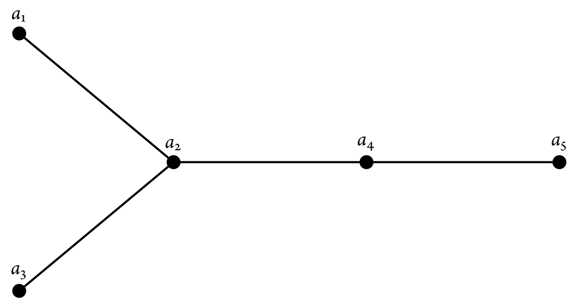


Figure 4.2.1: Tree for Example 4.2.2

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	R_{16}
a_1	a_1	a_2	a_2	a_2	a_2	a_2	a_3	a_3	a_4	a_4	a_4	a_4	a_4	a_5	a_5
a_2	a_2	a_1	a_6	a_3	a_4	a_4	a_7	a_2	a_2	a_2	a_6	a_5	a_5	a_4	a_6
a_6	a_4	a_4	a_1	a_7	a_5	a_7	a_2	a_6	a_7	a_5	a_2	a_6	a_2	a_6	a_4
a_3	a_7	a_7	a_4	a_1	a_6	a_5	a_6	a_4	a_6	a_3	a_3	a_2	a_6	a_2	a_2
a_4	a_3	a_6	a_3	a_4	a_3	a_1	a_4	a_1	a_1	a_6	a_7	a_7	a_1	a_3	a_1
a_5	a_6	a_5	a_5	a_5	a_7	a_3	a_5	a_5	a_3	a_1	a_5	a_1	a_3	a_7	a_7
a_7	a_5	a_3	a_7	a_6	a_1	a_6	a_1	a_7	a_5	a_7	a_1	a_3	a_7	a_1	a_3

Table 4.2.1: The single-peaked domain with respect to the tree in Figure 4.2.1

4.3 Choice functions and their properties

A *choice function* ϕ is a mapping from $\mathbb{D}^n$ to A. A choice function ϕ is *unanimous* if, whenever all the agents agree on their preferences, the top-ranked alternative of that common preference is chosen. More formally, $\phi : \mathbb{D}^n \to A$ is unanimous if for all profiles $p \in \mathbb{D}^n$ such that $p(i) = R$ for all agents $i \in N$ and some $R \in \mathbb{D}$, we have $\phi(p) = \tau(R)$. A choice function ϕ is called *anonymous* if it is symmetric in its arguments. In other words, anonymous choice functions disregard the identities of the agents. A choice function ϕ is *strategy-proof* if no agent can change its outcome in his/her favor by misreporting his/her sincere preference. More formally, $\phi : \mathbb{D}^n \to A$ is strategy-proof if for all agents $i \in N$ and all profiles $p, q \in \mathbb{D}^n$ with $p|_{N\setminus i} = q|_{N\setminus i}$, we have $\phi(p)\phi(q) \in p(i)$. A choice function ϕ is *group strategy-proof* if for all non-empty coalitions S of N and all profiles $p, q \in \mathbb{D}^n$ with $p|_{N\setminus S} = q|_{N\setminus S}$, we have either $\phi(p) = \phi(q)$ or $\phi(p)\phi(q) \in p(i)$ for some $i \in S$.

Next, we introduce the notion of symmetry. Symmetry has some resemblance with neutrality, however they are not the same.[3] Suppose that the agents are divided into two groups such that all agents in each group have the same preference. Suppose further that two alternatives a and b appear at the top two positions in each preference. Symmetry says that if the outcome of such a profile is a and the two groups interchange their preferences, then the outcome of the new profile will be b. In other words, symmetry ensures that if the roles of two alternatives are interchanged at certain type of profiles, the outcome is also interchanged accordingly. Note that symmetry is different from neutrality as it applies to a very specific class of profiles and only to the top-two ranked alternatives.

Definition 4.3.1 *We say that a choice function ϕ satisfies* symmetry *if for all $R \equiv ab\cdots$ and $R' \equiv ba\cdots$, and all subsets S of N, we have*

$$\phi((R)^S, (R')^{N\setminus S}) = a \text{ if and only if } \phi((R')^S, (R)^{N\setminus S}) = b.$$

4.3.1 Tree-median rule

The tree-median rule is an appropriate extension of the median rule defined in the context of single-peaked domains on lines. We first provide an verbal description of these rules. Suppose that the alternatives are named as $a_1, \ldots, a_m$ and that they are arranged on a line in the following order: $a_1 \prec \cdots \prec a_m$. Note that the median of a subset of alternatives B can be defined as the (unique) alternative a such that $|\{b \in B : b \prec a\}| < \frac{|B|}{2}$ and $|\{b \in B : b \succ a\}| < \frac{|B|}{2}$. For instance, if $B = \{a_1, a_3, a_4, a_9, a_{11}\}$, then a_4 is the unique alternative that satisfies the condition that $|\{b \in B : b \prec a_4\}| = |\{a_1, a_3\}| < 2.5$ and $|\{b \in B : b \succ a_4\}| = |\{a_9, a_{11}\}| < 2.5$. In other words, the number of alternatives which lie in any particular 'direction' of the median must be less than the half of the cardinality of the set. Here, two alternatives are said to be in the same direction with respect to an alternative a if they lie in the same component of the (possibly disconnected) graph that is obtained by deleting the alternative a from the line. We implement this idea on a tree.

Consider a tree $T = (V, E)$. For a vertex a of T, we denote by T^{-a} the graph that is obtained by deleting the alternative a (and all the edges involving a) from T, that is, $T^{-a} = \{\hat{V}, \hat{E}\}$, where $\hat{V} = V \setminus a$ and $\{x, y\} \in \hat{E}$ if and only if $\{x, y\} \in E$ and $a \notin \{x, y\}$. Note that T^{-a} is a disconnected graph unless a is a terminal node in T.[4] A component $\mathcal{C}$ of T^{-a} is defined as a maximal set of vertices of T^{-a} that are connected via some path in T^{-a}. Below, we provide an example of a tree T, and show the components of T^{-a} for some vertex a.

[3] [65] define a notion that is very similar to symmetry and call it neutrality. We use a different term to avoid confusion.
[4] A node is called terminal if it has degree 1.

Example 4.3.2 *Consider the tree T as given in Figure 4.3.1. Consider the vertex a_6. The components of T^{-a_6} are shown in Figure 4.3.2.*

Now, we are ready to define the notion of the median with respect to a tree. Let $T = (V, E)$ be a tree. For a subset $\widehat{V}$ of V, define the median of $\widehat{V}$ (with respect to T) as the unique vertex $a \in V$ such that for each component $\mathcal{C}$ of T^{-a}, we have

$$|\widehat{V} \cap \mathcal{C}| < \frac{|\widehat{V}|}{2}.$$

Whenever the tree T is clear from the context, we denote the median of a set $\widehat{V} \subseteq V$ with respect to T by median$(\widehat{V})$. The following example explains the idea of the median of a set. It should be clear from this example that the median of a set may not lie within the set.

Example 4.3.3 *Consider the tree T with $V(T) = \{a_1, \ldots, a_{10}\}$ as given in Figure 4.3.1. Consider the subset $\widehat{V} = \{a_1, a_4, a_7, a_8, a_9\}$ of V. We show that the median of $\widehat{V}$ is a_6. The components $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$ of T^{-a_6} are shown in Figure 4.3.2. Note that in each of these components, the number of elements from $\widehat{V}$ is less than the half of the cardinality of $\widehat{V}$. For instance, the elements of $\widehat{V}$ that are in Component $\mathcal{C}_1$ are a_1 and a_4. This proves that the median of $\widehat{V}$ is a_6. We proceed to show that a_6 is the unique vertex that satisfies this property. Note that since $\frac{n}{2} = 2.5$, a vertex v cannot be the median if a component in T^{-v} has more than two vertices. Consider the vertex a_4. Then, there is a component $C = \{a_6, a_7, a_8, a_9, a_{10}\}$ in T^{-a_4} that contains three elements a_7, a_8, a_9 from $\widehat{V}$. By using a similar logic, for any vertex v in $\{a_1, a_2, a_3, a_5\}$ there is a component in T^{-v} containing the vertices a_7, a_8, a_9, for any vertex v in $\{a_8, a_9, a_{10}\}$ there is a component in T^{-v} containing the vertices a_1, a_4, a_7, and for a_7 there is a component in T^{-a_7} containing the vertices a_1, a_4, a_8, a_9 from $\widehat{V}$. Since for each of these vertices, there is a component having more than two elements from $\widehat{V}$, none of them satisfies the requirement for being the median. This shows that a_6 is the unique median.*

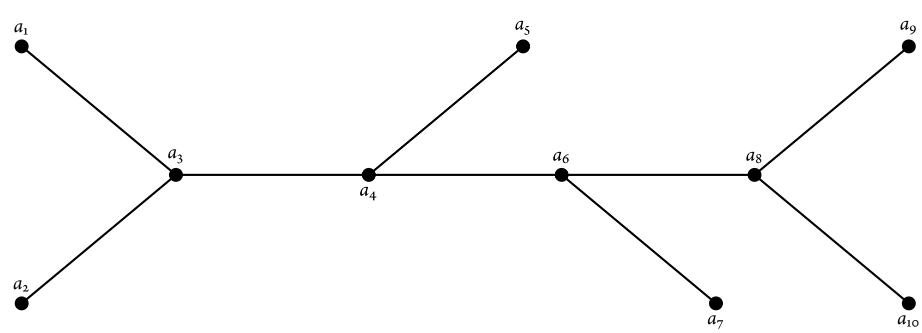


Figure 4.3.1: Tree for Example 4.3.2 and Example 4.3.3

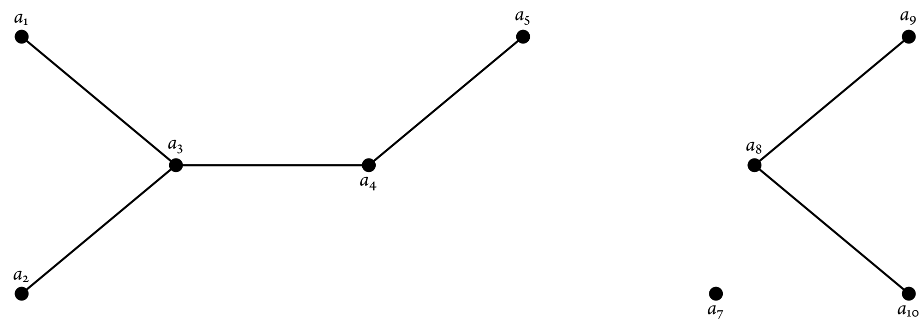


Figure 4.3.2: Components of T^{-a_6}

Now, we are ready to define the notion of the tree-median rule. It selects the median of the top-ranked alternatives at every profile.

Definition 4.3.4 *A choice function $\phi : \mathbb{D}^n \to A$ is called the tree-median rule with respect to a tree T with $V(T) = \tau(\mathbb{D})$ if for all $p \in \mathbb{D}^n$, $\phi(p) = median(\{\tau(p(i)) : i \in N\})$.*

Remark 4.3.5 *An alternative a is called pairwise majority winner at a profile if for all $b \neq a$, the number of agents who prefer a to b at that profile is more than $\frac{n}{2}$. It is worth noting that the outcome of a median rule at any profile is the pairwise majority winner (Condorcet winner) at that profile. To see this, suppose that the outcome of the tree-median rule is a at a profile p. Consider an alternative b other than a. Suppose b belongs to a component C of T^{-a}. By single-peakedness, every agent, whose top-ranked alternative is not in C, will prefer a to b. By the definition of the tree-median rule, the number of agents in component C is strictly less than $\frac{n}{2}$. Therefore, the number of agents who prefer a to b must be more than $\frac{n}{2}$, implying that a beats b by pairwise majority comparison.*

4.4 Results

Our first theorem characterizes the single-peaked domains on trees by means of choice functions that are unanimous, anonymous, symmetric, and group strategy-proof. It says that these domains are the only path-connected domains that admit such rules when the number of agents is odd.

Theorem 4.4.1 *Let $\mathbb{D}$ be a path-connected domain. Then, there exists a unanimous, anonymous, symmetric, and group strategy-proof choice function $\phi : \mathbb{D}^n \to A$ if and only if $\mathbb{D}$ is single-peaked on a tree and n is odd.*

The proof of this theorem is relegated to Appendix .1. In Section 4.4.1, we provide an idea of the proof of the only-if part of the theorem by considering the case of three alternatives.

Our next corollary says that if the number of agents is even, then there is no path-connected domain that admits a unanimous, anonymous, symmetric, and group strategy-proof rule. The intuition of this result is as follows. Since the number of agents is even, we can divide the agents into two groups N_1 and N_2 having equal size. Consider the profile where agents in N_1 have the same preference $ab\cdots$ and agents in N_2 have the same preference $ba\cdots$, for some $a, b \in A$. By unanimity and group strategy-proofness, the outcome at such a profile must be either a or b. Suppose that the outcome is a. Now, consider the profile where agents in N_1 have the same preference $ba\cdots$ and agents in N_2 have the same preference $ab\cdots$. By symmetry, the outcome at this profile must be b. However, this violates anonymity.

Corollary 4.4.1 *Let $\mathbb{D}$ be a path-connected domain and let n be even. Then, there is no unanimous, anonymous, symmetric, and group strategy-proof choice function $\phi : \mathbb{D}^n \to A$.*

Our next theorem characterizes the unanimous, anonymous, symmetric, and group strategy-proof rules on a single-peaked domain on a tree as the tree-median rules.

Theorem 4.4.2 *Let $\mathbb{D}$ be path-connected and single-peaked on a tree T and let n be odd. Then, a choice function $\phi : \mathbb{D}^n \to A$ is unanimous, anonymous, symmetric, and group strategy-proof if and only if it is the tree-median rule with respect to T.*

The proof of this theorem is relegated to Appendix .2. In Section 4.4.1, we provide an idea of the proof of the only-if part by considering the case of three alternatives.

4.4.1 An illustration of the proofs of Theorem 4.4.1 and Theorem 4.4.2

We illustrate the idea of the proof of the only-if parts of Theorem 4.4.1 and Theorem 4.4.2 by considering the case of three alternatives. Let $A = \{a, b, c\}$ be the set of three alternatives and let $N = \{1, \ldots, n\}$ be the set of agents. Suppose $\mathbb{D}$ is a path-connected domain and let ϕ be a unanimous, anonymous, symmetric, and group strategy-proof choice function from $\mathbb{D}^n$ to A. We show that

1. n is odd,
2. $\mathbb{D}$ is a set of single-peaked preferences on a tree, and
3. ϕ chooses the median of the top-ranked alternatives at any profile in $\mathbb{D}^n$.

Because $\mathbb{D}$ is path-connected we have, after a possible renaming of the alternatives, one of the following four cases

(i) $\mathbb{D} = \mathbb{L}(A)$

(ii) $\mathbb{D} = \mathbb{L}(A) \setminus \{acb\}$

(iii) $\mathbb{D} \subseteq \{abc, bac, bca, cba\}$ implying that $\mathbb{D}$ is single-peaked on a (sub)tree T_1 of the following tree

$$a \leftrightsquigarrow b \leftrightsquigarrow c.$$

(iv) $\mathbb{D} \subseteq \{abc, acb, cab, cba\}$ implying that $\mathbb{D}$ is single-peaked on a (sub)tree T_2 of the following tree[5]

$$a \leftrightsquigarrow c.$$

Consider a profile p and a coalition S such that $p(i) = xyz$ for all $i \in S$ and $p(i) = yxz$ for all $i \in N \setminus S$. By unanimity and group strategy-proofness, $\phi(p) \neq z$, as otherwise the agents in S will manipulate by reporting their preferences as yxz. By anonymity and group strategy-proofness, the outcome of any profile $\hat{p}$ such that $\hat{p}(i) \in \{xyz, yxz\}$ for all $i \in N$ and $|\{i : \hat{p}(i) = xyz\}| \geq |S|$ is x. By symmetry, this means $\phi(\hat{\hat{p}}) = y$ for any profile $\hat{\hat{p}}$ such that $\hat{\hat{p}}(i) \in \{xyz, yxz\}$ for all $i \in N$ and $|\{i : \hat{p}(i) = yxz\}| \geq |S|$. Therefore, it must be that $|S| > \frac{n}{2}$, as otherwise we can have a profile q such that both $|\{i : q(i) = xyz\}|$ and $|\{i : q(i) = yxz\}|$ are greater than or equal to $|S|$ and in view of the earlier observations, nothing can be defined as an outcome at q. Using similar logic, no outcome can be defined at a profile q such that $|\{i : q(i) = xyz\}| = |\{i : q(i) = yxz\}|$. This proves (1), that is, n is odd. This is formally proved in Lemma .1.2 (see Appendix .1).

Now, we proceed to prove (2). Consider a coalition S with $|S| > \frac{n}{2}$. We show that for any profile q such that $q(i) = q(j)$ for all $i, j \in S$ and $q(i) = q(j)$ for all $i, j \in N \setminus S$, the outcome is the top-ranked alternative of the agents in S. In the following table, we present such profiles where agents' preferences lie in the set $\{zxy, xzy, xyz, yxz\}$. We also present the outcomes of ϕ at the profiles where it can be obtained by unanimity and Lemma .1.2.

S \ $N \setminus S$	zxy	xzy	xyz	yxz
zxy	z	z		
xzy	x	x		
xyz			x	x
yxz			y	y

Table 4.4.1: Primary structure of a unanimous, anonymous, symmetric, and group strategy-proof choice function

[5]Such a set of preferences is known as a single-dipped domain in the literature.

We proceed to show that the outcome at any profile in the table will be the top-ranked alternative of the agents in S. Since the outcome at the profile (xzy, zxy) is x, by group strategy-proofness it must be x at (xyz, zxy).[6] In Table 4.4.2 we present the outcomes that can be obtained using similar logic.

S \ $N \setminus S$	zxy	xzy	xyz	yxz
zxy	z	z		
xzy	x	x	x	x
xyz	x	x	x	x
yxz			y	y

Table 4.4.2: Final structure of a unanimous, anonymous, symmetric, and group strategy-proof choice function

Consider the profile (yxz, xzy). Since the outcome at (xyz, xzy) is x, by group strategy-proofness the outcome at (yxz, xzy) must be x or y. Similarly, since the outcome at (yxz, xyz) is y, by group strategy-proofness the outcome at (yxz, xzy) must be y. Moreover, since the outcome at (yxz, xzy) is y and y is the bottom-ranked alternative for the agents in $N \setminus S$, by group strategy-proofness the outcome at (yxz, zxy) must be y. In Table 4.4.3 we present the outcomes that can be obtained using similar logic. Since S is arbitrary, Table 4.4.3 implies that the outcome of ϕ will be determined by the majority at any profile where the agents are partitioned into two groups such that agents in any group have the same preference.

S \ $N \setminus S$	zxy	xzy	xyz	yxz
zxy	z	z	z	z
xzy	x	x	x	x
xyz	x	x	x	x
yxz	y	y	y	y

Table 4.4.3: Additional structure of a unanimous, anonymous, symmetric, and group strategy-proof choice function

Since n is odd, there must be at least 3 agents. Therefore the set of agents can be partitioned into non-empty sets S_1, S_2, S_3 such that $|S_i \cup S_j| > \frac{n}{2}$ for all $i \neq j$. Consider the profile v such that $v(i) = xyz$

[6]For ease of presentation, by (xzy, zxy) we denote the profile where the agents in S have the preference xzy and the agents in $N \setminus S$ have the preference zxy. We continue to use similar notations.

for all $i \in S_1$, $v(i) = yzx$ for all $i \in S_2$, and $v(i) = zxy$ for all $i \in S_3$.[7] As $|S_1 \cup S_2| > \frac{n}{2}$, the outcome at the profile where agents in $S_1 \cup S_2$ have the preference yzx and the agents in S_3 have the preference zxy is y. Hence, by group strategy-proofness $\phi(v) \neq z$. Using a similar logic, $|S_2 \cup S_3| > \frac{n}{2}$ implies $\phi(v) \neq x$, and $|S_1 \cup S_3| > \frac{n}{2}$ implies $\phi(v) \neq y$. So, no outcome can be defined at the profile v, and hence a profile like v cannot lie in $\mathbb{D}^n$. Therefore, out of the four cases for $\mathbb{D}$ mentioned at the beginning, only Case (iii) and Case (iv) are possible. This proves that the domain $\mathbb{D}$ is a set of single-peaked preferences with respect to either the tree T_1 or the tree T_2. This completes the proof of (2).

We complete the sketch of the proof by showing (3). We deal with Case (iii) and Case (iv) separately. Case (iii): Here, $\mathbb{D}$ is a subset of single-peaked preferences $\{abc, bac, bca, cba\}$ with respect to the alphabetical order $a \prec b \prec c$ and $\mathcal{G}(\mathbb{D})$ is a (sub)graph of

$$a \leftrightsquigarrow b \leftrightsquigarrow c.$$

Let p be a profile in $\mathbb{D}^n$. We prove that $\phi(p)$ is the median of the top-ranked alternatives at p. We distinguish three cases.

Suppose $\phi(p) = b$. Consider the profile q such that $q(i) = bca$ if $p(i) = abc$, and $q(i) = p(i)$ otherwise. By group strategy-proofness, $\phi(q) = b$, as otherwise the agents i having preference bca at q will (group) manipulate at q by misreporting their preferences as $p(i)$. Next, consider the profile r such that $r(i) = bca$ if $q(i) = p(i) = bac$, and $r(i) = q(i) = cba$ otherwise. By group strategy-proofness, $\phi(r) = b$. Since agents have one of the two preferences bca and cba at the profile r, the outcome of ϕ at r will be the majority vote (winner) between b and c. As this outcome is b, it must be that majority of voters have the top-ranked alternative as b at the profile r. This implies that a majority of voters have top-ranked alternatives at p in the set $\{a, b\}$. Similarly, we can deduce that a majority of voters have top-ranked alternatives at p in the set $\{b, c\}$. Thus, it follows that at the profile p, there is a majority of voters having the top-ranked alternative in the set $\{a, b\}$ and a (possibly different) majority of voters having top-ranked alternatives in the set $\{b, c\}$, and hence, b is the median of the top-ranked alternatives at p.

Suppose $\phi(p) = a$. Consider the profile v such that $v(i) = bac$ if $p(i) \neq abc$, and $v(i) = p(i) = abc$ otherwise. Since agents have one of the two preferences abc and bac at the profile v, the outcome of ϕ at v will be the majority vote between a and b. In particular, $\phi(v) \in \{a, b\}$. Note that except for the preference abc, alternative b is strictly preferred to a at all other preferences in $\mathbb{D}$. So, group strategy-proofness implies that $\phi(v) \neq b$, as otherwise the agents i having preference bac at v will manipulate at p via $v(i)$. So, $\phi(v) = a$. Since the outcome of ϕ at v will be the majority vote between a and b, this means that there is a majority of voters having top-ranked alternative as a at p. So, a is the median of the top-ranked alternatives at p.

[7] Such a profile is called a Condorcet profile.

Suppose $\phi(p) = c$. This case is similar to the latter case where $\phi(p) = a$.

Case (iv): Here we have $\tau(\mathbb{D}) = \{a, c\}$. We have already argued that the outcome will be determined by the majority at profiles where agents are partitioned into two groups with each group having the same preference. Since $\tau(\mathbb{D}) = \{a, c\}$, by group strategy-proofness, this implies that the outcome of ϕ will be the majority vote between a and c at any profile. This in particular means that ϕ chooses the median of the top-ranked alternatives at any profile.

So, ϕ is the median rule and the only if parts of Theorem 4.4.1 and Theorem 4.4.2 are proved for the case of three alternatives.

4.5 Weakening group strategy-proofness to strategy-proofness

In this section, we show that group strategy-proofness cannot be replaced by strategy-proofness in Theorem 4.4.1, and consequently, provide a version of Theorem 4.4.1 with strategy-proofness. The following example shows that Theorem 4.4.1 does not hold under strategy-proofness.

Example 4.5.1 *Suppose that the set of alternatives is two-dimensional where each dimension/component has two elements: 0 and 1. More formally, the alternatives are $A = \{0, 1\}^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Agents' preferences are such that if a is the top-ranked alternative in a preference and b differs from a in both components, then b will be the bottom-ranked alternative in that preference.*[8] *For instance, if $(0, 1)$ is the top-ranked alternative in a preference, then $(1, 0)$ will be the bottom-ranked alternative in that preference. Therefore, there will be two preferences with $(0, 1)$ at the top for the two possible relative ordering of the remaining alternatives $(0, 0)$ and $(1, 1)$. The preferences are as follows: $(0, 1)(0, 0)(1, 1)(1, 0)$ and $(0, 1)(1, 1)(0, 0), (1, 0)$. In Table 4.5.1, we present all (eight) preferences satisfying this property. Consider the domain with these preferences.*

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8
$(0,0)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$	$(1,0)$	$(1,0)$	$(0,0)$
$(0,1)$	$(0,0)$	$(1,1)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,0)$	$(1,0)$
$(1,0)$	$(1,1)$	$(0,0)$	$(1,0)$	$(0,1)$	$(0,0)$	$(1,1)$	$(0,1)$
$(1,1)$	$(1,0)$	$(1,0)$	$(0,0)$	$(0,0)$	$(0,1)$	$(0,1)$	$(1,1)$

Table 4.5.1: Domain for Example 4.5.1

Suppose that there are three agents. We define an SCF called component-wise majority rule. For each component, it selects the element in that component that appears as the top-ranked element in that component for at least two agents. Note that the SCF depends only on the top-ranked alternatives in a profile. For an

[8] This is a special case of a more general condition known as separability in the literature.

illustration of the rule, consider a profile with top-ranked alternatives as $(1,0),(0,1),(1,0)$. *In the first component, element 1 appears at least two times as the top-ranked alternative, and hence, it is the outcome in that component. Similarly, 0 is the outcome in the second component. The final outcome of the rule is* $(1,0)$*, which is obtained by combining the outcomes in each component.*

It is shown in [76] (see Theorem 1) that the component-wise majority rule is strategy-proof. Unanimity and anonymity of the rule follow from the definition. For symmetry, consider a profile p where the agents in a group S, $\emptyset \neq S \neq N$*, have the preference* $R \equiv xy\cdots$ *and others have the preferences* $R' \equiv yx\cdots$ *for some x and y in A. By the definition of the domain, x and y can differ only over one component. So, assume without loss of generality,* $x=(0,0)$ *and* $y=(0,1)$*, and suppose that the outcome of the component-wise majority rule at this profile is* $(0,0)$*. Since the outcome in the second component is 0, by the definition of the component-wise majority rule, it must be that S contains at least 2 agents. Now, suppose that the agents in S interchange their preference with those in* $N \setminus S$*. The outcome in the first component will still be 0 as it is the top-ranked element of each agent in that component. Moreover, since S contains at least 2 agents, the outcome in the second component will now become 1, and hence, the final outcome will be* $(0,1)$*. This shows that the component-wise majority rule satisfies symmetry.*

Now, we argue that it is not group strategy-proof. Consider the profile of top-ranked alternatives $(0,0),(1,1),(1,0)$*. Suppose that both agents 1 and 2 prefer* $(0,1)$ *to* $(1,0)$*. Note that this assumption is compatible with our domain restriction. The outcome of the component-wise majority rule at this profile is* $(1,0)$*. However, if agents 1 and 2 together misreport their preferences as one having the top-ranked alternative as* $(0,1)$*, then the outcome of the component-wise majority rule will become* $(0,1)$*, which is preferred to* $(1,0)$ *for both agents 1 and 2. Therefore, the component-wise majority rule is not group strategy-proof.*

In what follows, we show that if we strengthen the notion of path-connectedness, then we can replace group strategy-proofness by strategy-proofness in Theorem 4.4.1.

Let a and b be two alternatives in $\tau(\mathbb{D})$. We say that a is *strongly top-connected to* b if there are R^a and R^b in $\mathbb{D}$ such that (i) $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$, and (ii) for all $x, y \notin \{a,b\}$, xR^ay if and only if xR^by. The notion of a strongly path-connected domain is defined in an obvious manner.

Our next theorem says that group strategy-proofness can be replaced by strategy-proofness if we strengthen path-connectedness by strongly path-connectedness.

Theorem 4.5.2 *Let* $\mathbb{D}$ *be a strongly path-connected domain. Then, there exists a unanimous, anonymous, symmetric, and strategy-proof choice function* $\phi : \mathbb{D}^n \to A$ *if and only if* $\mathbb{D}$ *is single-peaked on a tree and n is odd.*

The proof of this theorem is relegated to Appendix .3.

4.6 Relation to the literature

In this section, we discuss the connection of our results with some of the closely related papers.

4.6.1 [81]

[81] consider single-peaked domains on graphs (trees as a special case). Preferences are Euclidean with respect to the graph distance. They show that an SCF is strategy-proof and unanimous if and only if it is an extended generalized median voter scheme. Although tree median rules are special cases of extended generalized median voter scheme, our result does not follow from their result because of the following reasons.

(i) In their model, for each alternative there is exactly one preference with it as the top-ranked alternative. Thus, SCFs on such a domain become tops-only vacuously. However, in our case, there can be more than one preference with the same top-ranked alternative, and hence, tops-onlyness is required to be proved additionally. [84] shows that the maximal single-peaked domain on a line is tops-only, and recently, [2] generalize this result for arbitrary (that is, not necessary maximal) single-peaked domains on a line.[9] [20] provide a sufficient condition on a domain for it to be tops-only. None of these results applies to a path-connected single-peaked domain on a tree.

(ii) [81] use strategy-proofness whereas we use group strategy-proofness. To the best of our knowledge, it is not known in the literature whether extended generalized median voter schemes are group strategy-proof or not on domains that are single-peaked on a tree. [7] provide a sufficient condition on a domain for the equivalence of group strategy-proofness and strategy-proofness, however, their result also does not apply to such domains.

4.6.2 [65]

[65] consider a class of single-peaked domains based on an abstract betweenness property. They have analyzed the structure of strategy-proof and unanimous SCFs on such domains. Furthermore, they provide a characterization of such domains that admit SCFs satisfying strategy-proof, unanimous, neutral, and non-dictatorial/anonymity. Two particular results of [65] are closely related to our work, which we explain below.

(i) Corollary 5 in [65] says that a strategy-proof, unanimous, neutral, and anonymous SCF exists on a "rich" single-peaked domain if and only if n is odd and the domains is a "median space". On the other hand, Theorem 4.4.1 of our paper says that a group strategy-proof, unanimous, anonymous,

[9] A domain is tops-only if every unanimous and strategy-proof SCF on it is tops-only.

and symmetric SCF exists on a path-connected single-peaked domain if and only if n is odd and the domain is single-peaked on a tree. While neutrality and symmetry are similar in nature, the assumption of richness and the inclusion of median space make a significant difference between the two results as we explain below.

Richness: They assume the domains to be rich. In the context of domains that are single-peaked on a tree, this means the relative ordering of two alternatives that do not lie on the same path from the peak must be unrestricted. To see how strong this condition is, consider a single-peaked domain on a line. One implication of richness is that there must be preferences where the extreme left (or right) alternative is preferred to the "right-neighbor" (or the "left-neighbor") of the peak. For instance, if there are 100 alternatives $a_1, \ldots, a_{100}$ with the prior ordering $a_1 \prec \cdots \prec a_{100}$, then there must be a preference with a_2 at the top position where the "far away" alternative a_{100} is preferred to the neighboring one a_1. This is clearly a strong assumption for practical applications. Our notion of path-connectedness requires that for every two adjacent alternatives, say a_2 and a_3, there are two preferences where they swap their positions at the top two ranks, that is, preferences of the form $a_2 a_3 \cdots$ and $a_3 a_2 \cdots$ must be present. Thus, we do not require anything about the relative ordering of other alternatives.

Median space: A domain is a median space if the notion of median can be defined for any three alternatives in it, that is, for any three alternatives a, b, c, there is an alternative m called the "median" of a, b, c such that m lies between every pair of alternatives from a, b, c. Apart from domains that are single-peaked on a tree, there are several other domains that are median space (see Example 4 in [65]). Thus, domains that are single-peaked on a tree cannot be characterized by the properties used in [65]) and the use of group strategy-proofness does the job in our paper. As we have mentioned earlier, it is not yet known if group strategy-proofness and strategy-proofness are equivalent on domains that are single-peaked on a tree. Thus, (even the "if part" of) Theorem 4.4.1 of our paper does not follow from Corollary 5 of [65].

(ii) Theorem 4 of [65] says that an SCF on a rich median space is strategy-proof, unanimous, and neutral if and only if it is a particular type of voting by issues rules. Furthermore, if anonymity is imposed additionally, then these rules become tree median. Since the single-peaked domains on trees we consider do not satisfy richness, this result does not apply to these domains. Moreover, even if we additionally impose richness on such domains, since we work with group strategy-proofness, Theorem 4.4.1 of our paper does not follow from this result. The contribution of our result on these special class of rich domains is that it implies that strategy-proofness and group strategy-proofness are equivalent on those under unanimity, anonymity, and symmetry/neutrality. Such a result is not known in the literature and we feel it is not straightforward either.

Appendix

.1 Proof of Theorem 4.4.1

We introduce the following terminologies to facilitate the presentation of our proofs. For a coalition S, a preference R, and a profile q, we denote by $((R)^S, q|_{N\setminus S})$ the profile p where $p(i) = R$ for all $i \in S$ and $p(i) = q(i)$ for all $i \in N \setminus S$. We call such a profile S-*unanimous*. In a similar fashion, a profile of the form $((R)^S, (R')^{N\setminus S})$ is said to be $(S, N \setminus S)$-*unanimous*. Additionally, if $\tau(R) = a$ and $\tau(R') = b$, then such a profile is said to be (a, b)-$(S, N \setminus S)$-*unanimous*. Let V be a set of S-unanimous profiles in $\mathbb{D}^n$ for some coalition S. Given a choice function ϕ, we say that the coalition S is *decisive on* V (*for* ϕ), if $\phi(R^S, p|_{N\setminus S}) = \tau(R)$ for all $(R^S, p|_{N\setminus S}) \in V$. The coalition S is said to be *decisive* if it is decisive on the set of all S-unanimous profiles in $\mathbb{D}^n$. For instance, N is decisive for a unanimous choice function ϕ.

We are now ready to present the proof of Theorem 4.4.1.

(**If part**) Let T be a tree and let $\mathbb{D}$ be a single-peaked domain on T. Suppose n is odd. Consider the median rule $\phi : \mathbb{D}^n \to A$. By definition, ϕ satisfies unanimity, anonymity, and symmetry. In what follows, we show that it satisfies group strategy-proofness.

Consider a profile $p \in \mathbb{D}^n$. Suppose $\phi(p) = a$. Assume for contradiction that some coalition S manipulates ϕ at the profile p. First note that by the definition of single-peaked domain on T, if the top-ranked alternatives of the agents in S at the profile p are in different components of T^{-a}, then there is no alternative b that is strictly preferred to a by each agent in S. So, since the agents in S manipulate, it must be that their top-ranked alternatives in p are in some component $\mathcal{C}$ of T^{-a}. By the definition of the median rule, the number of agents who have top-ranked alternatives in $\mathcal{C}$ is less than $\frac{n}{2}$. Therefore, no matter how the agents in S misreport there preferences, the outcome at the misreported profile cannot be an alternative of $\mathcal{C}$. This, in turn, means that the agents in S will not prefer the outcome at the misreported profile, a contradiction. This completes the proof of the if part of Theorem 4.4.1.

(**Only-if part**) We prove the only-if part by means of the following lemmas. For all these lemmas, assume that $\mathbb{D}$ is a path-connected domain.

Lemma .1.1 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous and group strategy-proof choice function and let a coalition S be decisive on all $(S, N \setminus S)$-unanimous profiles. Then S is decisive.*

Proof: In order to prove that S is decisive, let $p \in \mathbb{D}^n$ be an S-unanimous profile such that $p(i) = R$ for all i in S, where $\tau(R) = a$. For an S-unanimous profile p, define $k(p) = |\{p(j) : j \in N\}|$ as the number of different preferences in p. We prove the lemma by using induction on k. Note that $k \geq 2$ by definition. Note that the base case where $k = 2$ follows from the definition of $(S, N \setminus S)$-unanimous profiles. Suppose S is decisive on all S-unanimous profiles p such that $k(p) \leq \bar{k}$, for some $\bar{k} \geq 2$. We show that S is decisive on all S-unanimous profiles p such that $k(p) \leq \bar{k} + 1$. Consider $p \in \mathbb{D}^n$ such that $k(p) = \bar{k} + 1$. Since $k(p) = \bar{k} + 1$, we can partition N as $T_1, \ldots, T_{k+1}$ such that for all $l \in \{1, \ldots, k+1\}$, there exists $R^l \in \mathbb{D}$ such that $p(i) = R^l$ for all $i \in T_l$. Since p is an S-unanimous profile, assume without loss of generality $S \subseteq T_1$. Assume for contradiction, $\phi(p) \neq a$. Suppose $\phi(p) = b$ for some $b \in A \setminus \{a\}$. Consider $q \in \mathbb{D}^n$ such that $q(i) = R^l$ for all $i \in T_l$ and all $l \in \{1, \ldots, k\}$ and $q(i) = R^k$ for all $i \in T_{k+1}$. Since $k(q) = \bar{k}$ by construction, we have by our induction hypothesis that $\phi(q) = a$. By means of group strategy-proofness for the agents in T_{k+1} at p via $q|_{T_{k+1}}$, we have

$$ba \in R^{k+1}. \tag{1}$$

Now consider the preference $r \in \mathbb{D}^n$ such that $r(i) = R^l$ for all $i \in T_l$ and all $l \in \{1, \ldots, k-2, k\}$ and $q(i) = R^{k+1}$ for all $i \in T_k$. Since $k(r) = \bar{k}$ by construction, we have by our induction hypothesis that $\phi(r) = a$. By means of group strategy-proofness for the agents in T_k at r via $p|_{T_k}$, we have

$$ab \in R^{k+1}. \tag{2}$$

Combining (1) and (2), $\phi(p) = a$. This completes the proof by induction. ■

Lemma .1.2 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function and let $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$ be two preferences in $\mathbb{D}$. Suppose a coalition S is such that $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$. Then $|S| > \frac{n}{2}$.*

Proof: Assume for contradiction $|S| \leq \frac{n}{2}$. By applying symmetry to $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$, we have

$$\phi((R^b)^S, (R^a)^{N\setminus S}) = b. \tag{3}$$

Since $|S| \leq \frac{n}{2}$, there exists $T \subseteq N \setminus S$ such that $|S| = |T|$. We write $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$ as

$$\phi((R^a)^S, (R^b)^T, (R^b)^{N\setminus(S\cup T)}) = a. \tag{4}$$

Now, applying anonymity to (4), since $|S| = |T|$,

$$\phi((R^b)^S, (R^a)^T, (R^b)^{N\setminus(S\cup T)}) = a. \tag{5}$$

This, together with (3) implies that agents in $N \setminus (S \cup T)$ manipulates at $((R^b)^S, (R^a)^{N\setminus S})$ via $(R^b)^{N\setminus(S\cup T)}$, a contradiction. ■

Lemma .1.3 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous and group strategy-proof choice function and let $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$ be two preferences in $\mathbb{D}$. Suppose a coalition S is such that $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$. Then S is decisive on all (a, b)-$(S, N \setminus S)$-unanimous profiles.*

Proof: Consider an (a, b)-$(S, N \setminus S)$-unanimous profile $p \in \mathbb{D}^n$. Assume for contradiction, $\phi(p) \neq a$. Suppose $\phi(p) = x$. Consider $q \in \mathbb{D}^n$ such that $q(i) = R^a$ for all $i \in S$ and $q|_{N\setminus S} = p|_{N\setminus S}$. We claim $\phi(q) = b$. If $\phi(q) = a$, then by means of unanimity agents in S manipulates at p via $q|_S$, a contradiction. If $\phi(q) \notin \{a, b\}$, then agents in S manipulates at q via some preference where b is the top-ranked alternative for all agents in S. So, $\phi(q) = b$. However, since $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$, this means agents in $N \setminus S$ manipulates at $((R^a)^S, (R^b)^{N\setminus S})$ via $q|_{N\setminus S}$, a contradiction. ■

Remark .1.1 *It follows from Lemma .1.2 and Lemma .1.3 that there is a unanimous, anonymous, symmetric, and group strategy-proof choice function $\phi : \mathbb{D}^n \to A$ only if n is odd. This completes the proof of Corollary 4.4.1.*

Lemma .1.4 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function and let a and b be top-connected alternatives. Then the following two are equivalent.*

(i) *S is decisive on all (a, b)-$(S, N \setminus S)$-unanimous profiles.*

(ii) $|S| > \frac{n}{2}$.

Proof: Consider $R^a \equiv ab\cdots$ and $R^b \equiv ba\cdots$. By group strategy-proofness and unanimity, $\phi((R^a)^S, (R^b)^{N\setminus S}) \in \{a, b\}$. If (i) holds, then $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$, and by Lemma .1.2, $|S| > \frac{n}{2}$.

Suppose (ii) holds. If $\phi((R^a)^S, (R^b)^{N\setminus S}) = b$, then by Lemma .1.2, we have $|N \setminus S| > \frac{n}{2}$, a contradiction to $|S| > \frac{n}{2}$. So, $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$. By Lemma .1.2, this implies S is decisive on all (a, b)-$(S, N \setminus S)$-unanimous profiles. ■

Lemma .1.5 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous and group strategy-proof choice function and let $x^1, \ldots, x^k$ be a path in $G(\mathbb{D})$ such that every three consecutive alternatives in the path are distinct. Suppose a coalition S is decisive on all (x^1, x^2)-$(S, N \setminus S)$-unanimous profiles. Then, S is decisive on all (x^s, x^t)-$(S, N \setminus S)$-unanimous profiles for all $1 \leq s < t \leq k$.*

Proof: We prove this by using induction on the value of $t - s$ for (x^s, x^t)-$(S, N \setminus S)$-unanimous profiles.

First, we prove that S is decisive on all (x^s, x^{s+1})-$(S, N \setminus S)$-unanimous profiles for all $0 \leq s < k$. Consider $p \in \mathbb{D}^n$ such that $p(i) \equiv x^2 x^3 \cdots$ for all $i \in S$ and $p(i) \equiv x^3 x^2 \cdots$ for all $i \in N \setminus S$. We show that $\phi(p) = x^2$. To ease the presentation of the proof, we use the notationR^{st} to denote a preference of the form $x^s x^t \cdots$. By our assumption, $\phi((R^{12})^S, (R^{23})^{N\setminus S}) = x^1$. Consider a profile $((R^{12})^S (R^{32})^{N\setminus S}) \in \mathbb{D}^n$. By group strategy-proofness, $\phi((R^{12})^S, (R^{32})^{N\setminus S}) \in \{x^1, x^2, x^3\}$. Since $\phi((R^{12})^S, (R^{23})^{N\setminus S}) = x^1$, by using group strategy-proofness for the agents in $N \setminus S$, $\phi((R^{12})^S, (R^{32})^{N\setminus S}) = x^1$. Consider a profile of the form $((R^{21})^S, (R^{32})^{N\setminus S})$. Since $\phi((R^{12})^S, (R^{32})^{N\setminus S}) = x^1$, by using group strategy-proofness for agents in S, $\phi((R^{21})^S, (R^{32})^{N\setminus S}) \in \{x^1, x^2\}$. If $\phi((R^{21})^S, (R^{32})^{N\setminus S}) = x^1$, then agents in $N \setminus S$ manipulates at $((R^{21})^S, (R^{32})^{N\setminus S})$ via $(R^{21})^{N\setminus S}$. So, $\phi((R^{21})^S, (R^{32})^{N\setminus S}) = x^2$. By group strategy-proofness, $\phi((R^{23})^S, (R^{32})^{N\setminus S}) = x^2$ and by Lemma .1.3, S is decisive on all (x^2, x^3)-$(S, N \setminus S)$-unanimous profiles. Continuing in this manner, it can be shown that S is decisive on all (x^s, x^{s+1})-$(S, N \setminus S)$-unanimous profiles for all $1 \leq s < k$. Suppose S is decisive on all (x^s, x^t)-$(S, N \setminus S)$-unanimous profiles where $t - s \leq l$ for some $l \leq k - 1$. We show that S is decisive on all (x^s, x^t)-$(S, N \setminus S)$-unanimous profiles where $t - s = l + 1$.

By our induction hypothesis, $\phi((R^{(s+1)s})^S, (R^{t+1})^{N\setminus S}) = x^{s+1}$. By group strategy-proofness, this means $\phi((R^{s(s+1)})^S, (R^{t+1})^{N\setminus S}) \in \{x^s, x^{s+1}\}$. Suppose $\phi((R^{s(s+1)})^S, (R^{t+1})^{N\setminus S}) = x^{s+1}$. Then, by group strategy-proofness $\phi((R^{s(s+1)})^S, (R^{(s+1)s})^{N\setminus S}) = x^{s+1}$, which contradicts our earlier step where we have shown that S is decisive on all (x^s, x^{s+1})-$(S, N \setminus S)$-unanimous profiles for all $0 \leq s < k$. So, $\phi((R^{s(s+1)})^S, (R^{t+1})^{N\setminus S}) = x^s$. By group strategy-proofness, this means $\phi((R^s)^S, (R^{t+1})^{N\setminus S}) = x^s$ implying that S is decisive on all (x^s, x^t)-$(S, N \setminus S)$-unanimous profiles for all $0 \leq s < t \leq k$. This completes the proof of the lemma. ■

Lemma .1.6 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function and let S be a coalition with $|S| > \frac{n}{2}$. Then S is decisive.*

Proof: By Lemma .1.4, S is decisive on all (a, b)-$(S, N \setminus S)$-unanimous profiles where $a \leftrightsquigarrow b$. By Lemma .1.5, S is decisive on all (x, y)-$(S, N \setminus S)$-unanimous profiles such that there exists a path in $G(\mathbb{D})$ connecting x and y. Since $G(\mathbb{D})$ is connected, this means S is decisive on all (x, y)-$(S, N \setminus S)$-unanimous profiles. Now, by Lemma .1.1, we have that S is decisive on all profiles. ■

The restriction of a preference $R \in \mathbb{L}(A)$ to a set $X \subseteq A$ is defined as $R|_X := \{xy : xy \in R \text{ and } xy \in X \times X\}$.

Lemma .1.7 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function. Consider a path $x^1, \ldots, x^k$ in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$. Then, for all $R \in \mathbb{D}$, $\tau(R|_X) = x^k$ implies $R|_X \equiv x^k x^{k-1} \cdots x^1$, where $X = \{x^1, \ldots, x^k\}$.*

Proof: Since n is odd, $n \geq 3$. Therefore, N can be partitioned into coalitions S, T and U such that $|S| = 1$, $|S \cup U| > \frac{n}{2}$, $|T \cup U| > \frac{n}{2}$ and $|S \cup T| > \frac{n}{2}$. Let a preference $R^1 \in \mathbb{D}$ be such that $\tau(R^1) = x^1$. Define the choice function $\psi : \mathbb{D}^{S \cup T} \to A$ such that for all $p \in \mathbb{D}^{S \cup T}$, $\psi(p) = \phi(\tilde{p})$ where $\tilde{p}(i) = R^1$ for all $i \in U$ and $\tilde{p}(i) = p(i)$ for all $i \in S \cup T$. Since ϕ is group strategy-proof, ψ is group strategy-proof. Further, since $|S \cup T| > \frac{n}{2}$, by Lemma .1.6, $S \cup T$ is decisive for ϕ. This, together with the fact that ϕ is unanimous implies ψ is unanimous. Since $|S \cup U| > \frac{n}{2}$, by Lemma .1.6, $\phi(\tilde{p}) = x^1$ where $\tilde{p}(i) = R^1$ for all $i \in S \cup U$ and $\tau(\tilde{p}(i)) = x^2$ for all $i \in T$. This means $\psi(p) = x^1$, where p is a (x^1, x^2)-(S, T)-unanimous profile. Since ψ is unanimous and group strategy-proof, by Lemma .1.5, we have for all $1 \leq s < t \leq k$ and all (x^s, x^t)-(S, T)-unanimous profiles q, $\psi(q) = x^s$. Using a similar logic, it follows that T is decisive on all (x^s, x^t)-(T, S)-unanimous profiles for all $1 \leq s < t \leq k$. Combining all these observations, we have

$$\phi(q) = x^{\min\{s,t\}}. \tag{6}$$

Now, we are ready to complete the proof of the lemma. Assume for contradiction that there exists $R \in \mathbb{D}$ such that $\tau(R|_X) = x^k$ and $x^r x^s \in R$ for some $r < s$. Then, by (6), $\psi(p) = x^s$ where $p(i) = R$ for all $i \in S$ and $\tau(p(i)) = x^s$ for all $i \in T$. Consider $q \in \mathbb{D}^n$ such that $\tau(q(i)) = x^r$ for all $i \in S$ and $q|_T = p|_T$. By (6), $\psi(q) = x^r$, which means that the agents in S manipulate at p via $q|_S$ contradicting the group strategy-proofness (also, strategy-proofness) of ψ. ■

Lemma .1.8 *Let $\phi : \mathbb{D}^n \to A$ be a unanimous, anonymous, symmetric, and group strategy-proof choice function. Then $G(\mathbb{D})$ must be a tree, $\mathbb{D}$ must be single-peaked on $G(\mathbb{D})$, and n must be odd.*

Proof: Assume for contradiction that there exists a cycle $x^1, \ldots, x^k, x^1$ in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$. Consider $R \in \mathbb{D}$ such that $\tau(R) = x^1$. Since $x^1, x^2, \ldots, x^k$ is a path in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$, by Lemma .1.7, $x^2 x^3 \in R$. Again, since $x^1, x^k, x^{k-1}, \ldots, x^2$ is a path in $G(\mathbb{D})$ such that $x^i \neq x^j$ for all $i \neq j$, by Lemma .1.7, $x^3 x^2 \in R$. However, this contradicts that R is a preference. So, $G(\mathbb{D})$ is a tree. Now, by means of Lemma .1.7 it follows that $\mathbb{D}$ is single-peaked on $G(\mathbb{D})$.

Now, we show n is odd. Suppose not. Take $|S| = \frac{n}{2}$. Let $R^a \equiv ab \cdots$ and $R^b \equiv ba \cdots$. By unanimity and group strategy-proofness, $\phi((R^a)^S, (R^b)^{N \setminus S}) \in \{a, b\}$. Assume without loss of generality,

$\phi((R^a)^S, (R^b)^{N\setminus S}) = a$. Since $|S| = \frac{n}{2}$, by anonymity of ϕ, this implies $\phi((R^b)^S, (R^a)^{N\setminus S}) = a$. On the other hand, since $\phi((R^a)^S, (R^b)^{N\setminus S}) = a$, by symmetry $\phi((R^b)^S, (R^a)^{N\setminus S}) = b$, which is a contradiction. ■

.2 Proof of Theorem 4.4.2

Proof: The proof of the if part of the Theorem follows from the same of the if part of Theorem 4.4.1. We proceed to prove the only-if part.

Assume for contradiction that $\phi(p) = x$ for some $p \in \mathbb{D}^n$ where x is such that there exists a component C in T^{-x} with $|\{i \in N : \tau(p(i)) \in C\}| \geq \frac{n}{2}$. Let $S = \{i \in N : \tau(p(i)) \in C\}$. Since n is odd, this means $|S| > \frac{n}{2}$. Consider $q \in \mathbb{D}^n$ such that $q(i) = p(i)$ for all $i \in S$ and $\tau(q(i)) = x$ for all $i \in N \setminus S$. By group strategy-proofness, $\phi(q) = x$. Let $y \in C$ be the (unique) vertex in C such that $\{x, y\}$ is an edge in T. Consider $r \in \mathbb{D}^n$ such that $r(i) \equiv yx\cdots$ for all $i \in S$ and $r(i) = q(i)$ for all $i \in N \setminus S$. By unanimity and group strategy-proofness, $\phi(r) \in \{x, y\}$. If $\phi(r) = y$, then, because preferences are single-peaked on T, agents in S manipulates at r via $q|_S$. So, $\phi(r) = x$. However, since r is a (x, y)-$(S, N \setminus S)$-unanimous profile with $|S| > \frac{n}{2}$, this contradicts Lemma .1.2. ■

.3 Proof of Theorem 4.5.2

The proof of Theorem 4.5.2 follows from following the steps in the proof of Theorem 4.4.1 with the following modifications.

Proof:[Proof of Lemma .1.5] Let $R, R' \in \mathbb{D}$ be such that $R \equiv x^1x^2\cdots$ and $R' \equiv x^3x^2\cdots$. Let S be a coalition and consider the profile p such that $p(i) = R$ for all $i \in S$ and $p(i) = R'$ for all $i \in N \setminus S$. In the proof of Lemma .1.5, we use the fact that by unanimity and group strategy-proofness, $\phi(p) \in \{x^1, x^2, x^3\}$. Clearly, this does not follow if we replace group strategy-proofness by strategy-proofness. However, since we additionally have the fact that the domain is strongly path-connected, this assertion follows. To see this, assume for contradiction that $\phi(p) \notin \{x^1, x^2, x^3\}$. Consider the preference $\bar{R}$ such that $\bar{R} \equiv x^2x^3\cdots$, and for all $a, b \notin \{x^2, x^3\}$, $ab \in \bar{R}$ if and only if $ab \in R$. We can move the agents in $N \setminus S$ sequentially to $\bar{R}$, and each time, by strategy-proofness we can claim that the outcome will remain the same as $\phi(p)$. Since $\phi(p) \notin \{x^1, x^2, x^3\}$, this contradicts the assumption of the lemma that S is decisive on all (x^1, x^2)-$(S, N \setminus S)$-unanimous profiles. This completes the proof of Lemma .1.5 for this case.

In every other place where group strategy-proofness is used, we can change the preferences of the agents in the corresponding group one by one (as discussed in the modified proof of Lemma .1.5), and apply strategy-proofness at each step to obtain the desired conclusion. ■

.4 Independence of axioms in Theorem 4.4.1

In this section, we establish the independence of the conditions that we have used in Theorem 4.4.1 Furthermore, we show how to modify Theorem 4.4.1 if we replace group strategy-proofness by strategy-proofness.

In what follows, we introduce some special type of choice functions and discuss their properties. We will use these functions to establish the mentioned independence.

Let $a \in A$ be an alternative and let $\mathbb{D}^{-a}$ be a domain such that $a \notin \tau(\mathbb{D}^{-a})$. A choice function $\phi^a : (\mathbb{D}^{-a})^n \to A$ is called *constant at a* if $\phi^a(p) = a$ for all profiles $p \in (\mathbb{D}^{-a})^n$. By definition, ϕ^a violates unanimity. Since the outcome of ϕ^a does not depend on the profiles, it satisfies anonymity, strategy-proofness, and group strategy-proofness. To apply symmetry, we need two preferences in the domain of the form $xy\cdots$ and $yx\cdots$ for some $x, y \in A$, and a profile where each agent has one of the two preferences such that the outcome at that profile is either x or y. Because a never appears at the top position in any preference in $\mathbb{D}^{-a}$, a cannot be one of x or y in the aforementioned preferences. Since both x and y are different from a, by definition the outcome of ϕ^a cannot be x or y. Thus, symmetry is vacuously satisfied by ϕ^a.

A choice function $\phi_j^{dict} : \mathbb{L}(A)^n \to A$, where $j \in N$, is called *dictatorial* if $\phi_j^{dict}(p) = \tau(p(j))$ for all $p \in \mathbb{L}(A)^n$. By definition, ϕ_j^{dict} satisfies unanimity, strategy-proofness, group strategy-proofness and violates anonymity. To see that ϕ_j^{dict} satisfies symmetry, consider a profile where a group S, $\emptyset \neq S \neq N$, of agents have a preference $P \equiv xy\cdots$ and other agents have the preference $P' \equiv yx\cdots$. Suppose that outcome of ϕ_j^{dict} at this profile is x. This means some agent in S is the dictator. Therefore, if agents in S and $N \setminus S$ interchange their preferences, then the top-ranked alternative of the dictator will be y, and consequently the outcome will be y, ensuring symmetry.

A choice function $\phi_a^{una} : (\mathbb{D}^{-a})^n \to A$, where $a \in A$, is called *unanimous with disagreement a*, if for all $p \in \mathbb{L}(A)^n$,

$$\begin{aligned} \phi_a^{una}(p) &= b \quad \text{if} \quad \tau(p(i)) = b \text{ for all } i \in N \\ &= a \quad \text{otherwise.} \end{aligned}$$

The rule ϕ_a^{una} satisfies unanimity by definition. Anonymity of ϕ_a^{una} follows from the fact that if agents interchange their preferences, then a unanimous profile will remain unanimous and a non-unanimous profile will remain non-unanimous, and hence by definition, the outcome of ϕ_a^{una} will not change. Since a does not appear at the top position in any preference in $\mathbb{D}^{-a}$, as we have explained in the case of ϕ^a, symmetry holds vacuously for ϕ_a^{una}. To see that ϕ_a^{una} is manipulable, consider a profile where some alternative b is the top-ranked alternative of every agent except agent 1 and b is preferred to a for agent 1. By definition, the outcome of ϕ_a^{una} at this profile is a. However, if agent 1 misreports her preference as one with b at the top position, then the outcome will become b and agent 1 will be strictly better off. So, ϕ_a^{una} is not strategy-proof, and hence, it is not group strategy-proof either.

For the next choice function and its (restricted) domain, let the alternatives be numbered as $a_1, \ldots, a_m$. To ease our presentation, whenever we use minimum or maximum of a set of alternatives, we mean it with respect to the ordering $a_1 \prec \cdots \prec a_m$. A domain $\mathcal{S}$ is said to be semi single-peaked domain if for all R in $\mathcal{S}$, $\tau(R) = a_k$ implies $R \equiv a_k \cdots a_{k-1} \cdots a_{k-2} \cdots a_2 \cdots a_1 \cdots$. Thus, each preference in a semi single-peaked domain maintains single-peakedness *only* on the left side of the peak (that is, the top-ranked alternative), that is, as one moves away from the peak on the left side, preference declines. Note that there is no restriction on the relative ordering of two alternatives if at least one of them is on the right of the peak.

A choice function $\phi^{low} : \mathcal{S}^n \to A$ is called *lowest peak* if for all $p \in \mathcal{S}^n$,

$$\phi^{low}(p) = \min\{\tau(p(i)) : i \in N\}.$$

As the name suggests, ϕ^{low} selects the minimum peak (with respect to the ordering $\prec$) at every profile. Unanimity and anonymity of ϕ^{low} follow from the definition. In what follows, we argue that ϕ^{low} satisfies group strategy-proofness (and hence strategy-proofness).

Suppose a group of agents S manipulate ϕ^{low} at a profile p. Let $\min(p)$ be the minimum peak of p. Since $\phi(p) = \min(p)$, the (sincere) peak of each agent in S must be strictly on the right of $\min(p)$. This in particular means that the peak of some agent outside S is $\min(p)$. Therefore, by the definition of ϕ, the only way the agents in S can change the outcome is to declare a peak which is on the (further) left of $\min(p)$. This will push the outcome to the left of $\min(p)$ as well. Since the sincere peaks of the agents in S are on the right of $\min(p)$ and the changed outcome is on the left of $\min(p)$, by the definition of semi single-peakedness, the changed outcome will become even worse for them. So, no group of agents can manipulate ϕ^{low} at any profile.

Finally, we explain that ϕ^{low} does not satisfy symmetry. Consider two preferences $R \equiv a_k a_{k+1} \cdots$ and $R' \equiv a_{k+1} a_k \cdots$, and consider a profile p where each agent in a group S, $\emptyset \neq S \neq N$, has the preference R and each remaining agent has the preference R'. By the definition of ϕ^{low}, $\phi^{low}(p) = a_k$. Now, consider the profile p' where each agent in S has the preference R' and each remaining agent has the preference R. In order to satisfy symmetry, the outcome at this profile must be x_{k+1}, but by the definition of ϕ^{low}, the outcome is x_k.

In the following table, we present the conditions that are satisfied by each of the above-mentioned choice functions. Note that this table establishes the independences of the conditions that are used in Theorem 4.4.1.

	ϕ^a on $(\mathbb{D}^{-a})^n$	ϕ_j^{dict} on $\mathbb{L}(A)^n$	ϕ^{low} on $\mathcal{S}^n$	ϕ_a^{una} on $(\mathbb{D}^{-a})^n$
unanimity	No	Yes	Yes	Yes
anonymity	Yes	No	Yes	Yes
symmetry	Yes	Yes	No	Yes
group strategy-proofness	Yes	Yes	Yes	No
strategy-proofness	Yes	Yes	Yes	No

Table .4.1: Independence of axioms in Theorem 4.4.1

5

Strategy-proof Random Voting Rules on Weak Domains

5.1 Introduction

We consider the standard social choice problem where a social planner has to choose an alternative from a feasible set based on the preferences of the agents in a society. Unanimity, Pareto optimality, strategy-proofness are considered as desirable properties of a deterministic social choice function (DSCF). A DSCF is unanimous if, whenever all agents agree on their top-ranked alternative, that alternative is chosen. It is Pareto efficient if its outcome cannot be improved in way so that no one is worse off and someone is better off. It is strategy-proof if no agent can benefit by misreporting her preference.

The horizon of social choice theory is expanded by introducing randomness in social choice functions. A random social choice function (RSCF) selects a probability distribution over the alternatives at every collection of preferences. The notions of unanimity and Pareto optimality remain the same for RSCFs, while that of strategy-proofness is formulated by means of stochastic dominance.

Importance of RSCFs over DSCFs is well-established in the literature (see [68], [67] and [37]). The appeal of RSCFs over DSCFs is that they allow for the introduction of fairness considerations and reasonable compromise in the decision-making process, facilitating the resolution of conflicts of interest (see [64] and [70]).

A preference is strict if it does not admit any indifference. The study of social choice functions when preferences are strict is quite extensive. In a seminal work, [40] shows that an RSCF on the strict unrestricted domain is unanimous and strategy-proof if and only if it is random dictatorial. Subsequently, several other random dictatorial domains are characterized in the literature. For domains with specific structure, unanimous and strategy-proof RSCFs on the strict single-peaked domain, the strict single-dipped domain, and strict single-crossing domains are characterized (see, [37], [74] and [68])

To our understanding, the assumption of strict preferences is quite strong for practical purpose. However, contrary to their strict counterpart, the literature of social choice theory for weak preferences is rather limited. Our objective of this paper is to explore this area.

We first analyse what happens when weak preferences are added to a random dictatorial domains. We introduce the notion of weak random dictatorial rules and provide the structure of unanimous and strategy-proof RSCFs on such domains by means of these rules. We use a result in [15] to prove our result. It is worth mentioning that our result generalizes the result of [40] for weak preferences. In the context of cardinal preferences, similar results are proved in [64], [35] and [47].

Next, we analyze what happens when weak preferences are added to the well-known single-peaked domain. We consider two types of weak domains in this context: one where indifference occurs only at the top-position, and the other one where it occurs only below the top-position. An important weak domain of the former type is the single-plateaued domain and that of the latter type is the single-peaked domain with outside option. We provide the structure of unanimous (or Pareto optimal) and strategy-proof RSCFs for each of these cases. [8] provide the structure of unanimous and strategy-proof DSCFs on the single-plateaued domain and [18] provide that for the single-peaked domain with outside option. We generalize these results for RSCFs. Also, we provide closed form presentation of these rules.

In Section 5.2 we introduce the basic model, notations and definitions. In Section 5.3 we investigate the the structure of unanimous and strategy-proof rules if weak preferences are added to a random dictatorial domains. We show that if agents have κ-single-plateaued preferences, then any RSCF that is unanimous and strategy-proof is also Pareto-optimal. In Section 5.4,we consider weak single-peaked domains and chracterize the Pareto optimal and strategy-proof RSCFs on them. In Section 5.5 we consider the case of single-plateaued domains, and show that unlike strict single-peaked domains, unanimity and strategy-proofness does not guarantee tops-onlyness. We also provide an axiomatic characterization of strategy-proof RSCFs under unanimity. We introduce the notion of generalized uncompromisingness which boils down to uncompromisingness when κ is equal to 1 and show that an unanimous RSCF is strategy-proof if and only if it is generalized uncompromising. Finally, in Section 5.6

we provide the conclusion.

5.2 Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of at least two agents, who collectively choose an element from a finite set $A = \{1, \ldots, m\}$ of at least two alternatives. For $x, y \in A$ such that $x \leq y$, we define the intervals $[x, y] = \{z \in A \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A (weak) *preference* R over A is a complete and transitive binary relation (also called a weak order) defined on A. We denote by P and I the anti-symmetric and the indifference part of R, respectively. That is, xPy means xRy and not yRx, and xIy means xRy and yRx. We denote by $\mathbb{W}(A)$ the set of all preferences over A. For a preference $R \in \mathbb{W}(A)$, we denote by $\tau(R)$ the set of alternatives that appear at the top-position of R, that is, $\tau(R) = \{x \in A \mid xRy \text{ for all } y \in A\}$. For a preference $R \in \mathbb{W}(A)$ and an alternative x, the upper contour set $U(x, R)$ of R at x is defined as the set of alternatives that are (weakly) preferred to x at R, that is, $U(x, R) = \{y \in A \mid yRx\}$. An antisymmetric preference is called *strict preference*. We denote a strict preference by P.

We denote a set of admissible preferences of an agent i by $\mathcal{D}_i$, and a set of admissible strict preferences by $\widehat{\mathcal{D}}_i$. Let $\mathcal{D}_N = \prod_{i \in N} \mathcal{D}_i$. An element of $\mathcal{D}_N$ is called a preference profile. For ease of presentation, we refer to sets $\mathcal{D}_i$ and $\mathcal{D}_N$ as domains, and sets $\hat{\mathcal{D}}_i$ and $\hat{\mathcal{D}}_N$ as strict domains. Furthermore, for a domain $\mathcal{D}_i$ or $\mathcal{D}_N$, we denote by strict$(\mathcal{D}_i)$ and strict$(\mathcal{D}_N)$ the set of strict preferences in $\mathcal{D}_i$ and $\mathcal{D}_N$, respectively.

A random social choice function assigns a probability distribution over the alternatives at each preference profile.

Definition 5.2.1 *A random social choice function (RSCF) ϕ on $\mathcal{D}_N$ is a mapping $\phi : \mathcal{D}_N \to \Delta A$.*

Unanimity says that whenever all the agents in a society have some alternative(s) common in their top position, those alternatives are chosen with probability 1.

Definition 5.2.2 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is unanimous if for all $R_N \in \mathcal{D}_N$ such that $\cap_{i \in N} \tau(R_i) \neq \emptyset$, we have $\sum_{x \in \cap_{i \in N} \tau(R_i)} \phi_x(R_N) = 1$.*

We say an alternative x Pareto dominates another alternative y at a preference profile R_N if every agent weakly prefers x to y and some agent strictly prefers x to y, that is, xR_iy for all $i \in N$ and xP_iy for some $i \in N$. An alternative is said to be Pareto dominated if some other alternative Pareto dominates it. Pareto optimality says that a Pareto dominated alternative cannot be selected with positive probability.

Definition 5.2.3 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ satisfies Pareto optimality if for all $R_N \in \mathcal{D}_N$, we have $\phi_x(R_N) = 0$ for all $x \in A$ such that x is Pareto dominated at R_N.*

An RSCF is strategy-proof if no agent can increase the probability of any upper contour set by misreporting his/her preferences.

Definition 5.2.4 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is called strategy-proof if for all $i \in N$, $(R_i, R_{N\setminus i}) \in \mathcal{D}_N$, $R'_i \in \mathcal{D}_i$, and $x \in A$*

$$\sum_{y \in U(x,R_i)} \phi_y(R_i, R_{N\setminus i}) \geq \sum_{y \in U(x,R_i)} \phi_y(R'_i, R_{N\setminus i}).$$

Remark 5.2.5 *An RSCF is called a deterministic social choice function (DSCF) if it selects a degenerate probability distribution at every preference profile. More formally, an RSCF $\phi : \mathcal{D}_N \to \Delta A$ is called a DSCF if $\phi_a(R_N) \in \{0, 1\}$ for all $a \in A$ and all $R_N \in \mathcal{D}_N$. The notions of unanimity, Pareto optimality, and strategy-proofness for DSCFs are special cases of the corresponding definitions for RSCFs.*

5.3 Weak random dictatorial domains

[23] provide a sufficient condition on strict domains so that every unanimous and strategy-proof random rule on it is random dictatorial. They call such domains random dictatorial domains. In this section, we investigate what happens to the unanimous and strategy-proof random rules if weak preferences are allowed to these domains.

First, we introduce the notion of random dictatorial rules on strict domains. An RSCF $\phi : \widehat{\mathcal{D}}_N \to \Delta A$ is called random dictatorial with coefficients $(\alpha_1, \ldots, \alpha_n) \in \Delta N$ if for each profile $P_N \in \hat{\mathcal{D}}_N$ and all $a \in A$, $\phi_a(P_N) = \sum_{\{i|\tau(P_i)=a\}} \alpha_i$. Thus, for a random dictatorial rule, each agent i has a weight α_i which he/she assigns to his/her top-ranked alternative. A strict domain $\widehat{\mathcal{D}}_N$ is called random dictatorial if every unanimous and strategy-proof random rule $\phi : \widehat{\mathcal{D}}_N \to A$ is random dictatorial.

A strict preference $\hat{P}$ is a strict extension of a preference R if for all $a, b \in A$, aPb implies $a\hat{P}b$. Note that a preference can have multiple strict extensions. For instance, if $R = a[bcd]e$, then the following preferences are strict extensions of R: *abcde*, *abdce*, *acbde*, *acdbe*, *adbce*, and *adcbe*. A domain $\mathcal{D}$ satisfies the strict extension property if it contains all strict extensions of all preferences in it.

It is shown in [40] that the strict unrestricted domain is random dictatorial. Note that every superset of the unrestricted domain (in particular, the universal domain) satisfies the strict extension property.

In what follows, we introduce the notion of weak random dictatorial rules. These are extensions of random dictatorial rules for arbitrary domains. Like random dictatorial rules, here to each agent i has a weight α_i, however now, this α_i weight is divided amongst the top-ranked alternatives of agent i.

Definition 5.3.1 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is weak random dictatorial with coefficients $(\alpha_1, \ldots, \alpha_n) \in \Delta N$ if for each profile $R_N \in \mathcal{D}_N$ there exists $\lambda(i, a) \in [0, 1]$ for all $(a, i) \in A \times N$ such that*

(i) $\lambda(a, i) = 0$ *if* $a \notin \tau(R_i)$,

(ii) $\sum_a \lambda(a, i) = \alpha_i$, *and*

(iii) $\sum_i \lambda(a, i) = \phi_a(R_N)$.

It is worth mentioning that a weak random dictatorial rule $\phi : \mathcal{D}_N \to \Delta A$ becomes random dictatorial if $|\tau(R_i)| = 1$ for all $R_i \in \mathcal{D}_i$. Our next theorem provides the structure of unanimous and strategy-proof RSCFs on minimally rich domains.

Theorem 5.3.2 *Let $\mathcal{D}_i$ satisfy the strict extensions property for all $i \in N$ and suppose strict$(\mathcal{D}_N)$ is a random dictatorial domain. If an RSCF $\phi : \mathcal{D}_N \to \Delta A$ is unanimous and strategy-proof, then it is weak random dictatorial.*

The proof of this theorem is relegated to Appendix .1.

Remark 5.3.3 *[77] introduces the notion of super dictatorial domains and provides a characterization of these domains. A domain is super dictatorial if its all supersets (including itself) are dictatorial. One can similarly define the notion of super random dictatorial domains: a domain is super random dictatorial if all its supersets are random dictatorial. It follows from Theorem 5.3.2 that the unrestricted domain (among others) is super random dictatorial.*

5.4 Random rules on single-peaked domains

In this section, we consider single-peaked domains and provide a characterization of Pareto optimal and strategy-proof RSCFs on these domains.

A preference is called single-peaked if there is a unique top-ranked alternative such that preference *weakly* declines as one moves away from the top-ranked alternative in any direction.

Definition 5.4.1 *A preference R is called single-peaked if it has a unique top-ranked alternative $\tau(R)$, called the peak, such that for all $a, b \in A$, $b < a < \tau(R)$ or $\tau(R) < a < b$ implies aRb. A domain is called single-peaked if each preference in it is single-peaked.*

A single-peaked preference is called strict single-peaked if it does not contain any indifference. A domain is called strict single-peaked if each preference in it is strict single-peaked.

Next, we introduce the notion of probabilistic fixed ballot rules. We need the following terminology. For a preference profile R_N and an alternative x, we denote the set of agents whose peaks are on the right of x by $S(x, R_N)$, that is, $S(x, R_N) = \{i \in N \mid \tau(R_i) \geq x\}$.

Definition 5.4.2 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is called a probabilistic fixed ballot rule if for all $S \subseteq N$, there exists $\beta_S \in \Delta A$ satisfying*

(i) $\beta_N(m) = 1$ and $\beta_\emptyset(1) = 1$, and

(ii) $\beta_S([x, m]) \leq \beta_T([x, m])$ for all $S \subseteq T$ and all $x \in A$

such that for all $R_N \in \mathcal{D}_N$ and all $x \in A$, we have

$$\phi_x(R_N) = \beta_{S(x,R_N)}[x, m] - \beta_{S(x+1,R_N)}[x + 1, m],$$

where $\beta_{S(m+1,R_N)}[m + 1, m] \equiv 0$.

Finally, we introduce the notion of extreme PFBRs. These are special cases of PFBRs where each β_S assigns positive probabilities only to the "extreme" alternatives 1 and m.

Definition 5.4.3 *A PFBR with respect to parameters $(\beta_S)_{S\subseteq N}$ is called extreme if $\beta_S(x) > 0$ implies $x \in \{1, m\}$.*

A single-peaked preference R is called left dichotomous if $\tau(R) = 1$ and $2Im$. In other words, except from alternative 1, which is ranked top at R, all other alternatives are indifferent to each other. Similarly, a preference R' is called right dichotomous if $\tau(R') = m$ and $(m - 1)I'1$. A single-peaked domain is minimally rich if it contains all strict single-peaked preferences, the left dichotomous, and the right dichotomous preference. Our next theorem says that every Pareto optimal and strategy-proof RSCF on the single-peaked domain is an extreme PFBR. [18] considers the single-peaked domain with outside options and characterizes Pareto optimal and strategy-proof DSCFs on those domain.[1] The single-peaked domains with outside option are special cases of minimally rich weak single-peaked domains, and hence a characterization of Pareto optimal and strategy-proof RSCFs on those domains follows as a corollary of our result.

Theorem 5.4.4 *Let $\mathcal{D}_i$ be a minimally rich single-peaked domain for each $i \in N$. An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is Pareto optimal and strategy-proof if and only if it is an extreme PFBR.*

The proof of the theorem is relegated to Appendix .2.

[1] A preference is single-peaked with outside options if there is a region around the peak such that the preference exhibits single-peakedness over the alternatives in that region and indifference over the ones outside it.

5.5 Random rules on single-plateaued domains

In this section, we introduce the notion of single-plateaued preferences. For these preferences, an interval of alternatives appear at the top-position, and as one go far from this interval (in any particular direction), preference declines *strictly*. Moreover, indifference occurs only at the top-position for such a preference. Note that single-plateaued preferences are the counter part of weak single-peaked preferences in the sense that for the former, indifference can occur only at the top, whereas for the latter, it can occur only below the top.

Throughout this section, we assume that admissible preferences are the same across agents.

Definition 5.5.1 *A preference $R \in \mathbb{W}(A)$ is called single-plateaued if there exist $x, y \in A$ with $x < y$ such that*

(i) $\tau(R) = [x, y]$,

(ii) for all $u, v \in A$, $[u < v \leq x$ or $y \leq v < u]$ implies vPu, and

(iii) for all $u, v \notin [x, y]$, either uPv or vPu.

In what follows, we introduce some particular type of single-plateaued preferences based on the size of the plateau.

Definition 5.5.2 *For $1 \leq \kappa_1 \leq \kappa_2 < m$, a single-plateaued preference $R \in \mathbb{W}(A)$ is called (κ_1, κ_2)-single-plateaued if $\kappa_1 \leq |\tau(R)| \leq \kappa_2$. A domain is called (κ_1, κ_2)-single-plateaued domain if it contains all (κ_1, κ_2)-single-plateaued preferences.*

For a single-plateaued preference R, we denote by $\tau^+(R)$ and $\tau^-(R)$ the right-end point and the left end-point of the plateau, respectively. More formally, if $\tau(R) = [x, y]$, then $\tau^+(R) = y$ and $\tau^-(R) = x$.

5.5.1 Equivalence of unanimity and Pareto optimality under strategy-proofness

In this section, we introduce the concepts of unanimity and Pareto optimality. Unanimity is a weaker notion of Pareto optimality, however we show that under strategy-proofness they are equivalent on a single-plateaued domain. It is worth mentioning that the same result holds on a single-peaked domain (see [63] and [84]).

From this section onward, we assume that all the agents have the same set of admissible preferences.

Our next theorem says that unanimity and Pareto optimality are equivalent for a strategy-proof RSCF on a (κ_1, κ_2)-single-plateaued domain.

Theorem 5.5.3 *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathcal{D}$ be a (κ_1, κ_2)-single-plateaued domain. Suppose $\phi : \mathcal{D}^n \to \Delta A$ is a strategy-proof RSCF. Then, ϕ is unanimous if and only if it is Pareto optimal.*

The proof of the theorem is relegated to Appendix .3.

5.5.2 Unanimity and almost plateau-onlyness

In this section, we analyze the connection between unanimity and a weaker version of plateau-onlyness called almost plateau-onlyness in the presence of strategy-proofness. An RSCF is called plateau-only if its outcome depends only on the plateaus at a preference profile.

Definition 5.5.4 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called plateau-only if for any two preference profiles $R_N, R'_N \in \mathcal{D}^n$, $\tau(R_i) = \tau(R'_i)$ for all $i \in N$ implies $\phi(R_N) = \phi(R'_N)$.*

On single-peaked domains, peaks-onlyness and unanimity are equivalent for random rules under strategy-proofness ([63] and [84]). However, as the following example suggests, the same does not hold even for deterministic rules on single-plateaued domains when peaks-onlyness is replaced by plateau-onlyness.

Example 5.5.5 *Consider the DSCF, say f, given in Table 5.5.1. It can be verified that ϕ is unanimous and strategy-proof. However, since $\tau([23]14) = \tau([23]41)$ and $f([23]14, [123]4) \neq f([23]41, [123]4)$, it is not plateau-only.* □

1 \ 2	1234	2134	2314	3214	3421	4321	[12]34	[23]14	[23]41	[34]21	[123]4	[234]1
1234	1	2	2	2	2	2	1	2	2	2	1	2
2134	2	2	2	2	2	2	2	2	2	2	2	2
2314	2	2	2	2	2	2	2	2	2	2	2	2
3214	2	2	2	3	3	3	3	3	3	3	3	3
3421	2	2	2	3	3	3	3	3	3	3	3	3
4321	2	2	2	3	3	4	2	3	3	4	3	4
[12]34	1	2	2	3	3	2	1	2	2	2	2	2
[23]14	2	2	2	3	3	3	2	3	2	3	3	2
[23]41	2	2	2	3	3	3	2	2	3	3	2	3
[34]21	2	2	2	3	3	4	2	3	3	4	3	3
[123]4	1	2	2	3	3	3	1	3	2	3	1	3
[234]1	2	2	2	3	3	4	2	3	2	4	2	3

Table 5.5.1

It is worth noting from Example 5.5.5 that if an agent changes his/her preference maintaining his/her plateau, then unanimity and strategy-proofness can never rule out the possibility of rearranging the probabilities of the alternatives in the plateau. In view of this fact, we weaken the notion of plateau-onlyness by almost plateau-onlyness. It says that if an agent changes his/her preference maintaining his/her plateau, then the probability of any alternative that lies *outside* his/her plateau must remain the same. In other words, the only change that can happen by this change of preference is that the probabilities of the alternatives in his/her plateau are rearranged.

Definition 5.5.6 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called almost plateau-only if for any two preference profiles $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in \mathcal{D}^n$, $\tau(R_i) = \tau(R'_i)$ implies $\phi_x(R_i, R_{N\setminus i}) = \phi_x(R'_i, R_{N\setminus i})$ for all $x \notin \tau(R_i)$.*

Theorem 5.5.7 *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathcal{D}$ be a (κ_1, κ_2)-single-plateaued domain. Let $\mathcal{D}$ be a (κ_1, κ_2)-single-plateaued domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. Then, ϕ is almost plateau-only.*

The proof of the theorem is relegated to Appendix .4.

5.5.3 A characterization of unanimous and strategy-proof rules on arbitrary single-plateaued domains

In this section, we provide a characterization of unanimous and strategy-proof RSCFs on arbitrary single-plateaued domains. We do this by identifying a property called generalized uncompromisingness of such rules. As the name suggests, this property is a generalization of the uncompromisingness property that exists in the literature in the context of single-peaked domains.

The notion of generalized uncompromisingness turns out to be relatively simpler for DSCFs. To help the reader, we first present this notion for DSCFs.

A DSCF satisfies generalized uncompromisingness if the following happens. Whenever an agent unilaterally moves his/her plateau in some direction, (i) if both the plateaus lie either strictly on the right of the outcome or strictly on the left of the outcome of the DSCF, then the outcome does not change, and (ii) if the right-end point or the left-end point of the plateau crosses the outcome, then the outcome moves in the direction to which the plateau has moved.

Definition 5.5.8 *An DSCF $f : \mathcal{D}^n \to A$ satisfies generalized uncompromisingness if for all $R_i, R'_i \in \mathcal{D}$, and all $R_{N\setminus i} \in \mathcal{D}^{n-1}$, we have*

(i) $[f(R_N), f(R'_N) \leq \min\{\tau^-(R_i), \tau^-(R'_i)\}]$ *or* $[\max\{\tau^+(R_i), \tau^+(R'_i)\} \leq f(R_N), f(R'_N)]$, *and*

(ii) $[\tau^+(R_i) < f(R_N) \leq \tau^+(R'_i)]$ *or* $[\tau^-(R_i) < f(R_N) \leq \tau^-(R'_i)]$ *implies* $f(R_N) \leq f(R'_i, R_{N\setminus i})$.

We illustrate the notion of generalized uncompromisingness by means of the following example. It should be noted that DSCFs satisfying generalized uncompromisingness can be constructed in a relatively easy manner.

Example 5.5.9 *Let the set of alternatives be* $A = \{1, 2, 3, 4, 5\}$ *and suppose that there are two agents* $N = \{1, 2\}$. *We consider an arbitrary single-plateaued domain* $\mathcal{D}$. *In Table 5.5.2, we present a DSCF, say f, that satisfies generalized uncompromisingness. To see that f satisfies part (i) of generalized uncompromisingness, consider, for instance, the preference profiles* (12345, [23]145) *and* (12345, [45]321). *Note that agent 2 changes his/her plateau from* [23] *to* [45] *from the former preference profile to the latter. The outcome at the former preference profile is 1, which lies strictly on the left of both the plateaus* [23] *and* [45]. *As required by (i), the outcome at the latter preference profile is also 1. To see that f satisfies part (ii) of generalized uncompromisingness, consider, for instance, the preference profiles* ([123]45, [123]45) *and* ([123]45, [45]321). *Note that the outcome at the former preference profile is 2, which lies (weakly) on the right of the former plateau 1 and strictly on the left of the latter plateau 4. As required by (ii), the outcome moves to its right from 2 to 3. It is worth mentioning that although the DSCF in this example is chosen to be unanimous, unanimity is not implied by generalized uncompromisingness. Later, we will make a formal remark to emphasize this fact.* □

1 \ 2	12345	[123]45	[23]145	[23]451	[234]51	32145	34521	[3245]1	[45]321	43215	43521	54321
12345	1	1	1	1	1	1	1	1	1	3	1	1
[123]45	1	2	3	3	3	3	3	3	3	3	3	3
[23]145	1	3	2	2	2	3	3	3	3	3	3	3
[23]451	1	3	2	2	2	3	3	3	3	3	3	3
[234]51	1	3	2	2	4	3	3	4	4	4	4	4
32145	1	3	3	3	3	3	3	3	3	3	3	3
34521	1	3	3	3	3	3	3	3	3	3	3	3
[3245]1	1	3	3	3	4	3	3	2	4	4	4	5
[45]321	1	3	3	3	4	3	3	4	4	4	4	5
43215	1	3	3	3	4	3	3	4	4	4	4	4
43521	1	3	3	3	4	3	3	4	4	4	4	4
54321	1	3	3	3	4	3	3	5	5	4	4	5

Table 5.5.2

Now, we present the notion of generalized uncompromisingness for RSCFs. It says that whenever an agent unilaterally moves his/her plateau in some direction, (i) if, for an alternative x, both the plateaus lie either strictly on the right of it or strictly on the left of it, then the probability of x does not change, and (ii) the probability of an interval $[x, m]$, where x lies in exactly one of the two plateaus, will weakly increase. In our formal definition, we present (i) by means of probabilities of sets of the form $[x, m]$, one can verify that it means exactly what we have explained above.

Definition 5.5.10 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ satisfies generalized uncompromisingness if for all $R_i, R_i' \in \mathcal{D}$, all $R_{N\setminus i} \in \mathcal{D}^{n-1}$, and all $x \in A$, we have*

(i) $[x \leq \min\{\tau^-(R_i), \tau^-(R_i')\}]$ *or* $[\max\{\tau^+(R_i), \tau^+(R_i')\} < x]$ *implies* $\phi_{[x,m]}(R_i', R_{N\setminus i}) = \phi_{[x,m]}(R_i, R_{N\setminus i})$, *and*

(ii) $[\tau^+(R_i) < x \leq \tau^+(R_i')]$ *or* $[\tau^-(R_i) < x \leq \tau^-(R_i')]$ *implies* $\phi_{[x,m]}(R_i', R_{N\setminus i}) \geq \phi_{[x,m]}(R_i, R_{N\setminus i})$.

Remark 5.5.11 *Note that as we have mentioned in the introduction of Definition 5.5.8, part (i) of generalized uncompromisingness implies that for all $(R_i, R_{N\setminus i}) \in \mathcal{D}^n$, all $i \in N$, and all $R_i' \in \mathcal{D}$, we have $\phi_x(R_i, R_{N\setminus i}) = \phi_x(R_i', R_{N\setminus i})$ for all $x \in A$ such that either $x < \{\tau^-(R_i), \tau^-(R_i')\}$ or $x > \max\{\tau^+(R_i), \tau^+(R_i')\}$.*

Remark 5.5.12 *It is worth noting that the notion of generalized uncompromisingness coincides with that of uncompromisingness ([63]) if we assume $\tau^+(R) = \tau^-(R)$ for all $R \in \mathcal{D}$, that is, if preferences are single-peaked.*

Remark 5.5.13 *By considering R_i and R_i' such that $\tau^-(R_i) = \tau^-(R_i')$ and $\tau^+(R_i) = \tau^+(R_i')$, it follows that generalized uncompromisingness implies almost plateau-onlyness.*

We illustrate the notion of generalized uncompromisingness by means of the following example. It is worth mentioning that, although the RSCF in the following example is chosen to be unanimous, the same is not implied by generalized uncompromisingness.

Example 5.5.14 *Let the set of alternatives, agents, and admissible preferences be the same as in Example 5.5.9. In Table 5.5.3, we present an RSCF, say ϕ, that satisfies generalized uncompromisingness. To see that ϕ satisfies part (i) of generalized uncompromisingness, consider, for instance, the preference profiles (12345, [123]45) and (12345, [234]51). Note that agent 2 changes his/her plateau from [123] to [234] from the former preference profile to the latter. As alternative 5 lies strictly to the right of both the plateaus [123] and [234], ϕ_5(12345, [123]45) $=$ ϕ_5(12345, [234]51) $= 0$. To see that ϕ satisfies part (ii) of generalized uncompromisingness, consider, for instance, the preference profiles ([2345]1, [234]51) and ([2345]1, [45]321). Alternative 3 lies (weakly) to the right of the plateau [234] and strictly to the left of the plateau [45]. As required by (ii), $\phi_{[3,5]}$([2345]1, [234]51) $<$ $\phi_{[3,5]}$([2345]1, [45]321). It is worth mentioning that although the RSCF in this example is chosen to be unanimous, unanimity is not implied by generalized uncompromisingness. Later, we will make a formal remark to emphasize this fact.* □

1 \ 2	12345	[123]45	[23]145	[23]451	[234]51	32145	34521	[2345]1	[45]321	43215	43521	54321
12345	(1, 0, 0, 0, 0)	(1, 0, 0, 0, 0)	(0.3, 0.7, 0, 0, 0)	(0.3, 0.7, 0, 0, 0)	(0.3, 0.7, 0, 0, 0)	(0.3, 0.3, 0.4, 0, 0)	(0.3, 0.3, 0.4, 0, 0)	(0.3, 0.3, 0.4, 0, 0)	(0.3, 0.3, 0.4, 0, 0)	(0.3, 0.3, 0.4, 0, 0)	(0.3, 0.3, 0.4, 0, 0)	(0.3, 0.3, 0.4, 0, 0)
[123]45	(1, 0, 0, 0, 0)	(0.3, 0.4, 0.3, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0.6, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)
[23]145	(0.3, 0.7, 0, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0.8, 0.2, 0, 0)	(0, 0.7, 0.3, 0, 0)	(0, 0.6, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)
[23]451	(0.3, 0.7, 0, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0.7, 0.3, 0, 0)	(0, 0.4, 0.6, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)
[234]51	(0.3, 0.7, 0, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0.6, 0.4, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0.2, 0.3, 0.5, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0.4, 0.4, 0.2, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)
32145	(0.3, 0.3, 0.4, 0, 0)	(0, 0.5, 0.5, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)
34521	(0.3, 0.3, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)
[2345]1	(0.3, 0.3, 0.4, 0, 0)	(0, 0.6, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0.4, 0.4, 0.2, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0.2, 0.3, 0.3, 0.2)	(0, 0, 0, 0.5, 0.5)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1)
[45]321	(0.3, 0.3, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 1, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 0.5, 0.5)	(0, 0, 0, 0.6, 0.4)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1)
43215	(0.3, 0.3, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 1, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)
43521	(0.3, 0.3, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 1, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)
54321	(0.3, 0.3, 0.4, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 1, 0)	(0, 0, 1, 0, 0)	(0, 0, 1, 0, 0)	(0, 0, 0, 0, 1)	(0, 0, 0, 0, 1)	(0, 0, 0, 1, 0)	(0, 0, 0, 1, 0)	(0, 0, 0, 0, 1)

Table 5.5.3

We are now ready to present the main theorem of this section. It provides a characterization of unanimous and strategy-proof RSCFs by saying that a unanimous RSCF is strategy-proof if and only if it satisfies generalized uncompromisingness.

Theorem 5.5.15 *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathcal{D}$ be a (κ_1, κ_2)-single-plateaued domain. If $\phi : \mathcal{D}^n \to \Delta A$ satisfies generalized uncompromisingness, then it is strategy-proof.*

The proof of the theorem is relegated to Appendix .5.

Theorem 5.5.16 *Let $1 \leq \kappa_1 \leq \kappa_2 \leq m$ and let $\mathcal{D}$ be a (κ_1, κ_2)-single-plateaued domain. Suppose $\phi : \mathcal{D}^n \to \Delta A$ is unanimous. Then, ϕ is strategy-proof if and only if it satisfies generalized uncompromisingness.*

The proof of the theorem is relegated to Appendix .6.

5.5.4 A functional form characterization of the unanimous and strategy-proof rules for the case of two agents

In this section, we consider the case where there are exactly two agents. We provide a complete characterization of the unanimous and strategy-proof rules in this scenario.

In what follows, we introduce the notion of probabilistic plateau rule when there are two agents. These rules are based on two parameters β_1 and β_2. Both these parameters represent some probability distributions over the set of alternatives. For instance, if the set of alternatives is $\{1, \ldots, 5\}$, then possible values of β_1 and β_2 are $(0.1, 0.2, 0.2, 0.3, 0.2)$ and $(0.4, 0, 0.1, 0.2, 0.3)$, respectively. Note that β_1 and β_2 are independent of each other.

Now, we explain how the the outcome of a probabilistic plateau rule ϕ is determined based on the parameter values β_1 and β_2. For any preference profile where $\tau(R_1) \cap \tau(R_2) \neq \emptyset$, define the outcome as an arbitrary probability distribution over $\tau(R_1) \cap \tau(R_2)$. Consider a preference profile where $\tau(R_1) \cap \tau(R_2) = \emptyset$. Suppose $\tau^+(R_1) < \tau^-(R_2)$. Consider an alternative x. If $x < \tau^+(R_1)$ or $\tau^-(R_2) < x$, then define $\phi_x(R_N) = 0$. If $\tau^+(R_1) < x < \tau^-(R_2)$, define $\phi_x(R_N) = \beta_2(x)$. Finally, if $x = \tau^+(R_1)$, then $\phi_x(R_N) = \beta_2[1, x]$, and if $x = \tau^-(R_2)$, then $\phi_x(R_N) = \beta_2[x, m]$. For the case where $\tau^+(R_2) < \tau^-(R_1)$, we use the probability distribution given by β_1 in place of β_2 to determine the outcomes.

For an example of probabilistic plateau rule , consider $\beta_1 = (0.1, 0.2, 0.2, 0.3, 0.2)$ and $\beta_2 = (0.4, 0, 0.1, 0.2, 0.3)$. Let ϕ be a probabilistic plateau rule with respect to (β_1, β_2). In Table 5.5.4 , we provide the values of ϕ at some preference profiles.

R_N	$\phi_1(R_N)$	$\phi_2(R_N)$	$\phi_3(R_N)$	$\phi_4(R_N)$	$\phi_5(R_N)$
$([12]345, 54321)$	0	0.4	0.1	0.2	0.3
$([234]15, [34]521)$	0	0	0.4	0.6	0
$([34]251, [12]345)$	0	0	0.3	0.7	0
$([543]21, 23451)$	0	0	0.3	0.7	0
$(54321, [23]145)$	0	0	0.5	0.3	0.2

Table 5.5.4

Below, we provide a formal definition of these rules.

Definition 5.5.17 *An RSCF $\phi : \mathcal{D}^2 \to \Delta A$ is called probabilistic plateau rule with respect to (β_1, β_2), where $\beta_i \in \Delta A$ for all $i \in N$, if for all $R_N \in \mathcal{D}^2$ and all $x \in A$, we have*

(i) $\tau(R_1) \cap \tau(R_2) \neq \emptyset$ implies $\phi(R_N)$ is an arbitrary probability distribution over $\tau(R_1) \cap \tau(R_2)$,

(ii) $\tau^+(R_1) < \tau^-(R_2)$ implies $\phi_{[x,m]}(R_N) = \beta_2[x, m]$ for all $\tau^+(R_1) < x \leq \tau^-(R_2)$ and $\phi_x(R_N) = 0$ for all $x \notin [\tau^+(R_1), \tau^-(R_2)]$, and

(iii) $\tau^+(R_2) < \tau^-(R_1)$ implies $\phi_{[x,m]}(R_N) = \beta_1[x, m]$ for all $\tau^+(R_2) < x \leq \tau^-(R_1)$ and $\phi_x(R_N) = 0$ for all $x \notin [\tau^+(R_2), \tau^-(R_1)]$.

Now, we present the main result of this section. It characterizes all unanimous and strategy-proof rules on (κ_1, κ_2)-single-plateaued domains for two agents.

Theorem 5.5.18 *Let $N = \{1, 2\}$ and $1 \leq \kappa_1 \leq \kappa_2 \leq m$. Suppose $\mathcal{D}$ is the (κ_1, κ_2)-single-plateaued domain. Then, an RSCF $\phi : \mathcal{D}^2 \to \Delta A$ is unanimous and strategy-proof if and only if it is a probabilistic plateau rule.*

The proof of this theorem is relegated to Appendix .7.

5.5.5 A class of unanimous and strategy-proof rules

In this section, we present a large class of RSCFs that are unanimous and strategy-proof. These RSCFs are extension of probabilistic plateau rules for arbitrary number of agents. Like probabilistic plateau rules, these rules too are based on a class of parameters that we call probability ballots.

Definition 5.5.19 *A collection $(\beta_S)_{S \subseteq N}$, where $\beta_S \in \Delta A$ for all $S \subseteq N$, is called probability ballots if for all $\emptyset \subseteq S \subset T \subseteq N$ and all $x \in A$, we have $\beta_S[x, m] \leq \beta_T[x, m]$.*

Now, we introduce the notion of generalized probabilistic ballot rule with parameters k and $(\beta_S)_{S \subseteq N}$, where $k \in \{0, \ldots, m\}$ and $(\beta_S)_{S \subseteq N}$ is a collection of probability ballots. We use the following notation: for a preference profile $R_N \in \mathcal{D}^n$, whenever $\cap_{i \in N} \tau(R_i) = \emptyset$ we denote by $I(R_N)$ the minimal interval I such that $I \cap \tau(R_i) \neq \emptyset$ for all $i \in N$, otherwise we define $I(R_N) = \cap_{i \in N} \tau(R_i)$. In other words, when $\cap_{i \in N} \tau(R_i) = \emptyset$, $I(R_N)$ denotes the minimal interval that contains some top-ranked alternative of each agent.

Definition 5.5.20 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called a generalized probabilistic ballot rule with parameters k and $(\beta_S)_{S \subseteq N}$, where $k \in \{0, \ldots, m\}$ and $(\beta_S)_{S \subseteq N}$ is a collection of probability ballots, if for all $R_N \in \mathcal{D}^n$ and all $x \in A$, we have*

(i) $\phi_{[x,m]}(R_N) = 1$ *for all* $x \leq I^-(R_N)$,

(ii) $\phi_{[x,m]}(R_N) = 0$, *for all* $x > I^+(R_N)$, *and*

(iii) $\cap_{i \in N} \tau(R_i) = \emptyset$ *implies that for all* $I^-(R_N) < x \leq I^+(R_N)$, $\phi_{[x,m]}(R_N) = \beta_S[x, m]$, *where S is such that $i \in S$ if and only if $\tau^-(R_i) \geq x - k$ and $\tau^+(R_i) \geq x$.*

Note that one can construct a large class of generalized probabilistic ballot rules by varying the values of βs and ks. In what follows, we present an example of generalized probabilistic ballot rule.

Example 5.5.21 *Let the set of alternatives be $A = \{1, 2, 3, 4, 5\}$ and the set of agents be $N = \{1, 2, 3\}$. Consider the generalized probabilistic ballot rule with parameters k and $(\beta_S)_{S \subseteq N}$, where $k = 1$ and $\beta_\emptyset = (1, 0, 0, 0, 0)$, $\beta_{\{1\}} = \beta_{\{2\}} = \beta_{\{3\}} = (0.5, 0.2, 0.2, 0.1, 0)$, $\beta_{\{1,2\}} = \beta_{\{1,3\}} = \beta_{\{2,3\}} = (0.2, 0.4, 0.3, 0.1, 0)$ and $\beta_{\{1,2,3\}} = (0.1, 0.3, 0.3, 0.2, 0.1)$. In Table 5.5.5, we provide the outcomes of ϕ at some particular preference profiles. We explain how the outcomes are calculated. Consider the first preference profile $R^1_N = ([12]345, [23]145, [23]415)$. Here, $I(R^1_N) = \{2\}$, therefore by Conditions (i) and (ii) of a generalized probabilistic ballot rule, we have $\phi_{[1,5]}(R^1_N) = 1$, $\phi_{[2,5]}(R^1_N) = 1$, $\phi_{[3,5]}(R^1_N) = \phi_{[4,5]}(R^1_N) = \phi_5(R^1_N) = 0$. Hence, $\phi_1(R^1_N) = 0$, $\phi_2(R^1_N) = 1$, $\phi_3(R^1_N) = \phi_4(R^1_N) = \phi_5(R^1_N) = 0$. Consider the second preference profile*

$R^2_N = ([12]345, [45]321, [34]521)$. *Here,* $\cap_{i \in N}\tau(R_i) = \emptyset$. *Hence,* $I(R^2_N) = \{2, 3, 4\}$. *Therefore by Conditions (i) and (ii) of a generalized probabilistic ballot rule, we have* $\phi_{[1,5]}(R^2_N) = 1$, $\phi_{[2,5]}(R^2_N) = 1$, $\phi_5(R^2_N) = 0$. *For ease of presentation, let the use Consider the alternative* 3. *Since* $\tau^-(R_2) \geq 2$, $\tau^-(R_3) \geq 2$, $\tau^+(R_2) \geq 3$ *and* $\tau^+(R_3) \geq 3$, *we have* $S = \{2, 3\}$. *By Condition (iii) of a generalized probabilistic ballot rule,* $\phi_{[3,5]}(R^2_N) = \beta_{\{2,3\}}[3, 5] = 0.4$. *Finally, consider the alternative* 4. *Since* $\tau^-(R_2) \geq 3$, $\tau^-(R_3) \geq 3$, $\tau^+(R_2) \geq 4$ *and* $\tau^+(R_3) \geq 4$, *we have* $S = \{2, 3\}$. *By Condition (iii) of a generalized probabilistic ballot rule,* $\phi_{[4,5]}([12]345, [45]321, [34]521) = \beta_{\{2,3\}}[4, 5] = 0.1$. *Hence,* $\phi_1([12]345, [45]321, [34]521) = 0$, $\phi_2([12]345, [45]321, [34]521) = 0.6$, $\phi_3([12]345, [45]321, [34]521) = 0.3$, $\phi_4([12]345, [45]321, [34]521) = 0.1$ *and* $\phi_3([12]345, [45]321, [34]521) = 0$. *Similarly, one can compute the outcome of* ϕ *at the other preference profiles mentioned in the table.*

(R_1, R_2, R_3)	$\phi_1(R_1, R_2, R_3)$	$\phi_2(R_1, R_2, R_3)$	$\phi_3(R_1, R_2, R_3)$	$\phi_4(R_1, R_2, R_3)$	$\phi_5(R_1, R_2, R_3)$
$R^1_N = ([12]345, [23]145, [23]415)$	0	1	0	0	0
$R^2_N = ([12]345, [45]321, [34]521)$	0	0.6	0.3	0.1	0
$R^3_N = ([123]45, [23]451, [234]51)$	0	0.7	0.3	0	0
$R^4_N = ([23]145, [234]51, [45]321)$	0	0	0.9	0.1	0

Table 5.5.5

In Section 5.5.4, we have introduced the notion of probabilistic plateau rules when there are two agents. It can be verified that those rules are special cases of generalized probabilistic ballot rules. We present the rules as probabilistic plateau rules rules as we find that more intuitive and reader friendly when there are two agents.

We now present the main theorem of this section.

Theorem 5.5.22 *Let* $\mathcal{D}$ *be a* (κ_1, κ_2)*-single-plateaued domain for some* $1 \leq \kappa_1 \leq \kappa_2 \leq m$. *Suppose* $\phi : \mathcal{D}^n \to \Delta A$ *is a generalized probabilistic ballot rule with parameters* k *and* $(\beta_S)_{S \subseteq N}$ *where* $k \in \{0, \ldots, \kappa_1 - 1\}$ *and* $(\beta_S)_{S \subseteq N}$ *is a collection of probability ballots. Then,* ϕ *is unanimous and strategy-proof.*

The proof of this theorem is relegated to Appendix .8.

In Section 5.5.6, we provide a characterization of unanimous, anonymous, plateau-only, and strategy-proof rules on a particular class of single-plateaued domains. It will be clear from that result the there are unanimous and strategy-proof rules other than generalized probabilistic ballot rules on a single-plateaued domain.

5.5.6 A functional form characterization of anonymous, plateau-only, and strategy-proof rules

In this section, we provide a functional form characterization of anonymous, plateau-only, and strategy-proof RSCFs on a class of single-plateaued domains. By Theorem 5.5.7, every unanimous and strategy-proof RSCF is almost plateau-only. We strengthen almost plateau-onlyness by plateau-onlyness in the interest of tractability. We also add anonymity for the same reason. Our characterization requires too many parameters even with these additional assumptions. Nevertheless, this characterization result can be extended by dropping anonymity in the same way as median rules are extended to min-max rules in the context of single-peaked domains ([63]) . To replace plateau-onlyness by almost plateau-onlyness (which is implied by unanimity and strategy-proofness), one would require many more parameters, which, we think, will be too technical for its practical use.

In what follows, we present a collection of parameters that we require for the description of our RSCFs. Let $\kappa \in \{1, \ldots, m\}$. In our subsequent discussion, this κ is going to represent the size of the plateau of a single-plateaued preference. Given a κ, by $\underset{\sim}{n}$ we denote a vector $(n_0, n_1, \ldots, n_{\kappa-1})$ such that $0 \leq n_{\kappa-1} \leq \cdots \leq n_1 \leq n_0 \leq n$. For instance, if $\kappa = 3$ and $n = 4$, then an example of $\underset{\sim}{n}$ would be $(3, 3, 1)$. Let $\underset{\sim}{N}$ be the collection of all such vectors. Consider a table of order $|\underset{\sim}{N}| \times |\{2, \ldots, m\}|$ where the rows are indexed by the vectors $\underset{\sim}{n}$ in $\underset{\sim}{N}$ and columns are indexed by the alternatives in $\{2, \ldots, m\}$. See Table 5.5.7, for an example of such a table when $\kappa = 3$, $n = 2$, and $m = 5$.

Now, we identify some cells of the table described above for which we will define the values of our parameters. Let $\kappa \in \{1, \ldots, m\}$. Call the cell (corresponding to the position) $(\underset{\sim}{n}, x)$ feasible for κ if $\underset{\sim}{n}_{\kappa-x} = n$ if $x \leq \kappa$ and $\underset{\sim}{n}_{m-x+1} = 0$ if $m - \kappa + 1 < x$. For instance, if $\kappa = 3$, $n = 3$ and $m = 10$, then the following are some feasible cells: $((3, 3, 2), 2)$, $((3, 3, 1), 2)$, $((3, 2, 2), 3)$, $((2, 2, 2), 5)$, $((3, 1, 1), 7)$, $((3, 2, 0), 9)$, $((2, 0, 0), 10)$ and the following are some infeasible cells: $((3, 1, 0), 2)$, $((1, 1, 1), 3)$, $((3, 3, 3), 9)$, $((2, 1, 1), 10)$. We denote by $\mathcal{F}(\kappa)$ the set of all feasible cells for κ.

We need the following terminologies to present some conditions on our parameters. For a vector $\underset{\sim}{n}$, we denote by $\underset{\sim}{n}^+$ the 'right-shifted' value of $\underset{\sim}{n}$, that is, $\underset{\sim}{n}^+_j = \underset{\sim}{n}_{j-1}$ if $1 \leq j \leq \kappa - 1$. For instance, if $\underset{\sim}{n} = (5, 3, 2, 2, 1)$, then $\underset{\sim}{n}^+ = (\cdot, 5, 3, 2, 2)$. Here, any number that is weakly bigger than 5 (and weakly smaller than n) can appear at the position of the dot. For two vectors $\underset{\sim}{n}$ and $\underset{\sim}{n}'$, we write $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$ if there is $l \in \{0, \ldots, \kappa - 1\}$ such that either $[\underset{\sim}{n}'_0 = \underset{\sim}{n}_0 + 1, \ldots, \underset{\sim}{n}'_l = \underset{\sim}{n}_l + 1$ and $\underset{\sim}{n}'_{l+1} = \underset{\sim}{n}_{l+1}, \ldots, \underset{\sim}{n}'_{\kappa-1} = \underset{\sim}{n}_{\kappa-1}]$ or $[\underset{\sim}{n}'_0 = \underset{\sim}{n}_0, \ldots, \underset{\sim}{n}'_l = \underset{\sim}{n}_l$ and $\underset{\sim}{n}'_{l+1} = \underset{\sim}{n}_{l+1} + 1, \ldots, \underset{\sim}{n}'_{\kappa-1} = \underset{\sim}{n}_{\kappa-1} + 1]$. In Table 5.5.6, we present some values of $\underset{\sim}{n}$ and $\underset{\sim}{n}'$ Note that when $\underset{\sim}{n}$ is $(3, 2, 2, 1, 0)$, the first, second and third components of $\underset{\sim}{n}$ are increased by 1, respectively, and the remaining are left unchanged. When $\underset{\sim}{n}$ is $(4, 3, 2, 0, 0)$, then the last and second-last components of $\underset{\sim}{n}$ are increased by 1 and the remaining are left unchanged.

Definition 5.5.23 *Let $\kappa \in \{1, \dots, m\}$. A collection $\{\beta(\underset{\sim}{n}, x)_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ of numbers in $[0, 1]$ is called plateau parameters for κ if for all $\underset{\sim}{n}$,*

(i) $\beta(\underset{\sim}{n}, x) \leq \beta(\underset{\sim}{n}^{+}, x - 1)$, *and*

(ii) $\beta(\underset{\sim}{n}, x) \leq \beta(\underset{\sim}{n} \oplus 1, x)$.

In Table 5.5.7, we present a collection of plateau parameters for the case when $\kappa = 3$, $n = 2$, and $m = 5$.

$\underset{\sim}{n}$	$\underset{\sim}{n}'$
$(3, 2, 2, 1, 0)$	$(4, 3, 3, 1, 0)$
$(4, 3, 2, 0, 0)$	$(4, 3, 2, 1, 1)$

Table 5.5.6

Example 5.5.24 *Let $n = 2$, $\kappa = 3$ and $A = \{1, 2, 3, 4, 5\}$. Here*

$$\begin{aligned}\mathcal{F}(3) = \{&((2,2,0),2),((2,2,1),2),((2,2,2),2),\\ &((2,0,0),3),((2,1,0),3),((2,2,0),3),((2,1,1),3),((2,2,1),3),((2,2,2),3),\\ &((0,0,0),4),((1,0,0),4),((1,1,0),4),((2,0,0),4),((2,1,0),4),((2,2,0),4),((1,1,1),4),\\ &((0,0,0),5),((1,0,0),5),((2,0,0),5)\}.\end{aligned}$$

Let $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)} \in \mathcal{F}(3)\}$ be a collection of plateau parameters as given in Definition 5.5.23. Consider $x = 3$, $\underset{\sim}{n} = (2, 0, 0)$ and $\underset{\sim}{n}' = (2, 2, 0)$. By Condition (i) in Definition 5.5.23, $0 \leq \beta((2, 0, 0), 3) \leq \beta((2, 2, 0), 2) \leq 1$. Take $x = 4$, $\underset{\sim}{n} = (0, 0, 0)$ and $\underset{\sim}{n}' = (1, 1, 0)$. By Condition (ii) of Definition 5.5.23, we must have $0 \leq \beta((0, 0, 0), 4) \leq \beta((1, 1, 0), 4) \leq 1$. Now, take $x = 4$, $\underset{\sim}{n} = (1, 1, 0)$ and $\underset{\sim}{n}' = (1, 1, 1)$. As βs are plateau parameters, by Condition (ii) of Definition 5.5.23, $0 \leq \beta((1, 1, 0), 4) \leq \beta((1, 1, 1), 4) \leq 1$. Table 5.5.7 provides an example of plateau parameters. □

Definition 5.5.25 *Let $\kappa \in \{1, \dots, m\}$. A collection of plateau parameters $\{\beta(\underset{\sim}{n}, x)_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ is called unanimous if $\beta(\underset{\sim}{n}, x) = 1$ whenever $\underset{\sim}{n}_0 = n$ and $\underset{\sim}{n}_{\kappa-1} > 0$, and $\beta(\underset{\sim}{n}, x) = 0$ whenever $\underset{\sim}{n}_0 < n$ and $\underset{\sim}{n}_{\kappa-1} = 0$.*

(n_0, n_1, n_2)	2	3	4	5
(0,0,0)			.4	.3
(1,0,0)			.5	.4
(1,1,0)			.45	
(1,1,1)				
(2,0,0)		.6	.6	.45
(2,1,0)		.55	.55	
(2,1,1)		.55		
(2,2,0)	.7	.6	.5	
(2,2,1)	.8	.7		
(2,2,2)	.9	.8		

Table 5.5.7

(n_0, n_1, n_2)	2	3	4	5
(0,0,0)			0	0
(1,0,0)			0	0
(1,1,0)			0	
(1,1,1)				
(2,0,0)		0.6	.6	.45
(2,1,0)		.55	.55	
(2,1,1)		1		
(2,2,0)	0.7	.6	.5	
(2,2,1)	1	1		
(2,2,2)	1	1		

Table 5.5.8

In Table 5.5.8, we provide a collection of unanimous plateau parameters.

In Example 5.5.24, for $\{\beta(\underset{\sim}{n}, x)_{(\underset{\sim}{n},x)\in\mathcal{F}(3)}\}$ to be unanimous plateau parameters we must have $\beta((2,2,1),2) = \beta((2,2,2),2) = \beta((2,1,1),3) = \beta((2,2,1),3) = \beta((2,2,2),3) = 1$ and $\beta((0,0,0),4) = \beta((1,0,0),4) = \beta((1,1,0),4) = \beta((0,0,0),5) = \beta((1,0,0),5) = 0$.

In what follows, we present the notion of κ-plateaued rules. We need the following terminology for our presentation. For an alternative $x \in A$, a number $l \in \{0, \ldots, \kappa - 1\}$, and a preference profile $R_N \in \mathcal{D}^n$, let $n_l^x(R_N) = |\{i \in N \mid \tau^+(R_i) \geq x + l\}|$ be the set of agents whose right-end point of the plateau at R_N is (weakly) on the right of $x + l$.

Definition 5.5.26 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is called a κ-plateaued rule for if there is a collection of plateau parameters $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ for κ such that for all $R_N \in \mathcal{D}^n$, $\phi_{[x,m]}(R_N) = \beta(\underset{\sim}{n}, x)$, where $\underset{\sim}{n}_l = n_l^x(R_N)$ for all $0 \leq l \leq \kappa - 1$.*

In Example 5.5.27, we present a κ-plateaued rule .

Example 5.5.27 *Let $n = 2$, $\kappa = 3$ and $A = \{1, 2, 3, 4, 5\}$. Let ϕ be a 3-plateaued rule with respect to a collection $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(3)}\}$ as given in Example 5.5.24. Consider the preference profile $R_N \in \mathcal{D}^n$ where $\tau(R_1) = [1, 3]$ and $\tau(R_2) = [3, 5]$. Take $x = 5$. Note that $n^5(R_N) = (1, 0, 0)$. Thus, $\phi_5(R_N) = \beta(5, (1, 0, 0)) = 0.4$. Similarly, $n^4(R_N) = (1, 1, 0)$, $n^3(R_N) = (2, 1, 1)$ and $n^2(R_N) = (2, 2, 1)$ (refer to Figure 5.5.1). As ϕ is a 3-plateaued rule , $\phi_{[4,5]}(R_N) = 0.45$, $\phi_{[3,5]}(R_N) = 0.55$, $\phi_{[2,5]}(R_N) = 0.8$ and $\phi_{[1,5]}(R_N) = 1$. Thus $\phi_5(R_N) = 0.4$, $\phi_4(R_N) = 0.05$, $\phi_3(R_N) = 0.1$, $\phi_2(R_N) = 0.25$ and $\phi_1(R_N) = 0.2$. The complete rule is given in Table 5.5.9.* □

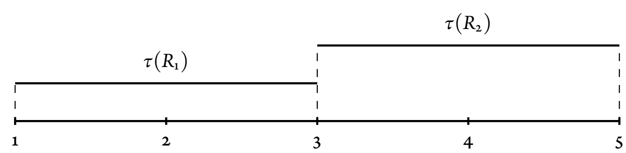


Figure 5.5.1: Single-plateaued preferences for Example 5.5.27

	$[1, 3]$	$[2, 4]$	$[3, 5]$
$[1, 3]$	$(0.3, 0.1, 0.2, 0.1, 0.3)$	$(0.2, 0.25, 0.05, 0.2, 0.3)$	$(0.2, 0.25, 0.1, 0.05, 0.4)$
$[2, 4]$	$(0.2, 0.25, 0.05, 0.2, 0.3)$	$(0.1, 0.3, 0, 0.3, 0.3)$	$(0.1, 0.2, 0.15, 0.15, 0.4)$
$[3, 5]$	$(0.2, 0.25, 0.1, 0.05, 0.4)$	$(0.1, 0.2, 0.15, 0.15, 0.4)$	$(0.1, 0.1, 0.3, 0.05, 0.45)$

Table 5.5.9

Now, we are ready to present the main results of this section. For ease of presentation, we call a (κ, κ)-singe-plateaued domain a κ-single-plateaued domain. Note that if $\kappa = 1$, then a $\mathcal{D}$ domain is the single-peaked domain. Theorem 5.5.28 characterizes all anonymous, plateau-only, and strategy-proof RSCFs on a κ-single-plateaued domain.

Theorem 5.5.28 *Let $\kappa \in \{1, \ldots, m\}$ and $\mathcal{D}$ be a κ-single-plateaued domain. An RSCF $\phi : \mathcal{D}^n \to \Delta A$ is anonymous, plateau-only, and strategy-proof if and only if it is κ-plateaued rule .*

The proof of the theorem is relegated to Appendix .9.

Corollary 5.5.1 *An RSCF is unanimous, anonymous, plateau-only, and strategy-proof if and only if it is a κ-plateaued rule with respect to some unanimous plateau parameters.*

The proof of the corollary is relegated to Appendix .10.

Our next theorem says that a κ-plateaued rule is strategy-proof (together with being anonymous and plateau-only) on any single-plateaued domain such that the size of the plateau for any preference in it is at least κ.

Theorem 5.5.29 *Let $\kappa \in \{1, \ldots, m\}$ and let $\mathcal{D}$ be a $(\kappa, \hat{\kappa})$-single-plateaued domain for some (arbitrary) $\hat{\kappa} \geq \kappa$. Suppose $\phi : \mathcal{D}^n \to \Delta A$ is a κ-plateaued rule . Then, ϕ is anonymous, plateau-only, and strategy-proof.*

The proof of the theorem is relegated to Appendix .11.

5.6 Conclusion

In this paper we study the structure of unanimous (or Pareto optimality) and strategy-proof random social choice functions when weak preferences are admissible. Theorem 5.3.2 shows that a weak random dictatorial RSCF is unanimous and strategy-proof on a domain satisfying the strict extensions property. Theorem 5.4.4 of this paper shows that under some minimal richness condition on the weak single-peaked domain an RSCF is Pareto optimal and strategy-proof if and only if it is an extreme PFBR. An interesting application of this result is the single-peaked domain with outside options (see [18]). Theorem Theorem 5.5.3 shows that on a single-plateaued domain under strategy-proofness, unanimity and Pareto optimality are equivalent for RSCFs. Theorem 5.5.7 shows that any unanimous and strategy-proof RSCF is almost plateau-only.

Next, in Theorems 5.5.15 and 5.5.16, we provide an axiomatic characterization of the unanimous and strategy-proof RSCFs. We show that an RSCF is unanimous and strategy-proof if and only if it satisfies a generalized version of uncompromisingness. Uncompromisingness says that as long as the plateau of an individual stay on one side of an alternative, the probability of that alternative cannot be changed. Generalized uncompromisingness additionally imposes some restriction on how the probability of an alternative can change when the plateau of an individual crosses it.

Finally, we proceed to present a functional form presentation of RSCFs. We provide a functional form characterization of the unanimous and strategy-proof rules on (κ_1, κ_2)-single-plateaued domains for two players. We also provide a class of unanimous and strategy-proof RSCFs on (κ_1, κ_2)-single-plateaued domains for more than two players. In Theorem 5.5.28 we strengthen almost plateau-onlyness by plateau-onlyness and provide a functional form characterization of the plateau-only, anonymous, and strategy-proof RSCFs.

Appendix

.1 Proof of Theorem 5.3.2

Proof: First, we present a lemma that provides a necessary and sufficient condition for weak random dictatorship.

Lemma .1.1 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is weak random dictatorial with coefficients $(\alpha_1, \ldots, \alpha_n)$ if and only if for all $R_N \in \mathcal{D}_N$, and all $B \subseteq A$, we have $\phi_B(R_N) \leq \sum_{\{i \mid B \cap \tau(R_i) \neq \emptyset\}} \alpha_i$.*

Proof: The only if part of the theorem follows from the definition of a weak random dictatorial rule (5.3.1). We prove the if part using a result from [15].

Consider a profile R_N. For each $i \in N$, let $A_i = \tau(R_i)$. Given the collection $(A_i)_{i \in N}$, we define an $m \times n$ dimensional matrix $\mathcal{B}$ in the following manner. The rows of $\mathcal{B}$ are indexed by alternatives and the columns are indexed by agents. The element $\mathcal{B}_{a,i}$ is zero if $a \notin A_i$.

Let R be the row vector indexed by the elements of A defined as $R_a = \phi_a(R_N)$ for all $a \in A$, and S is the column vector indexed by the elements of the set N defined as $S_i = \alpha_i$ for all $i \in N$. Given $\mathcal{B}$ and the vectors R, S, we define the class of matrices $\mathcal{B}(R, S)$ satisfying the conditions that $M \in \mathcal{B}(R, S)$ implies for all $a \in A$ and $i \in N$, (i) $M_{a,i} \geq 0$, (ii) $\mathcal{B}_{a,i} = 0$ implies $M_{a,i} = 0$, and (iii) $\sum_{a \in A} M_{a,i} = \phi_a(R_N)$ and $\sum_{i \in N} M_{a,i} = \alpha_i$. Note that the existence of a weak random dictatorial with co-efficients $(\alpha_1, \ldots, \alpha_n)$ is equivalent to the existence of a matrix M in $\mathcal{B}(R, S)$ as $M_{a,i}$ will serve as the value of $\lambda(a, i)$ for all $a \in A$ and $i \in N$.

In [15] (see Theorem 2.1), it shown that given $\mathcal{B}$ that is not decomposable[2], a matrix $\mathcal{B}(R, S)$ exists if and only if whenever the rows and columns of $\mathcal{B}$ can be permuted to the form

$$\begin{bmatrix} \mathcal{B}_1 & 0 \\ \mathcal{B}_2 & \mathcal{B}_3 \end{bmatrix} \tag{1}$$

where $\mathcal{B}_1$ is a non-vacuous $0, 1$-matrix formed from rows $i_1, \ldots, i_p$ and columns $j_1, \ldots, j_q$ of $\mathcal{B}$ and $\mathcal{B}_3$ is non-vacuous, then

$$r_{i1} + \cdots + r_{ip} < s_{j1} + \cdots + s_{jq}. \tag{2}$$

Consider a set of alternatives $B \subseteq A$ and the set of agents $i \in N$ such that $B \cap A_i \neq \emptyset$. Recall that by the restriction on $\mathcal{B}$, we have $\mathcal{B}_{a,i} = 0$ if $a \notin A_i$. Therefore, each set of alternatives B partitions the matrix

[2] The matrix $\mathcal{B}$ is decomposable if it can be written as $\begin{bmatrix} \mathcal{B}_1 & 0 \\ 0 & \mathcal{B}_3 \end{bmatrix}$.

(through some permutation of rows and columns) $\mathcal{B}$ in the following way: $\begin{bmatrix} \mathcal{B}_1 & 0 \\ \mathcal{B}_2 & \mathcal{B}_3 \end{bmatrix}$. So, by Theorem 2.1 of [15], $r_{i1} + \cdots + r_{ip} < s_{j1} + \cdots + s_{jq}$ implies that there exists a matrix in $\mathcal{B}(R, S)$. The proof is completed by showing that when $r_{i1} + \cdots + r_{ip} = s_{j1} + \cdots + s_{jq}$, then the matrix $\mathcal{B}$ is decomposable.[3] This follows from the fact that the condition $r_{i1} + \cdots + r_{ip} = s_{j1} + \cdots + s_{jq}$ implies $\mathcal{B}_{a,i} = 0$ for all $a \notin B$ and $i \in N$, and hence the matrix $\mathcal{B}_2 = 0$. ■

In view of Lemma .1.1, it is enough to show that for all $R_N \in \mathcal{D}_N$, and all $B \subseteq A$, we have $\phi_B(R_N) \leq \sum_{\{i \mid B \cap \tau(R_i) \neq \emptyset\}} a_i$. For all $R_i \in \mathcal{D}_i$ let $\hat{P}_i$ be a strict extension of R_i such that for all $b \in B$ and $c \in A \setminus B$, bI_ic implies $b\hat{P}_ic$.

Claim .1.1 $\phi_B(\hat{P}_i, R_{N\setminus i}) \geq \phi_B(R_i, R_{N\setminus i})$.

Proof: Consider agent i. Let $\{B_1, \ldots, B_k\}$ be a partition of B such that for all $l \in \{1, \ldots, k\}$ and all $b, b' \in B_l$, bI_ib', and $B_1P_i \cdots P_iB_k$. Let $l \in \{1, \ldots, k\}$. Consider the sets of alternatives $C_l = \{a \in A \mid aP_iB_l\}$ and $\bar{C}_l = C_l \cup B_l$. Note that C_l i an upper contour set in both R_i and $\hat{P}_i$, and $\bar{C}_l$ is an upper contour set in $\hat{P}_i$. Since C_l is an upper contour set in both R_i and $\hat{P}_i$, we have by strategy-proofness

$$\phi_{C_l}(\hat{P}_i, R_{N\setminus i}) = \phi_{C_l}(R_i, R_{N\setminus i}). \tag{3}$$

Again since $\bar{C}_l$ is an upper contour set in $\hat{P}_i$, we have by strategy-proofness

$$\phi_{\bar{C}_l}(\hat{P}_i, R_{N\setminus i}) \geq \phi_{\bar{C}_l}(R_i, R_{N\setminus i}). \tag{4}$$

Subtracting (3) from (4), we obtain

$$\phi_{B_l}(\hat{P}_i, R_{N\setminus i}) \geq \phi_{B_l}(R_i, R_{N\setminus i}). \tag{5}$$

Since $\phi_B(\hat{P}_i, R_{N\setminus i}) = \sum_{l \in \{1,\ldots,k\}} \phi_{B_l}(\hat{P}_i, R_{N\setminus i})$ and $\phi_B(R_i, R_{N\setminus i}) = \sum_{l \in \{1,\ldots,k\}} \phi_{B_l}(R_i, R_{N\setminus i})$, by (5) it follows that $\phi_B(\hat{P}_i, R_{N\setminus i}) \geq \phi_B(R_i, R_{N\setminus i})$. This completes the proof of the claim. ■

By applying Claim .1.1 for all $i \in N$, we obtain $\phi_B(\hat{P}_N) \geq \phi_B(R_N)$. ■

.2 Proof of Theorem 5.4.4

Proof: ("If" part) We show that an extreme PFBR is Pareto optimal and strategy-proof. Since an alternative receives positive probability in an extreme PFBR at a profile if and only if the alternative is

[3] If the matrix $\mathcal{B}$ is decomposable, then the problem reduces to a lower dimensional problem.

top-ranked by at least one agent, it follows that the outcome of an extreme PFBR at any profile cannot be Pareto dominated, which implies that such a rule is Pareto optimal. Let ϕ be an extreme PFBR. To show strategy-proofness, let us assume for contradiction that $\sum_{y\in U(x,R_i)} \phi_y(R'_i, R_{N\setminus i}) > \sum_{y\in U(x,R_i)} \phi_y(R_i, R_{N\setminus i})$ for some R_N and R'_i. Let $\widehat{P}_N \in \text{strict}(\mathcal{D}_N)$ and $P_i \in \text{strict}(\mathcal{D}_i)$ be such that $\tau(\widehat{P}_i) = \tau(R_i)$ for all $i \in N$, $\tau(P_i) = \tau(R'_i)$, and $U(x, R_i)$ is an upper contour set of $\widehat{P}_i$. By the definition of an extreme PFBR, we have $\phi(\widehat{P}_N) = \phi(R_N)$ and $\phi(P_i, \widehat{P}_{N\setminus i}) = \phi(R'_i, R_{N\setminus i})$. This implies $\sum_{y\in U(x,R_i)} \phi_y(P_i, \widehat{P}_{N\setminus i}) > \sum_{y\in U(x,R_i)} \phi_y(\widehat{P}_N)$, which in turn means that the restriction of ϕ on strict$(\mathcal{D}_N)$ is manipulable. However, this is a contradiction since the restriction of ϕ on strict$(\mathcal{D}_N)$ is a PFBR which is known to be strategy-proof (see [37]).

("Only-if" part) A profile $R_N \in \mathcal{D}_N$ is called a boundary profile if $\tau(R_i) \in \{1, m\}$ for all $i \in N$, that is, the top-ranked alternative of any preference in such a profile lies on the boundary 1 or m of the set of alternatives. We make extensive use of two particular types of strict single-peaked preferences in our proofs: a single-peaked preference P is called left (or right) if for all $x < \tau(P)$ and all $y > \tau(P)$, we have xPy (or yPx).

Lemma .2.1 *Let $i \in N$, $P_i \in$ strict$(\mathcal{D}_i)$, and $R_{N\setminus i} \in \mathcal{D}_{N\setminus i}$. Suppose $r < \tau(P_i) < s$ are such that sP_ir. Then,*

(i) there exists $\bar{P}_i \in \mathcal{D}_i$ with $\bar{P}_i \equiv \tau(P_i)\cdots sr\cdots$ such that $\phi_s(P_i, R_{N\setminus i}) \geq \phi_s(\bar{P}_i, R_{N\setminus i})$, and

(ii) $\bar{\bar{P}}_i \in \mathcal{D}_i$ with $\bar{\bar{P}}_i \equiv \tau(P_i)\cdots rs\cdots$ such that $\phi_s(P_i, R_{N\setminus i}) > \phi_s(\bar{\bar{P}}_i, R_{N\setminus i})$ implies $\phi_k(P_i, R_{N\setminus i}) < \phi_k(\bar{\bar{P}}_i, R_{N\setminus i})$ for some $k \in [r, \tau(P_i))$.

Proof: By the definition of single-peakedness, if sP_iaP_ir for some $a \in A$, then either $a > s$ or $a \in (r, \tau(P_i))$. Consider the strict single-peaked preference $\hat{P}_i \in \text{strict}(\mathcal{D}_i)$ such that $\tau(\hat{P}_i) = \tau(P_i)$, $U(s, P_i) = U(s, \hat{P}_i)$ and $s\hat{P}_ia\hat{P}_ir$ for some $a \in A$ if and only if $a \in (r, \tau(P_i)) \setminus U(s, P_i)$. Existence of such a preference is guaranteed by minimal richness. By strategy-proofness, $\phi_s(P_i, R_{N\setminus i}) = \phi_s(\hat{P}_i, R_{N\setminus i})$. By the definition of $\hat{P}_i$, there exists $l \geq 0$ such that $\hat{P}_i \equiv \cdots s(r+l)(r+l-1)\cdots(r+1)r\cdots$. Let $\hat{\hat{P}}_i$ be obtained by swapping the alternatives s and $(r+l)$ at $\hat{P}_i$. Thus, $\hat{\hat{P}}_i \equiv \cdots(r+l)s(r+l-1)\cdots(r+1)r\cdots$. Note that $\hat{\hat{P}}_i$ is strict single-peaked. By straightforward application of strategy-proofness, $\phi_s(\hat{P}_i, R_{N\setminus i}) \geq \phi_s(\hat{\hat{P}}_i, R_{N\setminus i})$. Continuing in this manner, we can arrive at a preference $\bar{P}_i$ such that $\bar{P}_i \equiv \tau(P_i)\cdots sr\cdots$ and $\phi_s(P_i, R_{N\setminus i}) \geq \phi_s(\bar{P}_i, R_{N\setminus i})$. This completes the proof of part (i) of the lemma.

Let $\bar{\bar{P}}_i$ be the strict single-peaked preference obtained by swapping s and r at $\bar{P}_i$. Since $\phi_s(P_i, R_{N\setminus i}) = \phi_s(\hat{P}_i, R_{N\setminus i})$ and we have arrived at the preference $\bar{\bar{P}}_i$ from $\hat{P}_i$ by a sequence of swaps between s and some alternatives in the set $\{r, \ldots, r+l\}$, if $\phi_s(\bar{\bar{P}}, R_{N\setminus i}) < \phi_s(P_i, R_{N\setminus i})$, then there must exist some $a \in \{r, \ldots, r+l\}$ such that $\phi_a(\bar{P}_i, R_{N\setminus i}) > \phi_a(P_i, R_{N\setminus i})$. This completes the proof of part (ii) of the lemma. ■

We prove the "only-if" part of the theorem in two steps. In the first step, we show that every Pareto optimal and strategy-proof RSCF on the domain strict$(\mathcal{D}_N)$ behaves like an extreme PFBR on the set of boundary profiles. In the next step, we show that the same happens on every profile.

Step 1. Let ϕ be a Pareto optimal and strategy-proof RSCF. We show that ϕ is an extreme PFBR. The following claim says that only the boundary alternatives 1 and m can get positive probability at boundary profiles.

Claim .2.1 *$\phi_x(R_N) = 0$ for all $x \in \{2, \ldots, m-1\}$ and all boundary profiles $R_N \in \mathcal{D}_N$.*

Proof of Claim. Assume for contradiction $\phi_x(R_N) > 0$ for some $x \in \{2, \ldots, m-1\}$ and for some boundary profile $R_N \in \mathcal{D}_N$. For each $R_i \in \mathcal{D}_i$, let $\bar{R}_i$ be the dichotomous preference with $\tau(\bar{R}_i) = \tau(R_i)$. Note that such preferences exist as the domain is minimally rich. By Pareto optimality, $\phi_x(\bar{R}_N) = 0$ for all $x \in \{2, \ldots, m-1\}$. For all $x \in \{1, m\}$, let N_x be the set of agents i whose top ranked alternative at $\bar{R}_i$ is x, that is $N_x = \{i \in N \mid \tau(\bar{R}_i) = x\}$. Take $i \in N$. Consider the profile $(R_i, \bar{R}_{N \setminus i})$. By strategy-proofness, $\phi_1(R_i, \bar{R}_{N\setminus i}) = \phi_1(\bar{R}_i, \bar{R}_{N\setminus i})$. Also, by Pareto optimality, $\phi_x(R_i, \bar{R}_{N\setminus i}) = 0$ for all $x \in \{2, \ldots, m-1\}$. This is because if $\phi_x(R_i, \bar{R}_{N\setminus i}) > 0$ for some $x \in \{2, \ldots, m-1\}$, then shifting this probability to 1 will be a Pareto improvement. Thus, $\phi(R_i, \bar{R}_{N\setminus i}) = \phi(\bar{R}_i, \bar{R}_{N\setminus i})$. Applying this logic repeatedly for all agents in N_1, we obtain $\phi(R_{N_1}, \bar{R}_{N\setminus N_1}) = \phi(\bar{R}_{N_1}, \bar{R}_{N\setminus N_1})$. Now, consider $i \in N_m$. Since $\tau(\bar{R}_i) = \tau(R_i) = m$, by strategy-proofness, $\phi_m(R_i, R_{N_1}, \bar{R}_{N\setminus N_1 \cup i}) = \phi_m(\bar{R}_i, R_{N_1}, \bar{R}_{N\setminus N_1\cup i})$. This, combined with the fact that $\phi(R_{N_1}, \bar{R}_{N\setminus N_1}) = \phi(\bar{R}_{N_1}, \bar{R}_{N\setminus N_1})$, yields

$$\phi_m(R_i, R_{N_1}, \bar{R}_{N\setminus N_1\cup i}) = \phi_m(\bar{R}_N). \tag{6}$$

Applying this argument for all agents in $N_m \setminus \{i\}$, we obtain $\phi_m(R_N) = \phi_m(\bar{R}_N)$. Since $\phi_x(R_N) > 0$ for some $x \in \{2, \ldots, m-1\}$, this implies

$$\phi_1(R_N) < \phi_1(\bar{R}_N). \tag{7}$$

Note that we can arrive at the profile R_N in a symmetrically opposite way: by changing the preferences of agents in N_m from $\bar{R}_i$ to R_i first, and then changing the preferences of agents in N_1 from $\bar{R}_i$ to R_i. Therefore, by using the same argument as for obtaining (6), we can conclude that $\phi_1(R_N) = \phi_1(\bar{R}_N)$, which contradicts (7). □

To prove that ϕ is an extreme PFBR, it remains to show that ϕ is monotonic over the boundary profiles, that is, for all $R_N, R'_N \in \mathcal{D}_N$ with $S(m, R_N) \subseteq S(m, R'_N)$, we have $\phi_m(R_N) \leq \phi_m(R'_N)$. This follows by straightforward application of strategy-proofness.

Step 2. Let strict$(\mathcal{D}_i) \subset \mathcal{D}_i$ be the strict single-peaked domain contained in $\mathcal{D}_i$. Since ϕ is Pareto optimal and strategy-proof, it must be a PFBR on strict$(\mathcal{D}_N)$. By Step 1, it follows that ϕ is an extreme PFBR. It is sufficient to show that ϕ is tops-only on $\mathcal{D}_N$. We prove this by using induction on the number of agents in a preference profile having non-strict preferences. We begin with the base case where there exists exactly one agent having non-strict preference.

Base case: Let $P_N \in$ strict$(\mathcal{D}_N)$ be a strict single-peaked preference profile and let R_i be a non-strict preference with $\tau(R_i) = \tau(P_i)$. We show that $\phi(P_N) = \phi(R_i, P_{N\setminus i})$. Assume for contradiction that $\phi(P_N) \neq \phi(R_i, P_{N\setminus i})$. Without loss of generality, let $s > \tau(P_i)$ be such that $\phi_s(R_i, P_{N\setminus i}) > \phi_s(P_i, P_{N\setminus i})$ and $\phi_t(R_i, P_{N\setminus i}) \leq \phi_t(P_i, P_{N\setminus i})$ for all $t \in [\tau(P_i), s)$.

Case B1. Suppose that there is no agent j such that $\tau(P_j) \in (\tau(P_i), s)$.

Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(\hat{P}_i) = \tau(R_i) = \tau(P_i)$. Since $(\hat{P}_i, P_{N\setminus i}) \in$ strict$(\mathcal{D}_N)$, by only-topsness $\phi_t(\hat{P}_i, P_{N\setminus i}) = 0$ for all $t \in (\tau(P_i), s)$. By strategy-proofness $\phi_{U(s,\hat{P}_i)}(\hat{P}_i, P_{N\setminus i}) \geq \phi_{U(s,\hat{P}_i)}(R_i, P_{N\setminus i})$ and $\phi_{\tau(P_i)}(\hat{P}_i, P_{N\setminus i}) = \phi_{\tau(P_i)}(R_i, P_{N\setminus i})$. Combining all these observations, it follows that

$$\phi_s(\hat{P}_i, P_{N\setminus i}) \geq \phi_s(R_i, P_{N\setminus i}). \tag{8}$$

Recall that

$$\phi_s(R_i, P_{N\setminus i}) > \phi_s(P_i, P_{N\setminus i}). \tag{9}$$

By (8) and (9) this implies

$$\phi_s(\hat{P}_i, P_{N\setminus i}) > \phi_s(P_i, P_{N\setminus i}). \tag{10}$$

Since $\tau(P_i) = \tau(\hat{P}_i)$, we have $\phi_s(P_i, P_{N\setminus i}) = \phi_s(\hat{P}_i, P_{N\setminus i})$, which contradicts (10).

Case B2. Suppose that Case 1 does not hold, that is, there are agents j such that $\tau(P_j) \in (\tau(P_i), s)$.

Consider $j \in N$ such that $\tau(P_j) \in (\tau(P_i), s)$ and there does not exist $l \in N$ such that $\tau(P_l) \in (\tau(P_i), \tau(P_j))$

Claim .2.2 *$\phi_s(R_i, \hat{P}_j, P_{N\setminus\{i,j\}}) = \phi_s(R_i, P_j, P_{N\setminus\{i,j\}})$ for some $\hat{P}_j \in$ strict$(\mathcal{D}_j)$ with $\tau(\hat{P}_j) = \tau(P_i)$.*

Proof of Claim .2.2

Case B2.1. Suppose $\tau(P_i)P_js$.

Since $\tau(P_i)P_js$, there exists $\hat{P}_j$ with $\tau(\hat{P}_j)$ such that $U(s, \hat{P}_j) = U(s, P_j)$. By strategy-proofness,

$\phi_s(R_i, \hat{P}_j, P_{N\setminus\{i,j\}}) = \phi_s(R_i, P_j, P_{N\setminus\{i,j\}})$.

Case B2.2. Suppose $sP_j\tau(P_i)$.

In view of Case 1 it is sufficient to show $\phi_s(R_i, P_j, P_{N\setminus\{i,j\}}) = \phi_s(R_i, \bar{P}_j, P_{N\setminus\{i,j\}})$ for some $\bar{P}_j$ with $\tau(\bar{P}_j) = \tau(P_j)$ and $\tau(P_i)\bar{P}_js$.

Since $\tau(P_i) < \tau(P_j) < s$, by Lemma .2.1, we can construct a preference $\tilde{P}_j \equiv \cdots s\tau(P_i)\cdots$ such that

$$\phi_s(R_i, P_j, P_{N\setminus\{i,j\}}) \geq \phi_s(R_i, \tilde{P}_j, P_{N\setminus\{i,j\}}). \tag{11}$$

Construct a preference $\tilde{\tilde{P}}_j$ by swapping s and $\tau(P_i)$ in $\tilde{P}_j$. By strategy-proofness,

$$\phi_s(R_i, \tilde{P}_j, P_{N\setminus\{i,j\}}) \geq \phi_s(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}}). \tag{12}$$

Combining (3) and (4), $\phi_s(R_i, P_j, P_{N\setminus\{i,j\}}) \geq \phi_s(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}})$. If $\phi_s(R_i, P_j, P_{N\setminus\{i,j\}}) = \phi_s(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}})$, then we take $\bar{P}_j = \tilde{\tilde{P}}_j$. Suppose $\phi_s(R_i, P_j, P_{N\setminus\{i,j\}}) > \phi_s(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}})$. It follows from Lemma .2.1 that there exists $r \in [\tau(P_i), \tau(P_j))$ such that $\phi_r(R_i, P_j, P_{N\setminus\{i,j\}}) < \phi_r(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}})$. Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(P_i) = \tau(\hat{P}_i)$. Since $(\hat{P}_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}}) \in \text{strict}(\mathcal{D}_N)$, by only-topsness and our assumption that there is no $l \in N$ such that $\tau(P_l) \in (\tau(P_i), \tau(P_j))$, we have $\phi_t(\hat{P}_i, \tilde{\tilde{P}}, P_{N\setminus\{i,j\}}) = 0$ for all $t \in (\tau(P_i), \tau(P_j))$. Consider $U(r, \hat{P}_i)$. By strategy-proofness, $\phi_{U(r,\hat{P}_i)}(\hat{P}_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}}) \geq \phi_{U(r,\hat{P}_i)}(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}})$. Because $\phi_r(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}}) > 0$ and $\phi_t(\hat{P}_i, \tilde{\tilde{P}}, P_{N\setminus\{i,j\}}) = 0$ for all $t \in U(r, \hat{P}_i) \setminus \{\tau(P_i)\}$, this implies $\phi_{\tau(P_i)}(\hat{P}_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}}) > \phi_{\tau(P_i)}(R_i, \tilde{\tilde{P}}_j, P_{N\setminus\{i,j\}})$. Since $\tau(P_i) = \tau(\hat{P}_i) = \tau(R_i)$, this is a contradiction to strategy-proofness. □

Consider the profile $(R_i, \hat{P}_j, P_{N\setminus\{i,j\}})$. Let $l \in N$ be such that $\tau(P_l) \in (\tau(P_i), s)$ and $\tau(P_{\hat{l}}) \notin (\tau(P_i), \tau(P_l))$ for all $\hat{l} \in N$. By using similar logic as for Claim .2.2, we can construct a preference $\hat{P}_l$ such that $\tau(\hat{P}_l) = \tau(P_l)$ and $\phi_s(R_i, \hat{P}_j, P_{N\setminus\{i,j\}}) = \phi_s(R_i, \hat{P}_j, \hat{P}_l, P_{N\setminus\{i,j,l\}})$. Continuing in this manner, we can construct a profile $(R_i, \hat{P}_{N\setminus i})$ such that $\tau(\hat{P}_j) = \tau(P_i)$ if $\tau(P_j) \in (\tau(P_i), s)$ and $\hat{P}_j = P_j$ if $\tau(P_j) \notin (\tau(P_i), s)$ and

$$\phi_s(R_i, \hat{P}_{N\setminus i}) = \phi_s(R_i, P_{N\setminus i}) \tag{13}$$

Note that by construction of $\hat{P}_{N\setminus i}$, there is no agent j such that $\tau(\hat{P}_j) \in (\tau(P_i), s)$. Thus, by applying the same logic as in Case B1, $\phi_s(R_i, \hat{P}_{N\setminus i}) = \phi_s(P_i, P_{N\setminus i})$. Combining this with (5), we have $\phi_s(P_i, P_{N\setminus i}) = \phi_s(R_i, P_{N\setminus i})$.

Induction step: Suppose that ϕ behaves like an extreme PFBR over all profiles at which at most k agents have non-strict preferences. We proceed to show that the same holds over all profiles where $k+1$ agents have non-strict preferences.

Consider $(R_S, P_{N\setminus S}) \in \mathcal{D}_N$ such that R_i is not strict for all $i \in S$, P_i is strict for all $i \in N \setminus S$ and $|S| = k+1$. Assume without loss of generality, agent $1 \in S$. Let P_1 be a strict preference with $\tau(P_1) = \tau(R_1)$. In view of our induction hypothesis, it is enough to show, $\phi(R_S, P_{N\setminus S}) = \phi(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Clearly, by strategy-proofness, $\phi_{\tau(R_1)}(R_S, P_{N\setminus S}) = \phi_{\tau(R_1)}(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Consider $l \in A$ such that

$\tau(R_i) = l$ for some $i \in S \setminus 1$. We show that $\phi_l(R_S, P_{N\setminus S}) = \phi_l(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Let P_i be a strict preference with $\tau(P_i) = \tau(R_i)$. By strategy-proofness,

$$\phi_l(P_i, R_{S\setminus i}, P_{N\setminus S}) = \phi_l(R_S, P_{N\setminus S}). \tag{14}$$

Since both the profiles $(P_i, R_{S\setminus i}, P_{N\setminus S})$ and $(P_1, R_{S\setminus 1}, P_{N\setminus S})$ have exactly k agents with non-strict preferences, by our induction hypothesis,

$$\phi_l(P_i, R_{S\setminus i}, P_{N\setminus S}) = \phi_l(P_1, R_{S\setminus 1}, P_{N\setminus S}). \tag{15}$$

Combining (14) and (15), we have $\phi_l(R_S, P_{N\setminus S}) = \phi_l(P_1, R_{S\setminus 1}, P_{N\setminus S})$.

Next, we proceed to show $\phi(R_S, P_{N\setminus S}) = \phi(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Assume for contradiction that $\phi(R_S, P_{N\setminus S}) \neq \phi(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Without loss of generality $s > \tau(P_1)$ be such that $\phi_s(R_S, P_{N\setminus S}) > \phi_s(P_1, R_{S\setminus 1} P_{N\setminus S})$ and $\phi_t(R_S, P_{N\setminus S}) \leq \phi_t(P_1, R_{S\setminus 1} P_{N\setminus S})$ for all $t \in (\tau(P_1), s)$. Let $i \in S$ be such that there does not exist $l \in S$ such that $\tau(R_l) \in (\tau(R_i), s]$. The rest of the proof follows by using similar logic as for the base case, but for the sake of completeness, we present it.

Case I1. Suppose there is no agent j such that $\tau(R_j) \in (\tau(R_i), s)$.

Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(\hat{P}_i) = \tau(R_i) = \tau(P_i)$. Since $(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S})$ has exactly k agents with non-strict preferences, by our induction hypothesis $\phi_t(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) = 0$ for all $t \in (\tau(P_i), s)$. By strategy-proofness $\phi_{U(s,\hat{P}_i)}(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) \geq \phi_{U(s,\hat{P}_i)}(R_S, P_{N\setminus S})$ and $\phi_{\tau(P_i)}(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) = \phi_{\tau(P_i)}(R_S, P_{N\setminus S})$. Combining all these observations,

$$\phi_s(\hat{P}_i, R_{S\setminus i}, P_{N\setminus i}) \geq \phi_s(R_S, P_{N\setminus S}). \tag{16}$$

Recall that

$$\phi_s(R_S, P_{N\setminus S}) > \phi_s(P_1, R_{S\setminus 1}, P_{N\setminus S}). \tag{17}$$

By (16) and (17) this implies

$$\phi_s(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) > \phi_s(P_1, R_{S\setminus 1}, P_{N\setminus S}). \tag{18}$$

Since at the profiles $(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S})$ and $(P_1, R_{S\setminus 1}, P_{N\setminus S})$, $\phi_s(\hat{P}_i, R_{S\setminus i}, P_{N\setminus S}) = \phi_s(P_1, R_{S\setminus 1}, P_{N\setminus S})$, which contradicts (18).

Case I2. Suppose that Case 1 does not hold, that is, there are agents j such that $\tau(P_j) \in (\tau(P_i), s)$.

Consider $j \in N$ such that $\tau(P_j) \in (\tau(P_i), s)$ and there does not exist $l \in N$ such that $\tau(P_l) \in (\tau(P_i), \tau(P_j))$.

Claim .2.3 $\phi_s(R_S, \hat{P}_j, P_{N\setminus S\cup j}) = \phi_s(R_S, P_j, P_{N\setminus S\cup j})$ *for some* $\hat{P}_j \in$ *strict*$(\mathcal{D}_j)$ *with* $\tau(\hat{P}_j) = \tau(P_i)$.

Proof of Claim .2.3

Case I2.1. Suppose $\tau(P_i)P_js$.

Since $\tau(P_i)P_js$, there exists $\hat{P}_j$ with $\tau(\hat{P}_j)$ such that $U(s, \hat{P}_j) = U(s, P_j)$. By strategy-proofness, $\phi_s(R_S, \hat{P}_j, P_{N\setminus S\cup j}) = \phi_s(R_S, P_j, P_{N\setminus S\cup j})$.

Case I2.2. Suppose $sP_j\tau(P_i)$.

In view of Case 1 it is sufficient to show $\phi_s(R_S, P_j, P_{N\setminus S\cup j}) = \phi_s(R_S, \bar{P}_j, P_{N\setminus S\cup j})$ for some $\bar{P}_j$ with $\tau(\bar{P}_j) = \tau(P_j)$ and $\tau(P_i)\bar{P}_js$.

Since $\tau(P_i) < \tau(P_j) < s$, by Lemma .2.1, we can construct a preference $\tilde{P}_j \equiv \cdots s\tau(P_i)\cdots$ such that

$$\phi_s(R_S, P_j, P_{N\setminus S\cup j}) \geq \phi_s(R_S, \tilde{P}_j, P_{N\setminus S\cup j}). \tag{19}$$

Construct a preference $\tilde{\tilde{P}}_j$ by swapping s and $\tau(P_i)$ in $\tilde{P}_j$. By strategy-proofness,

$$\phi_s(R_S, \tilde{P}_j, P_{N\setminus S\cup j}) \geq \phi_s(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j}). \tag{20}$$

Combining (19) and (20), $\phi_s(R_S, P_j, P_{N\setminus S\cup\{j\}}) \geq \phi_s(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup\{j\}})$. If $\phi_s(R_S, P_j, P_{N\setminus S\cup j}) = \phi_s(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j})$, then we take $\bar{P}_j = \tilde{\tilde{P}}_j$. Suppose $\phi_s(R_S, P_j, P_{N\setminus S\cup j}) > \phi_s(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j})$. It follows from Lemma .2.1 that there exists $r \in [\tau(P_i), \tau(P_j))$ such that $\phi_r(R_S, P_j, P_{N\setminus S\cup j}) < \phi_r(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j})$. Let $\hat{P}_i$ be the right strict single-peaked preference with $\tau(P_i) = \tau(\hat{P}_i)$. Since $(\hat{P}_i, R_{S\setminus i}, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j})$ has exactly k agents with non-strict preferences, by our induction hypothesis and assumption that there is no $l \in N$ such that $\tau(P_l) \in (\tau(P_i), \tau(P_j))$, we have $\phi_t(\hat{P}_i, R_{S\setminus i}, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j}) = 0$ for all $t \in (\tau(P_i), \tau(P_j))$. Consider $U(r, \hat{P}_i)$. By strategy-proofness, $\phi_{U(r,\hat{P}_i)}(\hat{P}_i, R_{S\setminus\{i\}}, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j}) \geq \phi_{U(r,\hat{P}_i)}(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j})$. Because $\phi_r(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j}) > 0$ and $\phi_t(\hat{P}_i, R_{S\setminus i}\tilde{\tilde{P}}, P_{N\setminus S\cup j}) = 0$ for all $t \in U(r, \hat{P}_i) \setminus \{\tau(P_i)\}$, this implies $\phi_{\tau(P_i)}(\hat{P}_i, R_{S\setminus i}, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j}) > \phi_{\tau(P_i)}(R_S, \tilde{\tilde{P}}_j, P_{N\setminus S\cup j})$. Since $\tau(P_i) = \tau(\hat{P}_i) = \tau(R_i)$, this is a contradiction to strategy-proofness. □

Consider the profile $(R_S, \hat{P}_j, P_{N\setminus S\cup j})$. Let $l \in N$ be such that $\tau(P_l) \in (\tau(P_i), s)$ and $\tau(P_{\hat{l}}) \notin (\tau(P_i), \tau(P_l))$ for all $\hat{l} \in N$. By using similar logic as for Claim .2.3, we can construct a preference $\hat{P}_l$ such that $\tau(\hat{P}_l) = \tau(P_l)$ and $\phi_s(R_S, \hat{P}_j, P_{N\setminus S\cup j}) = \phi_s(R_S, \hat{P}_j, \hat{P}_l, P_{N\setminus S\cup\{j,l\}})$. Continuing in this manner, we can construct a profile $(R_S, \hat{P}_{N\setminus S})$ such that $\tau(\hat{P}_j) = \tau(P_i)$ if $\tau(P_j) \in (\tau(P_i), s)$ and $\hat{P}_j = P_j$ if $\tau(P_j) \notin (\tau(P_i), s)$ and

$$\phi_s(R_S, \hat{P}_{N\setminus S}) = \phi_s(R_S, P_{N\setminus S}). \tag{21}$$

Note that by construction of $(R_S, \hat{P}_{N\setminus S})$, there is no agent j such that $\tau(\hat{R}_j) \in (\tau(P_i), s)$. Thus, by applying the same logic as in Case I1, $\phi_s(R_S, \hat{P}_{N\setminus S}) = \phi_s(P_1, R_{S\setminus 1}, P_{N\setminus S})$. Combining this with (21), we have $\phi_s(R_S, \hat{P}_{N\setminus S}) = \phi_s(P_1, R_{S\setminus 1}, P_{N\setminus S})$. ■

.3 Proof of Theorem 5.5.3

We make extensive use of two particular types of single-plateaued preferences in our proofs: a single-plateaued preference R is called left (or right) if for all $x < \tau^-(R)$ and all $y > \tau^+(R)$, we have xPy (or yPx).

Lemma .3.1 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ satisfies Pareto optimality if and only if it is unanimous and for all $R_N \in \mathcal{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$, we have $\phi_{I(R_N)}(R_N) = 1$, where $I(R_N)$ is the minimal interval such that $I(R_N) \cap \tau(R_i) \neq \emptyset$ for all $i \in N$.*

Proof:[Proof of Lemma .3.1] The proof of this lemma is somewhat straightforward. However, for the sake of completeness, we provide it here.

(If part) Suppose ϕ satisfies Pareto optimality. Then, it is straightforwardly unanimous. Take $R_N \in \mathcal{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$. We show that $\phi_{I(R_N)}(R_N) = 1$. It is sufficient to show $\phi_x(R_N) = 0$ for all $x \notin I(R_N)$. Take $x \notin I(R_N)$. Let $I(R_N) = [y, z]$. Assume without loss of generality $x < y$. Since $I(R_N) = [y, z]$, there exists $i \in N$ such that $\tau^+(R_i) = y$. Since R_N is not unanimous, there exists $j \in N$ such that $\tau^-(R_j) > y$. This means yP_jx. Moreover, since $I(R_N) = [y, z]$, $\tau^+(R_i) \geq y$ for all $i \in N$. This means yR_ix for all $i \in N$. Combining, we have y Pareto dominates x. So, $\phi_x(R_N) = 0$.
(Only-if part) Suppose an RSCF $\phi : \mathcal{D}^n \to \Delta A$ is unanimous and satisfies the property that for all $R_N \in \mathcal{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$, $\phi_{I(R_N)}(R_N) = 1$. We show that ϕ satisfies Pareto optimality. Take $R_N \in \mathcal{D}^n$. If R_N is a unanimous profile, then there is nothing to show. Suppose R_N is not unanimous. Take $x \in I(R_N)$. We show that there does not exist $y \in A$ such that yR_ix for all $i \in N$ and yP_jx for some $j \in N$. Assume for contradiction that there is such an alternative $y \in A$. Because $x \in I(R_N)$, there exist $i, j \in N$ such that $\tau^+(R_i) \leq x$ and $\tau^-(R_j) \geq x$. So, if $y > x$, then xP_iy. On the other hand, if $y < x$, then xP_jy. So, y cannot Pareto dominate x. ■

Proof: [Proof of Theorem 5.5.3] "Only if" part of the theorem is straightforward, we proceed to prove the "if" part. Let $\mathcal{D}$ be the (κ_1, κ_2)-single-plateaued domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. We show that ϕ is Pareto optimal. Take $R_N \in \mathcal{D}^n$ such that $\cap_{i\in N}\tau(R_i) = \emptyset$. In view of Lemma .3.1, it is sufficient to show that $\phi_{I(R_N)}(R_N) = 1$. Suppose not. Let $I(R_N) = [x, y]$. Assume without loss of generality, $\phi_{[1,x-1]}(R_N) > 0$. Take $i \in N$ such that $\tau^-(R_i) > x$. There must exist such an agent since R_N is not a unanimous profile. Let $R_i' \in \mathcal{D}$ be a left-single-plateaued preference such that $\tau^-(R_i') = x$ and $|\tau(R_i')| = \kappa_1$.

Claim. $\phi_{[1,x-1]}(R_i', R_{N\setminus i}) \geq \phi_{[1,x-1]}(R_N)$.

Proof of the claim. Note that since R_i' is a left-single-plateaued preference, for all $w \geq \tau^+(R_i')$ the interval $[1, w]$ is an upper contour set in R_i'. So, by strategy-proofness, $\phi_{[1,w]}(R_i', R_{N\setminus i}) \geq \phi_{[1,w]}(R_N)$, as otherwise agent i manipulates at $(R_i', R_{N\setminus i})$ via R_i. Now, assume for contradiction that $\phi_{[1,x-1]}(R_i', R_{N\setminus i}) < \phi_{[1,x-1]}(R_N)$. This implies for all $w \geq \tau(R_i')$,

$$\phi_{[x,w]}(R_i', R_{N\setminus i}) > \phi_{[x,w]}(R_N). \tag{22}$$

Consider the upper contour set $U(x, R_i)$. Since R_i is single-plateaued, there must be an alternative $z \geq \tau^+(R_i)$ such that $U(x, R_i) = [x, \tau^+(R_i)) \cup [\tau^+(R_i) + 1, z]$. By 22, we have $\phi_{[x,z]}(R_i', R_{N\setminus i}) > \phi_{[x,z]}(R_N)$. However, since $[x, z]$ is an upper contour set at R_i, this means agent i manipulates at R_N via R_i'. □

Continuing in this manner, we can construct a profile R_N' with $\phi_{[1,x-1]}(R_N') > 0$ where $\tau^-(R_i') = x$ and R_i' is left-single-plateaued for all $i \in \mathcal{D}_i$ with $\tau^-(R_i) > x$ and $R_i' = R_i$ for all other agents. Clearly, $x - 1 \notin \cap_{i\in N}\tau(R_i')$. Moreover, since $I(R_N) = [x, y]$, there must be $i \in N$ with $\tau^+(R_i) = x$. By the construction of R_N', $R_i' = R_i$ for such an agent i. This means $x + 1 \notin \tau(R_i')$ for such an agent, and consequently, $x + 1 \notin \cap_{i\in N}\tau(R_i')$. Thus, we have $\cap_{i\in N}\tau(R_i') = \{x\}$. By unanimity, $\phi_x(R_N') = 1$, which is a contradiction to $\phi_{[1,x-1]}(R_N') > 0$. ■

.4 Proof of Theorem 5.5.7

Proof: Let $\mathcal{D}$ be the (κ_1, κ_2)-single-plateaued domain and let $\phi : \mathcal{D}^n \to \Delta A$ be a unanimous and strategy-proof RSCF. We show ϕ is almost plateau-only. Take $i \in N$ and $(R_i, R_{N\setminus i}), (R_i', R_{N\setminus i}) \in \mathcal{D}^n$ such that $\tau(R_i) = \tau(R_i')$. Let $x \in A \setminus \tau(R_i)$ be arbitrary. It is enough to show $\phi_x(R_i, R_{N\setminus i}) = \phi_x(R_i', R_{N\setminus i})$. Without loss of generality, assume $x > \tau^+(R_i)$, $\phi_x(R_i, R_{N\setminus i}) < \phi_x(R_i', R_{N\setminus i})$, and $\phi_y(R_i, R_{N\setminus i}) = \phi_y(R_i', R_{N\setminus i})$ for all $y \in (\tau^+(R_i), x)$.

Let $j \in N$ be such that $\tau^+(R_j) \leq \tau^+(R_k)$ for all $k \in N$. Consider $R_j' \in \mathcal{D}$ such that $\tau^+(R_j') = \tau^+(R_i)$ and R_j' is left single-plateaued.

Claim .4.1 *$\phi_y(R_i, R_j', R_{N\setminus\{i,j\}}) = \phi_y(R_i, R_j, R_{N\setminus\{i,j\}})$ and $\phi_y(R_i', R_j', R_{N\setminus\{i,j\}}) = \phi_y(R_i', R_j, R_{N\setminus\{i,j\}})$ for all $y > \tau^+(R_i)$.*

Proof: It is sufficient to show $\phi_y(R_i, R'_j, R_{N\setminus\{i,j\}}) = \phi_y(R_i, R_j, R_{N\setminus\{i,j\}})$ for all $y > \tau^+(R_i)$. The proof of $\phi_y(R'_i, R'_j, R_{N\setminus\{i,j\}}) = \phi_y(R'_i, R_j, R_{N\setminus\{i,j\}})$ for all $y > \tau^+(R_i)$ follows from similar argument. Take $y \in A$ such that $y > \tau^+(R_i)$. Because $\tau^+(R'_j) = \tau^+(R_i)$ and $\tau^+(R_i) < y$, we have $\tau^+(R'_j) < y$. Since R'_j is left single-plateaued and $\tau^+(R'_j) < y$, $U(y, R'_j) = U(y, R_j) \cup [1, \tau^-(R_j))$. By strategy-proofness of ϕ, we have

$$\phi_{U(y,R_j)}(R_i, R_j, R_{N\setminus\{i,j\}}) \geq \phi_{U(y,R_j)}(R_i, R'_j, R_{N\setminus\{i,j\}}) \tag{23}$$

and

$$\phi_{U(y,R'_j)}(R_i, R'_j, R_{N\setminus\{i,j\}}) \geq \phi_{U(y,R'_j)}(R_i, R_j, R_{N\setminus\{i,j\}}). \tag{24}$$

As ϕ is unanimous and strategy-proof, by Theorem 5.5.3 it follows that ϕ satisfies Pareto optimality. By Pareto optimality and our assumption on R_j, $\phi_{[1,\tau^-(R_j))}(R_i, R_j, R_{N\setminus\{i,j\}}) = \phi_{[1,\tau^-(R_j))}(R_i, R'_j, R_{N\setminus\{i,j\}})$
$= 0$. This together with (23) implies

$$\phi_{U(y,R'_j)}(R_i, R'_j, R_{N\setminus\{i,j\}}) \leq \phi_{U(y,R'_j)}(R_i, R_j, R_{N\setminus\{i,j\}}) \tag{25}$$

Now, (24) and (25) imply

$$\phi_{U(y,R'_j)}(R_i, R'_j, R_{N\setminus\{i,j\}}) = \phi_{U(y,R'_j)}(R_i, R_j, R_{N\setminus\{i,j\}}). \tag{26}$$

Because $y > \tau^+(R'_j)$, using similar argument for $y-1$, we have

$$\phi_{U(y-1,R'_j)}(R_i, R_j, R_{N\setminus\{i,j\}}) = \phi_{U(y-1,R'_j)}(R_i, R'_j, R_{N\setminus\{i,j\}}). \tag{27}$$

Subtracting (27) from (26), we have $\phi_y(R_i, R_j, R_{N\setminus\{i,j\}}) = \phi_y(R_i, R'_j, R_{N\setminus\{i,j\}})$. This completes the proof of the claim. ■

Now we complete the proof of the theorem. Let $k \in N$ be such that $\tau^+(R_k) \leq \tau^+(R_l)$ for all $l \in N \setminus \{j\}$. Let $R'_k \in \mathcal{D}$ be such that $\tau^+(R'_k) = \tau^+(R_i)$ and R'_k is left single-plateaued. Using similar logic as for the proof of Claim .4.1, we have $\phi_y(R_i, R'_j, R_k, R_{N\setminus\{i,j,k\}}) = \phi_y(R_i, R'_j, R'_k, R_{N\setminus\{i,j,k\}})$ and $\phi_y(R_i, R'_j, R_k, R_{N\setminus\{i,j,k\}}) = \phi_y(R_i, R'_j, R'_k, R_{N\setminus\{i,j,k\}})$ for all $y > \tau^+(R_i)$.

Continuing in this manner, we construct profiles $(R_i, R'_{N\setminus i})$ and R'_N such that for all $l \in N \setminus i$, $\tau^+(R'_l) = \tau^+(R_i)$ if $\tau^+(R_l) < \tau^+(R_i)$ and $R'_l = R_l$ otherwise and $\phi_y(R_i, R'_{N\setminus i}) = \phi_y(R_i, R_{N\setminus i})$ and $\phi_y(R'_i, R'_{N\setminus i}) = \phi_y(R'_i, R_{N\setminus i})$ for all $y > \tau^+(R_i)$.

By Pareto optimality, $\phi_z(R_i, R'_{N\setminus i}) = \phi_z(R'_i, R'_{N\setminus i}) = 0$ for all $z < \tau^+(R_i)$. Also, by strategy-proofness of ϕ, $\phi_{\tau(R_i)}(R_i, R'_{N\setminus i}) = \phi_{\tau(R_i)}(R'_i, R'_{N\setminus i})$. Since $\phi_x(R_i, R_{N\setminus i}) < \phi_x(R'_i, R_{N\setminus i})$, by Claim .4.1 we have $\phi_{U(x,R_i)}(R_i, R'_{N\setminus i}) < \phi_{U(x,R_i)}(R'_i, R'_{N\setminus i})$. This means i manipulates at $(R_i, R'_{N\setminus i})$ via R'_i, which contradicts that ϕ is strategy-proof. ■

.5 Proof of Theorem 5.5.15

Proof: (If part) Let $\mathcal{D}$ be the (κ_1, κ_2)-single-plateaued domain and let $\phi : \mathcal{D}^n \to \Delta A$ be an RSCF satisfying generalized uncompromisingness. We show that ϕ is strategy-proof. Take $R_N \in \mathcal{D}^n$ and $R'_i \in \mathcal{D}$. By Remark 5.5.13, ϕ is almost plateau-only. Thus, if $\tau(R_i) = \tau(R'_i)$, $\phi_{U(x,R_i)}(R_i, R_{N\setminus i}) = \phi_{U(x,R_i)}(R'_i, R_{N\setminus i})$ for all $x \in A$. We distinguish the following cases based on the relative position of $\tau(R_i)$ and $\tau(R'_i)$ to complete the proof.

Case 1. Suppose $\tau^-(R_i) < \tau^-(R'_i)$ and $\tau^+(R_i) < \tau^+(R'_i)$

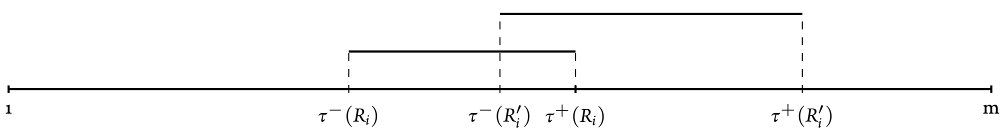


Figure .5.1: $\tau^-(R_i) < \tau^-(R'_i)$ and $\tau^+(R_i) < \tau^+(R'_i)$.

By Remark 5.5.11,

$$\phi_x(R_i, R_{N\setminus i}) = \phi_x(R'_i, R_{N\setminus i}) \text{ for all } x \notin [\tau^-(R_i), \tau^+(R'_i)]. \tag{28}$$

Because S is single-plateaued , for all $y \in A$, there are $y' \le \tau^-(R_i)$ and $y'' \ge \tau^+(R_i)$ such that $U(y, R_i) = [y', \tau^-(R_i) - 1] \cup [\tau^-(R_i), y'']$, where $[y', \tau^-(R_i) - 1] = \emptyset$ when $y' = \tau^-(R_i)$. This together with (28), implies that to show ϕ is strategy-proof it is sufficient to show $\phi_{[\tau^-(R_i),y]}(R_i, R_{N\setminus i}) \ge \phi_{[\tau^-(R_i),y]}(R'_i, R_{N\setminus i})$ for all $y \in [\tau^+(R_i), \tau^+(R'_i)]$.

Take $y = \tau^+(R_i)$. It follows from (28) that $\phi_{[\tau^-(R_i),\tau^+(R'_i)]}(R_i, R_{N\setminus i}) = \phi_{[\tau^-(R_i),\tau^+(R'_i)]}(R'_i, R_{N\setminus i})$. Now, take $y \in [\tau^+(R_i), \tau^+(R'_i) - 1]$. As $\tau^+(R_i) < y + 1$ and $\tau^+(R'_i) \ge y + 1$, by condition (ii) of Definition 5.5.10, we have

$$\phi_{[y+1,m]}(R_i, R_{N\setminus i}) \le \phi_{[y+1,m]}(R'_i, R_{N\setminus i}). \tag{29}$$

By condition (i) of Definition 5.5.10, $\phi_{[\tau^-(R_i),m]}(R_i, R_{N\setminus i}) = \phi_{[\tau^-(R_i),m]}(R'_i, R_{N\setminus i})$. This together with (29), implies

$$\phi_{[\tau^-(R_i),y]}(R_i, R_{N\setminus i}) \ge \phi_{[\tau^-(R_i),y]}(R'_i, R_{N\setminus i}).$$

Case 2. Suppose $\tau^-(R_i) > \tau^-(R'_i)$ and $\tau^+(R_i) > \tau^+(R'_i)$.

This case is very similar to Case 1, however for the sake of completeness we provide a formal proof. Take

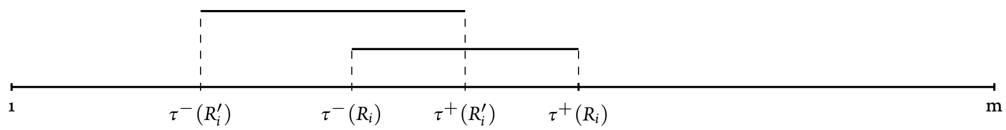


Figure .5.2: $\tau^-(R_i) > \tau^-(R_i')$ and $\tau^+(R_i) > \tau^+(R_i')$.

$x \notin [\tau^-(R_i') + 1, \tau^+(R_i)]$. By Condition (i) of Definition 5.5.10, we have

$$\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i}). \tag{30}$$

Using similar logic as in Case 1, to show ϕ is strategy-proof, it is sufficient to show $\phi_{[y,\tau^+(R_i)]}(R_i, R_{N\setminus i}) \geq \phi_{[y,\tau^+(R_i)]}(R_i', R_{N\setminus i})$ for all $y \in [\tau^-(R_i'), \tau^-(R_i)]$. First, take $y = \tau^-(R_i')$. By Condition (i) of Definition 5.5.10, it follows that $\phi_{[\tau^-(R_i'),\tau^+(R_i)]}(R_i, R_{N\setminus i}) = \phi_{[\tau^-(R_i'),\tau^+(R_i)]}(R_i', R_{N\setminus i})$. Next, take $y \in [\tau^-(R_i') + 1, \tau^-(R_i)]$. As $\tau^-(R_i) \geq y$ and $\tau^-(R_i') < y$, by Condition (ii) of Definition 5.5.10, we have

$$\phi_{[y,m]}(R_i, R_{N\setminus i}) \geq \phi_{[y,m]}(R_i', R_{N\setminus i}). \tag{31}$$

Taking $x = \tau^+(R_i) + 1$ in (30) and subtracting it from (31), we get

$$\phi_{[y,\tau^+(R_i)]}(R_i, R_{N\setminus i}) \geq \phi_{[y,\tau^+(R_i)]}(R_i', R_{N\setminus i}).$$

Case 3. Suppose $\tau^-(R_i) \geq \tau^-(R_i')$ and $\tau^+(R_i) \leq \tau^+(R_i')$.
The proof for this case follows by combining the arguments in Case 1 and Case 2. For the sake of completeness, we provide a formal proof here.

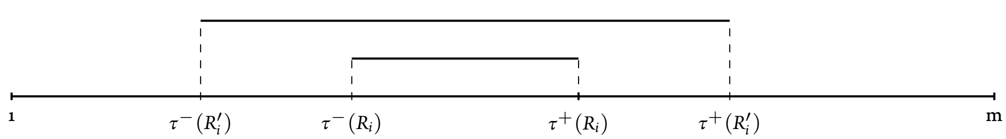


Figure .5.3: $\tau^-(R_i) \geq \tau^-(R_i')$ and $\tau^+(R_i) \leq \tau^+(R_i')$.

Case 4. Suppose $\tau^-(R_i) \leq \tau^-(R_i')$ and $\tau^+(R_i) \geq \tau^+(R_i')$.

By Remark 5.5.11, $\phi_x(R_i, R_{N\setminus i}) = \phi_x(R_i', R_{N\setminus i})$ for all $x \notin \tau(R_i)$. Therefore, agent i cannot manipulate at $(R_i, R_{N\setminus i})$ via R_i'.

■

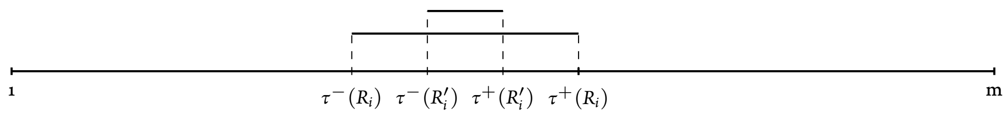


Figure .5.4: $\tau^-(R_i) \leq \tau^-(R_i')$ and $\tau^+(R_i) \geq \tau^+(R_i')$.

.6 Proof of Theorem 5.5.16

Proof: "If" part of the theorem follows from Theorem 5.5.15. We proceed to prove the "only if" part. Let $\phi : \mathcal{D}^n \to \Delta A$ be a strategy-proof and unanimous RSCF. We show ϕ satisfies generalized uncompromisingness. Take $i \in N$ and $R_i, R_i' \in \mathcal{D}$ with $\tau^+(R_i) < \tau^+(R_i')$, $R_{N\setminus i} \in \mathcal{D}^{n-1}$, and $x \in A$.

We show that $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$ if $\max\{\tau^+(R_i), \tau^+(R_i')\} < x$ or if $\min\{\tau^-(R_i), \tau^-(R_i')\} \geq x$. Suppose $\max\{\tau^+(R_i), \tau^+(R_i')\} < x$. In view of Theorem 5.5.7, we assume both R_i, R_i' to be left single-plateaued. By strategy-proofness, $\phi_{U(x-1,R_i)}(R_i, R_{N\setminus i}) \geq \phi_{U(x-1,R_i)}(R_i', R_{N\setminus i})$ and $\phi_{U(x-1,R_i')}(R_i', R_{N\setminus i}) \geq \phi_{U(x-1,R_i')}(R_i, R_{N\setminus i})$. However, since both R_i and R_i' are left single-plateaued, $U(x-1, R_i) = U(x-1, R_i') = [1, x-1]$. Therefore, $\phi_{[x,m]}(R_i', R_{N\setminus i}) = \phi_{[x,m]}(R_i, R_{N\setminus i})$.

For the case where $\min\{\tau^-(R_i), \tau^-(R_i')\} \geq x$, by means of Theorem 5.5.7 we can assume both R_i and R_i' to be right single-plateaued. Then $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$ follows by using similar argument as above.

Now we show (ii) in Definition 5.5.10. Suppose $\tau^+(R_i) < x \leq \tau^+(R_i')$. Since $x > \tau^+(R_i)$ in view of Theorem 5.5.7 , we assume R_i to be left single-plateaued. By strategy-proofness, $\phi_{U(x-1,R_i)}(R_i, R_{N\setminus i}) \geq \phi_{U(x-1,R_i)}(R_i', R_{N\setminus i})$. However, as R_i is a left single-plateaued preference, $U(x-1, R_i) = [1, x-1]$. This means $\phi_{[x,m]}(R_i, R_{N\setminus i}) \leq \phi_{[x,m]}(R_i', R_{N\setminus i})$.

Suppose $\tau^-(R_i) < x \leq \tau^-(R_i')$. Since $x \leq \tau^-(R_i')$ in view of Theorem 5.5.7, without loss of generality we assume R_i' to be right single-plateaued. Then, $U(x-1, R_i) = [x, m]$. By strategy-proofness, this means $\phi_{[x,m]}(R_i', R_{N\setminus i}) \geq \phi_{[x,m]}(R_i, R_{N\setminus i})$ which completes the proof. ■

.7 Proof of Theorem 5.5.18

Proof: (If part) In Section 5.5.5, we introduce a generalization of probabilistic plateau rule which we call generalized probabilistic ballot rules. Theorem 5.5.22 shows that those rules are unanimous and strategy-proof. Therefore, if part of the theorem follows from Theorem 5.5.22.
(Only-if part) Suppose an RSCF $\phi : \mathcal{D}^2 \to \Delta A$ is unanimous and strategy-proof. We show that ϕ is a probabilistic plateau rule. Condition (i) of Definition 5.5.17 follows from unanimity of ϕ.

Consider a preference profile $R_N \in \mathcal{D}^2$ with $\tau^+(R_1) < \tau^-(R_2)$. It follows from Theorem 5.5.3 that ϕ is Pareto optimal. By means of Pareto optimality, we have $\phi_x(R_N) = 0$ for all $x \notin [\tau^+(R_1), \tau^-(R_2)]$. We prcoceed to show $\phi_x(R_N) = \beta_2[x, m]$ for some $\beta_2 \in \Delta A$ where $x \in A$ such that $\tau^+(R_1) < x \leq \tau^-(R_2)$. Let $\bar{R}_N \in \mathcal{D}^2$ be such that $\tau(\bar{R}_1) = [1, \kappa_1]$ and $\tau(\bar{R}_2) = [m - \kappa_1 + 1, m]$. Define $\phi(\bar{R}_N) := \beta_2$. Consider the preference profile $(R_1, \bar{R}_2)$. Since $\tau^+(R_1) < x$, by strategy-proofness, $\phi_{[x,m]}(R_1, \bar{R}_2) = \beta_2[x, m]$. Again, since $x < \tau^-(R_2)$, $\phi_{[x,m]}(R_1, R_2) = \beta_2[x, m]$. This proves Condition (ii) of Definition 5.5.17. Condition (iii) of Definition 5.5.17 can be proved with a similar argument. ■

.8 Proof of Theorem 5.5.22

Proof: Let $k \in \{0, \ldots, \kappa_1 - 1\}$ and let $(\beta_S)_{S \subseteq N}$ be a collection of probability ballots. Suppose $\phi : \mathcal{D}^n \to \Delta A$ is a generalized probabilistic ballot rule with parameters k and $(\beta_S)_{S \subseteq N}$. We show that ϕ is unanimous and strategy-proof. Unanimity of ϕ follows from Condition (i) and Condition (ii) of Definition 5.5.20. To show that ϕ is strategy-proof, by Theorem 5.5.16, it is enough to show that it satisfies generalized uncompromisingness. Consider $R_i, R_i' \in \mathcal{D}_i$ and $R_{N\setminus i} \in \mathcal{D}^{n-1}$.

Case 1. Suppose $x \in A$ is such that $x \leq \min\{\tau^-(R_i), \tau^-(R_i')\}$. Assume without loss of generality that $\tau^-(R_i) \leq \tau^-(R_i')$. We further distinguish three cases based on location of x with respect to $I(R_i, R_{N\setminus i})$.

Case 1.1. $x \leq I^-(R_i, R_{N\setminus i})$.

As $x \leq I^-(R_i, R_{N\setminus i})$, we have by Condition (i) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 1$. Suppose $\tau^-(R_i) < I^-(R_i, R_{N\setminus i})$. Because $\tau^-(R_i) \leq \tau^-(R_i')$, this and the fact that $\tau^-(R_i) \leq I^-(R_i, R_{N\setminus i})$ imply $\tau^-(R_i) \leq I^-(R_i', R_{N\setminus i})$. Combining the two facts $\tau^-(R_i) \leq I^-(R_i, R_{N\setminus i})$ and $x \leq \tau^-(R_i)$, we have $x \leq I^-(R_i', R_{N\setminus i})$. Therefore, by Condition (i) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), we have $\phi_{[x,m]}(R_i', R_{N\setminus i}) = 1$. As $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 1$, it follows that $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i}) = 1$. Now, suppose $I^-(R_i, R_{N\setminus i}) < \tau^-(R_i)$. Since $\tau^-(R_i) \leq \tau^-(R_i')$, this implies $I^-(R_i, R_{N\setminus i}) = I^-(R_i', R_{N\setminus i})$. Because $x \leq I^-(R_i, R_{N\setminus i})$ by our assumption in Case 1.1, we have $x \leq I^-(R_i', R_{N\setminus i})$. By Condition (i) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), we obtain $\phi_{[x,m]}(R_i', R_{N\setminus i}) = 1$. As $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 1$, it follows that $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$.

Case 1.2. $I^+(R_i, R_{N\setminus i}) < x$.

By the definition of $I(R_i, R_{N\setminus i})$, we have $\tau^-(R_i) \leq I^+(R_i, R_{N\setminus i})$. As $x \leq \tau^-(R_i)$, this means we must have $x \leq \tau^-(R_i) \leq I^+(R_i, R_{N\setminus i})$, which in turn means that Case 1.2 is not possible.

Case 1.3. $I^-(R_i, R_{N\setminus i}) < x \leq I^+(R_i, R_{N\setminus i})$.

Consider the preference profile $(R_i, R_{N\setminus i})$. We first show that $\cap_{j \in N} \tau(R_j) = \emptyset$. Suppose not. Then by the definition of $I(R_N)$, we have $\tau^-(R_i) \leq I^-(R_i, R_{N\setminus i})$, which contradicts the assumption of Case 1 that $x \leq \tau^-(R_i)$. Therefore, we have $\cap_{j \in N} \tau(R_j) = \emptyset$. Combining this with the fact that $I^-(R_i, R_{N\setminus i}) < x \leq \tau^-(R_i)$, it follows that there must be some $l \in N$ such that $\tau^+(R_l) = I^-(R_i, R_{N\setminus i})$.

Consider the preference profile $(R'_i, R_{N\setminus i})$. By using $\tau^+(R_l) = I^-(R_i, R_{N\setminus i})$, $I^-(R_i, R_{N\setminus i}) < x$ and $x \leq \tau^-(R_i) \leq \tau^-(R'_i)$, we obtain $\tau^+(R_l) < \tau^-(R_i) \leq \tau^-(R'_i)$. As $\tau^+(R_l) < \tau^-(R'_i)$ we must have $\tau(R_l) \cap \tau(R'_i) = \emptyset$. This means $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) = \emptyset$. Because $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) = \emptyset$, by the definition of $I(R_i, R_{N\setminus i})$ and $I(R'_i, R_{N\setminus i})$ it follows that $I^-(R_i, R_{N\setminus i}) = I^-(R'_i, R_{N\setminus i})$. Since $\cap_{j\in N}\tau(R_j) = \emptyset$ and $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) = \emptyset$, we can use Condition (iii) of the definition of a generalized probabilistic ballot rule to calculate the outcome of ϕ at $(R_i, R_{N\setminus i})$ and $(R'_i, R_{N\setminus i})$. Let $S \subseteq N$ be such that $j \in S$ if and only if $x - k \leq \tau^-(R_j)$ and $x \leq \tau^+(R_j)$ and let $S' \subseteq N$ be such that $j \in S'$ if and only if $x - k \leq \tau^-(R'_j)x - k$ and $x \leq \tau^+(R'_j)$. Note that $S \setminus i = S' \setminus i$. Since $x \leq \tau^-(R_i) \leq \tau^-(R'_i)$, we have $i \in S$ and $i \in S'$, which, by means of the fact that $S \setminus i = S' \setminus i$, implies $S = S'$. By the definition of ϕ, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta_S[x, m]$ and $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = \beta_{S'}[x, m]$. Because $S = S'$, this implies $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Case 2. Suppose $x \in A$ is such that $\max\{\tau^+(R_i), \tau^+(R'_i)\} < x$. Assume without loss of generality that $\tau^+(R'_i) \leq \tau^+(R_i)$. We further distinguish three cases based on location of x with respect to $I(R_i, R_{N\setminus i})$.

Case 2.1. $x \leq I^-(R_i, R_{N\setminus i})$.

By the definition of $I(R_i, R_{N\setminus i})$, we have $I^-(R_i, R_{N\setminus i}) \leq \tau^+(R_i)$. As $\tau^+(R_i) < x$, this means we must have $I^-(R_i, R_{N\setminus i}) \leq \tau^+(R_i) < x$, which in turn means that Case 2.1 is not possible.

Case 2.2. $I^+(R_i, R_{N\setminus i}) < x$

As $I^+(R_i, R_{N\setminus i}) < x$, we have by Condition (ii) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 0$. Suppose $\tau^+(R_i) < I^+(R_i, R_{N\setminus i})$. Because $\tau^+(R'_i) \leq \tau^+(R_i)$, and $\tau^+(R_i) < I^+(R_i, R_{N\setminus i})$, we have $I^+(R'_i, R_{N\setminus i}) = I^+(R'_i, R_{N\setminus i})$. So, $I^-(R'_i, R_{N\setminus i}) < x$. Hence, $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = 0$. If $I^+(R_i, R_{N\setminus i}) \leq \tau^+(R_i)$, then $I^-(R'_i, R_{N\setminus i}) \leq \tau^+(R_i)$. So, $I^-(R'_i, R_{N\setminus i}) \leq \tau^+(R_i) < x$. Hence, $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = 0$. When $I^-(R_i, R_{N\setminus i}) < x$, we have $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Case 2.3. $I^-(R_i, R_{N\setminus i}) < x \leq I^+(R_i, R_{N\setminus i})$

Consider the preference profile $(R_i, R_{N\setminus i})$. We first show that $\cap_{j\in N}\tau(R_j) = \emptyset$. Suppose not. Then by the definition of $I(R_N)$, we have $I^+(R_i, R_{N\setminus i}) \leq \tau^+(R_i)$, which contradicts the assumption of Case 2 that $\tau^+(R_i) < x$. Therefore, we have $\cap_{j\in N}\tau(R_j) = \emptyset$. Combining this with the fact that $\tau^+(R_i) < x \leq I^+(R_i, R_{N\setminus i})$, it follows that there must be some $l \in N$ such that $\tau^-(R_l) = I^+(R_i, R_{N\setminus i})$. Consider the preference profile $(R'_i, R_{N\setminus i})$. By using $\tau^-(R_l) = I^+(R_i, R_{N\setminus i})$, $x \leq I^+(R_i, R_{N\setminus i})$ and $\tau^+(R'_i) \leq \tau^+(R_i) < x$, we obtain $\tau^+(R'_i) \leq \tau^+(R_i) < \tau^-(R_l)$. As $\tau^+(R'_i) < \tau^-(R_l)$ we must have $\tau(R_l) \cap \tau(R'_i) = \emptyset$. This means $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) = \emptyset$. Because $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) = \emptyset$, by the definition of $I(R_i, R_{N\setminus i})$ and $I(R'_i, R_{N\setminus i})$ it follows that $I^+(R_i, R_{N\setminus i}) = I^+(R'_i, R_{N\setminus i})$. Since $\cap_{j\in N}\tau(R_j) = \emptyset$ and $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) = \emptyset$, we can use Condition (iii) of the definition of a generalized probabilistic ballot rule to calculate the outcome of ϕ at $(R_i, R_{N\setminus i})$ and $(R'_i, R_{N\setminus i})$. Let $S \subseteq N$ be such that $j \in S$ if and only if $x - k \leq \tau^-(R_j)$ and $x \leq \tau^+(R_j)$ and let $S' \subseteq N$ be such that $j \in S'$ if and only if $x - k \leq \tau^-(R'_j)$ and $x \leq \tau^+(R'_j)$. Note that $S \setminus i = S' \setminus i$. Since $\tau^+(R'_i) \leq \tau^+(R_i) < x$, we have $i \in S$ and $i \in S'$, which, by means of the fact that $S \setminus i = S' \setminus i$, implies $S = S'$. By the definition of ϕ, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta_S[x, m]$ and $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = \beta_{S'}[x, m]$. Because $S = S'$, this implies $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Case 3. Suppose $x \in A$ is such that $\tau^+(R_i) < x \leq \tau^+(R'_i)$. We further distinguish three cases based on location of x with respect to $I(R_i, R_{N\setminus i})$.

Case 3.1. $x \leq I^-(R_i, R_{N\setminus i})$.

By the definition of $I(R_i, R_{N\setminus i})$, no matter whether $\cap_{i\in N}\tau(R_i)$ is empty or not, we have $I^-(R_i, R_{N\setminus i}) \leq \tau^+(R_i)$. Therefore, this case is not possible.

Case 3.2. $I^+(R_i, R_{N\setminus i}) < x$

By Condition (ii) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 0$. As $\phi_{[x,m]}(R'_i, R_{N\setminus i})$ cannot be negative, this implies that $\phi_{[x,m]}(R_i, R_{N\setminus i}) \leq \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Case 3.3. $I^-(R_i, R_{N\setminus i}) < x \leq I^+(R_i, R_{N\setminus i})$

Suppose $x \leq I^-(R'_i, R_{N\setminus i})$. Then, by Condition (i) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), we have $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = 1$. This means $\phi_{[x,m]}(R_i, R_{N\setminus i}) \leq \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Now suppose $I^-(R'_i, R_{N\setminus i}) < x$. By using the same logic as in Case 2.3, we can show that $\cap_{j\in N}\tau(R_j) = \emptyset$. By the assumption of Case 3, we have $\tau^+(R_i) < x$ and by the assumption of Case 3.3, we have $x \leq I^+(R_i, R_{N\setminus i})$. Combining we obtain $\tau^+(R_i) < x \leq I^+(R_i, R_{N\setminus i})$. As $\cap_{j\in N}\tau(R_j) = \emptyset$ and $\tau^+(R_i) < I^+(R_i, R_{N\setminus i})$, there must be some $l \in N$ such that $\tau^-(R_l) = I^+(R_i, R_{N\setminus i})$, and hence $I^+(R_i, R_{N\setminus i}) \leq I^+(R'_i, R_{N\setminus i})$. This, the assumption of Case 3.3, and the supposition of the current paragraph that $I^-(R'_i, R_{N\setminus i}) < x$, imply $I^-(R'_i, R_{N\setminus i}) < x \leq I^+(R'_i, R_{N\setminus i})$. Because $\tau^-(R_l) = I^+(R_i, R_{N\setminus i})$, the assumptions that $I^-(R_i, R_{N\setminus i}) < x \leq I^+(R_i, R_{N\setminus i})$ and $I^-(R'_i, R_{N\setminus i}) < x$, imply $I^-(R'_i, R_{N\setminus i}) < \tau^-(R_l)$, and hence $I^-(R'_i, R_{N\setminus i}) \notin \tau(R_l)$. We claim that $I(R'_i, R_{N\setminus i}) \neq \cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i)$. Assume for contradiction that $I(R'_i, R_{N\setminus i}) = \cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i)$. However, then $I(R'_i, R_{N\setminus i}) \subseteq \tau(R_j)$ for all $j \in N$, and in particular, $I^-(R'_i, R_{N\setminus i}) \in \tau(R_j)$ for all $j \in N$, which contradicts our earlier deduction that $I^-(R'_i, R_{N\setminus i}) \notin \tau(R_l)$. This proves our claim that $I(R'_i, R_{N\setminus i}) \neq \cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i)$. Because $I(R'_i, R_{N\setminus i}) \neq \cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i)$, by the definition of $I(R'_i, R_{N\setminus i})$ it must be that $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R'_i) \neq \emptyset$. Therefore, we can use Condition (iii) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20) to calculate the outcome of ϕ at the preference profiles $(R_i, R_{N\setminus i})$ and $(R'_i, R_{N\setminus i})$. Let $S \subseteq N$ be such that $i \in S$ if and only if $x - k \leq \tau^-(R_i)$ and $x \leq \tau^+(R_i)$ and let $S' \subseteq N$ be such that $i \in S'$ if and only if $x - k \leq \tau^-(R'_i)$ and $x \leq \tau^+(R'_i)$. By the assumption of Case 3, $\tau^+(R_i) < x \leq \tau^+(R'_i)$, and hence $S \subseteq S'$. By the definition of ϕ, we have $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta_S[x, m]$ and $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = \beta_{S'}[x, m]$, and by the definition of $(\beta_S)_{S\subseteq N}$, we have $\beta_S[x, m] \leq \beta_{S'}[x, m]$. Therefore, $\phi_{[x,m]}(R_i, R_{N\setminus i}) \leq \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Case 4. Suppose $x \in A$ is such that $\tau^-(R_i) < x \le \tau^-(R'_i)$. We further distinguish three cases based on location of x with respect to $I(R'_i, R_{N\setminus i})$.

Case 4.1. $x \le I^-(R'_i, R_{N\setminus i})$

By Condition (i) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 1$. As $\phi_x(R_i, R_{N\setminus i}) \le 1$, this implies that $\phi_{[x,m]}(R_i, R_{N\setminus i}) \le \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Case 4.2. $I^+(R'_i, R_{N\setminus i}) < x$

By the definition of $I(R_i, R_{N\setminus i})$, no matter whether $\cap_{i\in N}\tau(R_i)$ is empty or not, we have $\tau^-(R'_i) \le I^+(R'_i, R_{N\setminus i})$. Therefore, this case is not possible.

Case 4.3. $I^-(R'_i, R_{N\setminus i}) < x \le I^+(R'_i, R_{N\setminus i})$

Suppose $I^+(R_i, R_{N\setminus i}) < x$. Then, by Condition (ii) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20), we have $\phi_{[x,m]}(R_i, R_{N\setminus i}) = 0$. This means $\phi_{[x,m]}(R_i, R_{N\setminus i}) \le \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Now suppose $x \le I^+(R_i, R_{N\setminus i})$. By using the same logic as in Case 1.3, we can show that $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R_i) = \emptyset$. By the assumption of Case 4, we have $x \le \tau^-(R'_i)$ and by the assumption of Case 4.3, we have $I^-(R'_i, R_{N\setminus i}) < x$. Combining we obtain $I^-(R'_i, R_{N\setminus i}) < x \le \tau^-(R'_i)$. As $\cap_{j\in N\setminus i}\tau(R_j) \cap \tau(R_i) = \emptyset$ and $I^-(R'_i, R_{N\setminus i}) < \tau^-(R'_i)$, there must be some $l \in N$ such that $\tau^+(R_l) = I^-(R'_i, R_{N\setminus i})$, and hence $I^-(R_i, R_{N\setminus i}) \le I^-(R'_i, R_{N\setminus i})$. This, the assumption of Case 4.3, and the supposition of the current paragraph that $x \le I^+(R_i, R_{N\setminus i})$, imply $I^-(R_i, R_{N\setminus i}) < x \le I^+(R_i, R_{N\setminus i})$. Because $\tau^+(R_l) = I^-(R'_i, R_{N\setminus i})$, the assumptions that $I^-(R'_i, R_{N\setminus i}) < x \le I^+(R'_i, R_{N\setminus i})$ and $I^-(R_i, R_{N\setminus i}) < x$, imply $I^-(R_i, R_{N\setminus i}) < \tau^-(R_l)$, and hence $I^-(R_i, R_{N\setminus i}) \notin \tau(R_l)$. We claim that $I(R_i, R_{N\setminus i}) \ne \cap_{j\in N}\tau(R_j)$. Assume for contradiction that $I(R_i, R_{N\setminus i}) = \cap_{j\in N}\tau(R_j)$. However, then $I(R_i, R_{N\setminus i}) \subseteq \tau(R_j)$ for all $j \in N$, and in particular, $I^-(R_i, R_{N\setminus i}) \in \tau(R_j)$ for all $j \in N$, which contradicts our earlier deduction that $I^-(R_i, R_{N\setminus i}) \notin \tau(R_l)$. This proves our claim that $I(R_i, R_{N\setminus i}) \ne \cap_{j\in N}\tau(R_j)$. Because $I(R_i, R_{N\setminus i}) \ne \cap_{j\in N}\tau(R_j)$, by the definition of $I(R_i, R_{N\setminus i})$ it must be that $\cap_{j\in N}\tau(R_j) \ne \emptyset$. Therefore, we can use Condition (iii) of the definition of a generalized probabilistic ballot rule (Definition 5.5.20) to calculate the outcome of ϕ at the preference profiles $(R_i, R_{N\setminus i})$ and $(R'_i, R_{N\setminus i})$. Let $S \subseteq N$ be such that $i \in S$ if and only if $x - k \le \tau^-(R_i)$ and $x \le \tau^+(R_i)$ and let $S' \subseteq N$ be such that $i \in S'$ if and only if $x - k \le \tau^-(R'_i)$ and $x \le \tau^+(R'_i)$. By the assumption of Case 4, $\tau^-(R_i) < x \le \tau^-(R'_i)$, and hence $S \subseteq S'$. By the definition of ϕ, we have $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta_S[x, m]$ and $\phi_{[x,m]}(R'_i, R_{N\setminus i}) = \beta_{S'}[x, m]$, and by the definition of $(\beta_S)_{S\subseteq N}$, we have $\beta_S[x, m] \le \beta_{S'}[x, m]$. Therefore, $\phi_{[x,m]}(R_i, R_{N\setminus i}) \le \phi_{[x,m]}(R'_i, R_{N\setminus i})$. ■

.9 Proof of Theorem 5.5.28

Proof: (If part) Let $\mathcal{D}$ be a κ-single-plateaued domain for some $\kappa \in \{1, \dots, m\}$ and let $\phi : \mathcal{D}^n \to \Delta A$ be a κ-plateaued rule . By definition ϕ is anonymous and plateau-only. To show that ϕ is strategy-proof, in view of Theorem 5.5.15, it is enough to show that it satisfies generalized uncompromisingness. Consider $i \in N$, $R_i, R_i' \in \mathcal{D}$ with $\tau^+(R_i) < \tau^+(R_i')$, $R_{N\setminus i} \in \mathcal{D}^{n-1}$, and $x \in A$. First we show that Condition (i) in Definition 5.5.10 holds. Consider the case where $\tau^+(R_i') < x$ or $\tau^-(R_i) \geq x$. As $\tau^+(R_i) < \tau^+(R_i')$, it follows that when $\tau^+(R_i') < x$ we have $\tau^+(R_i) < x$, and when $\tau^-(R_i) \geq x$ we have $\tau^-(R_i') > x$. This implies $n_l^x(R_i, R_{N\setminus i}) = n_l^x(R_i', R_{N\setminus i})$ for all $0 \leq l \leq \kappa - 1$. By the definition of κ-plateaued rule , this yields $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$, which proves that ϕ satisfies Condition (i) in Definition 5.5.10.

Next we show ϕ satisfies Condition (ii) in Definition 5.5.10. We distinguish the following two cases.

Case 1. Suppose $\tau^+(R_i) < x \leq \tau^+(R_i')$.

By the definition of κ-plateaued rule, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta(\underset{\sim}{n}, x)$ and $\phi_{[x,m]}(R_i', R_{N\setminus i}) = \beta(\underset{\sim}{n}', x)$, where for all $0 \leq l \leq \kappa - 1$, $\underset{\sim}{n}_l = n_l^x(R_i, R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i', R_{N\setminus i})$. Since $\tau^+(R_i) < x$ and $\tau^+(R_i') \geq x$, there must exist $0 \leq l' \leq \kappa - 1$ such that $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $0 \leq l \leq l'$, and $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i})$ for all $l' < l \leq \kappa - 1$. This implies $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$. Therefore, by Condition (ii) of Definition 5.5.23, we have $\beta(\underset{\sim}{n}, x) \leq \beta(\underset{\sim}{n} \oplus 1, x)$, and hence it follows that $\phi_{[x,m]}(R_i', R_{N\setminus i}) \geq \phi_{[x,m]}(R_i, R_{N\setminus i})$.

Case 2. Suppose $\tau^-(R_i) < x \leq \tau^-(R_i')$.

By the definition of κ-plateaued rule, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta(\underset{\sim}{n}, x)$ and $\phi_{[x,m]}(R_i', R_{N\setminus i}) = \beta(\underset{\sim}{n}', x)$, where for all $0 \leq l \leq \kappa - 1$, $\underset{\sim}{n}_l = n_l^x(R_i, R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i', R_{N\setminus i})$. If $\tau^+(R_i) < x$, then $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $0 \leq l \leq \kappa - 1$. However, if $x \leq \tau^+(R_i) < x + \kappa - 1$, then there must exist l' such that $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i})$ for all $0 \leq l \leq l'$ and $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $l' < l \leq \kappa - 1$. In both these cases, $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$, and hence by using Condition (ii) of Definition 5.5.23, we have $\phi_{[x,m]}(R_i', R_{N\setminus i}) \geq \phi_{[x,m]}(R_i, R_{Nx\setminus i})$.

(Only-if part) Let ϕ be a strategy-proof, anonymous, and plateau-only RSCF on $\mathcal{D}^n$. We show that it is a κ-plateaued rule.

Lemma .9.1 *Let $x \in A$ and $R_N, R_N' \in \mathcal{D}^n$ be such that $n_l^x(R_N) = n_l^x(R_N')$ for all $0 \leq l \leq \kappa - 1$. Then, $\phi_{[x,m]}(R_N) = \phi_{[x,m]}(R_N')$.*

Proof: Note that for all $\bar{R}_N \in \{R_N, R_N'\}$, $|\{i \in N \mid \tau^+(\bar{R}_i) = x + l\}| = n_l^x(\bar{R}_N) - n_{l+1}^x(\bar{R}_N)$ for all $0 \leq l \leq \kappa - 2$ and $|\{i \in N \mid \tau^+(\bar{R}_i) < x\}| = n - n_0^x(\bar{R}_N)$. Because $n_l^x(R_N) = n_l^x(R_N')$ for all $0 \leq l \leq \kappa - 1$, this means $|\{i \in N \mid \tau^+(R_i) = x + l\}| = |\{i \in N \mid \tau^+(R_i') = x + l\}|$ for all $0 \leq l \leq \kappa - 2$, $|\{i \in N \mid \tau^-(R_i) \geq x\}| = |\{i \in N \mid \tau^-(R_i') \geq x\}|$ and $|\{i \in N \mid \tau^+(R_i) < x\}| = |\{i \in N \mid R_i'(1) < x\}|$. Let $n_l = |\{i \in N \mid \tau^+(R_i) = x + l\}|$ for all $0 \leq l \leq \kappa - 2$. Since ϕ is anonymous, assume without loss of generality that

(i) $\{i \in N \mid \tau^+(R_i) = x + l\} = \{i \in N \mid \tau^+(R_i') = x + l\} = \{i_{s_{l-1}} + 1, \ldots, i_{s_l}\}$, where for all $0 \leq l \leq \kappa - 2$, $s_l = n_0 + \ldots + n_l$ and $i_{s_{-1}} = 0$, and

(ii) for all $i \in N \setminus \{i_{s_{\kappa-2}}, \ldots, 1\}$, either $\tau^+(R_i), \tau^+(R'_i) < x$ or $\tau^-(R_i), \tau^-(R'_i) \geq x$. In view of this, it is enough to show that $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R'_i, R_{N\setminus i})$ for all $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in S$ such that either $\tau^+(R_i), \tau^+(R'_i) < x$ or $\tau^-(R_i), \tau^-(R_i) \geq x$. Suppose $\tau^+(R_i), \tau^+(R'_i) < x$. Since ϕ is plateau-only, assume without loss of generality that both R_i and R'_i are left single-plateaued. This means $U(x-1, R_i) = U(x-1, R'_i) = [1, x-1]$. Now, by straightforward application of strategy-proofness, it follows that $\phi_{[1,x-1]}(R_i, R_{N\setminus i}) = \phi_{[1,x-1]}(R'_i, R_{N\setminus i})$. Thus, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Next, suppose $\tau^-(R_i), \tau^-(R_i) \geq x$. Since ϕ is plateau-only, assume without loss of generality that both R_i and R'_i are right single-plateaued. This means $U(x, R_i) = U(x, R'_i) = [x, m]$. Now, by straightforward application of strategy-proofness, it follows that $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R'_i, R_{N\setminus i})$. This completes the proof of the lemma. ■

In view of Lemma .9.1, for all $(\underset{\sim}{n}, x) \in \mathcal{F}(\kappa)$, define $\beta(\underset{\sim}{n}, x) = \phi_{[x,m]}(R_N)$ where R_N is such that $n^x_l(R_i, R_{N\setminus i}) = \underset{\sim}{n}_l$ for all $0 \leq l \leq \kappa - 1$. In what follows we show that the parameters βs are plateau parameters.

We show that $(\beta(\underset{\sim}{n}, x))$ satisfies Condition (ii) in Definition 5.5.23. In view of Lemma .9.1, it is enough to show that $\phi_{[x,m]}(R_i, R_{N\setminus i}) \geq \phi_{[x,m]}(R'_i, R_{N\setminus i})$ where $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in \mathcal{D}^n$ are such that either $\left[\tau^+(R_i) < x \text{ and } \tau^+(R'_i) \geq x\right]$ or $\left[\tau^-(R_i) \geq x \text{ and } x \leq \tau^+(R'_i) < x + \kappa - 1\right]$. First we consider the case where $\tau^+(R_i) < x$ and $\tau^+(R'_i) \geq x$. In view of Lemma .9.1 , it is enough to show that $\phi_{[x,m]}(R_i, R_{N\setminus i}) \leq \phi_{[x,m]}(R'_i, R_{N\setminus i})$ where $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in S$ are such that $\tau^+(R_i) < x$ and $\tau^+(R'_i) \geq x$. Since ϕ is plateau-only, assume without loss of generality that R_i is left single-plateaued. Then $U(x-1, R_i) = [1, x-1]$. By strategy-proofness, $\phi_{U(x-1,R_i)}(R_i, R_{N\setminus i}) \geq \phi_{U(x-1,R_i)}(R'_i, R_{N\setminus i})$, which means $\phi_{[x,m]}(R_i, R_{N\setminus i}) \leq \phi_{[x,m]}(R'_i, R_{N\setminus i})$. Next we consider the case where $\tau^-(R_i) \geq x$ and $x \leq \tau^+(R'_i) < x + \kappa - 1$. Next we show that $(\beta(\underset{\sim}{n}, x))$ satisfies Condition (ii) in Definition 5.5.23. In view of Lemma .9.1, it is enough to show that $\phi_{[x,m]}(R_i, R_{N\setminus i}) \geq \phi_{[x,m]}(R'_i, R_{N\setminus i})$ where $(R_i, R_{N\setminus i}), (R'_i, R_{N\setminus i}) \in \mathcal{D}^n$ are such that $\tau^-(R_i) \geq x$ and $x \leq \tau^+(R'_i) < x + \kappa - 1$. Since ϕ is plateau-only, assume without loss of generality that R_i is right single-plateaued. Then $U(x, R_i) = [x, m]$. By strategy-proofness, $\phi_{U(x,R_i)}(R_i, R_{N\setminus i}) \geq \phi_{U(x,R_i)}(R'_i, R_{N\setminus i})$, which means $\phi_{[x,m]}(R_i, R_{N\setminus i}) \geq \phi_{[x,m]}(R'_i, R_{N\setminus i})$.

Finally, we show that $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ satisfies Condition (i) of Definition 5.5.23. Take $\underset{\sim}{n}, \underset{\sim}{n}^+ \in \underset{\sim}{N}$, $R_N \in \mathcal{D}^n$ and $x \in [3, m]$ such that $n^x_l(R_N) = \underset{\sim}{n}_l$ and $\underset{\sim}{n}^{x-1}_l(R_N) = \underset{\sim}{n}^+_l$ for all $0 \leq l \leq \kappa - 1$. It is easy to see that such a R_N and x exist for every possible choice of $\underset{\sim}{n}$ and $\underset{\sim}{n}^+$. Since ϕ is an RSCF, $\phi_{[x,m]}(R_N) \leq \phi_{[x-1,m]}(R_N)$. By the construction of $(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}$, we have $0 \leq \beta(\underset{\sim}{n}, x) \leq \beta(\underset{\sim}{n}^+, x-1) \leq 1$ for all $\underset{\sim}{n}, \underset{\sim}{n}^+ \in \underset{\sim}{N}$. ■

.10 Proof of Corollary 5.5.1

Proof: It remains to show that a κ-plateaued rule is unanimous if and only if it is based on a collection of unanimous plateau parameters.

(If part) Suppose $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ is unanimous and ϕ is a κ-plateaued rule with respect to $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$. We show ϕ is unanimous.

Let $R_N \in \mathcal{D}^n$ be such that $\cap_{i\in N}\tau(R_i) \neq \emptyset$. Let $\cap_{i\in N}\tau(R_i) = [y, z]$. Then, $n_0^y(R_N) = n$ and $n_{\kappa-1}^y(R_N) > 0$ (see Figure .10.1 for details). Since $\hat{\underset{\sim}{n}}_0 = n$ and $\hat{\underset{\sim}{n}}_{\kappa-1} > 0$, by unanimity of plateau parameters, we have $\beta(\hat{\underset{\sim}{n}}, y) = 1$. Because $\phi_{[y,m]}(R_N) = \beta(\hat{\underset{\sim}{n}}, y)$, this means $\phi_{[y,m]}(R_N) = 1$. As $\cap\tau(R_i) = [y, z]$, $n_0^{z+1}(R_N) < n$ and $n_{\kappa-1}^{z+1}(R_N) = 0$ (see Figure .10.1 for details). Since $\bar{\underset{\sim}{n}}_0 < n$ and $\bar{\underset{\sim}{n}}_{\kappa-1} = 0$, by unanimity of plateau parameters, we have $\beta(\bar{\underset{\sim}{n}}, z + 1) = 0$. Because $\phi_{[z+1,m]}(R_N) = \beta(\bar{\underset{\sim}{n}}, z + 1)$, this means $\phi_{[z+1,m]}(R_N) = 0$. So, we have $\phi_{[y,z]}(R_N) = 1$.

(Only-if part) Let ϕ be a strategy-proof, plateau-only, anonymous and unanimous RSCF on $\mathcal{D}^n$. We show it is a κ-plateaued rule with respect to unanimous plateau parameters. By Theorem 5.5.29, ϕ is a κ-plateaued rule. Let $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ be the plateau parameters of ϕ. We need to show that the collection $\{(\beta(\underset{\sim}{n}, x))_{(\underset{\sim}{n},x)\in\mathcal{F}(\kappa)}\}$ is unanimous.

Since, ϕ is unanimous and strategy-proof, by Theorem 5.5.3, it is Pareto optimal. Take $R_N \in \mathcal{D}^n$ and $x \in [2, m]$ such that $n_0^x(R_N) = n$ and $n_{\kappa-1}^x(R_N) > 0$. Since, $n_0^x(R_N) = n$, $\min_{i\in N}(\tau^+(R_i)) \geq x$. Also, since $n_{\kappa-1}^x(R_N) > 0$, there must exist $\hat{i} \in N$ such that $\tau^-(R_{\hat{i}}) \geq x$. Take $y < x$. As $\min_{i\in N}(\tau^+(R_i)) \geq x$, xR_iy for all $i \in N$. Moreover, because $\tau^-(R_{\hat{i}}) \geq x$, $xP_{\hat{i}}y$. By Pareto optimality of ϕ, $\phi_y(R_N) = 0$. Since, $y < x$ is arbitrary, this means $\phi_{[x,m]}(R_N) = 1$. Because $\phi_{[x,m]}(R_N) = \beta(\underset{\sim}{n}, x)$ where $(\underset{\sim}{n}, x)$ is such that $\underset{\sim}{n}_0 = n$ and $\underset{\sim}{n}_{\kappa-1} > 0$, it follows that $\beta(\underset{\sim}{n}, x) = 1$.

Now take $R_N \in \mathcal{D}^n$ such that $n_0^x < n$ and $n_{\kappa-1}^x = 0$. Since $n_0^x < n$, this means there exists $\bar{i} \in N$ such that $\tau^+(R_{\bar{i}}) < x$. Moreover, as $n_{\kappa-1}^x = 0$, $\min_{i\in N}\tau^-(R_i) < x$. Take $y \geq x$. As $\min_{i\in N}\tau^-(R_i) < x$, $\min_{i\in N}\tau^-(R_i)R_iy$ for all $i \in N$. Also, as $\tau^+(R_{\bar{i}}) < x$, this means $\tau^+(R_{\bar{i}})P_{\bar{i}}y$. By Pareto optimality, $\phi_y(R_N) = 0$. Since $y \geq x$ is arbitrary, we have $\phi_{[x,m]}(R_N) = 0$. Because $\phi_{[x,m]}(R_N) = \beta(\underset{\sim}{n}, x)$ where $(\underset{\sim}{n}, x)$ is such that $\underset{\sim}{n}_0 < n$ and $\underset{\sim}{n}_{\kappa-1} = 0$, it follows that $\beta(\underset{\sim}{n}, x) = 0$.

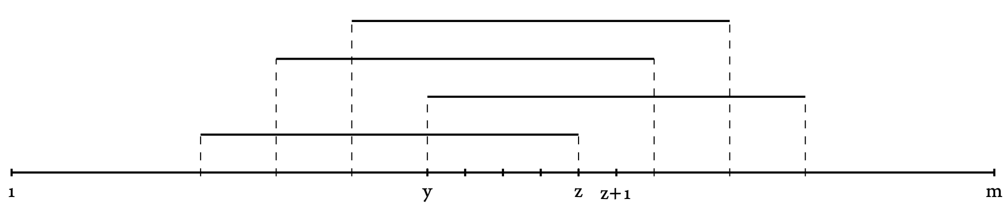


Figure .10.1: A preference profile $R_N \in \mathcal{D}^n$ such that $\cap_{i\in N}\tau(R_i) \neq \emptyset$

■

.11 Proof of Theorem 5.5.29

Proof: Let $\mathcal{D}$ be a $(\kappa, \hat{\kappa})$-single-plateaued domain for some $\kappa \in \{1, \ldots, m\}$ and some $\hat{\kappa} \geq \kappa$ and let $\phi : \mathcal{D}^n \to \Delta A$ be a κ-plateaued rule. By definition ϕ is anonymous and plateau-only. To show that ϕ is strategy-proof, by Theorem 5.5.15, it is enough to show that it satisfies generalized uncompromisingness. Consider $i \in N$, $R_i, R_i' \in \mathcal{D}$, $R_{N\setminus i} \in \mathcal{D}^{n-1}$, and $x \in A$. First we show that Condition (i) in Definition 5.5.10 holds. We distinguish the following two cases.

Case 1. Suppose $x \leq \min\{\tau^-(R_i), \tau^-(R_i')\}$.

As $\mathcal{D}$ is a $(\kappa, \hat{\kappa})$-single-plateaued domain, $|\tau(R_i)| \geq \kappa$ and $|\tau(R_i')| \geq \kappa$. This implies $n_l^x(R_i, R_{N\setminus i}) = n_l^x(R_i', R_{N\setminus i})$ for all $0 \leq l \leq \kappa - 1$. By the definition of κ-plateaued rule, this yields $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$.

Case 2. Suppose $\max\{\tau^+(R_i), \tau^+(R_i')\} < x$.

This implies $n_l^x(R_i, R_{N\setminus i}) = n_l^x(R_i', R_{N\setminus i})$ for all $0 \leq l \leq \kappa - 1$. By the definition of κ-plateaued rule, this yields $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$.

Next we show ϕ satisfies Condition (ii) in Definition 5.5.10. We distinguish the following two cases.

Case 1. Suppose $\tau^+(R_i) < x \leq \tau^+(R_i')$.

By the definition of κ-plateaued rule, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta(\underset{\sim}{n}, x)$ and $\phi_{[x,m]}(R_i', R_{N\setminus i}) = \beta(\underset{\sim}{n}', x)$, where for all $0 \leq l \leq \kappa - 1$, $\underset{\sim}{n}_l = n_l^x(R_i, R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i', R_{N\setminus i})$. Since $\tau^+(R_i) < x$ and $\tau^+(R_i') \geq x$, there must exist $0 \leq l' \leq \kappa - 1$ such that $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $0 \leq l \leq l'$, and $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i})$ for all $l' < l \leq \kappa - 1$. This implies $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$. Therefore, by Condition (ii) of Definition 5.5.23, we have $\beta(\underset{\sim}{n}, x) \leq \beta(\underset{\sim}{n} \oplus 1, x)$, and hence it follows that $\phi_{[x,m]}(R_i', R_{N\setminus i}) \geq \phi_{[x,m]}(R_i, R_{N\setminus i})$.

Case 2. Suppose $\tau^-(R_i) < x \leq \tau^-(R_i')$.

By the definition of κ-plateaued rule, $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \beta(\underset{\sim}{n}, x)$ and $\phi_{[x,m]}(R_i', R_{N\setminus i}) = \beta(\underset{\sim}{n}', x)$, where for all $0 \leq l \leq \kappa - 1$, $\underset{\sim}{n}_l = n_l^x(R_i, R_{N\setminus i})$ and $\underset{\sim}{n}_l' = n_l^x(R_i', R_{N\setminus i})$. If $\tau^+(R_i) < x$, then $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $0 \leq l \leq \kappa - 1$. However, if $x \leq \tau^+(R_i) < x + \kappa - 1$, then there must exist l' such that $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i})$ for all $0 \leq l \leq l'$ and $n_l^x(R_i', R_{N\setminus i}) = n_l^x(R_i, R_{N\setminus i}) + 1$ for all $l' < l \leq \kappa - 1$. In both these cases, $\underset{\sim}{n}' = \underset{\sim}{n} \oplus 1$, and hence by using Condition (ii) of Definition 5.5.23, we have $\phi_{[x,m]}(R_i', R_{N\setminus i}) \geq \phi_{[x,m]}(R_i, R_{Nx\setminus i})$. If $\tau^+(R_i) \geq x + \kappa - 1$, $n_l^x(R_i, R_{N\setminus i}) = n_l^x(R_i', R_{N\setminus i})$ for all $0 \leq l \leq \kappa - 1$. By the definition of κ-plateaued rule, this yields $\phi_{[x,m]}(R_i, R_{N\setminus i}) = \phi_{[x,m]}(R_i', R_{N\setminus i})$. ■

6

The Structure of (Local) Ordinal Bayesian Incentive Compatible Random Rules

6.1 Introduction

We consider social choice problems where a random social choice function (RSCF) selects a probability distribution over a finite set of alternatives at every collection of the agents' preferences in a society. An RSCF is dominant strategy incentive compatible (DSIC) if no agent can increase the probability of any upper contour set by misreporting her preference.[1] A random Bayesian rule (RBR) consists of an RSCF and a prior belief of each agent about the preferences of the others. We assume that the prior of an agent is "partially correlated": her belief about the preference of one agent may depend on that about another agent, but it does not depend on her own preference. Ordinal Bayesian incentive compatibility (OBIC) is the natural extension of the notion of incentive compatibility (IC) for RBRs. This notion is introduced in [29] and it captures the idea of Bayes-Nash equilibrium in the context of incomplete information game. An RBR is OBIC if no agent can increase the expected probability (with respect to her belief) of any upper contour set by misreporting her preference.

[1]An upper contour set at a preferences is a set "top k alternatives" for some number k, that is, alternatives having rank less than or equal to k.

The importance of Bayesian rules is well-established in the literature: on the one hand, they model real life situations where agents behave according to their beliefs, on the other hand, they are significant weakening of the seemingly too demanding requirement of DSIC that leads to dictatorship (or random dictatorships) unless the domain is restricted. It is worth mentioning that the RBRs are particularly important as randomization has long been recognized as a useful device to achieve fairness in allocation problems.

Locally DSIC (LDSIC) or locally OBIC (LOBIC) are weaker versions of the corresponding notions. As the name suggests, they apply to deviations/misreports to only "local" preferences (the notion of which is fixed a priori). The importance of these local notions is well-established in the literature. They are useful in modeling behavioral agents (see [19]). Furthermore, on many domains they turn out to be equivalent to their corresponding global versions, and thereby, they are used as a simpler way to check whether a given RSCF is DSIC (see [19], [80], [26], [54], etc.).

The main objective of this paper is to explore the structure of LOBIC RBRs on different domains. The structure of DSIC RSCFs is well-explored in the literature. On the unrestricted domain, they turn out to be random dictatorial, and on restricted domains such as single-peaked or single-crossing or single-dipped, they are some versions of probabilistic fixed ballot rules. However, to the best of our knowledge, the only thing known about the structure of LOBIC (or OBIC) RBRs is that if there are exactly two agents and at least four alternatives, then for almost all prior profiles (that is, for a set of prior profiles having full measure), a unanimous, neutral and OBIC RBR is random dictatorial ([57]).[2] Even for *deterministic* Bayesian rules (DBRs), not much is known. [58] show that for almost all prior profiles, a unanimous and OBIC DBR on the unrestricted domain is dictatorial, and later, [61] shows that for almost all prior profiles, an "elementary monotonic" and OBIC DBR on a swap-connected domain is DSIC. Recently, [46] extend these results for sparsely connected domains without restoration.[3]

Most of the existing literature consider the notion of localness that is derived from Kemeny distance (see [50] and [51] for further details about Kemeny distance).[4] According to this notion, two preferences are local if they differ by a swap of two adjacent alternatives. [19] and [80] provide the following motivations for using local strategy-proofness.

(i) Local notions of incentive compatibility makes it simpler for the designer to check if a given rule is DSIC.

(ii) Due to social stigma or self-guilt or bounded rationality, some behavioral agents consider

[2] A set of prior profiles is said to have full measure if its complement has Lebesgue measure zero.

[3] We provide a detailed discussion on the connection between our results and those in [46] in Section 6.9.3.

[4] [50] provides a characterization of the Kemeny distance with five axioms; metric, betweenness, neutrality, normalization and reducibility. Later, [17] provide another characterization where it is shown that the reducibility axiom is redundant for the characterization.

manipulations only for some particular deviations. Such deviations are captured by the notion of local preferences.

Following [54] and [55], we consider an arbitrary notion of localness which we formulate by a graph over preferences. The motivation behind this consideration is as follows. Firstly, when it comes to the task of checking whether a given rule is DSIC, which notion of localness will be suitable for this purpose totally depends on the device that the designer uses and the computational complexity in checking local DSIC. Secondly, when it comes to modeling behavioral agents there is no reason to assume such an agent will consider only manipulations by swapping two adjacent alternatives. Clearly, such local deviations depend on the agents, as well as on the particular context. For instance, an agent may try to manipulate by moving an alternative to the top of her sincere preference whenever she tries to make that alternative the outcome. Furthermore, the use of such a general notion enables us to apply our results on a large number of domains like multi-dimensional domains, domains under partitioning, domains under categorization, sequentially dichotomous domains, etc where it is not always possible to swap two adjacent alternatives without affecting the ranking of other alternatives.

We introduce the notion of lower contour monotonicity for an RBR and in Theorem 6.3.2 establish the equivalence between LOBIC and the much stronger (and well-studied) notion LDSIC on *any* domain for RBRs satisfying this property. The deterministic version of this result for the special case of swap-local domains is proved in [61].[5]

We show that under LOBIC, unanimity implies lower contour monotonicity on the unrestricted domain. Therefore, it follows as a corollary of Theorem 6.3.2 that for almost all prior profiles, unanimous and LOBIC (and hence OBIC) RBRs on the unrestricted domain are random dictatorial. Next, we move to restricted domains. It turns out that unanimity is not strong enough to ensure lower contour monotonicity for LOBIC RBRs on most well-known restricted domains. Therefore, we proceed to explore the relation of unanimity to another important property of a rule, namely tops-onlyness, on such domains.

Tops-onlyness is a strong property for a rule as it says that the designer can ignore any information about a preference beyond the top-ranked alternative. On the positive side, this property makes the structure of a rule quite simple, however, on the negative side, this property is not quite desirable as it ignores most part of a preference and thereby significantly restricts the scope for designing incentive compatible rules. Interestingly, the negative side of the tops-only property does not play any role for some domains as unanimity alone enforces it under DSIC. [20] provide a sufficient condition on a domain so that unanimity and DSIC imply tops-onlyness for DSCFs on it. Later, [22] show that the same sufficient condition does not work for RSCFs, and consequently, they provide a stronger sufficient condition on a

[5] A graph on a domain is swap-local if any two local preferences differ by a swap of consecutively ranked alternatives.

domain so that unanimity and DSIC imply tops-onlyness. We provide a sufficient condition on a domain so that for almost all prior profiles, unanimous and graph-LOBIC RBRs imply tops-onlyness. It is worth mentioning that establishing the tops-only property is a major step in characterizing unanimous and OBIC RBRs. Tops-onlyness significantly weakens the requirement of LCM on the RBR and hence plays a crucial role in generalizing our results for the equivalence of LOBIC and LDSIC (see Remark 6.3.6, Proposition 6.4.1 and Corollary 6.4.1)

Finally, we provide a discussion explaining why none of these results can be extended for fully correlated priors (that is, when the prior of an agent depends on her own preference). It is worth emphasizing that all the existing results for LOBIC DBRs ([58] and [61]) follow from our results. Furthermore, since every OBIC rule is LOBIC by definition, all our results hold for OBIC rules in particular.

The results in this paper hold for RBRs for almost all priors profiles, that is, for each prior profile in a set of prior profiles having full measure. It is worth mentioning the economic motivation of such results. Firstly, if the designer thinks all prior profiles are equally likely (or she does not have any particular information about prior profiles), then she knows that except for some "rare" cases (with Lebesgue measure zero), an RBR is LOBIC (or OBIC) if and only if its RSCF component is LDSIC (or DSIC). Since the structure of LDSIC (or DSIC) RSCFs is much simpler, she can use her knowledge about the same in dealing with the RBRs for such prior profiles. Secondly, if the objective of the designer is to maximize the expected total welfare (with respect to any prior distribution over preference profiles and the uniform distribution over prior profiles) of a society over LOBIC (or OBIC) RBRs, then she can restrict her attention (that is, the feasible set) to the LDSIC (or DSIC) RSCFs. This is because a non-LDSIC RSCF can be part of a LOBIC (or OBIC) RBR only for a (Lebesgue) measure zero set of cases which will not contribute to the expected value.

[58] introduce the notion of generic priors, the particularity of which is that they have full measure. It is shown in Example 1 of [57] that a unanimous and OBIC RBR with respect to a generic prior profile need not be random dictatorial, and therefore, it seemed that the dictatorial result does not extend (almost surely) for OBIC RBRs. However, it follows from our results that in fact it does, only thing is that one needs to construct the right class of priors ensuring the full measure.

We provide a wide range of applications of our results. We introduce the notion of betweenness domains and establish the structure of RBRs that are LOBIC for almost all prior profiles on these domains. Well-known restricted domains such as single-peaked on arbitrary graphs, hybrid, multiple single-peaked, single-dipped, single-crossing, and domains under partitioning are important examples of betweenness domains. We introduce a weaker version of lower contour monotonicity and obtain a characterization of unanimous RBRs or DBRs (depending on what is known in the literature regarding the equivalence of LDSIC and DSIC) that are LOBIC on these domains for almost all prior profiles.

Our consideration of arbitrary notion of localness allows us to deal with multi-dimensional domains. The importance of such domains is well understood in the literature; we provide a discussion on this in Section 6.8. We provide the structure of LOBIC RBRs on full separable multi-dimensional domains when the marginal domains satisfy the betweenness property, for instance, when the marginal domains are unrestricted or single-peaked on graphs or hybrid or multiple single-peaked or single-dipped or single-crossing.

The rest of the paper is organized as follows. Section 6.2 introduces the notions of domains, RSCFs, priors, RBRs, and their relevant properties. Sections 6.3 and 6.4 present our results for graph-connected and swap-connected domains. Sections 6.5, 6.6, 6.7 and 6.8 present the applications of our results on unrestricted, betweenness, non-regular and multi-dimensional domains. Finally, in Section 6.9 we provide a discussion on DBRs, (fully) correlated priors, and the relation of our paper with [46].

6.2 Preliminaries

We denote a finite set of alternatives by A and a finite set of n agents by N. A (strict) preference over A is defined as a linear order on A.[6] The set of all preferences over A is denoted by $\mathcal{P}(A)$. A subset $\mathcal{D}$ of $\mathcal{P}(A)$ is called a domain. Whenever it is clear from the context, we do not use brackets to denote singleton sets.

The weak part of a preference P is denoted by R. Since P is strict, for any two alternatives x and y, xRy implies either xPy or $x = y$. The kth ranked alternative in a preference P is denoted by $P(k)$.[7] The top-set $\tau(\mathcal{D})$ of a domain $\mathcal{D}$ is defined as the set of alternatives $\cup_{P \in \mathcal{D}} P(1)$. A domain $\mathcal{D}$ is regular if $\tau(\mathcal{D}) = A$. The upper contour set $U(x, P)$ of an alternative x at a preference P is defined as the set of alternatives that are strictly preferred to x in P, that is, $U(x, P) = \{a \in A \mid aPx\}$. A set U is called an upper contour set at P if it is an upper contour set of some alternative at P. The restriction of a preference P to a subset B of alternatives is denoted by $P|_B$, more formally, $P|_B \in \mathcal{P}(B)$ such that for all $a, b \in B$, $aP|_B b$ if and only if aPb. We use the following terminologies to ease the presentation: $P \equiv xy\cdots$ means $P(1) = x$ and $P(2) = y$; $P \equiv \cdots xy \cdots$ means x and y are consecutively ranked in P with xPy; $P \equiv \cdots x \cdots y \cdots$ means x is ranked above y. When the set of alternatives is precisely stated, say $A = \{a, b, c, d\}$, we write, for instance, $P = abcd$ to mean $P(1) = a$, $P(2) = b$, $P(3) = c$, and $P(4) = d$. We use similar notations without further explanations.

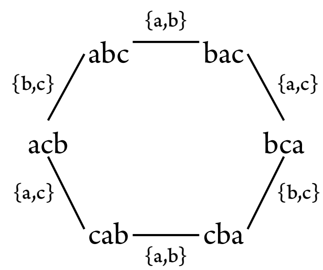
Each agent $i \in N$ has a domain $\mathcal{D}_i$ (of admissible preferences). We assume that each domain $\mathcal{D}_i$ is endowed with some graph structure $G_i = \langle \mathcal{D}_i, E_i \rangle$ where $E_i \subseteq \mathcal{D}_i \times \mathcal{D}_i$ is the set of edges. The graph G_i represents the proximity relation between the preferences: an edge between two preferences implies that

[6] A linear order is a complete, transitive, and antisymmetric binary relation.

[7] The rank of an alternative a in a preference P is k if and only if $|\{b \in A \mid bPa\}| = k - 1$.

they are close in some sense. The closeness plays the role that whenever an agent tries to manipulate: she only misreports her sincere preference as the one that is close to her sincere one. For instance, suppose $A = \{a, b, c\}$ and $\mathcal{D}_i$ is the set of all preferences over A. Suppose that two preferences are "close" if and only if they are swap-local, that is, differ by a swap of two consecutive alternatives. In other words, two preferences are close if their Kemeny distance is 1.[8] The graph G_i that represents this proximity relation is given in Figure 6.2.1. The alternatives that swap between two preferences are mentioned on the edge between the two.

Figure 6.2.1: The graph representing the proximity relation that two preferences are close if and only if they differ by a swap of two consecutive alternatives



We provide an example to explain why a proximity relation need not always be based on the Kemeny distance. Suppose that an agent cannot do complicated calculations in order to manipulate, she just moves an alternative to the top of her sincere preference whenever she tries to make that alternative the outcome by misreporting her preference. According to such a proximity, a preference P is close to a preference P' if P' is obtained by moving an alternative to the top position at P. For instance, *abc* is close to *bac* and *cab*. Another important instance that cannot be modelled by swap-localness is the one where the alternatives have multiple dimensions and preferences are separable.[9]

We denote by G_N a collection of graphs $(G_i)_{i \in N}$. Whenever we use some term involving the word "graph", we mean it with respect to a collection G_N. Two preferences P_i and P_i' of an agent i are graph-local if they form an edge in G_i, and a sequence of preferences $(P_i^1, \ldots, P_i^t)$ is a graph-local path if every two consecutive preferences in the sequence are graph-local. A domain $\mathcal{D}_i$ is graph-connected if there is a graph-local path between any two preferences in it. We denote by $\mathcal{D}_N$ the product set $\mathcal{D}_1 \times \cdots \times \mathcal{D}_n$ of individual domains. An element of $\mathcal{D}_N$ is called a preference profile. All the domains we consider in this paper are assumed to be graph-connected.

[8]The Kemeny distance between two preferences is the minimum number of adjacent flips that is required to reach one preference from the other.

[9]See Section 6.8 for a formal definition of a separable preference.

6.2.1 Random social choice functions and their properties

Let ΔA be the set of all probability distributions on A. A random social choice function (RSCF) is a mapping $\phi : \mathcal{D}_N \to \Delta A$. We denote the probability of an alternative x at $\phi(P_N)$ by $\phi_x(P_N)$. An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is called a deterministic social choice function (DSCF) if $\phi_x(P_N) \in \{0, 1\}$ for all $x \in A$ and all $P_N \in \mathcal{D}_N$.

An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is **unanimous** if for all $P_N \in \mathcal{D}_N$ such that for all $i \in N$, $P_i(1) = x$ for some $x \in A$, we have $\phi_x(P_N) = 1$. An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is **tops-only** if for all $P_N, P'_N \in \mathcal{D}_N$ such that $P_i(1) = P'_i(1)$ for all $i \in N$, we have $\phi(P_N) = \phi(P'_N)$.

A probability distribution ν stochastically dominates another probability distribution $\hat{\nu}$ at a preference P, denoted by $\nu P^{\text{sd}} \hat{\nu}$, if $\nu_{U(x,P)} \geq \hat{\nu}_{U(x,P)}$ for all $x \in A$ and $\nu_{U(y,P)} > \hat{\nu}_{U(y,P)}$ for some $y \in A$.[10] We write $\nu R^{sd} \hat{\nu}$ to mean either $\nu P^{sd} \hat{\nu}$ or $\nu = \hat{\nu}$. An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is *dominant strategy incentive compatible (DSIC) on a pair of preferences* (P_i, P'_i) of an agent $i \in N$, if $\phi(P_i, P_{-i}) R_i^{\text{sd}} \phi(P'_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}_{-i}$. An RSCF is **graph-locally dominant strategy incentive compatible (graph-LDSIC)** if it is DSIC on every pair of graph-local preferences of each agent, and it is called **dominant strategy incentive compatible (DSIC)** if it is DSIC on *every* pair of preferences of each agent. Note that the pair (P_i, P'_i) is ordered in the definition of DSIC on a pair of preference (P_i, P'_i), in particular, DSIC on the pair of preferences (P_i, P'_i) is different from DSIC on the pair of preferences (P'_i, P_i).

A set of alternatives B is a block in a pair of preferences (P, P') if it is a minimal non-empty set satisfying the following property: for all $x \in B$ and $y \notin B$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$. For instance, the blocks in the pair of preferences $(abcdefg, bcadegf)$ are $\{a, b, c\}$, $\{d\}$, $\{e\}$, and $\{f, g\}$. The lower contour set $L(x, P)$ of an alternative x at a preference P is $L(x, P) = \{a \in A \mid xPa\}$. A set L is a lower contour set at a preference P if it is a lower contour set of some alternative at P. Lower contour monotonicity says that whenever an agent i unilaterally deviates from P_i to a graph-local preference P'_i, the probability of each lower contour set at P_i restricted to any non-singleton block in (P_i, P'_i) will weakly increase. For instance, consider our earlier example $P_i = abcdefg$ and $P'_i = bcadegf$ with non-singleton blocks $\{a, b, c\}$ and $\{f, g\}$. The lower contour sets at P_i restricted to $\{a, b, c\}$ are $\{c\}$ and $\{b, c\}$, and that restricted to $\{f, g\}$ is $\{g\}$. Lower contour monotonicity says that the probability of each of the sets $\{c\}$, $\{b, c\}$, and $\{g\}$ will weakly increase if agent i unilaterally deviates from P_i to P'_i. The intuition behind lower contour monotonicity is simple, roughly speaking it says that whenever some alternatives are moved up in a preference, their probabilities also weakly increase. Clearly, it is a generalization of well-known monotonicity condition.

Definition 6.2.1 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is called **lower contour monotonic** if for all $i \in N$, all graph-local preferences $P_i, P'_i \in \mathcal{D}_i$, all non-singleton blocks B in (P_i, P'_i), and all $P_{-i} \in \mathcal{D}_{-i}$, we have*

[10] For a subset B of A, we denote by ν_B the total probability of the set B according to the probability distribution ν.

$\phi_L(P_i, P_{-i}) \leq \phi_L(P_i', P_{-i})$ *for each lower contour set* L *of* $P_i|_B$.

6.2.2 Priors, Random Bayesian rules, and their properties

A prior μ_i of an agent i is a probability distribution over $\mathcal{D}_{-i}$ which represents her belief about the preferences of the others, and a prior profile $\mu_N := (\mu_i)_{i \in N}$ is a collection of priors, one for each agent.

Definition 6.2.2 *Consider an RSCF* $\phi : \mathcal{D}_N \to \Delta A$. *A prior profile* μ_N *is called* ***compatible*** *with* ϕ *if for all* $i \in N$, *all* $P_i, P_i' \in \mathcal{D}_i$, *and all* $X \subsetneq A$,

$$\sum_{P_{-i}} \mu_i(P_{-i}) \big(\phi_X(P_i, P_{-i}) - \phi_X(P_i', P_{-i}) \big) = 0 \tag{6.1}$$
$$\implies \phi_X(P_i, P_{-i}) - \phi_X(P_i', P_{-i}) = 0 \text{ for all } P_{-i}.$$

Let $\mathcal{M}(\phi)$ denote the set of all prior profiles that are compatible with ϕ. It is worth noting that the prior μ_i of an agent i does not depend on her preference P_i.

A pair (ϕ, μ_N) consisting of an RSCF $\phi : \mathcal{D}_N \to \Delta A$ and a prior profile μ_N is called a random Bayesian rule (RBR) on $\mathcal{D}_N$. When the RSCF ϕ is a DSCF, then it is called a deterministic Bayesian rule (DBR).

The expected outcome with respect to the belief of an agent is called her interim expected outcome. More formally, the *interim expected outcome* $\phi(P_i, \mu_i)$ for an agent $i \in N$ at a preference $P_i \in \mathcal{D}_i$ from an RBR (ϕ, μ_N) on $\mathcal{D}_N$ is defined as the following probability distribution on A: for all $x \in A$,

$$\phi_x(P_i, \mu_i) = \sum_{P_{-i} \in \mathcal{D}_{-i}} \mu_i(P_{-i}) \phi_x(P_i, P_{-i}).$$

Example 6.2.3 *Let* $N = \{1, 2\}$ *and* $A = \{a, b, c\}$. *Consider the RBR* (ϕ, μ_N) *given in Table 6.2.2. Agent 1's belief* μ_1 *about agent 2's preferences is given in the top row and agent 2's belief* μ_2 *about agent 1's preferences in the leftmost column of the table. The outcomes of* ϕ *at different profiles are presented in the corresponding cells. Here, for instance,* $(0.7, 0, 0.3)$ *denotes the outcome where a, b, and c are given probabilities* 0.7, 0, *and* 0.3, *respectively. The rest of the table is self-explanatory. Consider the preference* $P_1 = abc$ *of agent 1. In what follows, we show how to compute her interim expected outcome* $\phi(P_1, \mu_1)$ *at this preference:* $\phi_a(P_1, \mu_1) = 0.2 \times 1 + 0.1 \times 1 + 0.05 \times 1 + 0.3 \times 0.5 + 0.15 \times 1 + 0.2 \times 1 = 0.85$. *Similarly, one can calculate that* $\phi_b(P_1, \mu_1) = 0.15$, *and* $\phi_c(P_1, \mu_1) = 0$, *and for agent 2's preference* $P_2 = bca$, $\phi_b(P_2, \mu_2) = 0.575$, $\phi_c(P_2, \mu_2) = 0.06$, *and* $\phi_a(P_2, \mu_2) = 0.365$.

Table 6.2.1: The RBR (ϕ, μ_N) in Example 6.2.3

	μ_1	0.2	0.1	0.05	0.3	0.15	0.2
μ_2	2 1	abc	acb	bac	bca	cba	cab
0.25	abc	(1,0,0)	(1,0,0)	(1,0,0)	(0.5,0.5,0)	(1,0,0)	(1,0,0)
0.2	acb	(1,0,0)	(1,0,0)	(1,0,0)	(0.7,0,0.3)	(1,0,0)	(1,0,0)
0.15	bac	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(1,0,0)
0.1	bca	(0,1,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
0.2	cba	(1,0,0)	(0,0,1)	(0,0.4,0.6)	(0,1,0)	(0,0,1)	(0,0,1)
0.1	cab	(1,0,0)	(0,0.4,0.6)	(1,0,0)	(1,0,0)	(0,0,1)	(0,0,1)

Table 6.2.2: The RBR (ϕ, μ_N) in Example 6.2.3

	μ_1	0.2	0.1	0.05	0.3	0.15	0.2
μ_2	1 \ 2	abc	acb	bac	bca	cba	cab
0.25	abc	(1,0,0)	(1,0,0)	(1,0,0)	(0.5,0.5,0)	(1,0,0)	(1,0,0)
0.2	acb	(1,0,0)	(1,0,0)	(1,0,0)	(0.7,0,0.3)	(1,0,0)	(1,0,0)
0.15	bac	(1,0,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(1,0,0)
0.1	bca	(0,1,0)	(1,0,0)	(0,1,0)	(0,1,0)	(0,1,0)	(0,1,0)
0.2	cba	(1,0,0)	(0,0,1)	(0,0.4,0.6)	(0,1,0)	(0,0,1)	(0,0,1)
0.1	cab	(1,0,0)	(0,0.4,0.6)	(1,0,0)	(1,0,0)	(0,0,1)	(0,0,1)

The notion of ordinal Bayesian incentive compatibility (OBIC) captures the idea of DSIC for an RBR by ensuring that no agent can improve her interim expected outcome by misreporting her preference.

Definition 6.2.4 *An RBR (ϕ, μ_N) on $\mathcal{D}_N$ is ordinal Bayesian incentive compatible (OBIC) on a pair of preferences (P_i, P'_i) of an agent $i \in N$ if $\phi_{\mu_i}(P_i) R_i^{sd} \phi_{\mu_i}(P'_i)$.*[11] *An RBR (ϕ, μ_N) is* ***graph-locally ordinal Bayesian incentive compatible (graph-LOBIC)*** *if it is OBIC on every pair of graph-local preferences in the domain of each agent, and it is* ***ordinal Bayesian incentive compatible (OBIC)*** *if it is OBIC on every pair of preferences in the domain of each agent.*

Note that OBIC is a weaker requirement than DSIC since if an RSCF ϕ is DSIC, then (ϕ, μ_N) is OBIC for all prior profiles μ_N.

For ease of presentation, given a property defined for an RSCF, we say an RBR (ϕ, μ_N) satisfies it, if ϕ satisfies the property.

[11] As in the case of DSIC on a pair of preferences, the pair (P_i, P'_i) is also ordered in the definition of OBIC on a pair of preferences (P_i, P'_i).

6.3 Results on graph-connected domains

In this section, we explore the structure of graph-LOBIC Bayesian rules on graph-connected domains. Since OBIC implies graph-LOBIC (by definition), all these results hold for OBIC RBRs as well.

Recall the definition of a block given in Page 10. The block preservation property says that if an agent unilaterally changes her preference to a graph-local preference, the total probability of any block in the two preferences will remain unchanged.

Definition 6.3.1 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ satisfies the **block preservation property** if for all $i \in N$, all graph-local preferences $P_i, P'_i \in \mathcal{D}_i$ of agent i, all blocks B in (P_i, P'_i), and all $P_{-i} \in \mathcal{D}_{-i}$, we have $\phi_B(P_i, P_{-i}) = \phi_B(P'_i, P_{-i})$.*

For two preferences P and P', $P \triangle P' = \{x \in A \mid U(x, P) \neq U(x, P')\}$ denotes the set of alternatives that change their relative ordering with some other alternative from P to P'. Note that the block preservation property implies $\phi_x(P_i, P_{-i}) = \phi_x(P'_i, P_{-i})$ for all $x \notin P_i \triangle P'_i$ as such an alternative forms a singleton block in (P_i, P'_i).

Our next proposition says that graph-LOBIC implies the block-preservation property almost surely (with probability one). In other words, for each RSCF ϕ, there is a set of prior profiles with full measure such that if it is graph-LOBIC with respect to any of the prior profiles in the set, it will satisfy the block-preservation property. The economic interpretation of this result is that if the designer thinks that all the priors of an agent are equally likely and wants to ensure that no agent can manipulate her RBR, then "almost surely" she needs to make the RSCF component of the RBR satisfy the block-preservation property.

Proposition 6.3.1 *For every RSCF $\phi : \mathcal{D}_N \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, if the RBR (ϕ, μ_N) is graph-LOBIC then ϕ satisfies the block-preservation property.*

The proof of this proposition is relegated to Appendix .2.

6.3.1 Equivalence of graph-LOBIC and graph-LDSIC under lower contour monotonicity

As we have mentioned in Section 6.1, Example 1 of [57] shows that in case of RBRs the equivalence of graph-LDSIC and graph-LOBIC does not hold for "generic priors" under lower contour monotonicity.[12,13] What we show in the following is that the set of generic priors for which the

[12] It is shown in Example 1 of [57] that a unanimous and OBIC RBR with respect to generic priors need not be DSIC. However, it can be verified that the RSCF they consider also satisfies lower contour monotonicity.

[13] See Section 6.9.1 for the definition of generic priors.

equivalence fails is actually rare in the sense that its Lebesgue measure is zero. More formally, we show that under lower contour monotonicity, the notion of graph-LDSIC becomes almost surely equivalent to the much weaker notion of graph-LOBIC.

Theorem 6.3.2 *For every lower contour monotonic RSCF $\phi : \mathcal{D}_N \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, the RBR (ϕ, μ_N) is graph-LOBIC if and only if ϕ is graph-LDSIC.*

The proof of this theorem is relegated to Appendix .3.1.

The economic interpretation of Theorem 6.3.2 is that if the designer wants to construct a graph-LOBIC RBR satisfying lower contour monotonicity, then for almost all prior profiles (that is, with full measure) she can restrict her attention to graph-LDSIC RSCF only.

Even though there is a measure zero set of prior profiles such that the RBR (ϕ, μ_N) is graph-LOBIC but ϕ is not graph-LDSIC, it is important to know the exact structure of that (measure zero) set. The structure of the set depends on the RSCF ϕ through a system of linear equations (see Equation (6.1) in Definition 6.2.2 and Claim .1.1).

It is worth emphasizing that Theorem 6.3.2 holds for *any* domain and for *any* graph structure on it (as long as it is connected). In Sections 6.5, 6.6 and 6.8, we discuss its applications on unrestricted, single-peaked on a graph (and on a tree or a line as special cases), multiple single-peaked, hybrid, multiple single-peaked, intermediate, single-dipped, single-crossing and multi-dimensional separable domains. One can also apply the theorem on domains under categorization, sequentially dichotomous domains, etc.

6.3.2 Relation between unanimity and tops-onlyness

In this section, we show that on any domain satisfying the path-richness property unanimity and graph-LOBIC for almost all prior profiles imply tops-onlyness.

Definition 6.3.3 *A domain $\mathcal{D}$ satisfies the* ***path-richness property*** *if for all preferences $P, P' \in \mathcal{D}$ such that $P(1) = P'(1)$,*

(i) if P and P' are not graph-local, then there is a graph-local path $(P^1 = P, \ldots, P^t = P')$ such that $P^l(1) = P(1)$ for all $l = 1, \ldots, t$, and

(ii) if P and P' are graph-local, then for each preference $\hat{P} \in \mathcal{D}$, there exists a graph-local path $(P^1 = \hat{P}, \ldots, P^t)$ with $P^t(1) = P(1)$ such that for all $l < t$ and all distinct $y, z \in P \triangle P'$, there is a common upper contour set U of P^l and P^{l+1} such that exactly one of y and z is contained in U.

A domain satisfies the path-richness property if for every two preferences P and P' having the same top-ranked alternative, say x, the following happens: (i) if P and P' are not graph-local then there is graph-local path from P to P' such that x appears as the top-ranked alternative in each preference in the path, and (ii) if P and P' are graph-local, then from any preference $\hat{P}$ there is a path to some preference $\bar{P}$ with x as the top-ranked alternative such that for any two alternatives a, b that change their relative ranking from P to P' and for any two consecutive preferences in the path, there is a common upper contour set of the preferences such that exactly one of a and b belongs to it. For an illustration of Condition (ii) of the path-richness property, suppose $A = \{a, b, c, d\}$, $P = abcd$ and $P' = adcb$, and assume that P and P' are graph-local. Consider a preference $\hat{P} = dbca$. Path-richness requires that a path of the following type must be present in the domain: $(dbca, dbac, dabc, adbc)$. To see that this path satisfies (ii), consider two alternatives that change their relative ordering from P to P', say b and c. Note that the upper contour set $\{d, b\}$ in P^1 and P^2 contains b but not c, the upper contour set $\{d, b, a\}$ in P^2 and P^3 contains b but not c, and so on. Path-richness requires that such a path must exist for every preference $\hat{P}$ in the domain.

Table 6.3.1: The domain in Example 6.3.4

P^1	P^2	P^3	P^4	P^5	P^6	P^7	P^8	P^9	P^{10}	P^{11}
a	a	a	c	c	c	c	e	e	e	e
b	c	c	a	b	b	e	c	c	c	d
c	b	b	b	a	e	b	b	b	d	c
d	d	e	e	e	a	a	a	d	b	b
e	e	d	d	d	d	d	d	a	a	a

Example 6.3.4 *Consider the domain in Table 6.3.1. We explain that this domain satisfies the path-richness property. Suppose that two preferences are graph-local if and only if they differ by a swap of two alternatives. Consider the preferences P^1 and P^3 having the same top-ranked alternative. Note that they are not graph-local. The path (P^1, P^2, P^3) is graph-local and a appears as the top-ranked alternative in each preference in the path. So, the path satisfies the requirement of (i). It can be verified that for other non graph-local preferences with the same top-ranked alternative (such as P^4 and P^7, or P^8 and P^{11}, etc.) such a path lies in the domain. Now, consider the preferences P^1 and P^2. Note that they are graph-local and the alternatives b and c are swapped in the two preferences (that is, $P^1 \triangle P^2 = \{a, b\}$). Consider any other preference, say P^7. The path $(P^7, P^6, P^5, P^4, P^3)$ has the property that (a) it ends with a preference that has the same top-ranked alternative a as P^1 and P^2, and (b) for every two consecutive preferences in the path, there is a common upper contour set of the two preferences that contains exactly one of b and c (for instance, the common upper contour set $\{a, c\}$ of P^3 and P^4 contains c but not b, and so on). It can be verified that such a path exists for every pair of graph-local preferences P and P′ having the same top-ranked alternative and for every preference $\hat{P}$. It is worth mentioning that for the kind of*

graph-localness we consider in this example, the requirement of (b) boils down to requiring that the swapping alternatives in the graph-local preferences maintain their relative ranking throughout the path.

The path-richness property may seem to be somewhat involved but we show in Sections 6.5, 6.6 and 6.7, most domains of practical importance like the unrestricted, single-peaked, single-dipped, single-crossing, domains under partitioning, etc. satisfy this property.

Our next theorem says that if the designer wants to construct a unanimous and graph-LOBIC RBR on a domain satisfying the path-richness property, then for almost all prior profiles she can restrict her attention to tops-only RSCFs. Clearly, this makes the construction considerably simpler. As we have mentioned in case of Theorem 6.3.2, the economic implication of this theorem is that if the designer thinks all the priors of an agent are equally likely, then she can be assured that a unanimous and graph-LOBIC RBR on a path-rich domain will be tops-only with probability one.

Theorem 6.3.5 *Suppose $\mathcal{D}$ satisfies the path-richness property. For every unanimous RSCF $\phi : \mathcal{D}_N \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, if the RBR (ϕ, μ_N) is graph-LOBIC then ϕ is tops-only.*

The proof of this theorem is relegated to Appendix .3.2

Remark 6.3.6 *Lower contour monotonicity can be weakened in a straightforward way under tops-onlyness. Let us say that an RSCF satisfies top lower contour monotonicity if it satisfies lower contour monotonicity only over (unilateral) deviations to graph-local preferences where the top-ranked alternative is changed. Thus, top lower contour monotonicity does not impose any restriction for graph-local preferences P and P' with $P(1) = P'(1)$. Clearly, under tops-onlyness, lower contour monotonicity will be automatically guaranteed in all other cases, and hence, top lower contour monotonicity will be equivalent to lower contour monotonicity. Since under graph-LOBIC, unanimity implies tops-onlyness on a large class of domains, this simple observation is of great help for practical applications.* □

6.4 The case of swap-connected domains

In this section, we consider graphs where two preferences are local if and only if they differ by a swap of two consecutively ranked alternatives. Formally, two preferences P and P' are swap-local if $P \triangle P' = \{x, y\}$ for some $x, y \in A$. For two swap-local preferences P and P', we say x overtakes y from P to P' if yPx and $xP'y$. A domain $\mathcal{D}_i$ is swap-connected if there is a swap-local path between any two preferences in it. We use terms like swap-LOBIC, swap-LDSIC, etc. (instead of graph-LOBIC, graph-LDSIC, etc.) to emphasize the fact that the graph is based on the swap-local structure.

When graphs are swap-connected, lower contour monotonicity boils down to the following condition called elementary monotonicity. An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is called **elementary monotonic** if for every $i \in N$, all swap-local preferences $P_i, P_i' \in \mathcal{D}_i$ of agent i, and all $P_{-i} \in \mathcal{D}_{-i}$, x overtakes some alternative from P_i to P_i' implies $\phi_x(P_i, P_{-i}) \leq \phi_x(P_i', P_{-i})$.

Remark 6.4.1 *As we have mentioned in Example 6.3.4, under swap-connectedness, Condition (ii) of the path-richness property (Definition 6.3.3) simplifies to the following condition: if there are two swap-local preferences having the same top-ranked alternative, say x, where two alternatives, say y and z, are swapped, then from every preference in the domain there must be a swap-local path to some preference with x as the top-ranked alternative such that the relative ranking of y and z remains the same along the path.*

6.4.1 Equivalence of swap-LDSIC and weak elementary monotonicity under tops-onlyness

Weak elementary monotonicity ([61]) is a restricted version of elementary monotonicity where the latter is required to be satisfied only for a particular type of profiles where all the agents agree on the ranking of alternatives from rank three onward.

Definition 6.4.2 *An RSCF $\phi : \mathcal{D}^n \to \Delta A$ satisfies **weak elementary monotonicity** if for all $i \in N$, and all (P_i, P_{-i}) and (P_i', P_{-i}) such that $P_i(k) = P_i'(k) = P_j(k)$ for all $j \in N \setminus i$ and all $k > 2$, we have $\phi_{P_i(1)}(P_i, P_{-i}) \geq \phi_{P_i(1)}(P_i', P_{-i})$.*

Our next result says that under tops-onlyness, for almost all priors, weak elementary monotonic and swap-LOBIC RBRs are swap-LDSIC.

Proposition 6.4.1 *For every tops-only and weak elementary monotonic RSCF $\phi : \mathcal{D}_N \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, the RBR (ϕ, μ_N) is swap-LOBIC if and only if ϕ is swap-LDSIC.*

The proof of this theorem is relegated to Appendix .3.3.

We obtain the following corollary from Theorem 6.3.5 and Proposition 6.4.1.

Corollary 6.4.1 *Suppose $\mathcal{D}$ satisfies the path-richness property. For every unanimous and weak elementary monotonic RSCF $\phi : \mathcal{D}_N \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, the RBR (ϕ, μ_N) is swap-LOBIC if and only if ϕ is swap-LDSIC.*

Remark 6.4.3 *[80] and [61] consider swap-connected domains without restoration. [80] shows that swap-DSIC and swap-LDSIC are equivalent on such domains and [61] shows that any unanimous and tops-only DBR on such domains is weak elementary monotonic and swap-LOBIC with respect to generic priors if and only if it is swap-LDSIC. [46] show that unanimous and swap-LOBIC with respect to generic priors DBRs on sparsely-connected domains without restoration are tops-only. [21] provide two conditions, namely, the interior property and the exterior property on the domain that are jointly sufficient for top-onlyness of unanimous and DSIC RSCFs. For the special case of swap-localness, Condition (i) in the definition of the path-richness property (Definition 6.3.3) is the same as the interior property, whereas Condition (ii) of the path-richness property is weaker than exterior property. In Appendix .4 we provide an example of a domain that is path-rich but does not satisfy the conditions provided in [80], [61] and [46].*

6.5 Application on the unrestricted domain

The domain $\mathcal{P}(A)$ containing all preferences over A is called the **unrestricted domain** (over A). Sincethe unrestricted domain satisfies the path-richness property, it follows from Theorem 6.3.5 that for almost all prior profiles, unanimity and swap-LOBIC implies tops-onlyness on the unrestricted domain. The following theorem further establishes that for almost all prior profiles, swap-LOBIC RBRs are in fact swap-LDSIC.

Theorem 6.5.1 *For every unanimous RSCF $\phi : \mathcal{P}^n \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, the RBR (ϕ, μ_N) is swap-LOBIC if and only if ϕ is swap-LDSIC.*

The proof of this theorem is relegated to Appendix .3.4.

[40] shows that every unanimous and DSIC RSCF on the unrestricted domain is *random dictatorial*. An RSCF is random dictatorial if it is convex combination of the dictatorial rules, that is, for each agent there is a fixed probability such that the agent is the dictator with that probability.

Definition 6.5.2 *An RSCF $\phi : \mathcal{D}_N \to \Delta A$ is **random dictatorial** if there exist non-negative real numbers β_i; $i \in N$, with $\sum_{i\in N} \beta_i = 1$, such that for all $P_N \in \mathcal{D}_N$ and $a \in A$, $\phi_a(P_N) = \sum_{\{i|P_i(1)=a\}} \beta_i$.*

Let us call a domain **swap random local-global equivalent (swap-RLGE)** if every swap-LDSIC RSCF on it is DSIC. It follows from [26] that the unrestricted domain is swap-RLGE. Since every OBIC RBR is swap-LOBIC by definition, it follows from Theorem 6.5.1 that the same result as [40] holds for almost all prior profiles even if we replace DSIC with the much weaker notion OBIC.

Corollary 6.5.1 *Let $|A| \geq 3$. For every unanimous RSCF $\phi : \mathcal{P}^n \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, the RBR (ϕ, μ_N) is swap-LOBIC if and only if ϕ is random dictatorial.*

6.6 Applications on domains satisfying the betweenness property

A **betweenness relation** β maps every pair of distinct alternatives (x, y) to a subset of alternatives $\beta(x, y)$ including x and y. We only consider betweenness relations β that are rational: for every $x \in A$, there is a preference P with $P(1) = x$ such that for all $y, z \in A$, $y \in \beta(x, z)$ implies yRz. Such a preference P is said to respect the betweenness relation β. A domain $\mathcal{D}$ respects a betweenness relation β if it contains all preferences respecting β. We denote such a domain by $\mathcal{D}(\beta)$. For a collection of betweenness relations $\mathcal{B} = \{\beta_1, \ldots, \beta_r\}$, we denote the domain $\cup_{l=1}^{r} \mathcal{D}(\beta_l)$ by $\mathcal{D}(\mathcal{B})$.

A pair of alternatives (x, y) is adjacent in β if $\beta(x, y) = \{x, y\}$. A betweenness relation β is **weakly consistent** if for all $x, \bar{x} \in A$, there is a sequence $(x^1 = x, \ldots, x^t = \bar{x})$ of adjacent alternatives in $\beta(x, \bar{x})$ such that for all $l < t$, we have $\beta(x^{l+1}, \bar{x}) \subseteq \beta(x^l, \bar{x})$. A betweenness relation β is **strongly consistent** if for all $x, \bar{x} \in A$, there is a sequence $(x^1 = x, \ldots, x^t = \bar{x})$ of adjacent alternatives in $\beta(x, \bar{x})$ such that for all $l < t$ and all $w \in \beta(x^l, \bar{x})$, we have $\beta(x^{l+1}, w) \subseteq \beta(x^l, \bar{x})$. A collection $\mathcal{B} = \{\beta_1, \ldots, \beta_r\}$ or a betweenness domain $\mathcal{D}(\mathcal{B})$ is strongly/weakly consistent if β_l is strongly/weakly consistent for all $l = 1, \ldots, r$.

Two betweenness relations β and β' are swap-local if for every $x \in A$, there are $P \in \mathcal{D}(\beta)$ and $P' \in \mathcal{D}(\beta')$ such that $P(1) = P'(1) = x$ and P and P' are swap-local. A collection $\mathcal{B}$ of betweenness relations is called swap-connected if for all $\beta, \beta' \in \mathcal{B}$, there is a sequence $(\beta^1 = \beta, \ldots, \beta^t = \beta')$ in $\mathcal{B}$ such that β^l and β^{l+1} are swap-local for all $l < t$.

We now define the local structure on a betweenness domain $\mathcal{D}(\mathcal{B})$ in a natural way. A preference P' is graph-local to another preference P if there is no preference $P'' \in \mathcal{D}(\mathcal{B}) \setminus \{P, P'\}$ that is "more similar" to P than P' is to P, that is, there is no P'' such that for all $x, y \in A$, $P|_{\{x,y\}} = P'|_{\{x,y\}}$ implies $P|_{\{x,y\}} = P''|_{\{x,y\}}$. Our next corollary follows from Theorem 6.3.5.

Corollary 6.6.1 *Let $\mathcal{B}$ be a collection of strongly consistent and swap-connected betweenness relations. For every unanimous RSCF $\phi : \mathcal{D}(\mathcal{B})^n \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, if the RBR (ϕ, μ_N) is graph-LOBIC then ϕ is tops-only.*

The proof of this corollary is relegated to Appendix .3.5.

A domain is called graph deterministic local-global equivalent (graph-DLGE) if every graph-LDSIC DSCF on it is DSIC.

Proposition 6.6.1 *Let $\mathcal{B}$ be a collection of weakly consistent and swap-connected betweenness relations. Then, $\mathcal{D}(\mathcal{B})$ is a graph-DLGE domain.*

The proof of this theorem is relegated to Appendix .3.6.

In what follows, we apply our results to explore the structure of LOBIC RBRs on well-known betweenness domains.

[69] introduce the notion of single-peaked domains on graphs and characterize all unanimous and DSIC RSCFs on these domains. We assume that the set of alternatives is endowed with an (undirected) graph $\mathcal{G} = \langle A, E \rangle$. For $x, \bar{x} \in A$ with $x \neq \bar{x}$, a path $(x^1 = x, \ldots, x^t = \bar{x})$ from x to $\bar{x}$ in $\mathcal{G}$ is a sequence of distinct alternatives such that $\{x^i, x^{i+1}\} \in E$ for all $i = 1, \ldots, t-1$. If it is clear which path is meant, we also denote it by $[x, \bar{x}]$. We assume that $\mathcal{G}$ is connected, that is, there is a path from x to $\bar{x}$ for all distinct $x, \bar{x} \in A$. If this path is unique for all $x, \bar{x} \in A$, then $\mathcal{G}$ is called a tree. A spanning tree of $\mathcal{G}$ is a tree $T = \langle A, E_T \rangle$ where $E_T \subseteq E$. In other words, spanning tree of $\mathcal{G}$ is a tree that can be obtained by deleting some edges of $\mathcal{G}$.

Definition 6.6.1 *A preference P is single-peaked on $\mathcal{G}$ if there is a spanning tree T of $\mathcal{G}$ such that for all distinct $x, y \in A$ with $P(1) \neq y$, $x \in [P(1), y] \implies xPy$, where $[P(1), y]$ is the path from $P(1)$ to y in T. A domain is called* ***single-peaked on*** $\mathcal{G}$ *if it contains all single-peaked preferences on $\mathcal{G}$.*

It follows from the definition that a single-peaked domain $\mathcal{D}_T$ on a tree T can be represented as a betweenness domain $\mathcal{D}(\beta^T)$ where β^T is defined as follows: $\beta^T(x, y) = [x, y]$. Single-peaked domains on graphs are well-known for the cases when the graph $\mathcal{G}$ is a line or a tree.[14] When the graph $\mathcal{G}$ is a line, then the corresponding domain is known in the literature as the **single-peaked domain**.[15]

We now argue that the betweenness relation β^T is strongly consistent. To see that β^T is strongly consistent consider two alternatives x and $\bar{x}$, and consider the unique path $[x, \bar{x}]$ between them in T. Let $[x, \bar{x}] = (x^1 = x, \ldots, x^t = \bar{x})$. By the definition of β^T, the path $[x, \bar{x}]$ lies in (in fact, is equal to) $\beta^T(x, \bar{x})$. Consider $x^l \in \beta^T(x, \bar{x})$ and $w \in \beta^T(x^l, \bar{x})$. Since both w and x^{l+1} lie on the path $[x^l, \bar{x}]$, it follows that $[x^{l+1}, w] \subseteq [x^l, \bar{x}]$, and hence $\beta^T(x^{l+1}, w) \subseteq \beta^T(x^l, \bar{x})$. This proves that β^T is the strongly consistent (and hence is also weakly consistent). Since a betweenness relation that generates a single-peaked domain on a tree is strongly consistent, it follows from the definition of a single-peaked domain on a graph that the betweenness relation that generates such a domain also satisfies the property. It is shown in [69] (see Lemma A.1 for details) that for all $x \in A$, the (sub)domain of $\mathcal{D}_{\mathcal{G}}$ containing all preferences with x as the top-ranked alternative is swap-connected, which implies that the betweenness relations generated by the spanning trees of a graph are swap-connected. Therefore, it follows from Corollary 6.6.1 that for almost all prior profiles, unanimous and swap-LOBIC RBRs on the single-peaked domain on a graph are tops-only. Consequently, we obtain from Corollary 6.4.1 that for almost all prior profiles, a unanimous and weak elementary monotonic RBR on the single-peaked domain on a graph is swap-LOBIC if and only if it is swap-LDSIC.

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[14] A tree is called a line if it has exactly two nodes with degree one (such nodes are called leafs).

[15] A line graph can be represented by a linear order $\prec$ over the alternatives in an obvious manner: if the edges in a line graph are $\{(a_1, a_2), \ldots, (a_{m-1}, a_m)\}$, then one can take the linear order $\prec$ as $a_1 \prec \cdots \prec a_m$.

It follows from Proposition 6.6.1 that the single-peaked domain on a graph is swap-DLGE. It is shown in [69] that a DSCF on the single-peaked domain on a graph is unanimous and DSIC if and only if it is a *monotonic collection of parameters based rule* (see Theorem 5.5 in [69] for details). Therefore, it follows as a corollary of Proposition 6.6.1 that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-peaked domain on a graph are monotonic collection of parameters based rule.[16]

[26] shows that the single-peaked domain is swap-RLGE. Moreover, [67] show that every unanimous and DSIC RSCF on the single-peaked domain is a *probabilistic fixed ballot rule (PFBR)*. Therefore, for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-peaked domain are PFBRs.

In what follows, we provide a discussion on the structure of unanimous and swap-LOBIC RBRs on the single-peaked domain that do not satisfy weak elementary monotonicity. The structure of such RBRs depends on the specific prior profile. In the following example, we present an RSCF for three agents that is unanimous and OBIC with respect to any independent prior profile (μ_1, μ_2, μ_3) where $\mu_2(abc) \geq \frac{1}{6}$.[17] By Corollary 6.6.1, we know that such an RSCF will be tops-only. In Table 6.6.1, the preferences in rows and columns belong to agents 1 and 2, respectively, and the preferences written at the top-left corner of any table belong to agent 3. Note that agent 3 is the dictator for this RSCF except when she has the preference *abc*. When she has the preference *abc*, the rule violates weak elementary monotonicity over the profiles (abc, bac, abc) and (bac, bac, abc). Note that except from such violations, the rule behaves like a PFBR.

Table 6.6.1: An RSCF on the single-peaked domain that is unanimous and OBIC with respect to any independent prior profile (μ_1, μ_2, μ_3) where $\mu_2(abc) \geq \frac{1}{6}$

abc	*abc*	*bac*	*bca*	*cba*
abc	(1, 0, 0)	(0.4, 0.6, 0)	(0.4, 0.6, 0)	(0.4, 0.6, 0)
bac	(0.5, 0.5, 0)	(0.5, 0.5, 0)	(0.5, 0.5, 0)	(0.5, 0.5, 0)
bca	(0.5, 0.5, 0)	(0.5, 0.5, 0)	(0.5, 0.5, 0)	(0.5, 0.5, 0)
cba	(0.5, 0.5, 0)	(0.5, 0.5, 0)	(0.5, 0.5, 0)	(0.5, 0.5, 0)

bac	*abc*	*bac*	*bca*	*cba*
abc	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)
bac	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)
bca	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)
cba	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)

bca	*abc*	*bac*	*bca*	*cba*
abc	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)
bac	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)
bca	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)
cba	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)	(0, 1, 0)

cba	*abc*	*bac*	*bca*	*cba*
abc	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)
bac	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)
bca	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)
cba	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)	(0, 0, 1)

[16] Although [69] provide the said characterization (Theorem 5.5) for RSCFs, we cannot apply it to obtain a characterization of LOBIC RSCFs as it is not known whether the single-peaked domain on a graph is RLGE or not.

[17] The rule is OBIC for dependent priors if: $5\mu_1(abc, abc) \geq \mu_1(bac, abc) + \mu_1(bca, abc) + \mu_1(cba, abc)$, where the first and the second preference in μ_1 denote the preferences of agents 2 and 3, respectively.

6.6.2 Hybrid domains

[25] introduce the notion of hybrid domains and discuss its importance. These domains satisfy single-peaked property only over a subset of alternatives. Let us assume that $A = \{1, \ldots, m\}$. Throughout this subsection, we assume that two alternatives $\underline{k}$ and $\overline{k}$ with $\underline{k} < \overline{k}$ are arbitrary but fixed.

Definition 6.6.2 *A preference P is called $(\underline{k}, \overline{k})$-hybrid if the following two conditions are satisfied:*

(i) *For all $r, s \in A$ such that either $r, s \in [1, \underline{k}]$ or $r, s \in [\overline{k}, m]$,*
$\left[r < s < P(1) \text{ or } P(1) < s < r \right] \Rightarrow [sPr]$.

(ii) $\left[P(1) \in [1, \underline{k}] \right] \Rightarrow \left[\underline{k}Pr \text{ for all } r \in (\underline{k}, \overline{k}] \right]$ *and*
$\left[P(1) \in [\overline{k}, m] \right] \Rightarrow \left[\overline{k}Ps \text{ for all } s \in [\underline{k}, \overline{k}) \right]$.[18]

A domain is $(\underline{k}, \overline{k})$-**hybrid** if it contains all $(\underline{k}, \overline{k})$-hybrid preferences. The betweenness relation β that generates a $(\underline{k}, \overline{k})$-hybrid domain is as follows: if $x < y$ then $\beta(x, y) = \{x, y\} \cup \big((x, y) \setminus (\underline{k}, \overline{k})\big)$ and if $y < x$ then $\beta(x, y) = \{x, y\} \cup \big((y, x) \setminus (\underline{k}, \overline{k})\big)$. In other words, an alternative other than x and y lies between x and y if and only if it lies in the interval $[x, y]$ or $[y, x]$ but not in the interval $(\underline{k}, \overline{k})$.

Using similar logic as we have used in the case of a single-peaked domain on a tree, it follows that the betweenness relation that generates a hybrid domain is strongly consistent. Therefore, Corollary 6.6.1 implies that for almost all prior profiles, unanimous and swap-LOBIC RBRs on the $(\underline{k}, \overline{k})$-hybrid domain are tops-only.

[25] show that every unanimous and DSIC RSCF on the hybrid domain is a $(\underline{k}, \overline{k})$-*restricted probabilistic fixed ballot rule* ($(\underline{k}, \overline{k})$-*RPFBR*). Since the hybrid domain is swap-RLGE (see [25] for details), Corollary 6.4.1 implies that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the $(\underline{k}, \overline{k})$-hybrid domain are $(\underline{k}, \overline{k})$-RPFBR.

6.6.3 Multiple single-peaked domains

The notion of multiple single-peaked domains is introduced in [72]. As the name suggests, these domains are union of several single-peaked domains. It is worth mentioning that these domains are different from hybrid domains–neither of them contains the other. For ease of presentation, we denote a single-peaked domain with respect to a prior ordering $\prec$ over A by $\mathcal{D}_{\prec}$.

Definition 6.6.3 *Let $\Omega \subseteq \mathcal{P}(A)$ be a swap-connected collection of prior orderings over A. A domain $\mathcal{D}$ is called* ***multiple single-peaked*** *with respect to Ω if $\mathcal{D} = \cup_{\prec \in \Omega} \mathcal{D}_{\prec}$.*

[18] For two alternatives x and y, by $(x, y]$ we denote the alternatives z such that $x < z \leq y$. The interpretation of the notation $[x, y)$ is similar.

Since the prior orders in a multiple single-peaked domain are assumed to be swap-connected, it follows that preferences with the same top-ranked alternative are swap-connected. This implies that the collection $\mathcal{B}$ of betweenness relations that generate a multiple single-peaked domain is swap-connected. Using similar logic as we have used in the case of a single-peaked domain on a tree, it follows that multiple single-peaked domains are both weakly and strongly consistent betweenness domains. Therefore, Corollary 6.6.1 implies that for almost all prior profiles, unanimous and swap-LOBIC RBRs on the multiple single-peaked domain are tops-only.

Let us assume without loss of generality that Ω contains the integer ordering $<$ over $A = \{1, \ldots, m\}$. For a class of prior ordering Ω over A, the left cut-off $\underline{k}$ is defined as the maximum (with respect to $<$) alternative with the property that $1 \prec 2 \prec \cdots \prec \underline{k} \prec x$ for all $x \notin \{1, \ldots, \underline{k}\}$ and all $\prec \in \Omega$. Similarly, define the right cut-off as the minimum alternative $\overline{k}$ such that $x \prec \overline{k} \prec \cdots \prec m - 1 \prec m$ for all $x \notin \{\overline{k}, \ldots, m\}$ and all $\prec \in \Omega$.

[72] shows that a DSCF is unanimous and DSIC on a multiple single-peaked domain with left cut-off $\underline{k}$ and right cut-off $\overline{k}$ if and only if it is a $(\underline{k}, \overline{k})$*-partly dictatorial generalized median voter scheme* (*$(\underline{k}, \overline{k})$-PDGMVS*). Moreover, by Proposition 6.6.1, a multiple single-peaked domain is a swap-DLGE domain. Combining all these results with Corollary Corollary 6.4.1, we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the multiple single-peaked domain are $(\underline{k}, \overline{k})$-PDGMVS.

6.6.4 Domains under partitioning

The notion of domains under partitioning is introduced in [62]. Such domains arise when a group of objects are to be partitioned based on the preferences of the agents over different partitions.

Let X be a finite set of objects and let A be the set of all partitions of X.[19] For instance, if $X = \{x, y, z\}$, then elements of A are $\{\{x\}, \{y\}, \{z\}\}$, $\{\{x\}, \{y, z\}\}$, $\{\{y\}, \{x, z\}\}$, $\{\{z\}, \{x, y\}\}$, and $\{\{x, y, z\}\}$. We say that two objects are together in a partition if they are contained in a common element (subset of X) of the partition. For instance, objects x and y are together in the partition $\{\{z\}, \{x, y\}\}$. If two objects are not together in a partition, we say they are separated. For three distinct partitions $X_1, X_2, X_3 \in A$, we say X_2 lies between X_1 and X_3 if for every two objects x and y, x and y are together in both X_1 and X_3 implies they are also together in X_2, and x and y are separate in both X_1 and X_3 implies they are also separate in X_2. For instance, any of the partitions $\{\{x\}, \{y, z\}\}$ or $\{\{y\}, \{x, z\}\}$ or $\{\{z\}, \{x, y\}\}$ lies between $\{\{x\}, \{y\}, \{z\}\}$ and $\{\{x, y, z\}\}$. This follows from the fact that no two objects are together (or separated) in both $\{\{x\}, \{y\}, \{z\}\}$ and $\{\{x, y, z\}\}$, so the betweenness condition is vacuously satisfied. For another instance, consider the partitions $\{\{x, y\}, \{z\}\}$ and $\{\{x, z\}, \{y\}\}$. The only partition that

[19] A partition of a set is a set of subsets of that set that are mutually exclusive and exhaustive.

lies between these two partitions is $\{\{x\},\{y\},\{z\}\}$. To see this, note that y are z are separate in both the partitions (and no two objects are together in both), and $\{\{x\},\{y\},\{z\}\}$ is the only partition (other than the two) in which y and z are separated.

Definition 6.6.4 *A domain $\mathcal{D}$ is **intermediate** if for all $P \in \mathcal{D}$ and every two partitions $X_1, X_2 \in A$, X_1 lies between $P(1)$ and X_2 implies $X_1 P X_2$.*

By definition, intermediate domains are betweenness domains. In Table 6.6.2, we present a preference in an intermediate domain over three objects, and in Table 6.6.3, we illustrate the localness structure by providing two local preferences in such a domain. Note that the betweenness relation does not specify the ordering of $\{\{a,b\},\{c\}\}$, $\{\{a,c\},\{b\}\}$, and $\{\{a\},\{b,c\}\}$ when $\{\{a\},\{b\},\{c\}\}$ is the top-ranked partition. Therefore, there are six preferences with $\{\{a\},\{b\},\{c\}\}$ as the top-ranked partition, P^1 is one of them. It is worth noting that an intermediate domain is not swap-connected. For instance, the preferences P^2 and P^3 are graph-local but not swap-local.

Table 6.6.2: An example of a preference in an intermediate domain over three objects

P^1
$\{\{a\},\{b\},\{c\}\}$
$\{\{a,b\},\{c\}\}$
$\{\{a,c\},\{b\}\}$
$\{\{a\},\{b,c\}\}$
$\{\{a,b,c\}\}$

Table 6.6.3: Two graph-local preferences in an intermediate domain over three objects

P^2	P^3
$\{\{a,b\},\{c\}\}$	$\{\{a,b,c\}\}$
$\{\{a,b,c\}\}$	$\{\{a,b\},\{c\}\}$
$\{\{a\},\{b\},\{c\}\}$	$\{\{a,c\},\{b\}\}$
$\{\{a,c\},\{b\}\}$	$\{\{a\},\{b,c\}\}$
$\{\{a\},\{b,c\}\}$	$\{\{a\},\{b\},\{c\}\}$

Proposition 6.6.2 *The intermediate domain is strongly consistent.*

The proof of this proposition is relegated to Appendix .3.7.

By Corollary 6.6.1 and Proposition 6.6.2, it follows that for almost all prior profiles, unanimous and DSIC RBRs on the intermediate domain are tops-only. This is a major step towards characterizing unanimous and OBIC RBRs for almost all prior profiles on the intermediate domain. It is worth mentioning that the structure of unanimous and DSIC RSCFs are yet not explored on the intermediate domain and it follows from Corollary 6.6.1 that every such rule is tops-only.

It is shown in [62] that a DSCF is unanimous and DSIC on the intermediate domain if and only if it is a *meet aggregator*. Moreover, by Proposition 6.6.1 and Proposition 6.6.2, every intermediate domain is graph-DLGE. Combining these results with Remark 6.3.6, we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the intermediate domain are meet aggregators.

6.7 Applications on non-regular domains

In this section, we consider two important non-regular domains, namely single-dipped and single-crossing domains. Let the alternatives be $A = \{1, \ldots, m\}$.

6.7.1 Single-dipped domains

A preference is single-dipped if there is a "dip" (the worst alternative) of it so that as one moves farther away from it, preference increases. These domains arise in the context of locating a "public bad" (such as garbage dump, nuclear plant, wind mill, etc.).

Definition 6.7.1 *A preference P is **single-dipped** if it has a unique minimal element $d(P)$, the* dip *of P, such that for all $x, y \in A$, $[d(P) \leq x < y \text{ or } y < x \leq d(P)] \Rightarrow yPx$. A domain is single-dipped if it contains all single-dipped preferences.*

In what follows, we argue that the single-dipped domain satisfies the path-richness property. Consider two preferences of the form $a \cdots xy \cdots$ and $a \cdots yx \cdots$. We need to show that from every preference of the form $b \cdots$, we can reach a preference with a as the top-ranked alternative through a swap-local path such that the relative ranking of x and y does not change along the path. Since only one of the alternatives 1 and m can be a top-ranked alternative in the single-dipped domain and the domain is symmetric with respect to 1 and m, it is sufficient to show this for $a = 1$ and $b = m$.

First suppose that for some $x, y \in \{2, \ldots, m\}$, there are preferences of the form $1 \cdots xy \cdots$ and $1 \cdots yx \cdots$ in the domain. Consider a preference $P \equiv m \cdots$. Suppose xPy. We can construct a swap-local path to a preference $P' \equiv m \cdots y$ such that no alternative overtakes y along the path. This can be done by shifting the dip of the preferences to y along the path, which is always possible by the definition of the single-dipped domain. Next, we go to a preference $P'' \equiv m1 \cdots y$ through a swap-local path such that y remains as the bottom ranked alternative in each preference in the path. Finally, we swap m and 1 to obtain a preference with 1 as the top-ranked alternative. By the construction of the whole path, no alternative overtakes y along the path. Since x is ranked above y in P, this, in particular, implies the relative ranking of x and y does not change along the path. Hence we obtain from Theorem 6.3.5 that almost all prior profiles, unanimous and swap-LOBIC RBRs on the single-dipped domain are tops-only.

It is shown in [68] that an RSCF on the single-dipped domain is unanimous and DSIC if and only if it is a *random committee rule*. By combining this result with Corollary 6.4.1 and the fact that every swap-LDSIC RSCF on the single-dipped domain is DSIC (see [26] for details) we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-dipped domain are random committee rules.

6.7.2 Single-crossing domains

A domain is single-crossing if its preferences can be ordered in a way so that no two alternatives change their relative ranking more than once along that ordering. Such domains are used in models of income taxation and redistribution, local public goods and stratification, and coalition formation (see [78] for details).

Definition 6.7.2 *A domain $\mathcal{D}$ is **single-crossing** if there is an ordering $\lhd$ over $\mathcal{D}$ such that for all $x, y \in A$ and all $P, P' \in \mathcal{D}$, $[x < y, P \lhd P', \text{ and } yPx] \implies yP'x$.*

To see that a single-crossing domain satisfies the path-richness property, consider an alternative a and suppose that there are two swap-local preferences $P \equiv a \cdots xy \cdots$ and $P' \equiv a \cdots yx \cdots$. Since P and P' are swap-local, they must be consecutive in the ordering $\lhd$. Assume without loss of generality that $P \lhd P'$. This means $x\hat{P}y$ for all $\hat{P}$ with $\hat{P} \lhd P$ and $y\bar{P}x$ for all $\bar{P}$ with $P' \lhd \hat{P}$. Consider any preference $\tilde{P}$. If $x\tilde{P}y$, then $\tilde{P} \lhd P$, and hence from $\tilde{P}$ one can go to the preference P following the path given by $\lhd$ maintaining the relative ordering between x and y. On the other hand, if $y\tilde{P}x$, then one can go from $\tilde{P}$ to the preference P' following the path given by $\lhd$. This shows that a single-crossing domain satisfies the path-richness property, and hence we obtain from Theorem 6.3.5 that almost all prior profiles, unanimous and swap-LOBIC RBRs on the single-crossing domain are tops-only.

[73] show that an RSCF on the single-crossing domain is unanimous and DSIC if and only if it is a *tops-restricted probabilistic fixed ballot rules (TPFBRs)*. Moreover, [26] shows that every swap-LDSIC RSCF on the single-crossing domain is DSIC. Combining these results with Corollary 6.4.1, we obtain that for almost all prior profiles, unanimous and weak elementary monotonic swap-LOBIC RBRs on the single-crossing domain are TPFBRs.

6.8 Applications on multi-dimensional separable domains

Multi-dimensional separable domains comprise the main application of our general model. Multi-dimensional models are used in political economy, as well as in public good location problems where an alternative represents the location of a political party/public good in the multi-dimensional political spectrum/Euclidean space (see [14] and [11] for details). Such models are also used to deal with the problem of forming a committee by taking members from a given set of candidates (see [76]). In a different context, this model is used in formulating the model of externalities in the context of the debate on liberalism (see [82] and [85]). In this setting, a social alternative has several components. Each component represents some aspect of the alternative. There is no dependence between the components, that is, the set of alternatives is a product set (of the alternatives available in different components). Separability implies that there is no interaction between the preferences of an agent (over the alternatives) in different components.

Let $K = \{1, \ldots, k\}$ with $k \geq 2$ be the set of components, and for each $l \in K$, let A^l be the set of at least two alternatives available in component l. We assume that the alternative set can be decomposed as a Cartesian product, i.e., $A = A^1 \times \cdots \times A^k$. Thus, an alternative x is a vector of k elements, which we denote by $(x^1, \ldots, x^k)$. For $l \in K$, we denote by A^{-l} the set $A^1 \times \cdots \times A^{l-1} \times A^{l+1} \times \cdots \times A^k$ and by x^{-l} an element of A^{-l}.

A preference $P \in \mathcal{P}(A)$ is *separable* if there exists a (unique) marginal preference P^l for each $l \in K$ such that for all $x, y \in A$, we have $[x^l P^l y^l \text{ for some } l \in K \text{ and } x^{-l} = y^{-l}] \Rightarrow [xPy]$. A domain is called separable if each preference in it is separable.

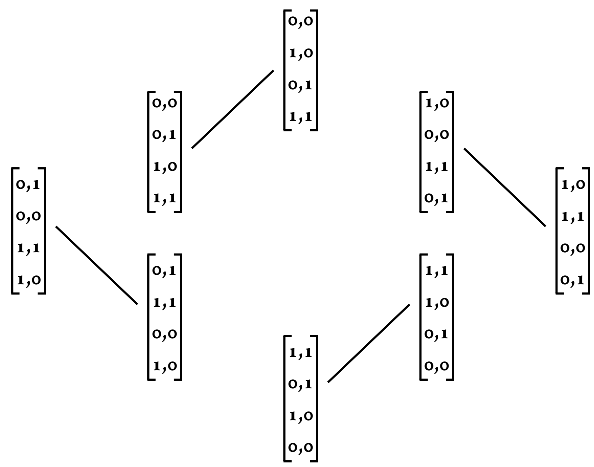
For a collection of marginal preferences $(P^1, \ldots, P^k)$, the collection of all separable preferences with marginals as $(P^1, \ldots, P^k)$ is denoted by $\mathcal{S}(P^1, \ldots, P^k)$. Similarly, for a collection of marginal domains $(\mathcal{D}^1, \ldots, \mathcal{D}^k)$, the set of all separable preferences with marginals in $(\mathcal{D}^1, \ldots, \mathcal{D}^k)$ is denoted by $\mathcal{S}(\mathcal{D}^1, \ldots, \mathcal{D}^k)$, that is, $\mathcal{S}(\mathcal{D}^1, \ldots, \mathcal{D}^k) = \cup_{(P^1,\ldots,P^k)\in(\mathcal{D}^1,\ldots,\mathcal{D}^k)} \mathcal{S}(P^1, \ldots, P^k)$. A separable domain of the form $\mathcal{S}(\mathcal{D}^1, \ldots, \mathcal{D}^k)$ is called a full separable domain. Throughout this subsection, we assume that the marginal domains are betweenness domains satisfying swap-connectedness and consistency, for instance, they can be any domain we have discussed so far except the intermediate domain. For $P_N \in \mathcal{S}(\mathcal{D}^1, \ldots, \mathcal{D}^k)$, we denote its restriction to a component $l \in K$ by P^l_N, that is, $P^l_N = (P^l_1, \ldots, P^l_n)$. We introduce the local structure in a full separable domain in a natural way.

Example 6.8.1 *Consider the situation where the set of possible candidates from which a committee has to be formed is $K = \{1, 2\}$, and for each candidate $l \in K$, the alternatives are $A^l = \{0, 1\}$ where 0 and 1 represents the corresponding candidate is excluded and included in the committee respectively. Thus, the set of alternatives $A = A^1 \times A^2 = \{(0, 0), (0, 1), (1, 0), (1, 1)\}$. Consider the preference $P = (0, 0)(0, 1)(1, 0)(1, 1)$. If we swap* *the top-two alternatives in this preference, we obtain the preference* $P' = (0,1)(0,0)(1,0)(1,1)$*, which is no more separable.*

Table 6.8.1: The separable domain in Example 6.8.1

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$
$(0,1)$	$(1,0)$	$(0,0)$	$(1,1)$	$(1,0)$	$(0,1)$	$(1,1)$	$(0,0)$
$(1,0)$	$(0,1)$	$(1,1)$	$(0,0)$	$(0,1)$	$(1,0)$	$(0,0)$	$(1,1)$
$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$

Figure 6.8.1: Swap-local structure of the domain in Table 6.8.1



The full separable domain on A is presented in Table 6.8.1 and the swap-local structure is shown in Figure 6.8.1 by means of a graph. Note that the graph is not connected. Thus, not only that swap-localness leads to non separable preferences, it cannot even connect every separable preference in the domain. This explains why swap-localness is not compatible with multi-dimensional separable preferences.

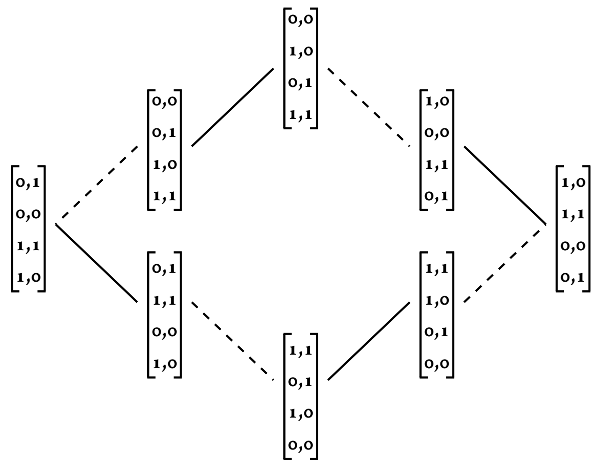
Definition 6.8.2 *Let* $\mathcal{D}^l$ *be swap-connected for all* $l \in K$*. Two preferences* $P, \bar{P} \in \mathcal{S}(\mathcal{D}^1, \dots, \mathcal{D}^k)$ *are* ***sep-local*** *if one of the following two holds:*

(i) $P \triangle \bar{P} = \{x, y\}$ *where* x, y *are such that* $|\{l \mid x^l \neq y^l\}| \geq 2$*.*

(ii) $P \triangle \bar{P} = \{((a^{-l}, x^l), (a^{-l}, y^l)) \mid a^{-l} \in A^{-l}\}$*, where* $l \in K$ *and* $x^l, y^l \in A^l$ *swap from* P^l *to* $\bar{P}^l$*.*

Thus, (i) in Definition 6.8.2 says that exactly one pair of alternatives (x,y), that vary over at least two components, swap from P to P', and (ii) in Definition 6.8.2 says that multiple pairs of alternatives of the form $((a^{-l}, x^l), (a^{-l}, y^l))$, where $a^{-l} \in A^{-l}$, swaps from P to P'. This structure makes the lower contour monotonicity property simpler: it imposes elementary monotonicity to every pair of swapping alternatives. We call it **sep-monotonicity**.

Figure 6.8.2: Sep-local structure of the domain in Table 6.8.1



In Example 6.8.1 we have shown that a multidimensional separable domain is not connected if we use swap-localness as the notion of localness. In Figure 6.8.2, we show how the same domain becomes connected under sep-localness as defined in Definition 6.8.2. We have used dotted lines to emphasize the edges that are newly added to the graph, that is, are sep-local but not swap-local.

For notational convenience, we denote a domain $\mathcal{S}(\mathcal{D}^1, \ldots, \mathcal{D}^k)$ by $\mathcal{S}$ in the following results. The following corollary is obtained from Theorem 6.3.2.

Corollary 6.8.1 *For every sep-monotonic RSCF $\phi : \mathcal{S}^n \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, the RBR (ϕ, μ_N) is sep-LOBIC if and only if ϕ is sep-LDSIC.*

It is worth mentioning that Corollary 6.8.1 holds as long as the marginal domains are swap-connected.

Our next two propositions are derived by using Theorem 6.3.5. An RSCF $\phi : \mathcal{S}^n \to \Delta A$ satisfies **component-unanimity** if for each component $l \in K$ and each $P_N \in \mathcal{S}^n$ such that $P_i^l(1) = x^l$ for all $i \in N$ and some $x^l \in A^l$, we have $\phi^l_{x^l}(P_N) = 1$.

Proposition 6.8.1 *For every unanimous RSCF $\phi : \mathcal{S}^n \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, if the RBR (ϕ, μ_N) is sep-LOBIC then ϕ satisfies component-unanimity.*

The proof of this proposition is relegated to Appendix .3.8.

Proposition 6.8.2 *For every unanimous RSCF $\phi : \mathcal{S}^n \to \Delta A$, there is a set of prior profiles $\mathcal{M}(\phi)$ with full measure such that for each $\mu_N \in \mathcal{M}(\phi)$, if the RBR (ϕ, μ_N) is sep-LOBIC then ϕ is tops-only.*

The proof of this proposition is relegated to Appendix .3.9.

For random rules, to the best of our knowledge, it is still not known whether sep-LDSIC implies DSIC or not. However, the same is shown for DSCFs on domains having unrestricted marginals (see [54] for details). Thus, it follows from Corollary 6.8.1 that for almost all prior profiles, sep-monotonic DSCFs, OBIC and DSIC are equivalent on such domains.

6.9 Discussion

6.9.1 The case of DBRs

A probability distribution ν on a finite set S is generic if for all subsets U and V of S, $\nu(U) = \nu(V)$ implies $U = V$. [58] show that on the unrestricted domain, every unanimous DBR that is OBIC with respect to a generic prior is dictatorial, and [61] shows that under elementary monotonicity, the notions DSIC and OBIC with respect to generic priors are equivalent. It can be verified that all our results hold for generic priors if we restrict our attention to DBRs.

6.9.2 Fully correlated priors

Note that the priors we consider in this paper are partially correlated: prior of an agent is independent of her own preference, while it may be correlated over the preferences of other agents. The natural question arises here as to what will happen if the prior of an agent depends on her own preferences too. Firstly, our proof technique for Theorem 6.3.2 will fail, but more importantly, Theorem 6.3.2 will not even hold anymore. It can be verified from the proof of Proposition 6.3.1 that if an RSCF is graph-LOBIC but not graph-LDSIC then it must satisfy a system of equations. The proof follows from the fact that the set of prior profiles that satisfy such a system of equations has Lebesgue measure zero. However, if an agent has two different priors for two local preferences, then we cannot obtain such a system of equations on a given prior (what we obtain are equations involving different priors), and consequently, nothing can be concluded about the Lebesgue measure of such priors. We illustrate this with the following example.

Suppose that there are two agents 1 and 2, and three alternatives a, b, and c. Consider two swap-local preferences *cab* and *cba* of agent 1. Consider the anti-plurality rule with the tie-breaking criteria as $a \succ b \succ c$. In Table 6.9.1, we present this rule when agent 1 has preferences *bac* and *bca*, and 2 has any preference. It is well-known (and also can be verified from the example) that anti-plurality rule is not swap-LDSIC. However, it is swap-LOBIC over the mentioned preferences of agent 1 if her prior satisfies the following conditions: $\mu_1(bca|cab) + \mu_1(cab|cab) - \mu_1(acb|cab) - \mu_1(cba|cab) \geq 0$ and $\mu_1(acb|cba) + \mu_1(cba|cba) - \mu_1(bca|cba) - \mu_1(cab|cba) \geq 0$. It is clear that the Lebesgue measure of such priors is not zero (this is because, as we have argued, the inequalities are imposed on two different priors $\mu_1(\cdot|cab)$ and $\mu_1(\cdot|cba)$). In a similar way, it follows that if one considers all possible restrictions arising from all possible swap-local preferences of each agent, the resulting priors for which the rule is LOBIC can have Lebesgue measure strictly bigger than zero.

Table 6.9.1: A DBR that is not swap-DSIC but is swap-LOBIC with respect to a class of correlated priors with positive Lebesgue measure

1 \ 2	abc	acb	bac	bca	cba	cab
cab	a	a	a	c	a	c
cba	b	c	b	b	c	b

6.9.3 Relation with [46]

[46] explore the structure of LOBIC DBRs with respect to generic priors (as defined in [58]) on sparsely connected domains without restoration. They show that if a unanimous DBR on a sparsely connected domain without restoration is LOBIC with respect to generic priors, then it will be tops-only. Since they consider sparsely connected domains without restoration, even the deterministic versions of our results for multi-dimensional domains and intermediate domains do not follow from their result. Coming to the unrestricted domain and single-peaked domains, which are sparsely connected domains without restoration (see [46] for details), Example 1 of [57] already shows that their results do not extend for RBRs on the unrestricted domain. Below, we provide an example to show that it does not extend for RBRs on single-peaked domains either.

Table 6.9.2: The RBR in Example 6.9.1

	μ_1	0.1	0.3	0.44	0.16
μ_2	1 \ 2	abc	bac	bca	cba
0.2	abc	(1,0,0)	(0.5,0.4,0.1)	(0.44,0.4,0.16)	(0.44,0.56,0)
0.24	bac	(0.4,0.3,0.3)	(0,1,0)	(0,1,0)	(0,0.56,0.44)
0.34	bca	(0.4,0.3,0.3)	(0,1,0)	(0,1,0)	(0,0.56,0.44)
0.22	cba	(0.4,0.3,0.3)	(0,0,1)	(0,0,1)	(0,0,1)

Example 6.9.1 *Consider the RBR in Table 6.9.2.*[20] *The priors of agents 1 and 2, μ_1 and μ_2 are generic. For instance, $\mu_1(abc)(= 0.1)$ is different from $\mu_1(S)$ for any set of preferences S other than $\{abc\}$, $\mu_1(abc) + \mu_1(bac)(= .4)$ is different from $\mu_1(S)$ for any set of preferences S other than $\{abc, bac\}$, etc. Preferences of agents 1 and 2 are depicted in the second column and the second row, and the outcome of the RSCF, say ϕ, is given by the corresponding cells. Clearly, the rule ϕ is unanimous. To see that ϕ is OBIC with respect to the given priors, consider, for instance, agent 1. Suppose her sincere preference is abc. If she reports this preference, she receives interim expected outcome $\phi(abc, \mu_1) = (0.514, 0.3856, 0.1004)$. If she misreports, say as the preference bac, then she receives interim expected outcome $\phi(bac, \mu_1) = (0.04, 0.8596, 0.1004)$. Since $\phi(abc, \mu_1)$ stochastically dominates $\phi(bac, \mu_1)$ at abc, agent 1 cannot manipulate by misreporting the preference abc as bac. In a similar fashion, it can be verified that no agent can manipulate ϕ. Now, consider the profiles (abc, bac) and (abc, bca). Each agent has the same top-ranked alternative in these two profiles. However, $\phi(abc, bac) \neq \phi(abc, bca)$, which means ϕ is not tops-only.*

Appendix

.1 Preliminaries for the proofs

Claim .1.1 *For every RSCF ϕ, the Lebesgue measure of the complement of $\mathcal{M}(\phi)$ is zero.*

Proof: [Proof of Claim .1.1] The proof of this claim follows from elementary measure theory; we provide a sketch of it for the sake of completeness. First note that for a given RSCF ϕ and for all $i \in N$, all $P_i, P_i' \in \mathcal{D}_i$, and all $X \subsetneq A$, (6.1) is equivalent to an equation of the form:

$$x_1 a_1 + \cdots + x_k a_k = 0, \tag{2}$$

where a's are some constants and x's are non-negative variables summing up to 1 (that is, probabilities). The question is if x's are drawn randomly (uniformly) from the space $\{(x_1, \ldots, x_k) \mid x_l \geq 0 \text{ for all } l \text{ and } \sum_l x_l = 1\}$, what is the Lebesgue measure of the priors for which (2) will be satisfied? Clearly, if a's are all zeros, (2) will be satisfied for all prior profiles. We argue that if a's are not all zeros, then (2) can be satisfied only for a set of prior profiles with Lebesgue measure zero, which will complete the proof by means of the fact that the number of agents, preferences, and alternatives are all finite. However, this follows from the facts that the solutions of (2) form a hyperplane and that the Lebesgue measure of a hyperplane is zero (because of dimensional reduction, such as the Lebesgue measure of a line in a plane is zero, that of a plane in a cube is zero, etc.).[21] ■

[20] See Example 6.2.3 for an explanation of the table.

[21] For a detailed argument, suppose that exactly one a, say a_1 is not zero. Note that this assumption gives maximum freedom

.2 Proof of Proposition 6.3.1

Proof: Let (ϕ, μ_N) be a graph-LOBIC RBR. Since we prove the claim for a set of prior profiles with full measure, in view of Claim .1.1, we assume that μ_N is compatible with ϕ. Consider graph-local preferences $P_i, P'_i \in \mathcal{D}_i$ and $P_{-i} \in \mathcal{D}_{-i}$. Suppose that B is a block in (P_i, P'_i). Let $U_B(P_i) = \{x \in A \mid xP_ib \text{ for all } b \in B\}$ be the set of alternatives that are strictly preferred to each element of B according to P_i. By the definition of a block in (P_i, P'_i), it follows that both $U_B(P_i)$ and $U_B(P_i) \cup B$ are upper contour sets in each of the preferences P_i and P'_i. Since P_i and P'_i are graph-local, by graph-LOBIC,

$$\sum_{P_{-i} \in \mathcal{D}_{-i}} \mu_i(P_{-i}) \phi_{U_B(P_i)}(P_i, P_{-i}) = \sum_{P_{-i} \in \mathcal{D}_{-i}} \mu_i(P_{-i}) \phi_{U_B(P_i)}(P'_i, P_{-i}) \tag{3}$$

and

$$\sum_{P_{-i} \in \mathcal{D}_{-i}} \mu_i(P_{-i}) \phi_{U_B(P_i) \cup B}(P_i, P_{-i}) = \sum_{P_{-i} \in \mathcal{D}_{-i}} \mu_i(P_{-i}) \phi_{U_B(P_i) \cup B}(P'_i, P_{-i}). \tag{4}$$

Subtracting (3) from (4), we have

$$\sum_{P_{-i} \in \mathcal{D}_{-i}} \mu_i(P_{-i}) (\phi_B(P_i, P_{-i}) - \phi_B(P'_i, P_{-i})) = 0. \tag{5}$$

Since μ_N is compatible with ϕ, this means $\phi_B(P_i, P_{-i}) = \phi_B(P'_i, P_{-i})$ for all $P_{-i} \in \mathcal{D}_{-i}$, which completes the proof. ■

Remark .2.1 *It is worth noting from the proof that an RBR (ϕ, μ_N) must satisfy (5) in order to be graph-LOBIC. If the RSCF ϕ is not LDSIC, then there will be at least one B such that $\phi_B(P_i, P_{-i}) - \phi_B(P'_i, P_{-i}) \neq 0$, in which case (5) can only be satisfied for set of prior profiles with measure zero.* □

.3 Other proofs

In view of Proposition 6.3.1, whenever we prove some statement for a class of RBRs (ϕ, μ_N) where μ_N belongs to a set with full measure, we assume that ϕ satisfies the block preservation property.

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for the values of x's and thereby maximize the Lebesgue measure of the solution space of (2). However, this means in any solution x_1 must be zero, the measure of which in the solution space is zero.

.3.1 Proof of Theorem 6.3.2

Proof: If part of the theorem follows from the definitions of graph-LDSIC and graph-LOBIC. We proceed to prove the only-if part. Let $\phi : \mathcal{D}_N \to \Delta A$ be an RSCF satisfying lower contour monotonicity and the block preservation property. We show that ϕ is graph-LDSIC. Consider graph-local preferences $P_i, P'_i \in \mathcal{D}_i, P_{-i} \in \mathcal{D}_{-i}$, and $x \in A$. We show $\phi_{U(x,P_i)}(P_i, P_{-i}) \geq \phi_{U(x,P_i)}(P'_i, P_{-i})$. Let $B_1, \ldots, B_t$ be the blocks in (P_i, P'_i) such that for all $l < t$ and all $b \in B_l$ and $b' \in B_{l+1}$, we have bP_ib'. Suppose that $x \in B_l$ for some $l \in \{1, \ldots, t\}$.

Let $\hat{B}_l = \{b \in B_l \mid bP_ix\}$ be the set of alternatives (possibly empty) in B_l that are (strictly) preferred to x. Note that the set $B_l \setminus \hat{B}_l$ is lower contour set of $P_i|_{B_l}$. Therefore, by lower contour monotonicity,

$$\phi_{B_l\setminus\hat{B}_l}(P'_i, P_{-i}) \geq \phi_{B_l\setminus\hat{B}_l}(P_i, P_{-i}). \tag{6}$$

Furthermore, by the block preservation property, we have

$$\phi_{B_l}(P'_i, P_{-i}) = \phi_{B_l}(P_i, P_{-i}). \tag{7}$$

Subtracting (6) from (7), we have

$$\phi_{\hat{B}_l}(P_i, P_{-i}) \geq \phi_{\hat{B}_l}(P'_i, P_{-i}). \tag{8}$$

Note that $U(x, P_i) = B_1 \cup \cdots \cup B_{l-1} \cup \hat{B}_l$. This means $\phi_{U(x,P_i)}(P_i, P_{-i}) = \phi_{B_1\cup\cdots\cup B_{l-1}}(P_i, P_{-i}) + \phi_{\hat{B}_l}(P_i, P_{-i})$ and $\phi_{U(x,P_i)}(P'_i, P_{-i}) = \phi_{B_1\cup\cdots\cup B_{l-1}}(P'_i, P_{-i}) + \phi_{\hat{B}_l}(P'_i, P_{-i})$. By the block preservation property, $\phi_{B_1\cup\cdots\cup B_{l-1}}(P_i, P_{-i}) = \phi_{B_1\cup\cdots\cup B_{l-1}}(P'_i, P_{-i})$, and by (8) , $\phi_{\hat{B}_l}(P_i, P_{-i}) \geq \phi_{\hat{B}_l}(P'_i, P_{-i})$. Combining these observations, we have $\phi_{U(x,P_i)}(P_i, P_{-i}) \geq \phi_{U(x,P_i)}(P'_i, P_{-i})$, which completes the proof. ■

.3.2 Proof of Theorem 6.3.5

We use the following lemma in our proof. The idea of the proof of this lemma is closely related to that of the proof of Lemma 2 in [46]. There are two key differences: first, [46] consider swap-localness whereas we consider graph-localness, and second, [46] consider deterministic rules whereas we consider random rules.

Lemma .3.1 *Suppose an RSCF $\phi : \mathcal{D}^n \to \Delta A$ satisfies unanimity and the block preservation property. Let $P_i, P'_i \in \mathcal{D}$ be graph-local and let $P_{-i} \in \mathcal{D}^{n-1}$ be such that $\phi_x(P_i, P_{-i}) \neq \phi_x(P'_i, P_{-i})$ for some $x \in P_i \triangle P'_i$. Consider an agent $j \neq i$ and suppose that there is a graph-local path $(P^1_j = P_j, \ldots, P^t_j = \bar{P}_j)$ such that for all*

$l < t$ and for every two alternatives $a, b \in P_i \triangle P'_i$, there is a common upper contour set U of both P_j^l and P_j^{l+1} such that exactly one of a and b is contained in U. Then $\phi_x(P_i, \bar{P}_j, P_{-\{i,j\}}) \neq \phi_x(P'_i, \bar{P}_j, P_{-\{i,j\}})$.

Proof:[Proof of Lemma .3.1] Suppose $\phi_x(P_i, P_j^l, P_{-\{i,j\}}) \neq \phi_x(P'_i, P_j^l, P_{-\{i,j\}})$ for some $l < t$ and some $x \in P_i \triangle P'_i$. It is enough to show that $\phi_x(P_i, P_j^{l+1}, P_{-\{i,j\}}) \neq \phi_x(P'_i, P_j^{l+1}, P_{-\{i,j\}})$. Let a and $\bar{a}$ be the alternatives, if exist, that are ranked just above and just below x, respectively, in $P_j^l|_{P_i \triangle P'_i}$. More formally, let $a \in P_i \triangle P'_i$ be such that $aP_j^l x$ and no alternative in $P_i \triangle P'_i$ is ranked between a and x, and let $\bar{a} \in P_i \triangle P'_i$ be such that $xP_j^l \bar{a}$ and no alternative in $P_i \triangle P'_i$ is ranked between x and a. Let U be the common upper contour set of P_j^l and P_j^{l+1} such that $U \cap \{a, x\} = a$, and $\widehat{U}$ be the common upper contour set of P_j^l and P_j^{l+1} such that $\widehat{U} \cap \{x, \bar{a}\} = x$. Here, U might be empty and $\widehat{U}$ might be A. Consider the set of alternatives $B = \hat{U} \setminus U$. Note that B can be expressed as a union of blocks in (P_j^l, P_j^{l+1}). Therefore, by applying the block preservation property to each block in B, we obtain $\phi_B(P_i, P_j^l, P_{-\{i,j\}}) = \phi_B(P_i, P_j^{l+1}, P_{-\{i,j\}})$ and $\phi_B(P'_i, P_j^l, P_{-\{i,j\}}) = \phi_B(P'_i, P_j^{l+1}, P_{-\{i,j\}})$. Moreover, since each $c \in B \setminus x$ is a block in (P_i, P'_i), we have by the block preservation property, $\phi_c(P_i, P_j^l, P_{-\{i,j\}}) = \phi_c(P'_i, P_j^l, P_{-\{i,j\}})$ and $\phi_c(P_i, P_j^{l+1}, P_{-\{i,j\}}) = \phi_c(P'_i, P_j^{l+1}, P_{-\{i,j\}})$ for all $c \in B \setminus x$. Combining these observations, it follows that $\phi_x(P_i, P_j^{l+1}, P_{-\{i,j\}}) \neq \phi_x(P'_i, P_j^{l+1}, P_{-\{i,j\}})$. ∎

Proof:[Proof of Theorem 6.3.5] Let $\mathcal{D}$ satisfy the path-richness property (see Definition 6.3.3) and suppose that $\phi : \mathcal{D}^n \to \Delta A$ is an RSCF satisfying unanimity and the block preservation property. We show that ϕ is tops-only. Assume for contradiction that $\phi(P_i, P_{-i}) \neq \phi(P'_i, P_{-i})$ for some $P_i, P'_i \in \mathcal{D}$ with $P_i(1) = P'_i(1)$ and some $P_{-i} \in \mathcal{D}^{n-1}$. By means of Condition (i) of the path-richness property, it is enough to assume that P_i and P'_i are graph-local. Therefore, by the block preservation property, it follows that $\phi_x(P_i, P_{-i}) \neq \phi_x(P'_i, P_{-i})$ for some $x \in P_i \triangle P'_i$.

Consider $j \in N \setminus \{i\}$. By Condition (ii) of the path-richness property, there is a path $(P_j^1 = P_j, \ldots, P_j^t = P'_j)$ with $P'_j(1) = P_i(1)$ such that for all $l < t$ and for every two alternatives $a, b \in P_i \triangle P'_i$, there is a common upper contour set U of both P_j^l and P_j^{l+1} such that exactly one of a and b is contained in U. By applying Lemma .3.1, it follows that $\phi_x(P_i, P'_j, P_{-i}) \neq \phi_x(P'_i, P'_j, P_{-i})$. By applying this logic to all agents except i, we construct $P'_{-i} \in \mathcal{D}^{n-1}$ such that $P'_j(1) = P_i(1)$ for all $j \neq i$ and $\phi_x(P_i, P'_{-i}) \neq \phi_x(P'_i, P'_{-i})$. However, since (P_i, P'_{-i}) and (P'_i, P'_{-i}) are unanimous preference profiles with the top-ranked alternative different from x, $\phi_x(P_i, P'_{-i}) = \phi_x(P'_i, P'_{-i}) = 0$, a contradiction. ∎

.3.3 Proof of Proposition 6.4.1

The proof of Proposition 6.4.1 is related to the proof of Proposition 5 in [46]. The similarity is that we both show that under tops-onlyness, weak elementary monotonicity is equivalent to elementary monotonicity for swap-LOBIC rules. The difference lies in our proof techniques. [46] first show that any unanimous and swap-LOBIC (with respect to generic priors) DBR on a sparsely connected domain without restoration is tops-only. Then they show that under tops-onlyness, weak elementary monotonicity is equivalent to elementary monotonicity for swap-LOBIC DBRs on a sparsely connected domain without restoration. On the other hand, we prove our result for tops-only and swap-LOBIC RBRs on *any* swap-connected domain. Since we work with arbitrary swap-connected domains, the techniques we employ in the proof are different from those in [46]. Another difference is that we work with random rules whereas [46] deal with deterministic rules.

Proof: If part of the theorem follows from the definitions of swap-LDSIC and swap-LOBIC. We proceed to prove the only-if part. Let $\mathcal{D}$ be swap-connected and suppose that $\phi : \mathcal{D}_N \to \Delta A$ is a tops-only RSCF satisfying weak elementary monotonicity and the block preservation property. We show that ϕ is swap-LDSIC.

Let P_i and P'_i be two swap-local preferences. If $P_i(1) = P'_i(1)$, then by tops-onlyness, $\phi(P_i, P_{-i}) = \phi(P'_i, P_{-i})$, and we are done. So, suppose $P_i \equiv ab\cdots$ and $P'_i \equiv ba\cdots$. Assume for contradiction that $\phi_a(P_i, P_{-i}) < \phi_a(P'_i, P_{-i})$. By the block preservation property, $\phi_{\{a,b\}}(P_i, P_{-i}) = \phi_{\{a,b\}}(P'_i, P_{-i})$, and hence our assumption for contradiction means $\phi_b(P_i, P_{-i}) > \phi_b(P'_i, P_{-i})$. Consider an agent $j \in N \setminus i$ such that $P_j(1) \notin \{a, b\}$. Note that since $\mathcal{D}_j$ is swap-connected one of the following two cases must hold for P_j: (i) there is a swap-local path from P_j to a preference $P'_j \equiv a\cdots$ such that b does not appear as the top-ranked alternative in any preference in the path, (ii) there is a swap-local path from P_j to a preference $P'_j \equiv b\cdots$ such that a does not appear as the top-ranked alternative in any preference in the path.

Suppose Case (i) holds. Let B be the set of alternatives that appear as the top-ranked alternative in some preference in the mentioned path. Consider the outcomes of ϕ when agent j changes her preferences along the path, while all other agents keep their preferences unchanged. By tops-onlyness, the outcome can change only when the top-ranked alternative changes along the path. Moreover, by the definition of swap-local path, the top-ranked alternative can change along the path only through a swap between two alternatives in B. By block preservation, this implies that the probability of the two swapping alternatives can only change in any such situations, and hence, the probability of the alternatives outside B will remain unchanged at the end of the path. Since $b \notin B$, this yields $\phi_b(P_i, P_j, P_{-\{i,j\}}) = \phi_b(P_i, P'_j, P_{-\{i,j\}})$ and $\phi_b(P'_i, P_j, P_{-\{i,j\}}) = \phi_b(P'_i, P'_j, P_{-\{i,j\}})$. This, together with our assumption for contradiction that $\phi_b(P_i, P_{-i}) > \phi_b(P'_i, P_{-i})$, implies $\phi_b(P_i, P'_j, P_{-\{i,j\}}) > \phi_b(P'_i, P'_j, P_{-\{i,j\}})$. Now, since $P_i \triangle P'_i = \{a, b\}$, we have by block preservation, $\phi_{\{a,b\}}(P_i, P'_j, P_{-\{i,j\}}) = \phi_{\{a,b\}}(P'_i, P'_j, P_{-\{i,j\}})$. Because $\phi_b(P_i, P'_j, P_{-\{i,j\}}) > \phi_b(P'_i, P'_j, P_{-\{i,j\}})$, this yields $\phi_a(P_i, P'_j, P_{-\{i,j\}}) < \phi_a(P'_i, P'_j, P_{-\{i,j\}})$. Using similar logic, we can conclude for Case (ii) that $\phi_a(P_i, P'_j, P_{-\{i,j\}}) < \phi_a(P'_i, P'_j, P_{-\{i,j\}})$.

Note that the preceding argument holds no matter what the preferences of the agents in $N \setminus \{i, j\}$ are.

Therefore, by repeated application of this argument for each agent $j \in N \setminus i$ with $P_j(1) \notin \{a, b\}$, we obtain $P'_{-i} \in \mathcal{D}_{-i}$ of the agents in $N \setminus i$ such that (i) $P'_j(1) \in \{a, b\}$ for each $j \in N \setminus i$, and (ii) $\phi_a(P_i, P'_{-i}) < \phi_a(P'_i, P'_{-i})$.

We now complete the proof by means of tops-onlyness. If $P'_j \equiv a \cdots$ then let $P''_j = P_i$, and if $P'_j \equiv b \cdots$ then let $P''_j = P'_i$. By tops-onlyness, $\phi(P_i, P'_{-i}) = \phi(P_i, P''_{-i})$ and $\phi(P'_i, P'_{-i}) = \phi(P'_i, P''_{-i})$, and hence, $\phi_a(P_i, P''_{-i}) < \phi_a(P'_i, P''_{-i})$. However, since for each $j \in N$, either $P''_j \equiv P_i$ or $P''_j \equiv P'_i$, this violates weak elementary monotonicity, a contradiction. ■

.3.4 Proof of Theorem 6.5.1

Pareto optimality is an efficiency property which requires that an alternative will receive zero probability if there is some other alternative that is preferred to it by each agent. More formally, an RSCF $\phi : \mathcal{D}_N \to \Delta A$ is **Pareto optimal** if for all $P_N \in \mathcal{D}_N$ and all $x \in A$ such that there exists $y \in A$ with yP_ix for all $i \in N$, we have $\phi_x(P_N) = 0$. Clearly, Pareto optimality is a much stronger requirement than unanimity. However, our next lemma says that they become equivalent under block preservation property.

Lemma .3.2 *Suppose an RSCF $\phi : \mathcal{P}(A)^n \to \Delta A$ satisfies unanimity and the block preservation property. Then ϕ is Pareto optimal.*

Proof:[Proof of Lemma .3.2] Consider $P_N \in \mathcal{P}(A)^n$ such that xP_iy for all $i \in N$ and some $x, y \in A$. We show that $\phi_y(P_N) = 0$. Assume for contradiction $\phi_y(P_N) > 0$. Consider $i \in N$. Since $\mathcal{P}(A)$ contains all preferences over A, there exists a swap-local path $(P_i^1 = P_i, \ldots, P_i^t)$ such that $P_i^t(1) = x$ and $U(P_i, y) = U(P_i^l, y)$ for all $l = 1, \ldots, t$. Since $U(y, P_i^1) = U(y, P_i^2)$, we have $y \notin P_i^1 \triangle P_i^2$, which implies that $\{y\}$ is a singleton block in (P_i^1, P_i^2). By the block preservation property, this implies $\phi_y(P_i^2, P_{-i}) = \phi_y(P_i, P_{-i})$. Continuing in this manner, we reach a preference profile (P_i^t, P_{-i}) such that $P_i^t(1) = x$ and $\phi_y(P_i^t, P_{-i}) > 0$. By applying the same argument to the agents $j \in N \setminus \{i\}$ we can construct a preference profile P'_N such that $P'_j(1) = x$ for all $j \in N$ and $\phi_y(P'_N) > 0$. Since $P'_j(1) = x$ for all $j \in N$, by unanimity we have $\phi_x(P'_N) = 1$, which contradicts that $\phi_y(P'_N) > 0$. ■

Proof:[Proof of Theorem 6.5.1] If part of the theorem follows from the definitions of swap-LDSIC and swap-LOBIC. We proceed to prove the only-if part. Let $\phi : \mathcal{P}(A)^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that ϕ is swap-LDSIC. By Lemma .3.2 and Theorem 6.3.5, ϕ is Pareto optimal and tops-only. To show that ϕ is swap-LDSIC, by Proposition 6.4.1, it is sufficient to show that ϕ is weak elementary monotonic. Consider swap-local preferences $P_i, \bar{P}_i \in \mathcal{P}(A)$ such that $P_i \equiv ab \cdots$ and $\bar{P}_i \equiv ba \cdots$. Assume for contradiction that $\phi_b(P_i, P_{-i}) > \phi_b(\bar{P}_i, P_{-i})$ for some $P_{-i} \in \mathcal{P}(A)^{n-1}$ such that $P_i(k) = \bar{P}_i(k) = P_j(k)$ for all $j \in N \setminus i$ and all $k > 2$. Let c be the alternative such that $P_i \equiv abc \cdots$. Because P_i and $\bar{P}_i$ are swap-local, this means $\bar{P}_i \equiv bac \cdots$. Consider $P_i^1 \in \mathcal{P}(A)$ such that $P_i^1 = acb \cdots$ and P_i^1 and P_i are swap-local, that is $P_i^1 \triangle P_i = \{b, c\}$. By tops-onlyness of ϕ, $\phi(P_i^1, P_{-i}) = \phi(P_i, P_{-i})$. Next, consider $P_i^2 \in \mathcal{P}(A)$ such that $P_i^2 \equiv cab \cdots$ and P_i^2 and P_i^1 are swap-local. By the block preservation property, $\phi_b(P_i^2, P_{-i}) = \phi_b(P_i^1, P_{-i})$. Now, consider $P_i^3 \in \mathcal{P}(A)$ such that $P_i^3 \equiv cba \cdots$ and P_i^3 and P_i^2 are swap-local. By tops-onlyness of ϕ, $\phi(P_i^3, P_{-i}) = \phi(P_i^2, P_{-i})$. Finally, consider $P_i^4 \in \mathcal{P}(A)$ such that $P_i^4 \equiv bca \cdots$ and P_i^4 and P_i^3 are swap-local. Since bP_i^4c and bP_jc for all $j \in N \setminus i$, we have by Pareto optimality, $\phi_c(P_i^4, P_{-i}) = 0$. Moreover, by the block preservation property, we have $\phi_b(P_i^4, P_{-i}) = \phi_b(P_i^3, P_{-i}) + \phi_c(P_i^3, P_{-i})$. This, together with the fact that $\phi_b(P_i^3, P_{-i}) = \phi_b(P_i, P_{-i})$, implies $\phi_b(P_i^4, P_{-i}) \geq \phi_b(P_i, P_{-i})$. By our assumption, this means that $\phi_b(P_i^4, P_{-i}) > \phi_b(\bar{P}_i, P_{-i})$. Since $P_i^4(1) = \bar{P}_i(1)$ which contradicts that ϕ is tops-only. ■

.3.5 Proof of Corollary 6.6.1

First, we state some important observations about betweenness domains which we will use in the proof.

Observation .3.1 *Consider an alternative $x \in A$ and let $\mathcal{D}^x(\beta)$ be the set of all preferences with top-ranked alternative x and satisfying the betweenness condition β. Then, the domain $\mathcal{D}^x(\beta)$ is swap-connected.*

Observation .3.2 *Let $x, y \in A$ and let $P \in \mathcal{D}(\beta)$ be such that $P(1) = x$ and $U(y, P) \cup y = \beta(x, y)$. Then, for all $\hat{P} \equiv x \cdots$, there is a swap-local path from $\hat{P}$ to P such that no alternative overtakes y along the path.*

Observation .3.3 *Let $\mathcal{D}(\beta)$ be strongly consistent. Let $x, \bar{x} \in A$ and let $(x^1 = x, \ldots, x^t = \bar{x})$ be a sequence of adjacent alternatives in $\beta(x, \bar{x})$ such that for all $l < t$ and all $w \in \beta(x^l, \bar{x})$, we have $\beta(x^{l+1}, w) \subseteq \beta(x^l, \bar{x})$. Then, for all $l < t$, there exist $P \equiv x^l \cdots$ and $P' \equiv x^{l+1} \cdots$ such that $\beta(x^l, x^t)$ is an upper contour set in both P and P'. To see this, consider x^l. Since $\mathcal{D}(\beta)$ is strongly consistent, there is a preference $P \in \mathcal{D}(\beta)$ such that $\beta(x^l, x^t)$ is an upper contour set of P. Name the alternatives in $\beta(x^l, x^t)$ as $w_1, \ldots, w_u$ such that $\beta(x^{l+1}, w_r) \subsetneq \beta(x^{l+1}, w_s)$ implies $r < s$. Since $\mathcal{D}(\beta)$ is strongly consistent, we have $\beta(x^{l+1}, w) \subseteq \beta(x^l, x^t)$ for all $w \in \beta(x^l, x^t)$, and hence there is a preference P', graph-local to P, satisfying the betweenness relation β such that $P' \equiv w_1 w_2 \cdots w_{u-1} w_u \cdots$. Therefore, $U(w_u, P') \cup w_u = \beta(x^l, x^t)$.*

We are now ready to start the proof. To ease the presentation, for a path π, we denote by π^{-1} the path π in the reversed direction, that is, if $\pi = (P^1, P^2, \ldots, P^t)$, then $\pi^{-1} = (P^t, P^{t-1}, \ldots, P^1)$.
Proof: [Proof of Corollary 6.6.1] Let $\mathcal{B}$ be a collection of strongly consistent and swap-connected betweenness relations. We show that $\mathcal{D}(\mathcal{B})$ satisfies the path-richness property.

First, we show $\mathcal{D}(\mathcal{B})$ satisfies Condition (i) of the path-richness property (see Definition 6.3.3). Consider P and P' with $P(1) = P'(1)$ that are not graph-local. If $P, P' \in \mathcal{D}(\beta)$ for some $\beta \in \mathcal{B}$, then by

Observation .3.1 there is a swap-local path from P to P' such that the top-ranked alternative does not change along the path. Suppose $P \in \mathcal{D}(\beta)$ and $P' \in \mathcal{D}(\hat{\beta})$ for some $\beta, \hat{\beta} \in \mathcal{B}$. Let $P(1) = P'(1) = x$ and let $(\beta^1 = \beta, \ldots, \beta^t = \hat{\beta})$ be a swap-local path. By the swap-connectedness of $\mathcal{B}$, there are swap-local preferences $P^1 \in \mathcal{D}(\beta^1)$ and $P^2 \in \mathcal{D}(\beta^2)$ with $P^1(1) = P^2(1) = x$. By Observation .3.1, there is a swap-local path π^1 from P to P^1 in $\mathcal{D}(\beta^1)$ such that x remains at the top-position in all the preferences in the path. Thus, the path (π^1, P^2) from P to P^2 satisfies Condition (i) of the path-richness property. Continuing in this manner, we can construct a path from P to P' that satisfies Condition (i) of the path-richness property.

Now, we show $\mathcal{D}(\mathcal{B})$ satisfies Condition (ii) of the path-richness property, that is, for all $P, P' \in \mathcal{D}(\mathcal{B})$ with $P(1) = P'(1)$, if P and P' are graph-local, then for each preference $\hat{P} \in \mathcal{D}(\mathcal{B})$, there exists a graph-local path $(P^1 = \hat{P}, \ldots, P^v)$ with $P^v(1) = P(1)$ such that for all $l < v$ and all distinct $a, b \in P \triangle P'$, there is a common upper contour set U of both P^l and P^{l+1} such that exactly one of a and b is contained in U. Since P and P' are graph-local with $P(1) = P'(1)$, by means of the fact that the collection $\mathcal{B}$ is swap-connected, it follows that P and P' are swap-local. So assume that $P \equiv w \cdots yz \cdots$ and $P' \equiv w \cdots zy \cdots$. Consider $\hat{P} \in \mathcal{D}(\mathcal{B})$. Suppose $\hat{P}(1) = x$ and $y\hat{P}z$. Let $\hat{P} \in \mathcal{D}(\beta)$ for some $\beta \in \mathcal{B}$. We construct a path from $\hat{P}$ to a preference with w as the top-ranked alternative maintaining Condition (ii) of the path-richness property with respect to y and z in two steps. For ease of presentation, we denote $\hat{P}$ by P^1.

Step 1: Since β is strongly consistent, there is a sequence $(x^1 = x, \ldots, x^t = y)$ of adjacent alternatives in $\beta(x^1, x^t)$ such that for all $l < t$ and all $u \in \beta(x^l, x^t)$, $\beta(x^l, x^t) \supseteq \beta(x^{l+1}, u)$. By Observation .3.2, there is a path π^1 from P^1 to a preference $\bar{P}^1$ with $\bar{P}^1(1) = x^1$ such that $U(x^t, \bar{P}^1) \cup x^t = \beta(x^1, x^t)$ and no alternative overtakes x^t along the path. Consider x^2. By Observation .3.3, there is a preference P^2 with $P^2(1) = x^2$ such that P^2 is graph-local to $\bar{P}^1$ and $\beta(x^1, x^t)$ is an upper contour set in P^2. Since $z \notin \beta(x^1, x^t)$ and $\beta(x^1, x^t)$ is a common upper contour set of $\bar{P}^1$ and P^2, Condition (ii) of the path-richness property is satisfied with respect to x^t and z on the path $(\bar{P}^1, P^2)$. As in the case for P^1 and $\bar{P}^1$, by Observation .3.2, we can construct a swap-local path π^2 from P^2 to some preference $\bar{P}^2$ with $\bar{P}^2(1) = x^2$ such that $U(x^t, \bar{P}^2) \cup x^t = \beta(x^2, x^t)$ and no alternative overtakes x^t along the path. As in the case for $\bar{P}^1$ and P^2, by Observation .3.3, there is a preference P^3 with $P^3(1) = x^3$ such that P^3 is graph-local to $\bar{P}^2$ and $\beta(x^2, x^t)$ is an upper contour set in P^3. It follows that the path (π^1, π^2, P^3) from P^1 to the preference P^3 satisfies Condition (ii) of the path-richness property with respect to x^t and z. Continuing in this manner, we can construct a path $\hat{\pi}$ in $\mathcal{D}(\beta)$ from $\hat{P}$ to a preference $\hat{\hat{P}}$ with $\hat{\hat{P}}(1) = y$ such that Condition (ii) of the path-richness property is satisfied along the path.

Step 2: Consider the preference $P \equiv w \cdots yz \cdots$. Let $P \in \mathcal{D}(\tilde{\beta})$ for some $\tilde{\beta} \in \mathcal{B}$. Using similar argument as in Step 1, we can construct a path $\tilde{\pi}$ in $\mathcal{D}(\tilde{\beta})$ from P to some $\tilde{P}$ with $\tilde{P}(1) = y$ such that Condition (ii) of the path-richness property is satisfied with respect to y and z.

Step 3: Since $\hat{\hat{P}}(1) = \tilde{P}(1) = y$ and the collection $\mathcal{B}$ is swap-connected, there is a swap-local path $\bar{\pi}$ in $\mathcal{D}(\mathcal{B})$ from $\hat{\hat{P}}$ to $\tilde{P}$ such that y stays as the top-ranked alternative in each preference of the path. Clearly, such a path will satisfy Condition (ii) of the path-richness property with respect to y and z.

Consider the path $(\hat{\pi}, \bar{\pi}, \tilde{\pi}^{-1})$ from $\hat{P}$ to P. By construction, this path satisfies Condition (ii) of the path-richness property with respect to y and z, which completes the proof. ■

.3.6 Proof of Proposition 6.6.1

Proof: [54] show that a domain $\mathcal{D}$ is graph-DLGE if and only if it satisfies the following property: for all distinct $P, P' \in \mathcal{D}$ and all $a \in A$, there exists a path π from P to P' with no (a, b)-restoration for all $b \in L(a, P)$. Here, a path is said to have no (a, b)-restoration if the relative ranking of a and b is reversed at most once along π. In what follows, we show that $\mathcal{D}(\mathcal{B})$ satisfies the above-mentioned property when $\mathcal{B}$ is weakly consistent and swap-connected. Consider two preferences $P \in \mathcal{D}(\beta)$ and $P' \in \mathcal{D}(\beta')$ for some $\beta, \beta' \in \mathcal{B}$ and $a \in A$. We show that there is a path π from P to P' that has no (a, x)-restoration for all $x \in L(a, P)$. By Observation .3.3, from P and P' there are graph-local paths $\hat{\pi}$ and $\bar{\pi}$, respectively, to some preferences $\hat{P}$ and $\bar{P}$ with a as the top-ranked alternatives such that no alternative overtakes a along each of the paths. Let $\tilde{\pi}$ be a swap-local path joining $\hat{P}$ and $\bar{P}$ such that a remains the top-ranked alternative throughout the path. Consider the path $(\hat{\pi}, \tilde{\pi}, \bar{\pi}^{-1})$. No alternative in $L(a, P)$ overtakes a along the path $\hat{\pi}$. So, if there is an (a, x)-restoration for some $x \in L(a, P)$ in the path $(\hat{\pi}, \tilde{\pi}, \bar{\pi}^{-1})$, then it must be that the restoration happens in the path $\bar{\pi}^{-1}$. However, then a must overtake x in this path, which means x overtakes a in the reversed path $\bar{\pi}$, which is not possible by the construction of the path $\bar{\pi}$. This completes the proof. ■

.3.7 Proof of Proposition 6.6.2

Proof: Consider $X, X \in A$. We show that there is a sequence $(X^1 = X, \ldots, X^t = X)$ of adjacent alternatives in $\beta(X, X)$ such that for all $l < t$ and all $W \in \beta(X^l, X^t)$, we have $\beta(X^{l+1}, W) \subseteq \beta(X^l, X^t)$. Let $l < t$ and consider $W \in \beta(X^l, X^t)$. We show $\beta(X^{l+1}, W) \subseteq \beta(X^l, X^t)$. Take $Z \notin \beta(X^l, X^t)$. Because Z does not lie in $\beta(X^l, X^t)$, there must be a pair (a, b) of objects such that either (i) a and b are together in both X^l and X^t, but separate in Z, or (ii) a and b are separate in both X^l and X^t, but together in Z. Because both X^{l+1} and W are in $\beta(X^l, X^t)$, it must hold that in case (i) a and b are together in both X^{l+1} and W, and in case (ii) they are separate in both X^{l+1} and W. In case (i), a and b are together in both X^{l+1} and W but they are separate in Z. Therefore, Z cannot lie in $\beta(X^{l+1}, W)$. On the other hand, in case (ii) a and b are separate in both X^{l+1} and W, but they are together in Z. Therefore, Z cannot lie in $\beta(X^{l+1}, W)$. This completes the proof. ■

.3.8 Proof of Proposition 6.8.1

We first prove some lemmas which we later use in the proof of the proposition. We use the following notions in the proofs. A preference P is **lexicographically separable** if there exists a (unique) component order $P^0 \in \mathcal{P}(K)$ and a (unique) marginal preference $P^j \in \mathcal{P}(A^j)$ for each $j \in K$ such that for all $x, y \in A$, we have $\left[x^l P^l y^l \text{ for some } l \in K \text{ and } x^j = y^j \text{ for all } jP^0 l\right] \Rightarrow [xPy]$. A lexicographically separable preference P can be uniquely represented by a $(k+1)$-tuple consisting of a lexicographic order P^0 over the components and marginal preferences $P^1, \ldots, P^k$.

Lemma .3.3 *Let $P \in \mathcal{S}$, $l \in K$, and $x, y \in A$ be such that $x^l P^l y^l$ and xPy. Then, for every component $j \neq l$ there is a sep-local path from P to a lexicographically separable preference $\bar{P} \in \mathcal{S}$ having same marginal preferences as P, and l and j as the lexicographically best and worst components, respectively, such that the x and y do not swap along the path.*

Proof: Assume without loss of generality, $l = 1$ and $j = m$. First, make the component 1 lexicographically best (without changing the marginal preferences of P) by swapping consecutively ranked alternatives multiple times in the following manner: each time swap a pair of consecutively ranked alternatives a and b where $a^1 P^1 b^1$ and bPa. Note that since $x^1 P^1 y^1$ and xPy, x and y are never swapped in this step. Having made 1 the lexicographically best component, the component 2 can be made lexicographically second-best in the following manner: each time swap a pair of consecutively ranked alternatives a and b in P where $a^1 = b^1$, $a^2 P^2 b^2$, and bPa. As we have explained for the case of component 1, alternatives x and y will not swap in this process. Continuing in this manner, we can finally obtain a preference $\bar{P}$ with lexicographic ordering over the components as $1\bar{P}^0 \cdots \bar{P}^0 k$ through a sep-local path along which the alternatives x and y are not swapped. ■

Lemma .3.4 *Let $P \in \mathcal{S}$ be a preference such that xPy for some alternatives x and y that differ in at least two components. Then, there is a sep-local path $(P^1 = P, \ldots, P^t = \hat{P})$ with $\hat{P}(1) = x$ such that $xP^l y$ for all $l < t$.*

Proof: Since xPy, there is a component l such that $x^l P^l y^l$. Assume without loss of generality $l = 1$. Consider component 2 . By Lemma .3.3, there is a sep-local path π^1 from P to a preference $\bar{P}$ having components 1 and 2 as the lexicographically best and the worst components, respectively, such that x and y do not swap along the path. Since 2 is the lexicographically worst component of $\bar{P}$, we can construct a sep-local path from $\bar{P}$ to a preference $\bar{\bar{P}}$ such that (i) the marginal preferences in each component other than 2 and the lexicographic ordering over the components of each preference in the path remains the same as $\bar{P}$, and (ii) x^2 appears at the top-position of $\bar{\bar{P}}^2$. Since component 1 is the lexicographically best component in all these preferences and x^1 is preferred to y^1 in the marginal preference in component 1 for all these preferences, it follows that x remains ranked above y along the path. Repeating this process for all the components $3, \ldots, k$, we can construct a path having no swap between x and y from P to a preference $\tilde{P}$ having (i) the same marginal preference as P in component 1, and (ii) x^t at the top-position of the marginal preference in component t for all $t > 1$.

Starting from the preference $\tilde{P}$, make component 1 lexicographically worst through a sep-local path without changing the marginal preferences. Since x^l is weakly preferred to y^l in each component l in each preference of this path, x will remain ranked above y throughout the path. Finally, move x^1 to the top-position in the marginal preference in component 1 thorough a(ny) swap-local path. Since x and y are different in at least two components, there is a component j lexicographically dominating component 1 (as it is the worst component) such that x^j is preferred to y^j in its marginal preference. Therefore, x will be ranked above y throughout the path. Note that in the final preference, for each component t, x^t appears at the top-position in the marginal preference in component t, and hence the alternative x appears at the top-position in it. ■

Lemma .3.5 *Let $\phi : \mathcal{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. Then $\phi(P_N) = \phi(\bar{P}_N)$ for all $P_N, \bar{P}_N$ such that $P^l_N = \bar{P}^l_N$ for all $l \in K$.*

Proof: It is enough to show that $\phi(P_i, P_{-i}) = \phi(\bar{P}_i, P_{-i})$ where $P^l_i = \bar{P}^l_i$ for all $l \in K$. Since preferences with the same marginals are swap-connected, we can assume without loss of generality that P_i and $\bar{P}_i$ are swap-local with the swap of alternatives x and y. Assume for contradiction $\phi(P_i, P_{-i}) \neq \phi(\bar{P}_i, P_{-i})$. By the block preservation property, this means $\phi_x(P_i, P_{-i}) \neq \phi_x(\bar{P}_i, P_{-i})$. By Lemma .3.4, for all $j \in N \setminus i$, there is a sep-local path $(P^1_j = P_j, \ldots, P^t_j = \bar{P}_j)$ with $\bar{P}_j(1) = P_i(1)$ satisfying the property that for all $l < t$ there is a common upper contour set U of both P^l_j and P^{l+1}_j such that exactly one of x and y is contained in U.[22] By Lemma .3.1, we have $\phi_x(P_i, \bar{P}_j, P_{-\{i,j\}}) \neq \phi_x(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. Continuing in this manner, we can construct $\bar{P}_{-i} \in \mathcal{S}^{n-1}$ such that $\bar{P}_j(1) = P_i(1)$ for all $j \neq i$ and $\phi_x(P_i, \bar{P}_{-i}) \neq \phi_x(\bar{P}_i, \bar{P}_{-i})$. However, since $(P_i, \bar{P}_{-i})$ and $(\bar{P}_i, \bar{P}_{-i})$ are unanimous preference profiles with the top-ranked alternative different from x, $\phi_x(P_i, \bar{P}_{-i}) = \phi_x(\bar{P}_i, \bar{P}_{-i}) = 0$, a contradiction. ■

Proof:[Proof of Proposition 6.8.1] Let $\phi : \mathcal{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that ϕ satisfies component-unanimity. Consider $P_N \in \mathcal{S}^n$ such that $P^l_i(1) = x^l$ for all $i \in N$, some $l \in K$, and some $x^l \in A^l$. Assume for contradiction $\phi^l_{x^l}(P_N) \neq 1$. Without loss of generality assume $l = 1$. By Lemma .3.3 and Lemma .3.5, we can assume that P_N is a profile of lexicographically separable preferences with each agent i having the component ordering $1P^\circ_i \cdots P^\circ_i k$. Fix some alternative x^k in component k and consider some agent i. As we have argued in the proof of Lemma

[22]Note that the statement of Lemma .3.4 is slightly different from what we mention here. Since any two consecutive preferences in a sep-local path differ by swaps of multiple pairs of consecutively ranked alternatives, these two statements are equivalent.

.3.4, there is a sep-local path from P_i to a preference $\bar{P}_i$ such that each preference in the path has the same lexicographic ordering over the components as P_i, $\bar{P}_i^k(1) = x^k$, and $\bar{P}_i^l = P_i^l$ for all $l \neq k$. By construction, for all $x^{-k} \in A^{-k}$ and $y^k, z^k \in A^k$, each pair of alternatives $((x^{-k}, y^k), (x^{-k}, z^k))$ forms a block for any two consecutive (sep-local) preferences in the path. This in particular implies $\phi^1_{x^1}(\bar{P}_i, P_{-i}) = \phi^1_{x^1}(P_N)$. Continuing this way, we can construct $\bar{P}_N \in \mathcal{S}^n$ such that $\bar{P}_i^k(1) = x^k$ for all $i \in N$ and $\phi^1_{x^1}(\bar{P}_N) = \phi^1_{x^1}(P_N)$.

Let $\bar{\bar{P}}_N$ be the profile of lexicographically separable preferences that has same marginal preferences as $\bar{P}$ and has lexicographic ordering over the components as $1\bar{\bar{P}}_i^0 \ldots \bar{\bar{P}}_i^0 k \bar{\bar{P}}_i^0 k-1$ for all $i \in N$. That is, the components $k-1$ and k are swapped from $\bar{P}_i^0$ to $\bar{\bar{P}}_i^0$. By Lemma .3.5, $\phi(\bar{\bar{P}}_N) = \phi(\bar{P}_N)$. Now, by using similar logic as for component k, we can construct $\hat{P}_N \in \mathcal{S}^n$ such that $\hat{P}_i^{k-1}(1) = x^{k-1}$ for all $i \in N$ and $\phi^1_{x^1}(\hat{P}_N) = \phi^1_{x^1}(P_N)$. Continuing in this manner, we can arrive at $\tilde{P}_N \in \mathcal{S}^n$ such that $\tilde{P}_i^t(1) = x^t$ for all $t \in K$ and all $i \in N$ and $\phi^1_{x^1}(\tilde{P}_N) = \phi^1_{x^1}(P_N)$. However, since $\tilde{P}_N$ is unanimous with $\tilde{P}_i(1) = x$ for all $i \in N$, we have $\phi_x(\tilde{P}_N) = 1$, which in particular implies $\phi^1_{x^1}(P_N) = 1$, a contradiction. ■

.3.9 Proof of Proposition 6.8.2

We use the following observation in the proof of Proposition 6.8.2.

Observation .3.4 *Let $l \in K$ and let $\pi^l = (\pi^l(1), \ldots, \pi^l(t))$ be a swap-local path in $\mathcal{D}^l$ such that the relative ordering of two alternatives $x^l, y^l \in A^l$ remains the same along the path. Then, for every component ordering $P^0 \in \mathcal{P}(K)$ having l as the worst component, and for every collection of marginal preferences $(P^1, \ldots, P^{l-1}, P^{l+1}, \ldots, P^k)$ over components other than l, the relative ordering of any two alternatives in the set $\{a \in A \mid a^l \in \{x^l, y^l\}\}$ will remain the same along the sep-local path $((P^0, P^1, \ldots, P^{l-1}, \pi^l(1), P^{l+1}, \ldots, P^k), \ldots,$ $(P^0, P^1, \ldots, P^{l-1}, \pi^l(t), P^{l+1}, \ldots, P^k))$ in the domain $\mathcal{S}(\mathcal{D}^1, \ldots, \mathcal{D}^k)$.*

Proof: Let $\phi : \mathcal{S}^n \to \Delta A$ be a unanimous RSCF satisfying the block preservation property. We show that ϕ is tops-only. Consider $P_N, \bar{P}_N \in \mathcal{S}^n$ with $P_i(1) = \bar{P}_i(1)$ for all $i \in N$. If $P_N^l = \bar{P}_N^l$ for all $l \in K$, then we are done by Lemma .3.5. It is sufficient to assume that only one agent, say i, changes her marginal preference to a swap-local preference in exactly one component, say t, and nothing else changes from P_N to $\bar{P}_N$. That is, P_i^t and $\bar{P}_i^t$ are swap-local with the swap of some y^t and z^t, $P_j^t = \bar{P}_j^t$ for all $j \in N \setminus i$, and $P_N^l = \bar{P}_N^l$ for all $l \neq t$. Assume without loss of generality, $t = k$. Furthermore, in view of Lemma .3.5, let us assume that all agents have the same component ordering Q^0 in both P_N and $\bar{P}_N$ where Q^0 is given by $1Q^0 \ldots Q^0 k$. We need to show $\phi(P_N) = \phi(\bar{P}_N)$. Assume for contradiction $\phi(P_N) \neq \phi(\bar{P}_N)$. Since k is the worst component in P_i^0, by block preservation property, this implies $\phi_{(x^{-k}, y^k)}(P_N) \neq \phi_{(x^{-k}, y^k)}(\bar{P}_N)$ for some (x^{-k}, y^k).

Consider P_j^k for some $j \neq i$. By our assumption on the marginal domains, there is a swap-local path $\pi^k = (\pi^k(1) = P_j^k, \ldots, \pi^k(t) = \hat{P}_j^k)$ in $\mathcal{D}^k$ with $\hat{P}_j^k(1) = P_i^k(1)$ such that for any two consecutive preferences in the path there is a common upper contour set U such that exactly one of y^k and z^k is contained in U. By Observation .3.4, the path $((P_j^0, P_j^1, \ldots, P_j^{k-1}, \pi^k(1)), \ldots,$ $(P_j^0, P_j^1, \ldots, P_j^{k-1}, \pi^k(t)))$ satisfies the property that for all $l < t$ and all $u, v \in P_i \triangle \bar{P}_i$ there is a common upper contour set U of both $(P_j^0, P_j^1, \ldots, P_j^{k-1}, \pi^k(l))$ and $(P_j^0, P_j^1, \ldots, P_j^{k-1}, \pi^k(l+1))$ such that exactly one of u and v is contained in U, and hence by Lemma .3.1, we have $\phi_{(x^{-k}, y^k)}(P_i, \hat{P}_j, P_{-\{i,j\}}) \neq \phi_{(x^{-k}, y^k)}(\bar{P}_i, \hat{P}_j, P_{-\{i,j\}})$, where $\hat{P}_j = (P_j^0, P_j^1, \ldots, P_j^{k-1}, \hat{P}_j^k)$. Continuing in this manner, we can construct $\hat{P}_{-i} \in \mathcal{S}^{n-1}$ such that for all $j \in N \setminus i$, $\hat{P}_j^k(1) = P_i^k(1)$ and $\hat{P}_j^l = P_j^l$ for all $l \neq k$, and $\phi_{(x^{-k}, y^k)}(P_i, \hat{P}_{-i}) \neq \phi_{(x^{-k}, y^k)}(\bar{P}_i, \hat{P}_{-i})$. Note that the preference profiles $(P_i, \hat{P}_{-i})$ and $(\bar{P}_i, \hat{P}_{-i})$ are component-unanimous for component k, and hence by Proposition 6.8.1, $\phi^k_{P_i^k(1)}(P_i, \hat{P}_{-i}) = \phi^k_{P_i^k(1)}(\bar{P}_i, \hat{P}_{-i}) = 1$. This implies $\phi_{(x^{-k}, y^k)}(P_i, \hat{P}_{-i}) = \phi_{(x^{-k}, y^k)}(\bar{P}_i, \hat{P}_{-i}) = 0$, which contradicts $\phi_{(x^{-k}, y^k)}(P_i, \hat{P}_{-i}) \neq \phi_{(x^{-k}, y^k)}(\bar{P}_i, \hat{P}_{-i})$. ■

.4 Connection with the related literature

Consider the domain $\widehat{\mathcal{D}}$ given in Table .4.1 and consider the localness structure to be swap-localness (see Figure .4.1). In what follows we show that $\widehat{\mathcal{D}}$ satisfies the path-richness property but violates the conditions provided in [80], [61] and [46].

Table .4.1: The domain $\widehat{\mathcal{D}}$ satisfying the path richness property but violating the conditions provided in [80], [61] and [46]

P_1	P_2	P_3	P_4	P_5	P_6	P_7
a	b	b	b	b	b	d
b	a	c	c	d	d	b
c	c	a	d	c	a	a
d	d	d	a	a	c	c

Figure .4.1: The localness structure of the domain $\widehat{\mathcal{D}}$

$$\underset{P_1}{abcd} \overset{\{a,b\}}{\text{———}} \underset{P_2}{bacd} \overset{\{a,c\}}{\text{———}} \underset{P_3}{bcad} \overset{\{a,d\}}{\text{———}} \underset{P_4}{bcda} \overset{\{c,d\}}{\text{———}} \underset{P_5}{bdca} \overset{\{c,a\}}{\text{———}} \underset{P_6}{bdac} \overset{\{b,d\}}{\text{———}} \underset{P_7}{dbac}$$

Path-richness: Condition (i) of the path-richness property (Definition 6.3.3) requires that for each (unordered) pair $\{P, P'\}$ such that $P(1) = P'(1)$ and P, P' are not swap-local, there is a swap-local path

$(P^1 = P, \ldots, P^t = P')$ such that $P^l(1) = P(1)$ for all $l = 1, \ldots, t$. Let us list all pairs of preferences in $\widehat{\mathcal{D}}$ such that $P(1) = P'(1)$ and P, P' are not swap-local: $\{P_2, P_4\}$, $\{P_2, P_5\}$, $\{P_2, P_6\}$, $\{P_3, P_5\}$, $\{P_3, P_6\}$ and $\{P_4, P_6\}$. Below, we present the path required in Condition (i) for each of the mentioned pairs:

- $\{P_2, P_4\} : (P_2, P_3, P_4)$
- $\{P_2, P_5\} : (P_2, P_3, P_4, P_5)$
- $\{P_2, P_6\} : (P_2, P_3, P_4, P_5, P_6)$
- $\{P_3, P_5\} : (P_3, P_4, P_5)$
- $\{P_3, P_6\} : (P_3, P_4, P_5, P_6)$
- $\{P_4, P_6\} : (P_4, P_5, P_6)$

Since $\widehat{\mathcal{D}}$ is swap-connected, by Remark 6.4.1, Condition (ii) of the path-richness property (Definition 6.3.3) requires the following for every pair of preferences $\{P, P'\}$ such that $P(1) = P'(1)$ and P, P' are swap-local: if P and P' are swap-local and $P \triangle P' = \{x, y\}$ then for each preference $\hat{P} \in \widehat{\mathcal{D}}$, there exists a swap-local path $(P^1 = \hat{P}, \ldots, P^t)$ with $P^t(1) = P(1)$ such that for all $l < t$, $xP^l y$ if and only if $xP^{l+1}y$. We list all pairs of preferences in $\widehat{\mathcal{D}}$ such that such that $P(1) = P'(1)$ and P, P' are swap-local: $\{P_2, P_3\}$, $\{P_3, P_4\}$, $\{P_4, P_5\}$ and $\{P_5, P_6\}$.

Consider the pair P_2, P_3. Note that $P_2(1) = P_3(1) = b$ and $P_2 \triangle P_3 = \{a, c\}$. Consider the preference $\hat{P} = P_1$. Then the path (P_1, P_2) satisfies Condition (ii) of the path-richness property as aP_1c, aP_2c and $P_2(1) = b$. Consider the preference $\hat{P} = P_7$. Then the path (P_7, P_6) satisfies Condition (ii) as aP_7c, aP_6c and $P_6(1) = b$. Similarly, Condition (ii) can be verified for other pairs of swap-local preferences.

Connected domains without restoration and sparsely connected domains without restoration: Now, we proceed to show that $\widehat{\mathcal{D}}$ is neither connected without restoration nor sparsely connected without restoration. It is shown in [46] that sparsely connected domains without restoration is a generalization of connected domains without restoration. Therefore, it is sufficient to show that $\widehat{\mathcal{D}}$ is not sparsely connected without restoration.

We begin with the definition of sparsely connected domains without restoration.

Definition .4.1 *A distinct sequence of swap-local preferences $(P^1, \ldots, P^k)$ in a domain is without $\{x, y\}$-restoration if there exists no distinct $l, l' \in \{1, \ldots, k-1\}$ such that $P_l \triangle P^{l+1} = P^{l'} \triangle P^{l'+1} = \{x, y\}$. A domain $\mathcal{D}$ is sparsely connected without restoration if for any $P, P' \in \mathcal{D}$ and any $x, y \in A$ such that $\{x, y\} \cap \tau(\mathcal{D}) \neq \emptyset$, P and P' are connected without $\{x, y\}$-restoration in $\mathcal{D}$, i.e., there exists a sequence of distinct preferences $(P = P^1, P^2, \ldots, P^k = P')$ in $\mathcal{D}$ without $\{x, y\}$-restoration.*

To show that the domain $\widehat{\mathcal{D}}$ is not sparsely connected without restoration we need to show that for some $P, P' \in \widehat{\mathcal{D}}$ and alternatives x, y there does not exist any sequence of distinct preferences $(P = P^1, P^2, \ldots, P^k = P')$ in $\widehat{\mathcal{D}}$ without restoration. Consider the preferences P_1 and P_6 and the alternatives a, c. The only swap-local path from P_1 to P_6 is $(P_1, P_2, P_3, P_4, P_5, P_6)$. Note that $P_2 \triangle P_3 = \{a, c\}$ and $P_5 \triangle P_6 = \{a, c\}$. This means there is no distinct sequence of swap-local preferences from P_1 to P_6 in $\widehat{\mathcal{D}}$ that is without $\{a, c\}$-restoration. Hence, $\widehat{\mathcal{D}}$ is not a sparsely connected domain without restoration.

Interior and exterior properties: We begin with defining the interior and exterior properties introduced in [22].

Definition .4.2 *A domain $\mathcal{D}$ satisfies the interior property if for all $a \in A$ and all distinct P, P' with $P(1) = P'(1) = x$ there exists a swap-local path $(P = P_1, \ldots, P_k = P')$ with $P_l(1) = x$ for all $l \leq k$.*

We need some terms and terminologies to define the exterior property. Given distinct $P, P' \in \mathcal{D}$, alternatives $x, y \in A$ are isolated in (P, P') if there exists $1 \leq k \leq m - 1$ such that

- $\cup_{t=1}^{k} P(t) = \cup_{t=1}^{k} P'(t)$,
- either $x \in \cup_{t=1}^{k} P(t)$ and $y \in A \setminus \cup_{t=1}^{k} P(t)$, or $x \in A \setminus \cup_{t=1}^{k} P(t)$ and $y \in \cup_{t=1}^{k} P(t)$.

Given distinct $P, P' \in \mathcal{D}$ and $x, y \in A$, let $(P = P_1, \ldots, P_k = P')$ be a sequence of preferences (not necessarily swap-local) such that x and y are isolated in (P_l, P_{l+1}) for all $l < k$. Then, $(P = P_1, \ldots, P_k = P')$ is referred to as an (x, y)-Is-path connecting P and P'.

Definition .4.3 *A domain $\mathcal{D}$ satisfies the exterior property if given $P, P' \in \mathcal{D}$ with $P(1) \neq P'(1)$ and $x, y \in A$ with xPy and $xP'y$, there exists an (x, y)-Is-path connecting P and P'.*

Now, we are ready to compare our results with that of [21]. [21] show that if a domain satisfies the interior and exterior properties then any unanimous and DSIC RSCF on it is tops-only. As we have stated in Remark 6.4.3, in case of swap-connected domains the Interior Property is same as Condition (i) of the path-richness property. We show that $\widehat{\mathcal{D}}$ does not satisfy the exterior property. Consider the preferences P_2 and P_6 and the alternatives a, c. We have aP_2c and aP_6c. In order to satisfy the exterior property, we need an (a, c)-Is-path connecting P_2 and P_6. However, there does not exist any such path connecting P_2 and P_6 (the only paths connecting P_2 and P_6 are $(P_2, P_3, P_4, P_5, P_6)$, (P_2, P_3, P_4, P_6), (P_2, P_3, P_5, P_6), (P_2, P_4, P_5, P_6), (P_2, P_3, P_6), (P_2, P_5, P_6), (P_2, P_4, P_6) and (P_2, P_6)).

7

Myopic-farsighted stability in many-to-one matching

7.1 Introduction

In matching literature, one of the most sought-after properties of a matching is stability. A matching is stable if no agent can be immediately better off by blocking the matching by forming/destroying the link with another agent. However, this notion of stability disregards the ability of agents to anticipate that a blocking can be countered by subsequent blockings. [42] introduced the notion of indirect dominance to capture such destabilizing effects. In this paper, we consider a many-to-one two-sided matching model where one side of the market is myopic and the other side is farsighted. A myopic agent tends to block a matching when the resulted matching is better for her. Whereas, a farsighted agent takes into account the possible counter-blocks that may follow after her blocking, and tends to block whenever the final outcome is better for her. [44] argue that there might be an asymmetry between the two sides of the market in their ability to foresee potential future changes. This can be further substantiated by [13] who shows that there are cases where one side of the market has some advantage over the other side of the market. We consider the college admissions problem where students have to matched to colleges. We assume the agents are heterogeneous with respect to their ability to foresee the consequences of a block, and thereby categorized as myopic and farsighted. We study the structure of stable matchings and stable sets in this setting.

A sequence of matchings constitutes a pairwise myopic-farsighted improving path if the farsighted agents are better off at the final matching compared to the matching they block and the myopic agents are better off at the immediate matching obtained after their blocking. A set of matchings forms a pairwise myopic-farsighted stable set if there is a pairwise myopic-farsighted improving path from every matching outside the set to it, and there is no pairwise myopic-farsighted improving path between two matchings in the set. A matching is stable if there is no pairwise myopic-farsighted improving path from the matching to any other matching. The objective of the paper is to explore the structure of stable sets and stable matchings in this setting.

A model involving both myopic and farsighted agents is first introduced in [44]. They consider a two-sided one-to-one (marriage problem) matching problem. In this setting, their main results show that when all agents are myopic then the pairwise myopic-farsighted stable set coincides with the core and when all agents are farsighted then the pairwise myopic-farsighted stable set coincides with a singleton subset of the core (see Theorems 2 and 3 in [44]). Moreover, when women are farsighted (and possibly some men too) or no men are farsighted, they identify some preference profiles where the women optimal matching constitutes a single pairwise myopic-farsighted stable set at some preference profiles (see Theorems 6 and 7 in [44]). [59] consider the many-to-one two-sided matching problem when all agents are farsighted. They show that a set of matchings constitutes a farsighted stable set if and only if it is a singleton subset of the strong core. They further show that the farsighted core, can be empty. [53] consider the roommate market where all agents are farsighted and show that a set of matchings is a farsighted stable set if and only if it is a singleton set containing a myopic stable matching. Several other papers consider the same model (with homogeneous agents, that is, either everyone is farsighted or everyone is myopic) and show that the core constitutes desirable outcomes (see [36], [59], [52], and [43]). However, to the best of our knowledge, the structure of pairwise myopic-farsighted stable sets and pairwise myopic-farsighted stable matchings are not known in the literature in matching models involving both myopic and farsighted agents.

As a standard many-to-one two-sided matching problem, we consider the college admission problem. We assume that colleges have a common preference over the students. This is a natural assumption in many situations as preferences over students are derived using their scores in some common exam. When students are farsighted and colleges are myopic, we provide a characterization of pairwise myopic-farsighted stable matchings, as well as the pairwise myopic-farsighted stable set at every preference profile. When students are myopic and colleges are farsighted, we provide a characterization of pairwise myopic-farsighted stable matchings at every preference profile and provide a class of pairwise myopic-farsighted stable sets at every preference profile.

The paper is organized as follows. Section 7.2 introduces the notions of matching, pairwise myopic-farsighted improving path, pairwise myopic-farsighted stable matching, and pairwise myopic-farsighted stable set. Section 7.3 presents our results when students are farsighted and colleges are myopic and Section 7.4 presents our results when students are myopic and colleges are farsighted. Finally, in Section 7.5 we provide the conclusion of the paper.

7.2 Preliminaries

We consider a two-sided many-to-one matching problem between a set of students $S = \{s_1, \dots, s_m\}$ and a set of colleges $C = \{c_0, c_1, \dots, c_n\}$. The set of college contains a specific college c_0 which is interpreted as a dummy college. Each college $c \in C$ has a quota q_c. We assume that $q_{c_0} = \infty$ whereas $1 \leq q_c < \infty$ for all $c \in C \setminus \{c_0\}$. We assume WLOG that $q_{c_i} \leq q_{c_j}$ for all $1 \leq i < j \leq n$. A matching μ is a mapping from S to C such that for all $c \in C$, $|\mu^{-1}(c)| \leq q_c$. For a student s, $\mu(s) = c_0$ implies that s is not matched with any college (stays "single"). For simplicity, we denote $\mu^{-1}(c)$ by $\mu(c)$. We denote by $\mathcal{M}$ the set of all matchings.

A strict preference P on a finite set A is linear order on A. The weak part of a strict preference P is denoted by R. Since P is strict, aRb for $a, b \in A$ means either aPb or $a = b$. For $B \subseteq A$ and $l \leq |B|$, $r_l(B, P)$ denotes the l-th ranked element of B according to P, that is, $r_l(B, P) = a$ if and only if $|\{a' \in B \mid a'Pa\}| = l - 1$.

Each student $s \in S$ has a (strict) preference P_s on C. Colleges in $C \setminus \{c_0\}$ have a *common* strict preference over individual students which we denote by P_c. Without loss of generality, we assume that $s_1 P_c s_2 P_c \cdots P_c s_{m-1} P_c s_m$, that is, according to P_c, s_1 is the best student, s_2 is the second best student, and so on. The common preference P_c of the colleges over individual students is extended to preference $\tilde{P}_c$ over sets of students in the following manner. For two sets of students S' and S'', a college always prefers the one with more students, that is, $|S'| > |S''|$ implies $S' \tilde{P}_c S''$. If $|S'| = |S''|$, then the preference on S' and S'' is derived in a lexicographic manner, that is, $S' \tilde{P}_c S''$ if and only if there exists $k \leq |S'| (= |S''|)$ such that $r_k(S', P_c) P_c r_k(S'', P_c)$ and $r_l(S', P_c) = r_l(S'', P_c)$ for all $l < k$. Note that by definition the empty set of students is the least preferred set of students for any college. To minimize notations, we use the notation P_c itself to denote the extension $\tilde{P}_c$.

We denote $S \cup C \setminus \{c_0\}$ by N. Elements of N are called agents. A collection of preferences of the agents in N is called a preference profile and is denoted by P_N. More formally, $P_N = \big((P_s)_{s \in S}, (P_c)_{c \in C \setminus \{c_0\}}\big)$.

7.2.1 Pairwise myopic-farsighted stable matchings and pairwise myopic-farsighted stable sets

The set of agents N is partitioned into two sets F and M. The agents in F are farsighted who anticipate that individual and coalitional deviations are countered by subsequent deviations and the agents in M are myopic in the sense that they do not anticipate such deviations. For a matching μ and a pair $(s, c) \in S \times C \setminus \{c_0\}$ such that $\mu(s) \in c$, we denote by $\mu - (s, c)$ the matching μ' obtained from μ by removing s from c and keeping everything else unchanged. More formally, μ' is such that $\mu'(s) = c_0, \mu'(c) = \mu(c) \setminus s$, and $\mu'(i) = \mu(i)$ for all $i \in N \setminus \{s, c\}$. For a matching μ, a pair $(s, c) \in S \times C$ such that $s \notin \mu(c)$, and a preference P_c, we denote by $\mu + (s, c)$ the matching μ' obtained from μ through matching s to c by removing the worst student in $\mu(c)$ according to P_c in case the quota of c was already full at μ and c is myopic (and keeping everything else unchanged). More formally, $\mu' = \mu + (s, c)$ is such that

(i) if $|\mu(c)| < q_c$, then $\mu'(c') = \mu(c') \setminus s$ for all $c' \in C \setminus c$ and $\mu'(c) = \mu(c) \cup s$, and

(ii) if $|\mu(c)| = q_c$, then $\mu'(c') = \mu(c) \setminus s$ for all $c' \in C \setminus c$ and $\mu'(c) = (\mu(c) \cup s) \setminus s'$ where s' is the worst student in $\mu(c)$ according to P_c if c is myopic, otherwise s' is an arbitrary student in $\mu(c)$. The assumption that a myopic college rejects the worst student whenever it is required to reject a student is consistent with the idea of the best response in game theory. In other words, a myopic agent moves towards the best possible immediate outcome. In the literature, any profitable deviation of a myopic agent is considered plausible. We feel, on top of the assumption of myopic type, this "greedy" deviation assumption becomes quite restrictive and somewhat impractical.

Whenever we say that a pair $(s, c) \in S \times C$ blocks a matching μ through μ', we mean that $\mu' = \mu + (s, c)$ or $\mu' = \mu - (s, c)$. Next, we introduce the notion of pairwise myopic-farsighted improving path at a profile P_N. A pairwise myopic-farsighted improving path from a matching μ to a matching μ' at a profile P_N is a sequence of matchings $\mu_0, \ldots, \mu_L$ starting from μ and ending at μ' satisfying the following properties. For all $l \in 0, \ldots, L-1$, μ_{l+1} is obtained through a block by a pair (s, c) to the matching μ_l.[1] Depending on whether s and c break their existing match or form a new match from μ_l to μ_{l+1}, at least one or both members of s and c need to take initiative in the block. If that member is myopic then she prefers her match in the immediate outcome μ_{l+1}, and if she is farsighted then she prefers her match at the final outcome μ'. More formally:

(i) If $\mu_{l+1} = \mu_l - (s, c)$, that is, if the match between s and c is broken from μ_l to μ_{l+1}, then either s or c takes the initiative. And, as we have just explained, a myopic member takes the initiative if she prefers her match at the immediate outcome, and a farsighted member takes initiative if she prefers her match at the final outcome μ_L.

(ii) If $\mu_{l+1} = \mu_l + (s, c)$, that is, a match between s and c is formed from μ_l to μ_{l+1}, then both of s and c take the initiative (as before, the reason for taking an initiative depends on whether the member is myopic or farsighted).

We now present the mathematical definition of a pairwise myopic-farsighted improving path path at a

[1]It is worth mentioning that if $\mu_{l+1} = \mu_l + (s, c)$ and c is myopic, then it must be that either s is preferred to some student in c at μ_l or the quota of c is not filled in μ_l.

profile P_N.

Definition 7.2.1 *A pairwise myopic-farsighted improving path from a matching $\mu \in \mathcal{M}$ to a matching $\mu' \in \mathcal{M} \setminus \{\mu\}$ at a profile P_N is a sequence of distinct matchings $(\mu_0, \dots, \mu_L)$ with $\mu_0 = \mu$ and $\mu_L = \mu'$ such that for every $l \in \{0, \dots, L-1\}$ either (i) or (ii) holds:*

(i) $\mu_{l+1} = \mu_l - (s, c)$ *for some* $(s, c) \in S \times C \setminus \{c_0\}$ *such that*

$$\begin{cases} \mu_{l+1}(s) P_s \mu_l(s) \text{ if } s \in M \\ \mu'(s) P_s \mu_l(s) \text{ if } s \in F. \end{cases}$$

or

$$\begin{cases} \mu_{l+1}(c) P_c \mu_l(c) \text{ if } c \in M \\ \mu'(c) P_c \mu_l(c) \text{ if } c \in F. \end{cases}$$

(ii) $\mu_{l+1} = \mu_l + (s, c)$ *for some* $(s, c) \in S \times C$ *such that*

$$\begin{cases} \mu_{l+1}(s) R_s \mu_l(s) \text{ if } s \in M \\ \mu'(s) R_s \mu_l(s) \text{ if } s \in F. \end{cases}$$

and

$$\begin{cases} \mu_{l+1}(c) R_c \mu_l(c) \text{ if } c \in M \\ \mu'(c) R_c \mu_l(c) \text{ if } c \in F. \end{cases}$$

with at least one of these preferences being strict.

We say a matching μ' dominates another matching μ at a profile P_N if there is a pairwise myopic-farsighted improving path from μ to μ' at P_N. If a matching μ' dominates another matching μ at a profile P_N then we write $\mu \xrightarrow{P_N} \mu'$. The set of all matchings that dominates a matching $\mu \in \mathcal{M}$ at a profile P_N is denoted by $h(\mu, P_N)$, that is, $h(\mu, P_N) = \{\mu' \in \mathcal{M} \setminus \{\mu\} \mid \mu' \xrightarrow{P_N} \mu\}$.

Definition 7.2.2 *A matching $\mu \in \mathcal{M}$ is pairwise myopic-farsighted stable at a profile P_N if if it is not dominated by any other matching at P_N, that is, $h(\mu, P_N) = \emptyset$.*

Definition 7.2.3 *A set of matchings $V \subset \mathcal{M}$ is a pairwise myopic-farsighted stable set at a profile P_N if it satisfies the following two conditions.*

(i) Internal stability: For every $\mu \in V$, it holds that $h(\mu, P_N) \cap V = \emptyset$.

(ii) External stability: For every $\mu \in \mathcal{M} \setminus V$, it holds that $h(\mu, P_N) \cap V \neq \emptyset$.

7.3 Students are farsighted

In this section, we assume that the students are farsighted and colleges are myopic, and explore the structure of pairwise myopic-farsighted stable matchings and pairwise myopic-farsighted stable sets.

For $k \in \{1, 2, \ldots, m\}$, let C_k denote the set of colleges that have quota more than or equal to k, that is, $C_k = \{c \in C \mid q_c \geq k\}$. Recall that by convention, $q_{c_0} = \infty$ and hence $c_0 \in C_k$ for all $k \in \{1, \ldots, m\}$. The option set of colleges of a student s_k is the set of colleges whose quota is at least k. For a preference profile P_N, let $\mathcal{M}(P_N) \subseteq \mathcal{M}$ denote the set of matchings μ where each student gets a college that is weakly better than any college in her option set, that is, $\mu(s_k) R_{s_k} r_1(C_k, P_{s_k})$ for all $k \in \{1, 2, \ldots, m\}$. Throughout this section, we assume that the number of students is at least as much as the number of positions in any college, that is $m \geq q_{c_n}$.

Our next proposition says that for all P_N each matching in $\mathcal{M}(P_N)$ dominates every other matching (be it in $\mathcal{M}(P_N)$ or not).

Proposition 7.3.1 *Suppose that the students are farsighted and the colleges are myopic. Then, for all $\bar{\mu} \in \mathcal{M}(P_N)$ and all $\mu \in \mathcal{M} \setminus \{\bar{\mu}\}$, we have $\bar{\mu} \in h(\mu, P_N)$ for all P_N.*

The proof of this proposition is relegated to Appendix .1.

In our next proposition, we show that a particular matching μ^* in $\mathcal{M}(P_N)$ is not dominated by any matching outside $\mathcal{M}(P_N)$, that is, there is no pairwise myopic-farsighted improving path from μ^* to any matching outside $\mathcal{M}(P_N)$. Define $\bar{\mu}^* \in \mathcal{M}(P_N)$ as $\bar{\mu}^*(s_k) = r_1(C_k, P_{s_k})$.

Proposition 7.3.2 *Suppose that the students are farsighted and the colleges are myopic and let P_N be an arbitrary preference profile. There is no pairwise myopic-farsighted improving path from $\bar{\mu}^*$ to any matching in $\mathcal{M} \setminus \mathcal{M}(P_N)$, that is, $h(\bar{\mu}^*, P_N) \subseteq \mathcal{M}(P_N)$.*

The proof of this proposition is relegated to Appendix .2.

We are now ready to present one of the two main theorems in this section. It says that there exists a pairwise myopic-farsighted stable matching at a profile P_N if and only if the set $\mathcal{M}(P_N)$ is singleton, and in that case, the singleton element of $\mathcal{M}(P_N)$ is the unique pairwise myopic-farsighted stable matching.

Theorem 7.3.1 *Suppose that the students are farsighted and the colleges are myopic and let P_N be an arbitrary profile. If $\mathcal{M}(P_N)$ is singleton, then the element in $\mathcal{M}(P_N)$ is the unique pairwise myopic-farsighted stable matching at P_N. If $\mathcal{M}(P_N)$ is not singleton, then there is no pairwise myopic-farsighted stable matching at P_N.*

The proof of this theorem is relegated to Appendix .3.

Our second main theorem of this section says that a set of matchings is stable at a profile P_N if and only if it is a singleton subset of $\mathcal{M}(P_N)$.

Theorem 7.3.2 *Suppose that the students are farsighted and the colleges are myopic and let P_N be an arbitrary preference profile. Then, a set of matchings V is pairwise myopic-farsighted stable set at P_N if and only if V is a singleton subset of $\mathcal{M}(P_N)$, that is, $V = \{\mu\}$ for some $\mu \in \mathcal{M}(P_N)$.*

The proof of this theorem is relegated to Appendix .4.

7.4 Colleges are farsighted

In this section, we assume that colleges are farsighted and students are myopic, and explore the structure of pairwise myopic-farsighted stable matchings and pairwise myopic-farsighted stable sets in this setting. Throughout this section, we assume that $\sum_{c \in C \setminus \{c_0\}} q_c = m$ and cP_sc_0 for all $s \in S$ and all $c \in C \setminus \{c_0\}$.

Let $S_c(P_N)$ denote that set of students whose most preferred college is c at the profile P_N, that is, $S_c(P_N) = \{s \mid r_1(P_s) = c\}$. The option sets $O_c(P_N)$ of a college c at a profile P_N is defined as the set of sets of students in the following manner:

$$O_c(P_N) = \{S' \subseteq S_c(P_N) \mid |S'| = q_c\} \text{ if } |S_c(P_N)| \geq q_c, \text{ and}$$
$$O_c(P_N) = \{S_c(P_N) \cup S \mid |S_c(P_N) \cup S| = q_c\} \text{ if } |S_c(P_N)| < q_c.$$

For an intuitive understanding of option set, let us consider the set of students $S_c(P_N)$ who rank a college c as their best at a profile P_N. If the number of students in $S_c(P_N)$ is more than the quota of c, then the option set consists of any subset of $S_c(P_N)$ containing q_c students. On the other hand, if the number of students in $S_c(P_N)$ is less than the quota of c, then the option set of c is any set of q_c students containing the students in $S_c(P_N)$.

Consider the set of matchings $\widehat{\mathcal{M}}(P_N)$ such that $\mu \in \widehat{\mathcal{M}}(P_N)$ if and only if

$$\mu(c) R_c S' \text{ for all } S' \in O_c(P_N) \text{ if } |S_c(P_N)| \geq q_c, \text{ and}$$
$$\mu(c) R_c S' \text{ for some } S' \in O_c(P_N) \text{ if } |S_c(P_N)| < q_c.$$

Our next proposition says that at a profile P_N each matching in $\widehat{\mathcal{M}}(P_N)$ dominates every other matching (be it in $\widehat{\mathcal{M}}(P_N)$ or not).

Proposition 7.4.1 *Suppose that the colleges are farsighted and the students are myopic. Then, for all $\hat{\mu} \in \widehat{\mathcal{M}}(P_N)$ and all $\mu \in \mathcal{M} \setminus \hat{\mu}$, we have $\hat{\mu} \in h(\mu, P_N)$ for all P_N.*

The proof of this proposition is relegated to Appendix .5.

Our next theorem characterizes all situations when a pairwise myopic-farsighted stable matching exists. It further characterizes the pairwise myopic-farsighted stable matchings whenever that exists.

Theorem 7.4.1 *Suppose that the students are myopic and the colleges are farsighted and let P_N be an arbitrary preference profile.*

(i) If $|S_c(P_N)| = q_c$ for all $c \in C \setminus \{c_0\}$, then $\mu^* \in \mathcal{M}$ such that $\mu^*(s) = r_1(P_s)$ for all $s \in S$ is the unique pairwise myopic-farsighted stable matching at P_N.

(ii) Otherwise, there is no pairwise myopic-farsighted stable matching at P_N.

The proof of this theorem is relegated to Appendix .6.

The following theorem provides a class of pairwise myopic-farsighted stable sets at an arbitrary preference profile.

Theorem 7.4.2 *Suppose that the colleges are farsighted and the student are myopic and let P_N be an arbitrary preference profile. Then, every singleton subset of $\widehat{\mathcal{M}}(P_N)$ is a myopic-farsighted stable set at P_N.*

The proof of the theorem is relegated to Appendix .7.

7.5 Conclusion

This paper considers situations where agents are not homogeneous with respect to their ability to foresee the consequences of a blocking. A myopic agent tends to block when the immediate outcome is better for her, while a farsighted agent does it when the final outcome after a sequence of possible counter-blocks is better for her. We provide characterizations of pairwise myopic-farsighted stable matchings and pairwise myopic-farsighted stable sets when students are farsighted and colleges are myopic. When students are myopic and colleges are farsighted, we characterize the pairwise myopic-farsighted stable matchings and provide a class of pairwise myopic-farsighted stable sets at arbitrary profiles.

In this paper, a myopic agent is completely naive to foresee even a single counter-block of her block, while a farsighted agent can foresee arbitrary number of such counter-blocks. A reasonable model would be something that does not assume agents to be either of the extremes. In particular, agents can be assumed to be boundedly rational who can foresee up to a limited number of counter-blocks of their blocks. The structure of stable matchings and stable sets in such a setting is an important future open problem in our view.

.1 Proof of Proposition 7.3.1

Proof: Let P_N be any arbitrary preference profile. Consider $\bar{\mu} \in \mathcal{M}(P_N)$ and an arbitrary $\mu \in \mathcal{M} \setminus \bar{\mu}$. We show that there is a pairwise myopic-farsighted improving path from μ to $\bar{\mu}$. We construct the following pairwise myopic-farsighted improving path from μ to $\bar{\mu}$. The path has two parts π_1 and π_2, and π_1 has some subparts.

The part π_1: The part π_1 has several sub-parts–one for each college. We denote the subpart corresponding to college c by π_1^c.

Sub-part $\pi_1^{c_1}$ corresponding to college c_1: Let $\{s_1, \ldots, s_{q_{c_1}}\} \setminus \mu(c_1)$ be the set of students in the set $\{s_1, \ldots, s_{q_{c_1}}\}$ who are not matched with c_1 at the matching μ. Let us index these students as $s_1^1, \ldots, s_{k_1}^1$ such that $s_{l-1}^1 P_c s_l^1$ for all $l = 2, \ldots, k_1$. If $\mu(c_1)$ is empty, then we go to the next subpart. Else, the path $\pi_1^{c_1} = (\mu_1^1, \mu_1^2, \ldots, \mu_1^{k_1})$ where $\mu_1^l = \mu_1^{l-1} + (s_l^1, c_1)$ for all $l = 1, \ldots, k_1$ and $\mu_1^0 = \mu$.

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Sub-part $\pi_1^{c_n}$ corresponding to college c_n: Let $\{s_1, \ldots, s_{q_{c_n}}\} \setminus \mu(c_1)$ be the set of students in the set $\{s_1, \ldots, s_{q_{c_n}}\}$ who are not matched with c_n at the matching $\mu_{n-1}^{k_{n-1}}$. Let us index these students as $s_1^n, \ldots, s_{k_1}^n$ such that $s_{l-1}^n P_c s_l^n$ for all $n = 2, \ldots, k_1$. If $\mu(c_n)$ is empty, then we go to the next subpart. Else, the path $\pi_1^{c_n} = (\mu_n^1, \mu_n^2, \ldots, \mu_n^{k_n})$ where $\mu_n^l = \mu_n^{l-1} + (s_l^n, c_n)$ for all $l = 1, \ldots, k_n$ and $\mu_n^0 = \mu_{n-1}^{k_{n-1}}$.

The part π_2: Let $\{\bar{s}_1, \cdots, \bar{s}_k\}$ be the set of students who are matched to different colleges at $\mu_n^{k_n}$ and $\bar{\mu}$ such that $\bar{s}_{l-1} P_c \bar{s}_l$ for all $l = 2, \ldots, k$. The path $\pi_2 = (\mu_{n+1}, \ldots, \mu_{n+k})$ where $\mu_{n+l} = \mu_{n+l-1} + (\bar{s}_l, \bar{\mu}(\bar{s}_l))$ for all $l = 1, \ldots, k$ and $\mu_n = \mu_n^{k_n}$.

Consider the path (μ, π_1, π_2). Note that a matching may appear along this path more than once consecutively. Consider the path obtained by replacing a number of successive occurrences of a matching by exactly one occurrence, and thus making all the matchings appearing along the path distinct. For notational simplicity, let us denote the obtained path by (μ, π_1, π_2) itself. We argue that this path is a pairwise myopic-farsighted improving path from μ to $\bar{\mu}$.

From students' point of view: Since students are farsighted, it is enough to show that each student is weakly better of at $\bar{\mu}$ compared to their matching where they block.

Students involved in first part of the path:

Students involved in $\pi_1^{c_1}$: Consider the students $s_1^1, \ldots, s_{k_1}^1$ who block with c_1. These students were matched with $\mu(s)$ when they participated in the blocking. We need to show that $\bar{\mu}$ is weakly better than μ for these students. As $q_{c_1} \leq q_c$ for all $c \in C$, $C_k = \{c_0, c_1, \ldots, c_n\}$ for all $k \in \{1, \ldots, q_{c_1}\}$. Since $\{s_1^1, \ldots, s_{k_1}^1\} \subseteq \{s_1, \ldots, s_{q_{c_1}}\}$, by the definition of $\bar{\mu}$, these students are matched to their most preferred colleges at $\bar{\mu}$. Therefore, they must weakly prefer $\bar{\mu}$ compared to their matching where they block.

Students involved in $\pi_1^{c_2}$: Consider the students $s_2^1, \ldots, s_{k_2}^2$. As we have argued earlier, the students $s_1, \ldots, s_{q_{c_1}}$ are matched with their most preferred college at $\bar{\mu}$, and hence they will be willing to participate in the blocking. If $q_{c_1} = q_{c_2}$, then we are done. Suppose $q_{c_1} < q_{c_2}$. Consider a student s in the set

$\{s_{q_{c_1}+1}, \ldots, s_{q_{c_2}}\} \cap \{s_2^1, \ldots, s_{k_2}^2\}$. If the student s was matched with c_1 at μ, then by the construction of $\pi_1^{c_1}$, she is currently matched to c_0. This means her matching when she blocks is some college in $c_0, c_2, \ldots, c_n$. As $q_{c_1} < q_{c_2}$ and $q_{c_2} \leq q_c$ for all $c \in C \setminus c_1$, $C_k = \{c_0, c_2, \ldots, c_n\}$ for all $k \in \{q_{c_1+1}, \ldots, q_{c_2}\}$. By the definition of $\bar{\mu}$, her matching at $\bar{\mu}$ will be weakly better than any of these colleges. Since s is farsighted she will be willing to participate in the blocking.

Continuing in this manner, it follows that students, who are involved in some blocking along the first part of the path, will be willing to participate in the blockings.

Students involved in the second part of the path: Note that at the end of the first part of the path, all the students in $\{s_1, \ldots, s_{q_{c_n}}\}$ are matched with c_n. Recall that for all $k = 1, \ldots, m$, $C_k = \{c \mid q_c \geq k\}$. Since $q_{c_1} \leq \cdots \leq q_{c_n}$, it follows that $c_n \in C_k$ for all $k = 1, \ldots, q_{c_n}$. By the definition of $\bar{\mu}$, the allocation of each student $s \in \{s_1, \ldots, s_{q_{c_n}}\}$ at $\bar{\mu}$ is weakly better than c_n. Therefore, the students in $\{s_1, \ldots, s_{q_{c_n}}\}$ will be willing to participate in the respective blockings. Similarly,at the end of the first part of the path, the students $s_{q_{c_n}+1}, \ldots, s_m$ are matched with c_0. Note that $c_0 \in C_k$ for all $k = 1, \ldots, m$. Hence, by the definition of $\bar{\mu}$, the allocation of each student $s \in \{s_{q_{c_n}+1}, \ldots, s_m\}$ at $\bar{\mu}$ is weakly better than c_0. Therefore, the students in $\{s_{q_{c_n}+1}, \ldots, s_m\}$ will also be willing to participate in the respective blockings.

Colleges' point of view:

Colleges involved in the first part of the path: Note that each college c in this path block (sequentially) with the top q_c students according to its (common) preference. Thus, the colleges will be better off by these blockings no matter what their initial matching was. Since colleges are myopic, they will be willing to participate in these blockings.

Colleges involved in the second part of the path: Note that at the end of the first part of the path, either no college is matched with any student or only c_n is matched to the top q_{c_n} students. Since preferences are lexicographic, all the colleges who are not matched to any student will be willing to participate in any blocking. If c_n is not matched to the top q_{c_n} students at $\bar{\mu}$, then this means that some of these students are matched to some other college at $\bar{\mu}$. By the construction of the path π_2, this means that whenever c_n participates in a blocking its quota is not exhausted. Hence, as preferences are lexicographic, c_n will be willing to participate in its respective blockings. ■

.2 Proof of Proposition 7.3.2

Proof: Let P_N be an arbitrary preference profile and let $\bar{\mu}^* \in \mathcal{M}(P_N)$ be such that $\bar{\mu}^*(s_k) = r_1(C_k, P_{s_k})$. Suppose $\mu \in h(\bar{\mu}^*, P_N)$. We show $\mu \in \mathcal{M}(P_N)$. Assume for contradiction that $\mu \notin \mathcal{M}(P_N)$, there must be a student s_k such that $\bar{\mu}^*(s_k) P_{s_k} \mu(s_k)$. Since the allocation of s_k is strictly worse in μ than that in $\bar{\mu}^*$, s_k will never participate in any blocking along any pairwise myopic-farsighted improving path that ends in the matching μ. Therefore, only way to reach the matching μ through a pairwise myopic-farsighted improving path is that the college $\bar{\mu}^*(s_k)$, say c, blocked with some student s at some stage t by removing s_k. Let μ^t be the matching before this block and (hence) μ^{t+1} is the matching obtained after (s, c) blocks μ^t. Since a college prefers to have more students as long as its quota allows, the fact that c removes s_k while blocking with s implies that μ_c has q_c students while doing this blocking. Moreover, since colleges are myopic, it must be that sP_cs_k and s_k is the worst student in $\mu^t(c)$ while c blocks with s. Since s_k is the worst student and c has q_c student at this stage, it must be that $\mu(c) = \{s_1, \ldots, s_k\}$, which in turn implies that there is no $s \notin S$ such that sP_cs_k, a contradiction. ■

.3 Proof of Theorem 7.3.1

Proof: Let P_N be an arbitrary preference profile and let $\bar{\mu} \in \mathcal{M}(P_N)$. By Proposition 7.3.1, we know that $\bar{\mu} \in h(\mu, P_N)$ for all $\mu \in \mathcal{M} \setminus \{\bar{\mu}\}$. This means no matching other than $\bar{\mu}$ can be a pairwise myopic-farsighted stable matching at P_N. Consider the case when P_N is such that $\mathcal{M}(P_N)$ is singleton. By Proposition 7.3.2, we have $h(\bar{\mu}, P_N) \subseteq \mathcal{M}(P_N)$. This means $h(\bar{\mu}, P_N) = \emptyset$ and hence, $\bar{\mu}$ is the unique pairwise myopic-farsighted stable matching at P_N. Now, consider the case when P_N is such that $\mathcal{M}(P_N)$ is not singleton. This means $|\mathcal{M}(P_N)| \geq 2$. Let $\bar{\bar{\mu}} \in \mathcal{M}(P_N)$. By using Proposition 7.3.1 once again, we know that $\bar{\bar{\mu}} \in h(\bar{\mu}, P_N)$, which means $\bar{\mu}$ cannot be pairwise myopic-farsighted stable either. This completes the proof. ■

.4 Proof of Theorem 7.3.2

Proof: ("If" part) Let P_N be a preference profile and let $V = \{\bar{\mu}\}$ where $\bar{\mu} \in \mathcal{M}(P_N)$. We show that V is a pairwise myopic-farsighted stable set at P_N. Since V is singleton, internal stability is vacuously satisfied. For external stability, we need to show that for every matching $\mu \notin V$, $h(\mu, P_N) \cap V \neq \emptyset$. This follows from Proposition 7.3.1. This completes the proof.

("Only-if" part) Let P_N be a preference profile and let $V \subseteq \mathcal{M}$ be a pairwise myopic-farsighted stable set at P_N. We show V is a singleton element from $\mathcal{M}(P_N)$. Let $\mu \in V$ and $\bar{\mu} \in V \cap \mathcal{M}(P_N)$ be two distinct matchings. By Proposition 7.3.1, $\bar{\mu} \in h(\mu, P_N)$. By internal stability, this implies that both μ and $\bar{\mu}$ cannot be in V. Therefore, if $V \cap \mathcal{M}(P_N) \neq \emptyset$, then V is a singleton set containing an element from $\mathcal{M}(P_N)$. To complete the proof, we need to rule out the possibility that $V \subseteq \mathcal{M} \setminus \mathcal{M}(P_N)$. Consider $\bar{\mu}^* \in \mathcal{M}(P_N)$ such that $\bar{\mu}^*(s_k) = r_1(C_k, P_{s_k})$. As $V \subseteq \mathcal{M} \setminus \mathcal{M}(P_N)$, this means that $\bar{\mu}^* \in \mathcal{M} \setminus V$. By Proposition 7.3.2, $h(\bar{\mu}^*, P_N) \subseteq \mathcal{M}(P_N)$. This means that $h(\bar{\mu}^*, P_N) \cap V = \emptyset$ which violates external stability. ■

.5 Proof of Proposition 7.4.1

Proof: Let P_N be any arbitrary preference profile. Consider a matching $\hat{\mu} \in \widehat{\mathcal{M}}(P_N)$ and a matching $\mu \in \mathcal{M} \setminus \hat{\mu}$. We construct a pairwise myopic-farsighted improving path from μ to $\hat{\mu}$. The path has two parts: the first part consists a sequence of pairs (not necessarily distinct) of matchings which we define one by one, and the second part consists of a sequence of matchings.

The first part of the path: To facilitate the presentation, we denote the top-ranked college of student s_l by $\hat{c}_l$.

(μ_0, μ'_0): Define $\mu_0 = \mu'_0 = \mu$.

(μ_1, μ'_1): If $s_1 \in \mu(\hat{c}_1)$, then define $\mu_1 = \mu'_1 = \mu_0$, otherwise define $\mu_1 = \mu'_1 = \mu_0 + (s_1, \hat{c}_1)$.

(μ_2, μ'_2): If $s_2 \in \mu(\hat{c}_2)$, then define $\mu_2 = \mu'_1$, otherwise define $\mu_2 = \mu'_1 + (s_2, \hat{c}_2)$. If $s_2 \notin \hat{\mu}(r_1(P_{s_2}))$, then $\mu'_2 = \mu_2 - (s_2, \hat{c}_2)$, otherwise $\mu'_2 = \mu_2$.

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(μ_m, μ'_m): If $s_m \in \mu(\hat{c}_m)$, then define $\mu_m = \mu'_{m-1}$, otherwise define $\mu_m = \mu'_{m-1} + (s_m, \hat{c}_m)$. If $s_m \notin \hat{\mu}(\hat{c}_m)$, then $\mu'_m = \mu_m - (s_m, \hat{c}_m)$, otherwise $\mu'_m = \mu_m$.

The second part of the path: Let $s_{l_1}, \ldots, s_{l_k}$ be the set of students who are unmatched at μ'_m and matched at $\hat{\mu}$. Let $\mu_{m+1} = \mu'_m + (s_{l_1}, \hat{\mu}(s_{l_1}))$, $\mu_{m+2} = \mu_{m+1} + (s_{l_2}, \hat{\mu}(s_{l_2})), \ldots, \mu_{m+k} = \mu_{m+(k-1)} + (s_{l_k}, \hat{\mu}(s_{l_k})) = \hat{\mu}$.

Consider the path $((\mu_0, \mu'_0), (\mu_1, \mu'_1), \ldots, (\mu_m, \mu'_m), \mu_{m+1}, \ldots, \mu_{m+k})$. As in the proof of Proposition 7.3.1, a matching may appear along this path more than once consecutively. Consider the path obtained by replacing a number of successive occurrences of a matching by exactly one occurrence, and thus making all the matchings appearing along the path distinct. For notational simplicity, let us denote the obtained path by $((\mu_0, \mu'_0), (\mu_1, \mu'_1), \ldots, (\mu_m, \mu'_m), \mu_{m+1}, \ldots, \mu_{m+k})$ itself. We argue that the path is pairwise myopic-farsighted improving path. We consider the two parts of the path separately.

The first part: The first part of the path has two types of blocking for each $k \in \{0, \ldots, m\}$: $(\mu'_{k-1} \xrightarrow{(s_k, \hat{c}_k)} \mu_k)$ and $(\mu'_{k-1} \xrightarrow{\hat{c}_k} \mu_k)$. We show that each of them constitutes a part of a pairwise myopic-farsighted improving path.

The block $(\mu'_{k-1} \xrightarrow{(s_k, \hat{c}_k)} \mu_k)$: If $\mu'_{k-1} = \mu_k$, there is nothing to show. Consider the case when $\mu_k = \mu'_{k-1} + (s_k, \hat{c}_k)$. Since s_k is myopic, she will be willing to participate in the block. To show that the college $\hat{c}_k$ will be willing to take part in the blocking, we distinguish the following two cases based on the quota of $\hat{c}_k$.

Case 1. Suppose $|S_{\hat{c}_k}(P_N)| \geq q_{\hat{c}_k}$. By the construction of the path, if there is a student s in $\mu_{k-1}(\hat{c}_k)$ who is better than s_k according to P_c, then it must be that $r_1(P_s) = \hat{c}_k$. Let $\hat{S} = \{s \in \mu_{k-1}(\hat{c}_k) \mid sP_c s_k\}$ be the set of

such students. Suppose $|\hat{S}| = q_{\hat{c}_k}$. By the definition of $O_{\hat{c}_k}(P_N)$, we have $\hat{S} \in O_{\hat{c}_k}(P_N)$, and hence $\hat{\mu}(\hat{c}_k)R_c\hat{S}$. Therefore, $\hat{c}_k$ will be willing to participate in the blocking. Now suppose $|\hat{S}| < q_{\hat{c}_k}$. Since $r_1(P_s) = \hat{c}_k$ for all $s \in \hat{S} \cup s_k$, there is an element $\tilde{S}$ in $O_{\hat{c}_k}(P_N)$ that contains $\hat{S} \cup s_k$. Because $s_k \notin \mu'_{k-1}(\hat{c}_k)$, by the definition of lexicographic preference, $\tilde{S}P_c\mu'_{k-1}(\hat{c}_k)$. Since $\tilde{S} \in O_{\hat{c}_k}(P_N)$, by the definition of $\hat{\mu}$, we have $\hat{\mu}(\hat{c}_k)R_c\tilde{S}$, and hence $\hat{\mu}(\hat{c}_k)P_c\mu'_{k-1}(\hat{c}_k)$. Therefore, $\hat{c}_k$ will be willing to participate in the blocking.
Case 2. Suppose $|S_{\hat{c}_k}(P_N)| < q_{\hat{c}_k}$. Let $\hat{S}$ be as defined in Case 1. As we have argued for Case 1, each $s \in \hat{S}$ is matched with her top-ranked college at μ'_{k-1}. By the definition of $\hat{\mu}$, there exists $\tilde{S} \in O_{\hat{c}_k}(P_N)$ such that $\hat{\mu}(\hat{c}_k)R_c\tilde{S}$. Since $r_1(P_s) = \hat{c}_k$ for all $s \in \hat{S} \cup s_k$, every element in $O_{\hat{c}_k}(P_N)$ contains $\hat{S} \cup s_k$, which in particular, means that $\tilde{S}$ contains ... as well. Because $s_k \notin \mu'_{k-1}(\hat{c}_k)$, by the definition of lexicographic preference, $\tilde{S}P_c\mu'_{k-1}(\hat{c}_k)$. This combined with the fact that $\hat{\mu}(\hat{c}_k)R_c\tilde{S}$ implies that $\hat{\mu}(\hat{c}_k)P_c\mu'_{k-1}(\hat{c}_k)$, and hence $\hat{c}_k$ will be willing to participate in the blocking.
The block $(\mu_k \xrightarrow{\hat{c}_k} \mu'_k)$: If $\mu_k = \mu'_k$, there is nothing to show. So, consider the situation where the college $\hat{c}_k$ removes a student, say s_k, at the matching s_k, that is, $\mu'_k = \mu_k - (s_k, \hat{c}_k)$. To show that $\hat{c}_k$ is willing to participate in the blocking we distinguish the following two cases.
Case 1. Suppose $|S_{\hat{c}_k}(P_N)| \geq q_{\hat{c}_k}$. By the construction of the path, if there is a student s in $\mu_k(\hat{c}_k)$ who is weakly better than s_k according to P_c, then it must be that $r_1(P_s) = \hat{c}_k$. Let $\hat{S} = \{s \in \mu_k(\hat{c}_k) \mid sR_cs_k\}$ be the set of such students. Suppose $|\hat{S}| = q_{\hat{c}_k}$. As $r_1(P_s) = \hat{c}_k$, this means $\hat{S} \in O_{\hat{c}_k}(P_N)$. By the definition $\hat{\mu}$, $\hat{\mu}(\hat{c}_k)R_c\hat{S}$. Therefore, $\hat{c}_k$ will be willing to participate in the blocking Now, suppose $|\hat{S}| < q_{\hat{c}_k}$. By the definition of $O_{\hat{c}_k}(P_N)$, there is $\tilde{S} \in O_{\hat{c}_k}(P_N)$ containing $\hat{S}$. By the definition of $\hat{\mu}$, we have $\hat{\mu}(\hat{c}_k)R_c\tilde{S}$. By the construction of the path we have $\hat{S} \setminus s_k \subseteq \hat{\mu}(\hat{c}_k)$. By combining the facts that $\hat{S} \subseteq \mu_k(\hat{c}_k)$, $\hat{S} \setminus s_k \subseteq \hat{\mu}(\hat{c}_k)$, and $s_k \notin \hat{\mu}(\hat{c}_k)$, the definition of lexicographic preference implies that there must be some student s with sP_cs_k who is not matched with $\hat{c}_k$ at μ_k but matched with $\hat{c}_k$ at $\hat{\mu}$. Again by using the definition of lexicographic preference, this implies that $\hat{\mu}(\hat{c}_k)P_c\mu_k(\hat{c}_k)$, and hence $\hat{c}_k$ will be willing to participate in the blocking.
Case 2. Suppose $|S_{\hat{c}_k}(P_N)| < q_{\hat{c}_k}$. By the construction of the path, if there is a student s in $\mu_k(\hat{c}_k)$ who is weakly better than s_k according to P_c, then it must be that $r_1(P_s) = \hat{c}_k$. Let $\hat{S} = \{s \in \mu_k(\hat{c}_k) \mid sR_cs_k\}$ be the set of such students. By the definition of $O_{\hat{c}_k}(P_N)$, every element in $O_{\hat{c}_k}(P_N)$ contains $\hat{S}$. By the definition of $\hat{\mu}$, we have $\hat{\mu}(\hat{c}_k)R_c\tilde{S}$ for some $\tilde{S} \in O_{\hat{c}_k}(P_N)$. By the construction of the path we have $\hat{S} \setminus s_k \subseteq \hat{\mu}(\hat{c}_k)$. By combining the facts that $\hat{S} \subseteq \mu_k(\hat{c}_k)$, $\hat{S} \setminus s_k \subseteq \hat{\mu}(\hat{c}_k)$, and $s_k \notin \hat{\mu}(\hat{c}_k)$, the definition of lexicographic preference implies that there must be some student s with sP_cs_k who is not matched with $\hat{c}_k$ at μ_k but matched with $\hat{c}_k$ at $\hat{\mu}$. Again by using the definition of lexicographic preference, this implies that $\hat{\mu}(\hat{c}_k)P_c\mu_k(\hat{c}_k)$, and hence $\hat{c}_k$ will be willing to participate in the blocking. ■

.6 Proof of Theorem 7.4.1

Proof: **Proof of (i)** Consider a profile P_N such that $|\{s \mid r_1(P_s) = c\}| = q_c$ for all $c \in C \setminus \{c_0\}$. We divide the proof in two parts: in the first part we show that $\mu^* \in \mathcal{M}$ such that $\mu^*(s) = r_1(P_s)$ for all $s \in S$ is pairwise myopic-farsighted stable at P_N, and the second part we show that there is no other pairwise myopic-farsighted stable matching at P_N.

μ^* is pairwise myopic-farsighted stable at P_N: Suppose not. Then, there is a pairwise myopic-farsighted improving path $(\mu^* = \mu_0, \ldots, \mu_L = \mu')$ from μ^* to some matching μ', that is, $\mu' \in h(\mu^*, P_N)$. Let $\hat{S} = \{s \mid \mu^*(s) \neq \mu'(s)\}$ be the set of students whose matchings are different in μ^* and μ'. Let $\hat{s}$ be the best student in the set $\hat{S}$ according to P_c. Suppose that $\hat{s}$ was matched with $\hat{c}$ at μ^*. Since $\mu^*(\hat{c}) \neq \mu'(\hat{c})$, there must be a stage $\hat{t}$ when the matching of $\hat{c}$ is changed for the first time, that is, $\mu^*(\hat{c}) = \mu_t(\hat{c})$ for all $t < \hat{t}$ and $\mu^*(\hat{c}) \neq \mu_{\hat{t}}(\hat{c})$. Each student $s \in \mu^*(\hat{c})$ is matched with her top-ranked college at stage $\hat{t} - 1$, so being myopic, $\hat{s}$ will not leave $\hat{c}$ by blocking with some other college (including c_0). This implies that $\hat{c}$ has participated in a blocking at μ^t. By the definition of $\hat{s}$, we have $\hat{s} P_c s$ for all $s \in \mu'(\hat{c}) \cap \hat{S}$. Because $\hat{s} \in \mu^*(\hat{c})$ and $\hat{s} \notin \mu'(\hat{c})$, by the definition of lexicographic preference, it must be that $\mu^*(\hat{c}) P_c \mu'(\hat{c})$. Since $\hat{c}$ is farsighted, this contradicts the fact that $\hat{c}$ has participated in any block at $\hat{t}$. Therefore, there is no pairwise myopic-farsighted improving path from μ^* to any μ', and hence μ^* is pairwise myopic-farsighted stable at P_N.

There is no pairwise myopic-farsighted stable matching other than μ^* at P_N: Assume for contradiction that $\tilde{\mu} \neq \mu^*$ is pairwise myopic-farsighted stable at P_N. Let $\tilde{S} = \{s \mid \mu'(s) \neq r_1(P_s)\}$ be the set of students who are not matched with their top-ranked college at $\tilde{\mu}$ and let $\tilde{s}$ be the best of them according to P_c, that is, $\tilde{s} P_c s$ for all $s \in \tilde{S}$. Suppose $\tilde{c} = r_1(P_{\tilde{s}})$. Since $|\{s \mid r_1(P_s) = \tilde{c}\}| = q_{\tilde{c}}$, there is $\bar{s} \in \tilde{\mu}(\tilde{c})$ such that $r_1(P_{\bar{s}}) \neq \tilde{c}$. This means $\bar{s} \in \tilde{S}$. Since $\tilde{s} P_c s$ for all $s \in \tilde{S}$, we have $\tilde{s} P_c \bar{s}$. Because $\tilde{s} P_c \bar{s}$, by the definition of lexicographic preference, we have $\left(\tilde{\mu}(\tilde{c}) \cup \{\tilde{s}\} \setminus \{\bar{s}\}\right) P_c \, \tilde{\mu}(\tilde{c})$. This implies that the college $\tilde{s}$ and the student $\tilde{s}$ block $\tilde{\mu}$ at P_N through the pairwise myopic-farsighted improving path $(\tilde{\mu}, \tilde{\mu} + (\tilde{s}, \tilde{c}))$. This is a contradiction to pairwise myopic-farsighted stability of $\tilde{\mu}$, which completes the proof.

This complete the proof of part (i) of the theorem.

Proof of (ii): Consider a profile P_N such that $|\{s \mid r_1(P_s) = c\}| \neq q_c$ for some $c \in C \setminus \{c_0\}$. We know from Proposition 7.3.2 that no matching outside $\widehat{\mathcal{M}}(P_N)$ can be pairwise myopic-farsighted stable at P_N. We show that no matching in $\widehat{\mathcal{M}}(P_N)$ is pairwise myopic-farsighted stable at P_N. Consider a matching $\hat{\mu}$ in $\widehat{\mathcal{M}}(P_N)$. Since $|\{s \mid r_1(P_s) = c\}| \neq q_c$ for some $c \in C \setminus \{c_0\}$ and $\sum_{c \in C \setminus \{c_0\}} q_c = m$, there must be a college $\tilde{c}$ such that $|\{s \mid r_1(P_s) = \tilde{c}\}| > q_{\tilde{c}}$. Let $\tilde{s} \in S_{\tilde{c}}(P_N)$ be such that $\tilde{s} \notin \hat{\mu}(\tilde{c})$, and let $\hat{s} \in S$ be the worst student in $\hat{\mu}(\tilde{c})$ according to P_c. Consider the block $(\tilde{s}, \tilde{c})$ where $\tilde{c}$ removes $\hat{s}$. Since $r_1(P_{\tilde{s}}) = \tilde{c}$, being myopic $\tilde{s}$ will be willing to participate in the block. So, if $\tilde{s} R_c \hat{s}$, then $(\hat{\mu}, \hat{\mu} + (\tilde{s}, \tilde{c}))$ is a pairwise myopic-farsighted improving path and hence $\hat{\mu}$ is not myopic-farsighted stable. Suppose $\hat{s}P_c\tilde{s}$. Consider the sequence of distinct matchings $(\mu_0 = \hat{\mu}, \mu_1 = \mu_0 + (\tilde{s}, \tilde{c}), \mu_2 = \mu_1 + (\hat{s}, \tilde{c}))$ where $\tilde{c}$ removes $\hat{s}$ at $\mu_0 + (\tilde{s}, \tilde{c})$ and $\tilde{s}$ at $\mu_1 + (\hat{s}, \tilde{c})$. We show that (μ_0, μ_1, μ_2) is a pairwise myopic-farsighted improving path. Since $r_1(P_{\tilde{s}}) = \tilde{c}$ and $\hat{s}$ is matched to c_0 at μ_1 which is her worst ranked college, being myopic both $\tilde{s}$ and $\hat{s}$ will be willing to participate in the respective blocks. Note that $\mu_0(\tilde{c}) = \mu_2(\tilde{c})$, therefore $\tilde{c}$ will be willing to participate in the blocking $(\tilde{s}, \tilde{c})$ at μ_0. Moreover, since $\hat{s}P_c\tilde{s}$, by the definition of lexicographic preferences $\left(\mu_1(\tilde{c}) \cup \{\tilde{s}\} \setminus \{\hat{s}\}\right) P_c \, \mu_1(\tilde{c})$. Hence, $\tilde{c}$ will be willing to participate in the blocking $(\hat{s}, \tilde{c})$ at μ_1. This proves that the path (μ_0, μ_1, μ_2) is pairwise myopic-farsighted improving path, and hence $\hat{\mu}$ is not pairwise myopic-farsighted stable at P_N. ■

.7 Proof of Theorem 7.4.2

Proof: Let P_N be a preference profile and let $V = \{\hat{\mu}\}$ where $\hat{\mu} \in \widehat{\mathcal{M}}(P_N)$. We show that V is a pairwise myopic-farsighted stable set at P_N. Since V is singleton, internal stability is vacuously satisfied. By Proposition 7.4.1, $\hat{\mu} \in h(\mu, P_N)$ for all $P_N \in \mathcal{M} \setminus \{\mu\}$. Hence, V also satisfies external stability as for all $\mu \notin V, h(\mu, P_N) \cap V = \hat{\mu}$. ■

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List of Publication(s)

- **Published Papers**

  (1) "On update monotone, continuous, and consistent collective evaluation rules", Social Choice and Welfare, 2020, 55(4), pp.759-776.

  (2) "A characterization of possibility domains under Pareto optimality and group strategy-proofness", Economic Letters, 2019, 183, p. 108567.

  (3) "Necessary and sufficient conditions for pairwise majority decisions on path-connected domains ", Theory and Decision, 2021, 91(3), pp.313-336.

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