Sphere fibrations over highly connected manifolds

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In the memory of my grandmother

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Contents

List of Notations

- $\mathbb{C}$ denotes the set of all complex numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{Z}$ denotes the set of all integers.
- $\mathbb{Z}^k$ denotes the direct sum of k-copies of $\mathbb{Z}$.
- $\mathbb{Q}$ denotes the set of all rational numbers.
- $\mathcal{D}^n$ denotes the n-th standered disk.
- ι_n denotes the homotopy class of the identity map $S^n \to S^n$.
- ΩX denotes the loop space of X.
- ΣX denotes the reduced suspension of a space X.
- $\Sigma^\infty X$ denotes the infinite suspension spectrum of a pointed space X.
- $X^{\vee k}$ denotes the wedge of k-copies of X.
- $\mathbb{C}P^n$ denotes the n-dimensional complex projective space .
- $\mathbb{H}P^n$ denotes the n-dimensional quaternionic projective space .
- $\mathbb{O}P^2$ denotes the 2-dimensional octonionic projective space.
- $\mathcal{T}op$ denotes the category of topological spaces.
- $\mathcal{T}op_{\frac{1}{2}}$ denotes the localised category of topological spaces, where 2 is inverted.
- $\mathcal{T}op_{\frac{1}{2},\frac{1}{3}}$ denotes the localised category of topological spaces, where 2 and 3 are inverted.
- π_m^s denotes the m-th stable stem .

Chapter 1

Introduction

This thesis explores the construction of certain sphere fibrations over highly connected manifolds. One may study consequences for the loop space decomposition of the highly connected manifolds.

1.1 Highly connected Poincaré duality complexes

From the point of view of classification problems in differential topology, apart from surfaces and 3-manifolds, the results for spheres appeared in the celebrated works of Milnor [31] and Kervaire and Milnor [30]. In dimension 4, classification problems have received a lot of attention from both topologists and geometers. The determination of simply connected 4-manifolds up to homotopy goes back to the early works of Whitehead [47] and Milnor [32]. In this case, the simply connected hypothesis on the 4-manifold M determines the homology groups up to an integer k, given by $H_2(M) \cong \mathbb{Z}^k$, and the homotopy type up to the classification of inner product spaces of rank k, given by the intersection form. Conversely, given a non-singular inner product space, the associated cell complex (which can be built by attaching the top cell using the given form to the wedge of 2-spheres) does satisfy Poincaré duality. However, not all of them are homotopy equivalent to smooth manifolds due to the restrictions proved by Rohlin [37] and Donaldson [18]. On the other hand, the topological classification problem for simply connected 4-manifolds solved by Freedman [21], does not carry the same restrictions.

A natural generalization of the simply connected 4-manifolds are the $(n-1)$-connected $2n$-manifolds. For these manifolds, the homology is again determined up to an integer k by $H_n(M) \cong \mathbb{Z}^k$. Their classification have been studied by Wall [44] via the approach of expressing these as a union of handlebodies. The intersection form is no longer sufficient to determine the homotopy type of M. Such an M with $H_n(M) \cong \mathbb{Z}^k$ possesses a minimal CW complex structure

$$M \simeq (S^n)^{\vee k} \cup_{L(M)} \mathcal{D}^{2n}, \text{ with } L(M) \in \pi_{2n-1}\big((S^n)^{\vee k}\big).$$

The homotopy group $\pi_{2n-1}\big((S^n)^{\vee k}\big)$ is computed via the Hilton-Milnor theorem [25] as

$$\pi_{2n-1}\big((S^n)^{\vee k}\big) \cong (\pi_{2n-1}(S^n))^{\oplus k} \oplus (\pi_{2n-1}S^{2n-1})^{\oplus\binom{k}{2}}.$$

The groups $\pi_{2n-1}(S^{2n-1}) \cong \mathbb{Z}$ occuring in the above description are mapped to $(S^n)^{\vee k}$ via Whitehead products of the different summands. Moreover, if n is even, the group $\pi_{2n-1}S^n$ contains a $\mathbb{Z}$-summand whose generator may be chosen either as the Whitehead product or the Hopf invariant one classes (which occur only when $n = 2, 4$, or 8 [1]). The projection of $L(M)$ onto these torsion-free summands are determined directly by the intersection form.

It is also an interesting question whether given $L(M) \in \pi_{2n-1}\big((S^n)^{\vee k}\big)$, there is a $(n-1)$-connected $2n$-manifold homotopy equivalent to the cell complex M. In this paper, we work around these issues by considering all such cell complexes M. These satisfy Poincaré duality in the sense that there is a degree $2n$ homology class $[M]$ which gives the Poincaré duality isomorphism via the cap product, and are called Poincaré duality complexes [45]. We write $\mathcal{PD}_k^m$ for the collection of Poincaré duality complexes that are k-connected and m-dimensional. In this notation, the above examples lie in $\mathcal{PD}_{n-1}^{2n}$.

The expression for $L(M) \in \pi_{2n-1}\big((S^n)^{\vee k}\big)$ shows that a general homotopical classification will rely on the knowledge of $\pi_{2n-1}S^n$, and thus, is not possible with our current knowledge of the homotopy groups of spheres. As a weaker classification, we consider the homotopy type of the loop space ΩM. If $k = \text{Rank}(H_n(M)) \geq 2$, one realizes that the homotopy type of ΩM depends only on k [13, 9]. One proves that the loop space is expressable as a weak product of the loop space of spheres which map to $\pi_* M$ via Whitehead products. If $k = 1$, this is not true as is observed in [9, §4.3].

The splitting results for the loop space of manifolds fall under the general framework of loop space decompositions. Such a decomposition for highly connected manifolds was first proved in [14] for the $(n-1)$-connected $(2n+1)$-manifolds. There have been a growing interest in results of this type [13, 9, 10, 6, 42, 26]. A general idea for producing loop space decompositions is given in [42]. Given a cofibration sequence $\Sigma A \to E \xrightarrow{h} J$ for which Ωh has a right homotopy inverse, there are equivalences

$$\Omega E \simeq \Omega J \times \Omega \text{Fib}(h), \; \text{Fib}(h) \simeq \Sigma A \rtimes \Omega J,$$

where Fib(h) is the homotopy fibre of h. While this technique may be applied in many examples, it is not very useful when the rank of the homology of E is small.

A different view of the loop space decompositions is given by fibre bundles. For example, in the case of $\mathbb{C}P^2$, the usual quotient is part of the principal bundle $S^1 \to S^5 \to \mathbb{C}P^2$ which yields the loop space decomposition $\Omega\mathbb{C}P^2 \simeq S^1 \times \Omega S^5$. Simply connected 4-manifolds also support principal S^1-bundles of the form $S^1 \to \#^{k-1}(S^2 \times S^3) \to M$ where Rank$(H_2(M)) = k \geq 2$ [19, 8]. The construction of such bundles have many geometric consequences. In the context of loop space decompositions, this implies $\Omega M \simeq S^1 \times \Omega(\#^{k-1}(S^2 \times S^3))$. The construction involves a choice of a primitive class in $H^2(M) \cong [M, \mathbb{C}P^\infty]$ using the fact that $\mathbb{C}P^\infty$ is the classifying space for S^1-bundles, and the classification of spin 5-manifolds by Smale [39]. In this paper, we search for generalizations of this construction for highly connected manifolds.

1.2 Existence result for sphere fibrations

Let $M_k \in \mathcal{PD}^{2n}_{n-1}$ be a Poincaré duality complex of dimension $2n$ which is $(n-1)$-connected and Rank$(H_n(M_k)) = k$. Let $E_k = \#^{k-1}(S^n \times S^{2n-1})$. We first observe that the existence of a fibration

$$S^{n-1} \to E_k \to M_k$$

puts some restrictions on n. As E_k is $(n-1)$-connected, we must have that the map $S^{n-1} \to E_k$ is null-homotopic. Now continuing the homotopy fibration sequence further, we find that $\Omega E_k \to \Omega M_k \to S^{n-1}$ is a principal fibration with a section and so, there is a splitting $\Omega M_k \simeq \Omega E_k \times S^{n-1}$. Therefore , S^{n-1} is a retract of an H-space, and hence is itself an H-space, which doesn't usually happen . We assume that n is even, and either $n = 2$, 4, or 8, or that we are working in the category $\mathcal{T}op_{1/2}$, which is the localized category of spaces after inverting 2. This hypothesis implies that S^{n-1} is an H-space.

We first notice that the classification results of Smale [39] and Barden [5] for 5-manifolds are not available in general. Additionally, the spheres are not loop spaces (other than $n = 2$, or 4), so it is not possible to obtain principal fibrations in general. We approach this problem from a homotopy theoretic point of view. The homology of the loop space is an associative algebra via the Pontrjagin product, and for both M_k and E_k, $H_*\Omega M_k$ and $H_*\Omega E_k$ may be computed as tensor algebras modulo a single relation [15]. It is then possible to produce a map between the associative algebras $H_*(\Omega E_k) \to H_*(\Omega M_k)$. Now the results of [9] imply that both ΩE_k and ΩM_k are a weak product of loop spaces on spheres, which enable us to construct a map $\Omega E_k \to \Omega M_k$ that realizes the above map on homology.

The next step is to try to construct a delooping of the map $\Omega E_k \to \Omega M_k$. This is done via obstruction theory using the cell structure of E_k as

$$E_k \simeq \big((S^n)^{\vee k-1} \vee (S^{2n-1})^{\vee k-1}\big) \cup_\phi \mathcal{D}^{3n-1},$$

where ϕ is a sum of Whitehead products. The map $\Omega E_k \to \Omega M_k$ already specifies choices for the map

$$(S^n)^{\vee k-1} \vee (S^{2n-1})^{\vee k-1} \to M_k.$$

Once the map $f : E_k \to M_k$ is appropriately constructed, a spectral sequence computation is used to show that $\mathsf{Fib}(f) \simeq S^{n-1}$. The first result that we prove is the following theorem (see Theorem 3.1.5)

Theorem 1.2.1. *For $k \geq 2$, let $M_k \in \mathcal{PD}^{2n}_{n-1}$ with n even and $H_n(M;\mathbb{Z}) \cong \mathbb{Z}^k$. After inverting the primes 2 and those which occur as torsion in $\pi_{2n-1}(S^n)$, there is a fibration $S^{n-1} \to E_k \to M_k$ such that E_k is homotopy equivalent to $\#^{k-1}(S^n \times S^{2n-1})$.*

Following this, we try for improved results which reduce the set of primes that are required to be inverted. The best case is when $n = 2$, 4, or 8, which contains a Hopf invariant one class. In the case $n = 2$, one already has a S^1-bundle over $M_k \in \mathcal{PD}^4_1$ as stated above. We also point this out from our homotopy theoretic techniques without using the classification results of Smale. For $n = 4$, we have $\pi_7(S^4) \cong \mathbb{Z}\oplus\mathbb{Z}/(12)$, so the primes that are required to be inverted in Theorem A are 2, and 3. We observe through direct computation (Example 4.2.3) that there is no principal S^3-bundle over $\mathbb{H}P^2\#\overline{\mathbb{H}P^2}$ with total space $S^4 \times S^7$. Here, the group structure on S^3 is by quaternionic multiplication identifying S^3 as the unit quaternions. However, we are able to prove integral versions of the sphere fibrations as stated in the following theorem. (see Theorems 4.1.2, 4.2.11)

Theorem 1.2.2. *a) Let M_k be a simply connected 4-manifold with $H_2 M_k \cong \mathbb{Z}^k$. Then, there is a principal S^1-fibration $S^1 \to E_k \to M_k$ where $E_k \simeq \#^{k-1}(S^2 \times S^3)$.*
b) Let $M_k \in \mathcal{PD}^8_3$, that is, $H_4(M_k) = \mathbb{Z}^k$ for $k \geq 2$. Such an M_k supports a S^3-fibration $S^3 \to E_k \to M_k$ with $E_k \simeq \#^{k-1}(S^4 \times S^7)$.

For $n = 8$, we have $\pi_{15}(S^8) \cong \mathbb{Z}\oplus\mathbb{Z}/(120)$ and so the primes that are required to be inverted in Theorem A are 2, 3, and 5. Through direct computations, we observe that even for $\mathbb{O}P^2\#\overline{\mathbb{O}P^2}$, it is not possible to construct the fibration $S^7 \to S^8 \times S^{15} \to \mathbb{O}P^2\#\overline{\mathbb{O}P^2}$. However, it appears that one may put down a list of criteria on $M \in \mathcal{PD}^{16}_7$ for which such fibrations do exist. We leave this question open for future research.

For general n, one may increase the value of k to obtain better results for spherical fibrations. The precise bound is given by the number of cyclic summands r in the stable stem π^s_{n-1}. The fibrations are then obtained in the category $\mathcal{T}op_{1/2}$ if $k > r$. If we further pass to $\mathcal{T}op_{1/2,1/3}$ (that is, also invert the prime 3), we have a homotopy associative multiplication on S^{n-1} which allows us to construct spaces $E_2(S^{n-1}) \simeq S^{n-1} * S^{n-1}$ and $P_2(S^{n-1})$, and a fibration

$$S^{n-1} \to E_2(S^{n-1}) \to P_2(S^{n-1}).$$

In the category $\mathcal{T}op_{1/2,1/3}$, we prove that the sphere fibrations are obtained as a pullback of the above fibration via a map $M \to P_2(S^{n-1})$. These results are summarized in the following theorem. (see Theorems 3.2.7, 3.2.15)

Theorem 1.2.3. *a) Let $M_k \in \mathcal{PD}^{2n}_{n-1}$ with $H_nM \cong \mathbb{Z}^k$, and $n > 8$ be an even integer. Let r be the number of cyclic odd torsion summands of π^s_{n-1}. If $k > r$, after inverting 2 there is a fibration $E_k \to M_k$ with fibre S^{n-1} where $E_k \simeq \#^{(k-1)}(S^n \times S^{2n-1})$.*
b) After inverting 2 and 3, the fibration $E_k \to M_k$ is a homotopy pullback of $M_k \to P_2(S^{n-1}) \leftarrow E_2(S^{n-1})$ for a suitable map $M_k \to P_2(S^{n-1})$.

1.2.1 Applications for Loop space decomposition

As applications of the spherical fibrations constructed here we consider connected sums $N\#M_k$ with M_k as above, and N a general simply connected $2n$-manifold (more precisely, $N \in \mathcal{PD}^{2n}_1$). Using the techniques in [42], we may observe that a loop space decomposition may be obtained using the fact that the attaching map of the top cell of M_k is inert. The condition *inert* means that the map $(S^n)^{\vee k} \to M_k$ has a right homotopy inverse after taking the loop space (that is $\Omega\big((S^n)^{\vee k}\big) \to \Omega M_k$ has a right homotopy inverse).

We present a fresh view for these loop space decompositions (Theorems 3.2.29, 3.2.30, 3.2.31). We input our fibrations into the arguments in [26] to realize fibrations over $N\#M$ with total space $G_\tau(N)\#E_k$ (Proposition 3.2.24). The manifold $G_\tau(N)$ is a union of $N_0 \times S^{n-1}$ and $S^{2n-1} \times \mathcal{D}^n$ (3.2.23) using an equivalence $S^{2n-1} \times S^{n-1} \to S^{2n-1} \times S^{n-1}$ associated to a map τ from S^{2n-1} to the space of homotopy equivalences of S^{n-1}. Earlier observations about connected sums of sphere products [9, Theorem B] imply that the attaching map of the top cell is inert. Thus, the homotopy type of $\Omega(G_\tau(N)\#E_k)$ depends only on $G_\tau(N) - *$. We identify this to be $N_0 \rtimes S^{n-1}$ and is thus independent of τ (Proposition 3.2.25). Finally, we point out that the loop space decompositions also yield results for the loop space of the configuration spaces of $N\#M_k$ (Theorem 3.2.35).

1.3 $SU(2)$-bundles over 8-manifolds

The last chapter explores $SU(2)$-bundles over 8-manifolds, aiming for results akin to those about circle bundles over 4-manifolds [19, 8]. In the case of simply connected 4-manifolds, the results are established by leveraging the classification of simply connected 5-manifolds achieved by Smale [39] and Barden [5].

A circle bundle $S^1 \to X \to M$ over a simply connected 4-manifold M is classified by $\alpha \in H^2(M)$, the total space $X(\alpha)$ is simply connected if α is primitive, and there are only two possibilities of $X(\alpha)$ via the classification of simply connected 5-manifolds. Explicitly, we have [19, Theorem 2]

1. For every simply connected 4-manifold M, there is a circle bundle α, such that $X(\alpha)$ is homotopy equivalent to a connected sum of $S^2 \times S^3$. If M is spin, among primitive α, this is the only possibility.

2. For a simply connected 4-manifold M which is not spin and a circle bundle α over it, $X(\alpha)$ is either homotopy equivalent to a connected sum of $S^2 \times S^3$, or to a connected sum of $S^2 \times S^3$ and another manifold B. The manifold B is (unique up to diffeomorphism) a non-spin simply connected 5-manifold whose homology is torsion-free, and Rank$(H_2(B)) = 1$.

The results of Smale and Barden are geometric in nature, and do not generalize easily to higher dimensions. Using homotopy theoretic methods, it was possible to construct sphere fibrations [11] over highly connected Poincaré-duality complexes possibly by inverting a few primes or in high enough rank. Among these sphere fibrations, the only case where they could be principal bundles was in dimension 8, and the question whether they may be realized as such was left unresolved.

In this chapter, we consider principal $SU(2)$-bundles, noting that $SU(2) = S^3$ is the only case apart from the circle where the sphere is a Lie group. The base space of the $SU(2)$-bundle which is appropriate for making a similar analysis is a highly connected 8-manifold. More precisely, we consider Poincaré duality complexes M (8-dimensional) that are 3-connected. These are obtained by attaching a single 8-cell to a buoquet of 4-spheres. We denote

$$\mathcal{PD}_3^8 = \text{ the collection of } 3\text{-connected } 8\text{-dimensional Poincaré duality complexes.}$$

The notation $M_k \in \mathcal{PD}_3^8$ assumes that Rank$(H_4(M_k)) = k$. The attaching map of the 8-cell is denoted by $L(M_k)$, and is of the form (once we have chosen a basis $\{\alpha_1, \dots, \alpha_k\}$ of

$\pi_4(M_k) \cong \mathbb{Z}^k$)

$$L(M_k) = \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \alpha_j] + \sum_{i=1}^{k} g_{i,i}\nu_i + \sum_{i=1}^{k} l_i\nu_i'. \tag{1.3.1}$$

The matrix $\big((g_{i,j})\big)$ is the matrix of the intersection form, and hence, is invertible. The notation ν_i stands for $\alpha_i \circ \nu$ and ν_i' for $\alpha_i \circ \nu'$. Here ν is the Hopf map, and $\nu' \in \pi_7(S^4)$ is the generator for the $\mathbb{Z}/(12)$ factor satisfying $[\iota_4, \iota_4] = 2\nu + \nu'$. For such complexes, we consider

$$\mathcal{P}(M_k) = \text{ the set of principal } SU(2)\text{-bundles } E(\psi) \xrightarrow{\psi} M_k \text{ such that } E(\psi) \text{ is 3-connected.}$$

The bundle ψ is classified by a primitive element $\psi \in H^4(M_k)$, which satisfies a criterion (see Proposition 5.3.9). In this context, we first encounter the question whether $\mathcal{P}(M_k)$ is non-empty. We prove (see Proposition 5.3.3 and Proposition 5.4.2)

Theorem 1.3.2. *For $k \geq 3$, the set $\mathcal{P}(M_k)$ is non-empty.*

For $k = 2$, there are examples where $\mathcal{P}(M_k)$ is empty. This means that for every principal $SU(2)$-bundle over such complexes, the total space has non-trivial π_3. The idea here is that the existence of ψ is given by a certain equation in k variables, and solutions exist once k is large enough.

In the case of simply connected 4-manifolds, the first kind of classification of circle bundles is the result of Giblin[22] which states

$$\text{If } M_2 = S^2 \times S^2, \text{ then } E(\psi) \simeq S^2 \times S^3 \text{ for any primitive } \psi.$$

We also have an analogous result in the 8-dimensional case

$$\text{If } \psi \in \mathcal{P}(S^4 \times S^4), \text{ then } E(\psi) \simeq S^4 \times S^7.$$

In fact, this fits into a more general framework. We call a manifold $M_k \in \mathcal{PD}_3^8$ *stably trivial* if $L(M_k)$ is stably null-homotopic (that is the stable homotopy class of $L(M_k) : S^7 \to (S^4)^{\vee k}$ is 0). In terms of (1.3.1), this means for every i, $g_{i,i} - 2l_i \equiv 0 \pmod{24}$. We have the following theorem (see Proposition 5.3.7)

Theorem 1.3.3. *Suppose M_k is stably trivial. Then, for every $\psi \in \mathcal{P}(M_k)$, $E(\psi) \simeq \#^{k-1} S^4 \times S^7$, a connected sum of $k-1$ copies of $S^4 \times S^7$.*

This directly generalizes the result for circle bundles over simply connected 4-manifolds that are spin (identifying the spin manifolds as those whose attaching map is stably null).

1.3.1 Classification of the homotopy type of 3-connected 11-manifolds

We proceed towards a more general classification of the homotopy type of the space $E(\psi)$ for $\psi \in \mathcal{P}(M_k)$. Let $\mathcal{PD}^{11}_{4,7}$ be the class of 3-connected 11-dimensional Poincaré duality complexes E such that $E \setminus \{pt\} \simeq$ a wedge of S^4 and S^7. We first observe that $E(\psi) \in \mathcal{PD}^{11}_{4,7}$ (see Proposition 5.2.2), and we try to address the question of the classification of complexes in $\mathcal{PD}^{11}_{4,7}$ up to homotopy equivalence. The homology of such complexes E is given by

$$H_m(E) \cong \begin{cases} \mathbb{Z} & m = 0, 11 \\ \mathbb{Z}^r & m = 4, 7 \\ 0 & \text{otherwise.} \end{cases}$$

We denote the number r by $\mathrm{Rank}(E)$. The classification works differently for $r = 1$, and for $r \geq 2$. Table 5.1 lists the various possibilities for $r = 1$. For $r \geq 2$, E is a connected sum of copies of $S^4 \times S^7$, and the complexes $E_{\lambda,\epsilon,\delta}$ defined below. Note that

$$\begin{aligned}\pi_{10}(S^4 \vee S^7) &\cong \pi_{10}(S^4) \oplus \pi_{10}(S^7) \oplus \pi_{10}(S^{10}) \\ &\cong \mathbb{Z}/(24)\{x\} \oplus \mathbb{Z}/(3)\{y\} \oplus \mathbb{Z}/(24)\{\nu_7\} \oplus \mathbb{Z}\{[\iota_4, \iota_7]\}.\end{aligned}$$

Here, $x = \nu \circ \nu_7$ and $y = \nu' \circ \nu_7$. Let

$$\phi_{\lambda,\epsilon,\delta} = [\iota_4, \iota_7] + \lambda(\iota_7 \circ \nu_7) + \epsilon(\iota_4 \circ x) + \delta(\iota_4 \circ y),$$

where $\iota_4 \colon S^4 \to S^4 \vee S^7$ and $\iota_7 \colon S^7 \to S^4 \vee S^7$ are the canonical inclusions, and define,

$$E_{\lambda,\epsilon,\delta} = (S^4 \vee S^7) \cup_{\phi_{\lambda,\epsilon,\delta}} D^{11}.$$

The attaching map of the top cell of E takes the form

$$L(E) : S^{10} \to \left(S^4 \vee S^7\right)^{\vee r}.$$

The stable homotopy class of $L(E)$ lies in

$$\pi^s_{10}\Big(\left(S^4 \vee S^7\right)^{\vee r}\Big) \cong \left(\mathbb{Z}/(24)\{\nu\} \oplus \mathbb{Z}/(2)\{\nu^2\}\right)^{\oplus r}.$$

This takes the form $\lambda_s \beta \circ \nu + \epsilon_s \alpha \circ \nu^2$ for some $\beta \in \pi_7\Big(\left(S^4 \vee S^7\right)^{\vee r}\Big)$ and $\alpha \in \pi_4\Big(\left(S^4 \vee S^7\right)^{\vee r}\Big)$. Up to a change of basis we may assume that $\lambda_s \mid 24$, and if λ_s is even, $\epsilon_s \in \mathbb{Z}/(2)$. These numbers are invariant over the homotopy equivalence class of E, and are denoted by $\lambda_s(E)$, and $\epsilon_s(E)$ (defined only if $\lambda_s(E)$ is even). We use these invariants to classify the homotopy types of elements in $\mathcal{PD}^{11}_{4,7}$ (see Theorem 5.1.17)

Theorem 1.3.4. *Let $E \in \mathcal{PD}^{11}_{4,7}$. Then the homotopy type of E is determined by the following.*

1. *If $\lambda_s(E)$ is even and $\epsilon_s(E) = 0$, then*

$$E \simeq \#^{r-1} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} \quad \textit{where } \epsilon \equiv 0 \pmod 2.$$

2. *If $\lambda_s(E)$ is even and $\epsilon_s(E) = 1$, then*

$$\begin{aligned} E &\simeq \#^{r-1} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} && \textit{where } \epsilon \equiv 1 \pmod 2 \\ \textit{or } E &\simeq \#^{r-2} E_{0,0,0} \# E_{0,1,0} \# E_{\lambda_s,\epsilon,\delta} && \textit{where } \epsilon \equiv 0 \pmod 2. \end{aligned}$$

3. *If $\lambda_s(E)$ is odd, then*

$$E \simeq \#^{r-1} E_{0,0,0} \# E_{\lambda_s,\epsilon,\delta} \quad \textit{or} \quad E \simeq \#^{r-2} E_{0,0,0} \# E_{0,1,0} \# E_{\lambda_s,\epsilon,\delta}.$$

Further given λ_s, the choices of ϵ and δ are those which are mentioned in Table 5.1.

We see that in the list given in Table 5.1, for certain cases the homotopy type of E is determined by $\lambda_s(E)$ and $\epsilon_s(E)$. This happens if $\lambda_s(E) = 0$, or 12. We also observe that the homotopy type of ΩE depends only on the rank r. Now, we look at $M_k \in \mathcal{PD}^8_3$, and try to determine the set of homotopy equivalence classes of $E(\psi)$ for $\psi \in \mathcal{P}(M_k)$. In this process, we determine a formula for $\lambda(\psi) := \lambda_s(E(\psi))$ (Proposition 5.2.13), and using this we determine the set of possible values of $\lambda_s(\psi)$ for $\psi \in \mathcal{P}(M_k)$. The stable homotopy class of $L(M_k)$ lies in

$$\pi_7^s\Big((S^4)^{\vee k}\Big) \cong \big(\mathbb{Z}/(24)\{\nu\}\big)^{\oplus k}.$$

This takes the form $\sigma_s \alpha \circ \nu$ for some $\alpha \in \pi_4\big((S^4)^{\vee k}\big)$, and up to a change of basis for $k \geq 2$, $\sigma(M_k) := \gcd(\sigma_s, 24)$ is an invariant of the stable homotopy type of M_k. Other than k and $\sigma(M_k)$, the explicit stable homotopy class of α above yields a linear map $\tau : H^4(M_k) \to \mathbb{Z}/(24)$ given by $\tau(\psi) = \psi(\sigma_s \alpha)$.

1.4 Constructing principal bundles of prescribed stable homotopy types

We use the invariants k, $\sigma(M_k)$, τ, and the intersection form to completely determine the possibilities of $\lambda(\psi)$ for $\psi \in \mathcal{P}(M_k)$. (see Theorem 5.2.14, Proposition 5.3.10, Theorem 5.3.11, Theorem 5.3.14, and Theorem 5.4.5)

Theorem 1.4.1. *For any $\psi \in \mathcal{P}(M_k)$, $\lambda(\psi)$ is a multiple of $\sigma(M_k)$ ($\pmod{24}$). Conversely, the multiples of $\sigma(M_k)$ that may be achieved are described as follows*

1. *If the intersection form of M_k is odd and $k \geq 7$, then $\{\lambda(\psi) \mid \psi \in \mathcal{P}(M_k)\}$ equals the set of multiples of $\sigma(M_k) \pmod{24}$.*
2. *If the intersection form of M_k is even, each $\psi \in \mathcal{P}(M_k)$ satisfies $\epsilon_s(\psi) \equiv 0 \pmod{2}$.*
3. *If $k \geq 7$, there are $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) = \sigma(M_k)$, and also there are $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) = 3\sigma(M_k)$.*
4. *If $\sigma(M_k) \equiv 2$, or $4 \pmod{8}$ for $k \geq 5$, there is a $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) \equiv 0 \pmod{8}$ if and only if the complex satisfies hypothesis (H_8).*
5. *If $\sigma(M_k) \equiv 2 \pmod{8}$ for $k \geq 5$, there is a $\psi \in \mathcal{P}(M_k)$ such that $\lambda(\psi) \equiv 4 \pmod{8}$ if and only if the complex satisfies hypothesis (H_4).*

For lower values of k, we do not get systematic results like the above. That is, the set $\{\lambda(\psi) \mid \psi \in \mathcal{P}(M_k)\}$ is not completely determined by $\sigma(M_k)$, k, τ, and the intersection form. Theorem 1.4.1 implies that there are certain M_k whose intersection form is even and there is no $\psi \in \mathcal{P}(M_k)$ such that $E(\psi) \simeq \#^{k-1}S^4 \times S^7$, however if the intersection form is odd, then for $k \geq 7$, there is a principal bundle $SU(2) \to \#^{k-1}(S^4 \times S^7) \to M_k$.

Chapter 2

Constructing maps between loop space homology algebras

In this chapter, we construct a map from the homology of the loop space of a connected sum of copies of $S^n \times S^{2n-1}$ to that of the loop space of a highly connected Poincaré duality complex . We use the fact that the latter is a quadratic algebra with a single relation which in turn comes from a non-singular intersection form. The results of this chapter appear in the paper [11].

2.1 Some algebraic results

Let V be a free module over a principal ideal domain R (in our applications $\mathbb{Z}[\frac{1}{p_1}, \cdots, \frac{1}{p_r}]$ for a finite set of primes $\{p_1, \cdots, p_r\}$) of finite rank k, and suppose $\alpha : V \to R$ is a non-zero linear function. Let $\mathcal{L}$ be a symmetric 2-tensor (that is an element of $Sym^2(V) = (V \otimes V)^{\Sigma_2}$) which is invertible (that is, with respect to any basis the corresponding $k \times k$ matrix with coefficients from R is invertible). We think of V as a graded vector space, also note that the associative algebra $T(V)$ has a graded Lie bracket given by

$$[v, w] = v \otimes w - (-1)^{|v||w|} w \otimes v \text{ for all } v, w \in T(V). \tag{2.1.1}$$

Proposition 2.1.2. *With V concentrated in a single grading m for m odd, α and $\mathcal{L}$ as above, for any basis $v_1, \cdots, v_{k-1}$ of Ker(α), there are $w_1, \cdots, w_{k-1} \in V \otimes V$ such that*

1. $\displaystyle\sum_{j=1}^{k-1} [v_j, w_j] = 0 \pmod{\mathcal{L}}$.
2. $\{w_1, \cdots, w_{k-1}\}$ *projects to a basis of* $V \otimes V / \big(R\{\mathcal{L}\} + V \otimes \text{Ker}(\alpha)\big)$.

Proof. Given a basis $v_1, \cdots, v_{k-1}$ of Ker(α), pick v_k such that $v_1, \cdots, v_k$ is a basis of V. This is possible as the image of α is a principal ideal (b) of R , and we may pick v_k such that $\alpha(v_k) = b$. As the collection $\{v_i \otimes v_j\}$ is a basis of $V \otimes V$, we have an expression

$$\mathcal{L} = \sum_{i=1}^{k} \sum_{j=1}^{k} g_{i,j} v_i \otimes v_j, \tag{2.1.3}$$

for a symmetric invertible matrix $\big((g_{i,j})\big)$ over R. Define w_i by

$$w_i = \sum_{j=1}^{k-1} g_{i,j}[v_j, v_k] + g_{i,k} v_k \otimes v_k. \tag{2.1.4}$$

Note that a basis of the free R-module $V \otimes V/\big(V \otimes \text{Ker}(\alpha)\big)$ is given by the images of the elements $v_j \otimes v_k$ for $1 \leq j \leq k$. It is clear that the coefficients of the w_i in terms of this basis are those of the first $(k-1)$ columns of the matrix $\big((g_{i,j})\big)$, with the last column corresponding to $\mathcal{L}$. This proves 2. For the statement 1, we compute using the graded Jacobi identity

$$[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]],$$

and the identity [36, §8.1]

$$[y, x \otimes x] = [[y, x], x],$$

for odd degree classes x. We have

$$\begin{aligned}
\sum_{i=1}^{k-1}[v_i, w_i] &= \sum_{i=1}^{k-1}\Big(\sum_{j=1}^{k-1} g_{i,j}[v_i, [v_j, v_k]] + g_{i,k}[v_i, v_k \otimes v_k]\Big) \\
&= \sum_{1 \leq i < j \leq k-1} g_{i,j}[[v_i, v_j], v_k] + \sum_{i=1}^{k-1} g_{i,i}[v_i \otimes v_i, v_k] + \sum_{i=1}^{k-1} g_{i,k}[[v_i, v_k], v_k] \\
&= [\mathcal{L}, v_k] - g_{k,k}[v_k \otimes v_k, v_k] \\
&= [\mathcal{L}, v_k].
\end{aligned}$$

The last step is true as $[v_k \otimes v_k, v_k] = 0$. □

We carry forward the analogy in Proposition 2.1.2 further using graded Lie algebras. First recall the definition of a graded Lie algebra [36]. A graded Lie algebra over a ring $\mathcal{R}$ in which 2 is not invertible carries an extra squaring operation on odd degree classes to encode the relation $x^2 = \frac{1}{2}[x, x]$ whenever $|x|$ is odd.

Definition 2.1.5. A graded Lie algebra $L = \oplus L_i$ is a graded $\mathcal{R}$-module together with a Lie bracket

$$[\ ,\ ] : L_i \otimes_{\mathcal{R}} L_j \to L_{i+j}$$

and a quadratic operation called *squaring* defined on odd degree classes

$$(\ )^2 : L_{2k+1} \to L_{4k+2}.$$

These operations are required to satisfy the identities

$$\begin{aligned}
[x, y] &= -(-1)^{\deg(x)\deg(y)}[y, x] \ \ (x \in L_i, y \in L_j),\\
[x, [y, z]] &= [[x, y], z] + (-1)^{\deg(x)\deg(y)}[y, [x, z]] \ \ (x \in L_i, y \in L_j, z \in L_k),\\
(ax)^2 &= a^2x^2 \ \ (a \in \mathcal{R}, x \in L_{2k+1}),\\
(x + y)^2 &= x^2 + y^2 + [x, y],\\
[x, x] &= 0 \ \ (x \in L_{2i}),\\
2x^2 &= [x, x],\ \ [x, x^2] = 0 \ \ (x \in L_{2k+1}),\\
[y, x^2] &= [[y, x], x] \ \ (x \in L_{2k+1}, y \in L_i).
\end{aligned}$$

Example 2.1.6. *Note that $T(V)$ is a graded Lie algebra with the Lie bracket described in (2.1.1). For $|u|$ odd, $(u)^2$ is defined to be $u \otimes u$. The identities above are easily verified. A symmetric 2-tensor $\mathcal{L}$ which is expressed in the form (2.1.3) may be written as*

$$\sum_{i=1}^{k}\sum_{j=1}^{k} g_{i,j}v_i \otimes v_j = \sum_{1\le i<j\le k} g_{i,j}[v_i, v_j] + \sum_{i=1}^{k} g_{i,i}v_i \otimes v_i.$$

Therefore, $\mathcal{L}$ belongs to the sub-Lie algebra of $T(V)$ generated by V if V is concentrated in a single odd degree as in the hypothesis of Proposition 2.1.2. Note that from the definition, we have that v^2 belongs to a graded Lie algebra for an odd degree class v.

It is possible to derive a Poincaré-Birkhoff-Witt theorem for graded Lie algebras under the extra assumption that the underlying module is free over $\mathcal{R}$.

Theorem 2.1.7. *[36, Theorem 8.2.2] If L is a graded Lie algebra over $\mathcal{R}$ which is a free $\mathcal{R}$-module in each degree, then $U(L)$ is isomorphic to the symmetric algebra on L. In terms of the multiplicative structure, the symmetric algebra on L is isomorphic to the associated graded of $U(L)$ with respect to the length filtration induced on $U(L)$.*

We also note the following result which states that the graded Lie algebra injects into the universal enveloping Lie algebra.

Proposition 2.1.8. *[9, Theorem 2.21] Suppose that $\mathcal{R}$ is a Principal Ideal Domain. Let L be a graded Lie algebra over $\mathcal{R}$ such that L_n is finitely generated for every n. Let $U(L)$ be its universal enveloping algebra. Then the natural map $\iota : L \to U(L)$ is injective.*

The graded Lie algebra of interest is $L(V, \mathcal{L})$ which is defined to be the graded Lie algebra $F(V)/(\mathcal{L})$. The notation $F(V)$ stands for the free Lie algebra generated by V and $\mathcal{L}$ being a symmetric 2-tensor, lies in $F(V)$ (as V is concentrated in odd degree). We may express the graded Lie algebra as

$$L(V, \mathcal{L}) \cong V \oplus \Big[[V, V] + (V)^2\Big]/(\mathcal{L}) \oplus \cdots$$

Example 2.1.9. *Note that $U(F(V)) \cong T(V)$, and [9, Proposition 2.9] implies that $U(L(V, \mathcal{L})) \cong T(V)/(\mathcal{L})$.*

Let $\dim(V) = k$. For a $(k-1)$-dimensional summand W of V, write

$$L^W(V, \mathcal{L}) \cong W \oplus \Big[[V, V] + (V)^2\Big]/(\mathcal{L}) \oplus \cdots$$

which becomes a Lie subalgebra of $L(V, \mathcal{L})$. We note that Proposition 2.1.2 actually identifies the Lie algebra $L^W(V, \mathcal{L})$.

Proposition 2.1.10. *Given any basis $v_1, \cdots, v_{k-1}$ of W, there are $w_1, \cdots, w_{k-1}$ satisfying the conditions of Proposition 2.1.2 such that the map $F(v_1, \cdots, v_{k-1}, w_1, \cdots, w_{k-1}) \to L(V, \mathcal{L})$ induces an isomorphism of graded Lie algebras*

$$\frac{F(v_1, \cdots, v_{k-1}, w_1, \cdots, w_{k-1})}{(\sum_{i=1}^{k-1} [v_i, w_i])} \cong L^W(V, \mathcal{L}).$$

Proof. Let $\mathcal{F}$ stand for the left hand side of the equation in the statement of the Proposition. We wish to show that $\mathcal{F} \cong L^W(V, \mathcal{L})$. We note that the universal enveloping algebra of $\mathcal{F}$ is $T(v_1, \cdots, v_{k-1}, w_1, \cdots, w_{k-1})/(\sum [v_i, w_i])$, and the universal enveloping algebra for $L(V, \mathcal{L})$ is $T(V)/\mathcal{L}$. It follows from Proposition 2.1.8 that both $\mathcal{F}$ and $L^W(V, \mathcal{L})$ are free of finite rank in each grading. Our first observation is that the Poincaré-Birkoff-Witt theorem (Theorem 2.1.7) implies that the ranks in each degree are the same. We then show that the map is surjective which will complete the proof.

Let V be concentrated in grading $n-1$ which is odd, and let l_d denote the degree d part of $L(V,\mathcal{L})$. The symmetric algebra on $L(V,\mathcal{L})$ then has Poincaré series

$$\frac{\prod_{d\ odd}(1+t^d)^{l_d}}{\prod_{d\ even}(1-t^d)^{l_d}}.$$

By Theorem 2.1.7, this is the Poincaré series of the universal enveloping algebra $T(V)/(\mathcal{L})$. As V is concentrated in a single degree $n-1$, and $\mathcal{L}$ is a quadratic element it follows from [9, §4.4] that

$$\frac{\prod_{d\ odd}(1+t^d)^{l_d}}{\prod_{d\ even}(1-t^d)^{l_d}} = \frac{1}{1-kt^{n-1}+t^{2n-2}}.$$

Analogously, for $\mathcal{F}$, we let f_d denote the degree d part of $\mathcal{F}$, and we apply the techniques from [9, §5.2] to deduce

$$\frac{\prod_{d\ odd}(1+t^d)^{f_d}}{\prod_{d\ even}(1-t^d)^{f_d}} = \frac{1}{1-(k-1)t^{n-1}-(k-1)t^{2n-2}+t^{3n-3}}.$$

We now have the factorization

$$1-(k-1)t^{n-1}-(k-1)t^{2n-2}+t^{3n-3} = (1+t^{n-1})(1-kt^{n-1}+t^{2n-2}),$$

which implies

$$f_d = \begin{cases} l_d & \text{if } d \neq n-1 \\ l_d - 1 & \text{if } d = n-1. \end{cases}$$

This implies that the degree-wise rank of $\mathcal{F}$ matches that of $L^W(V,\mathcal{L})$.

We now complete the proof by showing that the map

$$F(v_1,\cdots,v_{k-1},w_1,\cdots,w_{k-1}) \to L^W(V,\mathcal{L})$$

is surjective. We choose α such that Ker$(\alpha) = W$ in the notation of Proposition 2.1.2, and choose v_k so that $v_1,\cdots,v_k$ is a basis for V, and proceed by induction on the length of a bracket r, to show that any element of $L^W(V,\mathcal{L})$ of the form $[[\cdots[l_1,l_2],\cdots],l_r]$ belongs to the image (modulo $\mathcal{L}$), where each l_i is one of the basis elements v_j. It is enough to show this as, if λ lies in the image then so does λ^2, and a bracket whose entries contain squares may be rewritten in terms of those without the squares using the identity $[y,x^2] = [[y,x],x]$, and the Jacobi identity. For $r=1$, we are done by the choice that $v_1,\cdots,v_{k-1}$ is a basis of W. For $r=2$, we are done by the fact that $W\otimes V$, $w_1,\cdots,w_{k-1}$ and $\mathcal{L}$ together form a basis of $V\otimes V$ by 2) of Proposition 2.1.2.

In the general case, the proof follows from the induction hypothesis if $l_r = v_j$ for $j < k$. We only need to consider $l_r = v_k$. Now the term of interest is $[l_{r-1}, v_k]$ where l_{r-1} lies in the image. Here, we have that the l_{r-1} is the sum of iterated brackets on the v_j and w_j, so it suffices to show by the Jacobi identity that $[w_j, v_k]$ is thus representable. From the formula of w_j, we have

$$[w_j, v_k] = \sum_{l<k} g_{j,l}\big[[v_l, v_k], v_k] + g_{j,k}[v_k^2, v_k] = \sum_{l<k} g_{j,l}[v_l, v_k^2].$$

As v_k^2 lies in $V \otimes V$, the $r = 2$ argument expresses this in the image, and thus this expression belongs to the image. The proof is thus complete. □

2.2 The homology of highly connected Poincaré duality complexes

Recall that a Poincaré duality complex of dimension r is a cell complex, together with a homology class in degree r, the cap product with which induces Poincaré duality as in a manifold of dimension r. We write $\mathcal{PD}_k^r$ to be the collection of Poincaré duality complexes of dimension r that are k-connected. Let $M \in \mathcal{PD}_{n-1}^{2n}$ with n even. The Poincaré duality condition guarantees that $H_n(M) \cong \mathbb{Z}^k$ for some k, and the homology is of the form

$$H_i(M) = \begin{cases} \mathbb{Z} & \text{if } i = 0, 2n \\ \mathbb{Z}^k & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases} \tag{2.2.1}$$

We write M_k for an element of $\mathcal{PD}_{n-1}^{2n}$ having the homology described in (2.2.1). Now a minimal cell structure [23] on the space M_k implies the pushout square

$$\begin{tikzcd} S^{2n-1} \arrow[r, "i"] \arrow[d, "L(M_k)"] & \mathcal{D}^{2n} \arrow[d] \\ (S^n)^{\vee_k} \arrow[r] & M_k, \end{tikzcd} \tag{2.2.2}$$

where $L(M_k) \in \pi_{2n-1}\big((S^n)^{\vee k}\big)$. Let α_i denote the inclusion of the i^{th}-copy of S^n in M_k. Using Hilton's theorem [25], one has the decomposition

$$\pi_{2n-1}\big((S^n)^{\vee k}\big) = \big(\pi_{2n-1}(S^n)\big)^{\oplus k} \oplus \big(\pi_{2n-1}(S^{2n-1})\big)^{\oplus\binom{k}{2}}.$$

The first factor in the above decomposition is induced by the inclusion of the spherical wedge summands, and the second factor by the Whitehead products of a choice of two summands. We note down the primes appearing in the torsion subgroup of $\pi_{2n-1}(S^n)$ in the following notation.

Notation 2.2.3. *Define* $T_n = \{2\} \cup \{p \mid p$ *prime and* $\exists$ *non-trivial* p*-torsion in* $\pi_{2n-1}(S^n)\}$*.*

A key step in the computations of this paper is the loop space homology of M_k, that is, $H_*\Omega M_k$. This is a (associative) ring, which is a quadratic algebra if $k \geq 2$ [9]. More precisely, let $a_i \in H_{n-1}(\Omega M_k)$ denote the Hurewicz image of the adjoints of the $\alpha_i : S^n \to M_k$, and $l(M_k)$ denote the image of $L(M_k)$ under the composite

$$\rho\colon \pi_{2n-1}\big((S^n)^{\vee_k}\big) \xrightarrow{\cong} \pi_{2n-2}\Omega\big((S^n)^{\vee_k}\big) \to H_{2n-2}\Big(\Omega\big((S^n)^{\vee k}\big)\Big).$$

The classes a_i serve as algebra generators of $H_*\Omega M_k$ and in their terms $l(M_k)$ may be expressed as

$$l(M_k) = \sum_{i,j} -g_{i,j} a_i \otimes a_j.$$

To figure out the sign, observe that

$$\rho([\alpha_i, \alpha_j]) = (-1)^{|\alpha_i|-1}[a_i, a_j] = -[a_i, a_j] \text{ as } n \text{ is even [38].}$$

The matrix $\big((g_{i,j})\big)$ is the matrix of the intersection form of M_k, so in the notation of Proposition 2.1.2, this is a symmetric, invertible 2-tensor. The homology of the loop space may be computed as [9, 15]

$$H_*(\Omega M_k) \cong T(a_1, \cdots, a_k)/\big(l(M_k)\big). \tag{2.2.4}$$

2.3 Loop space homology of a connected sum of sphere products

A connected sum of sphere products has the form $T = S^{k_1} \times S^{n-k_1} \# \cdots \# S^{k_r} \times S^{n-k_r}$. This is a pushout

$$\begin{tikzcd}
S^{n-1} \arrow[r] \arrow[d, "\sum_{i=1}^{r}{[\iota_{k_i}, \iota_{n-k_i}]}"] & \mathcal{D}^n \arrow[d] \\
\vee_{1\leq i\leq r} S^{k_i} \vee S^{n-k_i} \arrow[r, "\vee_{i=1}^{k-1}(\mu_i \vee \delta_i)"] & S^{k_1} \times S^{n-k_1} \# \cdots \# S^{k_r} \times S^{n-k_r}.
\end{tikzcd}$$

In this expression $\mu_i : S^{k_i} \to T$ and $\delta_i : S^{n-k_i} \to T$ denotes the inclusion of the various factors. In these terms the loop space homology of T is given by [9]

$$H_*\Omega T \cong T(\tilde{\mu_1}, \tilde{\delta}_1, \cdots, \tilde{\mu_r}, \tilde{\delta}_r)/\Big(\sum_{i=1}^{r}[\tilde{\mu_i}, \tilde{\delta}_i]\Big), \tag{2.3.1}$$

where the superscript $\sim$ is used to denote the adjoint of the class in loop space homology. In this paper, the connected sum T used is of the form $\#^{k-1}(S^n \times S^{2n-1})$. We retain the notation μ_i, δ_i as above, and we have the pushout

$$\begin{tikzcd}
S^{3n-2} \arrow[r] \arrow[d, "\Sigma_{i=1}^{k-1}[\iota_i^n, \iota_i^{2n-1}]"'] & \mathcal{D}^{3n-1} \arrow[d] \\
(S^n \vee S^{2n-1})^{\vee_{k-1}} \arrow[r, "\vee_{i=1}^{k-1}(\mu_i \vee \delta_i)"] & \#^{(k-1)}(S^n \times S^{2n-1}).
\end{tikzcd} \tag{2.3.2}$$

The homology of the loop space is given by (2.3.1). Now we look at Proposition 2.1.2 from the perspective of the connected sum of sphere products above. We may map the generators $\tilde{\mu_i}$ of the loop space homology to the generators a_i of $H_*\Omega M$ for $1 \leq i \leq k-1$. Proposition 2.1.2 now tells us where to send the other generators $\tilde{\delta}_i$ to obtain an algebra map from $H_*\Omega T \to H_*\Omega M$. We summarize this in the following algebraic result.

Proposition 2.3.3. *Given a basis $a_1, \cdots, a_k$ of $H_{n-1}\Omega M_k$, there is a map of associative algebras $H_*\Omega\#^{k-1}(S^n \times S^{2n-1}) \to H_*\Omega M_k$ which sends $\tilde{\mu_i}$ to the classes a_i for $1 \leq i \leq k-1$.*

Our efforts in [11] involve

1) Realize the above algebra map by a map of spaces $\#^{k-1}(S^n \times S^{2n-1}) \to M_k$ for $k \geq 2$.

2) Identify the homotopy fibre of the corresponding map.

We show that it is possible to achieve 1) after inverting the primes in T_n or if the value of k is large. Once this is achieved the homotopy fibre is shown to be homotopy equivalent to S^{n-1} by a spectral sequence argument.

Chapter 3

Construction of Sphere fibrations

The objective of this chapter is to construct sphere fibrations over highly connected Poincaré duality complexes, after inverting finitely many primes. In this case, we also have a specific understanding of the primes that need to be inverted as the set T_n of Notation 2.2.3. We write $R_n = \mathbb{Z}[\{\frac{1}{p} \mid p \in T_n\}]$, and the homology computations throughout are taken with R_n-coefficients.

Then we improve the results of §3.1 in the sense that we reduce the number of primes that are needed in the localization. We show that for k greater than the number of cyclic summands in $\pi_{n-1}^s \otimes \mathbb{Z}[\frac{1}{2}]$, the sphere fibrations exist once 2 is inverted. Further if 3 is inverted, S^{n-1} is an A_3-space [4, 50], and in this case, these fibrations are obtained as a pullback of the S^{n-1}-fibration over the associated projective plane [40]. Finaly, we use the spherical fibrations of §3.2 to deduce new results for loop space decompositions. The spherical fibrations are complemented with the results of [26] which identify the pullback of a spherical fibration over a connected sum. The fibration splits over the loop space to produce loop space decompositions. The results of this chapter is a part of the paper [12].

3.1 Sphere fibrations in a localized category

For a space X, recall that ρ is the map

$$\rho : \pi_n(X) \cong \pi_{n-1}(\Omega X) \to H_{n-1}(\Omega X).$$

Recall the equation $\rho([\gamma_1, \gamma_2]) = (-1)^{|\gamma_1|-1}[\rho(\gamma_1), \rho(\gamma_2)]$ [38]. We prove the following main theorem.

Theorem 3.1.1. *For $k \geq 2$, let $M_k \in \mathcal{PD}^{2n}_{n-1}$ with n even, and $H_n(M_k;\mathbb{Z}) \cong \mathbb{Z}^k$. After inverting the primes in T_n, and the prime 3, there is a fibration $S^{n-1} \to E_k \to M_k$ such that E_k is homotopy equivalent to $\#^{k-1}(S^n \times S^{2n-1})$.*

Proof. Our strategy is to construct a map $f : E_k \to M_k$ such that the homotopy fibre of f is homotopy equivalent to S^{n-1}. We do this via the pushout description of E_k in (2.3.2).

For $1 \leq i \leq k-1$, we define f on the i^{th}-factor μ_i as the map $\alpha_i : S^n \to M_k$. It follows that the class $\tilde{\mu_i}$ in loop space homology maps to the class a_i, as $\rho(\alpha_i) = a_i$. Let $\big((g_{i,j})\big)$ be as in Proposition 2.1.2. The i^{th}-factor δ_i is mapped by f in accordance with (2.1.4) to

$$\beta_i = \sum_{j=1}^{k-1} g_{i,j}[\alpha_j, \alpha_k] + \frac{1}{2} g_{i,k}[\alpha_k, \alpha_k] \tag{3.1.2}$$

which belongs to the image of

$$\pi_{2n-1}\big((S^n)^{\vee k}\big) \otimes R_n \to \pi_{2n-1}(M_k) \otimes R_n$$

as $2 \in T_n$. We write $\rho(\beta_i) = -b_i$, and it follows that in loop space homology $\tilde{\delta_i}$ maps to the class $-b_i$. This defines

$$f : (S^n)^{\vee k-1} \bigvee (S^{2n-1})^{\vee k-1} \to M_k,$$

and to extend f all the way to E_k, we require to show that the attaching map of the $(3n-1)$-cell is mapped to 0 by the induced map f_* in homotopy groups. From Proposition 2.1.2, we have that

$$(\Omega f)_* \Big(\sum_{i=1}^{k-1} [\tilde{\mu_i}, \tilde{\delta_i}] \Big) = -\sum_{i=1}^{k-1} [a_i, b_i] = 0.$$

This implies

$$f_* \Big(\sum_{i=1}^{k-1} [\mu_i, \delta_i] \Big) \in \mathrm{Ker}(\rho).$$

Recall the definition of $L(M_k)$ in (2.2.2). From the definition of T_n, we see that the map

$$\pi_{2n-1}\big((S^n)^{\vee k}\big) \otimes R_n \xrightarrow{\rho} H_{2n-2}\Big(\Omega\big((S^n)^{\vee k}\big); R_n\Big) \cong T_{R_n}(a_1, \cdots, a_k)$$

is injective. We now write

$$\overline{L}(M_k) = \sum_{i<j} g_{i,j}[\alpha_i, \alpha_j] + \sum_i \frac{1}{2} g_{i,i}[\alpha_i, \alpha_i],$$

and observe that $\rho(L(M_k)) = \rho(\bar{L}(M_k)) = -l(M_k)$. This follows from the equation $\rho([\gamma_1, \gamma_2]) = (-1)^{|\gamma_1|-1}[\rho(\gamma_1), \rho(\gamma_2)]$ [38]. Moreover, we have the equation

$$\begin{aligned} f_*\Big(\sum_{i=1}^{k-1}[\mu_i, \delta_i]\Big) &= \sum_{i=1}^{k-1}[\alpha_i, \beta_i] \\ &= [\bar{L}(M_k), \alpha_k] + \frac{1}{2}g_{k,k}[[\alpha_k, \alpha_k], \alpha_k] \end{aligned}$$

from a computation analogous to the proof of Proposition 2.1.2 via the Jacobi identity for Whitehead products. As the prime 3 is inverted, we also have that $[[\alpha_k, \alpha_k], \alpha_k] = 0$. Now we apply the injectivity of ρ after tensoring with R_n to replace $\bar{L}(M_k)$ with $L(M_k)$ and obtain,

$$f_*\Big(\sum_{i=1}^{k-1}[\mu_i, \delta_i]\Big) = [L(M_k), \alpha_k] = 0$$

in $\pi_{3n-2}(M_k) \otimes R_n$. This constructs the map $f : E_k \to M_k$. The homotopy fibre $\mathrm{Fib}(f)$ of f has the same homology as S^{n-1} by the spectral sequence argument of Proposition 3.1.7, which completes the proof of the theorem as all the spaces are simply connected. $\square$

We now show that inverting 3 is not necessary in Theorem 3.1.1. We first consider the case $k = 2$ in the following example.

Example 3.1.3. *Let $M_2 \in \mathcal{PD}^{2n}_{n-1}$ with $H_n(M) = \mathbb{Z}^2$. We wish to construct a fibration $S^{n-1} \to S^n \times S^{2n-1} \to M_2$ after inverting the primes in T_n. Following the proof of Theorem 3.1.1, we are able to do this if under a choice of basis of $H_2(M)$, $g_{2,2}$ is divisible by 3. We know that the matrix $\begin{bmatrix} g_{1,1} & g_{2,1} \\ g_{1,2} & g_{2,2} \end{bmatrix}$ is symmetric and non-singular, and by the classification of such bilinear forms [33, Theorem 2.2], we may reduce the matrix to one of*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \text{ or } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In the latter two cases, we may arrange for a basis such that $g_{2,2}$ is divisible by 3 (by changing α_2 to $\alpha_1 + \alpha_2$ in the second case). In the situation of the first matrix, we note that there is a map $S^n \to M_2$ whose mapping cone has cohomology $\mathbb{Z}[x]/(x^3)$ with $|x| = n$. Then, n must be one of 2, 4, or 8. For the generator ι_2 of $\pi_2 S^2$, we have $\big[[\iota_2, \iota_2], \iota_2\big] = 0$, and for $n = 4$ or 8, we have $3 \in T_n$. Thus, we always have the required fibration when $k = 2$.

The case of Poincaré duality complexes M_k, when $k \geq 3$, is implied by the following lemma.

Lemma 3.1.4. *a) Let $\langle -,-\rangle$ be a symmetric bilinear form over $\mathbb{Z}$ of rank ≥ 3. Then, there is a primitive $v \neq 0$ such that $\langle v,v\rangle$ is divisible by 3.*

b) Let $\langle -,-\rangle$ be a symmetric bilinear form over $\mathbb{Z}$ of rank ≥ 5. Then, there is a primitive $v \neq 0$ such that $\langle v,v\rangle$ is divisible by 8.

Proof. This argument basically follows from [33, Ch.II, (3.2)-(3.4)]. We only demonstrate how the prime 3 argument translates to this case. Diagonalize the form over the field $\mathbb{F}_3$ to one which possesses diagonal entries $d_1, \cdots, d_k$ with $d_i = \pm 1$ or 0. If all the d_i are ± 1, clearly

$$\sum_{i=1}^{k} d_i x_i^2 = 0$$

has a non-zero solution $\underline{x}$ if $k \geq 3$. There exists a non-zero primitive v such that v reduces to $\underline{x}$ modulo 3. □

Example 3.1.3 and Lemma 3.1.4 allow us to choose α_k such that $g_{k,k}$ is divisible by 3. Thus, we conclude the proof of the following Theorem.

Theorem 3.1.5. *For $k \geq 2$, let $M_k \in \mathcal{PD}^{2n}_{n-1}$ with n even and $H_n(M;\mathbb{Z}) \cong \mathbb{Z}^k$. After inverting the primes in T_n, there is a fibration $S^{n-1} \to E \to M$ such that E is homotopy equivalent to $\#^{k-1}(S^n \times S^{2n-1})$.*

3.1.1 A spectral sequence argument

We recall a well known fact that all the classes in H_*E are transgressive in the (homology) Serre spectral sequence for $\Omega E \to PE \to E$ if the space E is a suspension. Recall that the space PE is defined to the space of all continuous paths in E with compact open topology.

Lemma 3.1.6. *Suppose $E \simeq \Sigma X$. Then, in the Serre spectral sequence for the path-space fibration $\Omega E \to PE \to E$, all elements are transgressive.*

Proof. This follows from the fact that the elements in the image of the homology suspension $H_*\Sigma\Omega E \to H_*E$ are transgressive, and that ΣX is a retract of $\Sigma\Omega\Sigma X$. □

We now note down the set up in which the spectral sequence argument is carried out. We are working with a localized category of spaces in which the primes in T_n are inverted, and the homology is computed with R_n-coefficients. The space $M_k \in \mathcal{PD}^{2n}_{n-1}$ satisfies $H_n(M_k) \cong \mathbb{Z}^k$. We assume that $E_k \simeq \#^{k-1}(S^n \times S^{2n-1})$, and that $f : E_k \to M_k$ is a map which satisfies

1. $f_* \colon H_n(E_k) \to H_n(M_k)$ is injective, and $H_n M_k \cong f_*(H_n E_k) \oplus R_n\{\lambda_k\}$ for some $\lambda_k \in H_n M_k$.

2. The map $\pi_{2n-1}E_k \xrightarrow{f_*} \pi_{2n-1}M_k \xrightarrow{\rho} H_{2n-2}(\Omega M_k)$ induces an isomorphism onto the quotient
$$H_{2n-2}(\Omega M_k)/\Big(\mathrm{Im}\,\big(\pi_n E_k \overset{\rho\circ f_*}{\to} H_{n-1}(\Omega M_k)\big)\cdot H_{n-1}(\Omega M_k)\Big),$$
where the product stands for the Pontrjagin product of $H_*\Omega M_k$.

Proposition 3.1.7. *With notations as above, let Fib(f) be the homotopy fibre of the map f. Then, $H_*\big(\text{Fib}(f)\big) \cong H_*\big(S^{n-1}\big)$.*

Proof. We compute the homology Serre spectral sequence for the fibration $\Omega M_k \to \mathrm{Fib}(f) \to E_k$ whose E^2-page is given by

$$E^2_{p,q} = H_p(E_k) \otimes H_q(\Omega M_k) \Rightarrow H_{p+q}(\mathrm{Fib}(f)). \tag{3.1.8}$$

We may also continue the fibration sequence further to obtain the fibration $\Omega E_k \to \Omega M_k \to \mathrm{Fib}(f)$. The homology of ΩE_k is described in (2.3.1) and the homology of ΩM_k is described in (2.2.4). These are given by

$$H_*(\Omega M_k) \cong T(a_1,\cdots,a_k)/\big(l(M_k)\big),\ H_*\Omega E_k \cong T(\tilde{\mu_1},\tilde{\delta_1},\cdots,\tilde{\mu}_{k-1},\tilde{\delta}_{k-1})/\Big(\sum_{i=1}^{k-1}[\tilde{\mu_i},\tilde{\delta_i}]\Big).$$

Note that [9, Theorem 2.8] implies that both $H_*(\Omega M_k)$ and $H_*(\Omega E_k)$ are torsion-free. Let $L(a_1,\cdots,a_k)$ be the free Lie algebra on $a_1,\cdots,a_k$. We also note from [9, Proposition 2.11] that the universal enveloping algebras are computed as

$$U\big(L(a_1,\cdots,a_k)/(l(M_k))\big) \cong T(a_1,\cdots,a_k)/\big(l(M_k)\big),$$

$$U\Big(L(\tilde{\mu_1},\tilde{\delta_1},\cdots,\tilde{\mu}_{k-1},\tilde{\delta}_{k-1})/\big(\sum_{i=1}^{k-1}[\tilde{\mu_i},\tilde{\delta_i}]\big)\Big) \cong T(\tilde{\mu_1},\tilde{\delta_1},\cdots,\tilde{\mu}_{k-1},\tilde{\delta}_{k-1})/\big(\sum_{i=1}^{k-1}[\tilde{\mu_i},\tilde{\delta_i}]\big).$$

Denote by ι the map from a Lie algebra L to it's universal enveloping algebra $U(L)$. We now apply Proposition 2.1.10 writing $W = (\Omega f)_*(H_{n-1}(\Omega E_k)) \subset H_{n-1}(\Omega M_k) = V$. Note the commutative diagram below of graded Lie algebras and their universal enveloping algebras.

$$\begin{tikzcd}
L(\tilde{\mu}_1, \tilde{\delta}_1, \cdots, \tilde{\mu}_{k-1}, \tilde{\delta}_{k-1})/\left(\sum_{i=1}^{k-1}[\tilde{\mu}_i, \tilde{\delta}_i]\right) \arrow[rr, "\iota"] \arrow[d, "f_*"', "\cong"] & & H_*(\Omega E_k) \arrow[dd, "(\Omega f)_*"] \\
L^W(V, l(M_k)) \arrow[r, "\subset"] \arrow[d, "\iota"] & L(V, l(M_k)) \arrow[d, "\iota"] & \\
U(L^W(V, l(M_k))) \arrow[r] & U(L(V, l(M_k))) \arrow[r, "\cong"] & H_*(\Omega M_k)
\end{tikzcd}$$

The top left vertical arrow is an isomorphism by Proposition 2.1.10. This diagram allows us to identify $H_*(\Omega E_k) \to H_*(\Omega M_k)$ as $U(\hat{f})$ where $\hat{f}$ is the inclusion

$$L^W(V, l(M_k)) \to L(V, l(M_k)).$$

By the Poincaré-Birkhoff-Witt theorem for graded Lie algebras stated in Theorem 2.1.7, we have that $(\Omega f)_* : H_*\Omega E_k \to H_*\Omega M_k$ is injective, and in each degree, it is a torsion-free summand of a torsion-free Abelian group. Now the universal coefficient theorem implies that $(\Omega f)^* : H^*(\Omega M_k) \to H^*(\Omega E_k)$ is surjective. The Leray-Hirsch theorem now implies that the cohomology spectral sequence for the fibration $\Omega E_k \to \Omega M_k \to \mathrm{Fib}(f)$ degenerates at the second page. The same result now holds for the homology spectral sequence. As a consequence, we have that the map $H_*\Omega M_k \to H_*\mathrm{Fib}(f)$ is surjective.

Now we turn our attention to the spectral sequence for the fibration $\Omega M_k \to \mathrm{Fib}(f) \to E_k$ (3.1.8). As the map $H_*(\Omega M_k) \to H_*\mathrm{Fib}(f)$ is surjective, the E^∞-page is concentrated in the 0^{th}-column. We now calculate all the differentials that hit the 0^{th}-column and compute the relevant cokernels.

Consider the commutative diagram

$$\begin{tikzcd}
PE_k \arrow[r, "P(f)"] \arrow[d] & PM_k \arrow[d] \\
E_k \arrow[r, "f"] & M_k.
\end{tikzcd}$$

which implies the map $PE_k \to \mathsf{Fib}(f)$, as $\mathsf{Fib}(f)$ is the homotopy pullback of $E_k \to M_k \leftarrow PM_k$. The differentials are computed via the following commutative diagram of fibrations.

$$\begin{tikzcd}
\Omega\overline{E_k} \arrow[r] \arrow[d] & \Omega E_k \arrow[r, "\Omega f"] \arrow[d] & \Omega M_k \arrow[d] \\
P\overline{E_k} \arrow[r] \arrow[d] & PE_k \arrow[r] \arrow[d] & \mathsf{Fib}(f) \arrow[d] \\
\overline{E_k} \arrow[r] & E_k \arrow[r, "id"] & E_k.
\end{tikzcd}$$

The space $\overline{E_k}$ is the $(3n-2)$-skeleton of E_k which is homotopy equivalent to $(S^n)^{\vee k-1} \vee (S^{2n-1})^{\vee k-1}$. In the Serre spectral sequence

$$E^2_{p,q} = H_p(\overline{E_k}) \otimes H_q(\Omega\overline{E_k}) \Rightarrow H_{p+q}(P\overline{E_k}),$$

the classes μ_i are transgressive by Lemma 3.1.6. It follows that in the spectral sequence

$$E^2_{p,q} = H_p(E_k) \otimes H_q(\Omega E_k) \Rightarrow H_{p+q}(PE_k),$$

the classes μ_i transgress to $\tilde{\mu_i}$ and the classes δ_i transgress to $\tilde{\delta_i}$. Therefore, in the spectral sequence (3.1.8), we have the formulas

$$d^n(\mu_i) = \Omega f_*(\tilde{\mu_i}),\ d^{2n-1}(\delta_i) = \Omega f_*(\tilde{\delta_i}),$$

and the remaining differentials on the classes μ_i and δ_i equals 0. Furthermore, from the commutative diagram

$$\begin{tikzcd}
\Omega M_k \arrow[r, "="] \arrow[d] & \Omega M_k \arrow[d] \\
\mathsf{Fib}(f) \arrow[r] \arrow[d] & P(M_k) \arrow[d] \\
E_k \arrow[r, "f"] & M_k,
\end{tikzcd}$$

we see that $\Omega M_k \to \mathsf{Fib}(f) \to E_k$ is a principal fibration. It follows that the differential in the spectral sequence (3.1.8) respects the graded (right) $H_*(\Omega M_k)$-module structure by a result of Moore [35]. More precisely, we have

$$d^k(\alpha \otimes hg) = \pm d^k(\alpha \otimes h)(1 \otimes g),$$

for $\alpha \in H_*(E_k)$ and $g, h \in H_*(\Omega M_k)$. Therefore,

$$\mathrm{Im}(d^n) + \mathrm{Im}(d^{2n-1}) \subset E^2_{0,*}$$

equals $\Omega f_* \Big(H_{n-1}(\Omega E_k) \Big) \cdot H_* \Omega M_k \, + \, \Omega f_* \Big(H_{2n-2}(\Omega E_k) \Big) \cdot H_*(\Omega M_k)$. This equals

$$W + W \cdot \tilde{H}_*(\Omega M_k) + \Omega f_* \Big(H_{2n-2}(\Omega E_k) \Big) \cdot H_*(\Omega M_k).$$

The hypothesis 2) stated before the proposition implies that

$$W \cdot H_{n-1}(\Omega M_k) + \Omega f_* \Big(H_{2n-2}(\Omega E_k) \Big) = H_{2n-2}(\Omega M_k).$$

Therefore, we have

$$E^2_{0,*} / \Big(\mathrm{Im}(d^n) + \mathrm{Im}(d^{2n-1}) \Big) \cong R_n\{\lambda_k\}.$$

Further, note that λ_k cannot be hit by any differential in the spectral sequence (3.1.8) other than the transgression. It follows that $E^\infty_{0,*} \cong R_n\{\lambda_k\}$, and as we have earlier seen that this is the only possible non-zero part of the E^∞-page, the result is proved. □

3.2 Sphere fibrations for manifolds with high Betti number

In this section, we improve the results of §3.1 in the sense that we reduce the number of primes that are needed in the localization. We show that for k greater than the number of cyclic summands in $\pi^s_{n-1} \otimes \mathbb{Z}[\frac{1}{2}]$, the sphere fibrations exist once 2 is inverted. Further if 3 is inverted, S^{n-1} is an A_3-space [4, 50], and in this case, these fibrations are obtained as a pullback of the S^{n-1}-fibration over the associated projective plane [40]. Throughout this section we work in the category of spaces with 2 inverted, and write $R_2 = \mathbb{Z}[1/2]$ and $R_{2,3} = \mathbb{Z}[1/2, 1/3]$.

3.2.1. Whitehead products in $\pi_{2n-1}S^n$. After inverting the prime 2, for n even, ΩS^n splits into a product $\Omega S^{2n-1} \times S^{n-1}$. The map $S^{2n-1} \to S^n$ which induces the inclusion of ΩS^{2n-1} may be chosen to be $\frac{1}{2}[\iota_n, \iota_n]$. In these terms we have

$$\pi_{2n-1}(S^n) \otimes R_2 \cong R_2\{[\iota_n, \iota_n]\} \oplus E(\pi_{2n-2}(S^{n-1}) \otimes R_2). \tag{3.2.2}$$

We now note the following formula for the Whitehead products for the generators in (3.2.2) using the Jacobi identity and [24, Theorem 6.1]

$$3\big[\iota_n, [\iota_n, \iota_n]\big] = 0, \quad [\iota_n, E\alpha] = [\iota_n, \iota_n] \circ \Sigma^n \alpha \; \forall \; \alpha \in \pi_{2n-2} S^{n-1} \otimes R_2. \tag{3.2.3}$$

The last equation is implied by the fact that the Hopf invariant of $E(\alpha)$ is 0. It is worthwhile to note here that $\Sigma^n\alpha \in \pi_{3n-2}(S^{2n-1})$ belongs to the stable range, so that the right hand side of the second equation of (3.2.3) is non-zero only when α represents a non-trivial stable homotopy class.

3.2.1 Constructing spherical fibrations after inverting 2

As in §3.1, by obstruction theory we construct a map $E_k \simeq \#^{k-1}(S^n \times S^{2n-1}) \to M_k$ for $M_k \in \mathcal{PD}^{2n}_{n-1}$, which means $H_n(M_k) \cong \mathbb{Z}^k$. One should note that unless we invert 2, the homomorphism

$$\pi_s(S^{n-1}) \to \pi_s(\Omega M_k) \cong \pi_{s+1}(M_k)$$

associated to a summand $S^n \to (S^n)^{\vee k} \to M_k$ has non-trivial kernel. This follows from the EHP-sequence for a sphere [29] and the fact that the inclusion of a sphere induces a summand on the level of homotopy groups [9, 13]. After inverting 2, the kernel vanishes. In this situation, we enumerate criteria under which it is possible to construct a map $E_k \to M_k$. Recall, the notations μ_i, δ_i of homology generators of E_k and α_i of M_k, and the attaching map $L(M_k)$ from §2.1.2.

Proposition 3.2.4. *Suppose that the attaching map $L(M_k) \in \pi_{2n-1}\big((S^n)^{\vee k}\big)$ of M_k takes the following form (for an invertible integer matrix $\big((g_{i,j})\big)$)*

$$L(M_k) = \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \alpha_j] + \sum_{i=1}^{k} \big(g_{i,i}(\frac{1}{2}[\alpha_i, \alpha_i]) + \alpha_i \circ \omega_i\big), \text{ for } \omega_i \in E(\pi_{2n-2}(S^{n-1})). \tag{3.2.5}$$

Assume that this satisfies

1. $g_{k,k} \equiv 0 \pmod 3$.
2. *ω_k lies in the kernel of $\Sigma : \pi_{2n-1}(S^n) \to \pi^s_{n-1}$.*

Then, there is a map $E_k \to M_k$ which sends μ_i to α_i, and that β_i satisfy the conditions of Proposition 3.1.7.

Proof. We define the β-classes as in (4.1.5)

$$\beta_i = \sum_{j=1}^{k-1} g_{i,j}[\alpha_j, \alpha_k] + \frac{1}{2} g_{i,k}[\alpha_k, \alpha_k] - \alpha_k \circ \omega_i. \tag{3.2.6}$$

Observe that the elements ω_i are in the kernel of $\rho : \pi_{2n-1}(S^n) \to H_{2n-2}(\Omega S^n)$. Thus, the β_i of (3.2.6) have the same image in $H_*\Omega M_k$ as those of (3.1.2), and so it satisfies the criteria of Proposition 3.1.7. We complete the proof by noting

$$\begin{aligned}\sum_{i=1}^{k-1}[\alpha_i,\beta_i] &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} g_{i,j}\big[\alpha_i,[\alpha_j,\alpha_k]\big] + \sum_{i=1}^{k-1}\frac{1}{2}g_{i,k}\big[\alpha_i,[\alpha_k,\alpha_k]\big] - \sum_{i=1}^{k-1}[\alpha_i,\alpha_k\circ\omega_i] \\ &= \sum_{1\leq i<j\leq k} -g_{i,j}\big[[\alpha_i,\alpha_j],\alpha_k\big] - \sum_{i=1}^{k-1}\frac{1}{2}g_{i,i}\big[[\alpha_i,\alpha_i],\alpha_k\big] - \sum_{i=1}^{k-1}[\alpha_i\circ\omega_i,\alpha_k] \\ &= -[L(M_k),\alpha_k] + \frac{1}{2}g_{k,k}\big[\alpha_k,[\alpha_k,\alpha_k]\big] + [\alpha_k\circ\omega_k,\alpha_k] \\ &= 0.\end{aligned}$$

The last equality follows from (3.2.3) and the hypothesis 1 and 2 of the proposition. ☐

Proposition 3.2.4 lays down the conditions we need to arrange in order to construct a map $E_k \to M_k$ whose homotopy fibre is S^{n-1}. For a finite Abelian group A, we define the number of cyclic summands to be the number r in its decomposition as

$$A \cong \mathbb{Z}/(a_1)\oplus\mathbb{Z}/(a_2)\cdots\oplus\mathbb{Z}/(a_r),\ \text{ with } a_i \mid a_{i+1}.$$

In this notation, we have the following theorem.

Theorem 3.2.7. *Let $M_k \in \mathcal{PD}^{2n}_{n-1}$ with $H_nM_k \cong \mathbb{Z}^k$, and $n > 8$. Let r be the number of cyclic summands of π^s_{n-1}. If $k > r$, after inverting 2 there is a fibration $E_k \to M_k$ with fibre S^{n-1} where $E_k \simeq \#^{(k-1)}(S^n \times S^{2n-1})$.*

Proof. We apply Proposition 3.1.7 and Proposition 3.2.4. We only need to show that one may choose a basis of $\pi_n(M_k)$ such that the hypotheses of Proposition 3.2.4 are satisfied. This is done for $r = 1$ in Example 3.2.8 and for $r > 1$ in Proposition 3.2.9 ☐

In the following example we work out the details when π^s_{n-1} is cyclic which is analogous to the methods of §4.

Example 3.2.8. *Suppose that π^s_{n-1} is cyclic with generator χ and of order d, and assume as in Proposition 3.2.4,*

$$L(M_k) = \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i,\alpha_j] + \sum_{i=1}^{k}\big(g_{i,i}(\frac{1}{2}[\alpha_i,\alpha_i]) + \alpha_i\circ\omega_i\big),\ \text{ for } \omega_i \in E(\pi_{2n-2}(S^{n-1})).$$

We assume $k \geq 2$, and show that it is possible to change the basis $\{\alpha_i\}$ so that the hypothesis 1 and 2 of Proposition 3.2.4 are satisfied. We may now write (for $\Sigma : \pi_{2n-1}(S^n) \to \pi^s_{n-1}$)

$$\Sigma\omega_i = x_i\chi \text{ for } x_i \in \mathbb{Z}/d.$$

We take x to be the greatest common divisor of the x_i in $\mathbb{Z}/d$ and write $x_i = c_i x$. The c_i may be arranged so that they have no common divisor over $\mathbb{Z}$. Now change the basis to $\{\alpha'_1, \cdots, \alpha'_k\}$ so that the first element is $\alpha'_1 = \sum c_i \alpha_i$. In this new basis, we have (for some representative $u \in \pi_{2n-1}(S^n)$ of $\chi \in \pi^s_{n-1}$)

$$\Sigma(\alpha'_1 \circ xu) - \Sigma(\sum_{i=1}^{k} \alpha_i \circ \omega_i) = 0.$$

This arranges the 2^{nd} hypothesis whenever $k \geq 2$. We now arrange for $g_{k,k} \equiv 0 \pmod 3$ without changing the first element. If $k \geq 4$, applying Lemma 3.1.4 to the summand spanned by the last $k-1$ elements yields a change of basis with $g_{k,k} \equiv 0 \pmod 3$. It remains to consider the cases $k = 2$ and $k = 3$. For $k = 3$, [33, Theorem 2.2] implies that there is a 1-dimensional summand with self intersection ±1. This implies $n \leq 8$, cases which are already dealt with in §4.

In the case $k = 2$, as $n \neq 2$, 4, or 8 we do not have a Hopf invariant one class, so that the matrix $((g_{i,j}))$ with respect to some basis is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Suppose that $c_1\alpha_1 + c_2\alpha_2$ is the element constructed above used to kill off the ω_i. As c_1 and c_2 don't have common divisors, one of them is not divisible by 3, say c_1. Then, we may choose a second element of the basis as one which is $\alpha_2 \pmod 3$ using the fact that $GL_n\mathbb{Z} \to GL_n\mathbb{F}_3$ is surjective. This satisfies $g_{2,2}$ is $0 \pmod 3$.

The general case with more than 1 cyclic summand is a repeated iteration of the argument in Example 3.2.8.

Proposition 3.2.9. *Suppose that M_k is as in Theorem 3.2.7. Then, there is a choice of basis that satisfies the hypothesis of Proposition 3.2.4.*

Proof. The case $r = 1$ is covered in Example 3.2.8. Note that the technique of Example 3.2.8 may be applied to a cyclic summand C of π^s_{n-1} to yield a basis where the choice of ω_i in (3.2.5) satisfies $pr_C(\Sigma\omega_i) = 0$ if $i > 1$ with pr_C standing for the projection of π^s_{n-1} onto the summand C. We now apply this fact one summand at a time to obtain a basis with the property that for $i > r$, $\Sigma\omega_i = 0$. We now arrange for $g_{k,k} \equiv 0 \pmod 3$ without changing the first r basis elements.

Let $\{\alpha_i\}$ be the basis obtained so far, and let $\langle -, - \rangle$ be the intersection form. Suppose $\langle \alpha_k, \alpha_k \rangle \not\equiv 0 \pmod 3$. If $k \geq r + 3$, then take the summand of $\mathbb{Z}^k$ spanned by $\alpha_{r+1}, \cdots, \alpha_k$ and apply Lemma 3.1.4 to get a change a basis so that $\langle \alpha_k, \alpha_k \rangle \equiv 0 \pmod 3$. In the other cases we change α_k by adding a linear combination of $\alpha_1, \cdots, \alpha_{k-1}$ so that $\langle \alpha_k, \alpha_k \rangle \equiv 0$

$\pmod 3$. We work over $\mathbb{F}_3$ and then lift the change of basis to an integral one. It is possible to choose a basis $\{v_1, \cdots, v_{k-1}\}$ of the summand B generated by $\alpha_1, \cdots, \alpha_{k-1}$ such that $\langle v_i, \alpha_k\rangle \equiv 0 \pmod 3$ for $i \leq k-2$. If in addition $\langle v_{k-1}, \alpha_k\rangle \equiv 0 \pmod 3$, then, the restriction of the intersection form to B is non-singular over $\mathbb{F}_3$. So, we may diagonalize it to one with non-zero entries. As $r \geq 2$, $k \geq 3$, so there are at least 2 of the v_i, so there is a combination $u = \alpha_k + \sum c_i v_i$ such that $\langle u, u\rangle \equiv 0 \pmod 3$. As the v_i involve only $\alpha_1, \cdots, \alpha_{k-1}$ so we may replace α_k with u and the proof is complete.

If $c = \langle v_{k-1}, \alpha_k\rangle \not\equiv 0 \pmod 3$, we note that for $t \not\equiv 0 \pmod 3$,

$$\langle \alpha_k + tv_{k-1}, \alpha_k + tv_{k-1}\rangle \equiv \langle \alpha_k, \alpha_k\rangle + \langle v_{k-1}, v_{k-1}\rangle + 2tc \pmod 3$$

which may be arranged to be 0 as long as

$$\langle \alpha_k, \alpha_k\rangle + \langle v_{k-1}, v_{k-1}\rangle \not\equiv 0 \pmod 3. \tag{3.2.10}$$

In case (3.2.10) does not hold consider v_j for $j < k-1$. There is at least one such j as $k \geq 3$. If $\langle v_j, v_j\rangle \not\equiv 0 \pmod 3$, then change α_k to $\alpha_k + v_j$ to ensure (3.2.10) holds. If further $\langle v_j, v_j\rangle \equiv 0 \pmod 3$ and $\langle v_j, v_{k-1}\rangle \not\equiv 0 \pmod 3$, then replace v_{k-1} with $v'_{k-1} = v_{k-1} + sv_j$ so that $\langle v'_{k-1}, v'_{k-1}\rangle \equiv 0 \pmod 3$ and this implies that (3.2.10) holds. Finally, if all the v_j satisfy $\langle v_j, v_{k-1}\rangle \equiv 0 \pmod 3$, then over $\mathbb{F}_3$, the form breaks up into orthogonal pieces spanned by $\{v_1, \cdots, v_{k-2}\}$ and $\{v_{k-1}, \alpha_k\}$. Thus, the restriction of the intersection form to the summand spanned by $v_1, \cdots, v_{k-2}$ is non-singular over $\mathbb{F}_3$. There is a linear combination v of these v_j which satisfies $\langle v, v\rangle \not\equiv 0 \pmod 3$. Again we may add this to α_k to ensure (3.2.10) holds. This completes the proof. ☐

3.2.2 Sphere fibrations as pullbacks

Now that we have sphere fibrations $S^{n-1} \to E_k \to M_k$ for large enough k after inverting 2, we may ask when these are realizable pullbacks. More precisely, we would like to build up an analogous story to §4 where the fibrations were principal fibrations. However, we know that the spheres do not usually possess a group structure except for S^1, S^3 or S^{n-1} after p-completion for $n \mid 2p-2$ as the Sullivan spheres [41, 48]. On the other hand, the odd spheres possess a homotopy associative multiplication once 2 and 3 are inverted [4, 50]. However, inverting 3 is necessary here otherwise we would only have a non-homotopy associative H-space structure [28]. Recall that $Q(S^n) = colim_k \Omega^k S^{n+k}$. We briefly recall the construction using the following [49, Corollary 3.2]

Proposition 3.2.11. *After inverting* 2 *and* 3*, the inclusion* $S^{n-1} \to Q(S^{n-1})$ *is a* $(5n+1)$*-equivalence.*

A direct corollary of Proposition 3.2.11 is that the E_∞-space structure on $Q(S^{n-1})$ yields a homotopy associative structure on S^{n-1}.

Corollary 3.2.12. *After inverting the primes* 2 *and* 3*,* S^{n-1} *has a homotopy associative multiplication.*

Proof. Since QS^{n-1} has a homotopy associative multiplication : $m : QS^{n-1} \times QS^{n-1} \to QS^{n-1}$. We define the multiplication on S^{n-1} by lifting

$$\phi : S^{n-1} \times S^{n-1} \overset{\iota\times\iota}{\to} QS^{n-1} \times QS^{n-1} \overset{m}{\to} QS^{n-1}$$

via the isomorphism $[S^{n-1} \times S^{n-1}, S^{n-1}] \cong [S^{n-1} \times S^{n-1}, QS^{n-1}]$ implied by Proposition 3.2.11. From this definition of multiplication on S^{n-1}, it follows that the map $S^{n-1} \to QS^{n-1}$ ia an H-map. Associativitiy is implied by the associativity of m and the isomorphism

$$[S^{n-1} \times S^{n-1} \times S^{n-1}, S^{n-1}] \cong [S^{n-1} \times S^{n-1} \times S^{n-1}, QS^{n-1}].$$

□

Following Stasheff [40], we consider the projective planes $P_2(X)$ for an A_3-space X which support a fibration $E_2(X) \to P_2(X)$ with an action $X \times E_2(X) \to E_2(X)$ identifying X as the fibre. In fact we have the diagram

$$\begin{tikzcd} E_1(X) \arrow[r] \arrow[d, "H(m_X)"] & E_2(X) \arrow[d] \\ P_1(X) \arrow[r] & P_2(X) \end{tikzcd}$$

with $H(m_X)$ standing for the Hopf construction on the multiplication m_X in the sequence of identifications

$$E_1(X) \simeq X * X \overset{H(m_X)}{\to} \Sigma X \simeq P_1(X),\ P_2(X) \simeq C(H(m_X)),\ E_2(X) \simeq X * X * X.$$

We now fix a model for QS^{n-1} as a A_∞-space such that there exist a homotopy equivalence $Q(S^{n-1}) \simeq \Omega Q(S^n)$ as H-spaces (see [7]). Therefore, Corollary 3.2.12 implies the following commutative diagram by the functoriality of the P_2-construction.

$$\begin{tikzcd}
E_2(S^{n-1}) \arrow[r] \arrow[d] & E_2(Q(S^{n-1})) \arrow[r] \arrow[d] & E\Omega QS^n \simeq * \arrow[d] \\
P_2(S^{n-1}) \arrow[r] & P_2(Q(S^{n-1})) \arrow[r] & B\Omega QS^n \simeq QS^n
\end{tikzcd} \tag{3.2.13}$$

We also note that the construction in Corollary 3.2.12 makes the multiplication m on S^{n-1} homotopy commutative. This implies using [27, Lemma 2.4] that

$$2H(m) = -[\iota_n, \iota_n]. \tag{3.2.14}$$

We now prove that the sphere fibration over M_k is obtained as a pullback of $E_2(S^{n-1}) \to P_2(S^{n-1})$.

Theorem 3.2.15. *With notations as in Theorem 3.2.7, after inverting 2 and 3, the fibration $E_k \to M_k$ is a homotopy pullback of $M_k \to P_2(S^{n-1}) \leftarrow E_2(S^{n-1})$ for a suitable map $M_k \to P_2(S^{n-1})$.*

Proof. We define the map $s_k : (S^n)^{\vee k} \to S^n$ which quotients out the first $k-1$ factors, and then compose it to $P_2(S^{n-1})$. Now consider the diagram

$$\begin{tikzcd}
S^{2n-1} \arrow[r] \arrow[d, "L(M_k)"] & \mathcal{D}^{2n} \arrow[d] & \\
(S^n)^{\vee k} \arrow[r] \arrow[rrd, "s_k"'] & M_k \arrow[rd, dashed] & \\
 & & P_2(S^{n-1})
\end{tikzcd}$$

Assume that $L(M_k)$ satisfies the form given in (3.2.5). In this form, applying Proposition 3.2.11, we see that $\Sigma\omega_k = 0$ implies that $\omega_k = 0$. We now observe via (3.2.14) that

$$s_k \circ L(M_k) = g_{k,k} H(m).$$

As $P_2(S^{n-1})$ is the mapping cone of $H(m)$, the dashed arrow exists in the above diagram, and hence we obtain a map $\phi_k : M_k \to P_2(S^{n-1})$. Let $E = \phi_k^*(E_2(S^{n-1}))$ be the homotopy pullback $M_k \to P_2(S^{n-1}) \leftarrow E_2(S^{n-1})$. We shall show that $E \simeq E_k$ by lifting the map $E_k \to M_k$ to E. Once we are able to do this, it follows that the map $E_k \to E$ is a homology equivalence proving the required result.

Consider the following diagram

$$\begin{tikzcd}
S^{3n-2} \arrow[d, "\chi"] & S^{n-1} \arrow[r, "="] \arrow[d] & S^{n-1} \arrow[r] \arrow[d] & QS^{n-1} \arrow[r] \arrow[d] & QS^{n-1} \arrow[d] \\
X \arrow[d] & E \arrow[r] \arrow[d] & E_2(S^{n-1}) \arrow[r] \arrow[d] & W \arrow[r] \arrow[d] & PQS^n \arrow[d] \\
E_k \arrow[ur, dashed] & M_k \arrow[r] & P_2(S^{n-1}) \arrow[r, "="] & P_2(S^{n-1}) \arrow[r] & QS^n.
\end{tikzcd} \tag{3.2.16}$$

In (3.2.16), W is defined so that the right-most square is a homotopy pullback. Note that Proposition 3.2.11 implies that $E_2(S^{n-1}) \to W$ is a $(5n+1)$-equivalence. Now the $(2n-1)$-skeleton X of E_k is $(S^n)^{\vee k-1} \vee (S^{2n-1})^{\vee k-1}$, and from the formula of α_i and β_i we check that the composite $X \to M_k \to P_2(S^{n-1})$ is null-homotopic. Therefore, the map of X all the way to QS^n lifts to PQS^n. Hence, it lifts to W being the pullback, and to $E_2(S^{n-1})$ from the connectivity of the map $E_2(S^{n-1}) \to W$. Again as E is the pullback, we get a lift of $X \to E$. Let χ stand for the attaching map of the $(3n-1)$-cell of E_k. The composite

$$S^{3n-2} \xrightarrow{\chi} X \to E \to M_k$$

is trivial, so the composite to E lifts to S^{n-1}. This implies that the composite to E is trivial as the inclusion $S^{n-1} \to E$ is null-homotopic. The last statement comes from the fact that the map $\Omega M_k \to S^{n-1}$ is surjective on π_{n-1}. Therefore, there is a lift $E_k \to E$. □

3.2.3 Applications for loop space decompositions

In this subsection, we use the spherical fibrations of §3.2 to deduce new results for loop space decompositions. The spherical fibrations are complemented with the results of [26] which identify the pullback of a spherical fibration over a connected sum. The fibration splits over the loop space to produce loop space decompositions.

3.2.17. Connected sum with highly connected manifolds. Given $2n$-dimensional Poincaré duality complexes M and N in which the attaching maps of the $2n$-cells are given by (respectively for M and N)

$$f : S^{2n-1} \to M_0, \quad g : S^{2n-1} \to N_0,$$

the connected sum is defined as the mapping cone of [45]

$$f + g : S^{2n-1} \to M_0 \vee N_0.$$

Theriault [42] has provided a method to transfer a loop space decomposition of M to one of $N\#M$ for arbitrary N. The hypothesis on M used to make the argument work is that the cell attachment is inert.

Definition 3.2.18. For a homotopy cofibration sequence

$$\Sigma A \xrightarrow{f} X_0 \xrightarrow{i} X,$$

f is said to be *inert* if Ωi has a homotopy right inverse.

Let $M_k \in \mathcal{PD}^{2n}_{n-1}$. For M_k, we have two different approaches for loop space decompositions [13] and [9]. In this case $M_0 \simeq (S^n)^{\vee k}$ and the attaching map is $L(M_k) : S^{2n-1} \to M_0$. As a consequence of the loop space decompositions of M_k, we readily observe

Proposition 3.2.19. *For $k \geq 2$, the map $L(M_k)$ is inert.*

This follows because ΩM_k is a product of loop spaces of spheres which are mapped via Whitehead products of the sphere inclusions into M_0 [9]. These clearly lift to ΩM_0. Now [42, Theorem 1.4] implies

Theorem 3.2.20. *Let $M \in \mathcal{PD}^{2n}_{n-1}$ with Rank$(H_n(M)) \geq 2$, and $N \in \mathcal{PD}^{2n}_1$. Then,*

$$\Omega(M\#N) \simeq \Omega M \times \Omega(\Omega M \ltimes N_0),$$

where $N_0 \simeq N - *$.

Now by [13, Theorem 1.4], we have

$$\Omega M \simeq \Omega(S^n \times S^n) \times \Omega(J \vee (J \wedge \Omega(S^n \times S^n)))$$

where $J = \vee_2^k S^n$ (where $H_n(M) \cong \mathbb{Z}^k$). Hence we have a decomposition of the loop space $\Omega(M\#N)$ in terms of simply connected spheres if N_0 is also a wedge of spheres.

3.2.21. Spherical fibrations over connected sums with highly connected manifolds. We now construct spherical fibrations over connected sums using the spherical fibrations $S^{n-1} \to E_k \to M_k$ proved in Theorems 3.1.5, 4.1.2, 4.2.11, and 3.2.7. Let $M_k \in \mathcal{PD}^{2n}_{n-1}$ with Rank$(H_n(M_k)) = k \geq 2$. For a $2n$-dimensional Poincaré duality complex N, we consider the quotient $M_k\#N \xrightarrow{q} M_k$, and the pullback $E_{k,N}$ of E_k to $M_k\#N$.

$$\begin{tikzcd}
E_{k,N} \arrow[r] \arrow[d] & E_k \arrow[d] \\
M_k\#N \arrow[r, "q"] & M_k,
\end{tikzcd} \tag{3.2.22}$$

We additionally observe that the loop space of $E_{k,N}$ depends only on N_0 and not on the attaching map of the top cell of N.

The homotopy type of $E_{k,N}$ is determined analogously as in [26, Lemma 3.1]. Let $F(n)$ denote the set of homotopy equivalences of S^{n-1}. For a $\tau : S^{n-1} \to F(2n)$ define $\mathcal{G}_\tau(N)$ as the pushout

$$\begin{tikzcd}
S^{2n-1} \times S^{n-1} \arrow[r, "i"] \arrow[d, "{(\phi,\pi_2)}"] & S^{2n-1} \times \mathcal{D}^n \arrow[d] \\
N_0 \times S^{n-1} \arrow[r] & \mathcal{G}_\tau(N),
\end{tikzcd} \tag{3.2.23}$$

where ϕ is the composite

$$S^{2n-1} \times S^{n-1} \xrightarrow{t} S^{2n-1} \times S^{n-1} \xrightarrow{\pi_1} S^{2n-1} \to N_0, \text{ with } t(x,s) = (\tau(s)x, s).$$

Using this notation, the pullback $E_{k,N}$ in (3.2.22) may be simplified using [26, Lemma 3.1].

Proposition 3.2.24. *Let $n \in \{2,4,8\}$. Then there is a $\tau : S^{n-1} \to F(2n)$ such that one has the equivalence*

$$E_{k,N} \simeq \mathcal{G}_\tau(N)\#E_k,$$

with $\mathcal{G}_\tau(N)$ defined as in (3.2.23).

Looking towards the loop space, we build up to an analogue of Theorem 3.2.20. For this, we require the knowledge of the homotopy type of $\mathcal{G}_\tau(N) - *$ for $* \in \mathcal{G}_\tau(N)$. The following identification implies that this is independent of τ and the attaching map of the top cell.

Proposition 3.2.25. *Let $\tau : S^{n-1} \to F(2n)$ be a map. Then, we have a homotopy equivalence*

$$\mathcal{G}_\tau(N) - * \simeq (N_0 \rtimes S^{n-1}).$$

Proof. Consider the (homotopy) pushout square (3.2.23). Express $S^{n-1} = S^{n-1}_U \cup S^{n-1}_L$ as the union of the upper and lower hemispheres, and the unit n-disk as the union of the disk of radius $1/2$ and the annulus as $\mathcal{D}^n = \mathcal{D}^n_{\leq 1/2} \cup \mathcal{D}^n_{\geq 1/2}$. We may then write

$$S^{2n-1} \times \mathcal{D}^n = S^{2n-1} \times \mathcal{D}^n_{\geq 1/2} \cup S^{2n-1}_U \times \mathcal{D}^n_{\leq 1/2} \cup S^{2n-1}_L \times \mathcal{D}^n_{\leq 1/2}.$$

Hence,

$$(S^{2n-1} \times \mathcal{D}^n) - \mathrm{int}(S_L^{2n-1} \times \mathcal{D}^n_{\leq 1/2}) \cong S^{2n-1} \times \mathcal{D}^n_{\geq 1/2} \cup S_U^{2n-1} \times \mathcal{D}^n_{\leq 1/2}.$$

Note that $S_L^{2n-1} \times \mathcal{D}^n_{\leq 1/2} \cong \mathcal{D}^{3n-1}$. We set $\mathcal{G}_\tau(N)_0 = \mathcal{G}_\tau(N) - \mathrm{int}(S_L^{2n-1} \times \mathcal{D}^n_{\leq 1/2})$ which is homotopy equivalent to $\mathcal{G}_\tau(N) - *$. The pushout diagram (3.2.23) induces the following homotopy pushout

$$\begin{tikzcd}
S^{2n-1} \times S^{n-1} \arrow[r, "i"] \arrow[d, "(\phi,\pi_2)"] & S^{2n-1} \times \mathcal{D}^n_{\geq 1/2} \cup S_U^{2n-1} \times \mathcal{D}^n_{\leq 1/2} \arrow[d] \\
N_0 \times S^{n-1} \arrow[r] & \mathcal{G}_\tau(N)_0,
\end{tikzcd}$$

which in turn induces the follwing diagram

$$\begin{tikzcd}
S_U^{2n-1} \times S^{n-1} \arrow[r, "i"] \arrow[d, "(i \times I)"] & S_U^{2n-1} \times \mathcal{D}^n_{\geq 1/2} \arrow[d] \\
S^{2n-1} \times S^{n-1} \arrow[r, "i"] \arrow[d, "(\phi,\pi_2)"] & S^{2n-1} \times \mathcal{D}^n_{\geq 1/2} \cup S_U^{2n-1} \times \mathcal{D}^n_{\leq 1/2} \arrow[d] \\
N_0 \times S^{n-1} \arrow[r] & \mathcal{G}_\tau(N)_0.
\end{tikzcd}$$

Note that the top square is a homotopy pushout square, and hence the outer square

$$\begin{tikzcd}
S_U^{2n-1} \times S^{n-1} \arrow[r, "i"] \arrow[d, "(\phi,\pi_2)\circ(i \times I)"] & S_U^{2n-1} \times \mathcal{D}^n_{\leq 1/2} \arrow[d] \\
N_0 \times S^{n-1} \arrow[r] & \mathcal{G}_\tau(N)_0
\end{tikzcd}$$

is also a homotopy pushout square. As $S_U^{2n-1} \times \mathcal{D}^n_{\leq 1/2}$ is contractible, $\mathcal{G}_\tau(N)_0$ is homotopy equivalent to the homotopy cofibre of the left vertical arrow, which is easily computed to be $N_0 \rtimes S^{n-1}$. □

As a consequence of Proposition 3.2.25, we obtain the following corollary using [42, Theorem 1.4].

Corollary 3.2.26. *Suppose $E \in \mathcal{PD}_1^{3n-1}$ in which the attaching map of the top cell is inert and $N \in \mathcal{PD}_1^{2n}$. Then, for any $\tau : S^{n-1} \to F(2n)$,*

$$\Omega(\mathcal{G}_\tau(N) \# E) \simeq \Omega E \times \Omega(\Omega E \ltimes J)$$

where $J = (N_0 \rtimes S^{n-1})$.

3.2.27. Loop space decompositions from spherical fibrations. We obtain loop space decompositions for $E_{k,N}$ using Corollary 3.2.26 and Proposition 3.2.24. Note that the calculations of [9] implies that the attaching map of the top cell of a connected sum of sphere products is inert. Therefore, we deduce the following result by applying Corollary 3.2.26 for $E = E_k$.

Proposition 3.2.28. *Let* $N \in \mathcal{PD}_1^{2n}$. *Then, for any* $\tau : S^{2n-1} \to F(n)$,

$$\Omega(\mathcal{G}_\tau(N)\#E_k) \simeq \Omega E_k \times \Omega(\Omega E_k \ltimes (N_0 \rtimes S^{n-1})).$$

Further, one has the equivalence $\Omega E_k \simeq \Omega(S^n \times S^{2n-1}) \times \Omega(J \vee J \wedge \Omega(S^n \times S^{2n-1}))$, *where* $J \simeq (S^n \vee S^{2n-1})^{\vee k-2}$.

Theorem 3.2.29. *Let* $n \in \{2, 4, 8\}$. *Suppose* $M_k \in \mathcal{PD}_{n-1}^{2n}$ *with* $H_n M_k \cong \mathbb{Z}^k$ *and* $k \geq 2$, *and* $N \in \mathcal{PD}_1^{2n}$. *Let* $E_k = \#^{k-1}(S^n \times S^{2n-1})$. *Then, we have the homotopy equivalence*

$$\Omega(N\#M_k) \simeq S^{n-1} \times \Omega E_k \times \Omega(\Omega E_k \ltimes Y)$$

where $Y \simeq (N_0 \rtimes S^{n-1})$, *after inverting the primes in* T_n *(that is, those occuring in the torsion part of* $\pi_{2n-1}(S^n)$ *together with the prime* 2*).*

Proof. The proof follows directly from Theorem 3.1.5 and Propositions 3.2.28 and 3.2.24.

□

Theorem 3.2.30. *Let* $n \in \{2, 4, 8\}$. *Suppose* $M_k \in \mathcal{PD}_{n-1}^{2n}$ *with rank* $(H_n M_k) = k > r$, *where* $r = \#$*cyclic torsion summands in* π_{n-1}^s, *and* $N \in \mathcal{PD}_1^{2n}$. *Then after inverting* 2, *we have the homotopy equivalence*

$$\Omega(N\#M_k) \simeq S^{n-1} \times \Omega E_k \times \Omega(\Omega E_k \ltimes Y)$$

where $Y \simeq (N_0 \rtimes S^{n-1})$.

Proof. The proof follows directly from the Theorem 3.2.7 and Propositions 3.2.28 and 3.2.24.

□

For $n = 2,\ 4$, we do not have to invert 2, and we have

Theorem 3.2.31. *Suppose* $M_k \in \mathcal{PD}_{n-1}^{2n}$ *with rank* $(H_n M_k) = k \geq 2$, *and* $n \in \{2, 4\}$, *and* $N \in \mathcal{PD}_1^{2n}$. *Then, we have the homotopy equivalence*

$$\Omega(N\#M_k) \simeq S^{n-1} \times \Omega E_k \times \Omega(\Omega E_k \ltimes Y)$$

where $Y \simeq (N_0 \rtimes S^{n-1})$.

3.2.32. Decomposition of looped configuration spaces. The loop space decomposition of a space has many applications, one of them being in the case of configuration spaces. We note this down in the examples treated above. Recall that the ordered configration space of X is given by $F_k(X) = \{(x_1, \ldots, x_k) \mid x_i \neq x_j \text{ if } i \neq j\}$.

Definition 3.2.33. [17] Let $\pi \colon F_k(M) \to M$ be the projection onto the first factor. The space M is said to be a σ_k-manifold if π admits a cross section.

If M is a σ_k-manifold, [17, Theorem 2.1] implies the homotopy equivalence

$$\Omega F_k(M) \simeq \Omega M \times \Omega(M - Q_1) \times \cdots \times \Omega(M - Q_k), \tag{3.2.34}$$

for any choice of k distinct points $q_1, \cdots, q_k$ of M with $Q_i = \{q_1, \cdots, q_i\}$. The hypothesis of being a σ_k-manifold is satisfied if M has a nowhere vanishing vector field [20, Theorem 5]. By the Poincaré-Hopf index theorem, this is satisfied if the Euler characteristic $\chi(M) = 0$. Putting all this together we obtain the following result for $N\#M$ where M is a $(n-1)$-connected $2n$-manifold.

Theorem 3.2.35. *Let $n \in \{2, 4, 8\}$. Suppose M is an $(n-1)$-connected $2n$-manifold for n even such that Rank$(H_n(M)) = r \geq 2$, and N is a simply connected $2n$-manifold with $\chi(N) = -r$. Then, after inverting the primes in T_n, the homotopy type of $\Omega F_k(N\#M)$ depends only on the homotopy type of $N - *$ and the integer r. More precisely we have the decomposition*

$$\Omega F_k(N\#M) \simeq S^{n-1} \times \Omega E \times \Omega(\Omega E \ltimes Y) \times \Omega(N_0 \vee (S^n)^{\vee r}) \times \prod_{i=1}^{k-1} \Omega(N_0 \vee (S^n)^{\vee r} \vee (S^{2n-1})^{\vee i})$$

in which $Y \simeq (N_0 \rtimes S^{n-1}), E = \#^{r-1}(S^n \times S^{2n-1})$.

Proof. The hypothesis $\chi(N) = -r$ implies $\chi(M\#N) = 0$. Thus, (3.2.34) applies to give

$$\Omega F_k(N\#M) \simeq \Omega(N\#M) \times \Omega(N\#M - Q_1) \times \cdots \times \Omega(N\#M - Q_k)$$

We now have the equivalence

$$\Omega(N\#M) \simeq S^{n-1} \times \Omega E \times \Omega(\Omega E \ltimes Y)$$

by Theorem 3.2.29. The other factors in the product are observed via the equivalences

$$N\#M - Q_i \simeq (N\#M - \text{pt}) \vee (S^{2n-1})^{\vee i-1} \simeq N_0 \vee (S^n)^{\vee r} \vee (S^{2n-1})^{\vee i-1}$$

for $i \geq 1$. □

Finally, one may observe that slightly more general versions of Theorem 3.2.35 are provable using Theorems 3.2.30 and 3.2.31.

Chapter 4

Sphere fibrations in low dimensional cases

In this chapter , a part of the paper [11], we inspect the conclusions of Theorem 3.1.5 in low dimensions, specifically when $n \leq 8$. The first case is when $n = 2$, where the complexes in $\mathcal{PD}_1^4$ are simply connected 4-manifolds. Throughout this chapter we use the Jacobi identity for Whitehead products [46, Cor 7.13]

$$\begin{aligned}(-1)^{pr}\big[[f,g],h\big] + (-1)^{pq}\big[[g,h],f\big] + (-1)^{rq}\big[[h,f],g\big] = 0,\\ \text{for } f \in \pi_p(X),\ g \in \pi_q(X),\ h \in \pi_r(X).\end{aligned} \tag{4.0.1}$$

Applying (4.0.1) together with the skew symmetry for Whitehead products ($[f,g] = (-1)^{pq}[g,f]$ for $f \in \pi_p(X)$, $g \in \pi_q(X)$), we obtain

$$\big[[\alpha_i,\alpha_j],\alpha_k\big] = -\big[\alpha_i,[\alpha_j,\alpha_k]\big] - \big[\alpha_j,[\alpha_i,\alpha_k]\big]. \tag{4.0.2}$$

4.1 Simply connected four manifolds

The simply connected 4-manifolds M_k for $k \geq 2$ (which are defined by $H_2(M_k) \cong \mathbb{Z}^k$), support a principal S^1-bundle whose total space is $E_k \simeq \#^{k-1}(S^2 \times S^3)$[8, 19]. The proof of this result relies on Smale's classification of spin 5-manifolds. Theorem 3.1.5 provides a homotopy theoretic method to approach the situation. We first point out the argument in a simple case.

Example 4.1.1. *Let $M = \mathbb{C}P^2\#\overline{\mathbb{C}P^2}$, which implies that M is the mapping cone of $S^3 \xrightarrow{\eta_1-\eta_2} S^2 \vee S^2$. Note that $\pi_3 S^2 \cong \mathbb{Z}$ implies that $T_2 = \{2\}$. The argument in Example 3.1.3 now implies that after inverting 2, we have a fibration $S^1 \to S^2 \times S^3 \to M$. This is two steps* *away from the geometric argument which implies that inverting* 2 *is not essential, and that the fibration is a principal* S^1*-fibration. Tracing out the formula in Example 3.1.3 and the fact that the triple Whitehead product of the identity map of* S^2 *is* 0*, we are supposed to consider the map*

$$S^2 \vee S^3 \to M, \text{ sending } \mu_1 \mapsto \alpha_1, \delta_1 \mapsto [\alpha_1, \alpha_2].$$

We readily compute (here $\eta_{(3)}$ *is the suspension of the Hopf map which satisfies* $2\eta_{(3)} = 0$*)*

$$\begin{aligned}[\mu_1, \delta_1] \mapsto & [\alpha_1, [\alpha_1, \alpha_2]] \\ & = [\alpha_1, \alpha_2] \circ \eta_{(3)} - [\eta_1, \alpha_2] \text{ by [24, Theorem 6.1]} \\ & = [\alpha_1, \alpha_2] \circ \eta_{(3)} - [\eta_1 - \eta_2, \alpha_2] - [\eta_2, \alpha_2] \\ & = [\alpha_1, \alpha_2] \circ \eta_{(3)} \text{ as } [\eta_2, \alpha_2] = 0 \\ & \neq 0.\end{aligned}$$

However, one may easily check that a slight tinkering of the formula :

$$\mu_1 \mapsto \alpha_1 + \alpha_2, \delta_1 \mapsto \eta_2$$

does indeed yield (using the fact that $[\eta_1, \alpha_2] = [\eta_1 - \eta_2, \alpha_2] = 0$ *and analogous formulas)*

$$\begin{aligned}[\mu_1, \delta_1] \mapsto & [\alpha_1 + \alpha_2, \eta_2] \\ & = [\alpha_1, \eta_2] \text{ as } [\alpha_2, \eta_2] = 0 \\ & = 0.\end{aligned}$$

Once we obtain the map $S^2 \times S^3 \to M$*, the spectral sequence argument of Proposition 3.1.7 implies that the homotopy fibre is* S^1*. In order to prove that this is indeed a principal fibration, note that a* $w \in H^2(M)$ *which satisfies* $\langle w, \alpha_1 + \alpha_2 \rangle = 0$*, is represented by a map* $M \to \mathbb{C}P^\infty$ *such that the composite* $S^2 \times S^3 \to M \to \mathbb{C}P^\infty$ *is null. It follows that* $S^2 \times S^3$ *maps to the homotopy fibre of* $w : M \to \mathbb{C}P^\infty$ *which is easily deduced to be an equivalence. Hence,* $S^1 \to S^2 \times S^3 \to M$ *becomes a principal fibration.*

In the general case we achieve the result by an analogous computation to Example 4.1.1. We are required to choose the basis $\alpha_1, \cdots, \alpha_{k-1}$ judiciously so that the formulas in Theorem 3.1.5 yield an integral result. This kind of choice was also made in [8, 19].

Theorem 4.1.2. *Let* M_k *be a simply connected* 4*-manifold with* $H_2 M_k \cong \mathbb{Z}^k$*. Then, there is a principal* S^1*-fibration* $S^1 \to E_k \to M_k$ *where* $E_k \simeq \#^{k-1}(S^2 \times S^3)$*.*

Proof. We prove for the even intersection here and the odd case is given in the next remark. Let $w \in H^2(M_k)$ be a class that reduces to $w_2(M_k)$ modulo 2, and choose $\alpha_1, \cdots, \alpha_{k-1}$ to be a linearly independent set spanning a summand of the kernel of the linear map $H_2(M_k) \to \mathbb{Z}$ induced by w. In the notation of (2.2.2), the matrix $\big((g_{i,j})\big)$ is the inverse of the matrix of the intersection form, and so by this choice $g_{i,i}$ is even for $i \leq k-1$. This implies $g_{i,i}[\eta_i, \alpha_j] = -g_{i,i}\big[\alpha_i, [\alpha_i, \alpha_j]\big]$ by [24, Theorem 6.1]. We define the β-classes as (where $\eta_j = \alpha_j \circ \eta$)

$$\beta_i = \sum_{j=1}^{k-1} g_{i,j}[\alpha_j, \alpha_k] + \sum_{j=1}^{i-1} g_{i,j}[\alpha_i, \alpha_j] + \sum_{j=i+1}^{k} g_{i,j}\eta_j. \tag{4.1.3}$$

Then, we have (noting that $L(M_k) = \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \alpha_j] + \sum_{i=1}^{k} g_{i,i}\eta_i$ (**??**) and the formula (4.0.2))

$$\begin{aligned}
\sum_{i=1}^{k-1}[\alpha_i, \beta_i] &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} g_{i,j}\big[\alpha_i, [\alpha_j, \alpha_k]\big] + \sum_{1\leq j<i\leq k-1} g_{i,j}\big[\alpha_i, [\alpha_i, \alpha_j]\big] + \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \eta_j] \\
&= \sum_{1\leq i<j\leq k-1} -g_{i,j}\big[[\alpha_i, \alpha_j], \alpha_k\big] - \sum_{i=1}^{k-1} g_{i,i}[\eta_i, \alpha_k] + \sum_{1\leq j<i\leq k-1} g_{i,j}\big[\alpha_i, [\alpha_i, \alpha_j]\big] \\
&\quad - \sum_{1\leq i<j\leq k} g_{i,j}\big[[\alpha_i, \alpha_j], \alpha_j\big] + \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \alpha_j] \circ \eta_{(3)} \text{ by [24, Theorem 6.1]} \\
&= -[L(M_k), \alpha_k] + L(M_k) \circ \eta_{(3)}] - g_{k,k}\eta_k \circ \eta_{(3)}, \text{ as } \big[[\alpha_k, \alpha_k], \alpha_k\big] = 0
\end{aligned}$$

Let $E_k = \#^{k-1}(S^2 \times S^3)$. Proceeding as in the proof of Theorem 3.1.1, we obtain a map $f : E_k \to M_k$ sending μ_i to α_i and δ_i to β_i. One directly observes that the image of the β_i in

$$H_{2n-2}(\Omega M_k)/\Big(\mathbb{Z}\{a_1, \cdots, a_{k-1}\} \cdot H_{n-1}(\Omega M_k)\Big)$$

equals $-b_i$ (of Theorem 3.1.1). The spectral sequence argument of Proposition 3.1.7 now applies to yield that the homotopy fibre of f is S^1. The argument in the last paragraph of Example 4.1.1 may now be repeated to deduce that this is a principal fibration. ☐

Remark 4.1.4. *Note that the formula for the* β_i *in* (4.1.3) *may be simplified in the case where* $w_2(M_k) \neq 0$. *Here, by choosing* α_k *such that* $\pmod 2$, α_k *is the Poincaré dual of* $w_2(M_k)$, *we have from [16, Lemma 2.4] that* $g_{i,i} \equiv g_{i,k} \pmod 2$ *for* $i \leq k-1$. *Then, the formula*

$$\beta_i = \sum_{j=1}^{k-1} g_{i,j}[\alpha_j, \alpha_k] + g_{i,k}\eta_k \tag{4.1.5}$$

gives

$$\sum_{i=1}^{k-1}[\alpha_i,\beta_i] = -[L(M_k),\alpha_k].$$

4.2 3-connected 8-manifolds

As in the case of simply connected 4-manifolds, we search for integral versions of the Theorem 3.1.5. For this we require some results about the Whitehead products in the homotopy groups of S^4 [43, 24]. Recall that ι_4 is the homotopy class of the identity map $S^4 \to S^4$.

$$\pi_7 S^4 \cong \mathbb{Z}\{\nu\} \oplus \mathbb{Z}/(12)\{\nu'\}, \quad [\iota_4,\iota_4] = 2\nu+\nu', \tag{4.2.1}$$

where $\nu =$ Hopf construction on the quaternionic multiplication. We also have using [24, §4] and [28, (3.7)]

$$\begin{aligned} \pi_{10} S^4 \cong \pi_{10}(S^7)\oplus\pi_9(S^3) = \mathbb{Z}/(24)\{x\}\oplus\mathbb{Z}/3\{y\}, \\ x = \nu\circ\nu_{(7)},\ y = \nu'\circ\nu_{(7)} = \nu'\circ\nu'_{(7)},\ \nu'_{(7)} = -2\nu_{(7)}, \\ [\iota_4,\iota_4]\circ\nu_{(7)} = 2x+y,\ [\nu',\iota_4] = -4x+y, \\ [\nu,\iota_4] = 2x,\ \big[[\iota_4,\iota_4],\iota_4\big] = y. \end{aligned} \tag{4.2.2}$$

We start with an example.

Example 4.2.3. *Let $M = \mathbb{H}P^2\#\overline{\mathbb{H}P^2}$, which is the mapping cone of $S^7 \overset{\nu_1-\nu_2}{\to} S^4\vee S^4$. We let α_1 and α_2 denote the two wedge summands of $S^4\vee S^4$. Form a map $E_2\to M$ ($E_2 = S^4\times S^7$) by*

$$\mu_1 \mapsto \alpha_1-\alpha_2, \text{ and } \delta_1 \mapsto \nu_2.$$

Observe that

$$\begin{aligned} [\mu_1,\delta_1] &\mapsto [\alpha_1-\alpha_2,\nu_2] \\ &= [\alpha_1,\nu_2]-[\alpha_2,\nu_2] \\ &= 2x_1-[L(M),\alpha_1]-2x_2 \\ &= -L(M)\circ\nu'_{(7)}-[L(M),\alpha_1], \end{aligned}$$

so that we get a map $E_2\to M$. Now it is easily checked that this satisfies the hypothesis of Proposition 3.1.7, and therefore, we deduce that the homotopy fibre of the map is S^3. We have thus constructed a fibration $S^3\to S^4\times S^7\to \mathbb{H}P^2\#\overline{\mathbb{H}P^2}$. We also note that this is a principal fibration as it is given by the pullback of the map $\mathbb{H}P^2\#\overline{\mathbb{H}P^2}\to\mathbb{H}P^\infty$ via the map which on π_4 sends both α_1 and α_2 to the generator.

The computations get more involved once we allow factors of ν' in the expression of $L(M)$. The following example deals with the rank 2 case for an even intersection form, which we may assume to be the hyperbolic form by the classification [33, Ch II (2.2)].

Example 4.2.4. *Consider $M_2 \in \mathcal{PD}_3^8$ determined by*

$$L(M_2) = [\alpha_1, \alpha_2] + l_1\nu_1' + l_2\nu_2'.$$

Note that if l_1 and l_2 are both 1, we cannot obtain a map $M_2 \to \mathbb{H}P^\infty$ which on π_4 sends $\alpha_1 \mapsto n_1\iota$ and $\alpha_2 \mapsto n_2\iota$ with $\gcd(n_1, n_2) = 1$. This is because $L(M_2) \mapsto n_1n_2\nu + (n_1n_2+n_1+n_2)\nu'$, and for coprime n_1, n_2, $n_1n_2 + n_1 + n_2$ must be odd. It follows that there is no principal S^3-fibration over M_2 in which the total space is 3-connected. However, we can verify that for all possible values of l_1 and l_2, there exist fibrations $S^3 \to S^4 \times S^7 \to M_2$.

For constructing the fibrations, we note it suffices to find μ_1, δ_1 such that $[\mu_1, \delta_1] = 0 \in \pi_{10}(M_2)$, and the image of δ_1 in the homology of the loop space satisfies the hypothesis of Proposition 3.1.7. We also note the symmetry between the factors α_1 and α_2, and also by replacing both by it's negative that $l_i \mapsto -l_i$ for $i = 1, 2$. Modulo these symmetries, the following formulas for μ_1 and δ_1 satisfy the required criteria.

$$l_1 \equiv 0 \pmod{6}, l_2 \equiv 0 \pmod{3} \;:\; \mu_1 = \alpha_2, \delta_1 = \nu_1.$$

$$l_1 \equiv 3 \pmod{6}, l_2 \equiv 0 \pmod{3} \;:\; \mu_1 = 6\alpha_1 + \alpha_2, \delta_1 = \nu_1.$$

$$l_1 \equiv 0 \pmod{6}, l_2 \equiv 2 \pmod{3} \;:\; \mu_1 = 4\alpha_1 + \alpha_2, \delta_1 = 63\nu_1 + 4\nu_2.$$

$$l_1 \equiv 3 \pmod{6}, l_2 \equiv 2 \pmod{3} \;:\; \mu_1 = 10\alpha_1 + \alpha_2, \delta_1 = 399\nu_1 + 4\nu_2.$$

$$l_1 \equiv 1 \pmod{6}, l_2 \equiv 1 \pmod{3} \;:\; \mu_1 = 2\alpha_1 + \alpha_2, \delta_1 = 175\nu_1 + 44\nu_2.$$

$$l_1 \equiv 4 \pmod{6}, l_2 \equiv 4 \pmod{6} \;:\; \mu_1 = 8\alpha_1 + \alpha_2, \delta_1 = 257\nu_1 + 4\nu_2.$$

$$l_1 \equiv 1 \pmod{6}, l_2 \equiv 2 \pmod{3} \;:\; \mu_1 = 10\alpha_1 + \alpha_2, \delta_1 = 401\nu_1 + 4\nu_2 - \nu_1'.$$

$$l_1 \equiv 4 \pmod{6}, l_2 \equiv 2 \pmod{3} \;:\; \mu_1 = 4\alpha_1 + \alpha_2, \delta_1 = 127\nu_1 + 8\nu_2 + \nu_1'.$$

We first construct the fibration when the intersection form is not even, so that an appropriate analogue of the formula in Remark 4.1.4 works. Let $M_k \in \mathcal{PD}_3^8$ with $H_4(M_k) = \mathbb{Z}^k$. Assume that $w_4(M_k) \neq 0$ which is equivalent to the intersection form not being even. Choose a basis

$\alpha_1, \cdots, \alpha_k$ of $H_4(M_k)$ such that

$$\langle \alpha_i, \alpha_k \rangle = \langle \alpha_i, \alpha_i \rangle \pmod 2 \text{ for } 1 \leq i \leq k-1, \tag{4.2.5}$$

where $\langle -, - \rangle$ is the intersection form. This is satisfied if α_k is the Poincaré dual of $w_4(M_k)$ $\pmod 2$ by an analogous argument to [16, Lemma 2.4]. We write $L(M_k)$ as

$$L(M_k) = \sum_{1 \leq i < j \leq k} g_{i,j}[\alpha_i, \alpha_j] + \sum_{i=1}^{k} g_{i,i}\nu_i + \sum_{i=1}^{k} l_i \nu_i'. \tag{4.2.6}$$

Proposition 4.2.7. *Suppose that $\alpha_1, \cdots, \alpha_k$ satisfy (4.2.5), and either*

(A) $6 \mid l_k$, or

(B) $4 \nmid g_{k,k}$ and $3 \mid l_k$,

with $g_{k,k}$ and l_k as in (4.2.6). Under this assumption we may choose $\beta_i \in \pi_{2n-1}(M_k)$ $1 \leq i \leq k-1$ such that $\sum_{i=1}^{k-1} [\alpha_i, \beta_i] = 0$, and the hypothesis of Proposition 3.1.7 is satisfied.

Proof. Consider the formula (4.1.5) and write

$$\beta_i' = \sum_{j=1}^{k-1} g_{ij}[\alpha_j, \alpha_k] + g_{i,k}\nu_k. \tag{4.2.8}$$

We then have using (4.2.2) and [24, Theorem 6.1]

$$\begin{aligned}
\sum_{i=1}^{k-1} [\alpha_i, \beta_i'] &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} g_{ij}\left[\alpha_i, [\alpha_j, \alpha_k]\right] + \sum_{i=1}^{k-1} g_{i,k}[\alpha_i, \nu_k] \\
&= -\sum_{1 \leq i < j \leq k} g_{i,j}\left[[\alpha_i, \alpha_j], \alpha_k\right] - \sum_{i=1}^{k-1} g_{i,i}[\nu_i, \alpha_k] + \sum_{i=1}^{k-1} (g_{i,i} + g_{i,k})[\alpha_i, \alpha_k] \circ \nu_{(7)} \\
&= -[L(M_k), \alpha_k] + g_{k,k}[\nu_k, \alpha_k] + \sum_{i=1}^{k} l_i[\nu_i', \alpha_k] - \sum_{i=1}^{k-1} \left(\frac{g_{i,i} + g_{i,k}}{2}\right)[\alpha_i, \alpha_k] \circ \nu_{(7)}' \\
&= -[L(M_k), \alpha_k] + 2g_{k,k}x_k - 4l_k x_k + l_k y_k + \sum_{i=1}^{k-1} l_i[\alpha_i, \nu_k'] - \sum_{i=1}^{k-1} \left(\frac{g_{i,i} + g_{i,k}}{2}\right)[\alpha_i, \nu_k'].
\end{aligned}$$

Note that $3 \mid l_k$ implies $l_k y_k = 0$. If $6 \mid l_k$, we also have $4 l_k x_k = 0$. Otherwise, the condition $4 \nmid g_{k,k}$ implies that $4l_k \equiv 2g_{k,k}r \pmod{24}$. We now rewrite using $l_k \nu'_k \circ \nu'_{(7)} = l_k y_k = 0$

$$\begin{aligned}(2g_{k,k} - 4l_k)x_k &= -(1-r)g_{k,k}\nu_k \circ \nu'_{(7)} \\ &= -(1-r)\Big[L(M_k) \circ \nu'_{(7)} - \sum_{1 \le i < j \le k} g_{i,j}[\alpha_i, \alpha_j] \circ \nu'_{(7)} - \sum_{i=1}^{k-1} g_{i,i}\nu_i \circ \nu'_{(7)} - \sum_{i=1}^{k-1} l_i \nu'_i \circ \nu'_{(7)}\Big] \\ &= -(1-r)\Big[L(M_k) \circ \nu'_{(7)} - \sum_{1 \le i < j \le k} g_{i,j}[\alpha_i, \nu'_j] + \sum_{i=1}^{k-1} g_{i,i}[\alpha_i, \nu_i] - \sum_{i=1}^{k-1} l_i \big[\alpha_i, [\alpha_i, \alpha_i]\big]\Big].\end{aligned}$$

Now we define β_i by perturbing the β'_i of (4.2.8)

$$\beta_i = \beta'_i - \big(l_i - \frac{g_{i,i} + g_{i,k}}{2}\big)\nu'_k - (1-r)\sum_{j=i+1}^{k} g_{i,j}\nu'_j + (1-r)g_{i,i}\nu_i - (1-r)l_i[\alpha_i, \alpha_i],$$

to get

$$\sum_{i=1}^{k-1}[\alpha_i, \beta_i] = -[L(M_k), \alpha_k] - (1-r)L(M_k) \circ \nu'_{(7)}.$$

We also verify easily that β_i satisfy the requirements of Proposition 3.1.7. ☐

Proposition 4.2.7 becomes applicable once we show that the hypotheses (4.2.5) and $3 \mid l_k$ are always achievable. This is the subject of the following lemma.

Lemma 4.2.9. *There is a choice of basis of $H_n(M_k)$ for $k \geq 2$ such that (4.2.5) holds, and either (A) or (B) of Proposition 4.2.7 is satisfied.*

Proof. We already have that if α_k is the Poincaré dual of $w_4(M_k) \pmod 2$, (4.2.5) is satisfied. We now assume that the basis α_i is chosen such that the induced inner product is diagonal modulo 3. We note that the change of l_i with a change of basis is not linear, however the following change of basis formulas hold by applying (4.2.2).

$$(\alpha_k \mapsto -\alpha_k) \to (l_k \mapsto -l_k + g_{k,k}).$$

$$\begin{pmatrix} \alpha_{k-1} \mapsto \alpha_{k-1} - \alpha_k \\ \alpha_k \mapsto \alpha_k \end{pmatrix} \to (l_k \mapsto l_k + l_{k-1} + g_{k,k-1}).$$

$$\begin{pmatrix} \alpha_{k-1} \mapsto \alpha_{k-1} - 2\alpha_k \\ \alpha_k \mapsto \alpha_k \end{pmatrix} \to (l_k \mapsto l_k + 2l_{k-1} + g_{k-1,k-1} + 2g_{k,k-1}).$$

It is now clear that a combination of the above manoeuvres allow us to arrange for $3 \mid l_k$.

Suppose that the basis that the numbers $g_{k,k}$ and l_k obtained above satisfy $4 \mid g_{k,k}$ and $2 \nmid l_k$. As the intersection form is odd, we must have i such that $g_{i,i}$ is odd. The transformation

$$\begin{pmatrix} \alpha_i \mapsto \alpha_i - 6\alpha_k \\ \alpha_k \mapsto \alpha_k \end{pmatrix} \to (l_k \mapsto l_k + 6l_i + 15g_{i,i} + 6g_{k,i}),$$

allows us to make $6 \mid l_k$. ☐

Now consider the case when the intersection form is even. Assuming $L(M_k)$ as in (4.2.6) we note that this implies that the $g_{i,i}$ are even for all i. We now adapt the formula (4.1.3) in this case to prove the following result.

Proposition 4.2.10. *Suppose that $k \geq 2$, $24 \mid g_{k,k}$, and $l_k = 0$. Under this assumption we may choose $\beta_i \in H_{2n-1}(M_k)$ such that the hypothesis of Proposition 3.1.7 is satisfied.*

Proof. We define the classes

$$\beta_i' = \sum_{j=1}^{k-1} g_{ij}[\alpha_j, \alpha_k] + \sum_{j=1}^{i-1} g_{i,j}[\alpha_i, \alpha_j] + \sum_{j=i+1}^{k} g_{i,j}\nu_j - l_i \nu_k'.$$

With this choice we compute using (4.2.2) as in Proposition 4.2.7

$$\begin{aligned}
\sum_{i=1}^{k-1}[\alpha_i, \beta_i'] &= \sum_{i=1}^{k-1}\sum_{j=1}^{k-1} g_{ij}\left[\alpha_i, [\alpha_j, \alpha_k]\right] + \sum_{1\leq j<i\leq k-1} g_{i,j}\left[\alpha_i, [\alpha_i, \alpha_j]\right] \\
&+ \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \nu_j] - \sum_{i=1}^{k-1} l_i[\alpha_i, \nu_k'] \\
&= \sum_{1\leq i<j\leq k-1} -g_{i,j}\left[[\alpha_i, \alpha_j], \alpha_k\right] - \sum_{i=1}^{k-1} g_{i,i}[\nu_i, \alpha_k] + \sum_{i=1}^{k-1} g_{i,i}[\alpha_i, \alpha_k] \circ \nu_{(7)} - \sum_{i=1}^{k-1} l_i[\nu_i', \alpha_k] \\
&+ \sum_{1\leq j<i\leq k-1} g_{i,j}\left[\alpha_i, [\alpha_i, \alpha_j]\right] - \sum_{1\leq i<j\leq k} g_{i,j}\left[[\alpha_i, \alpha_j], \alpha_j\right] + \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i, \alpha_j] \circ \nu_{(7)} \\
&= -[L(M_k), \alpha_k] + g_{k,k}[\nu_k, \alpha_k] + L(M_k) \circ \nu_{(7)} + \sum_{i=1}^{k-1} g_{i,i}[\alpha_i, \alpha_k] \circ \nu_{(7)} \\
&- \sum_{i=1}^{k-1} g_{i,i}x_i - \sum_{i=1}^{k-1} l_i y_i - g_{k,k}x_k \\
&= -[L(M_k), \alpha_k] + L(M_k) \circ \nu_{(7)} + \sum_{i=1}^{k-1} g_{i,i}[\alpha_i, \alpha_k] \circ \nu_{(7)} - \sum_{i=1}^{k-1} g_{i,i}x_i - \sum_{i=1}^{k-1} l_i y_i + g_{k,k}x_k.
\end{aligned}$$

As $24 \mid g_{k,k}$, $g_{k,k}x_k = 0$. We write

$$\begin{aligned} g_{i,i}[\alpha_i,\alpha_k]\circ\nu_{(7)} &= -\frac{g_{i,i}}{2}[\alpha_i,\alpha_k]\circ\nu'_{(7)} \\ &= -\frac{g_{i,i}}{2}[\alpha_i,\nu'_k]. \end{aligned}$$

We now define β_i by perturbing β'_i as

$$\beta_i = \beta'_i - \frac{g_{i,i}}{2}\cdot\nu_i + \frac{g_{i,i}}{2}\cdot\nu'_k - l_i[\alpha_i,\alpha_i],$$

and from the formulas (4.2.2), it follows that

$$\sum_{i=1}^{k-1}[\alpha_i,\beta_i] = -[L(M_k),\alpha_k] - L(M_k)\circ\nu_{(7)}.$$

Clearly the β_i satisfy the hypothesis of Proposition 3.1.7. □

We now summarize the computations in the following theorem.

Theorem 4.2.11. *Let $M_k \in \mathcal{PD}^8_3$, that is, $H_4(M_k) = \mathbb{Z}^k$ for $k \geq 2$. Such an M_k supports a S^3-fibration $S^3 \to E_k \to M_k$ with $E_k \simeq \#^{k-1}(S^4\times S^7)$.*

Proof. We follow the proof of Theorem 3.1.5. The choice of α_i and β_i are made in Proposition 4.2.7 and Lemma 4.2.9 if the intersection form is not even, and in Proposition 4.2.10 if the intersection form is even. In the latter case, we need to arrange that $24 \mid g_{k,k}$. For this, we use the fact that $g_{i,i}$ is even to rewrite

$$L(M_k) = \sum_{1\leq i<j\leq k} g_{i,j}[\alpha_i,\alpha_j] + \sum_{i=1}^{k}\frac{g_{i,i}}{2}[\alpha_i,\alpha_i] + \sum_{i=1}^{k} s_i\nu'_i.$$

Note that the s_i change linearly with $\alpha_i \pmod{12}$. Now we write

$$s_1\nu'_1 + \cdots + s_k\nu'_k = d\tau$$

where d equals the greatest common divisor of the s_i. Extending τ to a basis we assume that $s_i = 0$ if $i \geq 2$. Consider the summand spanned by the last $k-1$ basis elements. By Lemma 3.1.4, we obtain a primitive $v \neq 0$ such that $\langle v,v\rangle$ is divisible by 24 if $k \geq 6$. Extend τ, v to a basis, to verify the required criteria. For $k \leq 5$, we are done by the classification in [33, Ch II (2.2)]. More precisely, if $k \leq 5$, k can be 2 or 4, and in each case the intersection form is a direct sum of copies of the hyperbolic form. If $k = 2$, we are done by Example 4.2.4. If

$k = 4$, we consider the τ above and note that by adding multiples of 12 to the s_i, and the fact that their $\gcd$ is 1, we may choose two s_{i_1}, s_{i_2} among $s_1, \cdots s_4$ that are relatively prime. Choosing $v = s_j$, for $j \notin \{i_1, i_2\}$, allows us to proceed as before. Therefore, we have a fibration $E_k \to M_k$ for $E_k \simeq \#^{k-1}(S^4 \times S^7)$ with fibre S^3. ☐

4.3 Other examples

4.3.1. 7-**connected** 16-**manifolds.** One may approach results for $\mathcal{PD}_7^{16}$ in an analogous manner to those for $\mathcal{PD}_3^8$, and expect that the existence of the Hopf invariant one class $\sigma \in \pi_{15}(S^8)$ will allow us to construct integral versions of Theorem 3.1.5. However, this is not the case. First we note some formulas for Whitehead products and compositions in the homotopy groups of S^8. [43, (5.16)], [34, (7.4)]

$$\begin{aligned} \pi_{15}S^8 &\cong \mathbb{Z}\{\sigma\} \oplus \mathbb{Z}/(120)\{\sigma'\}, \quad [\iota_8, \iota_8] = 2\sigma - \sigma', \\ \pi_{22}S^8 \cong \pi_{22}(S^{15}) \oplus \pi_{21}(S^7) &= \mathbb{Z}/(240)\{z\} \oplus \mathbb{Z}/(24)\{u\} \oplus \mathbb{Z}/4\{v\}, \\ z &= \sigma \circ \sigma_{(15)}, \; u = \sigma' \circ \sigma_{(15)}, \\ [\sigma, \iota_8] &= 2z - u \pm 8u, \; \big[[\iota_8, \iota_8], \iota_8\big] = \pm 8u. \end{aligned} \tag{4.3.2}$$

From the formula (4.3.2), one can easily deduce via a direct calculation that it is not possible to construct a fibration $S^7 \to S^8 \times S^{15} \to \mathbb{O}P^2 \# \overline{\mathbb{O}P^2}$. For example, the formulas in Example 4.2.3 does not generalize here, as the terms $[\iota_8, \sigma]$ contain multiples of u which are not expressible in the form $\sigma \circ -$. It is possible to lay down conditions under which the formulas do yield the desired fibrations. We leave the study of these integral fibrations for a future publication.

4.3.3. 5-**connected** 12-**manifolds.** For manifolds in $\mathcal{PD}_5^{12}$ we know that the fibration $S^5 \to E_k \to M_k$ cannot exist over the integers as there is no Hopf invariant one class in $\pi_{11}(S^6)$. Therefore we have to invert 2. However, $\pi_{11}S^6 = \mathbb{Z}$, so it is not necessary to invert anything else.

Remark 4.3.4. *Using the calculation of the homotopy groups of sphere in Toda's range, we can say about the primes exactly we need to invert. In the cases of 9-connected 20-manifolds and 17-connected 36-manifolds, it only needs to invert 2. The other cases, upto 19-th stem, require at least one more prime to be inverted.*

Chapter 5

$SU(2)$-bundles over highly connected 8-manifolds

In this chapter, we analyze the possible homotopy types of the total space of a principal $SU(2)-$bundle over a 3-connected 8-dimensional Poincareé duality complex. Along the way, we also classify the 3-connected 11-dimensional complexes E formed from a wedge of S^4 and S^7 by attaching a 11-cell. The results of this chapter appear in the paper [12].

5.1 Homotopy classification of certain 3-connected 11-complexes

We study 3-connected 11-dimensional Poincaré duality complexes E such that $E \setminus \{pt\}$ is homotopic to a wedge of copies of S^4 and S^7. We write $\mathcal{PD}^{11}_{4,7}$ for the collection of such complexes. Our target in this section is to analyze them up to homotopy equivalence. We show that these are classified by numbers λ, ϵ and δ which are explained in detail below.

5.1.1. The rank one case. Let $E \in \mathcal{PD}^{11}_{4,7}$ be such that $E \setminus \{pt\} \simeq S^4 \vee S^7$, that is, Rank$(H_4(E)) = 1$. The homotopy type of E is determined by the attaching map of the top cell, which is an element of

$$\pi_{10}(S^4 \vee S^7) \cong \pi_{10}(S^4) \oplus \pi_{10}(S^7) \oplus \pi_{10}(S^{10}).$$

This must be of the form

$$\phi_{\lambda,\epsilon,\delta} = [\iota_4, \iota_7] + \lambda(\iota_7 \circ \nu_7) + \epsilon(\iota_4 \circ x) + \delta(\iota_4 \circ y), \tag{5.1.2}$$

where $x = \nu_4 \circ \nu_7$ and $y = \nu' \circ \nu_7$. The total space associated with $\phi_{\lambda,\epsilon,\delta}$ is denoted by

$$E_{\lambda,\epsilon,\delta} = (S^4 \vee S^7) \cup_{\phi_{\lambda,\epsilon,\delta}} D^{11}.$$

Note that as $(-\iota_4) \circ \nu = \nu + \nu'$, we have $(-\iota_4) \circ x = x + y$. For given any λ, ϵ and δ; we observe the effect of the self homotopy equivalences on $E_{\lambda,\epsilon,\delta}$ as follows

$$\begin{aligned} \iota_4 \mapsto -\iota_4, \quad \iota_7 \mapsto -\iota_7 \quad & \Longrightarrow E_{\lambda,\epsilon,\delta} \xrightarrow{\simeq} E_{-\lambda,\epsilon,\epsilon-\delta}, \\ \iota_4 \mapsto \iota_4, \quad \iota_7 \mapsto \iota_7 + a\iota_4 \circ \nu \quad & \Longrightarrow E_{\lambda,\epsilon,\delta} \xrightarrow{\simeq} E_{\lambda,\epsilon+(\lambda+2)a,\delta}, \\ \iota_4 \mapsto \iota_4, \quad \iota_7 \mapsto \iota_7 + b\iota_4 \circ \nu' \quad & \Longrightarrow E_{\lambda,\epsilon,\delta} \xrightarrow{\simeq} E_{\lambda,\epsilon-4b,\delta+(\lambda+1)b}. \end{aligned} \tag{5.1.3}$$

This leads us to homotopy equivalences between $E_{\lambda,\epsilon,\delta}$'s depending on the choice of $\lambda \in \pi_{10}(S^7) \cong \mathbb{Z}/24$. Note that the above equivalences imply that it is enough to consider for $\lambda = 0, 1, \ldots, 11$. Table 5.1 lists the different homotopy types in $\mathcal{PD}^{11}_{4,7}$ of rank 1.

λ	$\#E_{\lambda,\epsilon,\delta}$'s	$E_{\lambda,\epsilon,\delta}$'s
0	2	$E_{0,0,0}$, $E_{0,1,0}$
1	3	$E_{1,0,0}$, $E_{1,1,0}$, $E_{1,2,0}$
2	12	$E_{2,0,0}$, $E_{2,1,0}$, $E_{2,2,0}$, $E_{2,3,0}$, $E_{2,0,1}$, $E_{2,1,1}$, $E_{2,2,1}$, $E_{2,3,1}$, $E_{2,0,2}$, $E_{2,1,2}$, $E_{2,2,2}$, $E_{2,3,2}$
3	1	$E_{3,0,0}$
4	6	$E_{4,0,0}$, $E_{4,1,0}$, $E_{4,2,0}$, $E_{4,3,0}$, $E_{4,4,0}$, $E_{4,5,0}$
5	3	$E_{5,0,0}$, $E_{5,0,1}$, $E_{5,0,2}$
6	4	$E_{6,0,0}$, $E_{6,1,0}$, $E_{6,2,0}$, $E_{6,3,0}$
7	3	$E_{7,0,0}$, $E_{7,1,0}$, $E_{7,2,0}$
8	6	$E_{8,0,0}$, $E_{8,1,0}$, $E_{8,0,1}$, $E_{8,1,1}$, $E_{8,0,2}$, $E_{8,1,2}$
9	1	$E_{9,0,0}$
10	12	$E_{10,0,0}$, $E_{10,1,0}$, $E_{10,2,0}$, $E_{10,3,0}$, $E_{10,4,0}$, $E_{10,5,0}$, $E_{10,6,0}$, $E_{10,7,0}$, $E_{10,8,0}$, $E_{10,9,0}$, $E_{10,10,0}$, $E_{10,11,0}$
11	3	$E_{11,0,0}$, $E_{11,0,1}$, $E_{11,0,2}$
12	2	$E_{12,0,0}$, $E_{12,1,0}$

TABLE 5.1: Homotopy equivalence classes of $E_{\lambda,\epsilon,\delta}$.

5.1.4. A simplification of the attaching map. We simplify and reduce the attaching map of the top cell of E.

Proposition 5.1.5. *Let $E \in \mathcal{PD}^{11}_{4,7}$ with Rank$(E) = k-1$. The attaching map ϕ of the top cell of E as in (5.1.2) can be reduced, up to homotopy, to the following form*

$$\phi = \sum_{i=1}^{k-1}[\iota_4^i, \iota_7^i] + \sum_{i=1}^{k-1} \lambda_i \iota_7^i \circ \nu_{(7)} + \sum_{i=1}^{k-1} s_i \nu_i \circ \nu_{(7)} + \sum_{i=1}^{k-1} r_i \nu_i' \circ \nu_{(7)}.$$

Proof. By Hilton-Milnor decomposition, we have the following equivalence

$$\pi_{10}((S^4 \vee S^7)^{\vee k-1}) \cong \pi_{10}(S^4)^{\oplus(k-1)} \oplus \pi_{10}(S^7)^{\oplus(k-1)} \oplus \pi_{10}(S^7)^{\oplus\binom{k-1}{2}} \oplus \\ \pi_{10}(S^{10})^{\oplus(k-1)\times(k-1)} \oplus \pi_{10}(S^{10})^{\oplus\binom{k-1}{3}}.$$

We choose $\eta_1, \dots, \eta_{k-1} \in \pi_4(E)$ and $\gamma_1, \dots, \gamma_{k-1} \in \pi_7(E)$ such that they correspond to the homology generators, say $\tilde{\eta}_1, \dots, \tilde{\eta}_{k-1} \in H_4(E)$ and $\tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1} \in H_7(E)$ such that

$$\tilde{\eta}_i^* \cup \tilde{\gamma}_j^* = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{5.1.6}$$

Let $\tilde{f} \colon (S^4 \vee S^7)^{\vee k-1} \to E$ be the inclusion which sends $\iota_4^i \mapsto \eta_i$ and $\iota_7^i \mapsto \gamma_i$ for $1 \leq i \leq k-1$. Then $\tilde{f} \circ \phi \in \pi_{10}(E)$ whose image under the map $\rho \colon \pi_{10}(E) \to \pi_9(\Omega E) \to H_9(\Omega E)$ is 0 in the tensor algebra $T(\tilde{\eta}_1, \dots, \tilde{\eta}_{k-1}, \tilde{\gamma}_1, \dots, \tilde{\gamma}_{k-1})/(\sum_{i=1}^{k-1}[\tilde{\eta}_i, \tilde{\gamma}_i])$.

The attaching map ϕ may contain triple Whitehead products as $[\iota_i^4, [\iota_j^4, \iota_\ell^4]]$, the Whitehead products of the form $[\iota_i^4, \iota_j^7]$ for $i \neq j$, the terms involving $[\iota_i^4, \iota_j^4] \circ \nu_7$ and $[\iota_i^4, \iota_j^4] \circ \nu'_7$. The triple Whitehead product maps injectively to the loop homology of ΩE and hence they can not occur in the attaching map. The cup product formula in (5.1.6) implies that there is no Whitehead product of the form $[\iota_i^4, \iota_j^7]$ for $i \neq j$. If $[\iota_i^4, \iota_j^4] \circ \nu_7$ and $[\iota_i^4, \iota_j^4] \circ \nu'_7$ appear in the attaching map, we update the map $\tilde{f}$ by appropriately sending $\iota_i^7 \mapsto \gamma_i - \eta_j \circ \nu'$, $\iota_i^7 \mapsto \gamma_i - \eta_j \circ \nu$ and $\iota_j^7 \mapsto \gamma_j - [\eta_i, \eta_j]$ to get the desired form of the attaching map. □

We note that the composition is given by

$$S^{10} \to (S^4 \vee S^7)^{\vee k-1} \to (S^7)^{\vee k-1}$$

which is an element of $(\pi_{10} S^7)^{\oplus k-1} \cong (\mathbb{Z}/24\{\nu\})^{\oplus k-1}$. This can be calculated using the real e-invariant, see [3]. We use this to reduce Proposition 5.1.5 to the case $\lambda_i = 0$ for $i \leq k-2$.

Proposition 5.1.7. *Let $E \in \mathcal{PD}^{11}_{4,7}$ and Rank$(E) = k-1$. Then the attaching map ϕ of the top cell of E can be reduced to the following form*

$$\phi = \sum_{i=1}^{k-1} [\iota_4^i, \iota_7^i] + \lambda \iota_7^{k-1} \circ \nu_{(7)} + \sum_{i=1}^{k-1} \epsilon_i \nu_i \circ \nu_{(7)} + \sum_{i=1}^{k-1} \delta_i \nu'_i \circ \nu_{(7)}. \tag{5.1.8}$$

As a consequence $E \simeq \#_{i=1}^{k-2} E_{0,\epsilon_i,\delta_i} \# E_{\lambda,\epsilon_{k-1},\delta_{k-1}}$.

Proof. Let $\tau\colon H^7(E)\cong\mathbb{Z}\{\tilde{\gamma}_1^*,\tilde{\gamma}_2^*,\dots,\tilde{\gamma}_{k-1}^*\}\to\mathbb{Z}/24$ be the linear map defined by

$$\tau(\tilde{\gamma}_i^*)=e(r_i\circ\phi),\qquad \text{for } 1\leq i\leq k-1,$$

where e denotes for the real e-invariant and $r_i:(S^4\vee S^7)^{\vee k-1}\to S^7$ is the retraction onto the i-th factor. Let $\tilde{\tau}\colon H^7(E)\to\mathbb{Z}$ be the lift of τ and $\lambda=\gcd(\tilde{\tau}(\tilde{\gamma}_1^*),\dots,\tilde{\tau}(\tilde{\gamma}_{k-1}^*))$. Then we can change the generators $\tilde{\gamma}_1,\dots,\tilde{\gamma}_{k-1}$ such that $\tilde{\tau}(\tilde{\gamma}_i)=0$ for $1\leq i<k-1$ and $\tilde{\tau}(\tilde{\gamma}_{k-1})=\lambda$. So, for a suitable choice of dual bases we can have ϕ as in (5.1.8). ☐

5.1.9. A general classification up to homotopy. We now proceed to the classification of elements in $\mathcal{PD}_{4,7}^{11}$. In the connected sum $E_{\lambda_1,\epsilon_1,\delta_1}\#E_{\lambda_2,\epsilon_2,\delta_2}$, we transform $\alpha_1=\alpha_1'+\alpha_2'$ and $\alpha_2=\alpha_2'$ by $A=\begin{pmatrix}1&1\\0&1\end{pmatrix}$. If $2|\lambda_2$, then we transform

$$\beta_1=\beta_1'-\epsilon_1\nu_2-\frac{\lambda_2\epsilon_1}{2}\nu_2',\quad \beta_2=-\beta_1'+\beta_2'-\epsilon_1[\alpha_1',\alpha_2'].$$

Hence by the following expression

$$\begin{aligned}&[\alpha_1,\beta_1]+[\alpha_2,\beta_2]+\lambda_1\beta_1\circ\nu_{(7)}+\lambda_2\beta_2\circ\nu_7+\epsilon_1x_1+\epsilon_2x_2+\delta_1y_1+\delta_2y_2\\ =&[\alpha_1',\beta_1']+[\alpha_2',\beta_2']+(\lambda_1-\lambda_2)\beta_1'\circ\nu_{(7)}+\lambda_2\beta_2'\circ\nu_7+\epsilon_1x_1+(-\epsilon_1+\epsilon_2+2\lambda_2\epsilon_1-\lambda_1\epsilon_1)x_2\\ &\delta_1y_1+(\delta_1+\delta_2+\lambda_2\epsilon_1(1+\lambda_1))y_2,\end{aligned}$$

we conclude

$$E_{\lambda_1,\epsilon_1,\delta_1}\#E_{\lambda_2,\epsilon_2,\delta_2}\simeq E_{\lambda_1-\lambda_2,\epsilon_1,\delta_1}\#E_{\lambda_2,\epsilon_2+\epsilon_1(2\lambda_2-\lambda_1-1),\delta_1+\delta_2+(1+\lambda_1)\epsilon_1\lambda_2}\qquad \text{when } 2|\lambda_2. \tag{5.1.10}$$

Proposition 5.1.11. *For any unit $a\in\mathbb{Z}/24$, we have homotopy equivalence*

$$E_{\lambda,\epsilon,\delta}\#E_{0,0,0}\simeq\begin{cases}E_{a\lambda,\epsilon,\delta}\#E_{0,0,0} & \textit{if } a\equiv 1\pmod 3\\ E_{-a\lambda,\epsilon,\delta}\#E_{0,0,0} & \textit{if } a\equiv 2\pmod 3).\end{cases}$$

Proof. We transform

$$\begin{aligned}\alpha_1&=a\alpha_1'+b\alpha_2' &\qquad \beta_1&=a\beta_1'-24\beta_2'-b\epsilon\nu_2-24b\epsilon\nu_1\\ \alpha_2&=24\alpha_1'+a\alpha_2' & \beta_2&=-b\beta_1'+a\beta_2'-\epsilon b[\alpha_1',\alpha_2']\end{aligned}$$

with $a^2 - 24b = 1$ and calculate $[\alpha_1, \beta_1] + [\alpha_2, \beta_2] + \lambda\beta \circ \nu_{(7)} + \epsilon x_1 + \delta y_2$. This gives the homotopy equivalence

$$E_{\lambda,\epsilon,\delta}\#E_{0,0,0} \simeq E_{a\lambda,a^2\epsilon,a\delta+\binom{a}{2}\epsilon}\#E_{0,-b^2\epsilon-b\lambda\epsilon,b\delta+\binom{b}{2}\epsilon}. \tag{5.1.12}$$

This proves the proposition except for when $\lambda \equiv 0 \pmod 2$ and $\epsilon, b \equiv 1 \pmod 2$. In that case, we have $E_{0,1,0}$ instead of $E_{0,0,0}$ in the second component of the connected sum. From (5.1.10), we get

$$E_{\lambda,\epsilon,\delta}\#E_{0,1,0} \simeq E_{\lambda,\epsilon,\delta}\#E_{0,1+\epsilon(-\lambda-1),\delta} \tag{5.1.13}$$

which we apply following the equivalence in (5.1.12) for λ even and ϵ,b odd. This concludes the proof. ☐

The transformations above further simplify the possibilities of $E \in \mathcal{PD}^{11}_{4,7}$ listed in Proposition 5.1.7.

Proposition 5.1.14. *Let* $E \in \mathcal{PD}^{11}_{4,7}$ *and Rank*$(E) = k - 1$*. Then*

$$E \simeq \#^{k-3}E_{0,0,0}\#E_{0,\hat{\epsilon},0}\#E_{\lambda,\epsilon,\delta}$$

for some $\lambda, \epsilon \in \mathbb{Z}/24$, $\delta \in \mathbb{Z}/3$, $\hat{\epsilon} \in \mathbb{Z}/2$.

Proof. From Proposition 5.1.7, we have the attaching map ϕ as in (5.1.8). Repeated use of the homotopy equivalences in (5.1.3) and (5.1.13) gives the reduced form (5.1.8) as follows

$$\phi = \sum_{i=1}^{k-1}[\iota_4^i, \iota_7^i] + \lambda\iota_7^{k-1} \circ \nu_{(7)} + \hat{\epsilon}x_{k-2} + \epsilon x_{k-1} + \delta y_{k-1},$$

where $\lambda, \epsilon \in \mathbb{Z}/24$, $\delta \in \mathbb{Z}/3$, $\hat{\epsilon} \in \mathbb{Z}/2$. ☐

Corollary 5.1.15. *Let* $E \in \mathcal{PD}^{11}_{4,7}$ *and Rank*$(E) = k - 1$*. Then stably we have*

$$\Sigma^\infty E \simeq \Sigma^\infty(S^4 \vee S^7)^{\vee k-2} \vee \Sigma^\infty Cone(\lambda_s(E)\nu_{(11)} + \epsilon_s(E)x), \tag{5.1.16}$$

where x *is the generator of the stable homotopy group* $\pi_{14}(S^8) \cong \mathbb{Z}/2$ *and* $\lambda_s(E) \in \mathbb{Z}/24$, $\epsilon_s(E) \in \mathbb{Z}/2$*. Moreover, for* $k-1 \geq 2$ *if* $\lambda_s(E) \equiv 1 \pmod 2$ *in* (5.1.16)*, then* $\epsilon_s(E) = 0 \in \mathbb{Z}/2$*.*

Proof. From (5.1.8) and Proposition 5.1.14,

$$\Sigma^4\phi = \lambda\iota_{11}^{k-1} \circ \nu_{(11)} + \hat{\epsilon}\iota_8^{k-2} \circ \nu_{(11)} + \epsilon\iota_8^{k-1} \circ \nu_{(11)}.$$

Note that $\nu_{(11)} \in \pi_3^s(S^0) \simeq \mathbb{Z}/24$ and $\Sigma^4(\nu \circ \nu_{(7)}) \in \pi_6^s(S^0) \simeq \mathbb{Z}/2$ are the generators. If $\hat{\epsilon} = 0 \in \mathbb{Z}/2$, the result readily follows. Otherwise, if $\epsilon = 1, \hat{\epsilon} = 1 \in \mathbb{Z}/2$, we apply the transformation $\iota_8^{k-2} + \iota_8^{k-1} \mapsto \iota_8^{k-1}$. If $\epsilon = 0, \hat{\epsilon} = 1 \in \mathbb{Z}/2$, we interchange ι_8^{k-2} and ι_8^{k-1} to deduce the result. ☐

We see that the stable homotopy type of $E \in \mathcal{PD}_{4,7}^{11}$ is determined by $\lambda_s(E)$ which is a divisor of 24 and $\epsilon_s(E) \in \mathbb{Z}/2$. The following theorem classifies the different homotopy types of E given the values of λ_s and ϵ_s.

Theorem 5.1.17. *Let $E \in \mathcal{PD}_{4,7}^{11}$ and Rank$(E) = k-1$. Then depending on $\lambda_s = \lambda_s(E)$ and $\epsilon_s = \epsilon_s(E)$, the homotopy type of E is determined by the following.*

1. *If λ_s is even and $\epsilon_s = 0$, then*

$$E \simeq \#^{k-2}E_{0,0,0}\#E_{\lambda_s,\epsilon,\delta} \quad \text{where } \epsilon \equiv \epsilon_s \pmod 2.$$

2. *If λ_s is even and $\epsilon_s = 1$, then*

$$\begin{aligned} E &\simeq \#^{k-2}E_{0,0,0}\#E_{\lambda_s,\epsilon,\delta} && \text{where } \epsilon \equiv 1 \pmod 2) \\ \text{or } E &\simeq \#^{k-3}E_{0,0,0}\#E_{0,1,0}\#E_{\lambda_s,\epsilon,\delta} && \text{where } \epsilon \equiv 0 \pmod 2. \end{aligned}$$

3. *If λ_s is odd, then*

$$E \simeq \#^{k-2}E_{0,0,0}\#E_{\lambda_s,\epsilon,\delta} \quad \text{or} \quad E \simeq \#^{k-3}E_{0,0,0}\#E_{0,1,0}\#E_{\lambda_s,\epsilon,\delta}.$$

Further given λ_s, the choices of ϵ and δ are those which are mentioned in Table 5.1.

Proof. We write $Y_1 = \#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon,\delta}$ and $Y_2 = \#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon',\delta'}$, and let $Y_1 \xrightarrow{f} Y_2$ be a homotopy equivalence with homotopy inverse g. We show that in this case the pair (ϵ, δ) is related to (ϵ', δ') by the transformations (5.1.3). There exists a unique (up to homotopy) factorization $(S^7)^{\vee k-1} \to (S^7 \vee S^4)^{\vee k-1} \to Y_2$ through cellular approximation $\tilde{f}$ as in the following diagram.

$$\begin{tikzcd}
(S^7)^{\vee k-1} \arrow[r] \arrow[dr, "\tilde{f}"'] & \#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon,\delta} \arrow[r, "f"] & \#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon',\delta'} \\
& (S^7 \vee S^4)^{\vee k-1} \arrow[ur, dashed] &
\end{tikzcd}$$

Now we consider the composition $f_7\colon = \rho \circ \tilde{f}$ where ρ is the projection map.

$$\begin{tikzcd}
(S^7)^{\vee k-1} \arrow[r, "\vee_{i=1}^{k-1}\beta_i"] \arrow[dr, "\tilde{f}"'] \arrow[drr, "f_7"] & \#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon,\delta} \arrow[r, "f"] & \#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon',\delta'} \\
& (S^7 \vee S^4)^{\vee k-1} \arrow[r, "\rho"] & S^{7\vee k-1} \arrow[u, "\vee_{i=1}^{k-1}\gamma_i"']
\end{tikzcd}$$

From the stable homotopy type, we get an isomorphism

$$\begin{tikzcd}
f_7\colon \quad \pi_7((S^7)^{\vee k-1}) \arrow[r, "\cong"] \arrow[d, phantom, sloped, "\cong"] & \pi_7((S^7)^{\vee k-1}) \arrow[d, phantom, sloped, "\cong"] \\
\mathbb{Z}\{\beta_1, \dots, \beta_{k-1}\} & \mathbb{Z}\{\gamma_1, \dots, \gamma_{k-1}\}
\end{tikzcd}$$

where $f_7(\beta_{k-1}) \equiv \gamma_{k-1} \pmod{24}$. Hence, the corresponding matrix of f_7 is

$$\equiv \left(\begin{array}{ccc|c} & & & 0 \\ & A & & \vdots \\ & & & 0 \\ * & \dots & * & 1 \end{array}\right) \pmod{24}$$

for some $A \in GL_{k-2}(\mathbb{Z})$. From the inverse homotopy equivalence $g\colon Y_2 \to Y_1$, we can construct a corresponding block matrix of g_7 similar to that of f_7 for some $B \in GL_{k-2}(\mathbb{Z}) \pmod{24}$. Through suitable pre-composition of f_7 and post-composition of g_7 we may assume that $A = B = I$ where I is the identity matrix in $GL_{k-2}(\mathbb{Z})$. Thus both f_7 and g_7 are composition of shearing maps with $\beta_{k-1} \mapsto \gamma_{k-1}$, that is, they are composition of maps associated to $\beta_i \mapsto \gamma_i + c_i\gamma_{k-1}$ for some $c_i \in \mathbb{Z}$.

We now consider the map f on the 7-skeleton

$$\begin{tikzcd}
(S^4 \vee S^7)^{\vee k-1} \arrow[r, "f^{(7)}"] \arrow[d, "\vee_{i=1}^{k-1}(\alpha_i \vee \beta_i)"] & (S^4 \vee S^7)^{\vee k-1} \arrow[d, "\vee_{i=1}^{k-1}(\xi_i \vee \gamma_i)"] \\
Y_1 \arrow[r, "f"] & Y_2,
\end{tikzcd}$$

which takes the form

$$\beta_i \mapsto \gamma_i + c_i\gamma_{k-1} + \sum_{j=1}^{k-1} a_{i,j}\nu_j + \sum_{j=1}^{k-1} a'_{i,j}\nu'_j + \sum_{\substack{j=1\\ \ell=1\\ j\neq\ell}}^{\substack{j=k-1\\ \ell=k-1}} a_{i,j,\ell}[\xi_j, \xi_\ell], \quad \alpha_i \mapsto \xi_i, \quad \text{for } 1 \leq i \leq k-2,$$

$$\beta_{k-1} \mapsto \gamma_{k-1} + \sum_{j=1}^{k-1} b_j\nu_j + \sum_{j=1}^{k-1} b'_j\nu'_j + \sum_{\substack{j=1\\ \ell=1\\ j\neq\ell}}^{\substack{j=k-1\\ \ell=k-1}} b_{j,\ell}[\xi_j,\xi_\ell], \qquad \alpha_{k-1} \mapsto \xi_{k-1} - \sum_{j=1}^{k-2} c_j\xi_j.$$

As f is a homotopy equivalence, we must have that the attaching map of the 11-cell of Y_1 must be carried by $f^{(7)}$ to the attaching map of Y_2, that is, $f^{(7)} \circ L(Y_1) \simeq L(Y_2)$. We now look at the coefficients of $\xi_{k-1} \circ x$ and $\xi_{k-1} \circ y$ that arises in $f^{(7)} \circ L(Y_1)$ and note that the only terms which contribute to these coefficients are

$$f^{(7)}([\alpha_{k-1},\beta_{k-1}] + \lambda_s\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ x + \delta\alpha_{k-1} \circ y).$$

We now deduce

$$\epsilon' = (\lambda_s + 2)b_{k-1} - 4b'_{k-1} \qquad \text{and} \qquad \delta' = (\lambda_s + 1)b'_{k-1},$$

which verifies the result for complexes of the type $\#^{k-2}E_{0,0,0}\#E_{\lambda,\epsilon,\delta}$.

For the remaining cases, we follow the same argument with $Y_1 = \#^{k-3}E_{0,0,0}\#E_{0,1,0}\#E_{\lambda_s,\epsilon,\delta}$ with ϵ even if λ_s is, and $Y_2 = \#^{k-3}E_{0,0,0}\#E_{0,\hat{\epsilon},0}\#E_{\lambda_s,\epsilon',\delta'}$, where $\hat{\epsilon} = 0$ or 1. Note that the only terms which contribute to $\eta_{k-2} \circ x$ are

$$f^{(7)}([\alpha_{k-1},\beta_{k-1}] + [\alpha_{k-2},\beta_{k-2}] + \alpha_{k-2} \circ x + \lambda_s\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ x + \delta\alpha_{k-1} \circ y).$$

A direct computation implies

$$\lambda_s b_{k-2} + \epsilon c_{k-2}^2 + 1 \equiv \hat{\epsilon} \pmod 2. \tag{5.1.18}$$

First let λ_s be even and $\epsilon \equiv 0 \pmod 2$. This implies $\hat{\epsilon} = 1$. Finally, let λ_s is odd. We look at the coefficients of $[\eta_{k-1},[\eta_{k-2},\eta_{k-1}]]$ and $[\eta_{k-2},\eta_{k-1}] \circ \nu_{(7)}$ in $f^{(7)} \circ L(Y_1) \pmod 2$, which are $b_{k-2,k-1} + c_{k-2}b_{k-1} - a_{k-2,k-1}$ and $\lambda_s b_{k-2,k-1} + a_{k-2,k-1} + b_{k-2} - c_{k-2}b_{k-1} - \epsilon c_{k-2}$. Since both these coefficients are zero, we have $b_{k-2} \equiv \epsilon c_{k-2} \pmod 2$. Using the relation (5.1.18), we observe that $\hat{\epsilon} = 1$. The conditions on ϵ' and δ' are verified analogously as in the previous case. This completes the proof of the various implications in the theorem. □

5.1.19. The loopspace homotopy type. We study the loop space homotopy type of $E \in \mathcal{PD}^{11}_{4,7}$ with $E^{(7)} \simeq (S^4 \vee S^7)^{\vee k-1}$ and show that the loop space homotopy is independent of the λ, ϵ and δ occurring in the attaching map ϕ of E.

Theorem 5.1.20. *The homotopy type of the loop space of E is a weak product of loop spaces on spheres, and depends only on $k-1 = \text{Rank}(H_4(E))$. In particular,*

$$\Omega E \simeq \Omega(\#^{k-1}(S^4 \times S^7)).$$

Proof. This follows from the arguments in [9] and [10]. More explicitly, we first compute the homology of ΩE via cobar construction, see [2]. In this case, $H_*(E)$ is a coalgebra which is quasi-isomorphic to $C_*(E)$, and hence we may compute the cobar construction of $H_*(E)$ and deduce as in [10, Proposition 2.2]

$$H_*(\Omega E) \cong T(a_1, b_1, \ldots, a_{k-1}, b_{k-1})/(\sum [a_i, b_i])$$

where $\rho(\alpha_i) = a_i$ and $\rho(\beta_i) = b_i$ with ρ defined as

$$\rho\colon \pi_r(E) \cong \pi_{r-1}(\Omega E) \xrightarrow{Hur} H_{r-1}(\Omega E).$$

We then note that $H_*(\Omega E)$ is the universal enveloping algebra of the graded Lie algebra $L(a_1, b_1, \ldots, a_{k-1}, b_{k-1})/(\sum[a_i, b_i])$ where L is the free Lie algebra functor. Now we apply the Poincaré-Birkhoff-Witt theorem as in [10, Proposition 3.6] to deduce the result. ☐

5.2 Stable homotopy type of the total space.

In this section, we examine the possible stable homotopy types of the total space E for a principal $SU(2)$-fibration over $M_k \in \mathcal{PD}_3^8$. We relate this to the stable homotopy type of M_k. Let $f\colon M_k \to \mathbb{H}P^\infty$ be a map such that $\pi_4(f)\colon \pi_4(M_k) \to \pi_4(\mathbb{H}P^\infty) \cong \mathbb{Z}$ is surjective. This ensures that the homotopy fibre $E(f)$ is 3-connected and is a Poincaré duality complex of dimension 11. One easily deduces

$$H_i(E(f)) = \begin{cases} \mathbb{Z} & i = 0, 11 \\ \mathbb{Z}^{\oplus k-1} & i = 4, 7 \\ 0 & \text{otherwise} \end{cases}$$

from the Serre spectral sequence associated to the fibration $S^3 \to E(f) \to M_k$. We may now consider a minimal CW-complex structure on $E := E(f)$ with $(k-1)$ 4-cells, $(k-1)$ 7-cells, and one 11-cell, see [23, Section 2.2]. The 7-th skeleton $E^{(7)}$ is, therefore, a pushout

$$\begin{tikzcd}
(S^6)^{\vee(k-1)} \arrow[r] \arrow[d, "\vee_{i=1}^{k-1}\phi_i"'] & (D^7)^{\vee(k-1)} \arrow[d] \\
(S^4)^{\vee(k-1)} \arrow[r] & E^{(7)}
\end{tikzcd} \tag{5.2.1}$$

We now observe that the ϕ_i are all 0.

Proposition 5.2.2. *The maps $\phi_i \simeq 0$ for $1 \leq i \leq k-1$.*

Proof. Note that the homotopy class of each of the attaching maps is in $\pi_6(S^4)^{\oplus k-1}$ which lies in the stable range. Applying the Σ^∞ functor on the diagram (5.2.1), we get the cofibre sequence

$$\Sigma^\infty(S^6)^{\vee(k-1)} \xrightarrow{\Sigma^\infty(\vee_{i=1}^{k-1}\phi_i)} \Sigma^\infty(S^4)^{\vee(k-1)} \to \Sigma^\infty E^{(7)}$$

which in turn induces a long exact sequence (on the stable homotopy groups)

$$\cdots \to \pi_7^s(E^{(7)}) \to \pi_6^s(S^6)^{\oplus k-1} \xrightarrow{\Phi} \pi_6^s(S^4)^{\oplus k-1} \to \pi_6^s(E^{(7)}) \to \dots \tag{5.2.3}$$

where Φ is the induced map of $\vee_{i=1}^{k-1}\phi_i$. We have the following commutative diagram

$$\begin{tikzcd}
 & \pi_6^s(S^4)^{\oplus k-1} \arrow[r] \arrow[d] & \pi_6^s(E) \arrow[d] & \pi_6^s(E^{(7)}) \arrow[l, "\simeq"'] \\
\pi_6^s(S^7) \arrow[r] \arrow[d, equal] & \pi_6^s(S^4)^{\oplus k} \arrow[r] & \pi_6^s(M_k). & \\
0 & & &
\end{tikzcd}$$

The second row is a part of a long exact sequence and so, the map $\pi_6^s(S^4)^{\oplus k} \to \pi_6^s(M_k)$ is injective. Hence, the map $\pi_6^s(S^4)^{\oplus(k-1)} \to \pi_6^s(E)$ is injective, in (5.2.3) Φ is forced to be 0. The result follows. ☐

Proposition 5.2.2 implies that E fits into the pushout

$$\begin{tikzcd}
S^{10} \arrow[r] \arrow[d, "L(E)"'] & D^{11} \arrow[d] \\
(S^4 \vee S^7)^{\vee(k-1)} \arrow[r] & E
\end{tikzcd}$$

for some $[L(E)] \in \pi_{10}((S^4 \vee S^7)^{\vee(k-1)})$. Hence, E belongs to $\mathcal{PD}_{4,7}^{11}$. We consider

$$S^{10} \xrightarrow{L(E)} (S^4 \vee S^7)^{\vee(k-1)} \to (S^7)^{\vee(k-1)}$$

which is of the form

$$\sum_{i=1}^{k-1} \lambda_i \iota_7^i \circ \nu_7 \in \pi_{10}(S^7)^{\oplus k-1} \cong \bigoplus_{i=1}^{k-1} \mathbb{Z}/24\{\nu_7\}.$$

The coefficients λ_i can be computed via the e-invariant, see [3]. Recall that the e-invariant of a map $g\colon S^{11} \to S^8$ can be computed using Chern character. The complex K-theoretic e-invariant $e_{\mathbb{C}}$ is computed via the diagram

$$\begin{tikzcd}
 & \mathbb{Z}\{b_{12}\} \arrow[d, phantom, "\wr\Vert"] & & \mathbb{Z}\{b_8\} \arrow[d, phantom, "\wr\Vert"] & \\
0 \arrow[r] & \tilde{K}(S^{12}) \arrow[r] \arrow[d, "ch"] & \tilde{K}(\Sigma C_g) \arrow[r] \arrow[d, "ch"] & \tilde{K}(S^8) \arrow[r] \arrow[d, "ch"] & 0 \\
0 \arrow[r] & \tilde{H}^{ev}(S^{12};\mathbb{Q}) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & \tilde{H}^{ev}(\Sigma C_g;\mathbb{Q}) \arrow[r] & \tilde{H}^{ev}(S^8;\mathbb{Q}) \arrow[r] \arrow[d, phantom, "\wr\Vert"] & 0. \\
 & \mathbb{Q}\{a_{12}\} & & \mathbb{Q}\{a_8\} &
\end{tikzcd}$$

We obtain

$$ch(b_{12}) = a_{12}, \quad ch(b_8) = a_8 + ra_{12}.$$

If $g = \lambda\nu_{(7)}$, $e_{\mathbb{C}}(g) = r = \frac{\lambda}{12} \in \mathbb{Q}/\mathbb{Z}$, see [3]. We also have $e_{\mathbb{C}} = 2e$, where e is computed using the Chern character of the complexification $c\colon KO \to K$. Therefore, from [3, Proposition 7.14]

$$b_8 \in Im(c) \implies e(g) = \frac{r}{2} \in \mathbb{Q}/\mathbb{Z}. \tag{5.2.4}$$

5.2.5. K-theory of M_k. Consider the Atiyah-Hirzebruch spectral sequence

$$E_2^{**} = H^*(M_k; \pi_* K) \implies K^*(M_k).$$

As M has only even dimensional cells, this has no non-trivial differential for degree reasons. This gives the additive structure of $K^0(M_k)$. Let $H^4(M_k) \cong \mathbb{Z}\{\psi_1, \dots, \psi_k\}$ and $H^8(M_k) \cong \mathbb{Z}\{z\}$. Note that if $\alpha_1, \dots, \alpha_k \in \pi_4(M_k) \cong H_4(M_k) \cong \mathbb{Z}^k$ is dual to the basis $\psi_1, \dots, \psi_k$, in the expression (1.3.1), the matrix $((g_{i,j}))$ of the intersection form is related to the cup product via the equation $\psi_i \cup \psi_j = g_{i,j} z$. Let $\tilde{\psi}_1, \dots, \tilde{\psi}_k, \tilde{z}$ be classes in $K(M_k)$ corresponding to

$\psi_1, \ldots, \psi_k, z$ respectively, in the E^∞-page. We have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \tilde{K}^0(S^8) \arrow[r, "q^*"] \arrow[d, "ch"] & \tilde{K}^0(M_k) \arrow[r, "i^*"] \arrow[d, "ch"] & \tilde{K}^0((S^4)^{\vee k}) \arrow[r] \arrow[d, "ch"] & 0 \\
0 \arrow[r] & \tilde{H}^{ev}(S^8;\mathbb{Q}) \arrow[r] & \tilde{H}^{ev}(M_k;\mathbb{Q}) \arrow[r] & \tilde{H}^{ev}((S^4)^{\vee k};\mathbb{Q}) \arrow[r] & 0
\end{tikzcd}$$

where

$$ch(\tilde{z}) = z, \quad ch(i^*(\tilde{\psi}_j)) = \psi_j(\implies ch(\tilde{\psi}_j) = \psi_j + \tau_j z), 1 \le j \le k. \tag{5.2.6}$$

Note that in terms of the formula (1.3.1), $\psi_i\psi_j = g_{i,j}z$. We now use the fact that ch is a ring map to get

$$ch(\tilde{\psi}_i\tilde{\psi}_j) = (\psi_i + \tau_i z) \cup (\psi_j + \tau_j z) = \psi_i \cup \psi_j = g_{i,j}z.$$

As $ch\colon K(M_k) \to H^{ev}(M_k;\mathbb{Q})$ is injective, we deduce $\tilde{\psi}_i\tilde{\psi}_j = g_{i,j}\tilde{z}$.

Further let $q_i\colon (S^4)^{\vee k} \to S^4$ be the retraction onto the i-th factor, and note that $q_i \circ L(M)$ is stably equivalent to $(g_{i,i} - 2l_i)\alpha_i \circ \nu_{(7)}$. Thus the e-invariant

$$e_{\mathbb{C}}(q_i \circ L(M)) = (g_{i,i} - 2l_i)e_{\mathbb{C}}(\Sigma\nu_{(7)}) = \frac{g_{i,i} - 2l_i}{12} \in \mathbb{Q}/\mathbb{Z}.$$

We summarize these observations in the following proposition.

Proposition 5.2.7. *Let $M_k \in \mathcal{PD}_3^8$ with $L(M_k)$ as in (1.3.1). Then, $K^0(M_k) \cong \mathbb{Z}\{1, \tilde{\psi}_1, \tilde{\psi}_2, \ldots, \tilde{\psi}_k, \tilde{z}\}$. The ring structure is given by*

$$\tilde{\psi}_i\tilde{\psi}_j = g_{i,j}\tilde{z} \quad \text{for } 1 \le i, j \le k.$$

and

$$e_{\mathbb{C}}(q_i \circ L(M)) = \frac{g_{i,i} - 2l_i}{12} \in \mathbb{Q}/\mathbb{Z} \quad \text{for } 1 \le i \le k.$$

5.2.8. K-theory of $E(f)$. The space $E := E(f)$ is the total space of the sphere bundle associated to the quaternionic line bundle classified by f. We note that the quaternionic line bundle has a complex structure, and therefore, has a K-orientation.

Let $\gamma_{\mathbb{H}}$ be the canonical $\mathbb{H}$-bundle over $\mathbb{H}P^\infty$. The K-theoretic Thom class of $\gamma_{\mathbb{H}}$ is given by $\Phi_K\colon = \gamma_{\mathbb{H}} - 2 \in \tilde{K}^0(\mathbb{H}P^\infty)$. As the total space of the sphere bundle is contractible, the Thom space $Th(\gamma_{\mathbb{H}}) \simeq \mathbb{H}P^\infty$. Consider the map $\pi\colon \mathbb{C}P^\infty \to \mathbb{H}P^\infty$. The pullback bundle

$\pi^*(\gamma_{\mathbb{H}}) = \gamma_{\mathbb{C}} \oplus \bar{\gamma}_{\mathbb{C}}$, where $\gamma_{\mathbb{C}}$ is the canonical line bundle over $\mathbb{C}P^\infty$. Therefore,

$$\pi^* ch(\gamma_{\mathbb{H}} - 2) = ch(\pi^*(\gamma_{\mathbb{H}} - 2)) = ch(\gamma_{\mathbb{C}}) + ch(\bar{\gamma}_{\mathbb{C}}) - 2 = e^x + e^{-x} - 2,$$

where $H^*(\mathbb{C}P^\infty;\mathbb{Z}) \cong \mathbb{Z}[x]$ and $x = c_1(\gamma_{\mathbb{C}})$. Since π^* is injective on H^*, we may use this formula to deduce

$$ch(\Phi_K(\gamma_{\mathbb{H}})) = \Phi_H(\gamma_{\mathbb{H}})(1 + \frac{y}{12} + \frac{y2}{360} + \dots) \tag{5.2.9}$$

where $H^*(\mathbb{H}P^\infty) \cong \mathbb{Z}[y]$, $\pi^*(y) = x^2$ and $\Phi_H(\gamma_{\mathbb{H}})$ is the cohomological Thom class of $\gamma_{\mathbb{H}}$. We use the Thom isomorphism associated to $f^*(\gamma_{\mathbb{H}})$ to deduce the following.

Proposition 5.2.10. *Assume that $f^*(y) = \psi_k$. Then,*

$$\tilde{K}^0(Th(f^*(\gamma_{\mathbb{H}}))) \cong \tilde{K}^0(M)\{\Phi_K(f^*\gamma_{\mathbb{H}})\}$$

as a $\tilde{K}^0(M)$-module, and

$$ch(\Phi_K(f^*\gamma_{\mathbb{H}})) = \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \frac{\psi_k^2}{360} + \dots).$$

Proof. From the naturality of the Chern character as well as the Thom class, we have

$$ch(\Phi_K(f^*(\gamma_{\mathbb{H}})) = ch(f^*(\Phi_K((\gamma_{\mathbb{H}})) = f^* ch(\Phi_K(\gamma_{\mathbb{H}})).$$

The result follows from (5.2.9) and the fact $f^*(y) = \psi_k$. ☐

Notation 5.2.11. *Suppose we are in the situation of Proposition 5.2.10, that is, $f^*(y) = \psi_k$. We now assume that the basis $\{\psi_1, \dots, \psi_k\}$ is such that one of the following cases occur*

Case 1 If $\psi_k \cup \psi_k = \pm z$, then $\psi_j \psi_k = 0$ for $1 \le j \le k-1$.

Case 2 If $\psi_k \cup \psi_k = g_{k,k} z$ for some integer $g_{k,k} \neq \pm 1$, then assume $\psi_{k-1}\psi_k = 1$ and $\psi_j\psi_k = 0$ for $1 \le j \le k-2$.

In terms of these notations we prove the following calculation. Here we consider the cofibre sequence

$$E \to M_k \to Th(f^*(\gamma_{\mathbb{H}})) \to \Sigma E \to \Sigma M \to \dots$$

which demonstrates $K(\Sigma E)$ as a submodule of $K(Th(f^*\gamma_{\mathbb{H}}))$ because $K(\Sigma M_k) = 0$. The following proposition identifies this submodule.

Proposition 5.2.12. *(1) Suppose we are in Case 1 of Notation 5.2.11, then we have*

$$\tilde{K}^0(\Sigma E) \cong \mathbb{Z}\{\Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_1, \dots, \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_{k-1}, \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{z}\}.$$

(2) Suppose we are in Case 2 of Notation 5.2.11, then

$$\tilde{K}^0(\Sigma E) \cong \mathbb{Z}\{\Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_1, \dots, \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{\psi}_{k-2}, \Phi_K(f^*(\gamma_{\mathbb{H}}))(\tilde{\psi}_k - g_{k,k}\tilde{\psi}_{k-1}), \Phi_K(f^*(\gamma_{\mathbb{H}}))\tilde{z}\}.$$

Proof. We have the following short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \tilde{K}^0(\Sigma E) \arrow[r] & \tilde{K}^0(Th(f^*(\gamma_{\mathbb{H}})) \arrow[r, "s_0"] & \tilde{K}^0(M) \arrow[r] & 0 \end{tikzcd}$$

which implies that

$$\tilde{K}^0(\Sigma E) = \mathsf{Ker}(s_0 \colon \tilde{K}^0(Th(f^*(\gamma_{\mathbb{H}})) \to \tilde{K}^0(M))$$

where s_0 is the restriction along the zero section. Note that s_0 is a $\tilde{K}^0(M)$-module map. Hence,

$$\begin{aligned} s_0(\Phi_K(f^*\gamma_{\mathbb{H}}))\tilde{z}) &= e_K(f^*\gamma_{\mathbb{H}})\tilde{z} = 0 \\ s_0(\Phi_K(f^*\gamma_{\mathbb{H}}))\tilde{\psi}_i) &= e_K(f^*\gamma_{\mathbb{H}})\tilde{\psi}_i = g_{ik}\tilde{z}, 1 \leq i \leq k \end{aligned}$$

since $e_K(f^*\gamma_{\mathbb{H}})) = \tilde{\psi}_k + m\tilde{z}$ for some m. The result follows from a direct calculation of the kernel and the assumptions in the respective cases. □

We now choose the various maps $\chi_j \colon S^7 \to E$ for $1 \leq j \leq k-1$ such that on K-theory they precisely represent the choice of the first $(k-1)$-elements in the basis of Proposition 5.2.12. In these terms we calculate the $e_{\mathbb{C}}$-value of the composite

$$r_j \circ L(E) \colon S^{10} \overset{L(E)}{\to} (S^4 \vee S^7)^{\vee k-1} \overset{r_j}{\to} S^7$$

where r_j is the restriction onto the j-th factor, and $L(E)$ is the attaching map of the top cell of E.

Proposition 5.2.13. *(1) If we are in Case 1, then*

$$e_{\mathbb{C}}(r_j \circ L(E)) = \tau_j = \frac{g_{j,j} - 2l_j}{12} \in \mathbb{Q}/\mathbb{Z}, 1 \leq j \leq k-1.$$

(2) If we are in Case 2, then

$$e_{\mathbb{C}}(r_j \circ L(E)) = \tau_j = \frac{g_{j,j} - 2l_j}{12} \in \mathbb{Q}/\mathbb{Z}, 1 \leq j \leq k-2; \quad e_{\mathbb{C}}(r_{k-1} \circ L(E)) = \tau_k - g_{k,k}\tau_{k-1} \in \mathbb{Q}/\mathbb{Z}.$$

Proof. For the e-invariant, we calculate the Chern character $ch\colon \tilde{K}^0(Th(f^*\gamma_{\mathbb{H}})) \to H^{ev}(Th(f^*\gamma_{\mathbb{H}}); \mathbb{Q})$ in terms of (5.2.6) as follows

$$\begin{aligned} ch(\Phi_K((f^*\gamma_{\mathbb{H}}))\tilde{z}) &= \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \dots)z = \Phi_H(f^*\gamma_{\mathbb{H}})z, \\ ch(\Phi_K((f^*\gamma_{\mathbb{H}}))\tilde{\psi}_j) &= \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \dots)(\psi_j + \tau_j z) \\ &= \Phi_H(f^*\gamma_{\mathbb{H}})\psi_j + \tau_j\Phi_H(f^*\gamma_{\mathbb{H}})z, \\ ch((\Phi_K(f^*\gamma_{\mathbb{H}})(\tilde{\psi}_k - g_{k,k}\tilde{\psi}_{k-1})) &= \Phi_H(f^*\gamma_{\mathbb{H}})(1 + \frac{\psi_k}{12} + \dots)(\psi_k + \tau_k z - g_{k,k}\psi_{k-1} - g_{k,k}\tau_{k-1}z) \\ &= \Phi_H(f^*\gamma_{\mathbb{H}})(\psi_k - g_{k,k}\psi_{k-1}) + \Phi_H(f^*\gamma_{\mathbb{H}})(\tau_k - g_{k,k}\tau_{k-1})z. \end{aligned}$$

□

We now turn our attention to the attaching map

$$S^{10} \overset{L(E)}{\to} (S^4 \vee S^7)^{\vee k-1} \to (S^7)^{\vee k-1}.$$

In order to identify the composite we are required to compute the KO-theoretic e-invariant. We know that

$$KO^* = \mathbb{Z}[\eta, u][\mu^{\pm 1}]/(2\eta, \eta^3, \eta u, u^2 - 4\mu) \quad \text{and} \quad K^* = \mathbb{Z}[\beta^{\pm 1}],$$

with $|\eta| = -1,\ |u| = -4,\ |\mu| = -8$ and $|\beta| = -2$. The complexification map $c\colon KO \to K$ induces a graded ring homomorphism $c\colon KO^*(X) \to K^*(X)$ with

$$c(\eta) = 0, \quad c(u) = 2\beta^2, \quad c(\mu) = \beta^4.$$

Theorem 5.2.14. *Let $S^3 \to E \to M_k$ be the principal $SU(2)$-fibration classified by a given map $f\colon M_k \to \mathbb{HP}^\infty$. Suppose that $\Sigma L(M_k) \equiv 0 \pmod{\lambda}$. Then $\Sigma(r_j \circ L(E)) \equiv 0 \pmod{\lambda}, 1 \leq j \leq k-1$ where $r_j : (S^4 \vee S^7)^{\vee k-1} \to S^7$ is the retraction onto the j-th factor.*

Proof. We consider the Atiyah-Hirzebruch spectral sequences for $\mathbb{H}P^\infty$ and M_k

$$\begin{aligned} E_2^{*,*} &= H^*(\mathbb{H}P^\infty; \mathbb{Z}) \otimes KO^*(pt) \implies KO^*(\mathbb{HP}^\infty), \\ E_2^{*,*} &= H^*(M_k; \mathbb{Z}) \otimes KO^*(pt) \implies KO^*(M_k). \end{aligned}$$

The spectral sequences have no non-trivial differentials for degree reasons. Thus, $KO^*(\mathbb{H}P^\infty) \cong KO^*[\hat{y}]$, where $\hat{y} \in KO^4(\mathbb{H}P^\infty)$. The class $\hat{y}$ serves as KO-theoretic Thom class for $\gamma_{\mathbb{H}}$. We have

$$c(\Phi_{KO}(\gamma_{\mathbb{H}})) = c(\hat{y}) = \beta^{-2}\Phi_K(\gamma_{\mathbb{H}}).$$

Let $\hat{\psi}_j \in K\tilde{O}^4(M_k)$, $\hat{z} \in K\tilde{O}^8(M_k)$ be the class in the E^∞-page represented by $\psi_j \in H^4(M_k)$ and $z \in H^8(M_k)$. Then we get

$$c(\hat{\psi}_i) = \beta^{-2}\tilde{\psi}_i, \quad c(\hat{z}) = \beta^{-4}\tilde{z} \quad \text{and} \quad \hat{\psi}_i\hat{\psi}_j = g_{i,j}\hat{z}.$$

It follows that the K-theoretic generators $\Phi_K(f^*\gamma_{\mathbb{H}})\tilde{\psi}_j$ and $\Phi_K(f^*\gamma_{\mathbb{H}})(\tilde{\psi}_k - g_{k,k}\tilde{\psi}_{k-1})$ lie in the image of the map c. Therefore by (5.2.4), we get in Case 2 that

$$e(r_j \circ L(E)) \equiv \begin{cases} \frac{g_{j,j}-2l_j}{24} & (\mathrm{mod}\ \mathbb{Z}), \quad \text{for } j < k-1, \\ \frac{(g_{kk}-2l_k)-g_{k,k}(g_{k-1,k-1}-2l_{k-1})}{24} & (\mathrm{mod}\ \mathbb{Z}), \quad \text{for } j = k-1; \end{cases}$$

and in Case 1 that

$$e(r_j \circ L(E)) \equiv \frac{g_{j,j} - 2l_j}{24} \quad (\mathrm{mod}\ \mathbb{Z}), \quad \text{for } j \leq k-1.$$

The result now follows from Proposition 5.2.7. □

5.3 $SU(2)$-bundles over even complexes

We study the homotopy type of $E \in \mathcal{PD}^{11}_{4,7}$, the total space of a stable principal $SU(2)$-bundle over $M_k \in \mathcal{PD}^8_3$. In this section, we consider the case where the intersection pairing $\langle -, - \rangle \colon H_4(M_k) \times H_4(M_k) \to \mathbb{Z}$ is even i.e. $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in H_4(M_k)$. Note that in this case, k must be even. We observe the following.

1. If $k \geq 4$, M_k supports a principal $SU(2)$-bundle whose total space E is 3-connected.
2. The possible stable homotopy types of E can be determined directly from the stable homotopy type of M_k and the intersection form. In this regard, the formulas in Theorem 5.2.14 are used to demonstrate this connection.

5.3.1. Existence of $SU(2)$-bundles. We discuss the existence of principal $SU(2)$-bundle over $M_k \in \mathcal{PD}^8_3$ with an attaching map as in (1.3.1) whose intersection form is even. If

Rank($H_4(M)$) $= 2$, then up to isomorphism $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the only possible matrix for the intersection form. The attaching map $L(M) \in \pi_7(S^4 \vee S^4)$ of M is of the form

$$L(M) = [\alpha_1, \alpha_2] + l_1\nu_1' + l_2\nu_2', \tag{5.3.2}$$

where $l_1, l_2 \in \mathbb{Z}/12$. For a principal bundle $SU(2) \to E \to M_k$ where E is 3-connected, the classifying map $f_E \colon M_k \to \mathbb{H}P^\infty$ is such that $\pi_4(f)$ is surjective. Suppose $\alpha_i \mapsto n_i\iota_4, i = 1, 2$ such that $\gcd(n_1, n_2) = 1$ and $f|_{\overline{M}_k} \circ L(M_k) \simeq *$, where $\overline{M}_k$ denote the lower skeleton of M_k. If both l_1 and l_2 are odd, no such n_1 and n_2 exists. However, for $k \geq 4$, one may always construct a suitable map.

Proposition 5.3.3. *Suppose $M_k \in \mathcal{PD}_3^8$ such that Rank($H_4(M_k)$) $= k \geq 4$ and the intersection form is even. Then there exists a map $\psi \colon M_k \to \mathbb{H}P^\infty$ such that hofib(ψ) is 3-connected.*

Proof. From [11, Theorem 4.20], if $k \geq 6$, the attaching map of M_k can be expressed as

$$L(M_k) = \sum_{1 \leq i < j \leq k} g_{i,j}[\alpha_i, \alpha_j] + \sum_{1=1}^{k} \frac{g_{i,i}}{2}[\alpha_i, \alpha_i] + \sum_{i=1}^{k} s_i\nu_i'$$

such that $s_i = 0$ for $i \geq 2$ for a choice of basis $\{\alpha_1, \ldots, \alpha_k\}$ of $H_4(M_k)$. Then the map $\tilde{\psi} \colon (S^4)^{\vee k} \xrightarrow{(0,0,\ldots,0,1)} \mathbb{H}P^\infty$ extends to a map $\psi \colon M_k \to \mathbb{H}P^\infty$ such that $\pi_4(\psi)$ is surjective. Now if $k = 4$, by [33], the attaching map of M_k can be expressed as

$$L(M_k) = [\alpha_1, \alpha_2] + [\alpha_3, \alpha_4] + \sum_{i=1}^{4} l_i\nu_i'$$

for a choice of basis of $H_4(M_k)$. Choose two positive integers m, n such that $\gcd(m, n) = 1$ and $ml_1 + nl_3 \equiv 0 \pmod{12}$. Then the map $\tilde{\psi} \colon (S^4)^{\vee 4} \xrightarrow{(m,0,n,0)} \mathbb{H}P^\infty$ extends to a map $\psi \colon M_k \to \mathbb{H}P^\infty$ such that $\pi_4(\psi)$ is surjective. ☐

We now focus on the stable homotopy type of the total space $E(f)$ for $f \colon M_k \to \mathbb{H}P^\infty$ such that $\pi_4(f)$ is injective. From the attaching map of M_k as in (1.3.1) and even intersection form, we have

$$\Sigma^\infty M_k \simeq \Sigma^\infty(S^4)^{\vee k-1} \vee \Sigma^\infty(Cone(\sigma(M_k)\nu_{(7)}) \tag{5.3.4}$$

for some even $\sigma(M_k)$. Hence the stable homotopy type of M_k is determined by $\sigma(M_k)$.

Proposition 5.3.5. *Let $E(f_\psi)$ be the total space of a principal $SU(2)$-bundle over $M_k \in \mathcal{PD}_3^8$, classified by a map $f_\psi \colon M_k \to \mathbb{H}P^\infty$ for $k \geq 4$. Then*

$$\Sigma^\infty E(f_\psi) \simeq \Sigma^\infty (S^4 \vee S^7)^{\vee k-2} \vee \Sigma^\infty Cone(\lambda(\psi)\Sigma^4 \nu_{(7)}),$$

where $\lambda(\psi) := \lambda_s(E(f_\psi))$, is even and a multiple of $\sigma(M_k)$.

Proof. Note that $E(f_\psi) \in \mathcal{PD}_{4,7}^{11}$ and its stable homotopy type is given in the Corollary 5.1.15 where $\epsilon = \epsilon_s(E(f_\psi)) \in \mathbb{Z}/2$ and $\lambda(\psi) \in \mathbb{Z}/24$. So, it suffices to show that $2|\lambda(\psi)$ and $\epsilon = 0$. The fact $2|\lambda(\psi)$ follows from (5.3.4) and Theorem 5.2.14.

Now, the cofibre sequence obtained from cell structure of E and M_k induces the following commutative diagram

$$\begin{tikzcd}
\pi_{10}^s(S^{10}) \arrow[r, "\phi_*"] & \pi_{10}^s(S^4 \vee S^7)^{\bigoplus k-1} \arrow[r] \arrow[d] & \pi_{10}^s(E) \arrow[d] \\
\pi_{10}^s(S^7) \arrow[r, "\pi_{10}^s(\sum^\infty L(M))", "0"'] & \pi_{10}^s(S^4)^{\bigoplus k} \arrow[r] & \pi_{10}^s(M_k)
\end{tikzcd}$$

of stable homotopy groups where ϕ is the attaching map of top cell in E.

Since $\lambda(\psi)\beta_{k-1} \circ \nu_7 + \epsilon\alpha_{k-1} \circ x = 0$ in $\pi_{10}^s(E)$, its image in $\pi_{10}^s(M)$ is 0. Note that bottom left map $\pi_{10}^s(\sum^\infty L(M)) = 0$ because $2|\sigma(M_k)$ in (5.3.4) and hence the bottom right map is injective, where $\pi_{10}^s(S^4)^{\oplus k} \cong \mathbb{Z}/2\{\alpha_1 \circ \nu^2, \ldots, \alpha_k \circ \nu^2\}$. Since $2|\lambda(\psi)$, the middle vertical arrow sends $\Sigma^\infty \phi = \lambda(\psi)\beta_{k-1} \circ \nu_{(7)} + \epsilon\alpha_{k-1} \circ \nu^2$ to $\epsilon\alpha_{k-1} \circ \nu^2$ which is in turn mapped to 0 via the bottom right map as $\Sigma^\infty \phi = 0 \in \pi_{10}^s(E)$. Hence $\epsilon = 0 \in \mathbb{Z}/2$. □

5.3.6. Stably trivial manifolds. The following result states that the total space of a principal $SU(2)$-bundle over stably trivial $M_k \in \mathcal{PD}_3^8$ (i.e., $\sigma(M_k) \equiv 0 \pmod{24}$), is itself a connected sum of copies of $S^4 \times S^7$.

Proposition 5.3.7. *Let $E \in \mathcal{PD}_{4,7}^{11}$ be the total space of a principal $SU(2)$-bundle over stably trivial $M_k \in \mathcal{PD}_3^8$. Then $E \simeq \#^{k-1}(S^4 \times S^7)$.*

Proof. It follows from Proposition 5.3.5 for $k \geq 4$ that $\lambda_s \equiv 0 \pmod{24}$ and $\epsilon_s \equiv 0 \pmod 2$ for the total space $E \in \mathcal{PD}_{4,7}^{11}$. From Theorem 5.1.17, we have $E \simeq \#^{k-2}E_{0,0,0}\#E_{0,\epsilon,\delta}$, where ϵ is even . Note that using the self homotopy equivalences in 5.1.3, we have $E_{0,\epsilon,\delta} \simeq E_{0,0,0}$ when ϵ is even . Hence the result follows for $k \geq 4$. For $k = 2$, the attaching map of M_2 is of the form (5.3.2). Stably trivial condition implies $M_2 = S^4 \times S^4$ which further implies $E = S^4 \times S^7$. □

5.3.8. Possible stable homotopy types of the total space. Let $\psi \in H^4(M_k;\mathbb{Z})$ be a cohomology class represented by a map $\overline{\psi}\colon M_k \to K(\mathbb{Z},4)$ which has a unique lift $\tilde{\psi}\colon M_k \to S^4$ up to homotopy if $\overline{\psi}|_{(S^4)^{\vee k-1}} \circ L(M_k) \in \pi_7(S^4)$ is 0. As the inclusion $(S^4)^{\vee k} \to M_k$ induces an isomorphism on H_4 and H^4, a cohomology class $\psi \in H^4(M_k)$ always induces a map $\tilde{\psi}\colon (S^4)^{\vee k} \to S^4$. We consider the following diagram

$$\begin{tikzcd}
(S^4)^{\vee k} \arrow[r] \arrow[d, "\tilde{\psi}"] & M_k \arrow[d, dashed] \\
S^4 \arrow[r] & \mathbb{H}P^2
\end{tikzcd}$$

and formulate when the map $M_k \to \mathbb{H}P^2$ exists. Note that if the map exists then its homotopy fibre will be 3-connected.

For $\psi \in H^4(M_k)$, consider the composite

$$S^7 \xrightarrow{L(M_k)} (S^4)^{\vee k} \xrightarrow{\tilde{\psi}} S^4$$

and define $\tau(\psi) = [\tilde{\psi} \circ L(M)] \in \pi_7^s(S^4)$. Thus

$$\tau\colon H^4(M_k) \to \mathbb{Z}/24$$

and one can check it is a linear map.

Proposition 5.3.9. *Suppose $\psi \in H^4(M_k)$ is primitive. The map $\tilde{\psi}\colon (S^4)^{\vee k} \to S^4$ extends to a map $f_\psi : M_k \to \mathbb{H}P^2$ if and only if $\psi \cup \psi \equiv \tau(\psi)z \pmod{24}$ for some $z \in H^8(M_k)$.*

Proof. Consider a primitive element $\psi \in H^4(M_k)$. We extend ψ to a basis of $H^4(M_k)$, and use the dual basis of $\pi_4(M_k)$ to write down the attaching map of M_k is as in (1.3.1). In this notation, we have $\psi^2 = g_{k,k}z$ where $z \in H^8(M_k)$ is the chosen generator and $\tau(\psi) = g_{k,k} - 2l_k$. Thus $\tilde{\psi} \circ L(M)$ maps to $0 \in \pi_7(\mathbb{H}P^2)$ if and only if $l_k = 0$, that is $\psi \cup \psi \equiv \tau(\psi)z \pmod{24}$. Hence, the result follows. □

Proposition 5.3.9 gives us criteria for constructing the maps f_ψ out of cohomology classes ψ. Using this we determine which multiples of $\sigma(M_k)$ may occur as $\lambda_s(E)$ for $E \to M_k$ a principal $SU(2)$-bundle where E is 3-connected. We first show that there exist f_ψ such that $\lambda(\psi) = \sigma(M_k)$ if k is large enough.

Proposition 5.3.10. *Suppose the stable homotopy type of M_k is determined by $\sigma(M_k)$. Then there exists $f_\psi\colon M_k \to \mathbb{H}P^\infty$ such that*

1. *$\lambda(\psi) \equiv \sigma(M_k) \pmod 3$ for $k \geq 5$,*

2. *$\lambda(\psi) \equiv \sigma(M_k) \pmod 8$ for $k \geq 7$.*

Proof. We begin the proof with the first case. If $\tau \equiv 0 \pmod 3$, the proof follows from Theorem 5.2.14. Let $\tau \not\equiv 0 \pmod 3$ and ψ_0 a primitive cohomology class such that $\tau(\psi_0) \equiv \sigma(M_k) \not\equiv 0 \pmod 3$. We need to choose a $\psi \in \ker(- \cup \psi_0) \cap \ker(\tau)$ such that $\psi^2 \equiv \tau(\psi) \equiv 0 \pmod 3$. This we can do for $k - 2 \geq 3$, see [33, Chapter II, (3.2)-(3.4)]. By Poincaré duality, we get ψ' such that $\psi \cup \psi' = z$. We write $\psi' = \psi'_{\ker(\tau)} + t\psi_0$ for some t where $\psi'_{\ker(\tau)} \in \ker(\tau)$. Since $z = \psi' \cup \psi = \psi'_{\ker(\tau)} \cup \psi$, we may assume $\psi' = \psi'_{\ker \tau}$. Thus by assigning $\tau_1 = \tau(\psi_0) \equiv \sigma(M_k) \pmod 3$, $\tau_{k-1} = \tau(\psi') \equiv 0 \pmod 3$ and $\tau_k = \tau(\psi) \equiv 0 \pmod 3$; and choosing other ψ_i's such that $\tau_i = 0$ for $i = 2, \ldots, k-2$ we have

$$\lambda(\psi) \equiv \sigma(M_k) \pmod 3 \qquad \text{for } k \geq 5.$$

Now we look into the second case. The proof goes similarly to that of the above, except when we choose $\psi \in \ker(- \cup \psi_0) \cap \ker(\tau)$ such that $\psi^2 \equiv \tau(\psi) \equiv 0 \pmod 8$. We can choose such ψ for $k - 2 \geq 5$, see [33, Chapter II, (3.2)-(3.4)]. Then following similar arguments one can deduce

$$\lambda(\psi) \equiv \sigma(M_k) \pmod 8 \qquad \text{for } k \geq 7.$$

□

The following theorem constructs f_ψ with $\lambda(\psi) = 3\sigma(M_k)$ if $\sigma(M_k)$ is not divisible by 3.

Theorem 5.3.11. *Suppose $3 \nmid \sigma(M_k)$. Then for $k \geq 7$, there exists $f_\psi \colon M_k \to \mathbb{H}P^2$ such that $\lambda(\psi) = 3\sigma(M_k)$.*

Proof. Let

$$\tau^{(3)} \colon H^4(M_k, \mathbb{F}_3) \to \mathbb{F}_3$$

be the restriction of τ in modulo 3. As $\tau^{(3)}$ is surjective, there exists $\psi_0 \in H^4(M_k, \mathbb{F}_3)$ such that for all cohomology class ψ, $\tau^{(3)}(\psi)z \equiv \psi \cup \psi_0 \pmod 3$. In particular, $\tau^{(3)}(\psi_0)z \equiv \psi_0 \cup \psi_0 \pmod 3$. We consider two cases, $\psi_0^2 = 0$ or ψ_0^2 is unit.

First let ψ_0^2 is unit. Then we can choose $\psi_1, \ldots, \psi_{k-1}$ such that $\psi_0 \cup \psi_i = 0$. We take the dual basis α_i corresponding to ψ_i and α_k corresponding to ψ_0. Thus $\psi_0(\alpha_i) = 0$ for $i = 1, \ldots, k-1$ and $\psi_0(\alpha_k) = 1$. Hence

$$\tau_1 \equiv \cdots \equiv \tau_{k-1} \equiv 0 \pmod 3, \quad \text{and} \quad \tau_k \equiv g_{k,k} \equiv 1 \pmod 3,$$

which implies $\lambda(\psi) = \gcd(\tau_1, \ldots, \tau_{k-2}, \tau_{k-1}) \equiv 0 \pmod 3$.

Now let $\psi_0^2 = 0$. Then $\tau^{(3)}(\psi_0)z = 0$. We choose $\psi_1, \ldots, \psi_{k-1}$ such that $\psi_{k-1} \cup \psi_0 = 1$ and $\psi_i \cup \psi_0 = 0$ for $i = 1, \ldots, k-2$. After taking the dual basis, with similar argument we have

$$\tau_1 \equiv \cdots \equiv \tau_{k-2} \equiv 0 \pmod 3, \quad \tau_{k-1} \equiv \sigma(M_k) \pmod 3, \quad \text{and} \quad \tau_k \equiv g_{k,k} \equiv 0 \pmod 3.$$

Hence $\lambda(\psi) = \gcd(\tau_1, \ldots, \tau_{k-2}, \tau_k - g_{k,k}\tau_{k-1}) \equiv 0 \pmod 3$. □

Remark 5.3.12. *We note that Proposition 5.3.9, Proposition 5.3.10, and Theorem 5.3.11 does not use the fact that the intersection form is even, and also holds in the case where the intersection form is odd.*

Now we look to prove similar results modulo 8, which in turn provide us desired construction ψ as in Proposition 5.3.9 using the Chinese reminder theorem. However, in this case certain conditions are required for obtaining analogous f_ψ.

Definition 5.3.13.

- A complex M_k with $\sigma(M_k) = 2$ or 4, is said to satisfy hypothesis (H_8) if $(\ker \tau)^\perp = (\sigma(M_k)\psi) \pmod 8$ where $\psi \in H^4(M_k)$ (is the unique class $\pmod{\frac{8}{\sigma(M_k)}}$) satisfying
$$\begin{cases} \psi^2 \equiv 0 \pmod 8 & \text{if } \sigma(M_k) = 2 \\ \psi^2 \equiv 0 \pmod 4 & \text{if } \sigma(M_k) = 4. \end{cases}$$
- A complex M_k with $\sigma(M_k) = 2$ is said to satisfy hypothesis (H_4) if $(\ker \tau)^\perp = (2\psi) \pmod 4$ where $\psi \in H^4(M_k)$ (which is unique $\pmod 2$) satisfies
$$\psi^2 \equiv \tau(\psi) \equiv 0 \text{ or } 4 \pmod 8.$$

Note that the hypotheses (H_8) and (H_4) depends only on the intersection form and τ and not on the choice of ψ. We now prove the existence of f_ψ under the hypothesis defined above.

Theorem 5.3.14.

1. *Suppose $8 \nmid \sigma(M_k)$. For $k \geq 5$, there exist $f_\psi \colon M_k \to \mathbb{H}P^2$ such that $\lambda(\psi) \equiv 0 \pmod 8$ if and only if the complex satisfies hypothesis (H_8).*
2. *Suppose $\sigma(M_k) = 2$. Then for $k \geq 5$, there exist $f_\psi \colon M_k \to \mathbb{H}P^2$ such that $\lambda(\psi) \equiv 4 \pmod 8$ if and only if the complex satisfies hypothesis (H_4).*

Proof. The condition $k \geq 5$ comes from the fact that we are required to make certain choices modulo 3 using Proposition 5.3.10. First suppose in case (1), a f_ψ exists such that $\lambda(\psi) =$

$8\sigma(M_k)$. Then there exists a basis $\{\psi_1,\ldots,\psi_{k-2},\psi',\psi\}$ satisfying (working $\pmod 8$))

$$\begin{aligned} &\psi\cup\psi_i = 0 \text{ for } 1\le i\le k-2, && \psi\cup\psi' = 1, \\ &\tau(\psi_i) = 0 \text{ for } 1\le i\le k-2, && \tau(\psi') = \sigma(M_k) \quad\text{and}\quad \tau(\psi)=\psi^2=0. \end{aligned} \tag{5.3.15}$$

Note that $\langle\psi\rangle^\perp = \langle\psi_1,\ldots,\psi_{k-2},\psi\rangle \subset \ker(\tau) = \langle\psi_1,\ldots,\psi_{k-2},\psi,\frac{8}{\sigma(M_k)}\psi'\rangle$. This implies $(\ker(\tau))^\perp = (\sigma(M_k)\psi)$, and thus the hypothesis (H_8) is satisfied, where $\langle V\rangle^\perp = \{\psi'\colon \langle\psi\cup\psi',[M_k]\rangle\equiv 0 \pmod 8 \forall\psi\in V\}$.

For the converse part if the complex satisfies hypothesis (H_8), one can check that there is a choice of ψ such that (5.3.15) is satisfied. We look into the cases $\sigma(M_k)=2,4$.

First, let $\sigma(M_k)=2$. Then $(\ker(\tau))^\perp=(2\psi)$ and ψ is well defined modulo 4. We note that

$$\chi\cup(2\psi)\equiv\tau(\chi)z \pmod 8 \ \forall\chi\in H^4(M_k). \tag{5.3.16}$$

This implies

$$\tau(\psi)\equiv 2\psi^2\equiv 0 \pmod 8.$$

For any $\chi\in\ker(\tau)$, we have $(2\psi)\chi=\tau(\chi)\equiv 0 \pmod 8$. In particular $2\psi^2=\tau(\psi)\equiv 0 \pmod 8$. Together these two implies $\psi^2\equiv 0 \pmod 8$. The equation (5.3.16) implies that if $\chi\cup\psi=0$, $\tau(\chi)=0$. Now choosing a basis as in Case (2) of Proposition 5.2.13, we obtain the conditions in (5.3.15).

Now let $\sigma(M_k)=4$. Then $(\ker(\tau))^\perp=(4\psi)$ and ψ is determined modulo 2. We proceed analogously observing that

$$\chi\cup(4\psi)\equiv\tau(\chi)z \pmod 8 \ \forall\chi\in H^4(M_k),$$

which implies

$$\tau(\psi)\equiv 4\psi^2\equiv 0 \pmod 8.$$

Let ψ' be such that $\tau(\psi')=4$, and so we have that $\psi\cup\psi'$ is an odd multiple of z. If ψ^2 is $4 \pmod 8$ we change ψ to $\psi+2\psi'$ to ensure $\psi^2=\tau(\psi)\equiv 0 \pmod 8$. Now choosing a basis as in Case (2) of Proposition 5.2.13, we obtain the conditions in (5.3.15).

The case (2) also proceeds along analogous lines. For the existence (of f_ψ for some ψ) question, we need a basis satisfying

$$\begin{aligned}&\psi \cup \psi_i = 0 \text{ for } 1 \leq i \leq k-2, \quad \psi \cup \psi' = 1,\\ &\tau(\psi_i) \equiv 0 \pmod 4 \text{ for } 1 \leq i \leq k-2, \quad \tau(\psi') = \sigma(M_k), \quad \tau(\psi) = \psi^2 \equiv 0 \text{ or } 4 \pmod 8,\\ &\text{ such that at least one of } \tau(\psi_i) \text{ for } 1 \leq i \leq k-2, \text{ or } \tau(\psi) \equiv 4 \pmod 8.\end{aligned} \tag{5.3.17}$$

Note that $\langle\psi\rangle^\perp = \langle\psi_1, \ldots, \psi_{k-2}, \psi\rangle \subset \ker(\tau) = \langle\psi_1, \ldots, \psi_{k-2}, \psi, 2\psi'\rangle \pmod 4$ and $(\ker(\tau))^\perp = (2\psi) \pmod 4$. Hence, the hypothesis (H_4) is satisfied.
Conversely, if (H_4) is satisfied, we obtain a ψ such that $\psi^2 \equiv \tau(\psi) \equiv 0$ or $4 \pmod 8$. This ψ also satisfies

$$\chi \cup (2\psi) \equiv \tau(\chi)z \pmod 4 \; \forall \chi \in H^4(M_k).$$

Let ψ' be such that $\tau(\psi') = 2$, which implies that $\psi \cup \psi'$ is an odd multiple of z. Now replacing ψ by $\psi + 2\psi'$ if required we may assume that $\psi^2 \equiv \tau(\psi) \equiv 4 \pmod 8$. Now choosing a basis as in Case (2) of Proposition 5.2.13, we obtain the conditions in (5.3.17). ☐

If Rank$(H_4(M_k)) = k \geq 5$, the above results indicate a systematic computation of possible stable homotopy types of the total space depending on k, $\sigma(M_k)$ and the intersection form. In lower rank cases, the results depend on the explicit formula for the attachment $L(M_k)$, and not just on these variables. Hence the systematic description turns out to be cumbersome. We demonstrate some observations on the Rank$(H_4(M)) = 2$ case.

Example 5.3.18. *Recall that the attaching map $L(M) \in \pi_7(S^4 \vee S^4)$ of M_2 is of the form*

$$L(M) = [\alpha_1, \alpha_2] + l_1 \nu_1' + l_2 \nu_2'$$

where $l_1, l_2 \in \mathbb{Z}/12$. We already have

- *If both l_1 and l_2 are odd, there does not exist $f\colon M \to \mathbb{H}P^\infty$ such that $E =$ hofib(f) is 3-connected.*

If one of l_1 and l_2 is even, or both are even, there exists $f\colon M \to \mathbb{H}P^\infty$ such that $E =$ hofib(f) is 3-connected. Via an explicit calculation using Proposition 5.2.13, we observe the following.

1. *If none of l_1 and l_2 are divisible by 3, then we obtain $\lambda(E) \equiv 0 \pmod 3$.*
2. *If either of l_1 or l_2 or both are divisible by 3, then $\lambda(E) \not\equiv 0 \pmod 3$.*

3. *If* $\sigma(M) \equiv 4 \pmod 8$ *and* $l_1 l_2 \equiv 0 \pmod 8$ *where none of* l_1 *and* l_2 *are divisible by* 3, *then* $\lambda(E) \equiv 0 \pmod 8$.

4. *If* $\sigma(M) \equiv 2 \pmod 8$, *we can never obtain* $\lambda(E) \equiv 0, 4 \pmod 8$.

5.4 $SU(2)$-bundles over odd complexes

We now work out the case of $M_k \in \mathcal{PD}_3^8$ for which the intersection form is odd. Recall that the notation M_k means that Rank$(H_4(M_k)) = k$. The intersection form being odd implies that there are two possibilities of $\sigma(M_k)$, namely, 1 and 3 among the divisors of 24. Here, we prove
1) For $k \geq 3$, it is possible to obtain a $SU(2)$-bundle whose total space is 3-connected.
2) Further if $k \geq 7$, it is possible to obtain maps $\psi(j) : M_k \to \mathbb{H}P^\infty$ with $\lambda(\psi(j)) = j$ for every multiple $j \pmod{24}$ of $\sigma(M_k)$ which is also a divisor of 24.

5.4.1. Existence of $SU(2)$-bundles. Through an explicit computation, we demonstrate the existence of principal $SU(2)$-bundles over $M_k \in \mathcal{PD}_3^8$ if $k \geq 3$ when the intersection form is odd.

Proposition 5.4.2. *Suppose* $M_k \in \mathcal{PD}_3^8$ *such that Rank*$(H_4(M_k)) = k \geq 3$ *and the intersection form is odd. Then there exists a map* $\psi \colon M_k \to \mathbb{H}P^\infty$ *such that hofib*(ψ) *is* 3*-connected.*

Proof. Recall that the attaching map of M_k can be expressed as (1.3.1). Using Proposition 5.3.9, we are required to find a primitive element ψ such that $\tau(\psi)z \equiv \psi^2 \pmod{24}$, which is equivalent to checking that the coefficient of ν' in $\tilde{\psi} \circ L(M_k)$ is $0 \pmod{12}$. It suffices to find $\psi \pmod 8$ and $\psi \pmod 3$ separately.

We first work out the $\pmod 3$ case, where the base ring is a field of characteristic $\neq 2$ so that the form is diagonalizable. Considering the map $(S^4)^{\vee k} \xrightarrow{(0,\ldots,0,n_1,n_2,n_3)} \mathbb{H}P^\infty$ which sends $L(M_k)$ to

$$\Big(\pm\binom{n_1}{2} \pm \binom{n_2}{2} \pm \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k\Big) \cdot \nu' + \text{ multiple of } \nu,$$

where the $\pm$ correspond to the diagonal entries $\pmod 3$. We observe through a direct calculation that for every fixed choice $\epsilon_1, \epsilon_2, \epsilon_3$ of ± 1s, there is a n_1, n_2, n_3 with $\gcd(n_1, n_2, n_3) = 1$ such that

$$\Big(\epsilon_1\binom{n_1}{2} + \epsilon_2\binom{n_2}{2} + \epsilon_3\binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k\Big) \equiv 0 \pmod 3.$$

This completes the argument $\pmod 3$.

Working $\pmod 8$, the fact that the intersection form is odd implies that we may choose a basis such that $g_{k,k} \equiv \pm 1 \pmod 8$ and $g_{k,k-1} \equiv 0 \pmod 8$, see [33, Chapter II, (4.3)]. Also the intersection form can be written as the block matrix $\begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$.

If A is an even intersection form, then the result follows from Proposition 5.3.3 for $k \geq 5$. Now let A is not even, then the intersection form is $\begin{pmatrix} A' & 0 \\ 0 & B' \end{pmatrix}$ where B' is a diagonal matrix of order 2 with diagonal entries ± 1. If A' is an even intersection form, then for $k \geq 6$ the result follows from Proposition 5.3.3. If A' is not even, the intersection form updates to

$$\begin{pmatrix} A'' & 0 \\ 0 & B'' \end{pmatrix}$$

where B'' is a diagonal matrix of order 3 with diagonal entries ± 1. For $B'' = I_3$, the map $(S^4)^{\vee k} \xrightarrow{(0,\ldots,0,n_1,n_2,n_3)} \mathbb{H}P^\infty$ sends $L(M_k)$ to

$$\left(\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k\right) \cdot \nu' + \text{ multiple of } \nu,$$

where $\gcd(n_1, n_2, n_3) = 1$ ensures that the corresponding ψ is primitive. We may directly compute and observe that the equations

$$\left(\binom{n_1}{2} + \binom{n_2}{2} + \binom{n_3}{2} + n_1 l_{k-2} + n_2 l_{k-1} + n_3 l_k\right) \equiv 0 \pmod 8, \gcd(n_1, n_2, n_3) = 1,$$

have a common solution. A similar argument works for the other diagonal ± 1 matrix choices for B'''. This proves the result for $k \geq 6$.

If $3 \leq k \leq 5$, we know from [33, Chapter II, (3.2)-(3.4)] that the form is a direct sum of ± 1 and the hyperbolic form. The argument in the even case in Proposition 5.3.3 implies the result for a sum of two hyperbolic forms, and the above argument implies the result for a sum of 3 ± 1s and one hyperbolic form. The remaining cases are taken care of if we show the result for the intersection form

$$\begin{pmatrix} H & 0 \\ 0 & \pm 1 \end{pmatrix}$$

where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the hyperbolic matrix. We consider the map $(S^4)^{\vee 3} \xrightarrow{(n_1,n_2,n_3)} \mathbb{H}P^\infty$, and as above we need to find a common solution of

$$\binom{n_3}{2} + n_1 n_2 + \sum_{i=1}^{3} n_i l_i \equiv 0 \pmod 8, \gcd(n_1, n_2, n_3) = 1.$$

Through a direct calculation, we check that such solutions always exist. This completes the proof. □

The following example shows that Proposition 5.4.2 does not extend to the $k = 2$ case.

Example 5.4.3. *Consider M_2 such that*

$$L(M_2) = \nu_1 + \nu_2 + l_1\nu_1' + l_2\nu_2', \quad l_1 \equiv l_2 \equiv 2 \pmod 3.$$

A map $M_2 \to \mathbb{H}P^\infty$ which restricts to (n_1, n_2) on the 4-skeleton sends $L(M_2)$ to

$$\left(\binom{n_1}{2} + \binom{n_2}{2} + n_1 l_1 + n_2 l_2\right) \cdot \nu' + \text{ multiple of } \nu.$$

We may check directly that

$$\left(\binom{n_1}{2} + \binom{n_2}{2} + n_1 l_1 + n_2 l_2\right) \equiv 0 \pmod 3 \implies n_1 \equiv n_2 \equiv 0 \pmod 3.$$

Therefore, there is no map $M_2 \to \mathbb{H}P^\infty$ whose homotopy fibre is 3-connected.

5.4.4. Possible stable homotopy type of the total space. We note from §5.3 that Propositions 5.3.9, 5.3.10, and Theorem 5.3.11 are also valid when the intersection form is odd. We check that all stable homotopy types are achievable in the odd case if the rank k of $H_4(M)$ is ≥ 7. Applying the results from §5.3, it only remains to check that the different possibilities $\pmod 8$ are achievable. We do this in the theorem below.

Theorem 5.4.5. *Suppose $\sigma(M_k)$ is odd. Then for $k \geq 5$ and $j \in \{0, 2, 4\}$, there exists $\psi\colon M_k \to \mathbb{H}P^\infty$ such that $\lambda(\psi) \equiv j \pmod 8$.*

Proof. We work $\pmod 8$, knowing that if $k \geq 5$, Proposition 5.3.10 allows us to make a choice of ψ so that $\lambda(\psi)$ is as required $\pmod 3$. The proof is very similar to the proof of Theorem 5.3.11. If $\sigma(M_k)$ is odd, the linear map $\tau\colon (\mathbb{Z}/8)^k \to \mathbb{Z}/8$ is represented by some primitive class ψ (that is, $\tau(\chi) = \langle \chi, \psi \rangle \pmod 8$, and $\mathbb{Z}\{\psi\}$ is a summand of $H^4(M_k)$). In particular, $\psi^2 = \tau(\psi)$. Now, we have two cases.

First let $\tau(\psi)$ be odd, i.e. ψ^2 is unit in modulo 8. Then we can extend ψ to a basis $\psi, \psi_1, \ldots, \psi_{k-1}$ such that $\psi \cup \psi_i = 0$ for $1 \leq i \leq k-1$. We take the dual basis α_i corresponding to ψ_i and α_k corresponding to ψ. Thus $\psi(\alpha_i) = 0$ for $i = 1, \ldots, k-1$ and $\psi(\alpha_k) = 1$. Hence

$$\tau_1 \equiv \cdots \equiv \tau_{k-1} \equiv 0 \pmod 8, \quad \text{and} \quad \tau_k \equiv g_{k,k} \pmod 8,$$

which implies $\lambda(\psi) = \gcd(\tau_1, \ldots, \tau_{k-2}, \tau_{k-1}) \equiv 0 \pmod 8$ by Case (1) of Proposition 5.2.13. Now let ψ^2 be even. Extend ψ to a basis $\psi, \psi_1, \ldots, \psi_{k-1}$ such that $\psi_{k-1} \cup \psi = 1$ and $\psi_i \cup \psi = 0$ for $i = 1, \ldots, k-2$. After taking the dual basis, with similar argument we have

$$\tau_1 \equiv \cdots \equiv \tau_{k-2} \equiv 0 \pmod 8, \quad \tau_{k-1} \equiv 1 \pmod 8, \quad \text{and} \quad \tau_k \equiv g_{k,k} \pmod 8.$$

Hence $\lambda(\psi) = \gcd(\tau_1, \ldots, \tau_{k-2}, \tau_k - g_{k,k}\tau_{k-1}) \equiv 0 \pmod 8$ by Case (2) of Proposition 5.2.13. This proves the result for $j = 0$.

For $j = 2$, or 4, we can use $\psi(j) = \psi + j\psi_1$, and note that

$$\tau(\psi(j)) = \tau(\psi), \; \psi(j)^2 = \psi^2 + j^2\psi_1^2 \equiv \psi^2 \pmod 8.$$

The last equivalence comes from the fact that $\tau(\psi_1)z = \psi \cup \psi_1 \equiv 0 \pmod 8$, and so ψ_1^2 is forced to be an even multiple of z. We may now compute using the formulas of Proposition 5.2.13 to conclude that $\lambda(\psi(j)) \equiv j \pmod 8$. □

As in the even case, when the rank is high enough we have a systematic idea of the possibilities of the total space. However, in the low rank cases ($k \leq 6$) the results are not systematic, and may depend on individual cases rather than only on $\sigma(M_k), k$, and the intersection form.

Bibliography

[1] J. F. Adams, *On the non-existence of elements of Hopf invariant one*, Ann. of Math. (2), 72 (1960), pp. 20–104.

[2] ——, *On the cobar construction*, Proc. Nat. Acad. Sci. U.S.A., 42 (1956), pp. 409–412.

[3] ——, *On the groups $J(X)$. IV*, Topology, 5 (1966), pp. 21–71.

[4] ——, *The sphere, considered as an H-space* $\mathrm{mod}\, p$, Quart. J. Math. Oxford Ser. (2), 12 (1961), pp. 52–60.

[5] D. Barden, *Simply connected five-manifolds*, Ann. of Math. (2), 82 (1965), pp. 365–385.

[6] S. Basu, *The homotopy type of the loops on $(n-1)$-connected $(2n+1)$-manifolds*, in Algebraic topology and related topics, Trends Math., Birkhäuser/Springer, Singapore, 2019, pp. 1–25.

[7] M. Barratt, P. J. Eccles, $\Gamma+$*-structures—I: A free group functor for stable homotopy theory*, 1974, Topology.

[8] S. Basu and S. Basu, *Homotopy groups and periodic geodesics of closed 4-manifolds*, Internat. J. Math., 26 (2015), pp. 1550059, 34.

[9] ——, *Homotopy groups of highly connected manifolds*, Adv. Math., 337 (2018), pp. 363–416.

[10] ——, *Homotopy groups of certain highly connected manifolds via loop space homology*, Osaka J. Math., 56 (2019), pp. 417–430.

[11] S. Basu and A. K. Ghosh, *Sphere fibrations over highly connected manifolds*, (2023).

[12] S. Basu , A. K. Ghosh and S.Sau, *SU(2)-bundles over highly connected 8-manifolds*, (2024).

[13] P. Beben and S. Theriault, *The loop space homotopy type of simply-connected four-manifolds and their generalizations*, Adv. Math., 262 (2014), pp. 213–238.

[14] P. Beben and J. Wu, *The homotopy type of a Poincaré duality complex after looping*, Proc. Edinb. Math. Soc. (2), 58 (2015), pp. 581–616.

[15] A. Berglund and K. Börjeson, *Free loop space homology of highly connected manifolds*, Forum Math., 29 (2017), pp. 201–228.

[16] T. D. Cochran and N. Habegger, *On the homotopy theory of simply connected four manifolds*, Topology, 29 (1990), pp. 419–440.

[17] F. R. Cohen and S. Gitler, *On loop spaces of configuration spaces*, Trans. Amer. Math. Soc., 354 (2002), pp. 1705–1748.

[18] S. K. Donaldson, *An application of gauge theory to four-dimensional topology*, J. Differential Geom., 18 (1983), pp. 279–315.

[19] H. Duan and C. Liang, *Circle bundles over 4-manifolds*, Arch. Math. (Basel), 85 (2005), pp. 278–282.

[20] E. Fadell and L. Neuwirth, *Configuration spaces*, Math. Scand., 10 (1962), pp. 111–118.

[21] M. H. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom., 17 (1982), pp. 357–453.

[22] P. J. Giblin, *Circle bundles over a complex quadric*, J. London Math. Soc., 43 (1968), pp. 323–324.

[23] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[24] P. J. Hilton, *A certain triple Whitehead product*, Proc. Cambridge Philos. Soc., 50 (1954), pp. 189–197.

[25] ——, *On the homotopy groups of the union of spheres*, J. London Math. Soc., 30 (1955), pp. 154–172.

[26] R. Huang and S. Theriault, *Homotopy groups of manifolds stabilized by projective spaces*, Journal of Topology, pp. 1237-1257, doi = https://doi.org/10.1112/topo.12313,

[27] I. M. James, *Multiplication on spheres. I*, Proc. Amer. Math. Soc., 8 (1957), pp. 192–196.

[28] ——, *Multiplication on spheres. II*, Trans. Amer. Math. Soc., 84 (1957), pp. 545–558.

[29] ——, *On the suspension sequence*, Ann. of Math. (2), 65 (1957), pp. 74–107.

[30] M. A. Kervaire and J. W. Milnor, *Groups of homotopy spheres. I*, Ann. of Math. (2), 77 (1963), pp. 504–537.

[31] J. Milnor, *On manifolds homeomorphic to the* 7*-sphere*, Ann. of Math. (2), 64 (1956), pp. 399–405.

[32] ——, *On simply connected* 4*-manifolds*, in Symposium internacional de topología algebraica International symposium on algebraic topology, Universidad Nacional Autónoma de México and UNESCO, Mexico City, 1958, pp. 122–128.

[33] J. Milnor and D. Husemoller, *Symmetric bilinear forms*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 73, Springer-Verlag, New York-Heidelberg, 1973.

[34] M. Mimura, *The homotopy groups of Lie groups of low rank*, J. Math. Kyoto Univ., 6 (1967), pp. 131–176.

[35] J. C. Moore, *The double suspension and* p*-primary components of the homotopy groups of spheres*, Bol. Soc. Mat. Mexicana (2), 1 (1956), pp. 28–37.

[36] J. Neisendorfer, *Algebraic methods in unstable homotopy theory*, vol. 12 of New Mathematical Monographs, Cambridge University Press, Cambridge, 2010.

[37] V. A. Rohlin, *New results in the theory of four-dimensional manifolds*, Doklady Akad. Nauk SSSR (N.S.), 84 (1952), pp. 221–224.

[38] H. Samelson, *A connection between the Whitehead and the Pontryagin product*, Amer. J. Math., 75 (1953), pp. 744–752.

[39] S. Smale, *On the structure of* 5*-manifolds*, Ann. of Math. (2), 75 (1962), pp. 38–46.

[40] J. D. Stasheff, *Homotopy associativity of* H*-spaces. I, II*, Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid., 108 (1963), pp. 293–312.

[41] D. P. Sullivan, *Geometric topology: localization, periodicity and Galois symmetry*, vol. 8 of K-Monographs in Mathematics, Springer, Dordrecht, 2005. The 1970 MIT notes, Edited and with a preface by Andrew Ranicki.

[42] S. Theriault, *Homotopy fibrations with a section after looping*. to appear in Mem. Amer. Math. Soc.

[43] H. Toda, *Composition methods in homotopy groups of spheres*, Annals of Mathematics Studies, No. 49, Princeton University Press, Princeton, N.J., 1962.

[44] C. T. C. Wall, *Classification of $(n-1)$-connected $2n$-manifolds*, Ann. of Math. (2), 75 (1962), pp. 163–189.

[45] ——, *Poincaré complexes. I*, Ann. of Math. (2), 86 (1967), pp. 213–245.

[46] G. W. Whitehead, *Elements of homotopy theory*, vol. 61 of Graduate Texts in Mathematics, Springer-Verlag, New York-Berlin, 1978.

[47] J. H. C. Whitehead, *On simply connected, 4-dimensional polyhedra*, Comment. Math. Helv., 22 (1949), pp. 48–92.

[48] C. Wilkerson, *Spheres which are loop spaces* $\mod p$, Proc. Amer. Math. Soc., 39 (1973), pp. 616–618.

[49] A. Yamaguchi, *On the p-regularity of Stiefel manifolds*, Publ. Res. Inst. Math. Sci., 25 (1989), pp. 355–380.

[50] A. Zabrodsky, *Homotopy associativity and finite* CW *complexes*, Topology, 9 (1970), pp. 121–128.

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List of Publications

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