$\mathbb{A}^1$-homotopy types of $\mathbb{A}^2$ and $\mathbb{A}^2 \setminus \{(0,0)\}$

Biman Roy

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Author:
Biman Roy

Supervisor:
Utsav Choudhury

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Dedicated to my mother and my wife

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Biman Roy

Biman Roy

Contents

Chapter 1

Introduction

Let $\mathbb{A}^n_k$ be the affine n-space $Spec\ k[X_1, \ldots, X_n]$ over a field k. The complex analytic space associated to $\mathbb{A}^n_{\mathbb{C}}$ is the Euclidean space $\mathbb{C}^n$, which is topologically contractible. The set of complex points $X(\mathbb{C})$ of a complex variety X inherits complex analytic topology from the Euclidean space $\mathbb{C}^n$. Let us denote the topological space $X(\mathbb{C})$ with respect to the complex analytic topology by X^{an}. The implicit function theorem [68, Theorem 1.1.11] says that if the complex variety X is smooth, then X^{an} is a complex manifold. The smooth projective curves over $\mathbb{C}$ and the compact Riemann surfaces are essentially same [64, Theorem 3.1, Appendix B] and they are classified by their genus [64, Chapter IV]. A topologically contractible smooth complex affine variety has trivial Picard group [61, Theorem 1] and has only trivial group of units [56, Corollary 1.20]. Ramanujam, in his groundbreaking work showed that a smooth complex surface X is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ if and only if X^{an} is topologically contractible complex manifold and X^{an} is simply connected at infinity [108]. He also constructed a smooth complex surface to show that the topological contractibility is not enough to detect $\mathbb{A}^2_{\mathbb{C}}$ [108, Section 3].

Suppose Top is the category of the locally contractible topological spaces (sometimes it is the category of manifolds or the category of the CW complexes, according to the context) and $Ho(Top)$ is the associated homotopy category. A functor $\mathcal{F} : Top^{op} \to \mathcal{A}b$ ($\mathcal{A}b$ is the category of abelian groups, sometimes it will be category of sets, acccording to the context and Top^{op} is the opposite category of Top) is called I-invariant (I denotes the unit interval $[0,1]$) if the projection map $X \times I \to X$ induces an isomorphism $p^* : \mathcal{F}(X) \to \mathcal{F}(X \times I)$. There are several I-invariant functors on Top. A natural question is about the representability of the I-invariant functors in $Ho(Top)$. The singular cohomology functor $H^n(-;G)$ is I-invariant, where G is an abelian group. For a topological space X, $H^n(X, G)$ satisfies the Mayer-Vietoris property i.e. if X is a union of two open subsets U and V, then the sequence

$$H^n(X;G) \to H^n(U;G) \oplus H^n(V;G) \to H^n(U \cap V;G)$$

is exact (note that this property is very similar to the sheaf condition of a presheaf on a Grothendieck site). The classical Brown representibility establishes the representability of the singular cohomology in $Ho(Top)$ by the Eilenberg-Maclane spaces $K(G,n)$ [63, Theorem 4.57 and Section 4.E] i.e.

$$H^n(X;G) \cong [X, K(G,n)]_{Ho(Top)}.$$

The functor $Vect_n^{\mathbb{C}}(-)$ of the isomorphism classes of rank n complex vector bundles on the category of CW complexes is I-invariant and moreover it is representable in $Ho(Top)$ by the infinite Grassmannian BU_n [4, Theorem 8.5.13]. In particular for $n = 1$ we have,

$$Vect_1^{\mathbb{C}}(X) \cong [X, \mathbb{CP}^{\infty}]_{Ho(Top)} \cong H^2(X;\mathbb{Z}).$$

Complex K-theory functor $X \mapsto K(X)$ of the stably isomorphism classes of complex vector bundles over X is I-invariant and if X is a finite CW complex, then $K(-)$ is representable in $Ho(Top)$ by doubly infinite Grassmannian $BU \times \mathbb{Z}$ [4, Corollary 9.4.9] i.e.

$$K(X) \cong [X., BU \times \mathbb{Z}]_{Ho(Top)}.$$

Bott periodicity [4, Appendix B] and Brown representability [63, Theorem 4.58] imply that complex K-theory spectrum is indeed a reduced cohomology theory. Thus in particular complex K-theory satisfies Mayer-Vietoris property. This is a topological story.

Let Sm/k be the category of smooth, separated, finite type schemes over an algebraically closed field k. A presheaf of sets (or abelian groups or groups) $\mathcal{F} : (Sm/k)^{op} \to \mathcal{A}b$ is called $\mathbb{A}^1$-invariant if the projection map $X \times_k \mathbb{A}^1_k \to X$ induces an isomorphism $\mathcal{F}(X) \to \mathcal{F}(X \times_k \mathbb{A}^1_k)$ (see also Definition 3.2.1). Serre proved that finitely generated projective modules over a ring R are the essentially same as the algebraic vector bundles over $Spec\ R$, which are infact same with the locally free sheaf of $\mathcal{O}_{Spec\ R}$-modules ([116, §50], see also [117]). Any real or complex vector bundle over a topological contractible space is trivial. Celebrated Quillen-Suslin theorem is essentially the algebraic version of this, which says that any algebraic vector bundle over $\mathbb{A}^n_k$ is trivial ([116, Theorem 4], [107]). Inspired by the Quillen-Suslin theorem and the topological story, Bass-Quillen conjectured that if X is a regular Noetherian affine scheme of finite Krull dimension, then the projection map $X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}} \to X$ induces bijection

$$Vect_r(X) \to Vect_r(X \times_{\mathbb{Z}} \mathbb{A}^1_{\mathbb{Z}}),$$

where $Vect_r(X)$ denotes the set of all isomorphism classes of algebraic vector bundles of rank r over a scheme X [83, Section 6, Chapter 8]. Lindel in [85] proved Bass-Quillen conjecture is true if we restrict to smooth finite type affine k-schemes (see also [106], for a general version). The Picard group functor which is the isomorphism classes of the algebraic line bundles, is $\mathbb{A}^1$-invariant on Sm/k [64, Proposition 6.6]. More generally, the classical Chow group functor $CH^i(-)$ is $\mathbb{A}^1$-invariant on Sm/k [55, Theorem 3.3]. Classical Chow group (more generally, Bloch's higher Chow groups) shares similar properties as singular cohomology on topological spaces. Bloch's higher Chow group functor is $\mathbb{A}^1$-invariant and satisfies Mayer-Vietoris property in Zariski topology ([23, Theorem 2.1], [86, Theorem 3.3], see also [24],). For X is a smooth complex projective variety, there is a map $\eta : CH^i(X) \to H^{2i}(X, \mathbb{Z})$ which gives the fundamental class map

$$\eta : CH^*(X) \to H^*(X, \mathbb{Z}).$$

The map η takes the intersection products to the cup products ([52, Appendix C.2], [55, Chapter 19]). If X is the projective n-space $\mathbb{P}^n_{\mathbb{C}}$, then η is an isomorphism. The Grothendieck group $K_0(X)$ of the stably isomorphism classes of algebraic vector bundles over $X \in Sm/k$ is $\mathbb{A}^1$-invariant on Sm/k. In general, Quillen's higher K-groups $K_n(X)$ is $\mathbb{A}^1$- invariant [129, Lemma 12.8]. Thomason-Trobaugh proved that algebraic K-theory satisfies Nisnevich descent, equivalently algebraic K-theory satisfies Mayer-Vietoris property in the Nisnevich topology ([122, Theorem 10.8], see also [129, Section 10, Chapter V]) i.e. for an elementary distinguished square in Definition 2.2.2, the induced square of simplicial sets

$$\begin{array}{ccc} K(X) & \longrightarrow & K(U) \\ \downarrow & & \downarrow \\ K(V) & \longrightarrow & K(U \times_X V) \end{array}$$

is homotopy cartesian.

$\mathbb{A}^1$-homotopy theory, constructed by Morel and Voevodsky [94], is a way where we can apply the algebraic topology techniques in algebraic geometry. The homotopy category of topological spaces is constructed (or homotopy category of CW complexes or manifolds) making the unit interval equivalent to point. The topological contractibility of the unit interval $[0, 1]$ is given by the multiplication map which is a homotopy equivalence

$$\theta : I \times I \to I \text{ defined as } (x, y) \mapsto xy.$$

The multiplication map μ on the affine line $\mathbb{A}^1_k$ is a morphism of varieties

$$\mu : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to \mathbb{A}^1_k$$

. The 0-section and the 1-section of μ are the identity map and the constant map respectively. Extending the category Sm/k to the category of spaces $\Delta^{op}PSh(Sm/k)$ and using the standard model category techniques to invert the projection maps $X \times_k \mathbb{A}^1_k \to X$ and the Nisnevich local weak equivalences, Morel-Voevodsky constructed the unstable $\mathbb{A}^1$-homotopy category $\mathbf{H}(\mathbf{k})$. In $\mathbf{H}(\mathbf{k})$, naturally by the construction X and $X \times_k \mathbb{A}^1_k$ are isomorphic, for every $X \in Sm/k$. In particular, the affine line $\mathbb{A}^1_k$ is equivalent to the point $Spec\ k$ in $\mathbf{H}(\mathbf{k})$. The isomorphism classes of algebraic vector bundles of rank n over an affine scheme $X \in Sm/k$ is representable in $\mathbf{H}(\mathbf{k})$

by infinite Grassmannian ([93, Theorem 8.1], [123, Theorem 6.22], [12, Theorem 1]). The infinite projective space $\mathbb{P}^\infty_k$ represents the Picard group functor [94, Proposition 3.8, Section 4] i.e.

$$Pic(X) \cong Hom_{\mathbf{H(k)}}(X, \mathbb{P}^\infty_k).$$

In $\mathbf{H(k)}$, the algebraic K-theory $K_n(-)$'s is representable by doubly infinite Grassmannian ([94, Theorem 3.13, Section 4], see also [124, Remark 2]). There are the motivic Eilenberg-Maclane spaces $K(p, q, A)$ that represent the motivic cohomology in $\mathbf{H(k)}$ [127, Theorem 2]. Therefore the Bloch's higher Chow group is representable in $\mathbf{H(k)}$ [96, Lecture 17]. Motivic homotopy theory has successfully applications in algebraic geometry and number theory. It has been used to prove the Bloch-Kato conjecture and the norm residue isomorphism in characteristic 2 ([125], [126]). Motivic homotopy theory has been successfully used to obtain the geometric versions of the Grothendieck conjecture and the Konstevich-Zagier conjecture ([17], [18], [19], see also [21]) On the other hand, this new homotopical algebraic geometry gives us new insights in classical questions of algebraic topology ([50], [71]).

Characterisation problem is one of the central problems in affine algebraic geometry ([82]). There is also a topological story. The only topologically contractible open manifolds over $\mathbb{R}$ are the real line and the real plane in dimensions 1 and 2 respectively. However for $n \geq 3$, there are topologically contractible open manifolds over $\mathbb{R}$ in dimension n that are not homeomorphic to $\mathbb{R}^n$ ([130], [58]). The algebro-geometric picture goes in the similar way. The affine line $\mathbb{A}^1_k$ is the only $\mathbb{A}^1$-contractible smooth affine curve over a field k [6, Theorem 5.4.2.9]. Asok-Doran constructed infinitely many $\mathbb{A}^1$-contractible smooth varieties of dimension $n \geq 4$, which are the quotients of $\mathbb{A}^n_k$'s by a suitable $\mathbb{G}_a$-action [9, Theorem 5.1, Theorem 5.3]. However, these exotic $\mathbb{A}^1$-contractible varieties, constructed by Asok-Doran are strictly quasi-affine. The Koras-Russell threefolds of the first kind are also exotic $\mathbb{A}^1$-contractible varieties ([45, Theorem 1.1], [65, Theorem 4.2], [54, Theorem 9.9]). In dimension $n \geq 4$, no example of smooth $\mathbb{A}^1$-contractible affine k-varieties is known so far. The Ramanujam surface [108, Section 3] and the tom Dieck-Petrie surfaces [46, Theorem A] are topologically contractible but not isomorphic to $\mathbb{A}^2_{\mathbb{C}}$. In this situation, it is natural to ask whether $\mathbb{A}^2_k$ is the only $\mathbb{A}^1$-contractible variety of dimension 2 [6, Conjecture 5.5.2.3]. In this thesis we prove the following theorem which is the main theorem in this thesis.

Theorem 1.0.1. *(see Theorem 8.1.1) An $\mathbb{A}^1$-contractible smooth affine surface over a field k of characteristic zero is isomorphic to the affine plane $\mathbb{A}^2_k$.*

The study of the classification of varieties using the $\mathbb{A}^1$-homotopy types was initiated by Asok and Morel [11]. Asok-Morel proved that upto $\mathbb{A}^1$-weak equivalence, the only rational smooth proper surfaces over the field k are $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ or blow up of $\mathbb{P}^2$ at the finitely many points [11, Theorem 3.2.1]. They introduced $\mathbb{A}^1$-chain connectedness of a variety $X \in Sm/k$, which is very similar to the notion of path connectedness of a topological space. The $\mathbb{A}^1$-chain connected component sheaf $\pi_0^{ch}(X)$ associated to $X \in Sm/k$ is related to the $\mathbb{A}^1$-connected component sheaf $\pi_0^{\mathbb{A}^1}(X)$ (see Definition 2.2.11 and Definition 2.2.10). A variety X is $\mathbb{A}^1$-connected if it is $\mathbb{A}^1$-chain connected i.e. for every finitely generated separable field extension F/k, the section $\pi_0^{ch}(X)(Spec\ F)$ is trivial which means that any two F-points of X can be joined by a chain of $\mathbb{A}^1_F$'s in X. They proved a smooth proper scheme $X \in Sm/k$ is $\mathbb{A}^1$-connected if and only if it is $\mathbb{A}^1$-chain connected ([11, Theorem 2], see also [26, Corollary 3.10]). Near rationality property of a smooth proper variety is related to its $\mathbb{A}^1$-connectedness. The $\mathbb{A}^1$-connectedness is a birational property of a smooth proper variety over the field of characteristic zero [11, Corollay 2.4.6]. In particular, an $\mathbb{A}^1$-connected proper surface over a field of characteristic zero is rational. Infact any $\mathbb{A}^1$-connected surface over the field of characteristic zero is rational (see Theorem 5.2.1). Asok-Morel and Kahn-Sujatha proved that any retract k-rational proper variety is $\mathbb{A}^1$-chain connected ([11, Theorem 2.3.6], [72, Theorem 8.5.1 and Theorem 8.6.2]) and this result is recently improved by Balwe-Rani [34, Theorem 1.1]. The $\mathbb{A}^1$-connected component of a variety plays a crucial role in determining the existence of $\mathbb{A}^1$'s in a variety. In [26, Definition 3.2], Balwe-Hogadi-Sawant introduced the notion of $\mathbb{A}^1$-ghost homotopy. They extended Asok-Morel's $\mathbb{A}^1$-chain connected component of a variety to a general sheaf $\mathcal{F}$ on Sm/k and they considered the n-th iteration $\mathcal{S}^n(\mathcal{F})$. The sheaf $\mathcal{S}^n(\mathcal{F})$ carries the data of n-$\mathbb{A}^1-$ghost homotopies. They defined the universal $\mathbb{A}^1$-invariant sheaf $\mathcal{L}(\mathcal{F})$ by taking colimits of $\mathcal{S}^n(\mathcal{F})$'s [26, Theorem 2.13]. Though $\mathbb{A}^1$-connectedness of a smooth variety X does not imply that any two F-points of X are naively $\mathbb{A}^1$-homotopic, however if X is $\mathbb{A}^1$-connected, then any two F-points of X are n-$\mathbb{A}^1$-ghost homotopic [26, Corollary 2.18]. More generally, Balwe-Rani-Sawant showed that the section $\pi_0^{\mathbb{A}^1}(X)(Spec\ F)$ agrees with the section $\mathcal{L}(X)(Spec\ F)$, for any finitely generated separable field extensions F/k ([33, Theorem 2.2], see also Corollary 3.3.9). From this, it is tempting to think that $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant for $X \in Sm/k$. However, Ayoub constructed a space $\mathcal{X}$ for which $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not $\mathbb{A}^1$-invariant [20]. In this thesis, we study the $\mathbb{A}^1$-connected component sheaf of a smooth variety in great detail.

The successive works of Balwe-Hogadi-Sawant and Balwe-Sawant have established the usefulness of the ghost homotopy techniques in computing the $\mathbb{A}^1$-connected components of a smooth variety. Using ghost homotopy techniques, Balwe-Hogadi-Sawant [26, Corollary 3.10] gave a different proof of Asok-Morel's result [11, Theorem 2] on the comparison of the $\mathbb{A}^1$-chain connected component and the $\mathbb{A}^1$-connected component of a smooth proper variety [26, Corollary 3.10]. The $\mathbb{A}^1$-connected component and $\mathbb{A}^1$-chain connected component agree for a non-uniruled proper k-surface [26, Corollary 3.15] and they agree over the sections of the Henselian local schemes of dimension ≤ 1 ([30, Theorem 1], see also [29]), in case of a birationally ruled proper surface over a field of characteristic zero. The ghost homotopy techniques has beautiful application in determining the $\mathbb{A}^1$-connected component of a reductive algebraic group over a field of characteristic zero ([31, Theorem 3.4], [32, Theorem 2], see also [28, Theorem 1.3]), which was known to be $\mathbb{A}^1$-invariant [37, Corollary 4.18]. Balwe-Hogadi-Sawant proved that the $\mathbb{A}^1$-connectedness of a smooth proper variety over an algebraically closed field k is equivalent to the fact that the generic point can be joined to a k-rational point by a chain of $\mathbb{P}^1$'s [27, Corollary 2.4], which improves the Asok-Morel's result [11, Theorem 2.4.3]. Using the beautiful idea of ghost homotopy, we prove that the $\mathbb{A}^1$-connectedness of a smooth variety is indeed very much related to the existence of affine lines in the variety. We prove the following:

Theorem 1.0.2. *(see Theorem 6.1.2) An $\mathbb{A}^1$-connected smooth variety over an algebraically closed field k is $\mathbb{A}^1$-uniruled.*

The above theorem immediately implies the following corollary:

Corollary 1.0.3. *(see Corollary 6.1.3) An $\mathbb{A}^1$-connected smooth variety over an algebraically closed field k of characteristic zero has negative logarithmic Kodaira dimension.*

This is a result in the interface of $\mathbb{A}^1$-homotopy theory and birational geometry. The existence of the family of affine lines plays crucial role in the characterisation of the affine plane. In case of a smooth affine surface over an uncountable algebraically closed field k of characteristic 0, the properties of being $\mathbb{A}^1$-uniruled, $\mathbb{A}^1$-ruled and the negativity of logarithmic Kodaira dimension are equivalent ([80, Theorem 1.1] [91, §4, §5]). The logarithmic Kodaira dimension and existence of affine lines in a surface play a crucial role in studying the homology planes ([60], [90], [62]). Miyanishi proved the following algebraic characterisation of the affine plane which is a fundamental result in affine algebraic geometry:

Theorem 1.0.4. *[92, Section 4.1] A smooth affine surface $X = Spec\ A$ over an algebraically closed field k field of characteristic zero is isomorphic to the affine plane $\mathbb{A}^2_k$ if and only if A is a U.F.D. with only trivial units and X has negative logarithmic Kodaira dimension.*

Using the algebraic characterisation, we prove the main theorem in this thesis (Theorem 8.1.1). The key ingredient is the negativity of the logarithmic Kodaira dimension which we deduce from the $\mathbb{A}^1$-connectedness.

There is a topological realisation functor $\mathbf{H}(\mathbb{C}) \to Ho(Top)$ which takes a complex variety $X \in Sm/\mathbb{C}$ to $X(\mathbb{C})$ with respect to the complex analytic topology [43, Proposition 8.3]. Therefore an $\mathbb{A}^1$-contractible complex variety is always topologically contractible. However our main theorem implies that converse is not true.

Corollary 1.0.5. *(See Corollary 8.1.4) There are topologically contractible smooth complex surfaces which are not $\mathbb{A}^1$-contractible. For example, the Ramanujam surface [108, Section 3], the tom Dieck-Petie surfaces [46, Theorem A] are not even $\mathbb{A}^1$-connected.*

To study $\mathbb{A}^1$-connected component sheaf $\pi_0^{\mathbb{A}^1}(X)$ associated to a proper scheme $X \in Sm/k$, Asok-Morel introduced a birational, $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$ [11, Section 6.2]. There are canonical morphisms

$$\pi_0^{ch}(X) \to \pi_0^{\mathbb{A}^1}(X) \to \pi_0^{b\mathbb{A}^1}(X).$$

They proved that each of the canonical morphisms induces bijections over the sections $Spec\ F$, for every finitely generated separable field extensions F/k [11, Proposition 6.2.6]. We proved that the sheaf $\pi_0^{b\mathbb{A}^1}(X)$ is related to a homotopy theory on the category of spaces. It is the connected component sheaf of X in the birational model structure. We denote the connected component sheaf of $X \in Sm/k$ by $\pi_0^b(X)$ in the birational model structure (Definition 4.2.2).

Theorem 1.0.6. *(Theorem 4.2.3) Suppose, X is a smooth proper scheme over a field k. Then the canonical morphism $\eta : \pi_0^{b\mathbb{A}^1}(X) \to \pi_0^b(X)$ is an isomorphism.*

Poincarè conjecture is one of the classical questions in algebraic topology. The topological classification of compact surfaces [97, Theorem 5.1, Chapter 1] or the classical Uniformaization theorem of Riemann Surface ([2, Section 5], see also [1, Theorem 1.7.2], for existence of complex strucure in an oriented closed 2-manifold) tells us that any simply-connected closed 2-manifold is homeomorphic to the sphere. The Poincaré conjecture says that a simply connected closed three manifold is homeomorphic to the 3-sphere. The generalised Poincaré conjecture says that if a closed n-manifold has the homotopy type of the n-sphere in $\mathbb{R}^{n+1}$, then is homeomorphic to the n-sphere. The successive works of Smale [114], Fredman [53] and Hamilton-Perelman ([103], [104], [105]) established the affirmative solution to the generalised Poincaré conjecture. The Poincaré conjecture is a particular case of the Thurston's geometrization conjecture, which describes the fundamental geometries of a closed oriented 3-manifold and this was also proved by groundbreaking works of Perelman (see [35]).

There are two kinds of circles in $\mathbf{H}_\bullet(\mathbf{k})$: one is the simplicial circle S_s^1, which is defined to be the quotient of Δ^1 by its boundary and the other is the Tate circle S_t^1, which is defined to be the multiplicative group $\mathbb{G}_m$, pointed by 1 [94, Section 3.2]. The motivic spheres in $\mathbf{H}_\bullet(\mathbf{k})$ are the smash products of the copies of S_s^1 and the copies of S_t^1:

$$S^{i,j} := S_s^{i-j} \wedge S_t^j,$$

where S_s^p and S_t^q are the smash product of p-many copies of S_s^1 and the q-many copies of S_t^1 respectively. These motivic spheres are analogous to the spheres in algebraic topology. The quasi-affine varieties $\mathbb{A}_k^n \setminus \{(0,\dots,0)\}$ are the mixed motivic spheres in $\mathbf{H}_\bullet(k)$ [94, Example 2.20, Section 3.2]:

$$\mathbb{A}_k^n \setminus \{(0,\dots,0)\} \cong S_s^{n-1} \wedge S_t^n = S^{2n-1,n} \text{ in } \mathbf{H}_\bullet(k).$$

There is smooth affine $(2n-1)$-dimensional variety Q_{2n-1}

$$Q_{2n-1} := Spec(k[x_1,\dots,x_n,y_1,\dots,y_n]/(\sum_{i=1}^{n} x_i y_i - 1))$$

which is isomorphic to $\mathbb{A}^n_k \setminus \{(0,\dots,0)\}$ in $\mathbf{H}_\bullet(\mathbf{k})$ [7, Theorem 2]. Thus $\mathbb{A}^n_k \setminus \{(0,\dots,0)\}$ has $\mathbb{A}^1$-homotopy type of a smooth affine $(2n-1)$-fold. There is $2n$-dimensional smooth affine variety Q_{2n}

$$Q_{2n} := Spec(k[x_1,\dots,x_n,y_1,\dots,y_n,z]/(\sum_{i=1}^{n} x_i y_i - z(1+z))$$

such that Q_{2n} is isomorphic to the mixed motivic sphere $S^n_s \wedge S^n_t$ in $\mathbf{H}_\bullet(\mathbf{k})$ [7, Theorem 2]. More generally for a smooth affine k-variety X, the space $S^i_s \wedge S^i_t \wedge X$ has $\mathbb{A}^1$-homotopy type of a smooth affine k-variety [8, Theorem 4]. However, there is no smooth affine k-variety X isomorphic to $S^i_s \wedge S^j_t$ in $\mathbf{H}_\bullet(\mathbf{k})$, for $i > j$ [7, Proposition 4]. Moreover, Asok-Doran-Fasel also conjectured that for $i < j-1$, $S^i_s \wedge S^j_t$ can not be $\mathbb{A}^1$-weakly equivalent to a smooth k-variety. It is natural to ask that whether a pointed smooth k-variety X of dimension n, $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^n_k \setminus \{(0,\dots,0)\}$, is isomorphic to $\mathbb{A}^n_k \setminus \{(0,\dots,0)\}$ as k-varieties i.e.

Question 1.0.7. Suppose X is an n-dimensional smooth variety. If $X \cong S^{2n-1,n}$ in $\mathbf{H}_\bullet(\mathbf{k})$, then is X isomorphic to $\mathbb{A}^n_k \setminus \{(0,\dots,0)\}$?

In dimension 1, any smooth curve which is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^1_k \setminus \{0\}$, is isomorphic to $\mathbb{A}^1_k \setminus \{0\}$ (see Theorem 9.1.1). In dimension 2, we prove that over a field k of characteristic zero $\mathbb{A}^2_k \setminus \{(0,0)\}$ is the only open k-subvariety of a smooth affine k-surface, which is $\mathbb{A}^1$-weakly equivalent to the mixed motivic sphere $S^{3,2}$.

Theorem 1.0.8. *(see Theorem 9.1.2) Suppose X is a smooth affine surface over a field k of characteristic zero and $U \subset X$ is a non-empty open subscheme. Suppose that U is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^2_k \setminus \{(0,0)\}$. Then U is isomorphic to $\mathbb{A}^2_k \setminus \{(0,0)\}$ as k-varieties.*

However, there are smooth quasi-affine threefolds which are $\mathbb{A}^1$-weakly equivalent to the mixed motivic sphere $S^{5,3}$ but not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$. We prove the following theorem.

Theorem 1.0.9. *(see Theorem 9.2.3) Suppose X is a Koras-Russell threefold of the first kind and $p = (1,0,1,0)$ is a k-rational point of X. Then $X \setminus \{p\}$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$, but $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$.*

1.0.1 Arrangement of the Thesis

In Chapter 2 we discuss about the background materials in this thesis. In Section 2.1, we recall about the model categories. In Section 2.2, we recall the construction of the Morel-Voevodsky's $\mathbb{A}^1$-homotopy category $\mathbf{H}(\mathbf{k})$ and some of its properties. We recall Voevodsky's construction of the triangulated category of geometric motives $\mathbf{DM}_{gm}(k,\mathbb{Z})$ over a field k and some of its properties in Section 2.3. In Section 2.4, we recall the $\mathbb{A}^1$-derived category $D_{\mathbb{A}^1}(\mathcal{A}b(k))$, constructed by Morel and the $\mathbb{A}^1$-homology sheaves.

In Chapter 3 we discuss about the $\mathbb{A}^1$-invariance of the $\mathbb{A}^1$-connected component sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, of a space $\mathcal{X}$ over k. In Section 3.1, we recall the $\mathbb{A}^1$-rigid schemes (Definition 3.1.1), which are $\mathbb{A}^1$-invariant representable sheaves, along with their properties. A k-variety X has a local base of $\mathbb{A}^1$-rigid schemes at every points (Lemma 3.1.4). In Section 3.2, we review on Morel's Conjecture about the $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ and recall Balwe-Hogadi-Sawant's universal $\mathbb{A}^1$-invariant sheaf $\mathcal{L}(\mathcal{F})$, associated to a sheaf $\mathcal{F}$ (Subsection 3.2.2). In Section 3.3, we prove that the canonical surjection

$$\pi_0^{\mathbb{A}^1}(\mathcal{X})(Spec\ F) \to \mathcal{L}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(Spec\ F)$$

is a bijection, for every finitely generated separable field extension F/k (see Corollary 3.3.9). In Section 3.4, we provide some equivalent criterias on the $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ (Theorem 3.4.7).

In Chapter 4, we define the birational model structure on the category of spaces over k (Proposition 4.2.1). In Proposition 4.2.7, we prove that the birational model structure is indeed the left Bousfield localisation of the $\mathbb{A}^1$-model structure on the category of spaces over k at the class of the birational morphisms (see also Remark 4.2.8). We prove that the Asok-Morel's birational and $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$, associated to a proper scheme $X \in Sm/k$ is isomorphic the connected component sheaf of X in this birational model structure. We prove this in Theorem 4.2.3, which is the main theorem in this chapter.

Chapter 5 and Chapter 6 are the main chapters that provide the key ingredient to prove the main theorem (Theorem 8.1.1) in this thesis. In these two chapters we establish the fact that $\mathbb{A}^1$-connectedness of a smooth variety X over an algebraically closed field is related to the existence of affine lines in X. In Section 5.1, we recall several kinds of varieties containing the images of affine lines and how they are related to the negativity of the logarithmic Kodaira dimension. In Section 5.2, we prove that if a surface is $\mathbb{A}^1$-connected, then it is dominanted by images of $\mathbb{A}^1$ (Theorem 5.2.8, see Definition 5.1.1). In Theorem 6.1.2, we prove that any $\mathbb{A}^1$-connected variety is $\mathbb{A}^1$-uniruled. Therefore if X is $\mathbb{A}^1$-connected and the base field k is of characteristic zero, then X has negative logarithmic Kodaira dimension (Corollary 6.1.3). In Proposition 6.2, we prove that if X is a smooth affine k-surface with $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial, then either X contains a dominant family of affine lines or through every k-rational point there is an $\mathbb{A}^1$ in X along with two interesecting $\mathbb{A}^1$'s in X. We end Chapter 6 with some comments on the behaviours of the $\mathbb{A}^1$-connected component sheaf of an affine surface X over the field of positive characteristic with the geometric properties of X (Subsection 6.2.1).

The main result in Chapter 7 is Corollary 7.2.5, where we prove that if X is a smooth affine k-surface with $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial and $Sing_*(X)(Spec\ k)$ is Kan fibrant in degree 2, then X has negative logarithmic Kodaira dimension. We derive this result as a corollary of Proposition 6.2. A simplicial set is Kan fibrant in degree n if any l-th horn Λ_l^n can be filled to an

n-simplex (see Definition 7.2.1). Since $\mathbb{A}^n_k$ is an affine group scheme, $Sing_*(\mathbb{A}^n_k)(U)$ is simplicial abelian group for every $U \in Sm/k$, thus it is Kan fibrant [98, Definition 1.3]. In section 7.1 using [100, Theorem 3.1], we provide explicit formulas of horn filling of $Sing_*(\mathbb{A}^n_k)(U)$ (see Subsection 7.1.1) and using the formulas we prove that $Sing_*(\mathbb{A}^m_k)(U)$ is Kan fibrant in degree 2, for every $U \in Sm/k$ (see Lemma 7.2.3).

In Chapter 8, we prove the main theorem in this thesis. In Theorem 8.1.1, we prove that the affine plane $\mathbb{A}^2_k$ is the only $\mathbb{A}^1$-contractible smooth affine surface over any field k of characteristic zero. Theorem 8.1.3 is a consequence of the main theorem, which says that a topologically contractible smooth complex surface is isomorphic to the complex affine plane if and only if it is $\mathbb{A}^1$-connected. This gives Corollary 8.1.5 which establishes that for a smooth complex surface $\mathbb{A}^1$-contractibility is indeed a stronger notion than the topological contractibility. In particular, there are complex surfaces X such that the motive $M(X) \cong \mathbb{Z}$ in $\mathbf{DM}_{gm}(\mathbb{C}, \mathbb{Z})$, but X is not $\mathbb{A}^1$-contractible (Corollary 8.1.7). In Corollary 8.1.10, we provide characterisations of $\mathbb{A}^3_k$ and $\mathbb{A}^4_k$, as immediate consequences of the main theorem. In Subsection 8.1.1, we recall the locally nilpotent derivation on a k-algebra and as a corollary of the main theorem we prove that a smooth affine threefold over a field of charactertic zero is isomorphic to $\mathbb{A}^3_k$ if and only if it is $\mathbb{A}^1$-contractible and it has a locally nilpotent derivation with a slice (Corollary 8.1.13). In [111] Sathaye proved the following characterisation of the affine plane over a discrete valuation ring of equicharacteristic zero:

Theorem 1.0.10. *[111, Theorem 1] Let R be a discrete valuation ring (i.e. R is a Noetherian local domain of dimension 1 with the maximal ideal is principal) of equicharacteristic zero with the residue field and the field of fractions are k and K respectively. Suppose that A is an affine R-domain such that $A \otimes_R k \cong k[x, y]$ and $A \otimes_R K \cong K[x, y]$. Then A is isomorphic to $R[x, y]$ as R-algebras.*

Using Sathaye's theorem we obtain the following characterisation of $\mathbb{A}^2_R$ as a consequence of the main theorem in Subsection 8.1.2:

Theorem 1.0.11. *(see Theorem 8.1.14) Let R be a discrete valuation ring of equicharacteristic zero and X be a smooth affine scheme over R of relative dimension 2. Then X is $\mathbb{A}^1$-contractible if and only if X is isomorphic to $\mathbb{A}^2_R$.*

In [3, Theorem 5.1] Asanuma constructed an affine R-domain, where R is a discrete valuation ring with the residue field is of positive characteristic, such that the base extensions of A over the residue field and the fraction field of R are the polynomial rings in two variables and also $A[t] \cong R[x, y, z]$, but A is not isomorphic to $R[x, y]$. This example of Asanuma's pseudopolynomial domain shows that Theorem 8.1.14 is not true in case of discrete valuation ring of not equicharacteristic zero (see Remark 8.1.15).

In Chapter 9, the main theorem is Theorem 9.1.2 where we prove that if U is an open subscheme of a smooth affine k-surface, where k is any field of characteristic zero and U is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^2_k \setminus \{(0, 0)\}$, then U is isomorphic to $\mathbb{A}^2_k \setminus \{(0, 0)\}$. However, this is not true in dimension 3. In Theorem 9.2.3, we prove that if X is a Koras-Russell threefold of the first kind, then $X \setminus \{p\}$, where $p = (1,0,1,0)$ is the k-rational point of X, is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$, but $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$.

In Chapter 10, we define an algebraic invariant $\mathcal{O}_{ch}(X)$ associated to an affine k-variety X. The ring $\mathcal{O}_{ch}(X)$ consists the regular functions on X that are constant along every affine lines in X (see Definition 10.1.1). The ring $\mathcal{O}_{ch}(X)$ is related to the classical Makar-Limanov invariant of X (Proposition 10.2.6, see also [54, Section 2.5]). Unlike the Makar-Limanov invariant, $\mathcal{O}_{ch}(-)$ is a functorial invariant and the projection map $X \times_k \mathbb{A}^1_k \to X$ induces isomorphism (Proposition 10.2.7)

$$\mathcal{O}_{ch}(X) \to \mathcal{O}_{ch}(X \times_k \mathbb{A}^1_k),$$

however $\mathcal{O}_{ch}(-)$ is not representable in the $\mathbb{A}^1$-homotopy category $\mathbf{H}(\mathbf{k})$ (see Lemma 10.2.15). The ring $\mathcal{O}_{ch}(X)$ detects the affine lines in X. A smooth affine k-surface X with $\mathcal{O}(X)$ is a U.F.D. has dense set of affine lines if and only if $\mathcal{O}_{ch}(X) = k$ (see Theorem 10.2.5). In Proposition 10.2.10, we prove that

$$\mathcal{O}_{ch}(X) = Hom_{Sh(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k),$$

as k-subalgebras of $\mathcal{O}(X)$. In Theorem 10.2.19 we prove a straightforward characterisation of the affine k-plane using this invariant $\mathcal{O}_{ch}(-)$.

In the final chapter (Chapter 11), we define an universal $\mathbb{A}^1$-invariant sheaf of abelian groups $H_0^{\mathbb{A}^1\text{-naive}}(X)$ associated to a scheme $X \in Sm/k$ (see Definition 11.1.2). It is universal in the sense that given any morphism from X to an $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathcal{G}$ uniquely factors through $H_0^{\mathbb{A}^1\text{-naive}}(X)$ (Remark 11.1.3). Thus we have a canonical morphism

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1}(X).$$

In Theorem 11.1.5, we give several equivalent descriptions of $H_0^{\mathbb{A}^1\text{-naive}}(X)$, for example it is isomorphic to the universal $\mathbb{A}^1$-invariant sheaf $\mathcal{L}(\mathbb{Z}(X))$ (Proposition 11.1.4). In Corollary 11.1.6, we prove that if X is $\mathbb{A}^1$-connected, then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ isomorphic to the constant sheaf $\mathbb{Z}$. In Section 11.2, we describe some useful properties of $H_0^{\mathbb{A}^1\text{-naive}}(X)$, for a proper scheme $X \in Sm/k$. In Theorem 11.2.11, we prove that if $X \in Sm/k$ is a proper scheme, then the canonical morphism

$$\eta : \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)) \to H_0^{\mathbb{A}^1\text{-naive}}(X)$$

is an isomorphism of abelian groups, over the sections of every finitely generated separable field extension F/k. As a consequence in Corollary 11.2.12 we prove that a proper scheme X is $\mathbb{A}^1$-connected if and only if $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to the constant sheaf $\mathbb{Z}$. There is a canonical morphism

$$\theta : H_0^{\mathbb{A}^1\text{-naive}}(X) \to \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X)),$$

for a proper scheme $X \in Sm/k$ and in Corollary 11.2.14 we prove that θ induces isomorphism over the sections of every finitely generated separable field extensions F/k. As a consequence we have the canonical morphism induces isomorphism

$$H_0^{\mathbb{A}^1\text{-naive}}(X)(Spec\ F) \to H_0^{\mathbb{A}^1}(X)(Spec\ F),$$

for every finitely generated separable field extension F/k (see Remark 11.2.15).

Chapter 2

$\mathbb{A}^1$-homotopy theory: An Introduction

In this chapter we briefly recall the materials we use throughout the thesis. The main section in this chapter is Section 2.2, where we recall the $\mathbb{A}^1$-homotopy category $\mathbf{H}(\mathbf{k})$, constructed by Fabien Morel and Vladimir Voevodsky [94]. We also recall the model categories in Section 2.1. In Section 2.3 we recall Voevodsky's triangulated category of effective geometric motives over a field k from [96, Lecture 20], which we will use in the proof of Theorem 9.1.2. In Section 2.4, we recall the $\mathbb{A}^1$-derived category and the $\mathbb{A}^1$-homology sheaves from [93, Section 6.2]. In Chapter 11, we define universal $\mathbb{A}^1$-invariant sheaf of abelian groups $H_0^{\mathbb{A}^1\text{-naive}}(X)$ associated to $X \in Sm/k$, which is related to the 0-th $\mathbb{A}^1$-homology sheaf.

2.1 Model Categories

In this section we recall the model categories and the left Bousfield localisation. We refer [66], [67] and [87, Appendix] for the detailed discussion on the model categories.

Definition 2.1.1. Suppose, $\mathcal{M}$ is a category having small limits and small colimits and $\mathcal{M}$ is equipped with three classes of morphisms called weak equivalences, cofibrations and fibrations respectively. The category $\mathcal{M}$ along with the three classes of morphisms is called a model category if $\mathcal{M}$ satisfies the following:

1. If f, g are composable morphisms in $\mathcal{M}$ and any two of $f, g, g \circ f$ are weak equivalences, then the rest one is also a weak equivalence $\mathcal{M}$.

2. If a morphism $g : X' \to Y'$ is a retract of a morphism $f : X \to Y$ and f is a weak equivalence, cofibration or fibration, then g is also a weak equivalence, cofibration and fibration respectively i.e. if there is a commutative diagram

$$\begin{array}{ccccc} X' & \xrightarrow{i} & X & \xrightarrow{r} & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y' & \xrightarrow[i']{} & Y & \xrightarrow[r']{} & Y' \end{array}$$

such that $r \circ i$ and $r' \circ i'$ are the identity maps and if f is a weak equivalence, cofibration or fibration, then so is g.

3. The cofibrations have left lifting property with respect to the trivial fibrations (a morphism is called a trivial fibration if it is a fibration as well as a weak equivalence) i.e. the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle q}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ C & \longrightarrow & D \end{array}$$

has a lift (dotted arrow exists), where p is trivial fibration and q is cofibration.

4. The fibrations have right lifting property with respect to the trivial cofibrations (a morphism is called a trivial cofibration if it is a cofibration as well as a weak equivalence) i.e the following commutative diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ {\scriptstyle q}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle p} \\ C & \longrightarrow & D \end{array}$$

has a lift, where q is trivial cofibration and p is a fibration.

5. There are functorial factorisations of any morphism f as

$$f = \alpha(f) \circ \beta(f) = \gamma(f) \circ \delta(f)$$

such that $\alpha(f)$ is a fibration, $\beta(f)$ is a trivial cofibration and $\gamma(f)$ is a trivial fibration, $\delta(f)$ is a cofibration.

There are cofibrant and fibrant replacement functors $Q : \mathcal{M} \to \mathcal{M}$ and $R : \mathcal{M} \to \mathcal{M}$ respectively. Given a model category $\mathcal{M}$, there is an associated homotopy category $Ho\ \mathcal{M}$ which has objects same as $\mathcal{M}$ and morphisms

$$Hom_{Ho\ \mathcal{M}}(X, Y) = Hom_{\mathcal{M}}(QX, RY)/\sim,$$

where $Hom_{\mathcal{M}}(QX, RY)/\sim$ is the homotopy class of morphisms from QX to RY. There is a localisation functor $\gamma : \mathcal{M} \to Ho\ \mathcal{M}$. The category $Ho\ \mathcal{M}$ is indeed the localisation of $\mathcal{M}$ at the class of weak equivalences [57, Chapter 1].

Example 2.1.2. 1. The category of simplicial sets $\Delta^{op}Sets$ is a model category with respect to the weak equivalences defined as the weak equivalences of simplicial sets, fibrations are defined as the Kan fibrations and the cofibrations are defined as the monomorphisms.

2. The category $\Delta^{op}PSh(\mathcal{C})$ of simplicial presheaves on a small category $\mathcal{C}$ is a model category in which a morphism $\mathcal{X} \to \mathcal{Y}$ is a weak equivalence and fibration if the morphism

$\mathcal{X}(C) \to \mathcal{Y}(C)$ is a weak equivalence and a Kan fibration of simplicial sets respectively, for every $C \in \mathcal{C}$. The cobfibrations are the maps having left lifting property with respect to the trivial fibrations. This model structure is called the global projective model structure on $\Delta^{op}PSh(\mathcal{C})$. We will come again about this model structure in the next section.

3. The category $\Delta^{op}PSh(\mathcal{C})$ admits global injective model structure in which a morphism $\mathcal{X} \to \mathcal{Y}$ is a weak equivalence and cofibration if the map $\mathcal{X}(C) \to \mathcal{Y}(C)$ between simplicial sets is a weak equivalence and monomorphisms respectively, for every $C \in \mathcal{C}$. The fibrations are the maps having right lifting property with respect to the trivial cofibrations.

Definition 2.1.3. A functor $F : \mathcal{M} \to \mathcal{N}$ between the model categories is called a left Quillen functor if F preserves cofibrations and trivial cofibrations. The left Quillen functor F induces total left derived functor $Ho\, F : Ho\, \mathcal{M} \to Ho\, \mathcal{N}$ defined as $X \mapsto F(QX)$. Similarly a functor $G : \mathcal{N} \to \mathcal{M}$ between model categories is called a right Quillen functor if G preserves fibrations and trivial fibrations. An adjunction $(F, G, \phi) : \mathcal{M} \to \mathcal{N}$ (where, (F, G) are pairwise adjoint and $\phi : Hom_{\mathcal{N}}(FX, Y) \to Hom_{\mathcal{M}}(X, GY)$ is the bijection) is called Quillen adjunction if F is a left Quillen functor. The Quillen adjunction induces derived adjunction

$$Ho\, F : Ho\, \mathcal{M} \rightleftarrows Ho\, \mathcal{N} : Ho\, G$$

between the homotopy categories. A Quillen adjunction (F, G, ϕ) is called Quillen equivalence if for every cofibrant object X in $\mathcal{M}$ and fibrant object Y in $\mathcal{N}$ a morphism $FX \to Y$ is a weak equivalence in $\mathcal{N}$ if and only if the morphism $\phi(f) : X \to GY$ is a weak equivalence in $\mathcal{M}$. The Quillen adjunction gives an equivalence between the associated homotopy categories.

Remark 2.1.4. The identity functor on $\Delta^{op}PSh(\mathcal{C})$ induces an Quillen equivalence between the global projective model structure and the global injective model structure on $\Delta^{op}PSh(\mathcal{C})$ [87, Proposition A.3.3.8]. Therefore both the projective and injective model structures have equivalent homotopy categories.

Definition 2.1.5. A model category $\mathcal{M}$ which is also a simplicial category, enriched over simplicial sets i.e. for $X, Y \in \mathcal{M}$, there is a simplicial set $Map(X, Y)$ along with some compatibility conditions [67, Section 9.1.5], is called simplicial model category if it satisfies the following condition

- If $j : A \to B$ is a cofibration and $q : X \to Y$ is a fibration, then the map

$$Map(B, X) \xrightarrow{(j^*, q_*)} Map(A, X) \times_{Map(A,Y)} Map(B, Y)$$

is a Kan fibration between simplicial sets, which is a trivial fibration if j or q is also a weak equivalence in $\mathcal{M}$.

Remark 2.1.6. If $\mathcal{M}$ is a simplicial model category and $X, Y \in \mathcal{M}$, then

$$Hom_{Ho\, \mathcal{M}}(X, Y) \cong \pi_0(Map(QX, RY))$$

2.1.1 Left Bousfield localisation [87, Section A.3.7]

Definition 2.1.7. Suppose, $\mathcal{M}$ is a simplicial model category and S is a collection of morphisms in $\mathcal{M}$.

- An object Z is called S-local if for every map $f : X \to Y$ in S, the induced map

$$f^* : Map(QY, RZ) \to Map(QX, RZ)$$

  is a homotopy equivalence between the simplicial sets.

- A morphism $g : A \to B$ is called an S-equivalence if for every S-local object Z in $\mathcal{M}$, the induced map

$$g^* : Map(QB, RZ) \to Map(QA, RZ)$$

  is a homotopy equivalence between the simplicial sets.

Theorem 2.1.8. *Suppose $\mathcal{M}$ is a left proper, combinatorial and simplicial model category and S is a set of morphisms in $\mathcal{M}$. Then the left Bousfield localisation of $\mathcal{M}$ with respect to S exists i.e. there is a model category $L_S\mathcal{M}$ having the same underlying category as $\mathcal{M}$ with three distinguished classes of morphisms*

1. *Weak equivalences are S-local equivalences.*

2. *Cofibrations are same as in the model category $\mathcal{M}$.*

3. *Fibrations are the morphisms that have right lifting property with respect to the trivial cofibrations.*

The model category $L_S\mathcal{M}$ is also a left proper, simplicial and combinatorial model category. The fibrant objects in $L_S\mathcal{M}$ are the S-local objects which are the fibrant objects of $\mathcal{M}$.

Remark 2.1.9. The left Bousfield localisation $L_S\mathcal{M}$ is indeed a categorical localisation of $\mathcal{M}$ with respect to S [57, Chapter 1]. The identity map on $\mathcal{M}$ induces a left Quillen functor

$$i : \mathcal{M} \to L_S\mathcal{M}$$

which takes the morphisms in S to the weak equivalences in $L_S\mathcal{M}$ and given a left Quillen functor $F : \mathcal{M} \to \mathcal{N}$ (where, $\mathcal{N}$ is a model category) that takes the morphisms in S to weak equivalences, there is the unique left Quillen functor $\tilde{F} : L_S\mathcal{M} \to \mathcal{N}$ such that $F = \tilde{F} \circ i$.

2.2 The $\mathbb{A}^1$-homotopy category

In this section, first we discuss about $\Delta^{op}PSh(Sm/k)$, the category of simplicial presheaves on Sm/k, which is the underlying category of the Morel-Voevodsky's $\mathbb{A}^1$-homotopy category $\mathbf{H}(\mathbf{k})$.

In the subsection 2.2.1, we describe the $\mathbb{A}^1$-model structure, which gives the $\mathbb{A}^1$-homotopy category $\mathbf{H}(\mathbf{k})$. In subsection 2.2.2 and subsection 2.2.3, we discuss some importent properties of $\mathbf{H}(\mathbf{k})$ and the $\mathbb{A}^1$-homotopy sheaves associated to a space $\mathcal{X}$ respectively. We refer [94] and [99] to the reader for the detailed description of $\mathbf{H}(k)$. We refer [5] and [16] for a nice survey on $\mathbb{A}^1$-homotopy theory.

Let Sm/k be the category of smooth, finite type, separated schemes over a field k. There are several Grothendieck topologies on Sm/k (for example, Nisnevich topology, étale topology, Zariski topology etc.) for which Sm/k is a Grothendieck site. In Zariski topology, a Zariski covering of $X \in Sm/k$ is an open covering of X. A collection of morphisms $\{f_i : U_i \to X\}_i$ in Sm/k is an étale covering if each f_i is an étale morphism and every $x \in X$ has a preimage in some U_i. A collection of morphisms $\{f_i : U_i \to X\}_i$ in Sm/k is a Nisnevich covering of X if it satisfies the following:

- f_i is an étale morphism for every i.
- For every $x \in X$, there is some i and $y \in U_i$ such that $f_i(y) = x$ and the induced map $k(x) \to k(y)$ between the residue fields is an isomorphism.

Nisnevich topology is finer than Zariski topology and weaker than étale topology. The Zariski coverings of a scheme are open coverings. The covering

$$\{\mathbb{A}^1_{\mathbb{C}} \setminus \{1\} \hookrightarrow \mathbb{A}^1_{\mathbb{C}}, \mathbb{A}^1_{\mathbb{C}} \setminus \{0\} \xrightarrow{z \mapsto z^2} \mathbb{A}^1_{\mathbb{C}}\}$$

is a Nisnevich covering of $\mathbb{A}^1_{\mathbb{C}}$, but it is not a Zariski covering. By Sm/k we always mean the Grothendieck site Sm/k endowed with the Nisnevich topology. Similar to Zariski topology, for a scheme $X \in Sm/k$ of Krull dimension d and a Nisnevich sheaf of abelian groups $\mathcal{F}$ on Sm/k, the Nisnevich cohomology $H^i(X, \mathcal{F})$ vanishes for $i > d$ [94, Section 3.1, Proposition 1.8]. Similar to étale topology, in Nisnevich topology a closed immersion of smooth schemes over k locally looks like the inclusion $\mathbb{A}^n_k \to \mathbb{A}^m_k$ for some $n \leq m$ [94, Section 3.1]. In particular in Nisnevich topology, a variety $X \in Sm/k$ locally looks like affine spaces at every closed point. Like Zariski topology, algebraic K-theory satisfies descent in the Nisnevich topology [129, Section 10, Chapter V].

The category Sm/k is not cocomplete, colimit does not exist in Sm/k, in general. It is enlarged to the category of presheaves on Sm/k in which both the limits and colimits exist. The category of presheaves on Sm/k is denoted by $PSh(Sm/k)$. There is a fully faithful functor (Yoneda embedding) $i : Sm/k \to PSh(Sm/k)$ given by $X \in Sm/k \mapsto Hom_{Sm/k}(-, X)$. We denote the representable presheaf $Hom_{Sm/k}(-, X)$ by X, which is a sheaf in the Nisnevich topology [59, VII.2]. Every presheaf on Sm/k is a colimit of representables [43, §2.1.1]. We denote the category of Nisnevich sheaves on Sm/k by $Sh(Sm/k)$. There is a fully faithful functor $j : Sh(Sm/k) \to PSh(Sm/k)$. The functor j admits the left adjoint $j^{\#} : PSh(Sm/k) \to Sh(Sm/k)$, given by the Nisnevich sheafification.

For $X \in Sm/k$ and $x \in X$, a Nisnevich neighbourhood of x is a pair (U, ϕ), where $U \in Sm/k$ is irreducible and $\phi : U \to X$ is an étale morphism such that there is some $y \in U$ with $\phi(y) = x$ and the induced map $k(x) \to k(y)$ between the residue fields is an isomorphism. The category of Nisnevich neighbourhoods of x is denoted by Nbd_x, which is a filtered category. For a presheaf $\mathcal{F}$ on Sm/k, the stalk of $\mathcal{F}$ at x is denoted by $\mathcal{F}_x$ and it is defined as the filtered colimit:

$$\mathcal{F}_x := \operatorname*{colim}_{(U,\phi) \in Nbd_x} \mathcal{F}(U).$$

Remark 2.2.1. For a scheme $(X, \mathcal{O}_X)$ and a point x of X, $\mathcal{O}_{X,x}$ is the local ring at x. If X an affine scheme given by $(Spec\ A, \mathcal{O}_{Spec\ A})$ and x is a point of $Spec\ A$ given by a prime ideal P, then $\mathcal{O}_{X,x}$ is the local ring A_P. The Zariski stalk of $\mathcal{F}$ is the section $\mathcal{F}(Spec\ \mathcal{O}_{X,x})$. The Nisnevich stalk $\mathcal{F}_x$ is the section $\mathcal{F}(Spec\ \mathcal{O}^h_{X,x})$ (for a local ring R, R^h is the henselization of R). We refer [121, Tag 04GE] (see also [121, Tag 07QL]) for the definition and related properties of Henselian local rings.

For $X \in Sm/k$ and $x \in X$, the Nisnevich stalk at x gives a functor $x^* : Sh(Sm/k) \to Sets$, which commutes with finite limits and all colimits. Thus x^* is a point of the Nisnevich site. The Nisnevich topos $Sh(Sm/k)$ has enough points. Indeed, for every $X \in Sm/k$ and $x \in X$, the stalks x^* form a conservative set of points that detects isomorphisms of the Nisnevich sheaves. A morphism $\phi : \mathcal{F} \to \mathcal{G}$ of the Nisnevich sheaves of sets on Sm/k is an isomorphism if and only if $\phi : \mathcal{F}(Spec\ \mathcal{O}^h_{X,x}) \to \mathcal{G}(Spec\ \mathcal{O}^h_{X,x})$ is a bijection for every $X \in Sm/k$ and $x \in X$.

Definition 2.2.2. ([94, Definition 3.1.3], see also [12, Section 2]) An elementary distinguished square (in the Nisnevich topology) is a cartesian square in Sm/k of the form

$$\begin{array}{ccc} U \times_X V & \longrightarrow & V \\ \downarrow & & \downarrow{\scriptstyle p} \\ U & \xrightarrow{\ j\ } & X \end{array}$$

such that p is an étale morphism, j is an open embedding and $p^{-1}(X - U) \to (X - U)$ is an isomorphism (we put the reduced induced structure on the corresponding closed sets).

Remark 2.2.3. [94, Proposition 1.4, Section 3] A presheaf of sets $\mathcal{F}$ on Sm/k is a Nisnevich sheaf if and only if for every elementary distinguished square in Definition 2.2.2, the induced square of sets

$$\begin{array}{ccc} \mathcal{F}(X) & \longrightarrow & \mathcal{F}(U) \\ \downarrow & & \downarrow \\ \mathcal{F}(V) & \longrightarrow & \mathcal{F}(U \times_X V) \end{array}$$

is cartesian.

Let Δ be the category of finite ordinals and the morphisms in Δ are the order preserving maps. A simplicial presheaf is a functor $\mathcal{X} : \Delta^{op} \to PSh(Sm/k)$ (Δ^{op} is the opposite category of Δ). The category of simplicial presheaves on Sm/k is denoted by $\Delta^{op}PSh(Sm/k)$. We call it the category of spaces over k. Every simplicial presheaf is a homotopy colimit of the representables [43, Proposition 2.8]. Every presheaf can be regarded as a constant simplicial presheaf. There is a fully faithful functor from $PSh(Sm/k)$ to $\Delta^{op}PSh(Sm/k)$.

2.2.1 $\mathbb{A}^1$-model structure

The global projective model structure (Bousfield-Kan model structure [22, Proposition 8.1] or the universal model structure [43, Section 2]) on $\Delta^{op}PSh(Sm/k)$ is left proper, simplicial and cofibrantly generated. In the global projective model structure on $\Delta^{op}PSh(Sm/k)$, the weak equivalences are defined sectionwise weak equivalence between the simplicial sets, fibrations are defined sectionwise Kan fibration between the simplicial sets and cofibrations are defined as the maps satisfying the left lifting property with respect to the trivial fibrations. The global projective model structure satisfies the universal property [43, Proposition 2.3]: a functor $\gamma : Sm/k \to \mathcal{M}$ ($\mathcal{M}$ is a model category) factors through $\Delta^{op}PSh(Sm/k)$ uniquely upto a natural weak equivalence (where, $\Delta^{op}PSh(Sm/k)$ is the model category with respect to the global projective model structure).. For a space $\mathcal{X}$, there is a cofibrant space $Q\mathcal{X}$, which is the homotopy colimit of a diagram of representables, along with a weak equivalence $Q\mathcal{X} \to \mathcal{X}$ [43, Propoaition 2.8]. The left Bousfield localisation of the global projective model structure at the class of Nisnevich hypercovers gives the Nisnevich local model structure [44, Section 6]. Dugger-Hollander-Isaksen proved that the Nisnevich local model structure is the left Bousfield localisation of the global projective model structure at the collection of Čech hypercovers [44, Example A10]. The homotopy category of the Nisnevich local model structure is denoted by $\mathbf{H_s(k)}$. The fibrant objects of the Nisnevich local model structure are such spaces $\mathcal{X}$ which are sectionwise Kan fibrant and for every Čech hypercover $U_\bullet \to X$ associated to a Nisnevich cover $U \to X$, the induced map

$$\mathcal{X}(X) \to \operatorname*{holim}_{n \geq 0} \mathcal{X}(U_n)$$

(where U_n is the n-fold fibre product $U \times_X \cdots \times_X U$) is a weak equivalence ([44, Corollary 7.1])

There is a fibrant replacement functor for the Nisnevich local model structure

$$Ex : \Delta^{op}PSh(Sm/k) \to \Delta^{op}PSh(Sm/k)$$

along with a natural transformation $Id \to Ex$ such that for a space $\mathcal{X}$, the space $Ex(\mathcal{X})$ is fibrant in the Nisnevich local model structure and the map $\mathcal{X} \to Ex(\mathcal{X})$ is a trivial cofibration.

Definition 2.2.4. A simplicial presheaf $\mathcal{X}$ on Sm/k is said to satisfy Nisnevich excision if for any elementary distinguished square in the Nisnevich topology as in Definition 2.2.2, the induced square of simplicial sets

$$\begin{array}{ccc} \mathcal{X}(X) & \longrightarrow & \mathcal{X}(V) \\ \downarrow & & \downarrow \\ \mathcal{X}(U) & \longrightarrow & \mathcal{X}(U \times_X V) \end{array}$$

is homotopy cartesian [12, Section 3.2].

Remark 2.2.5. A fibrant space of the Nisnevich local model structure satisfies Nisnevich excision. A sectionwise Kan fibrant space is fibrant in the Nisnevich local model structure if and only if it satisfies Nisnevich excision ([44, Theorem 1.3], [12, Theorem 3.2.5], see also [94, Remark 3.1.15]).

We consider the left Bousfield localisation of the Nisnevich local model structure on the category of spaces $\Delta^{op}PSh(Sm/k)$ at the class of the projection maps $pr : X \times_k \mathbb{A}^1_k \to X$, following [94, Section 3.2]. The resulting model structure is called the unstable $\mathbb{A}^1$-model structure and the resulting homotopy category is denoted by $\mathbf{H}(\mathbf{k})$, called the unstable $\mathbb{A}^1$-homotopy category. If we start with the category of pointed simplicial presheaves, then the same construction yields the pointed unstable $\mathbb{A}^1$-homotopy category and it is denoted by $\mathbf{H}_\bullet(\mathbf{k})$. A morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ between the based spaces is an isomorphism in $\mathbf{H}_\bullet(\mathbf{k})$ if and only if the morphism $f : \mathcal{X} \to \mathcal{Y}$ forgetting the base points is an isomorphism in $\mathbf{H}(\mathbf{k})$.

Computation of the unstable motivic invariants of a space $\mathcal{X}$ requires an $\mathbb{A}^1$-fibrant model of $\mathcal{X}$. Before giving the description of $\mathbb{A}^1$-fibrant replacement functor, we recall the definition of $Sing_*$ functor from [94, Section 2.3].

Definition 2.2.6. For a space $\mathcal{X}$, the functor $Sing_* : \Delta^{op}PSh(Sm/k) \to \Delta^{op}PSh(Sm/k)$ is defined as

$$Sing_*(\mathcal{X})(U)_n = \mathcal{X}(\Delta^n_a \times_k U)_n,$$

where Δ^n_a is the cosimplicial object in $\Delta^{op}PSh(Sm/k)$, $\Delta^n_a = Spec\ \frac{k[x_0, x_1, .., x_n]}{(\sum_{i=0}^n x_i - 1)}$, which is isomorphic to $\mathbb{A}^n_k$.

Here the boundary maps

$$d_i : Sing_*(\mathcal{X})(U)_n \to Sing_*(\mathcal{X})(U)_{n-1},\ 0 \leq i \leq n$$

are induced by the maps $d^i : \Delta^{n-1}_a \to \Delta^n_a$ which are given by the morphism between k-algebras (we again denote it by d^i)

$$d^i : \frac{k[x_0, x_1, .., x_n]}{(\sum_{i=0}^n x_i - 1)} \to \frac{k[x_0, x_1, .., x_{n-1}]}{(\sum_{i=0}^{n-1} x_i - 1)}$$

defined as

$$\begin{cases} \overline{x_j} \mapsto \overline{x_j} & j < i \\ \overline{x_j} \mapsto \overline{0} & j = i \\ \overline{x_j} \mapsto \overline{x_{j-1}} & j > i \end{cases}$$

(here, for $f \in k[x_0,..,x_n]$ $\bar{f}$ denotes the class in $\frac{k[x_0,x_1,..,x_n]}{(\sum_{i=0}^{n} x_i - 1)}$). Similarly here the degeneracy maps

$$s_i : Sing_*(\mathcal{X})(U)_n \to Sing_*(\mathcal{X})(U)_{n+1},\ 0 \leq i \leq n$$

are induced by the maps $s^i : \Delta_a^{n+1} \to \Delta_a^n$ which are given by the morphism between k-algebras (we again denote it by s^i)

$$s^i : \frac{k[x_0,x_1,..,x_n]}{(\sum_{i=0}^{n} x_i - 1)} \to \frac{k[x_0,x_1,..,x_{n+1}]}{(\sum_{i=0}^{n+1} x_i - 1)}$$

defined as

$$\begin{cases} \overline{x_j} \mapsto \overline{x_j} & j < i \\ \overline{x_i} \mapsto \overline{x_i + x_{i+1}} & j = i \\ \overline{x_j} \mapsto \overline{x_{j+1}} & j > i \end{cases}$$

Remark 2.2.7.

1. The canonical morphism $\mathcal{X} \to Sing_*(\mathcal{X})$ is an $\mathbb{A}^1$-weak equivalence [94, Corollary 3.8, Section 2.3].
2. If $f : \mathcal{X} \to \mathcal{Y}$ is an $\mathbb{A}^1$-fibration, then the morphism $Sing_*(f) : Sing_*(\mathcal{X}) \to Sing_*(\mathcal{Y})$ is also an $\mathbb{A}^1$-fibration [94, Corollary 3.13, Section 2.3].

Theorem 2.2.8. *[94, Lemma 2.6, Section 3.2] The space $Ex \circ (Ex \circ Sing_*)^{\mathbb{N}} \circ Ex(\mathcal{X})$ is an $\mathbb{A}^1$-fibrant space (for a space $\mathcal{Y}$, $(Ex \circ Sing_*)^{\mathbb{N}}(\mathcal{Y}) = \operatorname{colim}_{n \in \mathbb{N}} (Ex \circ Sing_*)^n \mathcal{Y}$).*

Remark 2.2.9. Theorem 2.2.8 gives an $\mathbb{A}^1$-fibrant replacement functor

$$Ex^{\mathbb{A}^1} := Ex \circ (Ex \circ Sing_*)^{\mathbb{N}} \circ Ex : \Delta^{op}PSh(Sm/k) \to \Delta^{op}PSh(Sm/k)$$

along with a morphism $i : \mathcal{X} \to Ex^{\mathbb{A}^1}(\mathcal{X})$ such that i is an $\mathbb{A}^1$-weak equivalence along with a cofibration and $Ex^{\mathbb{A}^1}(\mathcal{X})$ Nisnevich fibrant along with $\mathbb{A}^1$-local [94, Lemma 2.6]. A space is $\mathbb{A}^1$–fibrant if and only if it is fibrant in the Nisnevich local model structure and it is $\mathbb{A}^1$–local [94, Proposition 3.19, Section 2.3]. Thus any $\mathbb{A}^1$-fibrant space satisfies Nisnevich excision (Definition 2.2.4, Remark 2.2.5).

2.2.2 Properties of $\mathbf{H}(k)$

The following are some classes of maps which are invertible in $\mathbf{H}(\mathbf{k})$:

1. A morphism $f : \mathcal{X} \to \mathcal{Y} \in \Delta^{op}PSh(Sm/k)$ such that the induced morphisms on the stalks (for the Nisnevich topology) i.e. smooth Henselian local schemes are weak equivalences of simplicial sets.
2. The projection morphism $pr : \mathcal{X} \times \mathbb{A}^1 \to \mathcal{X}$ for any $\mathcal{X} \in \Delta^{op}PSh(Sm/k)$.
3. The structure map of any vector bundle $\mathcal{V} \to X$, for $X \in Sm/k$.

Likewise spheres in algebraic topology, there are the notions of simplicial circle and Tate circle in $\mathbf{H}_\bullet(\mathbf{k})$ [94, Section 3.2]. The simplicial circle is the simplicial sheaf associated to the pointed simplicial set $\Delta^1/\partial\Delta^1$ and it is denoted by S^1_s. The Tate circle is the simplicial sheaf associated to $\mathbb{G}_m$ ($\mathbb{G}_m$ is the representable sheaf $\mathbb{A}^1_k \setminus \{0\}$), pointed by 1 and it is denoted by S^1_t. These two circles are related by following canonical isomorphism in $\mathbf{H}_\bullet(\mathbf{k})$ [94, Lemma 2.15, Section 3.2]:

$$S^1_s \wedge S^1_t \cong T,$$

where T is the quotient sheaf associated to $\mathbb{A}^1_k/(\mathbb{A}^1_k \setminus \{0\})$. The quotient sheaf T is isomorphic to the pointed projective line $\mathbb{P}^1_k$ in $\mathbf{H}_\bullet(\mathbf{k})$ [94, Corollary 2.18, Section 3.2]. Likewise differential topology of smooth manifolds, the tubular neighbourhood theorem holds in $\mathbf{H}_\bullet(\mathbf{k})$. For a closed immersion $i : Y \to X$ in Sm/k, there is a canonical isomorphism of the pointed spaces in $\mathbf{H}_\bullet(\mathbf{k})$

$$X/(X \setminus Y) \cong Th(N_{X,Y}),$$

where $Th(N_{X,Y})$ is the Thom space of the normal bundle $\nu : N_{X,Y} \to Y$ over Y [94, Theorem 2.23].

Several important theories are representable in $\mathbf{H}_\bullet(\mathbf{k})$. For example, the motivic cohomology (in particular Milnor K-theory, Picard group, Chow groups) is representable in $\mathbf{H}_\bullet(\mathbf{k})$ by the Eilenberg-Maclane objects $K(p,q,A)$ [127, Theorem 2]. In $\mathbf{H}_\bullet(\mathbf{k})$, algebraic K-theory is representable by the doubly infinite Grassmannian ([94, Theorem 3.13, Section 4], see also [124, Remark 2]).

2.2.3 $\mathbb{A}^1$-homotopy sheaves and $\mathbb{A}^1$-Contractibility

Suppose $\mathcal{X}$ is a space over a field k.

Definition 2.2.10. The $\mathbb{A}^1$-connected component sheaf of $\mathcal{X}$ is the Nisnevich sheaf associated to the presheaf

$$U \in Sm/k \mapsto Hom_{\mathbf{H(k)}}(U, \mathcal{X})$$

The $\mathbb{A}^1$-connected component sheaf of $\mathcal{X}$ is denoted by $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ which is a Nisnevich sheaf of sets on Sm/k. The space $\mathcal{X}$ is called $\mathbb{A}^1$-connected if $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is isomorphic to the trivial sheaf $Spec\ k$ on Sm/k.

We recall the definition of $\mathbb{A}^1$-chain connected component sheaf associated to $X \in Sm/k$, introduced by Asok-Morel [11, Definition 2.2.4].

Definition 2.2.11. The $\mathbb{A}^1$-chain connected component sheaf $\pi_0^{ch}(X)$ associated to $X \in Sm/k$ is the connected component sheaf of $Sing_*(X)$ i.e. the Nisnevich sheaf associated to the presheaf $\pi_0(Sing_*(X))$. The scheme X is called $\mathbb{A}^1$-chain connected if for every finitely generated separable field extension L/k, $\pi_0^{ch}(X)(Spec\ F)$ is trivial [11, Definition 2.2.2].

Remark 2.2.12. 1. The notion of $\mathbb{A}^1$-chain connectedness is analogous to the path connectedness of a topological space. By definition, a scheme $X \in Sm/k$ is $\mathbb{A}^1$-chain connected if for every finitely generated separable field extension F/k and two F-points x, y in X there are $H_1, \dots, H_n : \mathbb{A}^1_F \to X$ such that $H_1(0) = x$ and $H_n(1) = y$.

2. For a space $\mathcal{X}$, the morphism $i : \mathcal{X} \to Ex^{\mathbb{A}^1}(\mathcal{X})$ induces the canonical epimorphism

$$\theta : \pi_0^s(\mathcal{X}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$$

($\pi_0^s(\mathcal{X})$ is the Nisnevich sheaf associated to the presheaf $U \in Sm/k \mapsto Hom_{\mathbf{H_s(k)}}(U, \mathcal{X})$) [11, Corollary 2.1.5]. Thus if $\mathcal{X}$ is simplicially connected, then $\mathcal{X}$ is $\mathbb{A}^1$-connected.

3. The $\mathbb{A}^1$-connected component sheaf $\pi_0^{\mathbb{A}^1}(X)$ detects k-rational points in X. If X is $\mathbb{A}^1$-connected, then X has a k-rational point [11, Example 2.1.6].

4. The canonical morphism $X \to Sing_*(X)$ induces canonical epimorphism $X \to \pi_0^{ch}(X)$.

5. For $X \in Sm/k$, the morphism θ in (2) factors through the canonical epimorphism $X \to \pi_0^{ch}(X)$ (in this case $\pi_0^s(X) \cong X$) [11, Lemma 2.2.5]. Thus if X is $\mathbb{A}^1$-chain connected, then X is $\mathbb{A}^1$-connected [11, Proposition 2.2.7]. For a proper scheme $X \in Sm/k$, Asok-Morel [11, Theorem 2], (later Balwe-Hogadi-Sawant [26, Corollary 3.10]) proved that X is $\mathbb{A}^1$-chain connected if and only if X is $\mathbb{A}^1$-connected.

Definition 2.2.13. For a pointed space $(\mathcal{X}, x)$ over k, the i-th $\mathbb{A}^1$-homotopy sheaf of groups $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ $(i \geq 1)$ is defined to be the Nisnevich sheaf associated to the presheaf

$$U \in Sm/k \mapsto Hom_{\mathbf{H_\bullet(k)}}(S_s^i \wedge U_+, (\mathcal{X}, x)).$$

Here S_s^i is the simplicial i-th sphere, defined as the n-times smash product of the simplicial circle S_s^1 (i.e. $S_s^i = (S_s^1)^{\wedge n}$).

The sheaf of groups $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$ is a sheaf of abelian groups if $i \geq 2$. A space $\mathcal{X}$ over k is called $\mathbb{A}^1$-contractible if $\mathcal{X}$ is isomorphic to the trivial sheaf $Spec\ k$ in $\mathbf{H(k)}$. The space $\mathcal{X}$ is $\mathbb{A}^1$-contractible if and only if $\mathcal{X}$ is $\mathbb{A}^1$-connected and all $\pi_i^{\mathbb{A}^1}(\mathcal{X}, x)$'s are trivial, for every i and for any basepoint x of $\mathcal{X}$. For example, $\mathbb{A}^n_k$'s are $\mathbb{A}^1$-contractible. We recall two simple lemmas which we will use in the proof of Theorem 8.1.1.

Lemma 2.2.14. *Suppose, $X \in Sm/k$ is $\mathbb{A}^1$-connected. Then $\mathcal{O}(X)$ has trivial group of units ($\mathcal{O}(X) = \Gamma(X, \mathcal{O}_X)$ is the ring of regular functions on X).*

Proof. Since $\mathbb{G}_m$ is $\mathbb{A}^1$-local [94, Example 2.4], so $\pi_0^{\mathbb{A}^1}(\mathbb{G}_m)$ is isomorphic to $\mathbb{G}_m$. Thus a morphism $X \to \mathbb{G}_m$ uniquely factors through $\pi_0^{\mathbb{A}^1}(X)$. Hence,

$$\text{The group of units, } \mathcal{O}(X)^* = Hom_{Sm/k}(X, \mathbb{G}_m)$$
$$\cong Hom_{Sh(Sm/k)}(\pi_0^{\mathbb{A}^1}(X), \mathbb{G}_m)$$

Thus if X is $\mathbb{A}^1$-connected, then $\mathcal{O}(X)^* = k^*$. □

Lemma 2.2.15. *[94, Proposition 3.8, Section 4] Suppose, $X \in Sm/k$ is $\mathbb{A}^1$-contractible. Then the Picard group of X is trivial.*

We end this section mentioning the $\mathbb{A}^1$-excision property of $\mathbf{H(k)}$, which we will use in the proof of Theorem 9.2.3.

Theorem 2.2.16. *[10, Theorem 4.1] Suppose k is an infinite field and $X \in Sm/k$ is $\mathbb{A}^1$-connected. Let $U \subset X$ be an open subscheme such that the complement $X \setminus U$ is everywhere of codimension $d \geq 2$. Fix, $x \in U(k)$. If moreover, X is m-connected, $m \geq d-3$, then the canonical morphism*

$$j_* : \pi_i^{\mathbb{A}^1}(U, x) \to \pi_i^{\mathbb{A}^1}(X, x)$$

is an isomorphism for $0 \leq i \leq d-2$ and j_* is an epimorphism for $i = d-1$.

2.3 Triangulated Category of Motives over a field k

In this section we recall Voevodsky's construction of the triangulated category of geometric motives $\mathbf{DM}_{gm}(k, \mathbb{Z})$ over a field k with coffecients in $\mathbb{Z}$ in the Nisnevich topology [96, Lecture 20]. We refer to the reader [96] and [38, Section 2] for the details.

Let k be a field of characteristic zero and Cor_k be the additive category of finite correspondences. The objects of Cor_k are the smooth, separated, finite type schemes over k and morphisms in Cor_k from a connected scheme X to a scheme Y is the free abelian group on the set of all irreducible closed subschemes (called elementary correspondence) of $X \times Y$, which are finite and surjective over X. There is a faithful functor

$$i : Sm/k \to Cor_k.$$

An additive functor $F : Cor_k^{op} \to \mathcal{A}b$ ($\mathcal{A}b$ is the category of abelian groups) is called a presheaf with transfers. Let $PST(k)$ be the category of presheaves with transfers. The representable functor $Cor_k(-, X)$, associated to $X \in Sm/k$ is a sheaf with transfers in the Nisnevich topology and it is denoted by $\mathbb{Z}_{tr}(X)$, more precisely $\mathbb{Z}_{tr}(X)(U) = Cor_k(U, X)$, for every $U \in Sm/k$.

Let $K(PST(k))$ be the category of bounded above cochain complexes in $PST(k)$. Inverting the morphisms $f : A \to B$ in $K(PST(k))$ which are quasi-isomorphisms over the sections of smooth henselian local schemes (stalks in the Nisnevich topology), we obtain the derived category $\mathbf{D}^-(Sh(Cor_k))$ of Nisnevich sheaves with transfers. It is a tensor triangulated category with respect to the derived tensor product $\otimes_L^{tr}$. There is also derived hom functor $\underline{RHom}$ on $\mathbf{D}^-(Sh(Cor_k))$ which gives the adjunction

$$Hom_{\mathbf{D}^-(Sh(Cor_k))}(A \otimes_L^{tr} \mathbb{Z}_{tr}(X), B) \cong Hom_{\mathbf{D}^-(Sh(Cor_k))}(A, \underline{RHom}(\mathbb{Z}_{tr}(X), B)),$$

where $A, B \in K(PSh(k))$ and $X \in Sm/k$ [96, Lecture 8]. In $\mathbf{D}^{-}(Sh(Cor_k))$, again inverting the maps

$$\mathbb{Z}_{tr}(X \times_k \mathbb{A}^1_k)[n] \to \mathbb{Z}_{tr}(X)[n],$$

for every $X \in Sm/k$ and for every $n \in \mathbb{Z}$, we obtain Voevodsky's category of effective motives, denoted as $\mathbf{DM}^{eff,-}_{Nis}(k, \mathbb{Z})$. It is also a tensor triangulated category. There is a functor

$$Sm/k \to K(PST(k))$$

which takes $X \in Sm/k$ to the Suslin complex $C_*\mathbb{Z}_{tr}(X)$ associated to X. The Suslin complex $C_*\mathbb{Z}_{tr}(X)$ is the Moore complex (coboundary map is the alternating sum of the face maps) of the simplicial presheaf with transfers $C_\bullet\mathbb{Z}_{tr}(X)$. Here $C_\bullet\mathbb{Z}_{tr}(X)$ is given by

$$C_\bullet\mathbb{Z}_{tr}(X)_n(U) = \mathbb{Z}_{tr}(X)(U \times_k \Delta^n_a).$$

This functor $Sm/k \to K(PST(k))$ induces a functor $M : \mathbf{H(k)} \to \mathbf{DM}^{eff,-}_{Nis}(k, \mathbb{Z})$. The image $M(X)$ of $X \in Sm/k$ in $\mathbf{DM}^{eff,-}_{Nis}(k, \mathbb{Z})$ is called the motive of X. The motive of $Spec\ k$ is denoted by $\mathbb{Z}$. The thick subcategory generated by $M(X)$'s, for $X \in Sm/k$ is called the category of effective geometric motives over k and it is denoted by $\mathbf{DM}^{eff}_{gm}(k, \mathbb{Z})$. Inverting the Tate twist operation

$$M \mapsto M(1) = M \otimes^{tr}_L \mathbb{Z}(1),$$

we obtain the category of geometric motives, which is a tensor triangulated category. The category of geometric motives is denoted by $\mathbf{DM}_{gm}(k, \mathbb{Z})$. There are several properties of $\mathbf{DM}_{gm}(k, \mathbb{Z})$. Among them, we list three properties which we will use in the proof of Theorem 9.1.2.

1. (Gysin triangle) Suppose, $X, Z \in Sm/k$ and Z is a closed subscheme of X of codimension c. Then there is a distinguished triangle in $\mathbf{DM}_{gm}(k, \mathbb{Z})$:

$$M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1].$$

2. (Cancellation) The localisation functor $\mathbf{DM}^{eff}_{gm}(k, \mathbb{Z}) \to \mathbf{DM}_{gm}(k, \mathbb{Z})$ is fully faithful i.e. for $M, N \in \mathbf{DM}^{eff}_{gm}(k, \mathbb{Z})$ the map

$$Hom_{\mathbf{DM}^{eff}_{gm}(k,\mathbb{Z})}(M, N) \to Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(1), N(1))$$

3. Motivic cohomology is representable in $\mathbf{DM}_{gm}(k, \mathbb{Z})$:

$$H^{p,q}_{\mathcal{M}}(X, \mathbb{Z}) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(X), \mathbb{Z}(q)[p]).$$

Thus Bloch's higher Chow groups, in particular the classical Chow groups are representable in $\mathbf{DM}_{gm}(k,\mathbb{Z})$, since

$$H^{p,q}_{\mathcal{M}}(X,\mathbb{Z}) \cong CH^q(X, 2q-p) \text{ and}$$

$$H^{2i,i}_{\mathcal{M}}(X,\mathbb{Z}) \cong CH^i(X).$$

The category $\mathbf{DM}_{gm}(k,\mathbb{Z})$ is a rigid category. Suppose, $M \in \mathbf{DM}_{gm}(k,\mathbb{Z})$ with some twist $M(r) = M \otimes^{tr}_L \mathbb{Z}(r)$ is an effective motive. The dual of M is defined as

$$M^* := \underline{RHom}(M(r), \mathbb{Z}(i)(r-i)),$$

for large i (dual is independent of r, i) and M^* satisfies

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(L \otimes M, N) \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(L, M^* \otimes N)$$

Any object $M \in \mathbf{DM}_{gm}(k,\mathbb{Z})$ is reflexive i.e. the natural map $M \to M^{**}$ is an isomorphism. An object $M \in \mathbf{DM}_{gm}(k,\mathbb{Z})$ is called strongly dualisable if it is reflexive and the natural map

$$M^* \otimes M \to (M^* \otimes M)^*$$

is an isomorphism (see [41, Definition 1.2]). If F/k is a finite field extension, then $M(Spec\ F)^* = M(Spec\ F)$ and $M(Spec\ F)$ is a strongly dualisable object in $\mathbf{DM}_{gm}(k,\mathbb{Z})$.

We end this section with the following remark which we will use in the proof of Theorem 9.1.2.

Remark 2.3.1. Suppose $X \in Sm/k$ with $M(X)$ is isomorphic to $\mathbb{Z}$ in $\mathbf{DM}_{gm}(k,\mathbb{Z})$. Assume that Z is a singleton set consisting a closed point of X of codimension c ($c \geq 1$). Then the map $M(X) \to M(Z)(c)[2c]$ in the Gysin triangle

$$M(X \setminus Z) \to M(X) \to M(Z)(c)[2c] \to M(X \setminus Z)[1]$$

is the zero map in $\mathbf{DM}_{gm}(k,\mathbb{Z})$. Indeed, the object $M(Z)$ is a strongly dualisable object in $\mathbf{DM}_{gm}(k,\mathbb{Z})$ with $M(Z)^* = M(Z)$. So we have

$$\begin{aligned} & Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(\mathbb{Z}, M(Z)(c)[2c]) \\ & \cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(Z), \mathbb{Z}(c)[2c]) \\ & \cong H^{2c,2}_{\mathcal{M}}(Z,\mathbb{Z}) \\ & \cong CH^c(Z) \end{aligned}$$

Since Z has dimension zero, $CH^c(Z)$ is trivial and hence the map $M(X) \to M(Z)(c)[2c]$ is the zero map. Therefore the Gysin triangle splits. Hence by the property of a triangulated category

[101, Corollary 1.2.7], we have

$$M(X \setminus Z) \cong M(X) \oplus M(Z)(c)[2c-1].$$

2.4 $\mathbb{A}^1$-Derived Category

In this section we recall the construction of $\mathbb{A}^1$-derived category and the $\mathbb{A}^1$-homology sheaves. The construction is similar to the construction of $\mathbf{DM}^{eff,-}_{Nis}(k,\mathbb{Z})$ in the previous section. We refer [93, Section 6.2] to the reader for the details.

Let $\mathcal{A}b(k)$ be the category of presheaves of abelian groups on Sm/k and $C_*(\mathcal{A}b(k))$ be the category of chain complexes in $\mathcal{A}b(k)$. Nisnevich local model structure on the category of chain complexes $C_*(\mathcal{A}b(k))$ in $\mathcal{A}b(k)$ is defined as

1. A morphism $f : A_* \to B_*$ in $C_*(\mathcal{A}b(k))$ is a weak equivalence if it induces a quasi-isomorphism over the sections of the smooth Henselian local schemes (stalks in the Nisnevich topology).
2. A morphism $f : A_* \to B_*$ is a fibration if it is an epimorphism.
3. A morphism $f : A_* \to B_*$ is a cofibration if it has the left lifting property with respect to the trivial fibrations.

The associated homotopy category is the derived category $D(\mathcal{A}b(k))$ of $\mathcal{A}b(k)$. The left Bousfield localisation of the Nisnevich local model structure with respect to the projection maps

$$A_* \otimes \mathbb{Z}(\mathbb{A}^1_k) \to A_*,$$

(where $\mathbb{Z}(\mathbb{A}^1_k)$ is the free abelian sheaf on $\mathbb{A}^1_k$) gives the $\mathbb{A}^1$-model structure on $C_*(\mathcal{A})b(k)$. The associated homotopy category is called the $\mathbb{A}^1$-derived category and it is denoted by $D_{\mathbb{A}^1}(\mathcal{A}b(k))$. There is an $\mathbb{A}^1$-localisation functor

$$L^{ab}_{\mathbb{A}^1} : C_*(\mathcal{A}b(k)) \to C_*(\mathcal{A}b(k))$$

that takes a complex A_* to its fibrant replacement in the $\mathbb{A}^1$-model structure.
The normalised chain complex functor gives a functor

$$N_* : \Delta^{op}PSh(Sm/k) \to C_*(\mathcal{A}b(k))$$

which induces a functor

$$N^{\mathbb{A}^1}_* : \mathbf{H(k)} \to D_{\mathbb{A}^1}(\mathcal{A}b(k))$$

defined as $\mathcal{X} \mapsto L^{ab}_{\mathbb{A}^1}(N_*(\mathcal{X}))$. The n-th $\mathbb{A}^1$-homology sheaf associated to $\mathcal{X}$, denoted as $H^{\mathbb{A}^1}_n(\mathcal{X})$, is defined as the n-th homology sheaf of the complex $N^{\mathbb{A}^1}_*(\mathcal{X})$.

Definition 2.4.1. [93, Definition 1.7] A Nisnevich sheaf of abelian groups $\mathcal{F}$ on Sm/k is called strictly $\mathbb{A}^1$-invariant if the canonical map in the Nisnevich sheaf cohomology

$$H^i(X, \mathcal{F}) \to H^i(\mathbb{A}^1_k \times_k X, \mathcal{F})$$

is an isomorphism for every i.

Theorem 2.4.2. *[93, Corollary 6.31] For a space $\mathcal{X}$, $n \in \mathbb{Z}$, the $\mathbb{A}^1$-homology sheaf $H_n^{\mathbb{A}^1}(\mathcal{X})$ vanishes if $n < 0$ and for $n \geq 0$, $H_n^{\mathbb{A}^1}(\mathcal{X})$ is an strictly $\mathbb{A}^1$-invariant sheaf of abelian groups.*

Remark 2.4.3. [93, Section 6.3] Suppose, $X \in Sm/k$ and $n \geq 0$. There is a canonical morphism $\pi_n^{\mathbb{A}^1}(X) \to H_n^{\mathbb{A}^1}(X)$. The 0-th $\mathbb{A}^1$-homology sheaf $H_0^{\mathbb{A}^1}(X)$ satisfies the following universal property: the canonical morphism $X \to H_0^{\mathbb{A}^1}(X)$ induces a bijection

$$Hom_{\mathcal{A}b(k)}(H_0^{\mathbb{A}^1}(X), \mathcal{F}) \to Hom_{PSh(Sm/k)}(X, \mathcal{F}),$$

for every strictly $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathcal{F}$.

The canonical morphism between pointed spaces $\mathcal{X} \to \Sigma_s \mathcal{X}$ (where, $\Sigma_s \mathcal{X} := S^1_s \wedge \mathcal{X}$) induces isomorphism in $\mathbb{A}^1$-homology sheaves for every $n > 0$ [93, Remark 6.30]:

$$H_n^{\mathbb{A}^1}(\mathcal{X}) \cong H_{n+1}^{\mathbb{A}^1}(\Sigma_s \mathcal{X}), \text{ for every } n > 0.$$

We end this section with the following theorem which we will use in the proof of Theorem 9.2.3:

Theorem 2.4.4. *[119, Theorem 1.1] Let k be a perfect field. Suppose, $X, Y \in Sm/k$ are $\mathbb{A}^1$-simply connected schemes and $f : X \to Y$ is a morphism in Sm/k. Assume that f induces an isomorphism*

$$H_i^{\mathbb{A}^1}(X) \xrightarrow{\cong} H_i^{\mathbb{A}^1}(Y), \text{ for all } 2 \leq i < d \text{ and an epimorphism } H_d^{\mathbb{A}^1}(X) \to H_d^{\mathbb{A}^1}(Y),$$

where $d = max\{dimX + 1, dimY\}$, then f is an $\mathbb{A}^1$-weak equivalence.

Chapter 3

$\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(-)$

In this chapter we discuss about the $\mathbb{A}^1$-invariance of the $\mathbb{A}^1$-connected component sheaf. In Section 3.1, we recall the $\mathbb{A}^1$-rigid schemes over k. The $\mathbb{A}^1$-rigid k-schemes are fundamental in the sense that a locally finite type k-scheme has a local base consisting the $\mathbb{A}^1$-rigid k-schemes at every points (Lemma 3.1.4). In Section 3.2, we recall the $\mathbb{A}^1$-invariant presheaves (Definition 3.2.1). In Section 3.3, we give a comparison between the $\mathbb{A}^1$-conneted component sheaf and the universal $\mathbb{A}^1$-invariant sheaf. Finally in the last section (Section 3.4) we give some equivalent statements regarding the $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ (Theorem 3.4.7).

3.1 $\mathbb{A}^1$-Rigid Schemes

Let Sch/k be the category of finite type schemes over a field k. In this section we recall $\mathbb{A}^1$-rigid k-schemes and prove that for a smooth k-variety X, the $\mathbb{A}^1$-rigid k-schemes form a fundamental system of neighbourhood at each point of X (Lemma 3.1.4). The $\mathbb{A}^1$-rigid k-schemes can never be $\mathbb{A}^1$-connected, unless it is trivial. This in particular says that X cannot have a local base at any point consisting $\mathbb{A}^1$-connected k-varieties. This is different from the nature of a locally contractible topological space which has a local base consisting contractible spaces at every point. This section is taken from [39, Section 2].

Definition 3.1.1. [94, Example 2.4, Section 3.2] A k-scheme $X \in Sch/k$ is said to be $\mathbb{A}^1$-rigid if for each smooth k-scheme U, the natural map

$$Hom_{Sch/k}(U, X) \to Hom_{Sch/k}(\mathbb{A}^1_U, X)$$

induced by the projection map $\mathbb{A}^1_U \to U$ is a bijection.

Lemma 3.1.2. *(see also [11, Lemma 2.1.11]) A k-scheme X is $\mathbb{A}^1$-rigid if and only if for every finite separable field extension L/k, the map*

$$Hom_{Sch/k}(Spec\ L, X) \to Hom_{Sch/k}(\mathbb{A}^1_L, X),$$

induced by the projection $\mathbb{A}^1_L \to Spec\ L$ is a bijection.

Proof. The forward implication follows from the definition of an $\mathbb{A}^1$-rigid scheme. For the reverse implication, suppose that every morphism $\mathbb{A}^1_L \to X$ factors through the projection map $\mathbb{A}^1_L \to Spec\ L$. If possible assume that X is not an $\mathbb{A}^1$-rigid scheme. Then there is some $U \in Sm/k$ and $H : \mathbb{A}^1_U \to X$ such that $H \circ i_0 \neq H \circ i_1$, where $i_0, i_1 : U \to \mathbb{A}^1_U$ are the 0-section and the 1-section respectively, [96, Lemma 2.16]. The set

$$\{x \in U \text{ is a closed point} \mid k(x)/k \text{ is a finite separable extension}\}$$

($k(x)$ is the residue field of x) is dense in U [121, Tag 056U]. Thus there is some closed point x in U with $k(x)/k$ is finite extension, such that $H(0,x) \neq H(1,x)$. Let L be the residue field $k(x)$ at x. Define G as the composiotion

$$\mathbb{A}^1_L \xrightarrow{(Id_{\mathbb{A}^1_k}, x)} \mathbb{A}^1_U \xrightarrow{H} X.$$

Then $G(0) \neq G(1)$ (where, $G(0), G(1) : Spec\ L \to X$ are the 0-section and the 1-section respectively). So G does not factor through the projection $\mathbb{A}^1_L \to Spec\ L$. It is a contradiction. Therefore, X is $\mathbb{A}^1$-rigid. □

Remark 3.1.3.

1. The proof in Lemma 3.1.2 also shows that a k-scheme X is $\mathbb{A}^1$-rigid if and only if for every finite separable field extension L/k and $H : \mathbb{A}^1_L \to X$ is a morphism, then $H(0) = H(1)$, where $H(0), H(1) : Spec\ L \to X$ are the 0-section and 1-section respectively.

2. The $\mathbb{A}^1$-rigid k-schemes are examples of $\mathbb{A}^1$-fibrant spaces. These $\mathbb{A}^1$-fibrant objects have trivial $\mathbb{A}^1$-homotopy sheaf of groups [94, Example 2.4].

3. Two $\mathbb{A}^1$-rigid k-schemes are isomorphic in $\mathbf{H}(\mathbf{k})$ if and only if they are isomorphic as k-schemes [11, Lemma 2.1.9].

4. Abelian varieties, $\mathbb{G}_m$, any smooth projective curve of positive genus are the examples of $\mathbb{A}^1$-rigid k-schemes [11, Example 2.1.10].

5. For an $\mathbb{A}^1$-rigid scheme X, $\pi_0^{\mathbb{A}^1}(X)$ is isomorphic to X [11, Lemma 2.1.9]. Thus for an $\mathbb{A}^1$-rigid k-scheme X, $\pi_0^{\mathbb{A}^1}(X)$ is $\mathbb{A}^1$-invariant (see Definition 3.2.1).

6. Any open or closed subscheme of an $\mathbb{A}^1$-rigid k-scheme is $\mathbb{A}^1$-rigid. Indeed, if X is $\mathbb{A}^1$-rigid and $U \subset X$ is an open subscheme (or a closed subscheme) of X, then a morphism $H : \mathbb{A}^1_L \to U$ with $H(0) \neq H(1)$ gives a morphism $i \circ H : \mathbb{A}^1_L \to X$ ($i : U \hookrightarrow X$ is the inclusion) with $(i \circ H)(0) \neq (i \circ H)(1)$. This contradicts that X is $\mathbb{A}^1$-rigid by (1) of Remark 3.1.3. Thus U is also $\mathbb{A}^1$-rigid.

7. Finite product of $\mathbb{A}^1$-rigid k-schemes is $\mathbb{A}^1$-rigid. Indeed if $X_1, \ldots, X_n$ are the $\mathbb{A}^1$-rigid k-schemes and $H : \mathbb{A}^1_U \to X_1 \times_k \cdots \times_k X_n$ is a morphism, then each $p_i \circ H$ (p_i :

$X_1 \times_k \cdots \times_k X_n \to X_i$ is the i-th projection) factors through the projection $\mathbb{A}^1_U \to U$. Thus H factors through the projection $\mathbb{A}^1_U \to U$.

The next lemma shows that $\mathbb{A}^1$-homotopy theory of smooth schemes has as building blocks $\mathbb{A}^1$-rigid smooth k-schemes. These building blocks have no higher homotopies by Remark 3.1.3. This is different from the local nature of étale homotopy theory and also different from the usual homotopy theory of manifolds.

Lemma 3.1.4 (Local nature). *X is a locally finite type k-scheme, then X has a local base of $\mathbb{A}^1$-rigid k-schemes at each of its points.*

Proof. Since X is of locally finite type, X has an open covering by closed subschemes of $\mathbb{A}^n_k$. So it is enough to prove the theorem for $\mathbb{A}^n_k$ by Remark 3.1.3. For any point $P \in \mathbb{A}^n_k$, P is in a basic open set $D((x_1 - \alpha_1)(x_2 - \alpha_2)..(x_n - \alpha_n))$, for some k-rational points $\alpha_1, \ldots, \alpha_n$ in $\mathbb{A}^n_k$. This basic open set is a finite product of $\mathbb{G}_m$-s, so it is $\mathbb{A}^1$-rigid by Remark 3.1.3. Thus all open subsets of this basic open set form a local base at P of $\mathbb{A}^1$-rigid k-schemes by Remark 3.1.3. □

3.2 $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(-)$ and the Universal $\mathbb{A}^1$-invariant sheaf

We recall $\mathbb{A}^1$-invariant presheaves (Definition 3.2.1) in this section. In Subsection 3.2.1, we discuss the status of the Morel's conjecture related to $\mathbb{A}^1$-connected component sheaf. In Subsection 3.2.2, we recall the universal $\mathbb{A}^1$-invariant sheaf, introduced by Balwe-Hogadi-Sawant, which is the main ingredient from $\mathbb{A}^1$-homotopy theory we use to prove the main theorem (Theorem 8.1.1) in this thesis.

Definition 3.2.1. [93, Definition 7] A presheaf of sets $\mathcal{F}$ on Sm/k is said to be $\mathbb{A}^1$-invariant if for every $U \in Sm/k$, the map $\mathcal{F}(U) \to \mathcal{F}(\mathbb{A}^1_U)$ induced by the projection map $\mathbb{A}^1_U \to U$ is a bijection.

Remark 3.2.2. A presheaf of sets $\mathcal{F}$ on Sm/k is $\mathbb{A}^1$-invariant if and only if for every $U \in Sm/k$, $i_0^* = i_1^* : \mathcal{F}(\mathbb{A}^1_U) \to \mathcal{F}(U)$ (i_0^* is induced by the 0-section $U \to \mathbb{A}^1_U$ and i_1^* is induced by the 1-section $U \to \mathbb{A}^1_U$) [96, Lemma 2.16].

3.2.1 Conjecture of Morel

For a space $\mathcal{X}$, we denote the $\mathbb{A}^1$-connected component presheaf by $\tilde{\pi}_0{}^{\mathbb{A}^1}(\mathcal{X})$, which is defined as

$$U \in Sm/k \mapsto Hom_{\mathbf{H(k)}}(U, \mathcal{X}).$$

By definition, the presheaf $\tilde{\pi}_0{}^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant. Likewise in algebraic topology, there is connected component presheaf associated to a locally contractible topological space $X \in Top$, defined as

$$Y \mapsto Hom_{\mathbf{Ho(Top)}}(Y, X),$$

where the morphism in the homotopy category $Ho(Top)$ is the homotopy class of maps (we denote this presheaf by $\tilde{\pi}_0{}^{Top}(X)$). The topological connected component presheaf is I-invariant Due to its discreteness, its sheafification (we denote it by $\pi_0^{Top}(X)$) with respect to the usual open topology [94, Examples 2, Section 2.3] is also I-invariant Inspired by this, Morel conjectured that for a space $\mathcal{X}$, the $\mathbb{A}^1$-connected component sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, associated to $\mathcal{X}$ is $\mathbb{A}^1$-invariant [93, Conjecture 1.12]. Recently Ayoub has constructed a space $\mathcal{X}$ (this $\mathcal{X}$ is not a representable sheaf), for which $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is not $\mathbb{A}$-invariant [20]. Therefore, Morel's conjecture is false in general.

Remark 3.2.3. However, $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant for the following $\mathcal{X}$:

1. $\mathcal{X}$ is an $\mathbb{A}^1$-rigid smooth k-scheme [11, Lemma 2.1.9].
2. $\mathcal{X}$ is an $\mathbb{A}^1$-connected space.
3. $\mathcal{X}$ is a motivic H-group or a homogeneous space for motivic H-group [37, Theorem 4.18] over an infinite perfect field.
4. $\mathcal{X}$ is a smooth projective surface [26, Corollary 3.15, over any field in case of non-uniruled surface] [29, Theorem 1.2, over an algebraically closed field of characteristic zero in case of birationally ruled surface] or a smooth toric variety [128, Lemma 4.2, Lemma 4.4].

Therefore it is a natural to ask the following:

Question 3.2.4. Is $\pi_0^{\mathbb{A}^1}(X)$ an $\mathbb{A}^1$-invariant sheaf, for a quasi-projective variety $X \in Sm/k$ or in particular for a quasi-projective surface $X \in Sm/k$?

We end this section with the universal property of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$:

Lemma 3.2.5. *[26, Lemma 2.8] Suppose, $\mathcal{X}$ is a space and $\mathcal{F}$ is an $\mathbb{A}^1$-invariant presheaf of sets on Sm/k. Then a morphism $\mathcal{X} \to \mathcal{F}$ uniquely factors through the canonical morphism $\mathcal{X} \to \pi_0^{\mathbb{A}^1}(\mathcal{X})$.*

3.2.2 Universal $\mathbb{A}^1$-invariant sheaf

Definition 3.2.6. [26, Definition 2.9] Let $\mathcal{F}$ be a presheaf of sets on Sm/k, $\mathcal{S}(\mathcal{F})$ is defined as the Nisnevich sheaf associated to the presheaf $\mathcal{S}^{pre}(\mathcal{F})$ given by

$$\mathcal{S}^{pre}(\mathcal{F})(U) := \mathcal{F}(U)/\sim$$

for $U \in Sm/k$, where $\mathcal{F}(U)/\sim$ is the quotient of $\mathcal{F}(U)$ by the equivalence relation generated by $\sigma_0(z) \sim \sigma_1(z)$, $\forall z \in \mathcal{F}(\mathbb{A}^1_U)$ and $\sigma_0, \sigma_1 : \mathcal{F}(\mathbb{A}^1_U) \to \mathcal{F}(U)$ are induced by the 0-section and the 1-section $U \to \mathbb{A}^1_U$ respectively. For any $n > 1$, $\mathcal{S}^n(\mathcal{F})$ is defined inductively as the sheaves

$$\mathcal{S}^n(\mathcal{F}) := \mathcal{S}(\mathcal{S}^{n-1}(\mathcal{F})).$$

For any sheaf $\mathcal{F}$, there is a canonical epimorphism $\mathcal{F} \to \mathcal{S}(\mathcal{F})$. The sheaf $\mathcal{L}(\mathcal{F})$ is defined as

$$\mathcal{L}(\mathcal{F}) := \varinjlim_n \mathcal{S}^n(\mathcal{F}).$$

Therefore there is an induced epimorphism $\mathcal{F} \to \mathcal{L}(\mathcal{F})$.

Remark 3.2.7. 1. For $X \in Sm/k$, $\mathcal{S}(X)$ is the $\mathbb{A}^1$-chain connected component sheaf $\pi_0^{ch}(X)$ of X, defined by Asok-Morel (see Definition 2.2.11). Thus $\mathcal{S}(X)$ is the co-equalizer in $Sh(Sm/k)$ of

$$\underline{Hom}(\mathbb{A}^1_k, X) \underset{\theta_1}{\overset{\theta_0}{\rightrightarrows}} X$$

where θ_0 and θ_1 are induced by the 0-section and the 1-section $Spec\ k \to \mathbb{A}^1_k$ respectively.

2. There is a natural map

$$\mathcal{S}(X) \to \pi_0^{\mathbb{A}^1}(X)$$

which is an epimorphism.

3. For a presheaf $\mathcal{F}$ on Sm/k, $\mathcal{L}(\mathcal{F})$ is an $\mathbb{A}^1$-invariant sheaf [26, Theorem 2.13].

4. $\mathcal{L}(\mathcal{F})$ satisfies the same universal property as $\pi_0^{\mathbb{A}^1}(\mathcal{F})$: any morphism from $\mathcal{F}$ to an $\mathbb{A}^1$-invariant sheaf $\mathcal{G}$ uniquely factors through $\mathcal{L}(\mathcal{F})$ ([26, Remark 2.15]).

5. The canonical epimorphism $\mathcal{F} \to \mathcal{L}(\mathcal{F})$ uniquely factors through $\mathcal{F} \to \pi_0^{\mathbb{A}^1}(\mathcal{F})$ [26, Remark 2.15]. The morphism $\pi_0^{\mathbb{A}^1}(\mathcal{F}) \to \mathcal{L}(\mathcal{F})$ is an isomorphism if and only if $\pi_0^{\mathbb{A}^1}(\mathcal{F})$ is $\mathbb{A}^1$-invariant [26, Corollary 2.18].

3.3 Comparison of $\pi_0^{\mathbb{A}^1}(-)$ with $\mathcal{L}(-)$

In this section we give a comparison of the $\mathbb{A}^1$-connected sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ with the universal $\mathbb{A}^1$-invariant sheaf $\mathcal{L}(\mathcal{F})$, for a sheaf $\mathcal{F}$. This section is taken from [39, Section 2]. The main result in this section is Corollary 3.3.9. It is already proved in [33, Theorem 2.2]. However our proof works in a more general setting (see Remark 3.3.10).

Theorem 3.3.1. *[33, Theorem 2.2] Suppose $\mathcal{F}$ is a sheaf of sets on Sm/k and K/k is a finitely generated field extension. Then the natural map*

$$\pi_0^{\mathbb{A}^1}(\mathcal{F})(Spec\ K) \to \mathcal{L}(\mathcal{F})(Spec\ K)$$

is a bijection.

Definition 3.3.2. Suppose, $\mathcal{G} \in PSh(Sm/k)$. $\mathcal{G}$ is called homotopy invariant if for each finitely generated separable field extension F of k the map $\mathcal{G}(F) \to \mathcal{G}(\mathbb{A}^1_F)$ induced by projection is a bijection.

Example 3.3.3. 1. Any $\mathbb{A}^1$-invariant presheaf is homotopy invariant.

2. $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is homotopy invariant for any space $\mathcal{X}$ [37, Corollary 3.2].

Lemma 3.3.4. *Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on Sm/k which is homotopy invariant and F/k is a finitely generated separable field extension. Then the map $\mathcal{G}(F) \to \mathcal{S}(\mathcal{G})(F)$ is a bijection.*

Proof. Surjectivity follows because of the epimorphism $\mathcal{G} \to \mathcal{S}(\mathcal{G})$. For injectivity, suppose $a, b \in \mathcal{G}(F)$ such that a and b map to the same element of $\mathcal{S}(\mathcal{G})(F)$. Then there are chain of $\mathbb{A}^1_F$-s in $\mathcal{G}$ joining a and b. But any $H \in \mathcal{G}(\mathbb{A}^1_F)$ factors through $\mathcal{G}(F)$. Therefore $a = b$ in $\mathcal{G}(F)$. □

Lemma 3.3.5. *Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on Sm/k which is homotopy invariant and $X = Spec\ R$, spectrum of an essentially smooth discrete valuation ring. Then the map $\mathcal{G}(X) \to \mathcal{S}(\mathcal{G})(X)$ is surjective.*

Proof. Let α be an element of $\mathcal{S}(\mathcal{G})(X)$. The element α gives an element of $\mathcal{S}(\mathcal{G})(Spec\ R^h)$ (R^h is the Henselization of R). The map $\mathcal{G}(Spec\ R^h) \to \mathcal{S}(\mathcal{G})(Spec\ R^h)$ is surjective. So there is a Nisnevich neighbourhood $W \to X$ of the closed point of X and $\alpha' \in \mathcal{G}(W)$ such that α' maps to $\alpha|_W$. Suppose, $F = Frac(R)$ and $L = K(W)$. Since over F we have bijection $\mathcal{G}(F) \to \mathcal{S}(\mathcal{G})(F)$, there is $\beta \in \mathcal{G}(F)$ such that β maps to $\alpha|_F$. The following square is an elementary distinguished square in the Nisnevich topology (Definition 2.2.2):

$$\begin{array}{ccc} Spec\ L & \longrightarrow & W \\ \downarrow & & \downarrow \\ Spec\ F & \longrightarrow & X \end{array}$$

Since the morphism $\mathcal{G} \to \mathcal{S}(\mathcal{G})$ is bijection for sections over fields, we have $\beta|_L = \alpha'|_L$. As $\mathcal{G}$ is a sheaf, β and α' lift to an element $\tilde{\alpha} \in \mathcal{G}(X)$. This $\tilde{\alpha}$ maps to α. □

Theorem 3.3.6. *Let $\mathcal{G}$ be a Nisnevich sheaf of sets on Sm/k which is homotopy invariant. Then for each $X \in Sm/k$ with $dim(X) \leq 1$, the map $\mathcal{G}(X) \to \mathcal{S}(\mathcal{G})(X)$ is surjective.*

Proof. The proof of the theorem follows from Lemma 3.3.5 and the Zariski descent argument in the proof in [37, Theorem 3.1]. □

Corollary 3.3.7. *Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on Sm/k which is homotopy invariant. Then $\mathcal{S}(\mathcal{G})$ is also homotopy invariant.*

Proof. Using Lemma 3.3.4 and Theorem 3.3.6, we get the proof. □

Corollary 3.3.8. *Suppose $\mathcal{G}$ is a Nisnevich sheaf of sets on Sm/k which is homotopy invariant and F/k is a finitely generated separable field extension. Then the maps $\mathcal{G}(F) \to \mathcal{L}(\mathcal{G})(F)$ and $\mathcal{G}(\mathbb{A}^1_F) \to \mathcal{L}(\mathcal{G})(\mathbb{A}^1_F)$ are bijections.*

Proof. The map $\mathcal{G}(F) \to \mathcal{S}(\mathcal{G})(F)$ is a bijection by Lemma 3.3.4. Since $\mathcal{S}(\mathcal{G})$ is homotopy invariant by Corollary 3.3.7, the map $\mathcal{S}(\mathcal{G})(F) \to \mathcal{S}^2(\mathcal{G})(F)$ is a bijection. By induction we get, the map

$$\mathcal{S}^n(\mathcal{G})(F) \to \mathcal{S}^{n+1}(\mathcal{G})(F)$$

is a bijection $\forall\, n$. Since $\mathcal{L}(\mathcal{G})$ is the colimit of $\mathcal{S}^n(\mathcal{G})$, the map $\mathcal{G}(F) \to \mathcal{L}(\mathcal{G})(F)$ is a bijection. As $\mathcal{L}(\mathcal{G})$ is an $\mathbb{A}^1$-invariant sheaf ([26, Theorem 2.13]) and $\mathcal{G}$ satisfies $\mathcal{G}(F) \cong \mathcal{G}(\mathbb{A}^1_F)$, the map $\mathcal{G}(\mathbb{A}^1_F) \to \mathcal{L}(\mathcal{G})(\mathbb{A}^1_F)$ is a bijection. □

Corollary 3.3.9. *Suppose $\mathcal{X} \in \Delta^{op}PSh(Sm/k)$. The maps $\pi_0^{\mathbb{A}^1}(\mathcal{X})(F) \to \mathcal{L}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(F)$ and $\pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \to \mathcal{L}(\pi_0^{\mathbb{A}^1}(\mathcal{X}))(\mathbb{A}^1_F)$ are bijections for any finitely generated separable field extension F/k.*

Proof. The canonical morphism

$$\tilde{\pi}_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_F)$$

is a bijection for any finitely generated separable field extension F/k by [37, Corollary 3.2] (here $\tilde{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ is the presheaf on Sm/k defined as $U \in Sm/k \mapsto Hom_{\mathbf{H}(k)}(U, \mathcal{X})$). By definition, the presheaf $\tilde{\pi}_0^{\mathbb{A}^1}(\mathcal{X})$ is homotopy invariant. Therefore, the sheaf $\pi_0^{\mathbb{A}^1}(\mathcal{X})$ is homotopy invariant. Hence the corollary follows from Corollary 3.3.8. □

Remark 3.3.10. For a sheaf $\mathcal{F}$, Corollary 3.3.9 implies [33, Theorem 2.2]. Indeed in the commutative triangle,

$$\begin{array}{ccc} \pi_0^{\mathbb{A}^1}(\mathcal{F})(K) & \longrightarrow & \mathcal{L}(\mathcal{F})(K) \\ \downarrow & \swarrow & \\ \mathcal{L}(\pi_0^{\mathbb{A}^1}(\mathcal{F}))(K) & & \end{array}$$

where all morphisms are canonical epimorphisms, the upper horizontal map is an isomorphism if the left vertical map is an isomorphism.

We end this section with the following question.

Question 3.3.11. For a proper scheme $X \in Sm/k$, we have $\mathcal{S}(X)(F) \cong \mathcal{S}^n(X)(F)$ for each finitely generated separable extension F of k and for every $n \geq 1$ [26, Theorem 3.9]. Thus $\mathcal{S}(X)$ is homotopy invariant. On the other hand, the real sphere T in $\mathbb{A}^3_{\mathbb{R}}$ contains no non-constant $\mathbb{A}^1_{\mathbb{R}}$ but $\mathcal{S}^2(T)(\mathbb{R})$ is point ([113, Theorem 4.3.4], see also Remark 6.1.6). Therefore it is natural to ask whether $\mathcal{S}(X)$ is homotopy invariant for any scheme $X \in Sm/k$ with $k = \bar{k}$.

3.4 Equivalent Criterias of $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$

In this section we give some equivalent statements of $\mathbb{A}^1$-invariance of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, for a space $\mathcal{X}$.

Lemma 3.4.1. *Suppose $\mathcal{F}$ is an $\mathbb{A}^1$-invariant presheaf on Sm/k. Then the canonical map $\mathcal{F} \to Sing_*(\mathcal{F})$ is an isomorphism of simplicial presheaves.*

Proof. Since the cosimplicial object Δ_a^n is isomorphic to $\mathbb{A}_k^n$ and $\mathcal{F}$ is $\mathbb{A}^1$-invariant, so the canonical morphism

$$\mathcal{F}(U) \to \mathcal{F}(\Delta_a^n \times_k U)$$

is a bijection, for every $U \in Sm/k$. Hence, $\mathcal{F} \cong Sing_*(\mathcal{F})$, as simplicial presheaves. □

Definition 3.4.2. A commutative square in $\Delta^{op}PSh(Sm/k)$

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

is called a homotopy cartesian square (in the Nisnevich local model structure) if there is a factorization of the map $\mathcal{X} \to \mathcal{Z}$ as a Nisnevich local weak equivalence $\mathcal{X} \to \mathcal{X}'$ followed by a fibration (in the Nisnevich local model structure) $\mathcal{X}' \to \mathcal{Z}$ such that the induced map $\mathcal{W} \to \mathcal{X}' \times_{\mathcal{Z}} \mathcal{Y}$ is a Nisnevich local weak equivalence.

Lemma 3.4.3. *A commutative square in* $\Delta^{op}PSh(Sm/k)$

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

is homotopy cartesian (in the Nisnevich local model structure) if and only if for each Henselian local scheme U *(where* $U = Spec\ \mathcal{O}_{X,x}^h$*), the commutative square*

$$\begin{array}{ccc} \mathcal{W}(U) & \longrightarrow & \mathcal{X}(U) \\ \downarrow & & \downarrow \\ \mathcal{Y}(U) & \longrightarrow & \mathcal{Z}(U) \end{array}$$

is homotopy cartesian in the category of simplicial sets.

Proof. First suppose that the commutative square

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

is homotopy cartesian. Then there is a factorization of the map $\mathcal{X} \to \mathcal{Z}$ as as a local weak equivalence $\mathcal{X} \to \mathcal{X}'$ followed by a fibration $\mathcal{X}' \to \mathcal{Z}$ such that the induced map $\mathcal{W} \to \mathcal{X}' \times_{\mathcal{Z}} \mathcal{Y}$ is a local weak equivalence. Consider the commutative square of simplicial sets

$$\begin{array}{ccc} \mathcal{W}(U) & \longrightarrow & \mathcal{X}(U) \\ \downarrow & & \downarrow \\ \mathcal{Y}(U) & \longrightarrow & \mathcal{Z}(U) \end{array}$$

for sections over a Henselian local scheme $U = Spec\ \mathcal{O}^h_{X,x}$. The map $\mathcal{X}(U) \to \mathcal{Z}(U)$ has a factorization as $\mathcal{X}(U) \to \mathcal{X}'(U)$ followed by $\mathcal{X}'(U) \to \mathcal{Z}(U)$. Since the map $\mathcal{X} \to \mathcal{X}'$ is local weak equivalence, the map $\mathcal{X}(U) \to \mathcal{X}'(U)$ is an weak equivalence of simplicial sets. The map $U \times \Lambda^n_i \to U \times \Delta^n$, given by the inclusion of the i-th horn in Δ^n, is a trivial cofibration in the global projective model structure. Indeed the commutative square

$$\begin{array}{ccc} U \times \Lambda^n_i & \longrightarrow & \mathcal{Y}' \\ \downarrow & \nearrow & \downarrow \\ U \times \Delta^n & \longrightarrow & \mathcal{Z}' \end{array}$$

where the map $\mathcal{Y}' \to \mathcal{Z}'$ is a sectionwise Kan fibration of simplicial sets, has a lift since the map $\mathcal{Y}'(U) \to \mathcal{Z}'(U)$ is a Kan fibration. So the map $U \times \Lambda^n_i \to U \times \Delta^n$ is a trivial cofibration in the Nisnevich local model structure. Thus the commutative square

$$\begin{array}{ccc} U \times \Lambda^n_i & \longrightarrow & \mathcal{X}' \\ \downarrow & \nearrow & \downarrow \\ U \times \Delta^n & \longrightarrow & \mathcal{Z} \end{array}$$

has a lift. Therefore the map $\mathcal{X}'(U) \to \mathcal{Z}(U)$ is a Kan fibration. Since the map $\mathcal{W} \to \mathcal{X}' \times_{\mathcal{Z}} \mathcal{Y}$ is a local weak equivalence and the sections over the Henselian local scheme U are the stalks, the map $\mathcal{W}(U) \to \mathcal{X}'(U) \times_{\mathcal{Z}(U)} \mathcal{Y}(U)$ is a weak equivalence. Thus the commutative square of sections over U is homotopy cartesian diagram of simplicial sets.

Conversely, suppose the given commutative square is homotopy cartesian diagram of simplicial sets for every sections over Henselian local scheme $U = Spec\ \mathcal{O}^h_{X,x}$. Consider a factorization of $\mathcal{X} \to \mathcal{Z}$ as a Nisnevich local weak equivalence $\mathcal{X} \to \mathcal{X}'$ followed by a fibration $\mathcal{X}' \to \mathcal{Z}$. So the map $\mathcal{X}(U) \to \mathcal{X}'(U)$ is a weak equivalence of simplicial sets and since the map $\mathcal{X}' \to \mathcal{Z}$ is a fibration, the same argument in the previous paragraph shows that the map $\mathcal{X}'(U) \to \mathcal{Z}'(U)$ is a Kan fibration. Since the commutative square

$$\begin{array}{ccc} \mathcal{W}(U) & \longrightarrow & \mathcal{X}(U) \\ \downarrow & & \downarrow \\ \mathcal{Y}(U) & \longrightarrow & \mathcal{Z}(U) \end{array}$$

is homotopy cartesian diagram of simplicial sets and the map $\mathcal{X}(U) \to \mathcal{Z}(U)$ has a factorization as a weak equivalence $\mathcal{X}(U) \to \mathcal{X}'(U)$ followed by a Kan fibration $\mathcal{X}'(U) \to \mathcal{Z}(U)$, so the map $\mathcal{W}(U) \to \mathcal{X}'(U) \times_{\mathcal{Z}(U)} \mathcal{Y}(U)$ is a weak equivalence of simplicial sets for every $U \in Sm/k$ Henselian local scheme. Therefore the map $\mathcal{W} \to \mathcal{X}' \times_{\mathcal{Z}} \mathcal{Y}$ is a local weak equivalence. Hence the given square of simplicial presheaves is a homotopy cartesian diagram. $\square$

The following lemma is already in [37, Lemma 2.2]. We repeat it here for the sake of completeness.

Lemma 3.4.4. *[37, Lemma 2.2] Suppose a commutative square of simplicial sets*

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is homotopy cartesian. Then the induced map

$$\pi_0(W) \to \pi_0(X) \times_{\pi_0(Z)} \pi_0(Y)$$

is surjective.

Proof. We first replace the diagram

$$X \longrightarrow Z \longleftarrow Y$$

by a diagram

$$X' \xrightarrow{p} Z' \xleftarrow{q} Y'$$

where X', Y', Z' are Kan fibrant simplicial sets and p, q are Kan fibrations, along with a morphism of diagrams

$$\begin{array}{ccccc} X & \longrightarrow & Z & \longleftarrow & Y \\ \downarrow & & \downarrow & & \downarrow \\ X' & \xrightarrow{p} & Z' & \xleftarrow{q} & Y' \end{array} \tag{*}$$

where all the vertical maps the weak equivalences. Indeed, first we replace Z by Z' such that Z' is a Kan fibrant and $Z \to Z'$ is a weak equivalence. Then we factorize the map $X \to Z'$ as a weak equivalence $X \to X'$ followed by a Kan fibration $X' \to Z'$ and similarly we factorize the map $Y \to Z'$ as a weak equivalence $Y \to Y'$ followed by a Kan fibration $Y' \to Z'$. Since the square

$$\begin{array}{ccc} W & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Z \end{array}$$

is homotopy cartesian, thus from the commutative square

$$\begin{array}{ccc} W & \longrightarrow & X' \\ \downarrow & & \downarrow \\ Y' & \longrightarrow & Z' \end{array}$$

(here the maps are induced by $(*)$), the map $W \to X' \times_{Z'} Y'$ is a weak equivalence. Thus it remains to show that the map

$$\pi_0(X' \times_{Z'} Y') \to \pi_0(X') \times_{\pi_0(Z')} \pi_0(Y')$$

is a surjection. Suppose, α and β are the 0-simplices of X' and Y' respectively such that $p(\alpha) = q(\beta) \in \pi_0(Z')$. Thus there is a 1-simplex H of Z' such that $d_0H = p(\alpha)$ and $d_1H = q(\beta)$. Since p is a Kan fibration, the following diagram

$$\begin{array}{ccc} \Delta_0 & \xrightarrow{\alpha} & X' \\ \downarrow{\scriptstyle d_0} & \nearrow & \downarrow{\scriptstyle p} \\ \Delta^1 & \xrightarrow{H} & Z' \end{array}$$

has a lift $\tilde{H}$. Thus $p(d_1\tilde{H}) = d_1H = q(\beta)$ and $(\alpha, \beta) \in \pi_0(X') \times_{\pi_0(Z')} \pi_0(Y')$ has preimage $(d_1\tilde{H}, \beta) \in \pi_0(X' \times_{Z'} Y')$. Therefore, the map $\pi_0(W) \to \pi_0(X) \times_{\pi_0(Z)} \pi_0(Y)$ is surjective. □

Remark 3.4.5. A Nisnevich fibrant space $\mathcal{X}$ satisfies Nisnevich excision (Definition 2.2.4, Remark 2.2.5). Thus by Lemma 3.4.4, for an elementary distinguished square in Definition 2.2.2, the induced map

$$\pi_0(\mathcal{X}(X)) \to \pi_0(\mathcal{X}(U)) \times_{\pi_0(\mathcal{X}(U \times_X V))} \pi_0(\mathcal{X}(V))$$

is surjective. Thus $\mathcal{X}$ is $\mathbb{A}^1$-fibrant, then the map

$$\pi_0(\mathcal{X}(X)) \to \pi_0(\mathcal{X}(U)) \times_{\pi_0(\mathcal{X}(U \times_X V))} \pi_0(\mathcal{X}(V))$$

is surjective.

Lemma 3.4.6. *Suppose a commutative square*

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathcal{Y} & \longrightarrow & \mathcal{Z} \end{array}$$

of simplicial presheaves is homotopy cartesian. Then the induced map

$$\pi_0(\mathcal{W}) \to \pi_0(\mathcal{X}) \times_{\pi_0(\mathcal{Z})} \pi_0(\mathcal{Y})$$

is an epimorphism of Nisnevich sheaves.

Proof. Since the sections over the Henselian local schemes are the stalks, thus we need to show that for each Henselian local scheme U (where $U = Spec\ \mathcal{O}^h_{X,x}$), the map

$$\pi_0(\mathcal{W}(U)) \to \pi_0(\mathcal{X}(U)) \times_{\pi_0(\mathcal{Z}(U))} \pi_0(\mathcal{Y}(U))$$

is surjective. By Lemma 3.4.3, the commutative square of simplicial sets

$$\begin{array}{ccc} \mathcal{W}(U) & \longrightarrow & \mathcal{X}(U) \\ \downarrow & & \downarrow \\ \mathcal{Y}(U) & \longrightarrow & \mathcal{Z}(U) \end{array}$$

is homotopy cartesian and hence the result follows by Lemma 3.4.4. □

Here we give some equivalent statements regarding the homotopy invariance of $\pi_0^{\mathbb{A}^1}(\mathcal{X})$, for a space $\mathcal{X}$.

Theorem 3.4.7. *Let $\mathcal{X}$ be an $\mathbb{A}^1$-fibrant simplicial presheaf on Sm/k. Suppose,*

$$\mathcal{F} = \pi_0^{\mathbb{A}^1}(\mathcal{X}) = \pi_0(\mathcal{X}).$$

Then the following statements are equivalent:

1. *$\mathcal{F}$ is $\mathbb{A}^1$-invariant.*
2. *The canonical map $\mathcal{F} \to Sing_*(\mathcal{F})$ is an isomorphism.*
3. *The canonical map $\mathcal{F} \to Sing_*(\mathcal{F})$ induces a Kan fibration $\mathcal{F}(Spec\ \mathcal{O}^h_{X,x}) \to Sing_*(\mathcal{F})(Spec\ \mathcal{O}^h_{X,x})$ between the simplicial sets, for each $X \in Sm/k$ and $x \in X$.*
4. *The map $\mathcal{F} \to \pi_0(Sing_*(\mathcal{F}))$ induced by the canonical map $\mathcal{F} \to Sing_*(\mathcal{F})$ is an isomorphism of sheaves.*
5. *The canonical map $\mathcal{F} \to \mathcal{L}(\mathcal{F})$ is an isomorphism of sheaves.*
6. *The following cartesian square is homotopy cartesian in $\Delta^{op}PSh(Sm/k)$:*

$$\begin{array}{ccc} Sing_*(\mathcal{X}|\mathcal{F}) & \longrightarrow & \mathcal{F} \\ \downarrow & & \downarrow \\ Sing_*(\mathcal{X}) & \longrightarrow & Sing_*(\mathcal{F}) \end{array}$$

7. *The map $Sing_*(\mathcal{X}) \to Sing_*(\mathcal{F})$, induced by the map $\mathcal{X} \to \pi_0(\mathcal{X})$, induces a Kan fibration $Sing_*(\mathcal{X})(Spec\ \mathcal{O}^h_{X,x}) \to Sing_*(\mathcal{F})(Spec\ \mathcal{O}^h_{X,x})$, for each $X \in Sm/k$ and $x \in$.*
8. *The canonical map $\tilde{\pi_0}^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_{\mathcal{O}}) \to \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_{\mathcal{O}})$ between the sections over $\mathbb{A}^1_{\mathcal{O}}$ is a surjection, where $\mathcal{O} = \mathcal{O}^h_{X,x}$, for every $X \in Sm/k, x \in X$ (where, $\tilde{\pi_0}^{\mathbb{A}^1}(\mathcal{X})$ denotes the presheaf on Sm/k, which is defined as $U \mapsto Hom_{\mathbf{H(k)}}(U, \mathcal{X})$.).*

Proof. $(\mathbf{1}) \implies (\mathbf{2})$: It follows by Lemma 3.4.1.

$(\mathbf{2}) \implies (\mathbf{3})$: It follows since any isomorphism is a fibration.

$(\mathbf{3}) \implies (\mathbf{4})$: We need to show for each Henselian local scheme U (where $U = Spec\ \mathcal{O}^h_{X,x}$), the map $\mathcal{F}(U) \to \pi_0(Sing_*(\mathcal{F}))(U)$ is a bijection. Since the sections over Henselian local schemes are the stalks, so $\pi_0(Sing_*(\mathcal{F}))(U) = \pi_0(Sing_*(\mathcal{F})(U))$. But $Sing_*(\mathcal{F})(U)$ agrees with $\mathcal{F}(U)$ in 0-simplices, so it is already a surjection. For injectivity, suppose two sections $\alpha, \beta \in \mathcal{F}(U)$ are same in $\pi_0(Sing_*(\mathcal{F})(U))$. Thus there are $H_1, H_2, .., H_m \in \mathcal{F}(\mathbb{A}^1_U)$ such that

$H_1(0) = \alpha$ and $H_m(1) = \beta$. The map $\mathcal{F}(U) \to Sing_*(\mathcal{F})(U)$ is a Kan fibration. For every i, consider the commutative square

$$\begin{array}{ccc} \Lambda_0^1 & \xrightarrow{H_i(0)} & \mathcal{F}(U) \\ \downarrow & \nearrow & \downarrow \\ \Delta^1 & \xrightarrow[H_i]{} & Sing_*(\mathcal{F})(U) \end{array}$$

which has a lift $\Delta^1 \to \mathcal{F}(U)$ making the triangles commutative. Here the lift is $H_i(0)$, since $\mathcal{F}(U)$ is regarded as constant simplicial set. Since the lower triangle is commutative, we have $H_i = H_i(0) \circ pr$, where $pr : \Delta_a^1 \times_k U \to U$ is the projection map. Thus $H_i(0) = H_i(1)$, for all i. Hence, $\alpha = \beta$. So the map $\mathcal{F}(U) \to \pi_0(Sing_*(\mathcal{F}))(U)$ is injective, for every Henselian local scheme $U \in Sm/k$. Therefore, the map $\mathcal{F} \to \pi_0(Sing_*(\mathcal{F}))$ is an isomorphism of sheaves.

$(\mathbf{4}) \implies (\mathbf{5})$**:** The sheaf $\pi_0(Sing_*(\mathcal{F}))$ is the $\mathbb{A}^1$-chain connected component sheaf $\mathcal{S}(\mathcal{F})$ of $\mathcal{F}$. So if the canonical map $\mathcal{F} \to \pi_0(Sing_*(\mathcal{F}))$ is an isomorphism, then for each n the canonical map $\mathcal{S}^n(\mathcal{F}) \to \mathcal{S}^{n+1}(\mathcal{F})$ is an isomorphism. Therefore being colimit, $\mathcal{L}(\mathcal{F})$ is isomorphic to $\mathcal{F}$.

$(\mathbf{5}) \implies (\mathbf{1})$**:** It follows because $\mathcal{L}(\mathcal{F})$ is an $\mathbb{A}^1$-invariant sheaf, for any sheaf $\mathcal{F}$ on Sm/k [26, Theorem 2.13].

$(\mathbf{1}) \implies (\mathbf{6})$**:** If $\mathcal{F}$ is $\mathbb{A}^1$-invariant, then the right vertical map in the cartesian square in (6) is an isomorphism, thus in particular fibration. Hence the cartesian square in (5) is homotopy cartesian.

$(\mathbf{6}) \implies (\mathbf{4})$**:** Since the commutative diagram in (6) is a homotopy cartesian square, by Lemma 3.4.6 the induced morphism of sheaves

$$\phi : \pi_0(Sing_*(\mathcal{X}|\mathcal{F})) \to \pi_0(Sing_*(\mathcal{X})) \times_{\pi_0(Sing_*(\mathcal{F}))} \mathcal{F}$$

is an epimorphism. Since $\mathcal{X}$ is $\mathbb{A}^1$-fibrant, so $Sing_*(\mathcal{X})$ is also $\mathbb{A}^1$-fibrant [94, Corollary 3.13, Section 2.3]. The canonical map $\mathcal{X} \to Sing_*(\mathcal{X})$ is $\mathbb{A}^1$-weak equivalence, so $\mathcal{F} \cong \pi_0(Sing_*(\mathcal{X}))$. The map $\mathcal{X} \to \mathcal{F}$ induces a map $\psi : \mathcal{X} \to Sing_*(\mathcal{X}|\mathcal{F})$. Taking π_0, ψ gives a map

$$\theta : \mathcal{F} = \pi_0(\mathcal{X}) \to \pi_0(Sing_*(\mathcal{X}|\mathcal{F}))$$

The map θ is an epimorphism, since $\mathcal{X}$ and $Sing_*(\mathcal{X}|\mathcal{F})$ agree in 0-simplices. Suppose, $\mathcal{T} = \pi_0(Sing_*(\mathcal{F}))$. The morphism $\phi \circ \theta : \mathcal{F} \to \mathcal{F} \times_{\mathcal{T}} \mathcal{F}$ is the diagonal map and $\phi \circ \theta$ is an epimorphism i.e. for each Henselian local scheme U (where $U = Spec\ \mathcal{O}_{X,x}^h$) the diagonal map between sets

$$\mathcal{F}(U) \to \mathcal{F}(U) \times_{\mathcal{T}(U)} \mathcal{F}(U)$$

is surjective. This can only occur if the map $\mathcal{F}(U) \to \mathcal{T}(U)$ is injective. The map $\mathcal{F} \to \mathcal{T}$ is an epimorphism since $Sing_*(\mathcal{F})$ agrees with $\mathcal{F}$ in simplicial degree 0. Therefore the map $\mathcal{F} \to \mathcal{T}$ is an isomorphism. Thus $\mathcal{F} \cong \pi_0(Sing_*(\mathcal{F}))$.

$(\mathbf{1}) \implies (\mathbf{7})$**:** We have shown that (1) and (2) are equivalent. Since $\mathcal{F}$ is $\mathbb{A}^1$ invariant, the canonical map $\mathcal{F} \to Sing_*(\mathcal{F})$ is an isomorphism. This gives a morphism $\phi : Sing_*(\mathcal{X}) \to \mathcal{F}$, which is the composition of the map $Sing_*(\mathcal{X}) \to Sing_*(\mathcal{F})$ followed by the inverse of the morphism $\mathcal{F} \to Sing_*(\mathcal{F})$. Suppose $U = Spec\ \mathcal{O}^h_{X,x}$ is a Henselian local scheme. It is enough to show that the map $\phi : Sing_*(\mathcal{X})(U) \to \mathcal{F}(U)$ is a Kan fibration. Consider the commutative square of simplicial sets

$$\begin{array}{ccc} \Lambda^n_k & \longrightarrow & Sing_*(\mathcal{X})(U) \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \xrightarrow[\tau]{} & \mathcal{F}(U) \end{array}$$

Since $\mathcal{X}$ is $\mathbb{A}^1$-fibrant, so $Sing_*(\mathcal{X})$ is also $\mathbb{A}^1$-fibrant [94, Corollary 3.13, Section 2.3]. Thus $Sing_*(\mathcal{X})$ is stalkwise Kan fibrant. Therefore there is an n-simplex σ of $Sing_*(\mathcal{X})(U)$ such that the upper triangle commutes. We show that the lower triangle also commutes. The simplicial set $\mathcal{F}$ is a constant simplicial set, so all the boundary maps are the identity maps. Since the square is commutative and the upper triangle commutes, so for every $i \neq k, 0 \leq i \leq n$,

$$\phi(d_i\sigma) = d_i\tau = \tau.$$

Thus $d_i\phi(\sigma) = \phi(d_i\sigma) = \tau$. As the boundary maps in $\mathcal{F}(U)$ are the identity maps, so $\phi(\sigma) = \tau$. Therefore, the lower triangle also commutes. Hence ϕ is a Kan fibration, for the sections over U. Thus for every Henselian local scheme U, the map $Sing_*(\mathcal{X})(U) \to Sing_*(\mathcal{F})(U)$ is a Kan fibration.

$(\mathbf{7}) \implies (\mathbf{6})$**:** Since the commutative square is cartesian, it is cartesian diagram of simplicial sets for every section $U = Spec\ \mathcal{O}^h_{X,x}$ a Henselian local scheme. We have given that the map $Sing_*(\mathcal{X})(U) \to Sing_*(\mathcal{F})(U)$ is a Kan fibration. So the square is homotopy cartesian diagram of simplicial sets, for every section over Henselian local scheme U. Therefore the square of simplicial presheaves is simplicially homotopy cartesian by Lemma 3.4.3.

$(\mathbf{1}) \Leftrightarrow (\mathbf{8})$**:** Since the sections over the Henselian local schemes are the stalks, the sheaf $\mathcal{F}$ is $\mathbb{A}^1$-invariant if and only if the map

$$\mathcal{F}(U) \to \underline{Hom}(\mathbb{A}^1, \mathcal{F})(U),$$

is a bijections for every section over the Henselian local scheme U. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{\pi_0}^{\mathbb{A}^1}(\mathcal{X})(U) & \longrightarrow & \tilde{\pi_0}^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_U) \\ \downarrow & & \downarrow \\ \pi_0^{\mathbb{A}^1}(\mathcal{X})(U) & \longrightarrow & \pi_0^{\mathbb{A}^1}(\mathcal{X})(\mathbb{A}^1_U) \end{array}$$

Since the presheaf $\tilde{\pi_0}^{\mathbb{A}^1}(\mathcal{X})$ is $\mathbb{A}^1$-invariant, the upper horizontal map is an isomorphism. Since the sections over U are the stalks, the left vertical map is an isomorphism. The bottom horizontal map, induced by $\mathbb{A}^1_U \to U$, is an injection. Therefore the right vertical map is surjection if and only if the bottom horizontal map is a bijection. $\square$

Chapter 4

Birational Connected Components

In this chapter we define the birational model structure on $\Delta^{op}PSh(Sm/k)$, which is the left Bousfield localisation of the global projective model structure (Proposition 4.2.1) at the class of birational morphisms. The main theorem in this chapter is Theorem 4.2.3, where we prove that the connected component sheaf in the birational model structure associated to a proper scheme $X \in Sm/k$ is isomorphic to the birational $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$, defined by Asok-Morel [11, Definition 6.2.5]. This section is taken from [39, Section 3].

4.1 Birational $\mathbb{A}^1$-connected component sheaf

The first example of an $\mathbb{A}^1$-invariant sheaf associated to the $\mathbb{A}^1$-connected components of a scheme was constructed by Asok and Morel. In this short section, we recall the Asok-Morel's birational and $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$ [11, Section 6] associated to a proper scheme $X \in Sm/k$.

Definition 4.1.1. [11, Definition 6.1.1] A presheaf of sets $\mathcal{F}$ on Sm/k is called birational if it satisfies the following properties:

1. For $X \in Sm/k$ with irreducible components $X_1, X_2, \ldots, X_n$ the canonical map

$$\mathcal{F}(X) \to \prod_{i=1}^{n} \mathcal{F}(X_i)$$

   is a bijection.

2. For an open dense subscheme U of X, the canonical morphism $\mathcal{F}(X) \to \mathcal{F}(U)$ is a bijection.

Remark 4.1.2. A birational presheaf of sets is always a Nisnevich sheaf [11, Lemma 6.1.2].

Definition 4.1.3. [11, Section 6.2] Let $X \in Sm/k$ be a proper scheme. There is a birational and $\mathbb{A}^1$-invariant ([11, Theorem 6.1.7]) sheaf $\pi_0^{b\mathbb{A}^1}(X)$ associated to X [11, Section 6.2] such that its sections over any irreducible $U \in Sm/k$ is the $\mathbb{A}^1$-chain connected component of $k(U)$-rational points, i.e. $\mathcal{S}(X)(k(U)) = \pi_0^{b\mathbb{A}^1}(X)(U)$ ([11, Definition 6.2.5]).

Remark 4.1.4. 1. For a proper scheme $X \in Sm/k$, there is a canonical morphism

$$\pi_0^{\mathbb{A}^1}(X) \to \pi_0^{b\mathbb{A}^1}(X)$$

such that the composition

$$\pi_0^{ch}(X) \to \pi_0^{\mathbb{A}^1}(X) \to \pi_0^{b\mathbb{A}^1}(X)$$

is a bijection on sections over any finitely generated separable field extensions of k [11, Proposition 6.2.6], [26, Corollary 3.10] .

2. $\pi_0^{b\mathbb{A}^1}$ is a birational invariant of smooth proper schemes [72, Theorem 1]. However $\pi_0^{\mathbb{A}^1}$ is a not birational invariant sheaf of smooth proper schemes [26, Example 4.8].

4.2 Birational Model Structure

In this section in Theorem 4.2.3, we will prove that $\pi_0^{b\mathbb{A}^1}(X)$ is isomorphic to the connected component sheaf of X in the birational model structure (Proposition 4.2.1). This gives a proof of [72, Theorem 4]. Same line of argument is used in [36, Proposition 1.9]. In [102, Definition 2.6] Pelaez also constructed birational unstable motivic homotopy category (equivalent construction by Theorem 4.2.7)

Proposition 4.2.1. *The left Bousfield localisation of the global projective model structure on $\Delta^{op}PSh(Sm/k)$ with respect to the following set of maps*

$$\{U \xrightarrow{i} X \in Sm/k \mid i \text{ is an open immersion with dense image}\}$$

exists. It gives a model structure on $\Delta^{op}PSh(Sm/k)$ called the unstable birational model structure.

Proof. Existence of the left Bousfield localisation is proved in [87, Section A.3.7]. □

The resulting homotopy category associated to the birational model structure will be denoted by $\mathbf{H}_b(k)$.

Definition 4.2.2. For any space $\mathcal{X}$, the connected component presheaf associated to the birational model structure is defined as $U \mapsto Hom_{\mathbf{H}_b(k)}(U, \mathcal{X})$, for $U \in Sm/k$. It will be denoted by $\pi_0^b(\mathcal{X})$.

The aim of this section is to prove the following result:

Theorem 4.2.3. *There is an isomorphism of presheaves: $\pi_0^{b\mathbb{A}^1}(X) \cong \pi_0^b(X)$, for $X \in Sm/k$ a proper scheme.*

Let $f : U \to X \in Sm/k$ be a Nisnevich covering and $U_\bullet$ be the corresponding Čech simplicial scheme. Here U_n is the smooth scheme given by $U \times_X U \times_X \cdots \times_X U$ (the product is taken $n+1$ times). Let $f : U_\bullet \to X$ be the corresponding map of simplicial schemes. We show that inverting the birational morphisms in $\mathbb{A}^1$-homotopy category is equivalent to only inverting the birational morphisms in the global projective model structure (Theorem 4.2.7).

Lemma 4.2.4. *The map $f : U_\bullet \to X$ is a birational weak equivalence (i.e., it is an isomorphism in $\mathbf{H}_b(k)$).*

Proof. Any Nisnevich covering has a section over a dense open set. Therefore there is an open dense set $V \subset X$ such that the restriction $f^{-1}(V) \to V$ has a section. We have the following commutative diagram in $\Delta^{op}PSh(Sm/k)$:

$$\begin{array}{ccc} f^{-1}(V)_\bullet & \longrightarrow & U_\bullet \\ \downarrow & & \downarrow f \\ V & \longrightarrow & X \end{array}$$

where the left vertical map is induced by the restriction and the upper horizontal map is induced by the inclusion. The left vertical map is a sectionwise weak equivalence, since there is a section. The map $V \to X$ is an inclusion of dense open set, so it is a birational weak equivalence. As the map $f : U \to X$ is an étale map, for each n, the $(n+1)$-fold product $f^{-1}(V) \times_V f^{-1}(V) ... \times_V f^{-1}(V)$ is open and dense in $U \times_X U .. \times_X U$ fitting in the pullback square:

$$\begin{array}{ccc} f^{-1}(V) \times_V .. \times_V f^{-1}(V) & \longrightarrow & U \times_X U .. \times_X U \\ \downarrow & & \downarrow \\ V & \longrightarrow & X \end{array}$$

Therefore the morphism $f^{-1}(V)_\bullet \to U_\bullet$ is a birational weak equivalence [94, Proposition 2.14]. Hence $f : U_\bullet \to X$ is a birational weak equivalence. □

Corollary 4.2.5. *Any Nisnevich weak equivalence is a birational weak equivalence.*

Proof. The Nisnevich local model structure on $\Delta^{op}PSh(Sm/k)$ is the left Bousfield localisation of the projective model structure at the class of Čech hypercovers ([44, Theorem 6.2, Example A.11]),

$$\{U_\bullet \to X \mid U \to X \text{ is a Nisnevich covering}\}$$

Since the map $U_\bullet \to X$ is a birational weak equivalence by Lemma 4.2.4, the result follows. □

Lemma 4.2.6. *For every $X \in Sm/k$, the projection map $X \times \mathbb{A}^1 \to X$ is a birational weak equivalence.*

Proof. Suppose $\mathcal{X} \in \Delta^{op}PSh(Sm/k)$ and $X \in Sm/k$. Consider the presheaf of sets $\mathcal{F}_{\mathcal{X},X}$ on Sm/k defined as $Y \mapsto Hom_{\mathbf{H}_b(k)}(Y \times X, \mathcal{X})$. Then $\mathcal{F}_{\mathcal{X},X}$ is a birational sheaf on Sm/k.

Therefore we have a bijection $\mathcal{F}_{\mathcal{X},X}(Spec\ k) \to \mathcal{F}_{\mathcal{X},X}(\mathbb{P}^1_k)$ [72, Appendix A]. This implies the projection map $\mathbb{P}^1_k \times X \to X$ is an isomorphism in $\mathbf{H}_b(k)$. Composing it with the birational map $\mathbb{A}^1_k \times X \hookrightarrow \mathbb{P}^1_k \times X$ (induced by the natural open immersion $\mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$), we get that the projection map $X \times \mathbb{A}^1_k \to X$ is an isomorphism in $\mathbf{H}_b(k)$. □

Theorem 4.2.7. *Any $\mathbb{A}^1$-weak equivalence is a birational weak equivalence. Therefore the unstable birational model structure is equivalent to the motivic unstable birational model structure in [102, Definition 2.6].*

Proof. The left Bousfield localisation of the projective model structure (universal model structure) on $\Delta^{op}PSh(Sm/k)$ at the class of the Čech hypercovers and the projection maps $\mathbb{A}^1_X \to X \in Sm/k$ gives the $\mathbb{A}^1$-model structure [49, Proposition 8.1]. Therefore, the $\mathbb{A}^1$-weak equivalences are generated by the Čech covers and the projection maps $\mathbb{A}^1_X \to X$. Both are birational weak equivalences [Lemma 4.2.4 and Lemma 4.2.6]. Hence the result follows. □

Remark 4.2.8. Theorem 4.2.7 says that the birational model structure (Proposition 4.2.1) is the left Bousfield localisation of the $\mathbb{A}^1$-model structure on $\Delta^{op}PSh(Sm/k)$ at the class of the birational morphisms. Therefore, the identity map on $\Delta^{op}PSh(Sm/k)$ induces total left derived functor

$$\mathbf{H(k)} \to \mathbf{H}_b(k).$$

4.2.1 Proof of the Theorem 4.2.3

Proof. Suppose $U \in Sm/k$ is irreducible. Then, $\pi_0^{b\mathbb{A}^1}(X)(U) = \mathcal{S}(X)(k(U))$ by [11, Definition 6.2.5]. Recall the fine birational category of smooth k-schemes $S_b^{-1}Sm/k$ [72, Section 1.7] which is defined as the localisation of Sm/k with respect to the class of birational morphisms S_b. By [72, Theorem 6.6.3], we have the following natural bijection

$$Hom_{S_b^{-1}Sm/k}(U, X) \cong \pi_0^{b\mathbb{A}^1}(X)(U),$$

for each $U \in Sm/k$. The Yoneda embedding of Sm/k in $\Delta^{op}PSh(Sm/k)$ as representable constant simplicial presheaf induces a functor $\eta : S_b^{-1}Sm/k \to \mathbf{H}_b(k)$ because of the universal property of localisation [57, Section 1]). The functor η is universal and it factors the functor $\pi : Sm/k \to \mathbf{H}_b(k)$. This gives the map

$$Hom_{S_b^{-1}Sm/k}(U, X) \to Hom_{\mathbf{H}_b(k)}(U, X).$$

Thus we have a morphism $\eta : \pi_0^{b\mathbb{A}^1}(X) \to \pi_0^b(X)$.

Consider the following commutative diagram of presheaves on Sm/k,

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & \pi_0^{b\mathbb{A}^1}(X) \\ {\scriptstyle \pi}\downarrow & \overset{\theta}{\nearrow} & \downarrow{\scriptstyle \cong} \\ \pi_0^b(X) & \longrightarrow & \pi_0^b(\pi_0^{b\mathbb{A}^1}(X)) \end{array}$$

induced by the natural transformation $Id \to \pi_0^b(-)$. The top horizontal morphism of presheaves $\alpha : X \to \pi_0^{b\mathbb{A}^1}(X)$ is induced by the canonical functor $\alpha : Sm/k \to S_b^{-1}Sm/k$. The right most vertical map is an isomorphism, since $\pi_0^{b\mathbb{A}^1}(X)$ is a fibrant object in the birational model structure. This gives a morphism $\theta : \pi_0^b(X) \to \pi_0^{b\mathbb{A}^1}(X)$. The morphism $\eta \circ \theta \circ \pi$ is same as $\eta \circ \alpha$ and $\eta \circ \alpha$ is the natural morphism π. The morphism $\pi : X \to \pi_0^b(X)$ induces a bijection

$$Hom_{PSh(Sm/k)}(\pi_0^b(X), \pi_0^b(X)) \cong Hom_{PSh(Sm/k)}(X, \pi_0^b(X)),$$

since $\pi_0^b(X)$ is birational local. This gives $\eta \circ \theta$ is the identity morphism. So θ is a monomorphism. On the other hand, the morphism α factors through θ and the morphism α is sectionwise surjective, since $\pi_0^{b\mathbb{A}^1}(X)$ is a birational sheaf and its section over U is the $\mathbb{A}^1$-equivalence classes of $k(U)$-rational points of X. Hence θ is an epimorphism and consequently it is an isomorphism. □

Chapter 5

Existence of $\mathbb{A}^1$ and $\mathbb{A}^1$-Connectedness of a Surface

In this chapter we establish the fact that $\mathbb{A}^1$-connectedness of a smooth variety X over an algebraically closed field is related to the existence of affine lines in X. In Section 5.1, we recall several kinds of varieties containing the images of affine lines and how they are related to the negativity of the logarithmic Kodaira dimension. In Section 5.2, we prove that if a surface is $\mathbb{A}^1$-connected, then it is dominanted by images of $\mathbb{A}^1$ (Theorem 5.2.8, see Definition 5.1.1). This is the main result in this chapter. By the phrase "there is an $\mathbb{A}^1$ in X", we mean the existence of a non-constant morphism from $\mathbb{A}^1_k$ to X. For this chapter, we will assume that our base field k is an algebraically closed field. This chapter is taken from [39, Section 4].

5.1 Varieties Containing Affine Lines and Negative Logarithmic Kodaira Dimension

In this section we recall three important classes of varieties containing the images of the affine lines.

Definition 5.1.1. Suppose X is a k-variety.

1. X is said to be **dominated by images of** $\mathbb{A}^1$ if there is an open dense subset U of X such that for every $p \in U(k)$, there is an $\mathbb{A}^1$ in X through p [80, §1].

2. X is said to be $\mathbb{A}^1$**-uniruled or log-uniruled** if there is a dominant generically finite morphism $H : \mathbb{A}^1_k \times_k Y \to X$ for some k-variety Y.

3. X is said to be $\mathbb{A}^1$**-ruled** if there is a Zariski open dense subset U of X such that U is isomorphic to $\mathbb{A}^1_k \times_k Z$ for some k-variety Z [51, Definition 1].

Remark 5.1.2. Here we will describe few important relations between the above notions. Suppose, the base field k is of characteristic zero. By definition, the $\mathbb{A}^1$-ruled varieties are $\mathbb{A}^1$-uniruled and $\mathbb{A}^1$-uniruled varieties are dominated by images of $\mathbb{A}^1$. Suppose the field k is uncountable, then the smooth varieties dominated by images of $\mathbb{A}^1$ are $\mathbb{A}^1$-uniruled. This is essentially similar to the fact that a smooth projective variety dominated by images of $\mathbb{P}^1$ is uniruled [81, Chapter IV, Proposition 1.3]. Indeed, given a smooth k-variety X with fixed smooth completion $\bar{X}$ with boundary $D = \bar{X} \setminus X$, non-constant morphisms from $\mathbb{A}^1_k$ to X are parametrized by a certain subscheme $Mor((\mathbb{P}^1_k, \infty), (\bar{X}, D))$ of the hom scheme $Mor_k(\mathbb{P}^1_k, \bar{X})$ parametrizing morphisms $f : \mathbb{P}^1_k \to \bar{X}$ such that $f^{-1}(D) = \{\infty\}$. There is a canonical evaluation morphism $ev : Mor((\mathbb{P}^1_k, \infty), (\bar{X}, D)) \times_k (\mathbb{P}^1_k \setminus \{\infty\}) \to X$ which is dominant, as X is dominated by images of $\mathbb{A}^1$. Since k is uncountable and $Mor((\mathbb{P}^1_k, \infty), (\bar{X}, D))$ has only countably many irreducible components and there is a dense open subset U of X which is contained in the image of ev, we get an irreducible component Y of $Mor((\mathbb{P}^1_k, \infty), (\bar{X}, D))$ such that the restriction to Y of ev is a dominant morphism $\mathbb{A}^1_k \times_k Y \to X$. In case of k is countable field, the equivalence of (1) and (2) is not known.

Logarithmic Kodaira dimension is an useful invariant in birational geometry. Over the field of characteristic zero, an $\mathbb{A}^1$-uniruled variety and an $\mathbb{A}^1$-ruled variety have negative logarithmic Kodaira dimension. Before discussing about this, we recall the definition of logarithmic Kodaira dimension of a smooth quasi-projective variety.

Definition 5.1.3. ([70, Section 10.1, Section 11.2], [79]) Suppose V is a smooth proper k-variety and D is a divisor on V. The D-dimension $\kappa(V, D)$ is defined as following:

- If for some m,

  $$|mD| = \{D' \text{ is an effective divisor of } V \mid D' \text{ is linearly equivalent to } mD\} \neq \emptyset,$$

  then

  $$\kappa(V, D) := max\{dim(\Phi_{|mD|}(V)) \mid m \in \mathbb{N} \text{ and } |mD| \neq \emptyset\},$$

  where $\Phi_{|mD|} : V \dashrightarrow \mathbb{P}^N$ is the rational map associated to the complete linear system $|mD|$.

- If $|mD| = \emptyset$ for every m, then $\kappa(V, D) := -\infty$.

Suppose, X is a smooth quasi-projective k-variety. Assume that X can be embeded in a smooth proper k-variety V such that $D = V \setminus X$ is a divisor with simple normal crossings (this is always possible if the base field k is of characteristic zero or $dim(X) \leq 2$, by the resolution of singularities, [70, Theorem 7.21], [64, Theorem 3.9]). The logarithmic Kodaira dimension $\bar{\kappa}(X)$ of X is defined to be the $(K_V + D)$-dimension $\kappa(V, K_V + D)$, where K_V is the canonical divisor of V. The logarithmic Kodaira dimension is independent of choosing compactification of X.

Remark 5.1.4. 1. The logarithmic Kodaira dimension is a proper birational invariant i.e. if $f : X \to Y$ is a proper birational morphism between smooth quasi-projective varieties, then $\bar{\kappa}(X) = \bar{\kappa}(Y)$ [70, Theorem 10.2].

2. For a smooth quasi-projective variety X of dimension n, $\bar{\kappa}(X) \in \{-\infty, 0, 1, \ldots, n\}$ [70, Theorem 10.3].

3. For smooth quasi-projective k-varieties X, Y, $\bar{\kappa}(X \times_k Y) = \bar{\kappa}(X) + \bar{\kappa}(Y)$ [70, Theorem 11.3].

4. The affine n-space $\mathbb{A}^n_k$ and the projective n-space $\mathbb{P}^n_k$ have negative logarithmic Kodaira dimension.

5. Suppose X, Y are smooth quasi-projective k- surfaces over an algebraically closed field k and $f : X \to Y$ is a dominant separable morphism,. then $\bar{\kappa}(Y) \leq \bar{\kappa}(X)$ [109, Lemma 1.8].

6. If X is a k-surface, where k is an uncountable algebraically closed field of characteristic zero, then X is dominated by images of $\mathbb{A}^1$ if and only if X has negative logarithmic Kodaira dimension $\bar{\kappa}(X)$ [80, Theorem 1.1]. Moreover, if X is an affine k-surface, then $\bar{\kappa}(X) = -\infty$ if and only if X is $\mathbb{A}^1$-ruled [91, §4, §5]. However, it is not known in general whether varieties dominanted by images of $\mathbb{A}^1$ have negative logarithmic Kodaira dimension if k is countable.

Therefore, for a smooth affine k-surface X over an uncountable algebraically closed field k of characteristic zero, we have $(1), (2), (3)$ in Definition 5.1.1 are equivalent and these are equivalent to the negativity of the logarithmic Kodaira dimension. However in higher dimensions, being $\mathbb{A}^1$-ruled is a stronger notion than dominated by images of $\mathbb{A}^1$ [51, Proposition 9].

5.2 $\mathbb{A}^1$-Connectedness of a Surface and Surfaces dominated by images of $\mathbb{A}^1$

The $\mathbb{A}^1$-connectedness of a smooth variety X is strongly related to the existence of affine lines in X. The works of Asok, Morel, Balwe, Hogadi, Sawant and several others established the fact. We have discussed about this in the Introduction. In this section and also in Chapter 6, we again establish this fact. In Theorem 5.2.8, we prove that $\mathbb{A}^1$-connectedness of a surface X implies that either X contains family of affine lines or through every k-rational point of X there is an affine line in X. Before proceeding towards the main theorem, we recall one important result from the works of Asok and Morel in this regard.

Theorem 5.2.1. *(see also [6, Remark 5.4.2.11]) Let X be a smooth surface over an algebraically closed field k of characteristic zero. Suppose that the 0-th $\mathbb{A}^1$-homology sheaf $H_0^{\mathbb{A}^1}(X)$ is isomorphic to the constant sheaf $\mathbb{Z}$. Then X is a rational surface. Therefore, if X is $\mathbb{A}^1$-connected, then X is a rational surface.*

Proof. By Hironaka's resolution of singularities, there is a smooth proper k-surface $\overline{X}$ containing X as an open subvariety. Since the complement of X in $\overline{X}$ has codimension 1, by

[14, Proposition 3.8], the morphism $H_0^{\mathbb{A}^1}(X) \to H_0^{\mathbb{A}^1}(\overline{X})$ is an epimorphism. Thus by [78, Theorem 1.1] (see also [118, Theorem 2]) and from the structure theorem of finitely generated abelian groups, we conclude that $H_0^{\mathbb{A}^1}(\overline{X})(U)$ has no torsion elements, for every $U \in Sm/k$. Theorefore, $H_0^{\mathbb{A}^1}(\overline{X}) \cong \mathbb{Z}$. By [14, Theorem 5], $\bar{X}$ is $\mathbb{A}^1$-connected. Hence, $\overline{X}$ is a rational surface by [11, Corollary 2.4.7]. So X is a rational surface. If X is $\mathbb{A}^1$-connected, then $H_0^{\mathbb{A}^1}(X)$ is isomorphic to $\mathbb{Z}$, by [14, Proposition 1] and hence the same argument proves that X is a rational surface. □

Suppose $\mathcal{F}$ is a Nisnevich sheaf of sets on Sm/k and $W \in Sm/k$, $f \in \mathcal{F}(W)$.

Definition 5.2.2. An element $\alpha \in \mathcal{F}(Spec\ k)$ is in the image of f if $\exists\ \gamma \in W(Spec\ k)$ such that the composition $Spec\ k \to W \xrightarrow{f} \mathcal{F}$ is α.

Definition 5.2.3. A homotopy $H \in \mathcal{F}(\mathbb{A}^1_W)$ is said to be non-constant if $H(0) \neq H(1) \in \mathcal{F}(W)$, where $H(0)$ and $H(1)$ are induced by the 0-section and the 1-section from W to $\mathbb{A}^1_W$ respectively.

Remark 5.2.4. Let $\mathcal{F}$ be a sheaf and $X \in Sm/k$. A section $\alpha \in \mathcal{S}(\mathcal{F})(X)$ is given by a Nisnevich covering $W \to X$, a section $\gamma \in \mathcal{F}(W)$ and a Nisnevich covering $W' \to W \times_X W$ such that $p_1^*(\gamma)|_{W'}$ and $p_2^*(\gamma)|_{W'}$ in $\mathcal{F}(W')$ are joined by a chain of $\mathbb{A}^1$-homotopies (where $p_1, p_2 : W \times_X W \to W$ are the projection maps). If $p_1^*(\gamma)|_{W'} = p_2^*(\gamma)|_{W'}$, then γ can be lifted to some element $\alpha' \in \mathcal{F}(X)$ and in this case α' maps to α via the canonical morphism $\mathcal{F} \to \mathcal{S}(\mathcal{F})$. Otherwise, we will get an element $H \in \mathcal{F}(\mathbb{A}^1_{W'})$ such that $p_1^*(\gamma)|_{W'} = H(0) \neq H(1)$ as sections. This is essentially the data of $\mathbb{A}^1$-ghost homotopy mentioned in [26, Definition 3.2].

Condition 5.2.5. Suppose $X, W \in Sm/k$, $\alpha \in X(k)$ and $n \geq 0$. A homotopy $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$ is said to satisfy the condition $*(W, \alpha)$, if H satisfies the following properties:

1. H is a non-constant homotopy.
2. $H(0)$ factors through X i.e. there is a morphism $\psi : W \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} W & \xrightarrow{\psi} & X \\ \downarrow{\scriptstyle i_0} & & \downarrow \\ \mathbb{A}^1_W & \xrightarrow{H} & \mathcal{S}^n(X) \end{array}$$

   Here $i_0 : W \to \mathbb{A}^1_W$ is the 0-section and the right vertical map is the canonical epimorphism $X \to \mathcal{S}^n(X)$.
3. $\alpha \in \overline{\psi(W)}$ (By (2), $H \circ i_0$ factors through $\psi : W \to X$).

Proposition 5.2.6. *Suppose $X, W \in Sm/k$, $\alpha \in X(k)$ and $n \geq 1$. Let $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$ be a homotopy, where W is irreducible and H satisfies $*(W, \alpha)$. Then there is $W' \in Sm/k$ irreducible and a homotopy $H' \in \mathcal{S}^m(X)(\mathbb{A}^1_{W'})$ for some $m < n$ such that H' satisfies $*(W', \alpha)$.*

Proof. The morphism $X \to \mathcal{S}^n(X)$ is an epimorphism and $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$. Thus,

1. There is a Nisnevich covering $f : V \to \mathbb{A}^1_W$,

2. There is a morphism $\phi : V \to X$ such that the following diagram commutes:

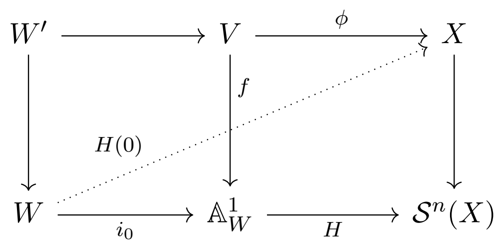
$$\begin{array}{ccc} V & \xrightarrow{\ \phi\ } & X \\ \downarrow{\scriptstyle f} & & \downarrow \\ \mathbb{A}^1_W & \xrightarrow{\ H\ } & \mathcal{S}^n(X) \end{array}$$

The morphism ϕ gives an element of $\mathcal{S}^{n-1}(X)(V)$ via the epimorphism $X \to \mathcal{S}^{n-1}(X)$. The elements $p_1^*(\phi)$ and $p_2^*(\phi)$ are same in $\mathcal{S}^n(X)(V \times_{\mathbb{A}^1_W} V)$ (where $p_1, p_2 : V \times_{\mathbb{A}^1_W} V \to V$ are the projection maps). Therefore, there is a Nisnevich covering $V' \to V \times_{\mathbb{A}^1_W} V$ and there is a chain of non-constant (since H is a non-constant homotopy, so $p_1^*(\phi)|_{V'} \neq p_2^*(\phi)|_{V'} \in \mathcal{S}^{n-1}(X)(V')$ by Remark 5.2.4) $\mathbb{A}^1$-homotopies $G_1, G_2, \ldots, G_k \in \mathcal{S}^{n-1}(X)(\mathbb{A}^1_{V'})$ such that

$$G_1(0) = p_1^*(\phi)|_{V'} \text{ and } G_k(1) = p_2^*(\phi)|_{V'}.$$

Suppose $V = \coprod_{i=1}^n V_i$, V_i-s are the irreducible components of V. Then $V \times_{\mathbb{A}^1_W} V$ is the union of $V_i \times_{\mathbb{A}^1_W} V_j$ varying i and j (note that, each $V_i \times_{\mathbb{A}^1_W} V_j$ is non-empty since W is irreducible) and for each irreducible component V_0 of V' which is also a connected component, there are dominant maps (étale maps) from V_0 to V_i (for some i) induced by the projection maps p_1 and p_2. We have the following cases.

Case 1: Suppose $\alpha \notin \overline{Im(\phi)}$. Consider the following diagram:



where $i_0 : W \to \mathbb{A}^1_W$ is the 0-section. Here the left square is cartesian and the lower triangle is commutative, since $H(0)$ factors through X. We have $\phi|_{W'} \neq H(0)|_{W'}$ as morphisms to X, since $\alpha \notin \overline{Im(\phi)}$. But they are the same in $\mathcal{S}^n(X)(W')$. Suppose $m \geq 0$ is the least such that these two maps are the same in $\mathcal{S}^{m+1}(X)(W')$. Thus there is a Nisnevich covering $W'' \to W'$ and there is a non-constant homotopy $H' \in \mathcal{S}^m(X)(\mathbb{A}^1_{W''})$, such that $H'(0) = H(0)|_{W''}$. There is an irreducible component (say W_0) of W'' such that $H'|_{\mathbb{A}^1_{W_0}}$ is non-constant. Since the map $W_0 \to W$ is dominant and $\alpha \in \overline{Im(H(0))}$, $\alpha \in \overline{Im(H'|_{\mathbb{A}^1_{W_0}}(0))}$.

Case 2: Suppose $\alpha \in \overline{Im(\phi)}$. Moreover assume that there is an irreducible component (say V_0) of V' that maps to $V_i \times_{\mathbb{A}^1_W} V_j$ (for some i and j) with $\alpha \in \overline{\phi(V_i)}$ and $p_1^*(\phi)|_{V_0} \neq p_2^*(\phi)|_{V_0}$. Then there is some t such that $G_t|_{\mathbb{A}^1_{V_0}}$ is the required non-constant homotopy (if for each t, $G_t|_{\mathbb{A}^1_{V_0}}$ is constant, then $p_1^*(\phi)$ and $p_2^*(\phi)$ agree in V_0). Since the map $V_0 \to V_i$ is dominant,

$\alpha \in \overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(0))}$. In particular, if $\alpha \in \overline{\phi(V_i)}$ for every i, then we can take any irreducible component V_0 of V' such that $G_1|_{\mathbb{A}^1_{V_0}}$ is the non-constant homotopy.

Case 3: Suppose $\alpha \in \overline{Im(\phi)}$ and there is a j such that $\alpha \notin \overline{\phi(V_j)}$. If needed, renumbering V_l-s, we can assume that $\alpha \in \overline{\phi(V_1)}, \overline{\phi(V_2)}, \ldots, \overline{\phi(V_i)}$ and $\alpha \notin \overline{\phi(V_{i+1})}, \ldots, \overline{\phi(V_n)}$. Moreover we can assume that for each irreducible component V_0 of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ with $m \leq i$ we have, $p_1^*(\phi)|_{V_0} = p_2^*(\phi)|_{V_0}$ in $\mathcal{S}^{n-1}(X)(V_0)$. Otherwise the conclusion follows from Case 2. Thus we have for every $m \leq i$,

$$p_1^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} = p_2^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} \in \mathcal{S}^{n-1}(X)(V_m \times_{\mathbb{A}^1_W} V_l).$$

Suppose there is a $t < n-1$ and there is an irreducible component W_0 of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ for some m, l with $m \leq i$ such that

$$p_1^*(\phi)|_{W_0} \neq p_2^*(\phi)|_{W_0} \in \mathcal{S}^t(X)(W_0).$$

Since $p_1^*(\phi)|_{W_0}$ and $p_2^*(\phi)|_{W_0}$ are the same in $\mathcal{S}^{n-1}(X)(W_0)$, we choose t such that $p_1^*(\phi)|_{W_0}$ is same with $p_2^*(\phi)|_{W_0}$ in $\mathcal{S}^{t+1}(X)(W_0)$. Then there is a Nisnevich covering $V'' \to W_0$ and a non-constant homotopy (by Remark 5.2.4) $H' \in \mathcal{S}^t(X)(\mathbb{A}^1_{V''})$ such that $H'(0) = p_1^*(\phi)|_{V''}$. So there is an irreducible component W_0' of V'' such that $H'|_{\mathbb{A}^1_{W_0'}}$ is non-constant. Since the map $W_0' \to V_m$ is dominant, we have $\alpha \in \overline{Im(H'|_{\mathbb{A}^1_{W_0'}})}$.

On the other hand, if there is no such t then for every irreducible component V_0 of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ for some $m \leq i$, we have $p_1^*(\phi)|_{V_0} = p_2^*(\phi)|_{V_0}$ as morphisms to X. Therefore we have,

$$p_1^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} = p_2^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l}, \ \forall m \leq i \ \forall l$$

as morphisms to X. But then all $\overline{\phi(V_l)}$ are the same for every l, since $p_1 : V_m \times_{\mathbb{A}^1_W} V_l \to V_m$ and $p_2 : V_m \times_{\mathbb{A}^1_W} V_l \to V_l$ are dominant maps. It is a contradiction, since we have assumed there is some j such that $\alpha \notin \overline{\phi(V_j)}$.

Therefore, the proposition is proved. □

Remark 5.2.7.

1. For any Nisnevich sheaf of sets $\mathcal{F}$, using the same argument as in the proof of Proposition 5.2.6, we have the following : Suppose there is a non-constant homotopy $H \in \mathcal{S}(\mathcal{F})(\mathbb{A}^1_W)$ for some $W \in Sm/k$ such that the image of $H(0)$ contains some $\alpha \in \mathcal{F}(Spec\ k)$. Then there is a non-constant homotopy $H' \in \mathcal{F}(\mathbb{A}^1_{W'})$ for some $W' \in Sm/k$ such that the image of $H'(0)$ contains α.

2. In the next chapter (Chapter 6) in Proposition 6.1.1, we will also give a way of constructing non-constant homotopy in $\mathcal{S}^m(X)$ starting from a non-constant homotopy in $\mathcal{S}^n(X)$ $(m < n)$. In the proof of Proposition 6.1.1, we will use generic argument instead of fixing a k-rational point in X.

The next theorem (Theorem 5.2.8) is the main theorem of this chapter. It shows that being $\mathbb{A}^1$-connected gives $\mathbb{A}^1$'s in a variety.

Theorem 5.2.8. *Let k be an algebraically closed field and $X \in Sm/k$ with $dim(X) \geq 2$. Suppose that $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial. Then one of the following holds :*

1. *$\forall\ x \in X(k)$, there is a non-constant $\mathbb{A}^1$ through x.*
2. *There is a non-constant homotopy $H : \mathbb{A}^1_Y \to X$, for some irreducible $Y \in Sm/k$, such that the dimension of the closure of the image of H is at least 2.*

In particular for a surface $X \in Sm/k$, if X is $\mathbb{A}^1$-connected, then X is dominated by images of $\mathbb{A}^1$.

Proof. Since $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial, so $\mathcal{L}(X)(Spec\ k)$ is trivial ([33, Theorem 2.2] or Corollary 3.3.9). Suppose that $\exists\ \alpha \in X(k)$ such that there is no non-constant $\mathbb{A}^1$ through α. Choose $\beta \in X(k)$ with $\beta \neq \alpha$. Also $\alpha \neq \beta \in \mathcal{S}(X)(Spec\ k)$, but $\alpha = \beta \in \mathcal{L}(X)(Spec\ k)$. Therefore, there is an $n \geq 1$ such that $\alpha = \beta \in \mathcal{S}^{n+1}(X)(Spec\ k)$ and $\alpha \neq \beta \in \mathcal{S}^n(X)(Spec\ k)$. So there is a non-constant homotopy $H \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$ such that $H(0) = \alpha$. Hence by applying Proposition 5.2.6 repeatedly, there exists some $Y \in Sm/k$ irreducible, along with a non-constant homotopy $H' : \mathbb{A}^1_Y \to X$, such that $\alpha \in \overline{Im(H'(0))}$. Since k is algebraically closed, the k-rational points are dense, so $H'(0) \neq H'(1)$ at some k-rational point. Therefore the image of H' contains a non-constant $\mathbb{A}^1$ and we have $\overline{Im(H')}$ contains α. Therefore $\overline{Im(H')}$ is of dimension at least 2, as we have assumed that there is no non-constant $\mathbb{A}^1$ through α.

Since H' is a non-constant homotopy, by shrinking Y we can assume that $H'(0, y) \neq H'(1, y)$, $\forall\ y \in Y(k)$ and the dimension of the closure of image is at least 2. Thus if X is a surface, the map H' is dominant. So there is a non-empty open subset U of X such that U is contained in the image of H'. Each $u \in U(k)$ has the preimage $(t, y) \in \mathbb{A}^1_Y$ for some k-point y in Y. Therefore, u is in the image of $H'|_{\mathbb{A}^1_k \times \{y\}}$. Thus X is dominated by images of $\mathbb{A}^1$. □

Remark 5.2.9.
1. Suppose X is a rationally connected smooth proper variety over an algebraically closed field k of characteristic zero. Then for any two k-points x, y in X, there is $f : \mathbb{P}^1_k \to X$ such that $x, y \in f(\mathbb{P}^1_k)$ [81, Chapter IV, Theorem 3.9]. So $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial. In the next chapter (Chapter 6) in Proposition 6.2.1, we will give a stronger version of Theorem 5.2.8 in case of smooth affine k-surfaces.
2. In the next chapter (Chapter 6) in Theorem 6.1.2, we will prove that any $\mathbb{A}^1$-connected smooth variety over an algebraically closed field k is $\mathbb{A}^1$-uniruled, which gives a stronger version of Theorem 5.2.8.

Corollary 5.2.10. *Suppose $X \in Sm/k$ is a surface with $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial and k is an uncountable algebraically closed field of characteristic zero. Then $\bar{\kappa}(X) = -\infty$.*

Proof. This follows by Theorem 5.2.8 and because of equivalence of (1) and (2) in Definition 5.1.1 (Remark 5.1.2). □

Corollary 5.2.11. *Suppose $X \in Sm/k$ is $\mathbb{A}^1$-connected surface and k is an uncountable algebraically closed field of characteristic zero. Then $\bar{\kappa}(X) = -\infty$.*

Corollary 5.2.12. *Suppose $X \in Sm/k$ is $\mathbb{A}^1$-connected and k is an algebraically closed field. Then there is a non-constant $\mathbb{A}^1$ in X.*

Remark 5.2.13. In Corollary 5.2.12, the assumption that k is an algebraically closed field, is necessary. The unit sphere T in $\mathbb{A}^3_{\mathbb{R}}$ given by the equation $x^2 + y^2 + z^2 = 1$ is $\mathbb{A}^1$-connected ([113, Theorem 4.3.4]), however there is no non-constant $\mathbb{A}^1_{\mathbb{R}}$ in T (see Remark 6.1.6). If $X \in Sm/k$ is an $\mathbb{A}^1$-connected surface and the base field is countable (e.g. $\bar{\mathbb{Q}}$), even though X is dominated by the images of $\mathbb{A}^1$'s, we don't know whether X has negative logarithmic Kodaira dimension.

We end the section making some comments on the $\mathbb{A}^1$-connectivity of the complex sphere in $\mathbb{A}^3_{\mathbb{C}}$.

5.2.1 Complex Sphere in $\mathbb{A}^3_{\mathbb{C}}$

We have seen that $\mathbb{A}^1$-connectivity of a smooth complex variety X implies X has logarithmic Kodaira dimension $-\infty$. For affine surfaces X, this implies X contains a cylinder. But $\mathbb{A}^1$-connectivity of a smooth complex surface X does not necessarily imply that $X(\mathbb{C})$ is simply connected at infinity. This subsection is taken from [40, Subsection 2.2]. Consider the complex sphere,

$$X = Spec \frac{\mathbb{C}[x, y, z]}{(x^2 + y^2 + z^2 - 1)}.$$

The complex sphere X is an $\mathbb{A}^1$-connected surface with non-trivial Picard group. Thus in particular, X is not isomorphic to $\mathbb{A}^2_{\mathbb{C}}$.
By change of variables, X is isomorphic to $Spec \frac{\mathbb{C}[x,y,z]}{(xy-z(1-z))}$. X is a smooth complex affine surface with non-trivial Picard group.

Lemma 5.2.14. *$\mathcal{O}(X)$ is not a U.F.D.*

Proof. The following product has two ways of factorization in $\mathcal{O}(X)$,

$$\bar{x}\bar{y} = \bar{z}\overline{1 - z}$$

The factorization is not unique. Therefore, $\mathcal{O}(X)$ is not a U.F.D. □

Remark 5.2.15. So the Picard group of X is non-trivial and hence $\mathbb{A}^1$-fundamental group of X is non-trivial. The surface X is in fact the Jouanolou device of $\mathbb{P}^1_{\mathbb{C}}$ [6, Example 5.3.1.6]. Thus the Picard group of X is isomorphic to $\mathbb{Z}$ and X is $\mathbb{A}^1$-connected.

Consider the morphisms

$$\phi, \psi : \mathbb{G}_m \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}} \to Spec \frac{\mathbb{C}[x, y, z]}{(xy - z(1 - z))}$$

given by $\phi(s,t) = (s, \frac{t(1-t)}{s}, t)$ *and* $\psi(s,t) = (\frac{t(1-t)}{s}, s, t)$. *Then* $\mathbb{G}_m \times_{\mathbb{C}} \mathbb{A}^1_{\mathbb{C}}$ *is isomorphic to the open subsets* $D(\bar{x})$ *and* $D(\bar{y})$ *of* X *by the morphisms* ϕ *and* ψ *respectively, where*

$$D(\bar{x}) = \left\{ P \in Spec\ \frac{\mathbb{C}[x,y,z]}{(xy - z(1-z))} \;\middle|\; \bar{x} \notin P \right\}$$

and

$$D(\bar{y}) = \left\{ P \in Spec\ \frac{\mathbb{C}[x,y,z]}{(xy - z(1-z))} \;\middle|\; \bar{y} \notin P \right\}$$

Thus X *contains cylinders. Hence* X *has negative logarithmic Kodaira dimension. The proof in [113, Theorem 4.3.4] shows that the complex sphere* X *is* $\mathbb{A}^1$*-chain connected, in particular* X *is* $\mathbb{A}^1$*-connected. Note that the closure* $\overline{X}$ *of* X *in* $\mathbb{P}^3_{\mathbb{C}}$ *is isomorphic to* $\mathbb{P}^1_{\mathbb{C}} \times \mathbb{P}^1_{\mathbb{C}}$ *and* $\overline{X} \setminus X = \mathbb{P}^1_{\mathbb{C}}$ *of degree* 2. *Therefore, the fundamental group at infinity of* X *is non trivial.*

Chapter 6

$\mathbb{A}^1$-homotopy theory and log-uniruledness

The goal of this chapter is very similar to the previous chapter (Chapter 5) to explore the behaviour of $\mathbb{A}^1$-homotopy theory with $\mathbb{A}^1$-uniruledness. In this chapter in Theorem 6.1.2, we show that an $\mathbb{A}^1$-connected variety contains a dominant family of affine lines, which is the main theorem in this chapter (compare this with Theorem 5.2.8). We also prove that a smooth affine k-surface X with $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ trivial, either contains a dominant family of affine lines in X or through every point there is an affine line in X and there exists a point $x \in X(k)$ through which intersecting $\mathbb{A}^1$'s pass (Proposition 6.2.1). Throughout the chapter we assume k to be an algebraically closed field unless otherwise mentioned. This chapter is taken from [40, Section 2].

6.1 $\mathbb{A}^1$-Connectedness of a Variety and its $\mathbb{A}^1$-uniruledness

In this section we prove that any $\mathbb{A}^1$-connected smooth variety is $\mathbb{A}^1$-uniruled (Theorem 6.1.2), which is the main theorem in this chapter.

Suppose $\mathcal{F}$ is a Nisnevich sheaf on Sm/k and $X, W \in Sm/k$. A non-constant homotopy (Definition 5.2.3) $H \in \mathcal{F}(\mathbb{A}^1_W)$ is such that $H(0) \neq H(1) \in \mathcal{F}(W)$, where $H(0)$ and $H(1)$ are induced by the 0-section and the 1-section from W to $\mathbb{A}^1_W$ respectively.

The next proposition (Proposition 6.1.1) gives a way of constructing non-constant $\mathbb{A}^1$-homotopy in $\mathcal{S}^m(X)$ starting from a non-constant homotopy in $\mathcal{S}^n(X)$ whenever $n > m$. In this respect the proposition is similar to Proposition 5.2.6. However, the difference is that in Proposition 5.2.6 the algorithm requires fixing a closed point but in the following proposition the argument is more generic in nature. We thank Prof. Chetan Balwe for giving suggestions to make the proof of Proposition 6.1.1 simpler.

Proposition 6.1.1. *Suppose $X, W \in Sm/k$ are irreducible schemes, $n \geq 1$ and $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$ is a non-constant homotopy such that $H(0)$ factors through X and $H(0) : W \to X$ is a dominant morphism. Then there is some $m < n$, $W' \in Sm/k$ irreducible and a non-constant homotopy $H' \in \mathcal{S}^m(X)(\mathbb{A}^1_{W'})$ such that $H'(0)$ factors through X and $H'(0) : W' \to X$ is a dominant morphism.*

Proof. The first part of the proof goes as in Proposition 5.2.6. The canonical morphism $\eta : X \to \mathcal{S}^n(X)$ is an epimorphism. So given $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$, there is a Nisnevich covering $f : V \to \mathbb{A}^1_W$ along with a morphism $\phi : V \to X$ making the diagram

$$\begin{array}{ccc} V & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \downarrow \\ \mathbb{A}^1_W & \xrightarrow{H} & \mathcal{S}^n(X) \end{array}$$

commutative and there is a Nisnevich covering $V' \to V \times_{\mathbb{A}^1_W} V$ along with a chain of non-constant (since H is a non-constant homotopy, so $p_1^*(\phi)|_{V'} \neq p_2^*(\phi)|_{V'} \in \mathcal{S}^{n-1}(X)(V')$ by Remark 5.2.4, here $p_1, p_2 : V \times_{\mathbb{A}^1_W} V \to V$ are the projection maps) $\mathbb{A}^1$-homotopies $G_1, G_2, \ldots, G_t \in \mathcal{S}^{n-1}(X)(\mathbb{A}^1_{V'})$ such that

$$G_1(0) = p_1^*(\phi)|_{V'} \text{ and } G_t(1) = p_2^*(\phi)|_{V'}.$$

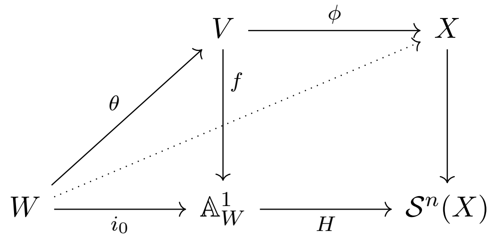
Suppose that V_i's, $1 \leq i \leq q$ are the irreducible components of V, which are also the connected components. Then each irreducible component V'_0 of V' maps to $V_i \times_{\mathbb{A}^1_W} V_j$ for some i, j (note that each $V_i \times_{\mathbb{A}^1_W} V_j$ is non-empty, since W is irreducible) and there are dominant maps (étale maps) from V'_0 to V_i (for some i) induced by p_1 and p_2. Consider the following commutative diagram induced by the 0-section of H:

$$\begin{array}{ccccc} U & \xrightarrow{\theta} & V & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f'} & {\scriptstyle H(0)} \nearrow & \downarrow{\scriptstyle f} & & \downarrow \\ W & \xrightarrow[i_0]{} & \mathbb{A}^1_W & \xrightarrow[H]{} & \mathcal{S}^n(X) \end{array}$$

where $i_0 : W \to \mathbb{A}^1_W$ is the 0-section and the left square is cartesian. Thus f' is a Nisnevich covering. Here the lower triangle is commutative, since $H(0)$ factors through X. The upper triangle is not commutative in general. However after shrinking W, we can always assume that $H(0)$ lifts to V i.e. there is a morphism $\theta' : W \to V$ such that $H(0)$ factors as the morphism $\theta' : W \to V$, followed the morphism $\phi : V \to X$. Indeed, suppose that $U_1, U_2, .., U_d$ are the connected components of U. If all the homotopies $H_i : \mathbb{A}^1_{U_i} \to \mathcal{S}^n(X)$ induced by the composition

$$\mathbb{A}^1_{U_i} \xrightarrow{(Id, f')} \mathbb{A}^1_W \xrightarrow{H} \mathcal{S}^n(X)$$

are constant i.e. $H_i(0) = H_i(1)$, then $H(0) = H(1)$. It is not possible, since H is a non-constant homotopy. Therefore there is some i such that H_i is non-constant. The 0-section $H_i(0) = H(0)|_{U_i}$, so $H_i(0)$ factors through X and $H_i(0) : U_i \to X$ is dominant. We then replace the homotopy H by H_i. Therefore we always have the following commutative diagram (after shrinking W):



Since W is irreducible and $H(0)$ is dominant, so W maps to some irreducible component of V (say V_l) and $\phi|_{V_l}$ is dominant.

We analyse the following cases:

Case 1: Suppose that $\overline{\phi(V_i)}$ are same for all i. In this case $\phi|_{V_i}$ is also dominant for every i. There is some irreducible component V_0' of V' such that $G_1|_{\mathbb{A}^1_{V_0'}}$ is a non-constant homotopy. If V_0' maps to $V_j \times_{\mathbb{A}^1_W} V_l$ for some j, l, the composition

$$V_0' \to V_j \times_{\mathbb{A}^1_W} V_l \xrightarrow{p_1} V_j \xrightarrow{\phi|_{V_j}} X$$

is dominant since the map $V_0' \to V_j$ is étale. Thus in this case $G_1|_{\mathbb{A}^1_{V_0'}}$ is the required homotopy.

Case 2: Suppose that there are i, j such that $\overline{\phi(V_i)} \neq \overline{\phi(V_j)}$. Thus if needed renumbering all V_m-s we can assume that $\overline{\phi(V_1)}, \overline{\phi(V_2)}, \ldots, \overline{\phi(V_s)} = X$ and $\overline{\phi(V_{s+1})}, \ldots, \overline{\phi(V_q)} \subsetneqq X$. If there is some irreducible component V_0' of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ with $m \leq s$ and $p_1^*(\phi)|_{V_0'} \neq p_2^*(\phi)|_{V_0'}$ in $\mathcal{S}^{n-1}(X)(V_0')$, then same as Case 2 there is some p such that $G_p|_{\mathbb{A}^1_{V_0'}}$ is a non-constant homotopy and $G_p|_{\mathbb{A}^1_{V_0'}}(0) = p_1^*(\phi)|_{V_0'}$.

Thus we can assume that for each irreducible component V_0' of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ with $m \leq s$ we have, $p_1^*(\phi)|_{V_0'} = p_2^*(\phi)|_{V_0'}$ in $\mathcal{S}^{n-1}(X)(V_0')$. Therefore we have for every $m \leq s$,

$$p_1^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} = p_2^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} \in \mathcal{S}^{n-1}(X)(V_m \times_{\mathbb{A}^1_W} V_l).$$

Here we have following two subcases:

Subcase 1: Suppose that there is a $t < n-1$ and an irreducible component W_0' of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ for some m, l with $m \leq s$ such that

$$p_1^*(\phi)|_{W_0'} \neq p_2^*(\phi)|_{W_0'} \in \mathcal{S}^t(X)(W_0').$$

Since $p_1^*(\phi)|_{W_0'}$ and $p_2^*(\phi)|_{W_0'}$ are same in $\mathcal{S}^{n-1}(X)(W_0')$, we choose t such that $p_1^*(\phi)|_{W_0'}$ is same with $p_2^*(\phi)|_{W_0'}$ in $\mathcal{S}^{t+1}(X)(W_0')$ and

$$p_1^*(\phi)|_{W_0'} \neq p_2^*(\phi)|_{W_0'} \in \mathcal{S}^t(X)(W_0').$$

Then there is a Nisnevich covering $V'' \to W_0'$ and a non-constant homotopy (by Remark 5.2.4) $H' \in \mathcal{S}^t(X)(\mathbb{A}^1_{V''})$ such that $H'(0) = p_1^*(\phi)|_{V''}$. So there is an irreducible component W_0'' of V'' such that $H'|_{\mathbb{A}^1_{W_0''}}$ is non-constant. Since the map $W_0'' \to V_m$ is étale and $\phi|_{V_m}$ is dominant, $H'|_{\mathbb{A}^1_{W_0''}}(0)$ is dominant.

Subcase 2: Suppose that for every irreducible component V_0' of V' that maps to $V_m \times_{\mathbb{A}^1_W} V_l$ for some $m \leq s$, we have $p_1^*(\phi)|_{V_0'} = p_2^*(\phi)|_{V_0'}$ as morphisms to X. Therefore we have,

$$p_1^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l} = p_2^*(\phi)|_{V_m \times_{\mathbb{A}^1_W} V_l}, \ \forall m \leq s \ \forall l$$

as morphisms to X. But then all $\overline{\phi(V_l)}$ are same for every l, since $p_1 : V_m \times_{\mathbb{A}^1_W} V_l \to V_m$ and $p_2 : V_m \times_{\mathbb{A}^1_W} V_l \to V_l$ are dominant (étale) maps. It is a contradiction, since we have assumed there are i, j such that $\overline{\phi(V_i)} \neq \overline{\phi(V_j)}$.

Therefore, the proposition is proved. □

Theorem 6.1.2. *Suppose $X \in Sm/k$ is an irreducible, $\mathbb{A}^1$-connected scheme. Then there is some $W \in Sm/k$ with W irreducible and a non-constant homotopy $H : \mathbb{A}^1_W \to X$ such that H is a dominant morphism.*

Proof. As X is $\mathbb{A}^1$-connected, the sheaf $\mathcal{L}(X)$ is trivial [26, Corollary 2.18] and X has a k-rational point (say $x_0 \in X(k)$). Consider two morphisms $Id : X \to X$ (the identity map on X) and $C_{x_0} : X \to Spec\ k \xrightarrow{x_0} X$. Since $\mathcal{L}(X)$ is trivial, Id and C_{x_0} are same in $\mathcal{L}(X)(X)$. Then there is a Nisnevich covering $f : Y \to X$ and some $m \geq 1$ such that f and $C_{x_0}|_Y$ are same in $\mathcal{S}^m(X)(Y)$. Choose least $n \geq 0$ such that f and $C_{x_0}|_Y$ are same in $\mathcal{S}^{n+1}(X)(Y)$and they are not same in $\mathcal{S}^n(X)(Y)$. Thus there is a Nisnevich covering $f' : Y' \to Y$ and there are chain of non-constant $\mathbb{A}^1$-homotopies $G_1, G_2, ..., G_t \in \mathcal{S}^n(X)(\mathbb{A}^1_{Y'})$ such that $G_1(0) = f \circ f'$ and $G_t(1) = C_{x_0}|_{Y'}$. There is some irreducible component Y_0 of Y' such that $G_1|_{\mathbb{A}^1_{Y_0}}$ is non-constant. Since the restriction $f \circ f'|_{Y_0} : Y_0 \to X$ is étale, $G_1|_{\mathbb{A}^1_{Y_0}}(0)$ is dominant. So applying Proposition 6.1.1 repeatedly, there is a non-constant homotopy $H : \mathbb{A}^1_W \to X$ for some $W \in Sm/k$ irreducible such that $H(0)$ is a dominant morphism. Therefore the morphism $H : \mathbb{A}^1_W \to X$ satisfies $H(0, -) \neq H(1, -) : W \to X$ and H is a dominant morphism. □

Corollary 6.1.3. *Suppose $X \in Sm/k$ is an $\mathbb{A}^1$-connected k-variety, where k is an algebraically closed field of characteristic zero. Then X has negative logarithmic Kodaira dimension.*

Proof. Since X is $\mathbb{A}^1$-connected, by Theorem 6.1.2, there is a non-constant homotopy $H : \mathbb{A}^1_k \times_k Y \to X$ such that Y is irreducible and H is dominant. Infact we can take $dim(Y) = dim(X) - 1$. By [70, Theorem 11.3], the variety $\mathbb{A}^1_k \times_k Y$ has logarithmic Kodaira dimension $-\infty$. Thus X has logarithmic Kodaira dimension $-\infty$ by [70, Proposition 11.4]. □

Recall that by the phrase "there is an $\mathbb{A}^1$ in W" for some $W \in Sm/k$, we mean that there is a non-constant morphism $\phi : \mathbb{A}^1_k \to W$. If the base field k is algebraically closed, then an

$\mathbb{A}^1$-connected variety X has always a non-constant $\mathbb{A}^1$ in X (Theorem 6.1.2, Corollary 5.2.12). But if k is not algebraically closed, then it is not true (Remark 6.1.6). However, over a general field k an $\mathbb{A}^1$-connected variety always admits a non-constant morphism $H : \mathbb{A}^1_L \to X$, for some finite separable field extension L/k (Proposition 6.1.5).

Lemma 6.1.4. *Let k be a field and $X \in Sm/k$. Suppose that there is a non-constant homotopy $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$ for some $W \in Sm/k$ such that W is irreducible. Then there is a non-constant homotopy $H' \in \mathcal{S}^{n-1}(X)(\mathbb{A}^1_{W'})$ for some $W' \in Sm/k$ such that W' is irreducible.*

Proof. The homotopy $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$ is given by a Nisnevich covering $V \to \mathbb{A}^1_W$ along with $G \in \mathcal{S}^{n-1}(X)(V)$ such that there is a Nisnevich covering $V' \to V \times_{\mathbb{A}^1_W} V$ with $p_1^*(G)|_{V'} = p_2^*(G)|_{V'}$ in $\mathcal{S}^{pre}(\mathcal{S}^{n-1}(X))(V')$ (here $p_1, p_2 : V \times_{\mathbb{A}^1_W} V \to V$ are the projection maps). Therefore there are $G_1, G_2, \ldots G_l \in \mathcal{S}^{n-1}(X)(\mathbb{A}^1_{V'})$ such that $G_1(0) = p_1^*(G)|_{V'}$ and $G_l(1) = p_2^*(G)|_{V'}$. Since H is non-constant, we can assume that G_1 is non-constant by Remark 5.2.4. Thus there is some irreducible component W' of V' such that $G_1|_{\mathbb{A}^1_{W'}}$ is non-constant. □

Proposition 6.1.5. *Let k be any field (not necessarily algebraically closed) and $X \in Sm/k$ be an $\mathbb{A}^1$-connected scheme. Suppose X has at least two k-rational points. Then there is a non-constant homotopy $H : \mathbb{A}^1_L \to X$ for some finite separable field extension L of k.*

Proof. Suppose, $X \in Sm/k$ is an $\mathbb{A}^1$-connected scheme and α, β are two k-rational points in X. If there is a non-constant $\mathbb{A}^1$ in X through α or β, we are done. Let us assume that there is no non-constant $\mathbb{A}^1$ in X through α and β. Since X is $\mathbb{A}^1$-connected, $\mathcal{L}(X)$ is trivial [26, Corollary 2.18]. Therefore, there is an $n \geq 1$ such that $\alpha = \beta \in \mathcal{S}^{n+1}(X)(Spec\ k)$ but $\alpha \neq \beta \in \mathcal{S}^n(X)(Spec\ k)$. Thus there is a chain of non-constant $\mathbb{A}^1$-homotopies (Remark 5.2.4) $H_1, H_2, \ldots, H_m \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$ such that $H_1(0) = \alpha$ and $H_m(1) = \beta$. Therefore applying Lemma 6.1.4 repeatedly, there is a non-constant homotopy $G : \mathbb{A}^1_W \to X$ for some $W \in Sm/k$ such that W is irreducible. Since G is the non-constant homotopy on $\mathbb{A}^1_W$ and the set

$$\{w \in W \text{ closed point} \mid k \subset k(w) \text{ is a finite separable extension}\}$$

is dense in W [121, Tag 056U], there is some $w_0 \in W$ such that $G(0)(w_0) \neq G(1)(w_0)$ as morphisms from W to X and $k(w_0)/k$ is a finite separable extension. Therefore the composition given by

$$\mathbb{A}^1_{k(w_0)} \xrightarrow{w_0 \times Id} \mathbb{A}^1_W \xrightarrow{G} X$$

is a non-constant morphism $H : \mathbb{A}^1_{k(w_0)} \to X$. □

Remark 6.1.6. If k is not an algebraically closed field, then the separable extension L in Proposition 6.1.5 can be a proper extension of k. For example, if X is the real sphere $Spec(\frac{\mathbb{R}[X,Y,Z]}{(X^2+Y^2+Z^2-1)})$, there is no non-constant morphism from $\mathbb{A}^1_{\mathbb{R}}$ to X. Indeed, if $\bar{X}$ is the projective closure of X, any morphism $\phi : \mathbb{A}^1_{\mathbb{R}} \to X$ extends to a morphism $\bar{\phi} : \mathbb{P}^1_{\mathbb{R}} \to \bar{X}$. Since X has no real points at infinity, the morphism $\bar{\phi}$ factors through $X \hookrightarrow \bar{X}$. As X is an affine scheme, the morphism $\bar{\phi}$ is constant. However the ring homomorphism $f : \frac{\mathbb{R}[X,Y,Z]}{(X^2+Y^2+Y^2-1)} \to \mathbb{C}[T]$ given by $X \mapsto 1,\ Y \mapsto T,\ Z \mapsto iT$ defines a non-constant morphism $\bar{f} : \mathbb{A}^1_{\mathbb{C}} \to X$.

We end this section with the following lemma (Lemma 6.1.7), which we will use to prove Proposition 6.2.1.

Lemma 6.1.7. *Suppose, X is a smooth affine k-variety and $H \in \mathcal{S}^n(X)(\mathbb{A}^1_W)$ is a non constant homotopy, where $n \geq 1$ and W smooth irreducible k-scheme. Let $f : V \to \mathbb{A}^1_W$ be a Nisnevich covering and suppose $\phi : V \to X$ is a morphism such that the epimorphism $\eta : X \to \mathcal{S}^n(X)$ maps ϕ to $H|_V$.*

Then there are irreducible components V_0 and V_0' of V such that there is no $\gamma : \mathbb{A}^1_k \to X$ so that the image $\gamma(\mathbb{A}^1_k)$ contains both $\overline{\phi(V_0)}$ and $\overline{\phi(V_0')}$.

Proof. Consider the following commutative diagram:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \downarrow \\ \mathbb{A}^1_W & \xrightarrow{H} & \mathcal{S}^n(X) \end{array}$$

Suppose $V = \coprod_{i=1}^m V_i$, V_i-s are the irreducible components of V.

If possible, for every i, j there is $\gamma_{i,j} : \mathbb{A}^1_k \to X$ such that its image (note that the image $\gamma_{i,j}(\mathbb{A}^1_k)$ is closed in X, since X is affine and we denote $Im(\gamma_{i,j})$ by $L_{i,j}$) contains both $\overline{\phi(V_i)}$ and $\overline{\phi(V_j)}$. Here are two cases.

Case 1: Suppose each $\overline{\phi(V_i)}$ is of dimension zero. Then $\overline{\phi(V_i)}$ is a singleton set for every i, since V_i is irreducible. Suppose, $\overline{\phi(V_i)} = \{\alpha_i\} \subset X(k)$. Then $\alpha_i = \alpha_j \in \mathcal{S}^n(X)(Spec\ k)$ for every i, j. Indeed, $\alpha_i, \alpha_j \in L_{i,j}$ and $\gamma_{i,j}(t) = \gamma_{i,j}(0) \in \mathcal{S}^n(X)(Spec\ k)$, for every $t \in \mathbb{A}^1_k$. To prove this, consider the naive $\mathbb{A}^1$-homotopy

$$\tilde{\gamma_{i,j}} : \mathbb{A}^1_k \to X \text{ defined as } s \mapsto \gamma(st).$$

Thus $H|_V = \alpha_i \in \mathcal{S}^n(X)(V)$ for all i (a k-point α is considered as an element of $\mathcal{S}^n(X)(V)$ as follows

$$V \to Spec\ k \xrightarrow{\alpha} X \to \mathcal{S}^n(X))$$

and hence $H = \alpha_i\ \forall i$. Therefore $H(0) = H(1) \in \mathcal{S}^n(X)(W)$, which is a contradiction since H is a non-constant homotopy.

Case 2: Suppose, there is some t such that $dim(\overline{\phi(V_t)}) \geq 1$. Then $\overline{\phi(V_t)} \subset L_{t,i}$ for every i. Since, $L_{t,i}$ is the image of $\mathbb{A}^1$, so $L_{t,i} = \overline{\phi(V_t)},\ \forall i$. Thus $\overline{\phi(V_t)}$, which is the image of an $\mathbb{A}^1$ in X, contains all the $\overline{\phi(V_i)}$'s. So there is an $\mathbb{A}^1$, say $\gamma : \mathbb{A}^1_k \to X$ such that the image $\gamma(\mathbb{A}^1_k)$

contains $\overline{\phi(V_i)}$ for every i (we denote the image $\gamma(\mathbb{A}^1_k)$ by $\mathbf{L}$, which is closed in X). We will prove that in this situation H is not a non-constant homotopy i.e. $H(0) = H(1) \in \mathcal{S}^n(X)(W)$. The 0-section $H(0)$ is given by the following commutative diagram:

$$\begin{array}{ccccc} U_0 & \xrightarrow{\theta_0} & V & \xrightarrow{\phi} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \eta} \\ W & \xrightarrow[i_0]{} & \mathbb{A}^1_W & \xrightarrow[H]{} & \mathcal{S}^n(X) \end{array}$$

Here i_0 is the 0-section and the left most square is a pullback square. Similarly the 1-section $H(1)$ is given by the following commutative diagram:

$$\begin{array}{ccccc} U_1 & \xrightarrow{\theta_1} & V & \xrightarrow{\phi} & X \\ {\scriptstyle f_1}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \eta} \\ W & \xrightarrow[i_1]{} & \mathbb{A}^1_W & \xrightarrow[H]{} & \mathcal{S}^n(X) \end{array}$$

Here i_1 is the 1-section and the left most square is a pullback square. Here are two possible cases regarding the closures of the images of $\phi \circ \theta_0$ and $\phi \circ \theta_1$.

Subcase 1: Assume that both the closures of $Im(\phi \circ \theta_0)$ and $Im(\phi \circ \theta_1)$ are $\mathbf{L}$. In this case, both $\phi \circ \theta_0$ and $\phi \circ \theta_1$ factor through the normalisation of $\mathbf{L}$ as $U_0 \xrightarrow{\tilde{\theta_0}} \mathbb{A}^1_k \to X$ and $U_1 \to \mathbb{A}^1_k \to X$ respectively. Therefore we have the following commutative diagram for $\phi \circ \theta_0$:

$$\begin{array}{ccccc} \mathcal{S}^n(U) & \longrightarrow & \mathcal{S}^n(\mathbb{A}^1_k) & \xrightarrow{\mathcal{S}^n(\gamma)} & \mathcal{S}^n(X) \\ \uparrow & & \uparrow{\scriptstyle \eta'} & & \uparrow{\scriptstyle \eta} \\ & \overset{\tilde{\theta_0}}{\nearrow} & \mathbb{A}^1_k & \overset{\gamma}{\searrow} & \\ U_0 & \xrightarrow{\theta_0} & V & \xrightarrow{\phi} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \eta} \\ W & \xrightarrow[i_0]{} & \mathbb{A}^1_W & \xrightarrow[H]{} & \mathcal{S}^n(X) \end{array}$$

From the above diagram we have,

$$\begin{aligned} H \circ i_0 \circ f_0 &= H \circ f \circ \theta_0 \\ &= \eta \circ \phi \circ \theta_0 \\ &= \eta \circ \gamma \circ \tilde{\theta_0} \\ &= \mathcal{S}^n(\gamma) \circ \eta' \circ \tilde{\theta_0} \end{aligned}$$

Since $\mathcal{S}^n(\mathbb{A}^1_k)$ is the trivial sheaf $Spec\ k$, therefore, $H(0)|_{U_0}$ is same as the composition of maps $U_0 \to Spec\ k \xrightarrow{\alpha} \mathcal{S}^n(X)$ for some $\alpha \in X(k)$. Now, $\gamma = \gamma(0) \in \mathcal{S}(X)(\mathbb{A}^1_k)$. Indeed, a naive

$\mathbb{A}^1$-homotopy between γ and $\gamma(0)$ is given by

$$\tilde{\gamma} : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to X \text{ defined as } (s,t) \mapsto \gamma(st).$$

Thus $\alpha = \gamma(0) \in \mathcal{S}^n(X)(Spec\ k)$ and $H(0)|_{U_0} = \gamma(0)$. Similarly, we have the same kind diagram for $\phi \circ \theta_1$ and $H(1)|_{U_1} = \gamma(0)$. Thus both $H(0) = H(1) = \gamma(0)$ and this contradicts the fact that H is a non-constant homotopy.

Subcase 2: Assume that one of the closures (say $\overline{Im(\phi \circ \theta_0)}$) consists finitely many points in $\mathbf{L}$. Suppose, $\overline{Im(\phi \circ \theta_0)} = \{\alpha_1, .., \alpha_m\} \subset X(k)$ where $\alpha_i = \gamma(t_i)$, for some $t_i \in \mathbb{A}^1_k$. So each irreducible component of U_0 maps to some α_i under $\phi \circ \theta_0$. Thus for each irreducible component (say U_0') of U_0, $\phi \circ \theta_0|_{U_0'}$ factors as $U_0' \xrightarrow{t_i} \mathbb{A}^1_k \xrightarrow{\alpha_i} X$. Hence in this case also $H(0) = \gamma(0)$, since $\gamma = \gamma(t_0) \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$, for every $t_0 \in \mathbb{A}^1_k$. Indeed, a naive $\mathbb{A}^1$-homotopy between γ and $\gamma(t_0)$ is given by

$$\tilde{\gamma} : \mathbb{A}^1_k \times_k \mathbb{A}^1_k \to X \text{ defined as } (s,t) \mapsto \gamma((st + (1-s)t_0)).$$

Similarly, $H(1) = \gamma(0)$. It is a contradiction to the fact that H is non-constant.
Hence the Lemma follows. □

6.2 Affine surfaces with $\pi_0^{\mathbb{A}^1}(-)(Spec\ k)$ is trivial

The following proposition (Proposition 6.2.1) gives a stronger version of Theorem 5.2.8 in case of smooth affine k-surfaces. From now on by "two $\mathbb{A}^1$'s (given by $\gamma_1, \gamma_2 : \mathbb{A}^1_k \to X$) in X intersect", we mean that $Im(\gamma_1) \cap Im(\gamma_2) \neq \emptyset$.

Proposition 6.2.1. *Let $X \in Sm/k$ be an affine surface such that $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial. Then one of the following holds:*

1. *There exists some $Y \in Sm/k$ such that Y is irreducible along with a non-constant homotopy $H : \mathbb{A}^1_k \times Y \to X$ which is dominant.*
2. *$\forall x \in X(k)$ there is an $\mathbb{A}^1$ in X through x and there are k-points α, β in X along with two distinct $\mathbb{A}^1$-s (the images are distinct) in X through α and β respectively such that they intersect.*

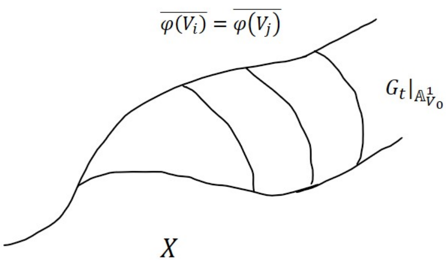
Proof. Suppose, $X \in Sm/k$ is an affine surface with $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial. If there is some $\alpha \in X(k)$ such that there is no non-constant $\mathbb{A}^1$ through α, according to the proof in [39, Theorem 4.9], there is a non-constant homotopy $H : \mathbb{A}^1_k \times Y \to X$ which is dominant for some $Y \in Sm/k$ such that Y is irreducible. If possible, the conclusion of this Proposition is false for X. Thus we assume that for each $x \in X(k)$ there is the unique $\mathbb{A}^1$ through x (this means given two $\mathbb{A}^1$'s γ_1, γ_2 through x, we have $Im(\gamma_1) = Im(\gamma_2)$) i.e. there are no intersecting $\mathbb{A}^1$'s in X and X does not admit such a dominant map as described in the conclusion of the Proposition. Fix $\alpha, \beta \in X(k)$ such that β lies outside the unique $\mathbb{A}^1$ through α. Since $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$

is trivial, $\mathcal{L}(X)(Spec\ k)$ is trivial ([33, Theorem 2.2] or Corollary 3.3.9). Therefore there is an $n \geq 1$ such that $\alpha = \beta \in \mathcal{S}^{n+1}(X)(Spec\ k)$, but $\alpha \neq \beta \in \mathcal{S}^n(X)(Spec\ k)$. Thus there is a chain of non-constant $\mathbb{A}^1$-homotopies (Remark 5.2.4) $H_1, H_2, \ldots, H_p \in \mathcal{S}^n(X)(\mathbb{A}^1_k)$ such that $H_1(0) = \alpha$ and $H_p(1) = \beta$. By applying Lemma 6.1.4 repeatedly, there is a non-constant homotopy $G \in \mathcal{S}(X)(\mathbb{A}^1_W)$ for some $W \in Sm/k$ such that W is irreducible. Since the morphism $X \to \mathcal{S}(X)$ is an epimorphism, there is a Nisnevich covering $f : V \to \mathbb{A}^1_W$ and a morphism $\phi : V \to X$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\phi} & X \\ \downarrow{\scriptstyle f} & & \downarrow \\ \mathbb{A}^1_W & \xrightarrow{G} & \mathcal{S}(X) \end{array}$$

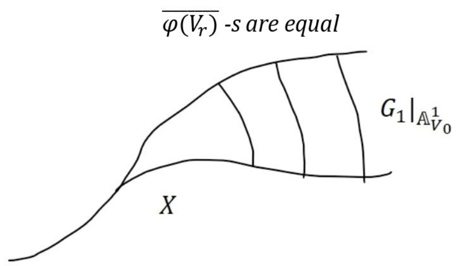
There is a Nisnevich covering $V' \to V \times_{\mathbb{A}^1_W} V$ such that $\phi \circ p_1|_{V'}$ and $\phi \circ p_2|_{V'}$ are $\mathbb{A}^1$-chain homotopic (where $p_1, p_2 : V \times_{\mathbb{A}^1_W} V \to V$ are the projections). Thus there is a chain of non-constant $\mathbb{A}^1$-homotopies (Remark 5.2.4) $G_1, G_2, \ldots, G_m : \mathbb{A}^1_{V'} \to X$ such that $G_1(0) = \phi \circ p_1|_{V'}$ and $G_m(1) = \phi \circ p_2|_{V'}$.

Suppose $V = \coprod_i V_i$, V_i-s are the irreducible components of V. Then each irreducible component of V' (which is also a connected component) maps to $V_i \times_{\mathbb{A}^1_W} V_j$ for some i, j (note that, each $V_i \times_{\mathbb{A}^1_W} V_j$ is non-empty since W is irreducible). If an irreducible component (say V_0) of V' maps to $V_q \times_{\mathbb{A}^1_W} V_s$ such that $\overline{\phi(V_q)}$ and $\overline{\phi(V_s)}$ are distinct, there is some t such that $G_t|_{\mathbb{A}^1_{V_0}}$ is a non-constant homotopy and $G_t|_{\mathbb{A}^1_{V_0}}(0) = \phi \circ p_1|_{V_0}$, since we have $G_1|_{\mathbb{A}^1_{V_0}}(0) = \phi \circ p_1|_{V_0}$ and $G_m|_{\mathbb{A}^1_{V_0}}(1) = \phi \circ p_2|_{V_0}$ and $\overline{Im(\phi \circ p_1|_{V_0})} = \overline{\phi(V_q)}$ and $\overline{Im(\phi \circ p_2|_{V_0})} = \overline{\phi(V_s)}$. As k is algebraically closed, there is a non-constant $\mathbb{A}^1$ contained in $Im(G_t|_{\mathbb{A}^1_{V_0}})$. Since G is a non-constant homotopy, there is some i and j such that there is no $\mathbb{A}^1$ in X that contains both $\overline{\phi(V_i)}$ and $\overline{\phi(V_j)}$ by Lemma 6.1.7. The rest of the proof follows from the following cases.



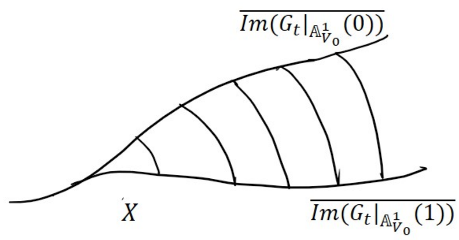
Case 1

Case 1: Assume that $\overline{\phi(V_i)}$ and $\overline{\phi(V_j)}$ are equal and there is some r such that $\overline{\phi(V_r)}$ and $\overline{\phi(V_i)}$ are distinct. There is some irreducible component of V' (say V_0) that maps to $V_i \times_{\mathbb{A}^1_W} V_r$. There is some t such that $G_t|_{\mathbb{A}^1_{V_0}}$ is non-constant and $G_t|_{\mathbb{A}^1_{V_0}}(0) = \phi \circ p_1|_{V_0}$. Here $\overline{Im(\phi \circ p_1|_{V_0})}$ is same with $\overline{\phi(V_i)}$. Since $G_t|_{\mathbb{A}^1_{V_0}}$ is non-constant and k is algebraically closed, $Im(G_t|_{\mathbb{A}^1_{V_0}})$ contains a non-constant $\mathbb{A}^1$ and $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}})}$ contains $\overline{\phi(V_i)}$. Therefore $G_t|_{\mathbb{A}^1_{V_0}}$ is dominant, since $\overline{\phi(V_i)}$ is not contained in a single $\mathbb{A}^1$. This is a contradiction since we have assumed that X does not admit such a dominant map.



Case 2

Case 2: Assume that for every r, $\overline{\phi(V_r)}$ are same. Choose an irreducible component of V' (say V_0) such that $G_1|_{\mathbb{A}^1_{V_0}}$ is non-constant. Suppose V_0 maps to $V_l \times_{\mathbb{A}^1_W} V_s$ for some l, s. Here $G_1(0) = \phi \circ p_1|_{V'}$ and $\overline{Im(\phi \circ p_1|_{V_0})}$ is same with $\overline{\phi(V_l)}$. Since $G_1|_{\mathbb{A}^1_{V_0}}$ is non-constant and k is algebraically closed, $Im(G_1|_{\mathbb{A}^1_{V_0}})$ contains a non-constant $\mathbb{A}^1$ and $\overline{Im(G_1|_{\mathbb{A}^1_{V_0}})}$ contains $\overline{\phi(V_l)}$. Therefore $G_1|_{\mathbb{A}^1_{V_0}}$ is dominant, since $\overline{\phi(V_l)}$ is not contained in a single $\mathbb{A}^1$. It is a contradiction.



Case 3

Case 3: Assume that $\overline{\phi(V_i)}$ and $\overline{\phi(V_j)}$ are distinct. There is an irreducible component (say V_0) of V' that maps to $V_i \times_{\mathbb{A}^1_W} V_j$. Then there is some t such that $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(1))}$ are distinct and there is no single $\mathbb{A}^1$ in X that contains both $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(1))}$. Indeed if possible, assume that for each l either $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(1))}$ are same or if $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(1))}$ are distinct, then there is an $\mathbb{A}^1$ in X that contains both the closures. Thus for each homotopy $G_l|_{\mathbb{A}^1_{V_0}}$, there is an $\mathbb{A}^1$ in X that contains both $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(1))}$, as $G_l(1) = G_{l+1}(0)$. Since there are no intersecting $\mathbb{A}^1$-s in X and $G_l(1) = G_{l+1}(0)$, there is a single $\mathbb{A}^1$ in X that contains all $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_l|_{\mathbb{A}^1_{V_0}}(1))}$ for any l. Thus there is an $\mathbb{A}^1$ in X that contains both $\overline{\phi(V_i)}$ and $\overline{\phi(V_j)}$. It is a contradiction. Since k is algebraically closed, $Im(G_t|_{\mathbb{A}^1_{V_0}})$ contains a non-constant $\mathbb{A}^1$ and $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(0))}$ and $\overline{Im(G_t|_{\mathbb{A}^1_{V_0}}(1))}$ are not contained in that $\mathbb{A}^1$. Therefore, $G_t|_{\mathbb{A}^1_{V_0}}$ is dominant. It is a contradiction.

Hence the Proposition follows. □

We end this chapter making some comments on the $\mathbb{A}^1$-connected component sheaf of an affine surface X over the field of positive characteristics and the existence of $\mathbb{A}^1$'s in X.

6.2.1 Comments in Positive Characteristics

Suppose, X is an affine surface over a field of positive charcteristic. In this subsection we list some behaviours of the $\mathbb{A}^1$-connected component sheaf of X with the geometric properties of X.

1. *There are smooth Frobenius sandwich surfaces ([84], [89, Section 4, Section 5]) X over an algebraically closed field k of characteristic $p > 0$ which are not rational, admit a finite surjective morphism $\phi : \mathbb{A}^2_k \to X$ and have non-negative logarithmic Kodaira dimension. Any two k-rational points in X can be joined by a chain of $\mathbb{A}^1_k$'s i.e. $\mathcal{S}(X)(Spec\ k)$ is trivial, since ϕ is surjective and $\mathbb{A}^2_k$ is $\mathbb{A}^1$-chain connected. Thus, $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial. However, over a field of characteristic zero any $\mathbb{A}^1$-connected surface is rational ([6, Proposition 5.4.2.7], [11, Corollary 2.4.7]). If X is a surface over an uncountable algebraically closed field k of characteristic zero and $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial, then X has negative logarithmic Kodaira dimension (Corollary 5.2.10).*

2. *Suppose the base field k is algebraically closed and is of infinite transcendence degree over the prime field $\mathbb{F}_p$ $(p > 0)$. Then any vector bundle over the Frobenius sandwich surface S in [89, Theorem 2.6] is trivial ([116], [95, Theorem 2]). The surface S is rational and S has non-negative logarithmic Kodaira dimension [89, Lemma 2.8]. Since such surface S admits a finite surjective morphism $\phi : \mathbb{A}^2_k \to X$, so similarly as in (1), $\pi_0^{\mathbb{A}^1}(S)(Spec\ k)$ is trivial.*

Chapter 7

Kan Fibrant Property of $Sing_*(X)(-)$

The simplicial set $Sing_*(\mathbb{A}^m_k)(U)$ is Kan fibrant for every $U \in Sm/k$, since $\mathbb{A}^n_k$ is an affine group scheme. In this chapter, in Section 7.1 we give an explicit formula for horn filling in $Sing_*(\mathbb{A}^m_k)(U)$ (Subsection 7.1.1). In Section 7.2, we define the notion of a simplicial set being Kan fibrant in degree n which is a weaker condition than a simplical set being Kan fibrant (Definition 7.2.1) and we use this condition to prove that if X is a smooth affine k-surface with $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial and $Sing_*(X)(Spec\ k)$ is Kan fibrant in degree 2, then X has negative logarithmic Kodaira dimension (Corollary 7.2.5).

7.1 Kan Fibrant Property of $Sing_*(\mathbb{A}^m_k)(-)$

Recall the definition of $Sing_*$ functor on $\Delta^{op}PSh(Sm/k)$ (Definition 2.2.6), for $X, U \in Sm/k$

$$Sing_*(X)(U)_n = Hom_{Sm/k}(\Delta^n_a \times_k U, X),$$

along the boundary maps d_i's and the degeneracy maps s_i's. In this section we provide a method of horn filling of the simplicial set $Sing_*(\mathbb{A}^m_k)(U)$, for every $U \in Sm/k$.

Definition 7.1.1. [98, Definition 1.3] A simplicial set X is called Kan fibrant if for every n, $0 \le l \le n$ and given l-th horn in X i.e. a map $\alpha : \Lambda^n_l \to X$, there is a map $\bar{\alpha} : \Delta^n \to X$ such that $\alpha = \bar{\alpha} \circ \theta$, where $\theta : \Lambda^n_l \to \Delta^n$ is the inclusion of the l-th horn in Δ^n. Equivalently, given n-many $(n-1)$ simplices $x_0, .., x_{l-1}, x_{l+1}, .., x_n$ of X satisfying the compatibility condition $d_i x_j = d_{j-1} x_i$ for $i < j, i, j \neq l$, there is an n-simplex x such that $d_i x = x_i$ for all $i \neq l$.

Example 7.1.2. Any simplicial group is Kan fibrant [98, Theorem 17.1]. Thus $Sing_*(\mathbb{A}^m_k)$ is sectionwise Kan fibrant since $\mathbb{A}^m_k$ is an affine group scheme. The retract of a Kan fibrant simplicial set is Kan fibrant.

In the next subsection (Subsection 7.1.1), using [100, Theorem 3.1], we write an explicit formula of horn filling of the sections of $Sing_*(\mathbb{A}^m_k)$.

7.1.1 Formula of horn filling of $Sing_*(\mathbb{A}^m_k)(U)$ $U \in Sm/k$:

$$\begin{aligned} Sing_*(\mathbb{A}^m_k)(U)_n &= Hom_{Sm/k}(\Delta^n_a \times_k U, \mathbb{A}^m_k) \\ &\cong Hom_{k-alg}(k[T_1, .., T_m], \frac{A[x_0, x_1, .., x_n]}{(\sum_{i=0}^n x_i - 1)}), \end{aligned}$$

where A is the ring of regular functions $\mathcal{O}(U)$. For $n \geq 1$ and $0 \leq l \leq n$, suppose we are given

$$\phi_0, .., \phi_{l-1}, \phi_{l+1}, .., \phi_n : k[T_1, .., T_m] \to \frac{A[x_0, x_1, .., x_{n-1}]}{(\sum_{i=0}^{n-1} x_i - 1)}$$

such that $d^i \circ \phi_j = d^{j-1} \circ \phi_i$, if $i < j$ and $i, j \neq l$, here we again denote the map $Id_A \otimes d^i$ by d^i (Id_A is the identity map on A):

$$Id_A \otimes d^i : A \otimes_k \frac{k[x_0, x_1, .., x_{n-1}]}{(\sum_{i=0}^{n-1} x_i - 1)} \to A \otimes_k \frac{k[x_0, x_1, .., x_{n-2}]}{(\sum_{i=0}^{n-2} x_i - 1)}$$

and denote the map $Id_A \otimes s^i$ by s^i again:

$$Id_A \otimes s^i : A \otimes_k \frac{k[x_0, x_1, .., x_{n-1}]}{(\sum_{i=0}^{n-1} x_i - 1)} \to A \otimes_k \frac{k[x_0, x_1, .., x_n]}{(\sum_{i=0}^{n} x_i - 1)}$$

Since T_p's are independent variables, suppose that each ϕ_i is given by

$$T_p \overset{\phi_i}{\longmapsto} \overline{f_i^{(p)}}, \ f_i^{(p)} \in A[x_0, x_1, .., x_{n-1}], \ 1 \leq p \leq m, \ 0 \leq i \leq n, i \neq l$$

and $\overline{f_i^{(p)}}$ is the class of $f_i^{(p)}$ in $\frac{A[x_0, x_1, .., x_{n-1}]}{(\sum_{i=0}^{n-1} x_i - 1)}$. We need to find some

$$\Phi : k[T_1, .., T_m] \to \frac{A[x_0, x_1, .., x_n]}{(\sum_{i=0}^{n} x_i - 1)}$$

such that $d^i \circ \Phi = \phi_i$, for all $i \neq l$.

Using [100, Theorem 3.1], we write the formulas of horn filling of $Sing_*(\mathbb{A}^m_k)(U)$. Define two k-algebra morphisms

$$T^j, S^j : \frac{A[x_0, x_1, .., x_n]}{(\sum_{i=0}^{n} x_i - 1)} \to \frac{A[x_0, x_1, .., x_n]}{(\sum_{i=0}^{n} x_i - 1)}$$

as $T^j = s^j \circ d^j$ and $S^j = s^j \circ d^{j+1}$, i.e.

$$\overline{f(x_0, .., x_n)} \overset{T^j}{\longmapsto} \overline{f(x_0, .., x_{j-1}, 0, x_j + x_{j+1}, x_{j+2}, .., x_n)}$$

$$\overline{f(x_0, .., x_n)} \overset{S^j}{\longmapsto} \overline{f(x_0, .., x_{j-1}, x_j + x_{j+1}, 0, x_{j+2}, .., x_n)}$$

Consider three cases.

Case 1: *Suppose, $l = 0$. We are given the $(n-1)$-simplices as $\phi_1, .., \phi_n$ and each ϕ_i is given by*

$$T_p \mapsto \overline{f_i^{(p)}},\ p = 1, \dots, m.$$

Define $\Phi : k[T_1, \dots, T_m] \to \frac{A[x_0,x_1,..,x_n]}{(\sum_{i=0}^{n} x_i - 1)}$ as

$$T_p \mapsto \overline{\sum_{i=0}^{n-1} f_{i+1}^{(p)}(x_0, .., x_{i-1}, x_i + x_{i+1}, x_{i+2}, .., x_n)}$$
$$+ \sum_{r=0}^{n-2} \sum_{\substack{t \neq 0 \\ 0 \le i_1 < .. < i_t \le r}} (-1)^t S^{i_1} \circ S^{i_2} \circ ..S^{i_t} (\overline{f_{r+2}^{(p)}(x_0, .., x_r, x_{r+1} + x_{r+2}, x_{r+3}, .., x_n)})$$

Thus Φ is defined as

$$T_p \longmapsto \sum_{i=0}^{n-1} s^i \overline{f_{i+1}^{(p)}} + \sum_{r=0}^{n-2} \sum_{\substack{t \neq 0 \\ 0 \le i_1 < \dots < i_t \le r}} (-1)^t S^{i_1} \circ \dots S^{i_t} s^{r+1} \overline{f_{r+2}^{(p)}},$$

for $p = 1, \dots, m$.

Case 2: *Suppose, $l = n$. We are given the $(n-1)$-simplices $\phi_0, \dots, \phi_{n-1}$ and ϕ_i and each ϕ_i is defined as*

$$T_p \mapsto \overline{f_i^{(p)}},\ p = 1, \dots, m.$$

Define $\Phi : k[T_1, \dots, T_m] \to \frac{A[x_0,x_1,..,x_n]}{(\sum_{i=0}^{n} x_i - 1)}$ as

$$T_p \longmapsto \overline{\sum_{i=0}^{n-1} f_i^{(p)}(x_0, .., x_{i-1}, x_i + x_{i+1}, x_{i+2}, .., x_n)}$$
$$+ \sum_{r=1}^{n-1} \sum_{\substack{t \neq 0 \\ r \le i_t < .. < i_1 \le n-1}} (-1)^t T^{i_1} \circ T^{i_2} \circ \dots T^{i_t} (\overline{f_{r-1}^{(p)}(x_0, .., x_{r-1} + x_r, x_{r+1}, .., x_n)})$$

Thus Φ is defined as

$$T \longmapsto \sum_{i=0}^{n-1} s^i \overline{f_i^{(p)}} + \sum_{r=1}^{n-1} \sum_{\substack{t \neq 0 \\ r \le i_t < \dots < i_1 \le n-1}} (-1)^t T^{i_1} \circ \dots T^{i_t} s^{r-1} \overline{f_{r-1}^{(p)}},$$

for $p = 1, \dots, m$.

Case 3: *Suppose, $0 < l < n$. We are given $\phi_0, \dots, \phi_{l-1}, \phi_{l+1}, \dots, \phi_n$ and each ϕ_i is defined as*

$$T_p \mapsto \overline{f_i^{(p)}},\ p = 1, \dots, m.$$

Define $\Phi : k[T_1, \dots, T_m] \to \frac{A[x_0,x_1,..,x_n]}{(\sum_{i=0}^{n} x_i - 1)}$ *as*

$$
\begin{aligned}
T_p \longmapsto & \overline{\sum_{i=0}^{l-1} f_i^{(p)}(x_0,..,x_{i-1},x_i+x_{i+1},x_{i+2},..,x_n)} + \overline{\sum_{i=0}^{n-l-1} f_{n-i}^{(p)}(x_0,..,x_{n-i-1}+x_{n-i},..,x_n)} \\
& + \sum_{s=1}^{l-1} \sum_{\substack{(t,v)\neq(0,0) \\ l\leq i_1<..<i_t\leq n-1 \\ s\leq j_v<..<j_1\leq l-1}} (-1)^{t+v} S^{i_1}\circ S^{i_2}\circ ..S^{i_t}\circ T^{j_1}\circ ..\circ T^{j_v}(\overline{f_{s-1}^{(p)}(x_0,..,x_{s-1}+x_s,..,x_n)}) \\
& + \sum_{\substack{r\neq 0 \\ l\leq i_1<..<i_r\leq n-1}} (-1)^r S^{i_1}\circ ..\circ S^{i_r} \overline{f_{l-1}^{(p)}(x_0,..,x_{l-1}+x_l,..,x_n)} \\
& + \sum_{t=1}^{n-1-l} \sum_{\substack{v\neq 0 \\ l\leq i_1<..<i_v\leq n-1-t}} (-1)^v S^{i_1}\circ S^{i_2}\circ \dots S^{i_v}(\overline{f_{n-t+1}^{(p)}(x_0,..,x_{n-t}+x_{n-t+1},..,x_n)}),
\end{aligned}
$$

for $p = 1, \dots m$. *This completes the formulas of horn filling of* $Sing_*(\mathbb{A}_k^m)(U)$.

In the next section in Lemma 7.2.3, using these formulas we will show that $Sing_*(\mathbb{A}_k^m)(U)$ *is Kan fibrant in degree* 2 *(Definition 7.2.1).*

7.2 Surfaces with $Sing_*(X)(Spec\ k)$ is Kan Fibrant

In this section we define the notion of a simplicial set being Kan fibrant in degree n *(Definition 7.2.1). In Lemma 7.2.3, using the formulas obtained in Subsection 7.1.1 we show that the sections of* $Sing_*(\mathbb{A}_k^m)$ *are Kan fibrant in degree* 2. *In Corollary 7.2.5, we prove that if* X *is a smooth affine surface over an algebraically closed field* k *of characteristic zero along with* $\pi_0^{\mathbb{A}^1}(X)(Spec\, k)$ *is trivial and* $Sing_*(X)(Spec\, k)$ *is Kan fibrant in degree* 2, *then* X *has negative logarithmic Kodaira dimension.*

Definition 7.2.1. A simplicial set X is called Kan fibrant in degree n if for each l-th horn Λ_l^n where $0 \leq l \leq n$, a map $\phi : \Lambda_l^n \to X$ can be extended to $\tilde{\phi} : \Delta^n \to X$.

Example 7.2.2. Any Kan fibrant simplicial set is Kan fibrant in degree n for any n. Any simplicial group is Kan fibrant in degree n [98, Theorem 17.1]. For example, the sections of $Sing_*(\mathbb{A}_k^m)$ are Kan fibrant in degree n for any n.

Lemma 7.2.3. *$Sing_*(\mathbb{A}_k^m)(U)$ is Kan fibrant in degree 2, for every $U \in Sm/k$.*

Proof. Suppose, we have given a morphism

$$\phi : \Lambda_l^2 \to Sing_*(\mathbb{A}_k^m)(U), \text{ where } U \in Sm/k, l \in \{0,1,2\}.$$

We will extend ϕ to a 2-simplex of $Sing_*(\mathbb{A}_k^m)(U)$. We have the following isomorphism:

$$Sing_*(\mathbb{A}_k^m)(U)_n = Hom_{Sm/k}(\Delta_a^n \times_k U, \mathbb{A}_k^m)$$
$$\cong Hom_{k-alg}(k[T_1,..,T_m], \frac{A[x_0,..,x_n]}{(\sum_{i=0}^n x_i - 1)}),$$

where A is the ring of regular functions $\mathcal{O}(U)$. Here are three cases according to the l-th horn, $l = 0, 1, 2$.

Case 1: $l = 0$ Suppose we are given

$$\phi_1, \phi_2 : k[T_1,..,T_m] \to \frac{A[x_0,x_1]}{(x_0+x_1-1)}$$

such that $d^1 \circ \phi_2 = d_1 \circ \phi_1$. Since T_j's are the independent variables, suppose ϕ_1, ϕ_2 are given by

$$T_j \overset{\phi_1}{\longmapsto} \overline{f_j},\ f_j \in A[x_0,x_1] \text{ and}$$

$$T_j \overset{\phi_2}{\longmapsto} \overline{g_j},\ g_j \in A[x_0,x_1]$$

and f_j, g_j satisfy

$$\overline{g_j(x_0,0)} = \overline{f_j(x_0,0)}.$$

Define $\Phi : k[T_1,..,T_m] \to \frac{A[x_0,x_1,x_2]}{(x_0+x_1+x_2-1)}$ as

$$T_j \mapsto \overline{g_j(x_0,x_1+x_2) + f_j(x_0+x_1,x_2) - g_j(x_0+x_1,x_2)}.$$

Then $d^1 \circ \Phi$ is given by

$$T_j \mapsto \overline{g_j(x_0,x_1) - g_j(x_0,x_1) + f_j(x_0,x_1)}$$

so, $T_j \overset{d^1\circ\Phi}{\longmapsto} \overline{f_j(x_0,x_1)}$.
The map $d^2 \circ \Phi$ is given by

$$T_j \mapsto \overline{g_j(x_0,x_1) - g_j(x_0+x_1,0) + f_j(x_0+x_1,0)}$$

. Since $\overline{g_j(x_0,0)} = \overline{f_j(x_0,0)} \in \frac{A[x_0]}{(x_0-1)}$, so $T_j \overset{d^2\circ\Phi}{\longmapsto} \overline{g_j(x_0,x_1)}$.

Case 2: $l = 1$ Suppose we are given

$$\phi_0, \phi_2 : k[T_1,..,T_m] \to \frac{A[x_0,x_1]}{(x_0+x_1-1)}$$

such that $d^0 \circ \phi_2 = d_1 \circ \phi_0$. Suppose ϕ_0, ϕ_2 are given by

$$T_j \overset{\phi_0}{\longmapsto} \overline{f_j},\ f_j \in A[x_0,x_1] \text{ and}$$

$$T_j \xmapsto{\phi_2} \overline{g_j},\ \ g_j \in A[x_0, x_1]$$

and f_j, g_j satisfy

$$\overline{g_j(0, x_0)} = \overline{f_j(x_0, 0)}.$$

Define $\Phi : k[T_1, .., T_m] \to \frac{A[x_0,x_1,x_2]}{(x_0+x_1+x_2-1)}$ as

$$T_j \mapsto \overline{f_j(x_0 + x_1, x_2) + g_j(x_0, x_1 + x_2) - f_j(x_0 + x_1 + x_2, 0)},$$

$$\text{so } T_j \xmapsto{\Phi} \overline{f_j(x_0 + x_1, x_2) + g_j(x_0, x_1 + x_2) - f_j(1, 0)}.$$

Similarly as in Case 1, we can check $d^0 \circ \Phi = \phi_0$ and $d^2 \circ \Phi = \phi_2$.

Case 3: $l = 2$ Suppose we are given

$$\phi_0, \phi_1 : k[T_1, .., T_m] \to \frac{A[x_0, x_1]}{(x_0 + x_1 - 1)}$$

such that $d^0 \circ \phi_1 = d^0 \circ \phi_0$. Suppose ϕ_0, ϕ_1 are given by

$$T_j \xmapsto{\phi_0} \overline{f_j},\ \ f_j \in A[x_0, x_1] \text{ and}$$

$$T_j \xmapsto{\phi_1} \overline{g_j},\ \ g_j \in A[x_0, x_1]$$

and f_j, g_j satisfy

$$\overline{g_j(0, x_0)} = \overline{f_j(0, x_0)}.$$

Define $\Phi : k[T_1, .., T_m] \to \frac{A[x_0,x_1,x_2]}{(x_0+x_1+x_2-1)}$ as

$$T_j \mapsto \overline{f_j(x_0 + x_1, x_2) + g_j(x_0, x_1 + x_2) - f_j(x_0, x_1 + x_2)}.$$

Similarly as in Case 1, we can check $d^0 \circ \Phi = \phi_0$ and $d^1 \circ \Phi = \phi_1$.
Therefore, $Sing_*(\mathbb{A}^m_k)(Spec\ k)$ is Kan fibrant in degree 2.

□

Remark 7.2.4. Thus in particular if we are given two morphisms $f, g : \mathbb{A}^1_k \to \mathbb{A}^m_k$ such that $f(1) = g(0)$, then the morphism $h : \mathbb{A}^2_k \to \mathbb{A}^m_k$ defined as

$$(t_1, t_2) \mapsto (f(1 - t_1) + g(t_2) - f(1))$$

satisfies $h(1 - x, 0) = f(x)$ and $h(0, x) = g(x)$. Therefore, if there are two intersecting $\mathbb{A}^1$'s in $\mathbb{A}^2_k$, then we can extend it to get a morphism from $\mathbb{A}^2_k$ to $\mathbb{A}^2_k$.

We end this section with the following corollary (Corollary 7.2.5) to Proposition 6.2.1. The following corollary is taken from [40, Corollary 2.12].

Corollary 7.2.5. *Let $X \in Sm/k$ be an affine surface, where k is an algebraically closed field of characteristic zero. Suppose that $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial and $Sing_*(X)(Spec\ k)$ is Kan fibrant in degree 2. Then X has negative logarithmic Kodaira dimension.*

Proof. Since $\pi_0^{\mathbb{A}^1}(X)(Spec\ k)$ is trivial, so by the Proposition 6.2.1 either there is some $Y \in Sm/k$ irreducible along with a non-constant homotopy $H : \mathbb{A}^1_k \times_k Y \to X$ which is dominant or there are two intersecting $\mathbb{A}^1$'s in X. For the first case, we proceed as in Corollary 6.1.3 to conclude that X has negative logarithmic Kodaira dimension. For the second case, since $Sing_*(X)(Spec\ k)$ is Kan fibrant in degree 2, there is a dominant morphism $\phi : \mathbb{A}^2_k \to X$. Therefore X has logarithmic Kodaira dimension $-\infty$ by [70, Proposition 11.4]. □

Chapter 8

Characterisation of the Affine Space

In this chapter we prove the main theorem (Theorem 8.1.1) in this thesis. In Theorem 8.1.1, we prove that over a field k of characteristic zero, the affine plane $\mathbb{A}^2_k$ is the only smooth affine surface which is $\mathbb{A}^1$-contractible. We provide a mixed characterisation of the affine complex plane $\mathbb{A}^2_{\mathbb{C}}$ (Theorem 8.1.3), which in particular says that $\mathbb{A}^1$-contractibility is indeed a stronger notion than topological contractibility (Corollary 8.1.5). We also give characterisations of $\mathbb{A}^n_k$'s for $n = 3, 4$ using $\mathbb{A}^1$-homotopy theory (Corollary 8.1.10). In Subsection 8.1.1, we recall the notion of locally nilpotent derivation and give a mixed characterisation of the affine space $\mathbb{A}^3_k$ using the locally nilpotent derivation (Corollary 8.1.13). In Subsection 8.1.2, using Sathaye's theorem [111, Theorem 1.1] we prove that $\mathbb{A}^2_R$ is the only $\mathbb{A}^1$-contractible smooth affine surface over a discrete valuation ring R of equicharacteristiz zero (Theorem 8.1.14).

8.1 Characterisation over a field of characteristic zero

In dimension 1, the affine line $\mathbb{A}^1_k$ is the only $\mathbb{A}^1$-contractible smooth affine curve over a field k [6, Theorem 5.4.2.9]. In this section in Theorem 8.1.1 we prove that the affine plane $\mathbb{A}^2_k$ is the only $\mathbb{A}^1$-contractible smooth affine surface over a field k of characteristic zero. This is the main theorem of this thesis. A variant of the main theorem (Theorem 8.1.3) and its consequences are also given in this section. We thank Prof. Amartya Kumar Dutta for this version of Theorem 8.1.1. This section is taken from [39, Section 5].

Theorem 8.1.1. *Let k be a field of characteristic zero and X be an smooth affine surface over the field k. Then X is $\mathbb{A}^1$-contractible if and only if X is isomorphic to $\mathbb{A}^2_k$.*

Proof. One direction is clear by definition of $\mathbb{A}^1$-contractibility. Conversely, suppose X is $\mathbb{A}^1$-contractible. We can consider k as a subfield of an uncountable algebraically closed field L. As L/k is the filtered colimit of its finitely generated sub-extensions over k, therefore by [94, Corollary 1.24] the base change $X_L := X \times_k L$ is $\mathbb{A}^1$-contractible. Then the Picard group of X_L is trivial and the group of units of X_L is L^*. Moreover by Corollary 5.2.11, the logarithmic Kodaira dimension of X_L is $-\infty$. Therefore, using [92, Section 4.1], we get $X_L \cong \mathbb{A}^2_L$. Thus $\mathcal{O}(X)$ is an $\mathbb{A}^2$-form over L/k. Using [76, Theorem 3] one can show that $\mathcal{O}(X)$ is a trivial

$\mathbb{A}^2$-form as k is of characteristic 0. Indeed, since $\mathcal{O}(X)$ is a finitely generated k-algebra, we can assume L to be a finitely generated field extension over k. In characteristic zero any algebraic extension is separable, therefore after the base change we can assume L to be a finitely generated purely transcendental extension (say, $L = k(X_1, X_2, .., X_n)$) over k by [76, Theorem 3]. Again since $\mathcal{O}(X)$ is a finitely generated k-algebra, there is some $f \in k[X_1, X_2, .., X_n]$ such that $\mathcal{O}(X) \otimes_k k[X_1, X_2, .., X_n]_f \cong k[X_1, X_2, .., X_n]_f[X, Y]$. Taking quotient by some maximal ideal, we can assume L is a separable algebraic extension. Therefore by [76, Theorem 3], we get $\mathcal{O}(X) \cong k[X, Y]$. □

Remark 8.1.2. Theorem 8.1.1 says that $\mathbb{A}^1$-contractibility of an affine surface detects the affine plane over the field of characteristic zero. However only $\mathbb{A}^1$-connectedness can not the affine plane. For example, the complex sphere in $\mathbb{A}^3_{\mathbb{C}}$ is $\mathbb{A}^1$-connected, but it is not isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ (it has non-trivial Picard group, see Subsection 5.2.1).

If the base field is $\mathbb{C}$ and $X(\mathbb{C})$ is topologically contractible, then X has trivial Picard group and trivial group of units. Moreover if X is $\mathbb{A}^1$-connected, then X has negative logarithmic Kodaira dimension by Corollary 5.2.11. Then by [92, Section 4.1], we have another characterisation of the affine complex plane.

Theorem 8.1.3. *A smooth complex surface X is isomorphic to $\mathbb{A}^2_{\mathbb{C}}$ if and only if it is topologically contractible and $\mathbb{A}^1$-connected.*

Corollary 8.1.4. *Suppose X is a smooth complex surface which is topologically contractible and of non-negative logarithmic Kodaira dimension. Then X is not $\mathbb{A}^1$-connected.*

Theorem 8.1.3 says that $\mathbb{A}^1$-contractibility is indeed a stronger notion than topological contractibility as Ramanujam surface [108, Section 3] is not $\mathbb{A}^1$-contractible but topologically contractible, so are the topologically contractible tom Dieck-Petrie surfaces [46, Theorem A]. Thus we have the following corollary which answers [9, Question 6.4].

Corollary 8.1.5. *There exists topologically contractible smooth algebraic surfaces which are not $\mathbb{A}^1$-contractible.*

Remark 8.1.6. There are topological contractible complex surfaces which are affine modifications of $\mathbb{A}^2_{\mathbb{C}}$, but they are not $\mathbb{A}^1$-contractible. The tom Dieck-Petrie surfaces are the affine modifications of $\mathbb{A}^2_{\mathbb{C}}$ [77, Example 3.1] and they are topologically contractible [46, Theorem A]. However the tom Dieck-Petrie surfaces are not even $\mathbb{A}^1$-connected (Corollary 8.1.4).

If X is the Ramanujam surface [108, Section 3] or the tom-Dieck Petrie surface [46, Theorem A], then the motive of X is trivial i.e. $M(X) \cong \mathbb{Z}$ in $\mathbf{DM}_{gm}(\mathbb{C}, \mathbb{Z})$ by [5, Theorem 1], since X is topologically contractible. However, X is not $\mathbb{A}^1$-contractible (Theorem 8.1.3).

Corollary 8.1.7. *There are complex surfaces X which have trivial motive i.e. $M(X) \cong \mathbb{Z}$ in $\mathbf{DM}_{gm}(\mathbb{C}, \mathbb{Z})$ but X is not $\mathbb{A}^1$-contractible.*

Corollary 8.1.8. *Any Koras-Russell threefolds of the first kind over a field of characteristic zero cannot be the product of two proper subvarieties.*

Proof. If possible, X is a Koras-Russell threefold of first kind and X is isomorphic to $Y \times_k Z$ where Y and Z are proper subvarieties of X. Then both Y and Z are smooth affine varieties. We can assume that Y is a curve and Z is a surface. Since X is $\mathbb{A}^1$-contractible [45, Theorem 1.1], being retract of X, both Y and Z are $\mathbb{A}^1$-contractible. Therefore, $Y \cong \mathbb{A}^1_k$ (Theorem **??**) and $Z \cong \mathbb{A}^2_k$ (Theorem 8.1.1). Thus X is isomorphic to $\mathbb{A}^3_k$ which is a contradiction [54, Thoerem 9.9]. □

Corollary 8.1.8 holds in a more general setting: Any Koras-Russell threefold (of any kind) cannot be the product of two other varieties. This can be proved using the properties of $\mathbb{G}_a$*-actions without using* $\mathbb{A}^1$*-homotopy theory. This was pointed by the referee.*

Corollary 8.1.9 (Generalised Zariski's cancellation)**.** *Let X and Y be varieties over a field k of charateristic zero. Suppose that X is a surface and $X \times_k Y \cong \mathbb{A}^N_k$. Then $X \cong \mathbb{A}^2_k$.*

Proof. If $X \times_k Y \cong \mathbb{A}^N_k$, then both X and Y are smooth affine k-varieties. Being retract of $\mathbb{A}^N_k$, X is $\mathbb{A}^1$-contractible. Thus $X \cong \mathbb{A}^2_k$ by Theorem 8.1.1. □

Theorem 8.1.1 has following immediate consequence in dimensions 3 and 4:

Corollary 8.1.10. *An $\mathbb{A}^1$-contractible smooth affine threefold X over a field k of characteristic zero is isomorphic to $\mathbb{A}^3_k$ if and only if it is isomorphic to a product of two proper k-subvarieties of lower dimension. Similarly an $\mathbb{A}^1$-contractible smooth affine fourfold X over a field k of characteristic zero is isomorphic to $\mathbb{A}^4_k$ if and only if it is isomorphic to a product of two proper k-subvarieties each of dimension two.*

The above corollaries (8.1.9, 8.1.10) can be stated and proved without an appeal to $\mathbb{A}^1$-homotopy theory. The main ingredient here is the negativity of the logarithmic Kodaira dimension of the surfaces appearing in the proof , which we have derived from Theorem 5.2.8.

8.1.1 Locally Nilpotent Derivation and Characterisation of the Affine 3-Space

In this section we recall locally nilpotent derivation. In Corollary 8.1.13, we give a characterisation of the affine 3-space, as a consequence of Theorem 8.1.1.

Let k be a field of characteristic zero and R is a k-algebra. The following definition is related to the property of being $\mathbb{A}^1$-ruled (Definition 5.1.1).

Definition 8.1.11. [54, Section 1.1.7] A locally nilpotent k-derivation $D : R \to R$ is a k-linear derivation such that for each $a \in R$ $\exists$ $n \in \mathbb{N}$ such that $D^n(a) = 0$. The derivation D has a slice if $\exists$ $s \in R$ with $D(s) = 1$. We denote the kernel of D by R^D which is a k-algebra and the set of all locally nilpotent k-derivations on R will be denoted by $LND_k(R)$.

Locally nilpotent derivation is an essential tool in affine algebraic geometry to characterise the polynomial rings. Miyanishi showed a two dimensional affine U.F.D. over an algebraically closed field k with no non-trivial units is isomorphic to $k[x, y]$ if it admits a non-trivial locally nilpotent k-derivation. [88, Theorem 1]. We want to also emphasise that locally nilpotent derivation corresponds to $\mathbb{G}_a$ -action only when $char(k) = 0$.

Remark 8.1.12.

1. Suppose $D \in LND_k(R)$ has a slice $s \in R$. Then $R = R^D[s]$ i.e. R is a polynomial ring over R^D of one variable and $D = \frac{d}{ds}$, derivative with respect to s [54, Corollary 1.26].

2. The locally nilpotent k-derivations on an affine k-domain B correspond to the algebraic $\mathbb{G}_a$-actions on $Spec\ B$ [54, Section 1.5].

3. Let X be an affine variety such that $\mathcal{O}(X)$ is a U.F.D. Then X is $\mathbb{A}^1$-ruled if and only if there is a non-trivial locally nilpotent derivation on $\mathcal{O}(X)$ [51, Proposition 2] .

The fact that $\mathbb{A}^2_k$ is the only $\mathbb{A}^1$-contractible smooth affine surface over a field k of characteristic zero has the following consequence.

Corollary 8.1.13. *A smooth affine threefold X over a field of characteristic zero is isomorphic to $\mathbb{A}^3_k$ if and only if X is $\mathbb{A}^1$-contractible and there exists a locally nilpotent derivation with a slice.*

Proof. One direction is straightforward. For the other direction, suppose X is $\mathbb{A}^1$-contractible and there is a locally nilpotent derivation on $\mathcal{O}(X)$ with a slice. Then by Remark 8.1.12, $X \cong U \times_k \mathbb{A}^1_k$, where U is a smooth affine k-surface. The surface U is $\mathbb{A}^1$-contractible, being a retract of X. Therefore by Theorem 8.1.1, $U \cong \mathbb{A}^2_k$ and hence $X \cong \mathbb{A}^3_k$. □

In this context, there is an algebraic characterisation of the polynomial ring $k[x, y, z]$ where k is an algebraically closed field of characteristic zero. A three dimensional finitely generated k-algebra, which is also a U.F.D., is isomorphic to $k[x, y, z]$ if and only if its Makar-Limanov invariant is trivial and it has a locally nilpotent derivation with a slice [42, Theorem 4.6].

8.1.2 Characterisation of Affine Plane over a DVR

In this subsection we prove that over a discrete valuation ring R of equicharacteristic zero, $\mathbb{A}^1$-contractibilty detects $\mathbb{A}^2_R$ (Theorem 8.1.14). However, if R is not of equicharacteristic zero, then it is not true (Remark 8.1.15). This subsection is taken from [40, Subsection 2.1].

Theorem 8.1.14. *Let R be an equicharacteristic zero discrete valuation ring and X be a smooth affine scheme over R of relative dimension 2. Then X is $\mathbb{A}^1$-contractible if and only if X is isomorphic to $\mathbb{A}^2_R$.*

Proof. Let K and k be the fraction field and residue field of R respectively. The base changes X_K and X_k of X over K and k respectively are $\mathbb{A}^1$-contractible [94, Corollary 1.24]. Thus X_K and X_k are isomorphic to $\mathbb{A}^2_K$ and $\mathbb{A}^2_k$ respectively by Theorem 8.1.1. Therefore X is isomorphic to $\mathbb{A}^2_R$ by [111, Theorem 1]. □

Remark 8.1.15. Theorem 8.1.14 is not true if R is not of equicharacteristic zero. The counterexample of such X appeared in [3, Theorem 5.1]. The affine scheme X is $\mathbb{A}^1$-contractible since it is a retract of $\mathbb{A}^3_R$ and the base extensions of X are isomorphic to $\mathbb{A}^2_K$ and $\mathbb{A}^2_k$ over the fraction field K and residue field k respectively. However, X is not isomorphic to $\mathbb{A}^2_R$.

The characterisation of affine surfaces using $\mathbb{A}^1$-homotopy theory yields the generalised Zariski cancellation:

Corollary 8.1.16. *Suppose X is a smooth affine scheme over R of relative dimension 2 and Y is a smooth scheme over R, where R is a discrete valuation ring of equicharacteristic zero. Suppose $X \times_R Y \cong \mathbb{A}^N_R$. Then $X \cong \mathbb{A}^2_R$.*

Proof. If $X \times_R Y \cong \mathbb{A}^N_R$, then X is a retract of $\mathbb{A}^N_R$. Thus X is an $\mathbb{A}^1$-contractible smooth affine scheme over R of relative dimension 2. Thus $X \cong \mathbb{A}^2_R$ by Theorem 8.1.14. □

Chapter 9

$\mathbb{A}^1$-homotopy type of $\mathbb{A}^2 \setminus \{(0,0)\}$

Let k be a field of characteristic 0. In this chapter we prove that if an open subscheme of a smooth affine k-surface has same $\mathbb{A}^1$-homotopy type as $\mathbb{A}^2_k \setminus \{(0,0)\}$, then it is isomorphic to $\mathbb{A}^2_k \setminus \{(0,0)\}$ (Theorem 9.1.2). This is the main theorem in this chapter. However, in dimension three we prove that a Koras-Russell threefold of the first kind minus a point is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$, but it is not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$ (Theorem 9.2.3). This chapter is taken from [40, Section 3].

9.1 $\mathbb{A}^1$-homotopy type of $S^{3,2}$

In this section in Theorem 9.1.2 we prove that over the field of characteristic zero $\mathbb{A}^2_k \setminus \{(0,0)\}$ is the only open subvariety of an affine surface $X \in Sm/k$, which is $\mathbb{A}^1$-weakly equivalent to the mixed sphere $S^{3,2} = S^1_s \wedge S^2_t$.

In Theorem 9.1.1, we prove that $\mathbb{G}_m$ is the only smooth curve which is $\mathbb{A}^1$-weakly equivalent to the Tate circle S^1_t. Theorem 9.1.1 is already in the literature. We include it here for the sake of completeness.

Theorem 9.1.1. *Suppose X is a smooth curve over a field k which is isomorphic to $\mathbb{G}_m$ in $\mathbf{H}(\mathbf{k})$. Then X is isomorphic to $\mathbb{G}_m$ as k-varieties.*

Proof. Suppose, X is a smooth curve $\mathbb{A}^1$-weakly equivalent to $\mathbb{G}_m$, then $\pi_0^{\mathbb{A}^1}(X) \cong \mathbb{G}_m$, since $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid (Remark 3.1.3). The canonical surjection of sheaves $X \to \pi_0^{\mathbb{A}^1}(X)$ gives a dominant morphism $\phi : X \to \mathbb{G}_m$. Now if X is not $\mathbb{A}^1$-rigid, then by Remark 3.1.3, there is a finite separable extension L/k along with a morphism $H : \mathbb{A}^1_L \to X$ such that $H(0) \neq H(1)$ (where $H(0), H(1) : Spec\ L \to X$ are the 0-section and the 1-section respectively). So H is dominant and since $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid, the morphism $\phi \circ H$ is constant. Thus ϕ is constant. It is a contradiction, since ϕ is dominant. Therefore, X is $\mathbb{A}^1$-rigid. Hence $X \cong \pi_0^{\mathbb{A}^1}(X)$, by Remark 3.1.3. So $X \cong \mathbb{G}_m$ as k-varieties. □

Theorem 9.1.2. *Let k be a field of characteristic 0 and X be a smooth affine k-surface such that $U \subset X$ is a non-empty open subscheme. Suppose that U is isomorphic to $\mathbb{A}^2_k \setminus \{(0,0)\}$ in $\mathbf{H}(\mathbf{k})$. Then U and $\mathbb{A}^2_k \setminus \{(0,0)\}$ are isomorphic as k-varieties.*

Proof. Let K/k be a field extension such that K is uncountable and algebraically closed.

Step 1 : The quasi-affine variety U is not affine. Indeed, if U is affine then $U_K := U \times_k Spec\ K$ is affine and $U_K \cong \mathbb{A}^2_K \setminus \{(0,0)\}$ in $\mathbf{H}(\mathbf{K})$. This implies that U_K is an affine variety with trivial Picard group and trivial group of units. Moreover, $\mathbb{A}^1$-connectedness of $\mathbb{A}^2_K \setminus \{(0,0)\}$ implies that U_K has logarithmic Kodaira dimension $-\infty$ (Corollary 6.1.3). Therefore, $U_K \cong \mathbb{A}^2_K$ as K-varieties by [92, Section 4.1]. As $\mathbb{A}^2_K \setminus \{(0,0)\}$ is not $\mathbb{A}^1$-simply connected [93, Theorem 6.40], this is absurd.

Step 2 : By Noetherian property, we can embed U in a smooth affine k-surface $\tilde{X}$, which is also smallest in the sense that there is no smooth affine surface in between U and $\tilde{X}$ contained in X. The closed subscheme $\tilde{X} \setminus U$ is finitely many closed points. Indeed, if there is an irreducible closed subset D of codimension 1 contained in $\tilde{X} \setminus U$, then D is an effective Cartier divisor which is a locally principal closed subscheme. Thus $\tilde{X} \setminus D$ is affine which contradicts that $\tilde{X}$ is the smallest. Threfore $U = \tilde{X} \setminus \{p_1, \ldots p_n\}$ where p_i's are the closed points of $\tilde{X}$.

Step 3 : The smooth K-scheme U_K is isomorphic to $\mathbb{A}^2_K \setminus \{(0,0)\}$ in $\mathbf{H}(\mathbf{k})$. Therefore U_K is $\mathbb{A}^1$-connected which implies that U_K is a connected open subset of $\tilde{X}_K := \tilde{X} \times_k Spec\ K$. Thus, $U_K = \tilde{X}_K \setminus \{p_1^K, \ldots, p_n^K\}$, where p_i^K is the extension of the closed point p_i to K, and therefore each p_i^K is a finite disjoint union of finitely many K points. As the connected components of the smooth scheme $\tilde{X}_K$ has dimension 2 and as $\tilde{X}_K \setminus \{p_1^K, \ldots, p_n^K\}$ is connected, therefore $\tilde{X}_K$ is a connected smooth scheme.

Step 4 : Since $\tilde{X}_K \setminus U_K$ is of pure codimension 2, $\tilde{X}_K$ has trivial Picard group by [64, Proposition 6.5]. By Theorem 6.1.2, since U_K is $\mathbb{A}^1$-connected, there is a dominant morphism $H : \mathbb{A}^1_K \times_K W \to U_K$ such that $H(0,-) \neq H(1,-)$ with W a smooth K-variety. Composing it with the inclusion $U_K \hookrightarrow \tilde{X}_K$, we get a dominant morphism $H : \mathbb{A}^1_K \times_K W \to \tilde{X}_K$. Therefore $\tilde{X}_K$ has logarithmic Kodaira dimension $-\infty$. The restriction map $\mathcal{O}(\tilde{X}_K) \to \mathcal{O}(U_K)$ is an isomorphism since $\tilde{X}_K \setminus U_K$ is of pure codimension 2. So $\tilde{X}_K$ has trivial group of units. Therefore, $\tilde{X}_K$ is isomorphic to $\mathbb{A}^2_K$ as K-varieties [92, Section 4.1]. As there is no non-trivial $\mathbb{A}^2$-form over the field characteristic 0 [76, Theorem 3], we get $\tilde{X} \cong \mathbb{A}^2_k$ and hence $U \cong \mathbb{A}^2_k \setminus \{p_1, \ldots p_n\}$, as k-varieties.

Step 5 : Next consider the Gysin trinagle [96, Theorem 15.15] in $\mathbf{DM}_{gm}(k, \mathbb{Z})$

$$M(U) \to M(\tilde{X}) \to \oplus_{i=1}^n M(\kappa(p_i))(2)[4] \to M(U)[1].$$

Note that each of $M(\kappa(p_i))$ are strongly dualizable with $M(\kappa(p_i))^* = M(\kappa(p_i))$ (for a object $M \in \mathbf{DM}_{gm}(k,\mathbb{Z})$, M^* is the dual [96, Definition 20.6, Example 20.11, Definition 20.15]). Therefore, $M(\tilde{X}) \cong \mathbb{Z} = M(Spec\ k)$ implies that the map $M(\tilde{X}) \to \oplus_{i=1}^n M(\kappa(p_i))(2)[4]$ is the zero map (Remark 2.3.1) and $M(U) \cong M(k) \oplus_{i=1}^n M(\kappa(p_i))(2)[3]$. But $M(U) \cong M(\mathbb{A}^2 \setminus \{(0,0)\} \cong M(k) \oplus M(k)(2)[3]$. This shows that $n = 1$ and p_1 is a k-rational point. Indeed, using the relation of motivic cohomology and higher Chow groups [96, Lecture 17], we have

$$H^{4,3}_{\mathcal{M}}(k,\mathbb{Z}) \cong CH^3(k,2) = 0.$$

Thus

$$\begin{aligned} H^{4,3}_{\mathcal{M}}(U,\mathbb{Z}) &\cong Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(k)(2)[3], \mathbb{Z}(3)[4]) \\ &\cong \oplus_{i=1}^n Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(\kappa(p_i))(2)[3], \mathbb{Z}(3)[4]). \end{aligned}$$

Using Voevodsky's cancellation in $\mathbf{DM}_{gm}(k,\mathbb{Z})$, we have

$$Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(k), \mathbb{Z}(1)[1]) \cong \oplus_{i=1}^n Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(\kappa(p_i)), \mathbb{Z}(1)[1]).$$

For $X \in Sm/k$, we have

$$\begin{aligned} Hom_{\mathbf{DM}_{gm}(k,\mathbb{Z})}(M(X), \mathbb{Z}(1)[1]) &\cong H^{1,1}_{\mathcal{M}}(X,\mathbb{Z}) \\ &\cong \mathcal{O}(X)^*, \text{ by [96, Corollary 4.2]}, \end{aligned}$$

Therefore, we must have $n = 1$ and p_1 is a k-rational point. This completes the proof. □

9.2 $\mathbb{A}^1$-homotopy type of $S^{5,3}$

Theorem 9.1.2 is not true in case of quasi-affine threefold. In this section in Theorem 9.2.3, we prove that if X is a Koras-Russell threefold of the first kind minus a point, then X is $\mathbb{A}^1$-weakly equivalent to the mixed sphere $S^{5,3} = S^2_s \wedge S^3_t$, but X is not isomorphic to $\mathbb{A}^3_k \setminus \{(0,0,0)\}$ as k-varieties.

Suppose that X is the Koras-Russell threefold of the first kind ([45], [54, Section 9.3]) which is given by the equation

$$x^m z = y^r + t^s + x \text{ in } \mathbb{A}^4_k,$$

where k is an algebraically closed field of characteristic 0, $m \geq 2$ and $r, s \geq 2$ are coprime integers. Consider the morphism,

$$\phi : X \to \mathbb{A}^3_k \text{ given by } (x,y,z,t) \mapsto (x,y,t).$$

Suppose, $p = (1,0,1,0)$ is a point in X. Then $\phi(p) = q = (1,0,0)$ and $\phi^{-1}(q) = p$. This gives the restriction morphism

$$\bar{\phi} : X \setminus \{p\} \to \mathbb{A}^3_k \setminus \{q\}.$$

We show here that $X \setminus \{p\} \cong \mathbb{A}^3_k \setminus \{q\}$ in $\mathbf{H}(\mathbf{k})$, but $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{q\}$ as k-varieties.

Lemma 9.2.1. *The induced map $d\phi_p : T_pX \to T_q\mathbb{A}^3_k$ is an isomorphism between the tangent spaces.*

Proof. Suppose, $f(x,y,z,t) = x^m z - y^r - t^s - x$. Then $\nabla f(x,y,z,t) = (mx^{m-1}z - 1, -ry^{r-1}, x^m, -st^{s-1})$, so $\nabla f(1,0,1,0) = (m-1,0,1,0)$. Thus the tangent space T_pX of X at p is given by

$$T_pX = \{(a,b,-(m-1)a,d) | a,b,d \in k\}$$

The map $d\phi_p : T_pX \to T_q\mathbb{A}^3_k$ is given by $(a,b,-(m-1)a,d) \mapsto (a,b,d)$. Therefore $d\phi_p$ is an isomorphism between the tangent spaces. □

The proof of the following lemma is same as in [47, Example 2.21]. We include it here for the sake of completeness.

Lemma 9.2.2. *[47, Example 2.21] The quasi-affine threefold $X' = X \setminus \{p\}$ is $\mathbb{A}^1$-chain connected.*

Proof. Suppose, F/k is a finitely generated field extension. Consider the projection map $p_x : X' \to \mathbb{A}^1_k$ given by $(x,y,z,t) \mapsto x$. The fiber over a point $\alpha \in \mathbb{G}_m$ is $\mathbb{A}^2_k$, if $\alpha \neq 1$ and $\mathbb{A}^2_k \setminus \{(0,0)\}$, if $\alpha = 1$. Thus the fiber over every point of $\mathbb{G}_m$ is $\mathbb{A}^1$-chain connected, in particular for every $\alpha \in \mathbb{G}_m$, any two F-points in $p_x^{-1}(\alpha)$ can be joined by a chain of $\mathbb{A}^1_F$'s. The fiber over 0 is $\mathbb{A}^1_k \times_k \Gamma_{r,s}$ (where, $\Gamma_{r,s}$ is the curve in $\mathbb{A}^2_k$ defined as $y^r + t^s = 0$). Here also any two F-points can be joined by $\mathbb{A}^1_F$'s. Indeed, for an F-point (t_1,t_2,t_3) in $\mathbb{A}^1_F \times_F \Gamma_{r,s}$, the naive $\mathbb{A}^1$-homotopy given by

$$\gamma : \mathbb{A}^1_F \to \mathbb{A}^1_k \times_k \Gamma_{r,s} \text{ as } v \mapsto (t_1 v, t_2 v^s, t_3 v^r)$$

joins $(0,0,0)$ with (t_1,t_2,t_3). To get the naive-$\mathbb{A}^1$-homotopy between the points in different fibers, we find the polynomials $y(v), t(v) \in k[v]$ such that v^m divides $y(v)^r + t(v)^s + v$. Indeed if r is even and s is odd (similarly for r is odd and s is even), suppose that $y(v)$ and $t(v)$ are given by

$$y(v) = 1 + a_0 v + a_1 v^2 + \cdots + a_{m-2} v^{m-1} \text{ and } t(v) = -1 - v \cdots - v^{m-1},$$

for some $a_i \in k$. We choose a_i according to the co-effecients of v^i is zero in $y(v)^r + t(v)^s + v$ for every $i \leq m-1$. The naive $\mathbb{A}^1$-homotopy $\theta : \mathbb{A}^1_F \to X'$ given by

$$v \mapsto (\alpha v, \frac{y(\alpha v)^r + t(\alpha v)^s + \alpha v}{(\alpha v)^m}, y(\alpha v), t(\alpha v))$$

connects a point in $p_x^{-1}(0)$ with $p_x^{-1}(\alpha)$, $\alpha \in \mathbb{G}_m(k)$. Note that, the point $p = (1, 0, 1, 0)$ does not lie in the image of θ. Indeed, if for some v, $\theta(v) = p$, then $\alpha v = 1, y(\alpha v) = 1, t(\alpha v) = 0$. But then $\frac{y(\alpha v)^r + t(\alpha v)^s + \alpha v}{(\alpha v)^m} = 2$. Therefore, X' is $\mathbb{A}^1$-chain connected. □

Theorem 9.2.3. *$X \setminus \{p\}$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{q\}$, however they are not isomophic as k-varieties.*

Proof. First we prove that $X \setminus \{p\}$ is $\mathbb{A}^1$-weakly equivalent to $\mathbb{A}^3_k \setminus \{q\}$. Consider the commutative diagram with rows are cofibre sequences:

$$\begin{array}{ccccc} X \setminus \{p\} & \longrightarrow & X & \longrightarrow & X/(X \setminus \{p\}) \\ \downarrow \bar{\phi} & & \downarrow \phi & & \downarrow \\ \mathbb{A}^3_k \setminus \{q\} & \longrightarrow & \mathbb{A}^3_k & \longrightarrow & \mathbb{A}^3_k/(\mathbb{A}^3_k \setminus \{q\}) \end{array}$$

The middle vertical map is an $\mathbb{A}^1$-weak equivalence, since X is $\mathbb{A}^1$-contractible [45, Theorem 1.1]. By homotopy purity, $X/(X \setminus \{p\})$ is isomorphic to the Thom space of the normal bundle over p [94, §3.2, Theorem 2.23]. Since p is a k-point of the smooth threefold X, the normal bundle over p in X is the trivial bundle of rank three over p. The right vertical map

$$(\mathbb{P}^1_k)^{\wedge 3} \cong X/(X \setminus \{p\}) \to \mathbb{A}^3_k/(\mathbb{A}^3_k \setminus \{q\}) \cong (\mathbb{P}^1_k)^{\wedge 3},$$

is induced by $d\phi_p$. Thus the right vertical map is also an $\mathbb{A}^1$-weak equivalence by Lemma 9.2.1 [126, Lemma 2.1]. Therefore, taking simplicial suspension, the map

$$\Sigma_s \bar{\phi} : \Sigma_s(X \setminus \{p\}) \to \Sigma_s(\mathbb{A}^3_k \setminus \{q\})$$

is an $\mathbb{A}^1$-weak equivalence. Now $X \setminus \{p\}$ is $\mathbb{A}^1$-connected by Lemma 9.2.2 and $\pi_1^{\mathbb{A}^1}(X \setminus \{p\})$ is also trivial [10, Theoem 4.1]. Thus $\bar{\phi}$ is a $\mathbb{A}^1$-homology equivalence [93, Remark 6.30]. Therefore by [119, Theorem 1.1], $\bar{\phi}$ is an $\mathbb{A}^1$-weak equivalence.

Now we show that $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{q\}$ as k-varieties. Suppose, if possible there is an isomorphism $\phi : \mathbb{A}^3_k \setminus \{q\} \to X \setminus \{p\}$ with its inverse ψ. Since q and p are the codimension 3 points of $\mathbb{A}^3_k$ and X respectively, both ϕ and ψ can be extended to a morphism $\bar{\phi} : \mathbb{A}^3_k \to X$ and $\bar{\psi} : X \to \mathbb{A}^3_k$. Both the maps $\bar{\psi} \circ \bar{\phi}$ and $\bar{\phi} \circ \bar{\psi}$ agree with the identity maps in a complement of a k-point. Therefore both $\bar{\phi}$ and $\bar{\psi}$ are isomorphisms. It is a contradiction since X has non-trivial Makar-Limanov invariant ([73], [54, Theorem 9.9]). Therefore, $X \setminus \{p\}$ is not isomorphic to $\mathbb{A}^3_k \setminus \{q\}$ as k-varieties. □

We end this chapter with the following question:

Question 9.2.4. Does there exist an n-dimensional smooth k-variety X such that $X \cong \mathbb{A}^n_k \setminus \{(0, \ldots, 0)\}$ in $\mathbf{H}_\bullet(\mathbf{k})$ but X is not isomorphic to $\mathbb{A}^n_k \setminus \{(0, \ldots, 0)\}$ as k-varieties, for $n > 3$?

Chapter 10

Regular Functions on $\mathcal{S}(X)$

So far we have used $\mathbb{A}^1$-homotopy theory to classify the affine varieties, specially the affine spaces. There are algebraic invariants associated to locally nilpotent derivation which also allow us to classify the affine spaces. One such invariant is the Makar-Limanov invariant. Makar-Limanov invariant of an affine variety X is a subring $ML(X)$ of $\mathcal{O}(X)$. The ring $ML(X)$ is the set of regular functions on X, constant along the orbits of all $\mathbb{G}_a$-actions on X [54, Section 2.5], affine variety X over a characteristic 0 field k. In particular for a k-domain R,

$$ML(R) := \bigcap_{D \in LND_k(R)} R^D.$$

In case of $R = \mathcal{O}(X)$, we write $ML(X)$ instead of $ML(R)$. This invariant is not functorial. A two dimensional affine U.F.D. over an algebraically closed field k of characteristic zero is isomorphic to $k[x, y]$ if and only if its Makar-Limanov invariant is trivial [54, Theorem 9.12]. In this chapter, we construct a new functorial invariant $\mathcal{O}_{ch}(X)$ (Definition 10.1.1) which is a subobject of $ML(X)$. It is homotopy invariant (Proposition 10.2.7) but it is not representable in $\mathbf{H}(k)$ (Lemma 10.2.15). In Proposition 10.2.10, we prove that it is the ring of regular functions on the $\mathbb{A}^1$-chain connected component sheaf of X i.e.

$$\mathcal{O}_{ch}(X) \cong Hom_{Sh(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1).$$

Theorem 10.2.5 provides evidence that the ring $\mathcal{O}_{ch}(X)$ detects $\mathbb{A}^1$-s in an affine variety X. From Section 4 onwards, In Theorem 10.2.5, we show that $\mathcal{O}_{ch}(X)$ is trivial i.e. $\mathcal{O}_{ch}(X) = k$ implies the existence of $\mathbb{A}^1$-s in X (Theorem 10.2.5). In this chapter we assume k to be an algebraically closed field. This chapter is taken from [39, Section 6].

10.1 Properties of $\mathcal{O}_{ch}(X)$

In this section we define $\mathcal{O}_{ch}(X)$ (Definition 10.1.1) and discuss its properties. Recall as in Chapter 5, by the phrase "a line $g : \mathbb{A}^1 \to X$", we mean non-constant morphism $g : \mathbb{A}^1_k \to X$.

Definition 10.1.1. Let X be an affine k-variety and $\mathcal{O}(X)$ be the ring of regular functions on X. For a fixed $g : \mathbb{A}^1_k \to X$, define

$$\mathcal{O}_{ch,g}(X) := \{f \in \mathcal{O}(X) \mid f \circ g \text{ is constant}\}$$

We define $\mathcal{O}_{ch}(X) = \bigcap_{g \in Hom_{Sch/k}(\mathbb{A}^1_k, X)} \mathcal{O}_{ch,g}(X)$, where Sch/k is the category of finite type k-schemes.

We get the following immediate properties of $\mathcal{O}_{ch,g}(X)$.

Lemma 10.1.2. *Suppose X is an affine k-variety and $g : \mathbb{A}^1_k \to X$ a k-morphism.*

1. *Suppose, A, B are k-algebras and B is finitely generated. A morphism $\phi : Spec\ B \to Spec\ A$ is constant if and only if in the induced k-algebra homomorphism $\tilde{\phi} : A \to B$, $\tilde{\phi}(f) \in k$ for all $f \in A$. In particular, a morphism $\phi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is constant if and only if the induced k-algebra homomorphism $\tilde{\phi} : k[T] \to k[T]$ takes T to an element of k.*

2. *$\mathcal{O}_{ch,g}(X)$ is a k-subalgebra of $\mathcal{O}(X)$. In particular, $\mathcal{O}_{ch}(X)$ is a k-subalgebra of $\mathcal{O}(X)$.*

3. *Suppose $f_1, f_2 \in \mathcal{O}(X)$. If the product $f_1 f_2 \in \mathcal{O}_{ch,g}(X)$ is non-zero, then $f_1 \in \mathcal{O}_{ch,g}(X)$ and $f_2 \in \mathcal{O}_{ch,g}(X)$.*

4. *The group of units of X, $\mathcal{O}(X)^* \subset \mathcal{O}_{ch,g}(X)$. Thus $\mathcal{O}(X)^* \subset \mathcal{O}_{ch}(X)$. If $\mathcal{O}_{ch}(X)$ is trivial, then X has trivial group of units.*

5. *Suppose, X is of dimension at least two. Then the morphism $\bar{i} : X \to Spec(\mathcal{O}_{ch,g}(X))$ induced by the inclusion $i : \mathcal{O}_{ch,g}(X) \to \mathcal{O}(X)$ is birational.*

Proof. **(1),(2) and (3):** The proofs are quite straightforward. For **(1)**, suppose $\phi : Spec\ B \to Spec\ A$ is constant. Then the image of ϕ is the singleton set, say $\{x_0\}$, where x_0 is a k-rational point of $Spec\ A$. Thus ϕ factors as

$$Spec\ B \to Spec\ k \xrightarrow{x_0} Spec\ A.$$

Hence $\tilde{\phi}$ factors as

$$A \to k \to B.$$

So $\tilde{\phi}(f) \in k$, for every $f \in A$. Conversely, suppose the k-algebra homomorphism $\tilde{\phi}A \to B$ takes every element of A to an element of k. Then for $P \in Spec\ B$, $\tilde{\phi}^{-1}(P) = Ker(\tilde{\phi})$, which is a prime ideal of A. Thus the image of ϕ is constant. For **(2)**, suppose that $f_1, f_2 \in \mathcal{O}_{ch,g}(X)$. Then $\tilde{g}(f_1), \tilde{g}(f_2) \in k$ ($\tilde{g} : \mathcal{O}(X) \to k[T]$ is induced by g) by **(1)**. Therefore $\tilde{g}(f_1 + f_2)$ and $\tilde{g}(f_1 f_2)$ are in k. Hence $(f_1 + f_2) \circ g$ and $(f_1 f_2) \circ g$ are constant by **(1)**. Therefore $f_1 + f_2, f_1 f_2 \in \mathcal{O}_{ch,g}(X)$. So it is a k-subalgebra of $\mathcal{O}(X)$. For **(3)**, suppose that the product $f_1 f_2 \in \mathcal{O}_{ch,g}(X)$ is non-zero. Thus by **(1)**, $\tilde{g}(f_1 f_2)$ is a non-zero constant. So both $\tilde{g}(f_1)$ and $\tilde{g}(f_2)$ are non-zero constants. Hence $f_1 \in \mathcal{O}_{ch,g}(X)$ and $f_2 \in \mathcal{O}_{ch,g}(X)$, by **(1)**.

(4): Suppose, $f \in \mathcal{O}(X)^*$, then f is a morphism from X to $\mathbb{G}_m$. Therefore $f \circ g$ is constant, since $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid (Remark 3.1.3). Hence $f \in \mathcal{O}_{ch,g}(X)$.

(5): Since X is of dimension at least 2, so $g : \mathbb{A}^1_k \to X$ is not dominant. The image of g is closed in X. Indeed, if g is non-constant, we extend g to a morphism $\bar{g} : \mathbb{P}^1_k \to \overline{X}$ ($\overline{X}$ is a compactification of X). Since $\mathbb{P}^1_k$ is a projective variety and X is an affine variety, there are no non-constant morphisms from $\mathbb{P}^1_k$ to X. Thus $\bar{g}$ maps the point at infinity of $\mathbb{P}^1_k$ to a point in $\overline{X} \setminus X$. The morphism $\bar{g}$ is the composition of two proper morphisms $\mathbb{P}^1_k \xrightarrow{\text{graph of } \bar{g}} \mathbb{P}^1_k \times_k \overline{X} \xrightarrow{\text{projection}} \overline{X}$, so $\bar{g}$ is a proper morphism. Thus the image of $\bar{g}$ is closed and $Im(g) = Im(\bar{g}) \cap X$ is closed in X. Therefore, image of g is given by some ideal I of $\mathcal{O}(X)$. So its complement is the union of basic open set $D(f)$-s, $f \in I$. Choose $f \in I$ with $D(f)$ is non-empty. Then $\tilde{g}(f) = 0$ (where $\tilde{g} : \mathcal{O}(X) \to k[T]$ is induced by g.). So $fh + \mu \in \mathcal{O}_{ch,g}(X)\ \forall\ h \in \mathcal{O}(X), \forall \mu\ \in k$ by Part (1). We have an injective homomorphism $i_* : \mathcal{O}_{ch,g}(X)_f \to \mathcal{O}(X)_f$ induced by the inclusion i. The map i_* is also surjective. Indeed for $\frac{h}{f^k} \in \mathcal{O}(X)_f$, $fh \in \mathcal{O}_{ch,g}(X)$ and the element $\frac{fh}{f^{k+1}}$ is mapped to $\frac{h}{f^k}$. Hence $\mathcal{O}_{ch,g}(X)_f$ and $\mathcal{O}(X)_f$ are isomorphic and therefore X and $Spec(\mathcal{O}_{ch,g}(X))$ are birational. □

Remark 10.1.3.

1. In the above Lemma 10.1.2(5), the assumption about the dimension of X is necessary. If $X = \mathbb{A}^1_k$ and g be the identity map on $\mathbb{A}^1_k$, then $\mathcal{O}_{ch,g}(X)$ is trivial.

2. The map $\bar{i} : X \to Spec\ \mathcal{O}_{ch,g}(X)$ induced by the inclusion $i : \mathcal{O}_{ch,g}(X) \hookrightarrow \mathcal{O}(X)$, takes the image of g to a k-rational point of $Spec\ \mathcal{O}_{ch,g}(X)$ i.e. $\bar{i} \circ g$ is constant. Indeed, the map $\tilde{g} : \mathcal{O}(X) \to k[T]$ takes an element of $\mathcal{O}_{ch,g}(X)$ to an element of k. Thus by Lemma 10.1.2(1), $\bar{i} \circ g$ is constant. Therefore the map $\bar{j} : X \to Spec\ \mathcal{O}_{ch}(X)$ induced by the inclusion $j : \mathcal{O}_{ch}(X) \hookrightarrow \mathcal{O}(X)$, takes the image of every $g : \mathbb{A}^1_k \to X$ to constant (given by a k-rational point of $Spec\ \mathcal{O}_{ch}(X)$). Therefore, we can think $Spec\ \mathcal{O}_{ch,g}(X)$ is obtained by collapsing the image of g in X and $Spec\ \mathcal{O}_{ch}(X)$ is obtained by collapsing all $\mathbb{A}^1$'s in X individually.

3. Property (3) of $\mathcal{O}_{ch,g}(X)$ in the above Lemma 10.1.2 is similar to a ring being factorially closed which is satisfied by kernel of a locally nilpotent derivation hence by the Makar-Limanov invariant [54, Section 1.4, Principle 1].

4. $\mathcal{O}_{ch,g}(X)$ may not always be finitely generated k-subalgebra of $\mathcal{O}(X)$. For instance, suppose $X = \mathbb{A}^2_k$ and g is the y-axis. Then $\mathcal{O}_{ch,g}(X) = k + xk[x, y]$. This subring of $k[x, y]$ is not Noetherian. For this, consider the chain of ideals $\{I_n\}_n$ in $\mathcal{O}_{ch,g}(X)$: I_n is the ideal generated by $\{x, xy, xy^2, .., xy^{n-1}\}$. This chain of ideals does not stabilize.

Remark 10.1.4. We can describe $\mathcal{O}_{ch,g}(X)$ explicitly. Any constant function in $\mathcal{O}(X)$ is in $\mathcal{O}_{ch,g}(X)$. The image of the affine line $g : \mathbb{A}^1_k \to X$ is closed in X. Let $\tilde{g} : \mathcal{O}(X) \to k[T]$ be the k-algebra homomorphism induced by g. A regular function on X is in $\mathcal{O}_{ch,g}(X)$ if and only if its image is in k under $\tilde{g}$. Thus for $\phi \in \mathcal{O}_{ch,g}(X)$, $\phi - \tilde{g}(\phi) \in Ker(\tilde{g})$. Therefore, $\mathcal{O}_{ch,g}(X) = k + Ker(\tilde{g})$. If $\phi = \lambda + \theta$ for some constant λ and $\theta \in Ker(\tilde{g})$, then ϕ takes value

λ along $Im(g)$.

Suppose g_1 and g_2 are two intersecting $\mathbb{A}^1$-s in X :

If $\phi \in \mathcal{O}_{ch,g_1}(X) \cap \mathcal{O}_{ch,g_2}(X)$, then $\phi = \lambda + \theta = \lambda' + \theta'$ for some constants λ, λ' and θ, θ' are in kernel of $\tilde{g_1}$ and $\tilde{g_2}$ respectively. Since g_1 and g_2 intersect, $\lambda = \lambda'$. Therefore $\mathcal{O}_{ch,g_1}(X) \cap \mathcal{O}_{ch,g_2}(X) = k + (Ker(\tilde{g_1}) \cap Ker(\tilde{g_2}))$, if g_1 and g_2 intersect.

Suppose g_1 and g_2 are parallel :

If the images of g_1 and g_2 are disjoint, then $\mathcal{O}_{ch,g_1}(X) \cap \mathcal{O}_{ch,g_2}(X)$ properly contains $k + (Ker(\tilde{g_1}) \cap Ker(\tilde{g_2}))$ from the following lemma (Lemma 10.2.2).

10.2 Triviality of $\mathcal{O}_{ch}(X)$ and Existence of $\mathbb{A}^1$'s in X

For an affine k-variety X, we say $\mathcal{O}_{ch}(X)$ is trivial if $\mathcal{O}_{ch}(X) = k$. In X is a smooth affine k-surface with $\mathcal{O}(X)$ is a U.F.D., then triviality of $\mathcal{O}_{ch}(X)$ detects $\mathbb{A}^1$'s in X (Theorem 10.2.5). In Proposition 10.2.6, we show that $\mathcal{O}_{ch}(X)$ is a k-subalgebra of X. Unlike Makar-limanov invariant, $\mathcal{O}_{ch}(X)$ is homotopy invariant (Theorem 10.2.7) and it is functorial. However, $\mathcal{O}_{ch}(-)$ is not representable in the $\mathbb{A}^1$-homotopy category (Theorem 10.2.15), since it does not satisfy the gluing property (Remark 3.4.5). In Proposition 10.2.10, we prove that $\mathcal{O}_{ch}(X)$ is the ring of regular functions on $\mathcal{S}(X)$.

Definition 10.2.1. A line $g : \mathbb{A}^1 \to X$ is called isolated if it does not intersect any other lines in X i.e. for any line $h : \mathbb{A}^1 \to X$, if $Im(h) \neq Im(g)$ then $Im(h) \cap Im(g) = \emptyset$.

Lemma 10.2.2. *Suppose X is an affine k-variety.*

1. *Suppose $g_1, g_2, .., g_n$ are pairwise parallel lines in X (i.e. $Im(g_i) \cap Im(g_j) = \emptyset$, $\forall i \neq j$) and $c_1, c_2, .., c_n$ are n many constants. Then there is $f \in \mathcal{O}(X)$ such that $f = c_i$ along $Im(g_i)$.*

2. *Let X be a smooth affine surface such that $\mathcal{O}(X)$ is a U.F.D. Suppose there is a line $g : \mathbb{A}^1 \to X$ which is isolated. Then $\mathcal{O}_{ch}(X)$ is non-trivial.*

Proof. 1. Since g_i and g_j are parallel, $Ker(\tilde{g_i}) + Ker(\tilde{g_j}) = \mathcal{O}(X)$. Indeed, if $Ker(\tilde{g_i}) + Ker(\tilde{g_j})$ is contained in some maximal ideal of $\mathcal{O}(X)$, then there is a common point of g_1 and g_2. So the ideals $Ker(\tilde{g_i})$ and $Ker(\tilde{g_j})$ are pairwise comaximal. Thus by Chinese remainder theorem, there exists $f \in \mathcal{O}(X)$ such that f is c_i along $Im(g_i)$.

2. Since $\mathcal{O}(X)$ is a U.F.D., there is a $f \in \mathcal{O}(X)$ irreducible such that the zero set of f is the closed set $Im(g)$. Then f is non-zero in the complement of $Im(g)$. So for any other line $h : \mathbb{A}^1 \to X$ with $Im(h) \neq Im(g)$, f is everywhere non-zero along $Im(h)$ as g is an isolated line. Hence f must be constant along $Im(h)$, since $\mathbb{G}_m$ is $\mathbb{A}^1$-rigid (Remark 3.1.3). Therefore f is a non-trivial element in $\mathcal{O}_{ch}(X)$.

∎

Therefore, for a smooth affine surface X with trivial Picard group, if $\mathcal{O}_{ch}(X)$ is trivial, then all lines in X cannot be parallel to each other (by "all lines in X are parallel to each other", we mean that given any two lines $g_1, g_2 : \mathbb{A}^1 \to X$ with $Im(g_1) \neq Im(g_2)$, we have $Im(g_1) \cap Im(g_2) = \emptyset$). Note that in $\mathbb{A}^1 \times \mathbb{G}_m$, any line parallel to x-axis is an isolated line and any polynomial of y is in $\mathcal{O}_{ch}(\mathbb{A}^1 \times \mathbb{G}_m)$.

Definition 10.2.3. A chain connected component of X is defined to be the largest subset of $X(k)$ such that any two points in it can be joined by a chain of $\mathbb{A}^1$-s.

Remark 10.2.4. Let $T \subset X(k)$ be a chain connected component. Then T is the union of lines in X such that the k-points in the images of the lines are in T.

Theorem 10.2.5. *Let X be a smooth affine surface such that $\mathcal{O}(X)$ is a U.F.D. Then $\mathcal{O}_{ch}(X)$ is trivial if and only if there is some dense chain connected component of X.*

Proof. Suppose there is some chain connected component of X which is dense in X. Let T be the union of all lines in that chain connected component and suppose that f is in $\mathcal{O}_{ch}(X)$. The function f is constant along T, since any two points of T can be joined by chain of lines. But T is dense in X. Therefore f is constant.

On the other hand, assume that $\mathcal{O}_{ch}(X)$ is trivial. If possible, there is a chain component (say T) which is the union of finitely many lines (i.e. finitely many distinct images). Then it is closed. There is some $f \in \mathcal{O}(X)$ such that its zero set is T. Then f is non-zero along every other line outside T. Therefore it is constant along each line outside T. But f is non-constant. This gives a contradiction. Therefore every chain connected component is a union of infinitely many lines (i.e. infinitely many distinct images). Choose any such chain connected component, say S. Its closure cannot be of dimension 1, since it contains infinitely many lines, hence S is dense in X. □

Functoriality

Suppose, $\alpha : Y \to X$ is a morphism of affine k-varieties. For an affine line g in Y, $\alpha \circ g$ is an affine line in X. So $f \circ (\alpha \circ g)$ is constant if $f \in \mathcal{O}_{ch}(X)$. Thus the morphism α induces $\alpha^* : \mathcal{O}(X) \to \mathcal{O}(Y)$ that restricts to a k-algebra homomorphism $\mathcal{O}_{ch}(X) \to \mathcal{O}_{ch}(Y)$. Therefore, $\mathcal{O}_{ch}(X)$ is functorial in X.

Proposition 10.2.6. *Let k be a field of characteristic 0. Suppose X is an affine k-variety. Then $\mathcal{O}_{ch}(X) \subset ML(X)$. Therefore, if $ML(X)$ is trivial then $\mathcal{O}_{ch}(X)$ is trivial.*

Proof. Suppose $f \in \mathcal{O}_{ch}(X)$ and D is a locally nilpotent k-derivation on $\mathcal{O}(X)$. We need to show that $f \in Ker(D)$. We have a k-algebra homomorphism $exp(D) : \mathcal{O}(X) \to \mathcal{O}(X)[T]$ defined as $exp(D)(g) = \sum_{n=0}^{\infty} \frac{D^n(g)}{n!} T^n$. Fix $x \in X(k)$. Consider the following composition of k-algebra homomorphisms (this is precisely considering f along each of the orbits of $x \in X(k)$):

$$k[T] \xrightarrow{T \mapsto f} \mathcal{O}(X) \xrightarrow{exp(D)} \mathcal{O}(X)[T] \xrightarrow{\text{evaluation at } x} k[T]$$

This composition takes T to an element of k by Lemma 10.1.2. Suppose, $exp(D)(f) = \sum_{n=0}^{\infty} f_n T^n \in \mathcal{O}(X)[T]$, where $f_n \in \mathcal{O}(X)$. Then for every $n \geq 1$, $f_n(x) = 0$, $\forall x \in X(k)$. Since $X(k)$ is dense in X, f_n-s are zero for every $n \geq 1$. Thus $exp(D)(f) = f$. Hence $f \in Ker(D)$. Therefore, $\mathcal{O}_{ch}(X) \subset ML(X)$. □

Homotopy invariance of $\mathcal{O}_{ch}(-)$

Proposition 10.2.7. *$\mathcal{O}_{ch}(-)$ is homotopy invariant i.e. $\mathcal{O}_{ch}(X) = \mathcal{O}_{ch}(X \times \mathbb{A}^1_k)$ (both are k-subalgebras of $\mathcal{O}(X \times \mathbb{A}^1_k)$), $\forall X \in Sm/k$.*

Proof. The projection map $p : X \times_k \mathbb{A}^1_k \to X$ induces an injective homomorphism $p^* : \mathcal{O}_{ch}(X) \to \mathcal{O}_{ch}(X \times \mathbb{A}^1_k)$. For the surjectivity, suppose $f \in \mathcal{O}_{ch}(X \times_k \mathbb{A}^1_k)$. For each $x \in X(k)$, consider the line $j_x : \mathbb{A}^1_k \to X \times_k \mathbb{A}^1_k$ defined as $t \mapsto (x, t)$. As $f \in \mathcal{O}_{ch}(X \times_k \mathbb{A}^1_k)$, $f \circ j_x$ is constant. Thus for every $x \in X(k)$ and $t \in \mathbb{A}^1_k(k)$, we have $f \circ i_0 \circ p(x, t) = f(x, t)$ (where $i_0 : X \to X \times_k \mathbb{A}^1_k$ is the 0-section). Since $X(k)$ is dense in X, $f = f \circ i_0 \circ p$. So $p^*(f \circ i_0) = f$, which proves the surjectivity of p^*. □

Remark 10.2.8. Makar-Limanov invariant is not homotopy invariant. If X is the Koras-Russell cubic threefold over $\mathbb{C}$, then $ML(X) = \mathbb{C}[T]$ ([54, Theorem 9.9]) and $ML(X \times \mathbb{A}^1_{\mathbb{C}}) = \mathbb{C}$ ([48, Section 1]). From Proposition 10.2.6 and homotopy invariance of $\mathcal{O}_{ch}(X)$, we have $\mathcal{O}_{ch}(X) = \mathcal{O}_{ch}(X \times \mathbb{A}^1_{\mathbb{C}}) = \mathbb{C}$.

Remark 10.2.9. For an affine variety $X \in Sm/k$, recall the sheaf $\mathcal{S}(X)$ of $\mathbb{A}^1$-chain connected components of X is the coequalizer of two morphisms

$$\underline{Hom}(\mathbb{A}^1_k, X) \underset{\theta_1}{\overset{\theta_0}{\rightrightarrows}} X$$

in $Shv(Sm/k)$ (Remark 3.2.7). There is a canonical isomorphism

$$Hom_{Sch/k}(X, \mathbb{A}^1_k) \cong \mathcal{O}(X).$$

The natural epimorphism $\pi : X \to \mathcal{S}(X)$ induces a monomorphism

$$Hom_{Sh(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k) \to Hom_{Sm/k}(X, \mathbb{A}^1_k).$$

This way we identify $Hom_{Sh(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k)$ as a k-subalgebra of $\mathcal{O}(X)$.

Proposition 10.2.10. *As k-subalgebras of $O(X)$, we have*

$$\mathcal{O}_{ch}(X) = Hom_{Sh(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k).$$

Proof. Suppose $\phi \in \mathcal{O}_{ch}(X)$, ϕ gives a morphism X to $\mathbb{A}^1_k$. We will show that $\phi \circ \theta_0 = \phi \circ \theta_1$ as morphisms from $\underline{Hom}(\mathbb{A}^1_k, X)$ to $\mathbb{A}^1_k$ i.e. to show that $\forall\ U \in Sm/k$ and $f : \mathbb{A}^1_U := \mathbb{A}^1_k \times_k U \to X$, $\phi \circ f \circ \sigma_0 = \phi \circ f \circ \sigma_1$ where $\sigma_0, \sigma_1 : U \to \mathbb{A}^1_U$ are the 0-section and the

1-section respectively. For any $x \in U(k)$, consider the morphism $i_x : \mathbb{A}^1_k \to \mathbb{A}^1_U$ defined as $t \mapsto (t, x)$. Composing f with i_x, we get a morphism from $\mathbb{A}^1_k$ to X. Since $\phi \in \mathcal{O}_{ch}(X)$, $\phi \circ f \circ i_x$ is constant. Thus $\phi(f(\sigma_0(x))) = \phi(f(\sigma_1(x)))$. The k-points are dense in U, so $\phi \circ f \circ \sigma_0 = \phi \circ f \circ \sigma_1$. Therefore, ϕ induces a unique morphism of sheaves from $\mathcal{S}(X)$ to $\mathbb{A}^1_k$. Thus using the identification we have observed $Hom_{Shv(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k)$ as k-subalgebra of $\mathcal{O}(X)$, we have $\phi \in Hom_{Shv(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k)$.

Conversely suppose, $\eta \in Hom_{Shv(Sm/k)}(\mathcal{S}(X), \mathbb{A}^1_k)$ and $g : \mathbb{A}^1_k \to X$ is a morphism . Then $\eta \circ \pi$ gives a morphism from X to $\mathbb{A}^1_k$. The morphism g is homotopic to the constant map. Indeed, there is a homotopy $H : \mathbb{A}^1_k \times \mathbb{A}^1_k \to X$ as the composition of g with the multiplication map $\mathbb{A}^1_k \times \mathbb{A}^1_k \to \mathbb{A}^1_k$ $((s,t) \mapsto st)$. Then $H\sigma_0$ is the constant map and $H\sigma_1 = g$, where $\sigma_0, \sigma_1 : \mathbb{A}^1_k \to \mathbb{A}^1_k \times \mathbb{A}^1_k$ are the 0-section and the 1-section (where we put 0 and 1 in the first coordinate) respectively. Since $\mathcal{S}(X)$ is the coequaliser of θ_0 and θ_1 (Remark 3.2.7, (1)), we have

$$\eta \circ \pi \circ H \circ \sigma_0 = \eta \circ \pi \circ H \circ \sigma_1$$

So $\eta \circ \pi \circ g$ is constant. This implies $\eta \circ \pi \in \mathcal{O}_{ch}(X)$. Now using the identification in remark 10.2.9 we get $\eta \in \mathcal{O}_{ch}(X)$. □

Remark 10.2.11. If any two k-points of X are joined by a chain of $\mathbb{A}^1$'s i.e. $\mathcal{S}(X)(Spec\ k)$ trivial, then any morphism $\mathcal{S}(X) \to \mathbb{A}^1_k$ factors through $Spec\ k$. Indeed, suppose that $\mathcal{S}(X)(Spec\ k)$ is trivial and $\phi : \mathcal{S}(X) \to \mathbb{A}^1_k$ is a morphism. Suppose that β is the singleton image of the map $\mathcal{S}(X)(Spec\ k) \to \mathbb{A}^1_k(k)$. Then for any $f \in \mathcal{S}(X)(U)$ ($U \in Sm/k$), the morphism $\phi(f) : U \to \mathbb{A}^1_k$ has the property that for every $x, y \in U(k)$, $\phi(f)(x) = \phi(f)(y) = \beta$. Since k is algebraically closed, $\phi(f)(x) = \beta$, for every $x \in U$. Hence ϕ factors as

$$\mathcal{S}(X) \to Spec\ k \xrightarrow{\beta} \mathbb{A}^1_k.$$

In the above argument (Remark 10.2.11), if we replace the condition $\mathcal{S}(X)$ is trivial by a sheaf $\mathcal{F}$ on Sm/k such that $\mathcal{F}(Spec\ k)$ is trivial and replace $\mathbb{A}^1_k$ by some $Y \in Sm/k$, then the same argument works. This gives the following lemma which is of independent interest.

Lemma 10.2.12. *Suppose $Y \in Sm/k$ and $\mathcal{F}$ is a sheaf on Sm/k such that $\mathcal{F}(Spec\ k)$ is trivial. Then the canonical map*

$$Y(Spec\ k) \to Hom_{Sh(Sm/k)}(\mathcal{F}, Y)$$

is a bijection.

Corollary 10.2.13. *Suppose X is a smooth affine k-surface such that $\mathcal{S}(X)(Spec\ k)$ is trivial or X is $\mathbb{A}^1$-chain connected. Then $\mathcal{O}_{ch}(X)$ is trivial.*

Proof. The corollary follows from Proposition 10.2.10 and Remark 10.2.11. □

Remark 10.2.14. Koras-Russell threefolds of the first kind are $\mathbb{A}^1$-chain connected [47, Example 2.21]. Therefore if X is a Koras-Russell threefold of the first kind, then $\mathcal{O}_{ch}(X)$ is trivial.

So far we have observed that $\mathcal{O}_{ch}(-)$, the presheaf of k-algebras on the category of affine varieties over k is homotopy invariant and it is directly related to $\mathbb{A}^1$-chain-connected component sheaf (Proposition 10.2.10). However $\mathcal{O}_{ch}(-)$ is not representable in $\mathbf{H}(\mathbf{k})$.

Lemma 10.2.15. *$\mathcal{O}_{ch}(-)$ is not representable in $\mathbf{H}(\mathbf{k})$.*

Proof. If possible, $\mathcal{O}_{ch}$ is given by $\mathbb{A}^1$-connected component presheaf of some $\mathbb{A}^1$-fibrant object $\mathcal{X}$. Then $\mathcal{O}_{ch}$ satisfies the gluing property (Remark 3.4.5), for any elementary distinguished square as in 2.2.2. Suppose, X is the affine line $\mathbb{A}^1_k$ and the elementary distinguished square (Definition 2.2.2) is given by Zariski open covering $U = \mathbb{A}^1_k - \{0\}$ and $V = \mathbb{A}^1_k - \{1\}$. Then both U and V contain no affine lines so, $\mathcal{O}_{ch}(U) = \mathcal{O}(U)$ and $\mathcal{O}_{ch}(V) = \mathcal{O}(V)$. So $\mathcal{O}(U) \times_{\mathcal{O}(U\cap V)} \mathcal{O}(V) = \mathcal{O}(\mathbb{A}^1_k)$. But $\mathcal{O}_{ch}(\mathbb{A}^1_k)$ is trivial. So the map $\mathcal{O}_{ch}(\mathbb{A}^1_k)$ to $\mathcal{O}_{ch}(U) \times_{\mathcal{O}_{ch}(U\cap V)} \mathcal{O}_{ch}(V)$ is not surjective. Hence $\mathcal{O}_{ch}$ does not satisfy the gluing property for this elementry distinguished square. Therefore, $\mathcal{O}_{ch}(-)$ is not representable in $\mathbf{H}(\mathbf{k})$. □

Remark 10.2.16. For an affine variety $X \in Sm/k$, let X_{Nis} be the small Nisnevich site. Then $\mathcal{O}_{ch}|_{X_{Nis}}$ is not a sheaf whenever $\mathcal{O}_{ch}(X) \neq \mathcal{O}(X)$ by Lemma 3.1.4. If $\mathcal{O}_{ch}(X) = \mathcal{O}(X)$, then $\mathcal{O}_{ch}|_{X_{Nis}}$ is a sheaf.

Question 10.2.17. Let $X \in Sm/k$ be an affine surface such that $\mathcal{O}(X)$ is a U.F.D. Suppose $\mathcal{O}_{ch}(X)$ is trivial. Is $X \cong \mathbb{A}^2_k$?

Remark 10.2.18. If X is a smooth affine surface with $\mathcal{O}(X)$ is a U.F.D. and $\mathcal{O}_{ch}(X)$ is trivial, then $\mathcal{O}(X)^* = k^*$ (by Lemma 10.1.2, (4)) and by Theorem 10.2.5, there is $T \subset X$ dense in X such that for each $x \in T(k)$ there is a non-constant morphism $g : \mathbb{A}^1_k \to X$ such that $x \in Im(g)$. But from this, we cannot conclude that X is dominated by images of $\mathbb{A}^1$ (Definition 5.1.1) which ensures the negativity of logarithmic Kodaira dimension of X.

However, there are singular affine surfaces with $\mathcal{O}(X)$ is U.F.D and $\mathcal{O}_{ch}(X)$ is trivial. Consider, for instance the singular Pham-Brieskorn surfaces $X_{p,q,r} = \{x^p + y^q + z^r = 0\} \subset \mathbb{A}^3_k$, where $p, q, r \geq 2$ are pairwise relatively prime integers. Then each $X_{p,q,r}$ is factorial [112, Section 4, Example (c)]. On the other hand, since $X_{p,q,r}$ is the affine cone over the closed curve $C_{p,q,r} = \{x^p + y^q + z^r = 0\}$ in the weighted projective space $\mathbb{P}(m/p, m/q, m/r)$ where $m = lcm(p, q, r)$, it is $\mathbb{A}^1$-chain connected in the naive sense that any two k-points of $X_{p,q,r}$ can be connected by finitely many non-constant images of $\mathbb{A}^1_k$, which implies $\mathcal{O}_{ch}(X_{p,q,r}) = k$.

However, we have the following characterisation of the affine plane using $\mathcal{O}_{ch}(-)$, which is somewhat straightforward.

Theorem 10.2.19. *Let k be an algebraically closed field of characteristic zero and X be a smooth affine surface over k. Suppose that $\mathcal{O}(X)$ is a U.F.D. and $\mathcal{O}_{ch}(X)$ is trivial. Then X is isomorphic to $\mathbb{A}^2_k$ if and only if $Sing_*(X)(Spec\ k)$ is Kan fibrant in degree 2 (see Definition 7.2.1).*

Proof. Since $\mathcal{O}_{ch}(X)$ is trivial, there are two distinct $\mathbb{A}^1$'s (the images are distinct) in X that intersect Lemma 10.2.2. There is a dominant morphism $\phi : \mathbb{A}^2_k \to X$, since $Sing_*(X)(Spec\ k)$ is Kan fibrant in degree 2. Thus X has negative logarithmic Kodaira dimension by [109, Lemma 1.8]. By Castelnuovo's rationality criterion, X is a rational surface [25, Theorem 13.27]. Therefore X is isomorphic to $\mathbb{A}^2_k$ by [109, Theorem 2]. □

Remark 10.2.20. We have seen $\mathcal{O}_{ch}(X)$ as the regular functions on $\mathbb{A}^1$-chain connected components of X [Proposition 10.2.10]. We define,

$$\mathcal{O}^{(n)}_{ch}(X) := Hom_{Sh(Sm/k)}(\mathcal{S}^n(X), \mathbb{A}^1_k).$$

Then $\mathcal{O}^{(n)}_{ch}(X)$ is a k-subalgebra of $\mathcal{O}(X)$ and if there is n such that any two k-points of X are n-$\mathbb{A}^1$-ghost homotopic i.e. $\mathcal{S}^n(X)(Spec\ k)$ is trivial, then $\mathcal{O}^{(n)}_{ch}(X) = k$ (Lemma 10.2.12). Since there is an epimorphism from $\mathcal{S}^n(X)$ to $\mathcal{S}^{n+1}(X)$, we have $\mathcal{O}^{(n+1)}_{ch}(X) \subset \mathcal{O}^{(n)}_{ch}(X)$. Thus inside $\mathcal{O}(X)$, we have a decreasing chain of k-subalgebras $\{\mathcal{O}^{(n)}_{ch}(X)\}_n$, not necessarily Noetherian. Does the above chain of k-subalgebras stabilize?

Remark 10.2.21. Suppose $X \in Sm/k$ is $\mathbb{A}^1$-connected. Then there is an n such that $\mathcal{S}^n(X)$ is trivial. Indeed, if X is $\mathbb{A}^1$-connected, then $\mathcal{L}(X)$ is trivial [26, Corollary 2.18]. Since $\mathcal{L}(X)$ is trivial, the identity map Id_X on X and the constant map C_{x_0} given by a k-rational point x_0 (X has a k-rational point, since X is $\mathbb{A}^1$-connected) are the same in $\mathcal{L}(X)(X)$. Thus there is a Nisnevich covering $f : Y \to X$ and an n such that $Id_X \circ f = C_{x_0} \circ f$ in $\mathcal{S}^n(X)(Y)$. Since $\mathcal{S}^n(X)$ is a sheaf, $Id_X = C_{x_0} \in \mathcal{S}^n(X)(X)$. Thus any map from $\phi : Spec\ \mathcal{O} \to X$ ($\mathcal{O}$ is a smooth Henselian local ring) is the same with the constant map $C_{x_0} \circ \phi$ in $\mathcal{S}^n(X)(Spec\ \mathcal{O})$. Thus $\mathcal{S}^n(X)(Spec\ \mathcal{O})$ is trivial, for every smooth Henselian local ring $\mathcal{O}$ and hence $\mathcal{S}^n(X)$ is the trivial sheaf. Therefore, if X is an $\mathbb{A}^1$-connected smooth affine k-variety, then the chain $\{\mathcal{O}^{(n)}_{ch}(X)\}_n$ in Remark 10.2.20 stabilizes for some n.

Question 10.2.22. Given an affine variety X, define X_i inductively as follows: $X_0 = X$ and $X_i = Spec(\mathcal{O}_{ch}(X_{i-1}))$. What is the relation between X_i and the spectrum of $\mathcal{O}^{(i)}_{ch}(X)$?

There is a canonical map from X_i to X_{i+1} for every i. Note that if X does not have any non-constant $\mathbb{A}^1_k$, then $\mathcal{O}_{ch}(X) = \mathcal{O}(X)$. Does there exist some n such that X_n has no non-constant $\mathbb{A}^1$ (see also Remark 10.1.3)?

Chapter 11

Naive 0-th $\mathbb{A}^1$-homology

In this chapter we define the universal $\mathbb{A}^1$-invariant sheaf of abelian groups $H_0^{\mathbb{A}^1\text{-naive}}(X)$ associated to a scheme $X \in Sm/k$ (Definition 11.1.2). Its section agrees with the section of the sheaf of free abelian groups $\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))$ over $Spec\ F$, for a proper scheme $X \in Sm/k$ and for every finitely generated separable field extension F/k (Theorem 11.2.11). This is the main theorem in this chapter. As a consequence in Corollary 11.2.12, we prove that a smooth proper scheme X is $\mathbb{A}^1$-connected if and only if $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to the constant sheaf $\mathbb{Z}$. There is a canonical morphism $H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1}(X)$ and if $X \in Sm/k$ is a proper scheme, it is an isomorphism over the sections $Spec\ F$, for every finitely generated separable field extension F/k (Remark 11.2.15). For any scheme $X \in Sm/k$, the canonical epimorphism $X \to \pi_0^{\mathbb{A}^1}(X)$ induces the isomorphism (Corollary 11.2.17, see also Remark 11.2.16)

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1\text{-naive}}(\pi_0^{\mathbb{A}^1}(X)).$$

Throughout this chapter we assume k to be an algebraically closed field.

11.1 Naive 0-th $\mathbb{A}^1$-homology sheaf

In this section we define naive 0-th $\mathbb{A}^1$-homology sheaf $H_0^{\mathbb{A}^1\text{- naive}}(X)$ associated to $X \in Sm/k$ (Definition 11.1.2). The sheaf $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is universal in the sense that a morphism from X to an $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathcal{G}$ uniquely factors through $H_0^{\mathbb{A}^1\text{-naive}}(X)$ (Remark 11.1.3). In Theorem 11.1.5, we give several equivalent descriptions of $H_0^{\mathbb{A}^1\text{-naive}}(X)$, for example it is isomorphic to the universal $\mathbb{A}^1$-invariant sheaf $\mathcal{L}(\mathbb{Z}(X))$ (Proposition 11.1.4, Definition 3.2.6). In Corollary 11.1.6, we prove that if X is $\mathbb{A}^1$-connected, then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ isomorphic to the constant sheaf $\mathbb{Z}$.

Suppose $\mathcal{F}$ is a Nisnevich sheaf on Sm/k. The presheaf of free abelian groups on $\mathcal{F}$, denoted by $\mathbb{Z}^{pre}(\mathcal{F})$, is defined as

$$U \in Sm/k \mapsto \text{the free abelian group on } \mathcal{F}(U).$$

The Nisnevich sheaf associated to $\mathbb{Z}^{\mathsf{pre}}(\mathcal{F})$ is called the sheaf of free abelian groups on $\mathcal{F}$ and it is denoted by $\mathbb{Z}(\mathcal{F})$. For example, the constant sheaf $\mathbb{Z}$ is the Nisnevich sheaf $\mathbb{Z}(Spec\ k)$. We will denote the category of sheaves of abelian groups on Sm/k by $\mathcal{A}b(k)$ and its full subcategory of $\mathbb{A}^1$-invariant sheaves of abelian groups on Sm/k is denoted by $\mathcal{A}b_{\mathbb{A}^1\text{-inv}}(k)$. The morphisms in $\mathcal{A}b(k)$ are sectionwise group homomorphisms.

Remark 11.1.1. There is a canonical morphism $\mathcal{F} \to \mathbb{Z}(\mathcal{F})$. This induces a map

$$Hom_{\mathcal{A}b(k)}(\mathbb{Z}(\mathcal{F}), \mathcal{G}) \to Hom_{Sh(Sm/k)}(\mathcal{F}, \mathcal{G})$$

which is an isomorphism, for any sheaf of abelian groups $\mathcal{G}$.

Definition 11.1.2. Suppose, $X \in Sm/k$. The naive 0-th $\mathbb{A}^1$-homology sheaf of X is defined to be the Nisnevich sheaf of abelian groups $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(X))$ and it is denoted by $H_0^{\mathbb{A}^1\text{-naive}}(X)$.

Remark 11.1.3. $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups on Sm/k [37, Corollary 5.2]. There is a canonical morphism $X \to H_0^{\mathbb{A}^1\text{-naive}}(X)$. This induces a map

$$Hom_{\mathcal{A}b_{\mathbb{A}^1\text{-inv}}(k)}(H_0^{\mathbb{A}^1\text{-naive}}(X), \mathcal{G}) \to Hom_{Sh(Sm/k)}(X, \mathcal{G})$$

which is a bijection, for any $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathcal{G}$. Equivalently, any morphism of sheaves $X \to \mathcal{G}$ uniquely factors through the morphism $H_0^{\mathbb{A}^1\text{-naive}}(X) \to \mathcal{G}$, for any $\mathbb{A}^1$-invariant sheaf of abelian groups $\mathcal{G}$. Thus the canonical morphism $X \to H_0^{\mathbb{A}^1}(X)$ uniquely factors through

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1}(X)$$

since $H_0^{\mathbb{A}^1}(X)$ is a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups [93, Corollary 6.31].

The following Proposition relates $H_0^{\mathbb{A}^1\text{-naive}}(X)$ with the universal $\mathbb{A}^1$-invariant sheaf (Definition 3.2.6).

Proposition 11.1.4. $H_0^{\mathbb{A}^1\text{-naive}}(X) \cong \mathcal{L}(\mathbb{Z}(X))$.

Proof. Since $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(X))$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups [37, Corollary 5.2], the canonical map

$$\pi_0^{\mathbb{A}^1}(\mathbb{Z}(X)) \to \mathcal{L}(\mathbb{Z}(X))$$

is an isomorphism by [26, Corollary 2.18]. □

There are also several descriptions of $H_0^{\mathbb{A}^1\text{-naive}}(X)$.

Theorem 11.1.5. *The following $\mathbb{A}^1$-invariant sheaves of abelian groups are pairwise isomorphic in $\mathcal{A}b_{\mathbb{A}^1\text{-inv}}(k)$:*

$$\begin{aligned} H_0^{\mathbb{A}^1\text{-naive}}(X) &\cong \mathcal{L}(\mathbb{Z}(X)) \cong \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X)))) \cong \mathcal{L}(\mathbb{Z}(\mathcal{L}(X))) \\ &\cong \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X))) \cong \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))) \cong \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))). \end{aligned}$$

Proof. 1. The first isomorphism follows from Proposition 11.1.4.

2. The natural morphism $\mathbb{Z}(X) \to \pi_0(Sing_*(\mathbb{Z}(X)))$ induces a morphism

$$\phi : \mathcal{L}(\mathbb{Z}(X)) \to \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X)))).$$

The morphism $\pi_0(Sing_*(\mathbb{Z}(X))) \to \mathcal{L}(\mathbb{Z}(X))$ factors uniquely through the morphism

$$\psi : \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X)))) \to \mathcal{L}(\mathbb{Z}(X))$$

(Remark 3.2.7). According to the construction of ϕ and ψ, the canonical morphism $\mathbb{Z}(X) \to \mathcal{L}(\mathbb{Z}(X))$ factors throgh $\psi \circ \phi$. Thus from the universal property of $\mathcal{L}(\mathbb{Z}(X))$, $\psi \circ \phi$ is the identity. Similarly, $\phi \circ \psi$ is the identity. Hence the second isomorphism follows.

3. The canonical morphism $X \to \mathcal{L}(X)$ induces a morphism $\mathbb{Z}(X) \to \mathbb{Z}(\mathcal{L}(X))$. This induces a morphism $\pi_0(Sing_*(\mathbb{Z}(X))) \to \mathcal{L}(\mathbb{Z}(\mathcal{L}(X)))$. This factors uniquely through the morphism

$$\phi : \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X)))) \to \mathcal{L}(\mathbb{Z}(\mathcal{L}(X))),$$

since $\mathcal{L}(\mathbb{Z}(\mathcal{L}(X)))$ is $\mathbb{A}^1$-invariant (Remark 3.2.7).

There is a canonical morphism $X \to \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X))))$. It factors through $\psi_0 : \mathcal{L}(X) \to \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X))))$. Since $\mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X))))$ is sheaf of abelian groups, ψ_0 factors through $\psi_1 : \mathbb{Z}(\mathcal{L}(X)) \to \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X))))$. Since $\mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X))))$ is $\mathbb{A}^1$-invariant, ψ_1 factors through

$$\psi : \mathcal{L}(\mathbb{Z}(\mathcal{L}(X))) \to \mathcal{L}(\pi_0(Sing_*(\mathbb{Z}(X)))).$$

From the construction of ϕ and ψ, the map $X \to \mathcal{L}(\mathbb{Z}(\mathcal{L}(X)))$ factors through $\phi \circ \psi$. Therefore by the universal property of $\mathcal{L}(\mathbb{Z}(\mathcal{L}(X)))$, $\phi \circ \psi$ is identity. Similarly, $\psi \circ \phi$ is identity. Hence the third isomorphism follows.

4. $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X)))$ is $\mathbb{A}^1$-invariant sheaf of abelian groups [37, Corollary 5.2]. Thus the canonical morphism $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X))) \to \mathcal{L}(\mathbb{Z}(\mathcal{L}(X)))$ is an isomorphism [26, Corollary 2.18].

5. There is a canonical morphism $X \to \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$. It factors uniquely through the morphism $\phi_1 : \mathcal{L}(X) \to \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$, since $\mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$ is $\mathbb{A}^1$-invariant. The morphism ϕ_1 factors through $\phi_2 : \mathbb{Z}(\mathcal{L}(X)) \to \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$, since $\mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$ is a sheaf of abelian groups. By the universal property of $\pi_0^{\mathbb{A}^1}(-)$ (Lemma 3.2.5), the morphism ϕ_2 uniquely factors through

$$\phi : \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X))) \to \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))).$$

The morphism $X \to \mathbb{Z}(\mathcal{L}(X))$ induces a morphism $\pi_0^{\mathbb{A}^1}(X) \to \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X)))$. It factors uniquely through the morphism $\psi_0 : \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)) \to \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X)))$. Since $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X)))$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups [37, Corollary 5.2], ψ_0 factors through the morphism

$$\psi : \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))) \to \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X))).$$

The canonical map $X \to \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{L}(X)))$ factors through $\psi \circ \phi$. Therefore $\psi \circ \phi$ is the identity map. Similarly, $\phi \circ \psi$ is also the identity. Hence the fifth isomorphism follows.

6. $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups [37, Corollary 5.2]. Therefore the canonical morphism $\pi_0^{\mathbb{A}^1}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))) \to \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)))$ is an isomorphim [26, Corollary 2.18].

□

Corollary 11.1.6. *Suppose, $X \in Sm/k$ is $\mathbb{A}^1$-connected. Then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to the constant sheaf $\mathbb{Z}$.*

Proof. By Theorem 11.1.5, there is an isomorphism

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to \mathcal{L}(\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))).$$

Since X is $\mathbb{A}^1$-connected and the constant sheaf $\mathbb{Z}$ is $\mathbb{A}^1$-invariant, so $H_0^{\mathbb{A}^1\text{-naive}}(X) \cong \mathbb{Z}$. □

11.2 Sections of $H_0^{\mathbb{A}^1\text{-naive}}(X)$, for a proper scheme X

In this section we describe some useful properties of $H_0^{\mathbb{A}^1\text{-naive}}(X)$, for a proper scheme $X \in Sm/k$. In Theorem 11.2.11, we prove that if X is a smooth proper k-scheme, then the canonical epimorphism

$$\eta : \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)) \to \mathcal{L}(\mathbb{Z}(X))$$

induces isomorphism over the sections $Spec\ F$, for every finitely generated separable field extensions F/k.

Definition 11.2.1. A sheaf of sets $\mathcal{F}$ on Sm/k is called injective over dominant morphisms if for any dominant morphism $U \to X$ in Sm/k, the restriction map $\mathcal{F}(X) \to \mathcal{F}(U)$ is injective.

Lemma 11.2.2. *Suppose, $X \in Sm/k$ and U is any k-scheme. If two morphisms $f, g : U \to X$ agree on a dense subset V of U, then $f = g$.*

Proof. Since X is separated, the image of the diagonal morphism $\Delta : X \to X \times_k X$ is closed in $X \times_k X$. Consider the morphism

$$(f, g) : U \to X \times_k X \text{ defined as } x \mapsto (f(x), g(x))$$

Then $(f, g)^{-1}(\Delta(X))$ is closed in X and contains V, so $(f, g)^{-1}(\Delta(X)) = U$. Thus, for each $x \in U$, we have $f(x) = g(x)$. □

Remark 11.2.3. Therefore any representable sheaf $X \in Sm/k$ is injective over dominant morphisms.

The proof of Lemma 11.2.4 and Theorem 11.2.5 are taken from [37, Lemma 5.3] and [37, Lemma 5.4] respectively. We include these here for the sake of completeness.

Lemma 11.2.4. *Suppose, a Nisnevich sheaf $\mathcal{F}$ is injective over dominant morphisms. Then for a dominant morphism $\phi : U \to V$ in Sm/k, the induced map*

$$\phi^*_{\mathbb{Z}} : \mathbb{Z}^{pre}(\mathcal{F})(V) \to \mathbb{Z}^{pre}(\mathcal{F})(U)$$

is an injective group homomorphism.

Proof. Suppose $\alpha \in \mathbb{Z}^{pre}(\mathcal{F})(V)$ such that $\phi^*_{\mathbb{Z}}(\alpha) = 0$. Moreover assume that α is given by $\sum_{i=1}^{l} n_i\alpha_i$, where $\alpha_i \in \mathcal{F}(V)$ and $\alpha_i \neq \alpha_j$ for all $i \neq j$. Then $\phi^*_{\mathbb{Z}}(\alpha) = \sum_{i=0}^{l} n_i\phi^*(\alpha_i)$ (where $\phi^* : \mathcal{F}(V) \to \mathcal{F}(U)$ is induced by ϕ). Since $\mathcal{F}$ is injective over dominant morphisms, $\phi^*(\alpha_i) \neq \phi^*(\alpha_j)$ for all $i \neq j$. Thus $\phi^*_{\mathbb{Z}}(\alpha) = 0$ implies that $n_i = 0$, for every i. Hence $\alpha = 0$ and consequently $\phi^*_{\mathbb{Z}}$ is injective. □

Theorem 11.2.5. *Suppose $\mathcal{F}$ is a Nisnevich sheaf which is injective over dominant morphisms. Then $\mathbb{Z}^{pre}(\mathcal{F})$ is an almost sheaf i.e. $\mathbb{Z}(\mathcal{F})$ is given by*

$$\mathbb{Z}(\mathcal{F})(U) = \mathbb{Z}^{pre}(\mathcal{F})(U) \text{ for } U \in Sm/k \text{ irreducible and}$$

$$\mathbb{Z}(\mathcal{F})(\coprod_{i=1}^{n} U_i) = \prod_{i=1}^{n} \mathbb{Z}^{pre}(\mathcal{F})(U_i), \text{ for } U = \coprod_{i=1}^{n} U_i \text{ with } U_i \text{ 's are connected components of } U.$$

Proof. We show that the above description of $\mathbb{Z}(\mathcal{F})$ is a Nisnevich sheaf. We need to show that for a elementary distinguished square in Sm/k (Definition 2.2.2)

$$\begin{array}{ccc} U \times_X V & \longrightarrow & U \\ \downarrow & & \downarrow p \\ V & \xrightarrow{j} & X \end{array}$$

where p is an étale morphism and j is an open embedding, the corresponding square of abelian groups

$$\begin{array}{ccc} \mathbb{Z}(\mathcal{F})(X) & \xrightarrow{p^*} & \mathbb{Z}(\mathcal{F})(U) \\ \downarrow j^* & & \downarrow \\ \mathbb{Z}(\mathcal{F})(V) & \longrightarrow & \mathbb{Z}(\mathcal{F})(U \times_X V) \end{array}$$

is cartesian i.e. the map $(p^*, j^*) : \mathbb{Z}(\mathcal{F})(X) \to \mathbb{Z}(\mathcal{F})(U) \times_{\mathbb{Z}(\mathcal{F})(U\times_X V)} \mathbb{Z}(\mathcal{F})(V)$ between abelian groups is an isomorphism (Remark 2.2.3). We can assume that $X, U, V \in Sm/k$ are irreducibles. Since j is dominant, thus j^* is injective and hence (p^*, j^*) is injective. For surjevtivity, suppose $\beta \in \mathbb{Z}(\mathcal{F})(U)$ and $\gamma \in \mathbb{Z}(\mathcal{F})(V)$ such that $\beta|_{U\times_X V} = \gamma|_{U\times_X V}$. Suppose

$\beta = \sum_{i=1}^{t} n_i\beta_i,\ \gamma = \sum_{j=1}^{l} m_j\gamma_j$ with $\beta_i \in \mathcal{F}(U), \gamma_j \in \mathcal{F}(V)$ for every i, j and $\beta_i \neq \beta_{i'}, \gamma_j \neq \gamma_{j'}$ for every $i \neq i', j \neq j'$. Then

$$\sum_{i=1}^{t} n_i\beta_i|_{U\times_X V} = \sum_{j=1}^{l} m_j\gamma_j|_{U\times_X V} \qquad (*)$$

Since $\mathcal{F}$ is injective over dominant morphisms and the morphisms $U \times_X V \to U$ and $U\times_X V \to V$ are dominant, thus $\beta_i|_{U\times_X V}$'s are distinct and $\gamma_j|_{U\times_X V}$'s are distinct. Therefore in the expression (*), $t = l$ and for every i there is unique $j(i)$ such that $n_i = m_{j(i)}$ and $\beta_i|_{U\times_X V} = \gamma_{j(i)}|_{U\times_X V}$. Since $\mathcal{F}$ is a Nisnevich sheaf, for such $i, j(i)$ we can glue β_i and $\gamma_{j(i)}$ to get a section $\delta_{i,j(i)} \in \mathcal{F}(X)$ such that $\delta_{i,j(i)}|_U = \beta_i$ and $\delta_{i,j(i)}|_V = \gamma_{j(i)}$. Then $\sum_{i=1}^{t} n_i\delta_{i,j(i)}$ maps to (β, γ) under (p^*, j^*). Therefore, $\mathbb{Z}^{pre}(\mathcal{F})$ is an almost sheaf. □

Remark 11.2.6. Thus if $\mathcal{F}$ is a sheaf which is injective over dominant morphisms, then the section $\mathbb{Z}(\mathcal{F})(U)$ for $U \in Sm/k$ irreducible, is the free abelian group on the set $\mathcal{F}(U)$. For a representable sheaf $X \in Sm/k$, $\mathbb{Z}^{pre}(X)$ is an almost sheaf (Remark 11.2.3).

The canonical morphism $\mathcal{F} \to \mathbb{Z}(\mathcal{F})$ induces a morphism

$$\theta : \mathcal{S}^n(\mathcal{F}) \to \mathcal{S}^n(\mathbb{Z}(\mathcal{F})).$$

The morphism θ factors through

$$\phi : \mathbb{Z}(\mathcal{S}^n(\mathcal{F})) \to \mathcal{S}^n(\mathbb{Z}(\mathcal{F}))$$

since $\mathcal{S}^n(\mathbb{Z}(\mathcal{F}))$ is a sheaf of abelian groups (Remark 11.1.1). There is a commutative triangle consisting morphism of sheaves of abelian groups

$$\begin{array}{ccc} \mathbb{Z}(\mathcal{S}^n(\mathcal{F})) & \xrightarrow{\ \phi\ } & \mathcal{S}^n(\mathbb{Z}(\mathcal{F})) \\ & \nwarrow \quad \nearrow_{\psi} & \\ & \mathbb{Z}(\mathcal{F}) & \end{array}$$

Since ψ is an epimorphism, ϕ is an epimorphism.

Theorem 11.2.7. *Suppose $\mathcal{F}$ is a Nisnevich sheaf which is injective over dominant morphisms. Then the epimorphism $\phi : \mathbb{Z}(\mathcal{S}(\mathcal{F})) \to \mathcal{S}(\mathbb{Z}(\mathcal{F}))$ is an isomorphism.*

Proof. We show that for every smooth Henselian local ring $\mathcal{O}$, the map

$$\phi_{\mathcal{O}} : \mathbb{Z}(\mathcal{S}(\mathcal{F}))(Spec\ \mathcal{O}) \to \mathcal{S}(\mathbb{Z}(\mathcal{F}))(Spec\ \mathcal{O}),$$

induced by ϕ, over the sections $Spec\ \mathcal{O}$, is an isomorphism of abelian groups. The map $\phi_{\mathcal{O}}$ is surjective, since ϕ is an epimorphism and the sections over $Spec\ \mathcal{O}$ are the stalks. For injectivity, suppose $\alpha \in \mathbb{Z}(\mathcal{S}(\mathcal{F}))(Spec\ \mathcal{O})$ such that $\phi_{\mathcal{O}}(\alpha) = 0$. Suppose that $\alpha = \sum_{i=1}^{k} n_i\alpha_i$, where

$\alpha_i \in \mathcal{F}(Spec\ \mathcal{O})$ for every i and $\alpha_i \neq \alpha_j$ in $\mathcal{S}(\mathcal{F})(Spec\ \mathcal{O})$ for every $i \neq j$. Since $\phi_{\mathcal{O}}(\alpha) = 0$, there is a chain of $\mathbb{A}^1$-homotopies $G_1, .., G_r \in \mathbb{Z}(\mathcal{F})(\mathbb{A}^1_{\mathcal{O}})$ such that $G_1(0) = \phi_{\mathcal{O}}(\alpha)$ and $G_r(1) = 0$. Since $\mathcal{F}$ is injective over dominant morphisms by Theorem 11.2.5, $\mathbb{Z}(\mathcal{F})(\mathbb{A}^1_{\mathcal{O}})$ is the free abelian group on $\mathcal{F}(\mathbb{A}^1_{\mathcal{O}})$. Suppose

$$G_j = \sum_{t=1}^{l_j} m_t^{(j)} H_t^{(j)},$$

then $G_1(0) = \sum_{t=1}^{l_1} m_t^{(1)} H_t^{(1)}(0) = \sum_{i=1}^{k} n_i \alpha_i$ and $G_r(1) = 0 = \sum_{t=1}^{l_r} m_t^{(r)} H_t^{(r)}(1)$.

For simplicity, we complete the rest of the proof for $r = 1$. Suppose, $G = \sum_{j=1}^{l} m_j H_j$ along with $G(0) = \sum_{j=1}^{l} m_j H_j(0) = \sum_{i=1}^{k} n_i \alpha_i \in \mathbb{Z}(\mathcal{F})(Spec\ \mathcal{O})$ and $G(1) = \sum_{j=1}^{l} m_j H_j(1) = 0$. Thus there is a partition of I (we can assume it is increasing)

$$I = \{1, 2, .., l\} = \coprod_{s=1}^{t} I_s,$$

where $I_s = \{r_{s-1}+1, r_{s-1}+2, .., r_s\}$, $1 \leq s \leq t$ and $r_0 = 0$, $r_t = l$. $1 \leq r_1 < r_2 < .. < r_p = l$ such that $\sum_{i \in I_s} m_i = 0$, for every s and for each fixed s, $H_i(1)$'s are equal for all $i \in I_s$. Now, for every s, $H_p(0) = H_q(0)$, if $p, q \in I_s$. Indeed, if there are $p, q \in I_s$ such that $H_p(0) \neq H_q(0)$, for some s, then from the expression $G(0) = \sum_{i=1}^{k} n_i \alpha_i \in \mathbb{Z}(\mathcal{F})(Spec\ \mathcal{O})$, we have $H_p(0) = \alpha_i$ and $H_q(0) = \alpha_j$ for some i, j and $i \neq j$. Thus $\alpha_i = \alpha_j \in \mathcal{S}(\mathcal{F})(Spec\ \mathcal{O})$, which is not possible. Therefore, for each i, there are $I_{s_1}^{(i)}, I_{s_2}^{(i)}, .., I_{s_{d_i}}^{(i)}$ such that $n_i = \sum_{j \in \cup_{v=1}^{d_i} I_{s_v}} m_j$. Thus $n_i = 0$, for every i and hence $\alpha = 0$. Therefore, $\phi_{\mathcal{O}}$ is a monomorphism and thus ϕ is an isomorphism. □

Using Remark 11.2.3 and Theorem 11.2.7, we have the following:

Corollary 11.2.8. *Suppose, $X \in Sm/k$. Then the canonical morphism*

$$\phi : \mathbb{Z}(\mathcal{S}(X)) \to \mathcal{S}(\mathbb{Z}(X))$$

is an isomorphism.,

Recall the notion of almost proper sheaf defined in [26, Definition 3.6]. Proper k-schemes are almost proper sheaves. We prove that for a proper scheme $X \in Sm/k$ and a finitely generated separable field extension F/k, the section of $H_0^{\mathbb{A}^1\text{-naive}}(X)$ over $Spec\ F$ is the free abelian group on $\pi_0^{\mathbb{A}^1}(X)(Spec\ F)$ (Theorem 11.2.11).

Lemma 11.2.9. *Suppose $X \in Sm/k$ is a proper scheme. Then $\mathbb{Z}(X)$ is an almost proper sheaf.*

Proof. Suppose $U \in Sm/k$ is an irreducible variety of dimension ≤ 2. An element $s \in \mathbb{Z}(X)(U)$ is given by $\sum_{i=1}^{n} m_i f_i$, where $m_i \in \mathbb{Z}$ and $f_i : U \to X$ is a morphism (Remark 11.2.6). Since X is proper, the morphism $F : U \to X \times_k X \times_k .. \times_k X$ given by $F(x) = (f_1(x), f_2(x), .., f_n(x))$ can be extended to a morphism $\bar{F} : \bar{U} \to X \times_k .. \times_k X$, for some compactification $\bar{U} \in Sm/k$. Thus (AP1) in [26, Definition 3.6] is satisfied. For (AP2), suppose $U \in Sm/k$ is an irreducible curve and U' is an open subscheme. Suppose that $s_1 = \sum_{i=1}^{p} a_i f_i,\ s_2 = \sum_{j=1}^{q} b_j g_j \in \mathbb{Z}(X)(U)$ and $s_1|_{U'} = s_2|_{U'}$. Since X is separated, so for every $i \neq i', j \neq j'$, $f_i|_{U'} \neq f_{i'}|_{U'}$ and $g_j|_{U'} \neq g_{j'}|_{U'}$ (Remark 11.2.3). Therefore, $p = q$ and for every i, a_i is same with exactly one b_j along with $f_i = g_j$. Thus $s_1 = s_2$. Thus (AP2) in [26, Definition 3.6] is satisfied. Hence $\mathbb{Z}(X)$ is an almost proper sheaf. □

Corollary 11.2.10. *Suppose $X \in Sm/k$ is a proper scheme. Then for any $n \geq 1$*

$$\mathcal{S}^n(\mathbb{Z}(X))(Spec\ F) = \mathbb{Z}(\mathcal{S}(X))(Spec\ F)$$

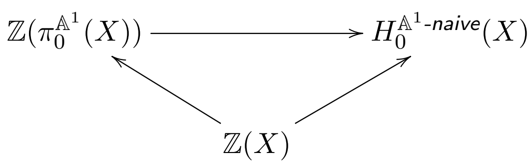
for every finitely generated separable field extension F/k and $n \geq 1$.

Proof. The Corollary follows from Lemma 11.2.9 and [26, Theorem 3.9]. □

The canonical morphism $X \to H_0^{\mathbb{A}^1\text{-naive}}(X)$ factors through $\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))$, By Proposition 11.1.4, we have an epimorphism of sheaves of abelian groups

$$\eta : \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X)) \to \mathcal{L}(\mathbb{Z}(X))$$

There is a commutative triangle consisting all epimorphisms of sheaves of abelian groups



Theorem 11.2.11. *Suppose, $X \in Sm/k$ is a proper scheme. Then for every finitely generated separable field extension F/k,*

$$\eta : \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))(Spec\ F) \to H_0^{\mathbb{A}^1\text{-naive}}(X)(Spec\ F)$$

is an isomorphism of abelian groups.

Proof. Using Proposition 11.1.4 we need to show, the canonical map

$$\eta : \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))(Spec\ F) \to \mathcal{L}(\mathbb{Z}(X))(Spec\ F)$$

is an isomorphism. Since sections over F/k are stalks,

$$\mathcal{L}(\mathbb{Z}(X))(Spec\ F) = \varinjlim_n \mathcal{S}^n(\mathbb{Z}(X))(Spec\ F)$$

in the category of abelian groups. By Corollary 11.2.10,

$$\mathcal{L}(\mathbb{Z}(X))(Spec\ F) = \mathbb{Z}(\mathcal{S}(X))(Spec\ F).$$

The morphism η is a surjection. For injectivity, suppose $\alpha = \sum_{i=1}^{k} n_i x_i$ is such that $\eta(\alpha) = 0 \in \mathbb{Z}(\mathcal{S}(X))(Spec\ F)$, where $n_i \in \mathbb{Z}$, $x_i \in X(F)$ for all i such that $x_i \neq x_j \in \pi_0^{\mathbb{A}^1}(X)(Spec\ F)$ for $i \neq j$. Then $x_i \neq x_j \in \mathcal{S}(X)(Spec\ F)$ for all $i \neq j$ by [11, Theorem 2.4.3]. Therefore $\eta(\alpha) = 0 \in \mathbb{Z}(\mathcal{S}(X))(Spec\ F)$ implies $n_i = 0$ for all i. Therefore, $\alpha = 0 \in \mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))(Spec\ F)$. Hence, η is injective. Therefore, η is an isomorphism over the section $Spec\ F$ for every finitely generated separable field extension F/k.. □

Corollary 11.2.12. *Let $X \in Sm/k$ be a proper scheme. Then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to $\mathbb{Z}$ if and only if X is $\mathbb{A}^1$-connected.*

Proof. The reverse implication follows from Theorem 11.1.6. For the forward implication, suppose that $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to the constant sheaf $\mathbb{Z}$. Then Theorem 11.2.11 the abelian group $\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))(Spec\ F)$ has rank 1, for every finitely generated separable field extension F/k. Thus $\pi_0^{\mathbb{A}^1}(X)(Spec\ F)$ is trivial. Therefore, X is $\mathbb{A}^1$-connected by [99, Lemma 3.3.6].

□

Corollary 11.2.13. *Suppose, X is a smooth proper surface over an algebraically closed field k of characteristic zero. Then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to $\mathbb{Z}$ if and only if X is a rational surface.*

Proof. The corollary follows from Corollary 11.2.12 and [11, Corollary 2.4.7]. □

Recall the birational and $\mathbb{A}^1$-invariant sheaf $\pi_0^{b\mathbb{A}^1}(X)$ introduced by Asok and Morel [11, Section 6], associated to a proper scheme $X \in Sm/k$. The section of $\pi_0^{b\mathbb{A}^1}(X)$ over $Spec Ł$ is the set of $\mathbb{A}^1$-equivalence classes of L-rational points [11, Proposition 6.2.6]. The free abelian presheaf $\mathbb{Z}^{pre}(\pi_0^{b\mathbb{A}^1}(X))$ is a birational and a strictly $\mathbb{A}^1$-invariant sheaf of abelian groups on Sm/k ([11, Section 6] and [13, Lemma 2.4]). So the canonical morphism $X \to \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$ uniquely factors through the morphism (Remark 11.1.3)

$$\theta : H_0^{\mathbb{A}^1\text{-naive}}(X) \to \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X)).$$

Corollary 11.2.14. *Suppose, $X \in Sm/k$ is a proper scheme. Then for every finitely generated separable field extension F/k, the canonical morphism θ induces*

$$\theta : H_0^{\mathbb{A}^1\text{-naive}}(X)(Spec\ F) \to \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))(Spec\ F)$$

an isomorphism of abelian groups.

Proof. Consider, the composition

$$\mathbb{Z}(\pi_0^{\mathbb{A}^1}(X))(Spec\ F) \xrightarrow{\eta} H_0^{\mathbb{A}^1\text{-naive}}(X)(Spec\ F) \xrightarrow{\theta} \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))(Spec\ F),.$$

By Theorem 11.2.11 and [11, Proposition 6.2.6], the morphism θ is an isomorphism over the section $Spec\ F$, for every finitely generated separable field extension F/k. □

Remark 11.2.15. Therefore if $X \in Sm/k$ is a proper scheme, then by [11, Theorem 2.4.3, Theorem 6.2.1] and from the isomorphism $H_0^{\mathbb{A}^1}(X) \to \mathbb{Z}(\pi_0^{b\mathbb{A}^1}(X))$ [78, Theorem 1.1] (see also [118, Theorem 2]), the canonical morphism (Remark 11.1.3)

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1}(X)$$

is an isomorphism over the sections of every finitely generated separable field extensions F/k.

Remark 11.2.16. Suppose, $\mathcal{F}$ is a Nisnevich sheaf of sets on Sm/k. We can extend the Definition 11.1.2 for any Nisnevich sheaf

$$H_0^{\mathbb{A}^1\text{-naive}}(\mathcal{F}) := \pi_0^{\mathbb{A}^1}(\mathbb{Z}(\mathcal{F})).$$

Then by [37, Corollary 5.2], $H_0^{\mathbb{A}^1\text{-naive}}(\mathcal{F})$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups on Sm/k and $H_0^{\mathbb{A}^1\text{-naive}}(\mathcal{F})$ satisfies the same universal property as Remark 11.1.3. The proof in Theorem 11.1.5 also holds if we replace $X \in Sm/k$ by the sheaf $\mathcal{F}$.

The above remark (Remark 11.2.16) and Theorem 11.1.5 give the following corollary:

Corollary 11.2.17. *For $X \in Sm/k$, the canonical morphism $X \to \pi_0^{\mathbb{A}^1}(X)$ induces isomorphism (compare with [14, Proposition 1])*

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1\text{-naive}}(\pi_0^{\mathbb{A}^1}(X)).$$

We end this chapter with the following questions.

Question 11.2.18. Suppose, $X \in Sm/k$.

1. How far is the sheaf $H_0^{\mathbb{A}^1}(X)$ from $H_0^{\mathbb{A}^1\text{-naive}}(X)$. The sheaf $H_0^{\mathbb{A}^1}(X)$ is strictly $\mathbb{A}^1$-invariant sheaf of abelian groups [93, Corollary 6.31] and the sheaf $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups [37, Corollary 5.2]. Is the canonical map

$$H_0^{\mathbb{A}^1\text{-naive}}(X) \to H_0^{\mathbb{A}^1}(X)$$

an isomorphism or atleast it induces an isomorphism over the sections of every finitely generated separable field extensions F/k? We know that this is true if X is a smooth proper k-scheme (Remark 11.2.15).

2. If $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to the constant sheaf $\mathbb{Z}$, then is X $\mathbb{A}^1$-connected? By Theorem 11.1.6, if X is $\mathbb{A}^1$-connected, then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to $\mathbb{Z}$ and if X is proper, then $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is isomorphic to $\mathbb{Z}$ if and only if X is $\mathbb{A}^1$-connected, by Corollary 11.2.12. We also ask that if X is a smooth affine complex surface and $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is trivial, then whether X has negative logarithmic Kodaira dimension (see also, Corollary 5.2.11).

3. If $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is trivial, then $\mathcal{O}(X)$ has only trivial group of units. Indeed, the group of units of X

$$\begin{aligned}\mathcal{O}(X)^* &\cong Hom_{Sm/k}(X, \mathbb{G}_m)\\ &\cong Hom_{\mathcal{A}b_{\mathbb{A}^1\text{-inv}}(k)}(H_0^{\mathbb{A}^1\text{-naive}}(X), \mathbb{G}_m) \text{ (By Remark 11.1.3)},\end{aligned}$$

since $\mathbb{G}_m$ is an $\mathbb{A}^1$-invariant sheaf of abelian groups. Suppose, X is a smooth affine complex surface such that $\mathcal{O}(X)$ is a U.F.D. If moreover $H_0^{\mathbb{A}^1\text{-naive}}(X)$ is trivial, then is X isomorphic to $\mathbb{A}^2_{\mathbb{C}}$?

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