Elliptic Harnack Inequality, Conformal Walk Dimension and Martingale Problem for Geometric Stable Processes

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Introduction

Our central object of study in this thesis is the class of *geometric stable processes*, which are pure jump Lévy processes on $\mathbb{R}^d, d \geq 1$ that exhibit distinctive jump kernel and Green function behavior in comparison to other pure jump Lévy processes such as the α-stable process. In this thesis, we will prove three important results relating to such processes. We first define the *elliptic Harnack inequality*(EHI) and prove that it holds for the geometric stable process. In the second part of this thesis, we introduce the *conformal walk dimension* and show that it is infinite for geometric stable processes. We conclude the thesis by proving the *well-posedness of the martingale problem* associated to Lévy-type operators given by a mixture of the infinitesimal generators of geometric stable processes.

The EHI has been an important inequality in the fields of probability and differential equations for many decades. It was first proved for the Laplacian, and has since been proved for a large class of differential and pseudo-differential operators. Many of these proofs use properties of the Markov processes associated to the operators. Notable works include that of Moser[66] and De Giorgi[29] for elliptic perturbations of the Laplacian, and Bass and Levin[12] to perturbations of α-stable processes. Their method was significantly generalised by Song and Vondraček[79]. For diffusion processes on metric measure spaces, Barlow and Murugan[6] proved that the EHI is completely characterised by a family of Poincaré and cut-off Sobolev inequalities.

In this work, we consider the question of whether the EHI holds for geometric stable processes. The articles of Sikic, Song and Vondracek[82], Mimica and Kim[54] and Mimica and Kassmann[53] are significant works on geometric stable processes and other processes with similar jump kernels. We shall now describe the objectives of this thesis.

In Chapter 1, we define the EHI precisely and prove that it holds for geometric stable processes. The key steps involved in this computation are an estimate for the Green function on the boundary of a ball, and a Harnack-type estimate for the Poisson kernel associated to the process. See Theorem 1.1.2 for the precise statement. This chapter is based on the proof of [2, Theorem 2.2], an ongoing joint work with Siva Athreya and Mathav Murugan.

The motivation behind proving the EHI is to use geometric stable processes, informally, as a barrier between characterisations of the EHI and the parabolic Harnack inequality(PHI). To elaborate, Kajino and Murugan[54] define the notion of *conformal walk dimension* and prove that for a strongly local Dirichlet form on a metric measure Dirichlet space, the EHI holds if and only if the conformal walk dimension is equal to 2 (see Theorem 2.2.1). The conformal walk dimension of a process is the infimum of all $\beta > 0$ such that, informally, a suitable time change of the process satisfies the PHI with scale function $r \mapsto r^{\beta}$. Our main result in Chapter 2 is that the conformal walk dimension of the geometric stable process is infinite, and we refer the reader to Theorem 2.1.4 for the precise statement. In particular, no time change of a geometric stable process can satisfy the PHI with a scale function of the form $r \mapsto r^{\beta}, \beta > 0$. This chapter is based on the proof of [2, Theorem 2.6].

Lastly, in Chapter 3, we define the martingale problem for an operator and prove that for Lévy-type operators given by a mixture of the infinitesimal generators of geometric stable processes, the martingale problem is well-posed. We refer to Theorem 3.1.2 for the precise result. To underline the importance of this work, we note that many methods in the literature, including the original method of Stroock and Varadhan[79] make use of Green function and heat kernel estimates of Lévy processes in their results, which are not known to exist for geometric stable processes. The key step in our proof is a resolvent perturbation bound. This chapter is based on the main result in [47].

Chapter 1

Elliptic Harnack inequality and geometric stable processes

The elliptic and parabolic Harnack inequalities, abbreviated as EHI and PHI respectively, are important inequalities in the fields of harmonic analysis, probability, and differential equations. Of these, the EHI is weaker and is known for its consequences in the classical setting of the Laplacian such as the Hölder regularity of harmonic functions and Liouville's theorem on the constancy of bounded harmonic functions.

The EHI for harmonic functions on $\mathbb{R}^d$ was proved by Harnack[43]. It may be formulated as follows. Define the Laplacian operator on $\mathbb{R}^d$ by $\Delta f(x) = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}(x)$. Given $D \subset \mathbb{R}^d$ and $f : D \to \mathbb{R}$, f is said to be harmonic on $D \subset \mathbb{R}^d$ if

$$\Delta f(x) = 0 \quad \text{for all } x \in D.$$

For any $B \subset D \subset \mathbb{R}^d$ such that $\overline{B} \subset D$, the EHI states that there exists a constant $C(d, B, D) > 0$ such that for every non-negative harmonic function f on D,

$$f(x) \leq C f(y) \quad \text{for all } x, y \in B. \tag{1.1}$$

The infinitesimal generator of the Brownian motion $\{B_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is $\frac{1}{2}\Delta$. Therefore, it is possible to rephrase the definition of harmonicity in terms of the Brownian motion i.e. we call $f : \mathbb{R}^d \to \mathbb{R}$ as harmonic on $D \subset \mathbb{R}^d$ if

$$\mathbb{E}_x f(X_{\tau_{D'}}) = f(x) \text{ for all } D' \subset D, x \in D', \tag{1.2}$$

where $\tau_{D'} = \inf\{t > 0 : B_t \notin D'\}$. Under some regularity conditions, the two notions of harmonicity above coincide. Thus, the EHI may be said to hold for the Brownian motion on $\mathbb{R}^d$.

Given an arbitrary Feller process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^d$ with generator $(A, D(A))$, we call $f : \mathbb{R}^d \to \mathbb{R}$ as harmonic with respect to $\{X_t\}_{t\geq 0}$ on $D \subset \mathbb{R}^d$ if (1.2) holds with

$\{B_t\}_{t\geq 0}$ replaced by $\{X_t\}_{t\geq 0}$. The EHI is said to hold for $\{X_t\}_{t\geq 0}$ if (1.1) holds for all harmonic functions with respect to $\{X_t\}_{t\geq 0}$.

One class of processes for which the EHI has been studied is that of pure jump Lévy processes i.e. those with Lévy triples of the form $(0, 0, k(h)\mathrm{d}h)$ for some non-negative function $k : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ such that $\int_{\mathbb{R}^d\setminus\{0\}}(1 \wedge \|h\|^2)k(h)\mathrm{d}h < \infty$. Bogdan and Sztonyk[17] provide a characterisation of the EHI when $k(h)$ is a symmetric locally bounded function that is homogeneous of degree $-d-\alpha$ for some $\alpha \in (0, 2)$. Sufficient conditions for the inequality to hold were also provided by Grzywny[40], where it was assumed that $k(h) \equiv k(\|h\|)$ is a radially symmetric function decreasing in $\|h\|$ satisfying a weak lower scaling condition(WLSC). For a large class of subordinate Brownian motions, including geometric stable processes, Mimica and Kim[54] prove that the EHI holds at small scales. Our main result in this chapter(see Theorem 1.1.2) is that the EHI holds for the geometric stable process at *all* scales.

The chapter is organised as follows. We will first begin by defining the geometric stable process and stating the EHI in Section 1.1. In Section 1.2, we discuss the motivation, literature and the idea of the proof of the main result. In Section 1.3 we state two key propositions and prove the main result assuming these propositions. In Section 1.4 we state and prove some preliminary lemmas. Finally, in Section 1.5 we prove the two key propositions. The final section 1.6 contains the proof of an auxiliary proposition.

1.1 Main Result

In this section we will define geometric stable processes and state the main result of this chapter, which is a scale invariant Harnack inequality for such processes. We begin with the definition of geometric stable processes. Fix $d \geq 3$ throughout the rest of this section. Let $\|x\| = \left(\sum_{i=1}^d |x_i|^2\right)^{\frac{1}{2}}$ denote the Euclidean norm on $\mathbb{R}^d$.

We will first define a Lévy process on $\mathbb{R}^d$. For this, let $D([0, \infty))$ denote the set of all cádlág functions $f : [0, \infty) \to \mathbb{R}^d$ i.e. those that are right continuous and admit left limits at every $t \geq 0$, equipped with the Skorokhod topology (see [49, Chapter 6] for the definition). Consider the coordinate process $\{X_t\}_{t\geq 0}$ defined by $X_t(\omega) = \omega(t)$ for all $\omega \in D([0, \infty)), t \geq 0$. Given a probability measure $\mathbb{P}$ on $D([0, \infty))$, $\{X_t\}_{t\geq 0}$ is a stochastic process on $\mathbb{R}^d$. We call $\{X_t\}_{t\geq 0}$ a *Lévy process* on $\mathbb{R}^d$ if

1. $X_0 = 0$ $\mathbb{P}$-almost surely.

2. For all $n \geq 1$, $0 = t_0 < t_1 < \ldots < t_n$, the random variables $\{X_{t_i} - X_{t_{i-1}}, 1 \leq i \leq n\}$ are mutually independent.

3. For all $t > s$, the distribution of $X_t - X_s$ depends only on $t - s$.

4. $\{X_t\}_{t\geq 0}$ is stochastically continuous i.e. for all $t \geq 0$ and $\epsilon > 0$,

$$\lim_{s\to t} \mathbb{P}[\|X_s - X_t\| > \epsilon] = 0.$$

We refer the reader to the monographs [15] and [74] for a detailed exposition on Lévy processes.

Next, we define subordinators and subordinate Brownian motions, which are an important class of Lévy processes that include geometric stable processes. An $\mathbb{R}$-valued Lévy process $\{S_t\}_{t\geq 0}$ is called a subordinator if the sample path $t \to S_t(\omega)$ is an increasing function almost surely. Given a standard Brownian motion $\{B_t\}_{t\geq 0}$ on $\mathbb{R}^d$ and an independent subordinator $\{S_t\}_{t\geq 0}$, the process $\{X_t\}_{t\geq 0}$ defined by $X_t = B_{S_t}$ for all $t \geq 0$ is called a subordinated Brownian motion subordinated by $\{S_t\}_{t\geq 0}$. For more details on subordinators, we refer the reader to [15, Chapter III] and [74, Chapter 6]. We refer the reader to [71] and [16, Chapter 5] for an exposition on subordinate Brownian motions.

We will define the Laplace exponent next. This will be followed by the definition of the geometric stable process of index $\alpha, \alpha \in (0, 2)$. Given a Lévy process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^d$, the Laplace exponent $\Psi : \mathbb{R}^d \to \mathbb{C}$ of $\{X_t\}_{t\geq 0}$ is defined by

$$\Psi(\lambda) = -\log \mathbb{E}[e^{i(\lambda\cdot X_1)}]. \tag{1.3}$$

For $\alpha \in (0, 2)$, let $\{S_t^\alpha\}_{t\geq 0}$ be a subordinator whose Laplace exponent is given by

$$\phi_\alpha(\lambda) = \log(1 + \lambda^{\alpha/2}), \tag{1.4}$$

for all $\lambda > 0$.

The geometric stable process of index $\alpha \in (0, 2)$, $\{X_t^\alpha\}_{t\geq 0}$ is the subordinated Brownian motion subordinated by $\{S_t^\alpha\}_{t\geq 0}$. That is,

$$X_t^\alpha = B_{S_t^\alpha} \tag{1.5}$$

for all $t \geq 0$.

Fix $\alpha \in (0, 2)$ and $d \geq 3$ throughout the rest of this subsection. For every $x \in \mathbb{R}^d$, let $\mathbb{P}_x$ be the probability measure on $(\Omega, \mathcal{F})$ such that $(\{X_t^\alpha\}_{t\geq 0}, \mathbb{P}_x)$ is a geometric stable process of index α satisfying $\mathbb{P}_x(X_0^\alpha = x) = 1$. Let $\mathbb{E}_x$ denote the expectation under $\mathbb{P}_x$. We will now define the concept of a harmonic function with respect to $\{X_t^\alpha\}_{t\geq 0}$. For any Borel set $B \subset \mathbb{R}^d$, let

$$\tau_B = \inf\{t > 0 : X_t^\alpha \notin B\} \tag{1.6}$$

denote the exit time of $\{X_t^\alpha\}_{t\geq 0}$ from B.

Definition 1.1.1 (Harmonic function) *Given $D \subset \mathbb{R}^d$ open, $h : \mathbb{R}^d \to \mathbb{R}$ is said to be harmonic on D with respect to the process $\{X_t^\alpha\}_{t\geq 0}$ if, for every bounded open set $B \subset \bar{B} \subset D$ and $x \in B$,*

$$h(x) = \mathbb{E}_x[h(X^\alpha_{\tau_B})]. \tag{1.7}$$

Remark 1 *There is an alternate definition of harmonic functions which may be found in [50, (2.4), Definition 2.4] and [35, Definition 3.1], that uses the language of Dirichlet forms and MMD spaces (see Section 2.1 for the relevant definitions). Namely, suppose that $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ is the MMD space associated to a Hunt process $\{X_t\}_{t\geq 0}$ and $(\mathscr{E}_e, D(\mathscr{E}_e))$ denotes the extended Dirichlet space. A function $h \in D(\mathscr{E}_e)$ is called harmonic on an open set $U \subset X$ if $\mathscr{E}(h, f) = 0$ for all $f \in D(\mathscr{E}) \cap C_c(X)$ such that $supp_\mu[f] \subset U$, where $supp_\mu[f]$ denotes the essential support of f under the measure μ.*

However, by [19, Theorem 2.11(i)] and remarks on page 3501 of the same paper, the notions of harmonicity described above and in Definition 1.1.1 coincide for the geometric stable process of index α, $\{X_t^\alpha\}_{t\geq 0}$ whenever h is bounded. In particular, the EHI for $\{X_t^\alpha\}_{t\geq 0}$ is unaffected by the choice of definition used for harmonic functions.

We are now ready to state our first main result, the scale invariant EHI for geometric stable processes with index $\alpha \in (0, 2)$. For $x_0 \in \mathbb{R}^d$, let $B(x_0, a)$ be the ball of radius $a > 0$ centered at x_0 in $(\mathbb{R}^d, \|\cdot\|)$.

Theorem 1.1.2 (Scale invariant elliptic Harnack inequality) *For any $\alpha \in (0, 2)$ and $d \geq 3$, there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^d$, $r > 0$ and $h : \mathbb{R}^d \to [0, \infty)$ harmonic on $B(x_0, r)$ with respect to $\{X_t^\alpha\}_{t\geq 0}$,*

$$h(x) \leq Ch(y) \quad \text{for all } x, y \in B\left(x_0, \frac{r}{2}\right). \tag{1.8}$$

The assumption $d \geq 3$ is crucial in ensuring that the process $\{X_t\}_{t\geq 0}$ is transient (see Section 1.2.3) and in the proof of Lemma 1.4.3, which is used in the lower bound of the Green function estimate Proposition 1.3.1. However, using a dimension reduction argument as in the proof of [54, Theorem 1.2], this assumption can be relaxed to include all dimensions $d \geq 1$.

Corollary 1.1.3 *Theorem 1.1.2 holds for all $\alpha \in (0, 2)$ and $d \geq 1$.*

In the next section, we shall discuss the motivation and significance of Theorem 1.1.2 along with the idea of the proof.

1.2 Motivation, Literature and Idea of Proof

We shall divide this section into three parts. In Section 1.2.1 we shall briefly review the wide literature on the EHI. In Section 1.2.2, we shall discuss the motivation for the main result. Finally, in Section 1.2.3 we conclude with an overview of Theorem 1.1.2.

1.2.1 Literature on the EHI

The EHI is a fundamental estimate in the field of potential theory and differential equations. We refer the reader to [52] for a general survey on this inequality. Historically, the EHI was first proved by Harnack[43] for the Laplacian $\Delta = \sum_{i=1}^{d} \frac{\partial^2 f}{\partial x_i^2}$ and proved for the fractional Laplacian $(-\Delta)^{\alpha/2}, \alpha \in (0,2)$ by Riesz[72].

In a landmark series of works, Moser [66, 67] proved that the EHI holds for elliptic perturbations of the Laplacian, which are operators of the form

$$\mathcal{A} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right), \tag{1.9}$$

where $(a_{ij})_{1 \leq i,j \leq n}$ is a uniformly elliptic bounded function on $\mathbb{R}^d$.

We shall now briefly comment on various techniques and tools developed to prove the EHI. Moser's technique involves two key steps. The first is an iteration argument involving a Sobolev inequality, which is proved using a Poincaré inequality, while the second is a version of the John-Nirenberg lemma. Fabes and Stroock[32] use a modified Nash inequality(instead of the Poincaré inequality) to prove the Sobolev inequality. A completely different approach known as the expansion of positivity was introduced by De Giorgi [29] to prove the EHI. These methods with appropriate modifications (for example, the Bomberi-Giusti argument[18, Theorem 4] can replace the John-Nirenberg lemma) are the most commonly used methods to prove the EHI for families of operators that are stable under perturbations similar to that described in (1.9).

There are many implications of the EHI. Of these, the most notable is the interior Hölder regularity of harmonic functions. That is, let $\mathcal{A}$ be as in (1.9), u satisfy $\mathcal{A}u = 0$ in a domain $D \subset \mathbb{R}^n$ and $B \subset \overline{B} \subset D$. Then, there exist $C > 0$ and $\gamma \in (0,1)$ depending only upon $n, \mathcal{A}, B$ and D such that

$$|u(x) - u(y)| \leq C\|x - y\|^{\gamma}$$

for all $x, y \in B$. For more applications of the EHI, we refer to [66, Sections 5-7].

In the setting of pure-jump Lévy processes, Bass and Levin[12] proved that the EHI holds for operators of the form

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d \setminus \{0\}} (f(x+h) - f(x) - (\nabla f(x) \cdot h) 1_{\|h\|<1}) \frac{n(x,h)}{\|h\|^{d+\alpha}} \mathrm{d}h,$$

where $\alpha \in (0,2), d \geq 1$ are fixed, and n is a function such that $n(x,h) = n(x,-h)$ and $c_1 \leq n(x,h) \leq c_2$ for some $c_1, c_2 > 0$. These operators are referred to as "stable-like" since the choice $n(x,h) \equiv 1$ yields the infinitesimal generator of an α-stable process,

$\alpha \in (0,2)$. The proof uses three key steps : exit time estimates, a Krylov-Safonov hitting estimate, and an estimate on the exit measure of a ball evaluated at sets far from the ball. Song and Vondracek[76] use the argument of Bass and Levin to prove the EHI for a large class of Markov processes.

A characterisation of the EHI for jump processes possessing a homogenous jump kernel on $\mathbb{R}^d$ with at least stable-like decay was provided by Bogdan and Sztonyk[17]. Bass [9] provides a characterisation of the EHI which holds in the setting of strongly local Dirichlet forms on graphs but employs strong assumptions such as capacity estimates, expected occupation time estimates and a geometric assumption on the graph. Such assumptions are significantly relaxed in Barlow and Murugan[6], where a characterisation of the EHI is proved in the setting of metric measure spaces with a strongly local Dirichlet form under the existence of a well behaved Green function. This assumption was also relaxed in the most general version of the argument for which we refer the reader to [8]. We also refer the reader to [24] for a characterisation of the EHI that applies to non-local operators.

1.2.2 The geometric stable processes

The geometric stable distribution was introduced for the first time in [55]. The authors proved that the distribution answered a question of Zolotarev, who sought an analog for infinitely divisible and stable distributions in the context of summing a random number of random variables. The geometric stable processes on the other hand were introduced in the context of applied economics in [65] to model asset returns. We refer to [82] for a study on the potential theory of geometric stable processes.

An operator closely related to the infinitesimal generator of geometric stable processes is the logarithmic Laplacian, which possesses a similar Fourier symbol. Correa and De Pablo[28] also study the Dirichlet problem for a similar class of operators on various domains.

The geometric stable processes differ from α-stable processes in that they tends to make small jumps with higher intensity. Recall from (1.5) that $X_t^\alpha = B_{S_t^\alpha}$ is a geometric stable process of index α. By [74, Theorem 30.1], $\{X_t^\alpha\}_{t\geq 0}$ is a pure jump Lévy process whose Lévy measure possesses a radially symmetric density $j_\alpha : \mathbb{R}_+ \to \mathbb{R}$. The jump kernel j_α satisfies the bound

$$\frac{c_1}{r^d} \leq j_\alpha(r) \leq \frac{c_2}{r^d} \quad \text{for all } 0 < r \leq 1$$

for some $c_1, c_2 > 0$. This is asymptotically smaller than that of any α-stable process as $r \to 0$. Due to this, they do not satisfy the weak lower scaling(WLSC) assumption(see [40]). The exit time estimates (see Lemma 1.4.5) show that these processes spend more time at the origin, and their Green function (see (1.34) for an estimate) is not integrable, unlike the Green function of the α-stable process which is integrable.

For this reason, arguments used to prove the EHI for stable-like processes may fail when applied to geometric stable processes. For example, perturbations of the α-stable process satisfy a Krylov-Safonov type estimate (see [12, Proposition 3.4] for such an estimate) which fails for geometric stable processes, as proved in [64, Proposition 1.2]. We remark that a *scale variant* EHI (see [71, Theorem 4.20] and [82, Theorem 6.7]) for the geometric stable processes can be proved using these methods.

However, geometric stable processes are subordinate Brownian motions. A major advantage that such processes enjoy is the monotonicity of the jump kernel density and Green function (see [16, Chapter 5], [71]). Using such results in a strong fashion and avoiding the arguments in [12], Mimica and Kim[54] proved that the EHI holds for geometric stable processes at small scales: that is, Theorem 1.1.2 holds for $r \in (0, 1)$. In Theorem 1.1.2, this is extended to all scales.

Since the conclusion and proof of [50, Theorem 1.2] bear a strong resemblance to our results, a discussion on the differences between the results and proofs follows. As remarked in the previous paragraph, [50, Theorem 1.2] only proves the existence of the constant C in the EHI estimate for all $r \in (0, 1)$ for the geometric stable process, which can be interpreted as the EHI being invariant only under small scalings. On the other hand, we extend the method to show that the EHI is invariant under all possible scalings, which is a stronger result.

The key lemmas such as the jump kernel estimate ([50, Proposition 3.2]) and Green function estimate ([50, Proposition 3.5]) employed in the proof of [50, Theorem 1.2] hold only for $r \in (0, 1)$. Ultimately, it is this limitation that results in their main result asserting scale-invariance of the Harnack inequality only in the range $r \in (0, 1)$. Such estimates are not available as $r \to \infty$ (for example, contrast the proofs of [82, Theorem 3.5,Theorem 3.6] which obtain jump kernel bounds as $r \to \infty$ for the geometric stable processes including $\alpha = 2$ using different techniques). We restrict ourselves to the geometric stable processes with $\alpha \in (0, 2)$, where all required estimates are available to us.

We include a proof of the continuity of Green function ($y \to G^{\alpha}_{B(x_0,r)}(x, y)$) and exit time ($y \to \mathbb{E}_y \tau_{B(x_0,r)}$) which was assumed in the proof of [54, Theorem 1.2], see Proposition 1.4.9.

1.2.3 Overview of the proof of Theorem 1.1.2

We shall now discuss the proof of Theorem 1.1.2. The theorem states that the geometric stable process satisfies the scale invariant elliptic Harnack inequality. There are two key steps in the proof, Propositions 1.3.1 and 1.3.2. Proposition 1.3.1 establishes Green function estimates on the boundary of a ball while Proposition 1.3.2 is a Harnack-type estimate for the Poisson kernel of a ball. Our proof is inspired by the techniques used in the proofs of [17, Theorem 1] and [54, Theorem 1.2].

We will now discuss the proof of Proposition 1.3.1. The key tool in the proof is a maximum principle for a Dynkin-like operator, which is applied to suitable classes of functions. For all $\alpha \in (0,2)$ and suitable $x \in \mathbb{R}^d, r > 0$ and $f : \mathbb{R}^d \to \mathbb{R}$, consider the operator.

$$(\mathcal{U}_r f)(x) = \frac{\mathbb{E}_x[f(X^{\alpha}_{\tau_{B(x,r)}})] - f(x)}{\mathbb{E}_x[\tau_{B(x,r)}]}. \tag{1.10}$$

See Definition 1.4.6 for the precise definition of $(\mathcal{U}_r f)(x)$. Similar operators are also defined in [17, Section 3, Page 140] and [54, Section 5, Page 12].

The operator $\mathcal{U}_r f(x)$ satisfies the following maximum principle(see [54, Proposition 5.6]) : if $\mathcal{U}_r f(x_0) < 0$ for some $x_0 \in \mathbb{R}^d$ and $r > 0$, then $f(x_0) > \inf_{y \in \mathbb{R}^d} f(y)$. We use a specific version of the principle in our proof (see Proposition 1.4.8).

We shall now define the Green function, and then state the suitable class of functions which we use in the maximum principle. Fix $x_0 \in \mathbb{R}^d, r > 0$. Let $\tau_{B(x_0,r)}$ be the exit time as defined in (1.6). Define the measures

$$G^{\alpha}(x_0, A) = \int_0^{\infty} \mathbb{P}_{x_0}(X^{\alpha}_t \in A)\mathrm{d}t, \tag{1.11}$$

and

$$G^{\alpha}_{B(x_0,r)}(x, A) = \mathbb{E}_x\left[\int_0^{\tau_{B(x_0,r)}} 1_{\{X^{\alpha}_t \in A\}}\mathrm{d}t\right], \tag{1.12}$$

for all Borel sets $A \subset \mathbb{R}^d$.

Since the geometric stable process is rotation-invariant, it is a "genuinely d-dimensional" Lévy process (see [74, Definition 24.18] for the definition of a genuinely d-dimensional process). Since $d \geq 3$, by [74, Theorem 37.8] we have that $\{X_t\}_{t \geq 0}$ is transient. Finally, by [74, Theorem 35.4(iv)], if $d \geq 3$ then $G^{\alpha}(x_0, A) < \infty$ for every bounded set A.

By [16, Corollary 5.3] and [16, (5.47), p.111], $G^{\alpha}(x_0, \cdot)$ is absolutely continuous with respect to the Lebesgue measure, and admits a density of the form $G^{\alpha}(x_0, y) = g_{\alpha}(\| x_0 - y \|)$ for some function $g_{\alpha} : \mathbb{R}_+ \to \mathbb{R}$. We will refer to either of $G^{\alpha}(\cdot, \cdot)$ or $g_{\alpha}(\cdot)$ as the Green function. The measure $G^{\alpha}_{B(x_0,r)}(x, \cdot)$ also admits a density which we continue to denote by $G^{\alpha}_{B(x_0,r)}(x, \cdot) : \mathbb{R}^d \to \mathbb{R}$ and call the Green function on $B(x_0, r)$.

The estimate in Proposition 1.3.1 consists of an upper and a lower bound. For the upper bound, we show that for suitable $h_1, h_2 : \mathbb{R}_+ \to \mathbb{R}_+$ and fixed $x \in B(0, r)$, the functions

$$f(y) = h_1(r)\mathbb{E}_y \tau_{B(0,r)} - h_2(r) G^{\alpha}_{B(0,r)}(x, y)$$

are such that $\mathcal{U}_r f(y)$ can be estimated for y near the boundary of $B(0, r)$, and the maximum principle can be applied. On the other hand, for the lower bound, we consider instead the family

$$f(y) = h_1(r)\mathbb{E}_y \tau_{B(0,r)} - h_2(r) G^{\alpha}_{B(0,r)}(x, y) \wedge g_{\alpha}(Cr),$$

where C is a well-chosen constant independent of r, x and y. A suitable modification of these families of functions are also used in the proofs of [54, Proposition 5.7, Proposition 5.8] and [17, Lemma 9, Lemma 12].

The usage of the maximum principle requires the continuity of the above function f on a suitable domain in $\mathbb{R}^d$. Fix $x_0 \in \mathbb{R}^d, r > 0$ and $x \in B(x_0, r)$. We prove Proposition 1.4.9, which states that $y \to G^\alpha_{B(x_0,r)}(x, y)$ is continuous on $\mathbb{R}^d \setminus \{x\}$ for all $x \in B(x_0, r)$, and $y \to \mathbb{E}_y \tau_{B(x_0,r)}$ is continuous on $\mathbb{R}^d$(also see the discussion at the end of the previous section).

The proof of Proposition 1.3.2 requires a decomposition of the Poisson kernel into an interior and a boundary part. More precisely, fix $x_0 \in \mathbb{R}^d, r > 0$. By [46, Theorem 2], for $x \in B(x_0, r)$, the exit measure $\mathbb{P}_x(X^\alpha_{\tau_{B(x_0,r)}} \in \cdot)$ has a density $K^\alpha_{B(x_0,r)}(x, \cdot)$ given by

$$K^\alpha_{B(x_0,r)}(x, z) = \int_{B(x_0,r)} G^\alpha_{B(x_0,r)}(x, y) j_\alpha(\| z - y \|) \mathrm{d}y \quad \text{for all } z \in \overline{B(x_0, r)}^c. \quad (1.13)$$

This density is referred to as the Poisson kernel. To prove Proposition 1.3.2 we write

$$\begin{aligned} K^\alpha_{B(x_0,r)}(x, z) =& \int_{A(x_0,br,r)} G^\alpha_{B(x_0,r)}(x, y) j_\alpha(\| z - y \|) \mathrm{d}y + \int_{B(x_0,br)} G^\alpha_{B(x_0,r)}(x, y) j_\alpha(\| z - y \|) \mathrm{d}y \\ :=& I_1 + I_2, \end{aligned}$$

where $A(x_0, br, r) = B(x_0, r) \setminus B(x_0, br)$. The boundary part I_1 is estimated using Proposition 1.3.1, while the interior part I_2 is estimated using the preliminary estimates for the Green function and exit time. Harnack[43] used an estimate similar to Proposition 1.3.2 to derive the Harnack inequality.

Finally, the proof of Theorem 1.1.2 follows straightforwardly from the definition of the Poisson kernel and Proposition 1.3.2.

1.3 Proof of Theorem 1.1.2 and Corollary 1.1.3

In this section we prove Theorem 1.1.2 by assuming two key propositions (see Proposition 1.3.1 and Proposition 1.3.2 below) whose proofs we shall present subsequently. This will be followed by the proof of Corollary 1.1.3. Fix $\alpha \in (0, 2)$ and $d \geq 3$ throughout the rest of this section.

Recall the jump kernel j_α from Section 1.2.2, and the Green function g_α from Section 1.2.3. By [74, Theorem 30.1], j_α and g_α are positive, decreasing and continuous functions.

For fixed $x_0 \in \mathbb{R}^d, r > 0$ and $x \in B(x_0, r)$, recall the Green function $G^\alpha_{B(x_0,r)}(x, y)$ from Section 1.2.3. The Dynkin-Hunt formula (see [33, (4.4.3), Section 4.4] : this is the "0"-order version of the usual Dynkin formula [33, (4.1.6), Section 4.1]) states that

$$G^{\alpha}_{B(x_0,r)}(x,y) = g_{\alpha}(\| x-y \|) - \mathbb{E}_x[g_{\alpha}(\| X^{\alpha}_{\tau_{B(x_0,r)}} - y \|)] \tag{1.14}$$

for all $x, y \in B(x_0, r)$.

We need a couple of definitions to present the first proposition. For $0 \leq r' < r$, let

$$A(x_0, r', r) = B(x_0, r) \setminus B(x_0, r') \tag{1.15}$$

denote the annulus centered at x_0 of inner radius r' and outer radius r. Define $L : (0, \infty) \to \mathbb{R}^+$ by

$$L(s) = \begin{cases} \frac{1}{\log(1+\frac{1}{s})} & 0 < s < 1, \\ 1 & s \geq 1. \end{cases} \tag{1.16}$$

Proposition 1.3.1 *There exist constants $0 < b_1 < b_2 < \frac{1}{2}$ and $C > 1$ such that $2b_1 < b_2$ and for all $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$C^{-1} r^{-d} L(r) \mathbb{E}_y[\tau_{B(x_0,r)}] \leq G^{\alpha}_{B(x_0,r)}(x,y) \leq C r^{-d} L(r) \mathbb{E}_y[\tau_{B(x_0,r)}], \tag{1.17}$$

for every $x \in B(x_0, b_1 r)$ and $y \in A(x_0, b_2 r, r)$. Furthermore, the upper bound in (1.17) holds for every $x \in B(x_0, \frac{b_2}{2} r)$ and $y \in A(x_0, b_2 r, r)$.

The proposition above provides an estimate for $G^{\alpha}_{B(x_0,r)}(x,y)$ when x is close to the center of $B(x_0, r)$ and y is close to the boundary. It mirrors analogous propositions for the Brownian motion (see [27, Lemma 6.7]), however the proof varies (see Section 1.2.3 where we have presented an overview). We will use Proposition 1.3.1 to show a Harnack-type estimate for the Poisson kernel.

Recall the Poisson kernel from Section 1.2.3. By definition of the Poisson kernel, for any non-negative Borel measurable function $f : \mathbb{R}^d \to [0, \infty)$ we have

$$\mathbb{E}_x[f(X^{\alpha}_{\tau_{B(x_0,r)}})] = \int_{\overline{B(x_0,r)}^c} K^{\alpha}_{B(x_0,r)}(x,z) f(z) \mathrm{d}z. \tag{1.18}$$

We now state a Harnack-type estimate for the Poisson kernel.

Proposition 1.3.2 *Let $b_1 > 0$ be as in Proposition 1.3.1. Then, there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$K^{\alpha}_{B(x_0,r)}(x_1, z) \leq C K^{\alpha}_{B(x_0,r)}(x_2, z)$$

for all $x_1, x_2 \in B(x_0, \frac{b_1}{2} r)$ and $z \in \overline{B(x_0, r)}^c$.

Assuming Propositions 1.3.1 and 1.3.2, we now prove Theorem 1.1.2. The proof follows by standard techniques and is similar to the proof presented in [54, Theorem 1.2].

Proof of Theorem 1.1.2 : Let $x_0 \in \mathbb{R}^d$, $r > 0$. Suppose that h is a non-negative harmonic function in $B(x_0, r)$. As $B(x_0, \frac{r}{2}) \subset B(x_0, r)$, for all $x \in B(x_0, r/2)$ by (1.7) we have

$$h(x) = \mathbb{E}_x \left[h \left(X^\alpha_{\tau_{B(x_0, \frac{r}{2})}} \right) \right]$$
$$= \int_{\overline{B(x_0, \frac{r}{2})}^c} K^\alpha_{B\left(x_0, \frac{r}{2}\right)}(x, z) h(z) \mathrm{d}z, \tag{1.19}$$

where we have used (1.18) in the last equality. Let $b_1 > 0$ be as in Proposition 1.3.1. By Proposition 1.3.2, for $x_1, x_2 \in B(x_0, \frac{b_1}{2} r)$ there is a $C > 0$ such that

$$K^\alpha_{B(x_0, \frac{r}{2})}(x_1, z) \leq C K^\alpha_{B(x_0, \frac{r}{2})}(x_2, z)$$

for all $z \in \overline{B\left(x_0, \frac{r}{2}\right)}^c$.Multiplying both sides of the above inequality by the non-negative quantity $h(z)$, integrating over $z \in \overline{B(x_0, \frac{r}{2})}^c$ and using (1.19), we obtain $h(x_1) \leq Ch(x_2)$ for all $x_1, x_2 \in B(x_0, \frac{b_1}{2} r)$. The result now follows by a standard Harnack chaining argument. □

Finally, we shall present the proof of Corollary 1.1.3.

Proof of Corollary 1.1.3. Let $d \geq 2$, $\alpha \in (0, 2)$ be arbitrary. For a vector $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$, let $Ux = (x_1, x_2, \ldots, x_{d-1})$ be the projection map onto the first $d-1$ coordinates.

If $(\{X^\alpha_t\}_{t \geq 0}, \mathbb{P}_x)$ is a geometric stable process of index α started at $x \in \mathbb{R}^d$, then an application of [74, Proposition 11.10] shows that $(\{UX^\alpha_t\}_{t \geq 0}, \mathbb{P}_{Ux})$ is a geometric stable process of index α started at $Ux \in \mathbb{R}^{d-1}$.

Suppose that Theorem 1.1.2 holds for dimension d and α. We will now show that it holds for dimension $d-1$ and α. Let $x_0 \in \mathbb{R}^{d-1}$ and $r > 0$ be arbitrary. Suppose that $h : \mathbb{R}^{d-1} \to [0, \infty)$ is harmonic on $B(x_0, r)$ with respect to $\{UX^\alpha_t\}_{t \geq 0}$. Define $f : \mathbb{R}^d \to [0, \infty)$ by $f(x) = h(Ux)$.

We claim that f is harmonic on $B(x_0, r) \times \mathbb{R}$. Assuming this claim, we shall complete the proof of the corollary; we shall subsequently prove the claim. Note that f is harmonic on $B((x_0, 0), r) \subset B(x_0, r) \times \mathbb{R}$. Applying (1.8) for the harmonic function f on $B((x_0, 0), r)$ and recalling that $f(x) = h(Ux)$,

$$\mathrm{esssup}_{B(x_0, r/2)} h \leq \mathrm{esssup}_{B((x_0, 0), r/2)} f \leq C \mathrm{essinf}_{B((x_0, 0), r/2)} f \leq C \mathrm{essinf}_{B(x_0, r/2)} h,$$

where C is independent of f, x_0 and r, and hence of h, x_0 and r. This shows that (1.8) holds for dimension $d-1$ and α.

By Theorem 1.1.2, the corollary holds for $d = 3$ and any $\alpha \in (0, 2)$. Applying the above argument first with $d = 3$ and $d = 2$ shows that Theorem 1.1.2 holds for any $d \geq 1$ and $\alpha \in (0, 2)$, completing the proof of the corollary.

We are left to prove the claim that $f(x) = h(Ux)$ is harmonic on $B(x_0, r) \times \mathbb{R}$ when h is harmonic on $B(x_0, r)$. For simplicity, denote $B(x_0, r) \times \mathbb{R}$ by D. Let $B \subset D$ be a bounded open set such that $\overline{B} \subset D$.

Let $\tilde{\tau}_{B(x_0,r)} = \inf\{t > 0 : UX_t \notin B(x_0, r)\}$. Then, observe that

$$\tilde{\tau}_{B(x_0,r)} = \tau_D. \tag{1.20}$$

Let $y \in D$ be arbitrary. Noting that $f(y) = h(Uy)$ and using the harmonicity of h,

$$f(y) = h(Uy) = \mathbb{E}_{Uy} h((UX)_{\tilde{\tau}_{B(x_0,r)}}) = \mathbb{E}_y h(UX_{\tau_D}) = \mathbb{E}_y f(X_{\tau_D}), \tag{1.21}$$

where we used (1.20) to obtain the third equality. Observe that (1.21) also holds when $y \notin D$.

Note that $\tau_B \leq \tau_D$. Thus, for any $x \in B$, by the Strong Markov property of $\{X_t\}_{t \geq 0}$ we have

$$\mathbb{E}_x f(X_{\tau_D}) = \mathbb{E}_x \mathbb{E}_{X_{\tau_B}} f(X_{\tau_D}) = \mathbb{E}_x f(X_{\tau_B}),$$

where we used (1.21) for $y = X_{\tau_B}$ to obtain the second equality. Combining the above equation with (1.21) applied for $y = x$, we see that $f(x) = \mathbb{E}_x f(X_{\tau_B})$. By Definition 1.1.1, f is harmonic on D, completing the proof of the claim. □

1.4 Green function, exit time and maximum principle

In this section, we list the key preliminary lemmas required to prove Proposition 1.3.1 and Proposition 1.3.2. In Section 1.4.1, we prove some preliminary estimates. In Section 1.4.2, we mention the required estimates on the Green function, jump kernel and the exit time. In Section 1.4.3, we discuss the maximum principle. Throughout the rest of this section, let $\alpha \in (0, 2)$ and $d \geq 3$ be fixed.

1.4.1 Function estimates

Recall the function L from (1.16). In this section, we shall define another function $\tilde{L}$ required for the proof of Propositions 1.3.1 and 1.3.2 and prove some preliminary estimates about L and $\tilde{L}$. Define $\tilde{L} : (0, \infty) \to \mathbb{R}_+$ by

$$\tilde{L}(s) = \begin{cases} 1 & 0 < s < 1, \\ s^{-\alpha} & s \geq 1. \end{cases} \tag{1.22}$$

Prior to stating our first preliminary estimate, we require the following lemma. A function $F : (0,\infty) \to \mathbb{R}^+$ is said to be regularly varying of index $s \in \mathbb{R}$ at 0 if

$$\lim_{x\to 0} \frac{F(\lambda x)}{F(x)} = \lambda^s, \quad \text{for all } \lambda > 0.$$

Similarly, it is said to be regularly varying of order $s \in \mathbb{R}$ at ∞ if

$$\lim_{x\to \infty} \frac{F(\lambda x)}{F(x)} = \lambda^s, \quad \text{for all } \lambda > 0.$$

Lemma 1.4.1 *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be any function such that*

(a) There exist $s, s' \in \mathbb{R}$ such that f is regularly varying at 0 and ∞ with indices s, s' respectively.

(b) For every $0 < a < b < \infty$, there exists $C > 0$ such that

$$C^{-1} \leq f(x) \leq C \text{ for all } x \in [a,b].$$

Then, for every $\mu > 0$, there exists a constant $C_\mu > 1$ such that

$$C_\mu^{-1} \leq \frac{f(\mu r)}{f(r)} \leq C_\mu \quad \text{for all } r > 0.$$

Proof. Let f and $\mu > 0$ be fixed. Assuming (a) we have

$$\lim_{x\to 0} \frac{f(\mu x)}{f(x)} = \mu^s. \tag{1.23}$$

and

$$\lim_{x\to \infty} \frac{f(\mu x)}{f(x)} = \mu^{s'}. \tag{1.24}$$

Let $G(r) = \frac{f(\mu r)}{f(r)}$. By (1.23), there exists $\delta > 0$ such that $2\mu^s \geq G(r) \geq \frac{1}{2}\mu^s$ for $r \in (0,\delta)$. By (1.24), there exists $\delta' > 0$ such that $2\mu^{s'} \geq G(r) \geq \frac{1}{2}\mu^{s'}$ for $r \in (\delta',\infty)$. Increasing δ' if necessary, we assume $\delta' > \delta$.

Assuming (b), there exist $m, M > 0$ such that $m \leq G(x) \leq M$ for $x \in [\delta,\delta']$. Along with the bounds on G obtained in the previous paragraph, this proves the lemma. □

We are now ready to state our first preliminary estimate.

Lemma 1.4.2 *(a) There exists $C > 1$ such that*

$$C^{-1}\frac{L(s)}{\tilde{L}(s)} \leq \int_{B(0,s)} \frac{L^2(\| x \|)}{\tilde{L}(\| x \|) \| x \|^d} \mathrm{d}x \leq C\frac{L(s)}{\tilde{L}(s)}, \quad \text{for all } s > 0.$$

(b) Let $\mu > 0$ be given. Then, there exists $C_1 \equiv C_1(\mu, \alpha, d) > 1$ such that

$$C_1^{-1} \leq \frac{L(\mu r)}{L(r)} \leq C_1, \text{ and}$$
$$C_1^{-1} \leq \frac{\tilde{L}(\mu r)}{\tilde{L}(r)} \leq C_1$$

for all $r > 0$.

We will prove Lemma 1.4.2(a) first.

Proof of Lemma 1.4.2(a): By a change of variable, we may write

$$\int_{B(0,s)} \frac{L^2(\| x \|)}{\tilde{L}(x) \| x \|^d} \mathrm{d}x = C \int_0^s \frac{L^2(r)}{r\tilde{L}(r)} \mathrm{d}r, \tag{1.25}$$

where $C > 0$ is some constant. Define $K : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$K(s) = \begin{cases} \frac{1}{s \log^2\left(\frac{2}{s}\right)} & 0 < s < 1 \\ s^{\alpha-1} & s \geq 1 \end{cases}. \tag{1.26}$$

By the definitions (1.16) and (1.22) of L and $\tilde{L}$ respectively, there exists $C > 1$ such that for all $s > 0$,

$$C^{-1}K(s) \leq \frac{L^2(s)}{s\tilde{L}(s)} \leq CK(s). \tag{1.27}$$

Combining the above with (1.25),

$$C^{-1} \int_0^s K(s)\mathrm{d}s \leq \int_{B(0,s)} \frac{L^2(\| x \|)}{\tilde{L}(x) \| x \|^d} \mathrm{d}x \leq C \int_0^s K(s)\mathrm{d}s. \tag{1.28}$$

We will now prove that Lemma 1.4.2(a) holds for $s < 1$. Let $s_0 < 1$ be fixed. By the definition (1.26) of K,

$$\int_0^{s_0} K(s)\mathrm{d}s = \int_0^{s_0} \frac{\mathrm{d}s}{s \log^2\left(\frac{2}{s}\right)} = \frac{1}{\log\left(\frac{2}{s_0}\right)} = \sqrt{s_0 K(s_0)}. \tag{1.29}$$

By (1.27),

$$C_1^{-1}\frac{L(s_0)}{\tilde{L}(s_0)} \leq \sqrt{s_0 K(s_0)} \leq C_1 \frac{L(s_0)}{\tilde{L}(s_0)}.$$

Combining (1.28),(1.29) and the above proves Lemma 1.4.2(a) the region $s < 1$.

We will now prove that the Lemma 1.4.2(a) holds for $s \geq 1$. Let $s_0 \geq 1$ be fixed. By (1.29), $\int_0^1 K(s)ds = \frac{1}{\log(2)} = I$ is a finite positive quantity. Thus,

$$\int_0^{s_0} K(s)\mathrm{d}s = I + \int_1^{s_0} s^{\alpha-1}\mathrm{d}s = I + \frac{s_0^\alpha - 1}{\alpha}. \tag{1.30}$$

By the definitions (1.16) and (1.22) of L and $\tilde{L}$,

$$C\frac{L(s_0)}{\tilde{L}(s_0)} \leq I + \frac{s_0^\alpha - 1}{\alpha} \leq \left(I + \frac{1}{\alpha}\right)\frac{L(s_0)}{\tilde{L}(s_0)}.$$

where $C > 0$ is some finite positive constant. Combining (1.28),(1.30) and the above inequality shows that Lemma 1.4.2(a) holds for $s \geq 1$. □

We will now prove Lemma 1.4.2(b).

Proof of Lemma 1.4.2(b) : We will show that the hypothesis of Lemma 1.4.1 hold for each of the functions L and $\tilde{L}$.

From the definition (1.16), L is regularly varying with index 0 at both 0 and ∞, and is bounded and bounded away from 0 on every finite interval. Hence, it satisfies both hypotheses of Lemma 1.4.1.

Similarly, from the definition (1.22), $\tilde{L}$ is regularly varying at 0 with index 0 and at ∞ with index $-\alpha$, and is bounded and bounded away from 0 on every finite interval. Hence, it satisfies both hypotheses of Lemma 1.4.1.

Lemma 1.4.2(b) follows by applying Proposition 1.4.1 to each of the functions L and $\tilde{L}$ respectively. □

We shall now state and prove our second preliminary estimate.

Lemma 1.4.3 *For every $K > 0$, there exists $B > 0$ such that for all $b < B$,*

$$\left(b^{-d}\frac{L^2(br)}{L^2(r)}\frac{\tilde{L}(r)}{\tilde{L}(br)}\right) \geq K, \quad \textit{for all } r > 0.$$

Proof. Let $K > 0$ be given and $B < 1$ be a constant that will be chosen later. By the definition (1.22) of $\tilde{L}$,

$$\frac{\tilde{L}(r)}{\tilde{L}(Br)} \geq B^\alpha \quad \text{for all } r > 0. \tag{1.31}$$

By the definition (1.16) of L we have $L(Br) = L(r)$ for all $r > \frac{1}{B}$. Furthermore, the function $\frac{L^2(Br)}{L^2(r)}$ is decreasing on $(0, \frac{1}{B})$. Therefore, for every $r > 0$,

$$B^{-d}\frac{L^2(Br)}{L^2(r)} \geq \min\left\{\log^2(2)\frac{B^{-d}}{L^2(\frac{1}{B})}, B^{-d}\right\}.$$

Combining (1.31) and the above equation we have

$$\left(B^{-d}\frac{L^2(Br)}{L^2(r)}\frac{\tilde{L}(r)}{\tilde{L}(Br)}\right) \geq \min\left\{\log^2(2)\frac{B^{\alpha-d}}{L^2(\frac{1}{B})}, B^{\alpha-d}\right\}, \quad \text{for all } r > 0. \tag{1.32}$$

Note that the above inequality holds for all $B > 0$, and

$$\lim_{B\to 0}\min\left\{\log^2(2)\frac{B^{\alpha-d}}{L^2(\frac{1}{B})}, B^{\alpha-d}\right\} = +\infty.$$

Thus, by choosing $0 < B < 1$ small enough, we can ensure that the right hand side of (1.32) exceeds K for all $b < B$. This completes the proof of the lemma. □

1.4.2 Green function and exit time estimates

In this section, we will state some well known estimates for the jump kernel $j_\alpha(r), r > 0$ and the Green function $g_\alpha(r), r > 0$ (see Section 1.3 for the definitions). We will then prove interior lower bounds for the Green function (see Lemma 1.4.4) and upper and lower bounds for the exit time (see Lemma 1.4.5).

Recall the function $\tilde{L}$ from (1.22). By [82, Theorem 3.4, Theorem 3.5], there is a $C > 1$ such that

$$C^{-1}r^{-d}\tilde{L}(r) \leq j_\alpha(r) \leq Cr^{-d}\tilde{L}(r), \quad \text{for all } r > 0. \tag{1.33}$$

Recall L from (1.16). By [54, Proposition 4.5] and [71, Theorem 3.3], there exists a $C > 1$ such that

$$C^{-1}r^{-d}\frac{L^2(r)}{\tilde{L}(r)} \leq g_\alpha(r) \leq Cr^{-d}\frac{L^2(r)}{\tilde{L}(r)}, \quad \text{for all } r > 0. \tag{1.34}$$

Our first result of this section is the lower bound on the Green function on $B(x_0, r)$ where $x_0 \in \mathbb{R}^d$ and $r > 0$.

Lemma 1.4.4 *There exists $a \in (0, \frac{1}{3})$ such that for every $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$G^\alpha_{B(x_0,r)}(x, y) \geq \frac{1}{2}g_\alpha(\| x - y \|) \quad \textit{for all } x, y \in B(x_0, ar).$$

Proof. For some fixed $M > 1$ which will be chosen later, let $f : (0, \infty) \to \mathbb{R}_+$ be defined by

$$f(r) = \frac{g_\alpha(Mr)}{g_\alpha(r)},$$

for $r > 0$. As g_α is continuous and positive on $(0, \infty)$, so is f. By (1.34) we have

$$f(r) \le CM^{-d} \frac{L^2(Mr)}{L^2(r)} \frac{\tilde{L}(r)}{\tilde{L}(Mr)}. \tag{1.35}$$

Note that, by the definition of $\tilde{L}$,

$$\frac{\tilde{L}(r)}{\tilde{L}(Mr)} \le M^\alpha$$

for all $r > 0$. Using the above in (1.35) we have that

$$f(r) \le CM^{\alpha-d} \frac{L^2(Mr)}{L^2(r)}, \tag{1.36}$$

for all $r > 0$.

By the definition of L (see (1.16)) we have

$$\frac{L^2(Mr)}{L^2(r)} = \begin{cases} \frac{\log^2(1+\frac{1}{r})}{\log^2(1+\frac{1}{Mr})} & r < \frac{1}{M} \\ \log^2(1+\frac{1}{r}) & 1/M \le r < 1 \\ 1 & r \ge 1 \end{cases}$$

Observe that $\frac{\log^2(1+\frac{1}{r})}{\log^2(1+\frac{1}{Mr})}$ is an increasing function on $[0, \frac{1}{M})$ and $\log^2(1+\frac{1}{r})$ is a decreasing function on $[\frac{1}{M}, 1)$. Using these observations in the above equation,

$$\frac{L^2(Mr)}{L^2(r)} \le \max\left\{\frac{\log^2(1+M)}{\log^2(2)}, \log^2(1+M), 1\right\} = \frac{\log^2(1+M)}{\log^2(2)}.$$

Combining the above with (1.36), we have that there exists $C > 0$ such that

$$f(r) \le CM^{\alpha-d} \frac{\log^2(1+M)}{\log^2(2)} \tag{1.37}$$

for all $M > 1$ and $r > 0$. Observe that as $\alpha < d$, we can choose $M > 1$ large enough to obtain

$$f(r) \le \frac{1}{2}, \quad \text{for all } r > 0. \tag{1.38}$$

Let $a = \frac{1}{1+2M}$. Then, $a \in (0, \frac{1}{3})$ and $\frac{1-a}{2a} = M$. Fix $x_0 \in \mathbb{R}^d, r > 0$ and $x, y \in B(x_0, ar)$. As $X^\alpha_{\tau_{B(x_0,r)}} \in B(x_0, r)^c$ we have

$$\| X^\alpha_{\tau_{B(x_0,r)}} - y \| \ge (1-a)r \ge (1-a)\frac{\| x-y \|}{2a} = M \| x-y \| .$$

Since g_α is a decreasing function, we have

$$g_\alpha\left(\| X^\alpha_{\tau_{B(x_0,r)}} - y \|\right) \le g_\alpha(M \| x - y \|).$$

Using the above in (1.14) we hav

$$\begin{aligned} G^\alpha_{B(x_0,r)}(x,y) &\ge g_\alpha(\| x - y \|) - g_\alpha\left(M \| x - y \|\right) \\ &= g_\alpha(\| x - y \|)\left(1 - f(\| x - y \|)\right) && \text{(by definition of } f\text{)}, \\ &\ge \frac{1}{2} g_\alpha(\| x - y \|) && \text{(by (1.38))}. \end{aligned}$$

□

We now state and prove exit time bounds.

Lemma 1.4.5 (Upper and interior lower bounds for exit time) *Let a be chosen so that the conclusion of Lemma 1.4.4 is satisfied. Then, there exists $C > 1$ such that for every $x_0 \in \mathbb{R}^d$ and $r > 0$,*

$$C^{-1}\frac{L(r)}{\tilde{L}(r)} \le \mathbb{E}_x[\tau_{B(x_0,r)}] \le C\frac{L(r)}{\tilde{L}(r)}, \quad \textit{for all } x \in B\left(x_0, \frac{ar}{2}\right). \tag{1.39}$$

In addition, the upper bound in (1.39) holds for all $x \in B(x_0, r)$.

Proof. Let $r > 0, x_0 \in \mathbb{R}^d$. We derive the lower bound in (1.39) first.

Let $x \in B(x_0, \frac{ar}{2})$. Then, $B(x, \frac{ar}{2}) \subset B(x_0, ar) \subset B(x_0, r)$. Using the definition of $G^\alpha_{B(x_0,r)}(x,y)$ we have

$$\mathbb{E}_x[\tau_{B(x_0,r)}] = \int_{B(x_0,r)} G^\alpha_{B(x_0,r)}(x,y)\mathrm{d}y \ge \int_{B(x,\frac{ar}{2})} G^\alpha_{B(x_0,r)}(x,y)\mathrm{d}y. \tag{1.40}$$

Applying Lemma 1.4.4 first, followed by (1.34), we have

$$\begin{aligned} \int_{B(x,\frac{ar}{2})} G^\alpha_{B(x_0,r)}(x,y)\mathrm{d}y &\ge \frac{1}{2}\int_{B(x,\frac{ar}{2})} g_\alpha(\| x - y \|)\mathrm{d}y \\ &\ge C\int_{B(x,\frac{ar}{2})} \frac{L^2(\| x - y \|)}{\tilde{L}(\| x - y \|) \| x - y \|^d}\mathrm{d}y, \end{aligned} \tag{1.41}$$

By Lemma 1.4.2(a), we have that

$$\int_{B(x,\frac{ar}{2})} \frac{L^2(\| x - y \|)}{\tilde{L}(\| x - y \|) \| x - y \|^d}\mathrm{d}y \ge \frac{L(\frac{ar}{2})}{\tilde{L}(\frac{ar}{2})} = \frac{L(\frac{ar}{2})}{L(r)}\frac{\tilde{L}(r)}{\tilde{L}(\frac{ar}{2})}\frac{L(r)}{\tilde{L}(r)}.$$

Using the bounds provided in Lemma 1.4.2(b) with $\mu = \frac{a}{2}$ for $L, \tilde{L}$ in the above we have

$$\int_{B(x,\frac{ar}{2})} \frac{L^2(\parallel x-y \parallel)}{\tilde{L}(\parallel x-y \parallel) \parallel x-y \parallel^d} \mathrm{d}y \geq C \frac{L(r)}{\tilde{L}(r)}, \tag{1.42}$$

for some $C > 0$. From, (1.40),(1.41) and (1.42) we obtain the lower bound in (1.39).

We will now prove the upper bound in (1.39). Let $x \in B(x_0, r)$. Again, by the definition of $G^{\alpha}_{B(x_0,r)}(x, y)$ we have

$$\mathbb{E}_x[\tau_{B(x_0,r)}] = \int_{B(x_0,r)} G^{\alpha}_{B(x_0,r)}(x,y)\mathrm{d}y \leq \int_{B(x_0,r)} g_\alpha(\parallel x-y \parallel)\mathrm{d}y. \tag{1.43}$$

Since $B(x_0, r) \subset B(x, 2r)$,

$$\begin{aligned} \int_{B(x_0,r)} g_\alpha(\parallel x-y \parallel)\mathrm{d}y &\leq \int_{B(x,2r)} g_\alpha(\parallel x-y \parallel)\mathrm{d}y \\ &\leq C \int_{B(x,2r)} \frac{L^2(\parallel x-y \parallel)}{\tilde{L}(\parallel x-y \parallel) \parallel x-y \parallel^d} \mathrm{d}y, \end{aligned} \tag{1.44}$$

where we used (1.34) in the last step. By Lemma 1.4.2(a), we have that

$$\int_{B(x,2r)} \frac{L^2(\parallel x-y \parallel)}{\tilde{L}(\parallel x-y \parallel) \parallel x-y \parallel^d} \mathrm{d}y \leq \frac{L(2r)}{\tilde{L}(2r)} = \frac{L(2r)}{L(r)} \frac{\tilde{L}(r)}{\tilde{L}(2r)} \frac{L(r)}{\tilde{L}(r)}.$$

Using the bounds in Lemma 1.4.2(b) with $\mu = 2$ for $L, \tilde{L}$ in the above we have

$$\int_{B(x,2r)} \frac{L^2(\parallel x-y \parallel)}{\tilde{L}(\parallel x-y \parallel) \parallel x-y \parallel^d} \mathrm{d}y \leq C \frac{L(r)}{\tilde{L}(r)}, \tag{1.45}$$

for some $C > 0$. Combining (1.43) - (1.45), we obtain the upper bound in (1.39), completing the proof. □

1.4.3 The maximum principle

In this section, we will define the key analytic tool in our proof, which is a Dynkin-like operator. We mention how this operator behaves when applied to harmonic functions and the exit time function, and provide a maximum principle satisfied by it. Finally, we state a technical condition required for the usage of the maximum principle.

Definition 1.4.6 (Dynkin-like operator) *Let $x \in \mathbb{R}^d$. Define the vector space*

$$V_x = \left\{ f : \mathbb{R}^d \to \mathbb{R} : f \textit{ is Borel measurable and } \mathbb{E}_x \left[f(X^{\alpha}_{\tau_{B(x,r)}}) \right] < \infty \textit{ for all } r > 0 \right\}. \tag{1.46}$$

For any $r > 0$, the operator $(\mathcal{U}_r(\cdot))(x) : V_x \to \mathbb{R}$ is defined by

$$(\mathcal{U}_r f)(x) = \frac{\mathbb{E}_x[f(X^\alpha_{\tau_{B(x,r)}})] - f(x)}{\mathbb{E}_x[\tau_{B(x,r)}]}.$$

Using (1.13) and (1.18), we can rewrite $(\mathcal{U}_r f)(x)$ as

$$(\mathcal{U}_r f)(x) = \frac{1}{\mathbb{E}_x[\tau_{B(x,r)}]} \int_{\overline{B(x,r)}^c} \int_{B(x,r)} G^\alpha_{B(x,r)}(x,y) j_\alpha(\| y - z \|)(f(z) - f(x)) \mathrm{d}y\mathrm{d}z. \tag{1.47}$$

Observe that $(\mathcal{U}_r(\cdot))(x)$ is a linear operator for all $x \in \mathbb{R}^d, r > 0$. The next proposition contains the result of evaluating this operator on harmonic functions and the exit-time function.

Proposition 1.4.7 (Harmonic functions and exit time) *Let $D \subset \mathbb{R}^d$ be an open set, $x \in D$ and $r > s > 0$ be such that $s < dist(x, \partial D)$.*

(a) If h is harmonic in D, then $h \in V_x$ and $(\mathcal{U}_s h)(x) = 0$.

(b) Define $\eta(z) = \mathbb{E}_z \tau_{B(x,r)}$ for $z \in \mathbb{R}^d$. Then, for all $y \in B(x, r - s)$ we have

$$\eta \in V_y \text{ and } (\mathcal{U}_s \eta)(y) = -1.$$

Proof. See [54, Remark 5.5] for the proof of part (a) and [54, Example 5.4] for the proof of part (b). □

Recall $A(x_0, r', r)$ from (1.15). The operator $(\mathcal{U}_r(\cdot))(x)$ admits the following maximum principle.

Proposition 1.4.8 (Maximum Principle) *Let a be as in Lemma 1.4.4. Let $x_0 \in \mathbb{R}^d$, $r > 0$. Suppose $f : \mathbb{R}^d \to \mathbb{R}$ such that*

(i) f is a continuous on $\overline{A(x_0, \frac{ar}{2}, r)}$.

(ii) $f \geq 0$ on $A(x_0, \frac{ar}{2}, r)^c$,

(iii) for all $y \in A(x_0, \frac{ar}{2}, r)$, $f \in V_y$, and

(iv) for all $y \in A(x_0, \frac{ar}{2}, r)$, there exists $s_y > 0$ such that $(\mathcal{U}_{s_y} f)(y) < 0$.

Then, $f \geq 0$ on $\mathbb{R}^d$.

Proof. Suppose, by way of contradiction, that $f \geq 0$ on $\mathbb{R}^d$ is false. Since $f \geq 0$ on $A(x_0, \frac{ar}{2}, r)^c$ by assumption, f may only take negative values inside $A(x_0, \frac{ar}{2}, r)$. By continuity of f on $\overline{A(x_0, \frac{ar}{2}, r)}$, there exists $y \in A(x_0, \frac{ar}{2}, r)$ such that $f(y) = \inf_{x \in \mathbb{R}^d} f(x)$.

By assumption, $f \in V_y$ and there exists $s_y > 0$ such that $\mathcal{U}_{s_y} f(y) < 0$. Applying [54, Proposition 5.6] with the choices of $r > 0, y \in \mathbb{R}^d, s_y > 0$ and f as above, $f(y) > \inf_{x \in \mathbb{R}^d} f(x)$. This contradicts the definition of y. It follows that $f \geq 0$ on $\mathbb{R}^d$. □

Remark 2 *In Proposition 1.4.8, the set $A(x_0, \frac{ar}{2}, r)$ may be replaced by any bounded open subset of $\mathbb{R}^d$ with appropriate changes to the assumptions. However, the proposition will only be applied to annuli of the form $A(x_0, \frac{ar}{2}, r)$, hence it is stated only for these sets.*

Finally, we will need the following technical result which is key to the application of the maximum principle.

Proposition 1.4.9 (Continuity of Green function and Exit time) *Let $x_0 \in \mathbb{R}^d$, $r > 0$ and $x \in B(x_0, r)$. Then,*

(i) the function $y \mapsto G^{\alpha}_{B(x_0,r)}(x, y)$ is continuous on $\mathbb{R}^d \setminus \{x\}$, and

(ii) the function $y \mapsto \mathbb{E}_y[\tau_{B(x_0,r)}]$ is a continuous function on $\mathbb{R}^d$.

The proof of Proposition 1.4.9 is given in Section 1.6 at the end of this chapter. We conclude this section with the following Lemma that is needed in the proof of the upper bound in (1.17) of Proposition 1.3.1.

Lemma 1.4.10 *Let $r > 0$ and $x \in B(0, \frac{ar}{4})$. Define $h : \mathbb{R}^d \to \mathbb{R}$ by*

$$h(z) = G^{\alpha}_{B(0,r)}(x, z).$$

Then, h is harmonic for all $y \in A(0, \frac{ar}{2}, r)$ and $h \wedge g_\alpha\left(\frac{ar}{16}\right) - h \in V_y$. Further for $s < \min(r- \| y \|, \frac{ar}{8})$ there exists $C_L > 0$ such that

$$\mathcal{U}_s\left(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h\right)(y) \geq -C_L r^{-d} L(r).$$

Proof. Fix $y \in A(0, \frac{ar}{2}, r)$ and $s < \min(r- \| y \|, \frac{ar}{8})$. Recall the domain of $(\mathcal{U}_r(\cdot))(y)$, which is the set V_y defined in (1.46). We will now prove that $\left(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h\right) \in V_y$.

Clearly the function is Borel measurable. By the triangle inequality,

$$\begin{aligned}\left|\mathbb{E}_y\left[h\left(X^{\alpha}_{\tau_{B(y,r)}}\right) \wedge g_\alpha\left(\frac{ar}{16}\right) - h\left(X^{\alpha}_{\tau_{B(y,r)}}\right)\right]\right| &\leq \left|\mathbb{E}_y\left[h\left(X^{\alpha}_{\tau_{B(y,r)}}\right) \wedge g_\alpha\left(\frac{ar}{16}\right)\right]\right| \\ &\quad + \left|\mathbb{E}_y\left[h\left(X^{\alpha}_{\tau_{B(y,r)}}\right)\right]\right|. \qquad (1.48)\end{aligned}$$

The first term above is bounded by $g_\alpha(\frac{ar}{16})$. For the second term, we note that h is a non-negative function which is harmonic at y : this is proved in Lemma 1.6.2 for completeness. By (1.7), $\mathbb{E}_y\left[h\left(X^\alpha_{\tau_{B(y,r)}}\right)\right] = h(y)$. Combining this observation with (1.48),

$$\left|\mathbb{E}_y\left[h\left(X^\alpha_{\tau_{B(y,r)}}\right) \wedge g_\alpha\left(\frac{ar}{16}\right) - h\left(X^\alpha_{\tau_{B(y,r)}}\right)\right]\right| \leq g_\alpha\left(\frac{ar}{16}\right) + h(y) < \infty,$$

proving that $(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h) \in V_y$, as desired.

Suppose that we are given any $z \in B(x, \frac{ar}{16})^c$. Note that $h(z) \leq g_\alpha(\| x - z \|)$. Since g_α is a decreasing function, $g_\alpha(\| x - z \|) \leq g_\alpha(\frac{ar}{16})$. It follows that

$$\left(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h\right)(z) = 0 \text{ for all } z \in B\left(x, \frac{ar}{16}\right)^c. \tag{1.49}$$

In particular, since

$$\| y - x \| \geq \| y \| - \| x \| > \frac{ar}{2} - \frac{ar}{4} = \frac{ar}{4} > \frac{ar}{16},$$

we conclude from (1.49) that $\left(h \wedge \frac{ar}{16} - h\right)(y) = 0$. Therefore, by (1.47),

$$\begin{aligned}&\left(\mathcal{U}_s\left(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h\right)\right)(y)\\ =&\frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{\overline{B(y,s)}^c}\int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|)\left(h(z) \wedge g_\alpha\left(\frac{ar}{16}\right) - h(z)\right) \mathrm{d}v\mathrm{d}z.\end{aligned} \tag{1.50}$$

Let $z \in B(x, \frac{ar}{16})$ be given. Then, by the triangle inequality,

$$\| y - z \| \geq \| y \| - \| x \| - \| x - z \| > \frac{ar}{2} - \frac{ar}{4} - \frac{ar}{16} > \frac{ar}{8}.$$

Note that $s < \frac{ar}{8}$. Hence, we have that $z \in \overline{B(y,s)}^c$. Therefore, the containment $B(x, \frac{ar}{16}) \subset \overline{B(y,s)}^c$ holds. As a consequence of this relation and (1.49),

$$\begin{aligned}&\frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{\overline{B(y,s)}^c}\int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|)\left(h(z) \wedge g_\alpha\left(\frac{ar}{16}\right) - h(z)\right) \mathrm{d}v\mathrm{d}z\\ =&\frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{B(x,\frac{ar}{16})}\int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|)\left(h(z) \wedge g_\alpha\left(\frac{ar}{16}\right) - h(z)\right) \mathrm{d}v\mathrm{d}z.\end{aligned} \tag{1.51}$$

Observe that $h(z) \wedge g_\alpha(\frac{ar}{16}) - h(z) \geq -h(z)$ since h is non-negative. Thus,

$$\frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]} \int_{B(x,\frac{ar}{16})} \int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|) \left(h(z) \wedge g_\alpha \left(\frac{ar}{16} \right) - h(z) \right) \mathrm{d}v\mathrm{d}z$$
$$\geq - \frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]} \int_{B(x,\frac{ar}{16})} \int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|) h(z) \mathrm{d}v\mathrm{d}z. \tag{1.52}$$

Combining (1.50),(1.51) and (1.52),

$$\left(\mathcal{U}_s \left(h \wedge g_\alpha \left(\frac{ar}{16} \right) - h \right) \right)(y)$$
$$\geq - \frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]} \int_{B(x,\frac{ar}{16})} \int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|) h(z) \mathrm{d}v\mathrm{d}z. \tag{1.53}$$

Now, given any $z \in B(x, \frac{ar}{16})$ and $v \in B(y,s)$, by the triangle inequality,

$$\| z - v \| \geq \| y \| - \| y - v \| - \| x \| - \| x - z \|$$
$$\geq \frac{ar}{2} - \frac{ar}{8} - \frac{ar}{4} - \frac{ar}{16} > \frac{ar}{16}.$$

Since j_α is a decreasing function,

$$- \frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]} \int_{B(x,\frac{ar}{16})} \int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) j_\alpha(\| z - v \|) h(z) \mathrm{d}v\mathrm{d}z$$
$$\geq - \frac{j_\alpha(\frac{ar}{16})}{\mathbb{E}_y[\tau_{B(y,s)}]} \int_{B(x,\frac{ar}{16})} \int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) h(z) \mathrm{d}v\mathrm{d}z. \tag{1.54}$$

However, observe that

$$- \frac{j_\alpha(\frac{ar}{16})}{\mathbb{E}_y[\tau_{B(y,s)}]} \int_{B(x,\frac{ar}{16})} \int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) h(z) \mathrm{d}v\mathrm{d}z$$
$$= - \frac{j_\alpha(\frac{ar}{16})}{\mathbb{E}_y[\tau_{B(y,s)}]} \left(\int_{B(x,\frac{ar}{16})} G^\alpha_{B(0,r)}(x,z) \mathrm{d}z \right) \left(\int_{B(y,s)} G^\alpha_{B(y,s)}(y,v) \mathrm{d}v \right)$$
(by the definition of h)
$$= - \frac{j_\alpha(\frac{ar}{16})}{\mathbb{E}_y[\tau_{B(y,s)}]} \left(\int_{B(x,\frac{ar}{16})} G^\alpha_{B(0,r)}(x,z) \mathrm{d}z \right) \mathbb{E}_y[\tau_{B(y,s)}]$$
$$= - j_\alpha \left(\frac{ar}{16} \right) \left(\int_{B(x,\frac{ar}{16})} G^\alpha_{B(0,r)}(x,z) \mathrm{d}z \right). \tag{1.55}$$

Observe that $B(x, \frac{ar}{16}) \subset B(0,r)$. As a consequence,

$$-j_\alpha\left(\frac{ar}{16}\right)\left(\int_{B(x,\frac{ar}{16})} G^\alpha_{B(0,r)}(x,z)\mathrm{d}z\right) \geq -j_\alpha\left(\frac{ar}{16}\right)\left(\int_{B(0,r)} G^\alpha_{B(0,r)}(x,z)\mathrm{d}z\right)$$
$$= -j_\alpha\left(\frac{ar}{16}\right)\mathbb{E}_x\tau_{B(0,r)}. \tag{1.56}$$

Combining (1.53) - (1.56),

$$\left(\mathcal{U}_s\left(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h\right)\right)(y) \geq -j_\alpha\left(\frac{ar}{16}\right)\mathbb{E}_x[\tau_{B(0,r)}]. \tag{1.57}$$

Recall that $x \in B(0, \frac{ar}{4}) \subset B(0, ar)$. Applying (1.33) and Lemma 1.4.5,

$$-j_\alpha\left(\frac{ar}{16}\right)\mathbb{E}_x[\tau_{B(0,r)}] \geq C\left(\frac{ar}{16}\right)^{-d}\tilde{L}\left(\frac{ar}{16}\right)\frac{L(r)}{\tilde{L}(r)}. \tag{1.58}$$

By Lemma 1.4.2(b) applied to $\tilde{L}$,

$$C\left(\frac{ar}{16}\right)^{-d}\tilde{L}\left(\frac{ar}{16}\right)\frac{L(r)}{\tilde{L}(r)} = Cr^{-d}L(r)\frac{\tilde{L}\left(\frac{ar}{16}\right)}{\tilde{L}(r)} \geq Cr^{-d}L(r). \tag{1.59}$$

Combining (1.57)-(1.59), for some constant $C_L > 0$,

$$\left(\mathcal{U}_s\left(h \wedge g_\alpha\left(\frac{ar}{16}\right) - h\right)\right)(y) \geq C_L r^{-d}L(r),$$

which completes the proof of the lemma. □

1.5 Proof of Proposition 1.3.1 and Proposition 1.3.2

As $\{X^\alpha_t\}_{t\geq 0}$ is a Lévy process, and therefore is translation invariant in distribution, we may assume without loss of generality that $x_0 = 0$ in the statement of Proposition 1.3.1 and Proposition 1.3.2. We will choose b_2 first in the proof of the upper bound of (1.17) and then $b_1 < 2b_2$ in the proof of the lower bound of (1.17).

Proof of upper bound in (1.17) *of Proposition 1.3.1 :* Let $r > 0$, a be as in Lemma 1.4.4, $b_2 = \frac{a}{2}$ and $x \in B(0, \frac{ar}{4}) \equiv B(0, \frac{b_2 r}{2})$, and define $u : \mathbb{R}^d \to \mathbb{R}$ by

$$u(z) = Kr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}] - G^\alpha_{B(0,r)}(x,z) \wedge g_\alpha\left(\frac{ar}{16}\right),$$

where $z \in \mathbb{R}^d$ and $K > 0$ is a constant to be chosen later. Assume, for the moment, that u satisfies the hypothesis (i)-(iv) of Proposition 1.4.8. We will complete the proof under this assumption, and prove the assumption subsequently.

By Proposition 1.4.8 we have $u \geq 0$ on $\mathbb{R}^d$. Fix $y \in A(0, \frac{ar}{2}, r)$ as $u(y) \geq 0$ this implies

$$Kr^{-d}L(r)\mathbb{E}_y\tau_{B(0,r)} \geq g_\alpha\left(\frac{ar}{16}\right) \wedge G^\alpha_{B(0,r)}(x,y). \tag{1.60}$$

Observe that $\| x - y \| > \frac{ar}{4} > \frac{ar}{16}$ whenever $x \in B(0, \frac{ar}{4})$ and $y \in A(0, \frac{ar}{2}, r)$. As $G^\alpha_{B(0,r)}(x,y) \leq g_\alpha(\| x - y \|)$ and $g_\alpha(\cdot)$ is a decreasing function we have

$$g_\alpha\left(\frac{ar}{16}\right) \wedge G^\alpha_{B(0,r)}(x,y) = G^\alpha_{B(0,r)}(x,y). \tag{1.61}$$

Therefore from (1.60) and (1.61), the upper bound in (1.17) holds with $b_2 = \frac{a}{2}$. To complete the proof we will now prove that u satisfies the hypothesis (i) - (iv) of Proposition 1.4.8.

(i) By Proposition 1.4.9, u is a continuous function on $\mathbb{R}^d \backslash \{x\}$. Since $x \notin \overline{A(0, \frac{ar}{2}, r)}$, u is continuous on $\overline{A(0, \frac{ar}{2}, r)}$ and so hypothesis (i) holds.

(ii) Using Lemma 1.4.5, for $z \in B(0, \frac{ar}{2})$, there is a $C_1 > 0$ such that

$$\begin{aligned} u(z) &= Kr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}] - G^\alpha_{B(0,r)}(x,z) \wedge g_\alpha\left(\frac{ar}{16}\right) \\ &\geq C_1 K r^{-d}\frac{L^2(r)}{\tilde{L}(r)} - G^\alpha_{B(0,r)}(x,z) \wedge g_\alpha\left(\frac{ar}{16}\right) \\ &\geq C_1 K r^{-d}\frac{L^2(r)}{\tilde{L}(r)} - g_\alpha\left(\frac{ar}{16}\right). \end{aligned}$$

Using (1.34) in the above we have that there is a $C_2 > 0$ such that

$$u(z) \geq C_1 K r^{-d}\frac{L^2(r)}{\tilde{L}(r)} - C_2 r^{-d}\frac{L^2(\frac{ar}{16})}{\tilde{L}(\frac{ar}{16})}.$$

Using Lemma 1.4.2(b), there exists $C_3 > 0$ such that

$$u(z) \geq (C_1 K - C_3)\frac{L^2(r)}{\tilde{L}(r)} r^{-d}.$$

Thus if we choose $K \geq \frac{C_3}{C_1}$, $u(z) \geq 0$ for $z \in B(0, \frac{ar}{2})$. As $u(z) \geq 0$ for $B(0,r)^c$ we have shown that u satisfies hypothesis (ii) of Proposition 1.4.8.

(iii) We will prove that u is a bounded function. For any $z \in \mathbb{R}^d$,

$$|u(z)| = \left| Kr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}] - G^{\alpha}_{B(0,r)}(x,z) \wedge g_\alpha\left(\frac{ar}{16}\right)\right|$$
$$\le Kr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}] + \left|G^{\alpha}_{B(0,r)}(x,z) \wedge g_\alpha\left(\frac{ar}{16}\right)\right|$$
$$\le Kr^{-d}\frac{L^2(r)}{\tilde{L}(r)} + g_\alpha\left(\frac{ar}{16}\right) < \infty,$$

where we used Lemma 1.4.5 in the last inequality. Consequently, it follows that $u \in V_y$ (see (1.46)), thus satisfying hypothesis (iii) of Proposition 1.4.8.

(iv) Let $s_y = \frac{1}{2}\min(r - \parallel y \parallel, \frac{ar}{8})$ and using linearity of $U_{s_y}(\cdot)(y)$ we have

$$\mathcal{U}_{s_y}u(y) = Kr^{-d}L(r)\mathcal{U}_{s_y}(\mathbb{E}_{\cdot}[\tau_{B(0,r)}])(y) - \mathcal{U}_{s_y}\left(G^{\alpha}_{B(0,r)}(x,\cdot) \wedge g_\alpha\left(\frac{ar}{16}\right)\right)(y).$$

Applying Proposition 1.4.7(b) and Lemma 1.4.10 in the above we have that there is a $C_4 > 0$

$$\mathcal{U}_{s_y}u(y) \le r^{-d}L(r)\left(-K + C_4\right).$$

If we choose $K \ge \max\{C_4, \frac{C_3}{C_1}\}$ then $\mathcal{U}_{s_y}u(y) < 0$ for $y \in A(0, \frac{a}{2}r, r)$. So hypothesis (iv) of Proposition 1.4.8 is also satisfied.

□

1.5.1 Proof of lower bound in Proposition 1.3.1

Throughout this section, we assume that the upper bound in (1.17) holds with constant (say) $M > 1$ i.e.

$$G^{\alpha}_{B(0,r)}(x,z) \le Mr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}], \quad \text{for all } x \in B\left(0, \frac{ar}{4}\right), z \in A\left(0, \frac{ar}{2}, r\right). \tag{1.62}$$

As in the proof of the upper bound, we shall define a specific function and apply the maximum principle given by Proposition 1.4.8. We will need the upper bound and the below technical lemma for verifying the hypothesis of the proposition. The lemma provides bounds on the Green function and the operator $\mathcal{U}$ from (1.47).

Lemma 1.5.1 *Let $a \in (0, \frac{1}{3})$ be as in Lemma 1.4.4, M be as in (1.62), b_1 be as in Lemma 1.5.1(b).*

(a) There exists a constant $C_1 > 0$ such that for all $r > 0$ and $x, v \in B\left(0, \frac{ar}{2}\right)$,

$$G^{\alpha}_{B(0,r)}(x,v) \ge C_1 r^{-d}L(r)\mathbb{E}_v[\tau_{B(0,r)}]. \tag{1.63}$$

(b) There exists $b_1 \in (0, \frac{a}{8})$ *such that for all* $r > 0$, $x \in B(0, b_1 r)$ *and* $v \in B(x, b_1 r)$, *we have*

$$\frac{1}{2} G^{\alpha}_{B(0,r)}(x, v) \geq M r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}]$$

(c) Fix $r > 0$, $x \in B(0, b_1 r)$ *and define* $U : \mathbb{R}^d \to \mathbb{R}$ *by*

$$U(v) = G^{\alpha}_{B(0,r)}(x, v) \wedge M r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}]. \tag{1.64}$$

Then, there exists a constant $C > 0$ *such that*

$$\mathcal{U}_s U(y) \leq -C r^{-d} L(r),$$

for all $y \in A\left(0, \frac{ar}{2}, r\right), s < \min\left(r- \parallel y \parallel, \frac{b_1 r}{8}\right)$.

We first finish the proof of the lower bound in (1.17) of Proposition 1.3.1 assuming the Lemma and subsequently provide the proof.

Proof of lower bound in (1.17) *of Proposition 1.3.1 :* Recall the constant b_1 from Lemma 1.5.1(b). Let $r > 0$ and a be as in Lemma 1.4.4, and fix $x \in B(0, b_1 r)$. Define $H : \mathbb{R}^d \to \mathbb{R}$ by

$$H(v) = \left(G^{\alpha}_{B(0,r)}(x, v) \wedge M r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}]\right) - k r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}], \tag{1.65}$$

where $v \in \mathbb{R}^d$ and $k > 0$ is a constant to be chosen later. We will finish the proof of the proposition by assuming that H satisfies the hypothesis (i) - (iv) of Proposition 1.4.8. Indeed, by Proposition 1.4.8, it follows that $H \geq 0$ on $\mathbb{R}^d$. Thus, for every $x \in B(0, b_1 r)$ and $y \in A(0, \frac{ar}{2}, r)$,

$$G^{\alpha}_{B(0,r)}(x, y) \wedge M r^{-d} L(r) \mathbb{E}_y[\tau_{B(0,r)}] \geq k r^{-d} L(r) \mathbb{E}_y[\tau_{B(0,r)}].$$

By (1.62), the left hand side of the above expression is equal to $G^{\alpha}_{B(0,r)}(x, y)$, proving the lower bound of (1.17) with b_1 as in Lemma 1.5.1(b) and $b_2 = \frac{a}{2}$.

To complete the proof we will now prove that H satisfies the hypothesis (i) - (iv) of Proposition 1.4.8.

(i) By Proposition 1.4.9, H is a continuous function on $\mathbb{R}^d \setminus \{x\}$. By Lemma 1.5.1(b), $b_1 < \frac{a}{8}$. Hence, $x \notin \overline{A(0, \frac{ar}{2}, r)}$, and H is continuous on $\mathbb{R}^d \setminus \{x\}$. Thus, hypothesis (i) holds.

(ii) Observe that $\mathbb{E}_v[\tau_{B(0,r)}] = G^{\alpha}_{B(0,r)}(x, v) = 0$ for $v \in B(0, r)^c$. Hence, $H \equiv 0$ on $B(0, r)^c$. On the other hand, by Lemma 1.5.1(a), for any fixed $v \in B(0, \frac{ar}{2})$ we have

$$G^{\alpha}_{B(0,r)}(x, v) \wedge M r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}] \geq (C_1 \wedge M) r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}].$$

Therefore, if $k < C_1 \wedge M$, then by definition of H and the above inequality, it follows that $H(v) \geq 0$ for $v \in B(0, \frac{ar}{2})$. Thus, H satisfies hypothesis (ii) of Proposition 1.4.8.

(iii) We will prove that H is a bounded function. For any $z \in \mathbb{R}^d$,

$$\begin{aligned}|H(z)| &= \left|\left(G^\alpha_{B(0,r)}(x,z) \wedge Mr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}]\right) - kr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}]\right| \\ &\le \left|G^\alpha_{B(0,r)}(x,z) \wedge Mr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}]\right| + \left|kr^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}]\right| \\ &\le (M+k)r^{-d}L(r)\mathbb{E}_z[\tau_{B(0,r)}] \le (M+k)r^{-d}\frac{L^2(r)}{\tilde{L}(r)} < \infty,\end{aligned}$$

where we used Lemma 1.4.5 in the last inequality. Consequently, it follows that $H \in V_y$ (see (1.46)), thus satisfying hypothesis (iii) of Proposition 1.4.8.

(iv) Let $s_y = \frac{1}{2}\min(r - \|y\|, \frac{b_1 r}{8})$. By linearity of $(\mathcal{U}_{s_y}(\cdot))(y)$,

$$\begin{aligned}(\mathcal{U}_{s_y}H)(y) = (\mathcal{U}_{s_y}(G^\alpha_{B(0,r)}(x,\cdot) \wedge C_U r^{-d}L(r)\mathbb{E}_\cdot[\tau_{B(0,r)}]))(y) \\ - kr^{-d}L(r)\left(\mathcal{U}_{s_y}\mathbb{E}_\cdot[\tau_{B(0,r)}]\right)(y).\end{aligned}$$

We apply Lemma 1.5.1(c) to the first term on the right hand side and Proposition 1.4.7(b) to the second term to obtain

$$(\mathcal{U}_{s_y}H)(y) \le -(C' - k)r^{-d}L(r),$$

where C' is as in Lemma 1.5.1(c). If $k < C'$ then $(\mathcal{U}_{s_y}H)(y) < 0$. Therefore, the function H satisfies hypothesis (iv) of Proposition 1.4.8.

If we choose $k < \min\{C_1 \wedge M, C'\}$ then the proof is complete. □

We now provide the proof of Lemma 1.5.1. It will be done in two parts. We prove (a) and (b) together followed by (c).

Proof of Lemma 1.5.1 (a) and Lemma 1.5.1(b): Let $a \in (0, \frac{1}{3})$ be as in Lemma 1.4.4. Fix any $r > 0$ and $0 < \epsilon \le \frac{1}{2}$. We will first assume the following general statement that for all $x, v \in B(0, \epsilon a r)$ there is a $C_1 > 0$ such that

$$G^\alpha_{B(0,r)}(x,v) \ge C_1 \epsilon^{-d} \frac{L^2(2\epsilon a r)}{L^2(r)} \frac{\tilde{L}(r)}{\tilde{L}(2\epsilon a r)} \left(r^{-d}L(r)\mathbb{E}_v[\tau_{B(0,r)}]\right) \tag{1.66}$$

We will now prove (a). Set $\epsilon = \frac{1}{2}$ in (1.66) to obtain

$$G^\alpha_{B(0,r)}(x,v) \ge C_1 \frac{L^2(ar)}{L^2(r)} \frac{\tilde{L}(r)}{\tilde{L}(ar)} \left(r^{-d}L(r)\mathbb{E}_v[\tau_{B(0,r)}]\right)$$

An application of Lemma 1.4.2(b) with the choice $\mu = a$ to the functions L and $\tilde{L}$ yields part (a). We will now prove part (b). By Lemma 1.4.3, for any $K > 0$, there is an $0 < \epsilon_0 < a$ such that

$$\epsilon_0^{-d} \frac{L^2(\epsilon_0 r)}{L^2(r)} \frac{\tilde{L}(r)}{\tilde{L}(\epsilon_0 r)} \geq K \quad \text{for all } r > 0.$$

Choose $K = \frac{2M}{(2a)^d C_1}$ in the above and $\epsilon = \frac{\epsilon_0}{2a}$ in (1.66) to obtain

$$\frac{1}{2} G^{\alpha}_{B(0,r)}(x, v) \geq M r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}]$$

for all $x, v \in B(0, \frac{\epsilon_0}{2} r)$. As $\epsilon_0 < \frac{1}{3}$, choosing $b_1 = \frac{\epsilon_0 a}{8}$ completes the proof of part (b).

To finish the proof we prove (1.66). Fix an $x, v \in B(0, \epsilon a r)$. By Lemma 1.4.4,

$$G^{\alpha}_{B(0,r)}(x, v) \geq \frac{1}{2} g_{\alpha}(\| x - v \|). \tag{1.67}$$

As $\| x - v \| \leq 2\epsilon a r$ and g_α is a decreasing function, we have

$$\frac{1}{2} g_{\alpha}(\| x - v \|) \geq \frac{1}{2} g_{\alpha}(2\epsilon a r) \tag{1.68}$$

From (1.34), we have that there is a $C > 0$ such that

$$g_{\alpha}(2\epsilon a r) \geq C (2\epsilon a r)^{-d} \frac{L^2(2\epsilon a r)}{\tilde{L}(2\epsilon a r)} \tag{1.69}$$

Using (1.68) and (1.69) in (1.67) we have that there is a $C > 0$ such that

$$G^{\alpha}_{B(0,r)}(x, v) \geq C \epsilon^{-d} r^{-d} \frac{L^2(2\epsilon a r)}{\tilde{L}(2\epsilon a r)} = C \left(\epsilon^{-d} \frac{L^2(2\epsilon a r)}{L^2(r)} \frac{\tilde{L}(r)}{\tilde{L}(2\epsilon a r)} \right) \left(r^{-d} \frac{L^2(r)}{\tilde{L}(r)} \right) \tag{1.70}$$

By Lemma 1.4.5, we have that there is a $C > 1$ such that

$$\mathbb{E}_v[\tau_{B(0,r)}] \leq C \frac{L(r)}{\tilde{L}(r)}, \quad \text{for all } v \in B(0, \frac{ar}{2}).$$

Using the above in (1.70) we have (1.66). □

Proof of Lemma 1.5.1(c): Fix $r > 0$, $x \in B(0, b_1 r)$, $y \in A(0, \frac{ar}{2}, r)$ and $s < \min(r - \| y \|, \frac{b_1 r}{8})$. Since $z \to \mathbb{E}_z \tau_{B(0,r)}$ is bounded above by Lemma 1.4.5, it follows from (1.64) that $U : \mathbb{R}^d \to \mathbb{R}$ given by

$$U(v) = G^{\alpha}_{B(0,r)}(x, v) \wedge M r^{-d} L(r) \mathbb{E}_v[\tau_{B(0,r)}]$$

is non-negative and bounded above on $\mathbb{R}^d$. Consequently, it follows that $U \in V_y$ (see (1.46)). As the function $z \to G^{\alpha}_{B(0,r)}(x,z)$ is harmonic at y, (see Lemma 1.6.2), $\mathcal{U}_s h(y) = 0$ by Proposition 1.4.7(a). Therefore by linearity of $\mathcal{U}_s(\cdot)(y)$,

$$\mathcal{U}_s U(y) = (\mathcal{U}_s(U-h))(y).$$

By (1.47) and (1.62)

$$\begin{aligned}\mathcal{U}_s U(y) &= \frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{\overline{B(y,s)}^c}\int_{B(y,s)} G^{\alpha}_{B(y,s)}(y,v)j_\alpha(\| z-v \|)(U(z)-G^{\alpha}_{B(0,r)}(x,z))\mathrm{d}v\mathrm{d}z\\ &= \frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{B(0,\frac{ar}{2})\cap\overline{B(y,s)}^c}\int_{B(y,s)} G^{\alpha}_{B(y,s)}(y,v)j_\alpha(\| z-v \|)(U(z)-G^{\alpha}_{B(0,r)}(x,z))\mathrm{d}v\mathrm{d}z.\end{aligned}$$

As $b_1 < \frac{a}{8}$, (specified in Lemma 1.5.1(b)), we have $B(x,b_1r) \subset \overline{B(y,s)}^c \cap B\left(0,\frac{ar}{2}\right)$. Using this and the non-negativity of U above we have

$$\begin{aligned}\mathcal{U}_s U(y) &\le \frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{B(x,b_1r)}\int_{B(y,s)} G^{\alpha}_{B(y,s)}(y,v)j_\alpha(\| z-v \|)(U(z)-h(z))\mathrm{d}v\mathrm{d}z\\ &\le -\frac{1}{\mathbb{E}_y[\tau_{B(y,s)}]}\int_{B(x,b_1r)}\int_{B(y,s)} G^{\alpha}_{B(y,s)}(y,v)j_\alpha(\| z-v \|)G^{\alpha}_{B(0,r)}(x,z)\mathrm{d}v\mathrm{d}z,\end{aligned}$$

By choice of s, $B(y,s) \subset B(0,r)$. Therefore, for $v \in B(y,s)$ and $z \in B(x,b_1r)$, $\| v-z \| \le 2r$ by the triangle inequality. Using this containment and the fact that j_α is decreasing above we have

$$\begin{aligned}\mathcal{U}_s U(y) &\le -\frac{j_\alpha(2r)}{\mathbb{E}_y[\tau_{B(y,s)}]}\left(\int_{B(y,s)} G^{\alpha}_{B(y,s)}(y,v)\mathrm{d}v\right)\left(\int_{B(x,b_1r)} G^{\alpha}_{B(0,r)}(x,z)\mathrm{d}z\right)\\ &\le -j_\alpha(2r)\int_{B(x,b_1r)} G^{\alpha}_{B(0,r)}(x,z)\mathrm{d}z\\ &\le -C^{-1}r^{-d}\tilde{L}(2r)\int_{B(x,b_1r)} G^{\alpha}_{B(0,r)}(x,z)\mathrm{d}z, \qquad (1.71)\end{aligned}$$

for some $C > 0$ by (1.33). As $b_1 < \frac{a}{2}$, by Lemma 1.4.4 followed by (1.34) there is a $C > 0$ such that

$$\begin{aligned}\int_{B(x,b_1r)} G^{\alpha}_{B(0,r)}(x,z)\ \mathrm{d}z &\ge \frac{1}{2}\int_{B(x,b_1r)} g_\alpha(\| x-z \|)\mathrm{d}z\\ &\ge C\int_{B(x,b_1r)} \frac{L^2(\| x-z \|)}{\tilde{L}(\| x-z \|)\|x-z\|^d}\mathrm{d}z\end{aligned}$$

Using Lemma 1.4.2(a) above we have that there is a $C > 0$ such that

$$\int_{B(x,b_1 r)} G^{\alpha}_{B(0,r)}(x,z)\,\mathrm{d}z \geq C\frac{L(b_1 r)}{\tilde{L}(b_1 r)}.$$

Using the above in (1.71) we have that there is a $C > 0$ such that

$$\mathcal{U}_s U(y) \leq -Cr^{-d}\tilde{L}(2r)\frac{L(b_1 r)}{\tilde{L}(b_1 r)} = -Cr^{-d}\frac{\tilde{L}(2r)}{\tilde{L}(r)}\frac{\tilde{L}(r)}{\tilde{L}(b_1 r)}\frac{L(b_1 r)}{L(r)}L(r).$$

We apply Lemma 1.4.2(b) to the functions L and $\tilde{L}$ on the right hand side with the appropriate choices of μ to obtain

$$\mathcal{U}_s U(y) \leq -Cr^{-d}L(r),$$

for some $C > 0$. This completes the proof.

□

1.5.2 Proof of Proposition 1.3.2

Recall, $a \in (0, \frac{1}{3})$ from Lemma 1.4.5, $b_1 \in (0, \frac{a}{8})$ from Lemma 1.5.1(b) and choice of $b_2 = \frac{a}{2}$ so that Proposition 1.3.1 holds.

Proof of Proposition 1.3.2 : Let $r > 0$ and $z \in B(0,r)^c$ be arbitrary. Suppose that $x_1, x_2 \in B(0, b_1 r)$. By the definition of the Poisson kernel, for $i = 1, 2$,

$$K_{B(0,r)}(x_i, z) = \int_{B(0,r)} G^{\alpha}_{B(0,r)}(x_i,y) j_\alpha(\| z-y \|)\mathrm{d}y$$
$$= \int_{A(0,b_2 r,r)} G^{\alpha}_{B(0,r)}(x_i,y) j_\alpha(\| z-y \|)\mathrm{d}y + \int_{B(0,b_2 r)} G^{\alpha}_{B(0,r)}(x_i,y) j_\alpha(\| z-y \|)\mathrm{d}y.$$

So the proof Proposition 1.3.1 will be complete if we show that there is a $C > 1$ such that :

$$\int_{A(0,b_2 r,r)} G^{\alpha}_{B(0,r)}(x_1,y) j_\alpha(\| z-y \|)\mathrm{d}y \leq C\int_{A(0,b_2 r,r)} G^{\alpha}_{B(0,r)}(x_2,y) j_\alpha(\| z-y \|)\mathrm{d}y, \tag{1.72}$$

$$\int_{B(0,b_2 r)} G^{\alpha}_{B(0,r)}(x_1,y) j_\alpha(\| z-y \|)\mathrm{d}y \leq C\int_{B(0,b_2 r)} G^{\alpha}_{B(0,r)}(x_i,y) j_\alpha(\| z-y \|)\mathrm{d}y. \tag{1.73}$$

We first prove (1.72). Fix $y \in A(0, b_2 r, r)$. Then, using the upper and lower bounds in (1.17) of Proposition 1.3.1, there is a $C > 1$ such that

$$G^\alpha_{B(0,r)}(x_1, y) \le C r^{-d} L(r) \mathbb{E}_y[\tau_{B(0,r)}] \le C^2 G^\alpha_{B(0,r)}(x_2, y).$$

Multiplying the inequality above by $j_\alpha(\| z - y \|)$ and integrating over $y \in A(0, b_2 r, r)$, completes the proof of (1.72).

We will now prove (1.73). By choice of b_1, b_2, it is easy to see that

$$B(x_2, b_1 r) \subset B(0, b_2 r) \subset B(x_1, (b_1 + b_2) r). \tag{1.74}$$

So, using (1.14) and (1.74) we have

$$\int_{B(0,b_2 r)} G^\alpha_{B(0,r)}(x_1, y) \mathrm{d}y \le \int_{B(0,b_2 r)} g_\alpha(\| x_1 - y \|) \mathrm{d}y \le \int_{B(x_1,(b_1+b_2)r)} g_\alpha(\| x_1 - y \|) \mathrm{d}y. \tag{1.75}$$

Further using Lemma 1.4.4 and (1.74) we have

$$\int_{B(0,b_2 r)} G^\alpha_{B(0,r)}(x_2, y) \mathrm{d}y \ge \frac{1}{2} \int_{B(0,b_2 r)} g_\alpha(\| x_2 - y \|) \mathrm{d}y \ge \frac{1}{2} \int_{B(x_2,b_1 r)} g_\alpha(\| x_2 - y \|) \mathrm{d}y.$$

Combining the above with (1.75) we obtain

$$\frac{\int_{B(0,b_2 r)} G^\alpha_{B(0,r)}(x_2, y) \mathrm{d}y}{\int_{B(0,b_2 r)} G^\alpha_{B(0,r)}(x_1, y) \mathrm{d}y} \ge \frac{1}{2} \frac{\int_{B(x_2,b_1 r)} g_\alpha(\| x_2 - y \|) \mathrm{d}y}{\int_{B(x_1,(b_1+b_2)r)} g_\alpha(\| x_1 - y \|) \mathrm{d}y} \tag{1.76}$$

We will now bound the right hand side from below. Using (1.34) followed by Lemma 1.4.2(a) there is a $C > 0$ such that

$$\begin{aligned} \frac{\int_{B(x_2,b_1 r)} g_\alpha(\| x_2 - y \|) \mathrm{d}y}{\int_{B(x_1,(b_1+b_2)r)} g_\alpha(\| x_1 - y \|) \mathrm{d}y} &\ge C \frac{\int_{B(x_2,b_1 r)} \frac{L^2(\|x_2 - y\|)}{\tilde{L}(\|x_2 - y\|) \|x_2 - y\|^d} \mathrm{d}y}{\int_{B(x_1,(b_1+b_2)r)} \frac{L^2(\|x_1 - y\|)}{\tilde{L}(\|x_1 - y\|) \|x_1 - y\|^d} \mathrm{d}y} \\ &\ge C \frac{L(b_1 r) \tilde{L}((b_1 + b_2) r)}{\tilde{L}(b_1 r) L((b_1 + b_2) r)} \end{aligned} \tag{1.77}$$

By elementary algebra observe that for $r > 0$

$$\frac{L(b_1 r) \tilde{L}((b_1 + b_2) r)}{\tilde{L}(b_1 r) L((b_1 + b_2) r)} = \frac{L(b_1 r)}{L(r)} \frac{L(r)}{L((b_1 + b_2) r)} \frac{\tilde{L}(b_1 r)}{\tilde{L}(r)} \frac{\tilde{L}(r)}{\tilde{L}((b_1 + b_2) r)}.$$

Using this identity and applying Lemma 1.4.2(b) to each of the four terms with appropriate μ, we have that there is a $C > 0$ such that

$$\frac{\int_{B(x_2,b_1r)} g_\alpha(\| x_2 - y \|)\mathrm{d}y}{\int_{B(x_1,(b_1+b_2)r)} g_\alpha(\| x_1 - y \|)\mathrm{d}y} \geq C.$$

So, from (1.76) it follows that there is a $C > 0$ such that

$$\int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_1, y)\mathrm{d}y \leq C \int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_2, y)\mathrm{d}y. \tag{1.78}$$

As $z \in B(0, r)^c$ for any $y \in B(0, b_2r)$ we have

$$(1 - b_2) \| z \| \leq \| z - y \| \leq (1 + b_2) \| z \| .$$

As j_α is a decreasing function we have

$$j_\alpha((1 + b_2) \| z \|) \leq j_\alpha(\| z - y \|) \leq j_\alpha((1 - b_2) \| z \|). \tag{1.79}$$

On the other hand, by (1.33), we have

$$\frac{j_\alpha((1 - b_2) \| z \|)}{j_\alpha((1 + b_2) \| z \|)} \leq C\frac{(1 - b_2)^d}{(1 + b_2)^d} \frac{\tilde{L}((1 - b_2) \| z \|)}{\tilde{L}((1 + b_2) \| z \|)} \leq C, \tag{1.80}$$

for some $C > 0$, where we used Lemma 1.4.2(b) with $\tilde{L}$. Now (1.73) is an easy consequence of (1.78), (1.79) and (1.80) as indicated below,

$$\begin{aligned}
\int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_1, y) j_\alpha(\| z - y \|)\mathrm{d}y &\overset{(1.79)}{\leq} j_\alpha((1 - b_2) \| z \|) \int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_1, y)\mathrm{d}y \\
&\overset{(1.78)}{\leq} C j_\alpha((1 - b_2) \| z \|) \int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_2, y)\mathrm{d}y \\
&\overset{(1.80)}{\leq} C j_\alpha((1 + b_2) \| z \|) \int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_2, y)\mathrm{d}y \\
&\overset{(1.79)}{\leq} C \int_{B(0,b_2r)} G^\alpha_{B(0,r)}(x_2, y) j_\alpha(\| z - y \|)\mathrm{d}y.
\end{aligned} \tag{1.81}$$

This completes the proof. □

1.6 Proof of Proposition 1.4.9

Recall that Proposition 1.4.9 was used to prove the upper and lower bounds of Proposition 1.3.1. In this section we shall present the proof of Proposition 1.4.9. This proof was communicated to us in the note [20]. We begin by proving some preliminary lemmas. The first, Lemma 1.6.1 is an analogue of Zaremba's cone condition (see [51, Theorem 2.19, Chapter 4] for the original statement). Next, in Lemma 1.6.2 we show that the Green function is harmonic. Finally, in Lemma 1.6.3 we discuss sufficient conditions for the harmonic function with Dirichlet boundary conditions to be continuous.

Recall from (1.5) that $X_t^\alpha = B_{S_t^\alpha}$ is the geometric stable process of index α, where $\{S_t^\alpha\}_{t\geq 0}$ is the subordinator with Laplace exponent given by (1.4). Throughout this section, fix $d \geq 3$, $\alpha \in (0,2)$, $r > 0$ and $x_0 \in \mathbb{R}^d$. Denote the ball $B(x_0, r)$ by B and its boundary $\partial B(x_0, r)$ by ∂B.

Lemma 1.6.1 *For every $x \in \partial B$, $\mathbb{P}_x(\tau_B = 0) = 1$.*

Proof. Let $x \in \partial B$. By the geometry of $B \subset \mathbb{R}^d$, there exists $L > 0$ and a set $\Gamma_x \subset B^c$ such that Γ_x is a spherical sector of $B(x, L)$ and $\frac{m(\Gamma_x)}{m(B(x,L))} = C_\Gamma > 0$(where $m(A)$ is the Lebesgue measure of $A \subset \mathbb{R}^d$). We have

$$\begin{aligned} \mathbb{P}_x(\tau_B = 0) = \lim_{t\downarrow 0} \mathbb{P}_x(\tau_B \leq t) &\geq \lim_{t\downarrow 0} \mathbb{P}_x(X_t^\alpha \in \Gamma_x) \\ &= \lim_{t\downarrow 0} C_\Gamma \mathbb{P}_x(X_t^\alpha \in B(x, L)) = C_\Gamma > 0, \end{aligned} \tag{1.82}$$

where the equality between the lines follows by the rotational invariance of $\{X_t^\alpha\}_{t\geq 0}$.

Let $\mathcal{F}_{0+}$ be the germ σ-algebra corresponding to $\{X_t^\alpha\}_{t\geq 0}$ with $X_0^\alpha = x$. Observe that $\{\tau_B = 0\} \in \mathcal{F}_{0+}$. It follows from Blumenthal's $0-1$ law (see [74, Proposition 40.4] and [15, Proposition 4]) that $\mathbb{P}_x(\tau_B = 0) \in \{0, 1\}$, which combined with (1.82) shows that $\mathbb{P}_x(\tau_B = 0) = 1$, as desired. □

Recall the measure $G_B^\alpha(x, \cdot)$ from (1.12) and its density, the Green function $G_B^\alpha(x, y)$. By [74, (30.5),Theorem 30.1] and Tonelli's theorem, one sees that

$$G_B^\alpha(x, y) = \int_0^{\tau_B} \int_0^\infty p(s, x, y)\mathbb{P}(S_t^\alpha \in \mathrm{d}s)\mathrm{d}t \quad \text{for all } x, y \in B, \tag{1.83}$$

where $p(s, x, y)$ is the transition density of Brownian motion on $\mathbb{R}^d$.

We will now prove that the Green function is harmonic. Recall the definition of harmonic functions from Definition 1.1.1.

Lemma 1.6.2 (Harmonicity) *Given $x \in B$, $G_B^\alpha(x, \cdot)$ is harmonic on $B \setminus \{x\}$.*

Proof. Let $D \subset B \setminus \{x\}$ be given, and let $y \in D$. We have

$$\begin{aligned}
\mathbb{E}_y[G_B^\alpha(x, X_{\tau_D}^\alpha)] &= \int_{\mathbb{R}^d} G_B^\alpha(x,z)\mathbb{P}_y(X_{\tau_D}^\alpha \in \mathrm{d}z) \\
&\overset{(1.83)}{=} \int_{\mathbb{R}^d}\int_0^{\tau_B}\int_0^\infty p(s,x,z)\mathbb{P}(S_t^\alpha \in \mathrm{d}s)\ \mathrm{d}t\ \mathbb{P}_y(X_{\tau_D}^\alpha \in \mathrm{d}z) \\
&= \int_0^{\tau_B}\int_0^\infty\int_{\mathbb{R}^d} p(s,x,z)\mathbb{P}_y(X_{\tau_D}^\alpha \in \mathrm{d}z)\mathbb{P}(S_t^\alpha \in \mathrm{d}s)\mathrm{d}t,
\end{aligned} \tag{1.84}$$

where we used Tonelli's theorem in the last equality. Note that $\tau_D \leq \tau_B < \infty$ since $\{X_t^\alpha\}_{t\geq 0}$ is transient. By the Strong Markov property applied to τ_D we have

$$\begin{aligned}
\int_0^{\tau_B}\int_0^\infty\int_{\mathbb{R}^d} p(s,x,z)\mathbb{P}_y(X_{\tau_D}^\alpha \in \mathrm{d}z)\mathbb{P}(S_t^\alpha \in \mathrm{d}s)\mathrm{d}t &= \int_0^{\tau_B}\int_0^\infty p(s,x,y)\mathbb{P}(S_t^\alpha \in \mathrm{d}s)\mathrm{d}t \\
&\overset{(1.83)}{=} G_B^\alpha(x,y).
\end{aligned}$$

Combining the above with (1.84), $G_B^\alpha(x,\cdot)$ is harmonic in $B \setminus \{x\}$. □

Finally, we state and prove the following lemma, a sufficient condition for harmonic functions to be continuous under Dirichlet boundary conditions. Before stating it, we note that by [74, (30.5),Theorem 30.1] and [15, Exercise 4, Section 1], $\{X_t^\alpha\}_{t\geq 0}$ is a *strong Feller* process i.e. for every bounded measurable f and $t > 0$, $x \mapsto \mathbb{E}_x[f(X_t^\alpha)]$ is a continuous function on $\mathbb{R}^d$.

Lemma 1.6.3 *Let $\phi : B^c \to \mathbb{R}$ be a bounded function such that ϕ is continuous at $x \in \partial B$. Then,*

$$\lim_{y\to x, y\in B} \mathbb{E}_y[\phi(X_{\tau_B}^\alpha)] = \phi(x).$$

Proof. The following claim will be used to prove our result.
Claim: For any $r > 0$,

$$\lim_{y\to x, y\in B} \mathbb{P}_y[\| X_{\tau_B}^\alpha - y \| < r] = 1. \tag{1.85}$$

Assuming the claim, let ϕ be bounded and continuous at $x \in \partial B$. For any fixed $r > 0$, by the triangle inequality and boundedness of ϕ we have

$$\begin{aligned}
\left|\mathbb{E}_y[\phi(X_{\tau_B}^\alpha)] - \phi(x)\right| &\leq 2\mathbb{P}_y(\|X_{\tau_B}^\alpha - y\| > r)\sup_{B^c}|\phi| \\
&\quad + \mathbb{P}_y(\|X_{\tau_B}^\alpha - y\| \leq r) \sup_{z\in B(x,\|y-x\|+r)\cap B^c} |\phi(z) - \phi(x)|.
\end{aligned}$$

In the above inequality we first let $y \to x, y \in B$. Then, we let $r \to 0$. By (1.85) and the continuity of ϕ at x, it follows that $\mathbb{E}_y[\phi(X_{\tau_B}^\alpha)] \to \phi(x)$, as desired.

We now prove the claim. For any $0 < \delta < \epsilon$, define

$$g_\delta(x) = \mathbb{P}_x(X_s^\alpha \in B, \delta \le s \le \epsilon) = \int_{\mathbb{R}^d} \mathbb{P}_y(\tau_B > \epsilon - \delta)\mathbb{P}_x(X_\delta^\alpha \in \mathrm{d}y).$$

Since $\{X_t^\alpha\}_{t\ge0}$ is a strong Feller Process, g_δ is a convolution of a continuous and integrable function, hence continuous. Furthermore, as $\delta \downarrow 0$, g_δ decreases to the function $g(y) = \mathbb{P}_y(\tau_B > \epsilon)$.

Thus, g is an upper-semicontinuous function, being the decreasing limit of a sequence of continuous functions. Furthermore, $g(x) = 0$ by Lemma 1.6.1. By the definition of upper-semicontinuity,

$$0 = g(x) \ge \overline{\lim}_{y\to x} g(y) \ge \overline{\lim}_{y\to x, y\in B} g(y) \ge \underline{\lim}_{y\to x, y\in B} g(y) \ge 0.$$

Thus, $\lim_{y\to x} g(y) = \lim_{y\to x} \mathbb{P}_y(\tau_B > \epsilon) = 0$. Now, if $r > 0$ is fixed, then

$$\begin{aligned}
\mathbb{P}_y(\| X_{\tau_B}^\alpha - y \| < r) &\ge \mathbb{P}_y\left(\max_{0\le t\le\epsilon} \| X_t^\alpha - X_0^\alpha \| < r; \tau_B \le \epsilon\right) \\
&\ge \mathbb{P}_y\left(\max_{0\le t\le\epsilon} \| X_t^\alpha - X_0^\alpha \| < r\right) - \mathbb{P}_y\left(\tau_B > \epsilon\right)) \\
&= \mathbb{P}_0\left(\max_{0\le t\le\epsilon} \| X_t^\alpha \| < r\right) - \mathbb{P}_y\left(\tau_B > \epsilon\right)
\end{aligned}$$

Letting $y \to x, y \in B$ followed by $\epsilon \to 0$, we obtain the proof of (1.85), as desired. □

Now we are ready to prove Proposition 1.4.9. We begin with the proof of Proposition 1.4.9(a).

Proof of Proposition 1.4.9(a). Fix $x \in B$. We will first show that $G_B^\alpha(x,\cdot)$ is continuous in $B \setminus \{x\}$ using [53, Theorem 1.4]. Let $y \ne x, y \in B$ and $r' = \min\{\|y - x\|, \|r - y\|, \frac{1}{4}\}$. Then, $B(y, r') \subset B$, hence $G_B^\alpha(x,\cdot)$ is harmonic in $B(y, r')$ by Lemma 1.6.2.

We will now verify that the hypotheses of [53, Theorem 1.4] hold with the choice $K(y,h) = j_\alpha(h)$, where j_α is the jump kernel defined in Section 1.3. Hypotheses $(K1)$ and $(K2)$ hold since $\{X_t^\alpha\}_{t\ge0}$ is a symmetric Lévy process, while $(K3)$ holds by the jump kernel bound (1.33). Applying [53, Theorem 1.4] to $G_B^\alpha(x,\cdot)$ in the ball $B(y, r')$, it follows that $G_B^\alpha(x,\cdot)$ is continuous at y, as desired.

We will now show that $G_B^\alpha(x,\cdot)$ is continuous on ∂B. Suppose that $z \in \partial B$. Let B' be a larger ball that contains $\overline{B}$ so that $z \in B'$. As $\{X_t^\alpha\}_{t\ge0}$ is transient, $\tau_B \le \tau_{B'} < \infty$

almost surely. For all $y \in B$ we have

$$\begin{aligned}
&G^{\alpha}_{B'}(x,y)\\
&\overset{(1.83)}{=}\int_0^{\tau_{B'}}\int_0^{\infty} p(s,x,y)\mathbb{P}(S^{\alpha}_t\in \mathrm{d}s)\mathrm{d}t\\
&=\int_0^{\tau_B}\int_0^{\infty} p(s,x,y)\mathbb{P}(S^{\alpha}_t\in \mathrm{d}s)\mathrm{d}t+\int_{\tau_B}^{\tau_{B'}}\int_0^{\infty} p(s,x,y)\mathbb{P}(S^{\alpha}_t\in \mathrm{d}s)\mathrm{d}t\\
&=G^{\alpha}_B(x,y)+\mathbb{E}_x[G^{\alpha}_{B'}(X^{\alpha}_{\tau_B},y)],
\end{aligned}$$

where we used (1.83) and the strong Markov property for τ_B in the last equality.

We let $y\to z, y\in B$ on the both sides of the above equality. Since $z\in B'$, it follows from the first part of this proof applied to the ball B' that $G^{\alpha}_{B'}(x,y)\to G^{\alpha}_{B'}(x,z)$. By the same assertion and Lemma 1.6.3, it follows that

$$\lim_{y\to z, y\in B}\mathbb{E}_x[G^{\alpha}_{B'}(X^{\alpha}_{\tau_B},y)]=G^{\alpha}_{B'}(x,z).$$

Thus, $\lim_{y\to x,y\in B} G^{\alpha}_B(x,y)=0$. On the other hand, clearly $G^{\alpha}_{B'}(x,z')=0$ for $z'\in B^c$. It follows that $G^{\alpha}_B(x,\cdot)$ is continuous on ∂B.

That $G^{\alpha}_B(x,\cdot)$ is continuous on $\overline{B}^c$ is trivial since it is identically zero in this region. This completes the proof of the proposition. □

We will now prove Proposition 1.4.9(b).

Proof of Proposition 1.4.9(b). Let $M_s=\mathbb{E}_0[\tau_{B(0,s)}]$ for $s>0$. Then $M_s<\infty$ for all $s>0$ by Lemma 1.4.5. Let $f(x):=\mathbb{E}_x[\tau_B]$. Note that f is bounded on $\overline{B}$. Furthermore, by the Markov property,

$$\mathbb{E}_x[\tau_B^2]=2\int_0^{\tau_B}\mathbb{E}_x[\tau_B]\mathrm{d}t\le 2M_r^2. \tag{1.86}$$

We will first prove that f is continuous in B. Fix $x\in B$ and let $r(x)>0$ be such that $B(x,r(x))\subset B$. We will show that f coincides with a uniform limit of continuous functions in a neighbourhood of x and is therefore continuous. To do this, start with a decreasing sequence $\{b_n\}_{n\ge1}$ such that $b_1\le r(x)$ and $b_n\to 0$. For fixed $n\ge 1$, clearly $\mathbb{P}_x(\tau_{B(x,b_n)}\le c)\to\mathbb{P}_x(\tau_{B(x,b_n)}=0)=0$. Thus, there exists $c_n>0$ small enough such that

$$\mathbb{P}_x(\tau_{B(x,b_n)}\le c_n)<\frac{1}{n}. \tag{1.87}$$

Since $b_n\to 0$, the above condition forces $c_n\to 0$.

We will now prove that

$$\mathbb{E}_y[f(X^{\alpha}_{c_n})]\to f(y) \text{ uniformly in } y\in B(x,r(x)). \tag{1.88}$$

To prove this, for any $n \geq 1$ and $y \in B(x, r(x))$ we have

$$\begin{aligned} |f(y) - \mathbb{E}_y[f(X^\alpha_{c_n})]| &= |f(y) - \mathbb{E}_y[f(X^\alpha_{c_n}); \tau_{B(y,b_n)} > c_n] - \mathbb{E}_y[f(X^\alpha_{c_n}); \tau_{B(y,b_n)} \leq c_n]| \\ &\leq |f(y) - \mathbb{E}_y[f(X^\alpha_{c_n}); \tau_{B(y,b_n)} > c_n]| + |\mathbb{E}_y[f(X^\alpha_{c_n}); \tau_{B(y,b_n)} \leq c_n]| \\ &:= I_1 + I_2. \end{aligned} \tag{1.89}$$

Note that $I_2 \leq \frac{M_r}{n}$ by (1.87). We shall now bound I_1. By the definition of f,

$$\begin{aligned} I_1 &= \left|\mathbb{E}_y[\tau_B; \tau_{B(y,b_n)} > c_n] + \mathbb{E}_y[\tau_B; \tau_{B(y,b_n)} \leq c_n] - \mathbb{E}_y[\mathbb{E}_{X^\alpha_{c_n}}[\tau_B]; \tau_{B(y,b_n)} > c_n]\right| \\ &\leq \left|\mathbb{E}_y[\tau_B - \mathbb{E}_{X^\alpha_{c_n}}[\tau_B]; \tau_{B(y,b_n)} > c_n]\right| + \left|\mathbb{E}_y[\tau_B; \tau_{B(y,b_n)} \leq c_n]\right| \\ &\leq c_n + \mathbb{E}_y[\tau_B^2]\mathbb{P}_y(\tau_{B(y,b_n)} \leq c_n) \leq c_n + 2\frac{M_r^2}{n}. \end{aligned}$$

where we used the Cauchy-Schwarz inequality in the second last inequality, and (1.39) and (1.87) in the last inequality. By the above bounds and (1.89),

$$|f(y) - \mathbb{E}_y[f(X^\alpha_{c_n})]| \leq c_n + \frac{2M_r^2 + M_r}{n} \quad \text{for all } y \in B(x, r(x)).$$

From here, (1.88) follows. Furthermore, by the strong Feller property, $y \to \mathbb{E}_y[f(X^\alpha_{c_n})]$ is continuous. It follows that f is continuous at every $x \in B$.

Now, we will prove that f is continuous in ∂B. Let $x \in \partial B$. Pick a ball B' containing $\overline{B}$ so that $x \in B'$. By the first part of this argument, $y \to \mathbb{E}_y[\tau_{B'}]$ is continuous at x. However, by the strong Markov property applied to τ_B, for $y \in B$ we have

$$\mathbb{E}_y[\tau_B] = \mathbb{E}_y[\tau_{B'}] - \mathbb{E}_y[\mathbb{E}_{X^\alpha_{\tau_B}}[\tau_{B'}]]. \tag{1.90}$$

On both sides, we let $y \to x, y \in B$. We have $\mathbb{E}_y[\tau_{B'}] \to \mathbb{E}_x[\tau_{B'}]$ by the first part of this argument. Further, by the same assertion and Lemma 1.6.3 we have

$$\mathbb{E}_y[\mathbb{E}_{X^\alpha_{\tau_{B'}}}[\tau_B]] \to \mathbb{E}_x[\mathbb{E}_{X^\alpha_{\tau_{B'}}}[\tau_B]] = \mathbb{E}_x[\tau_{B'}].$$

Combining this with (1.90), $\lim_{y \to x, y \in B} \mathbb{E}_y[\tau_B] = 0$. Since $f \equiv 0$ on B^c, We have shown that f is continuous on ∂B. Finally, since $f \equiv 0$ on $\overline{B}^c$, it follows that f is continuous everywhere, as desired. □

Chapter 2

The conformal walk dimension of geometric stable processes

Recall the geometric stable processes from (1.5). In this chapter, we will define the notion of conformal walk dimension and prove that it is infinite for any geometric stable process.

Recall the elliptic Harnack inequality from Theorem 1.1.2. The parabolic Harnack inequality(PHI) is a generalization of the EHI in the following sense: the EHI only constrains the behaviour of a stochastic process in space, while the PHI constrains the joint space-time behaviour of the process and therefore has stronger implications when it holds. A natural question is to ask how the inequalities are related.

In this direction, Murugan and Kajino[54] showed that one can relate the EHI and PHI for symmetric diffusion processes by suitable reparametrisations of time and space (see Theorem 2.2.1). Roughly speaking, if the EHI holds then for any $\epsilon > 0$, the PHI with space time scaling $r \mapsto r^{2+\epsilon}$ can be made to hold by a reparametrisation. In the language of the same paper, this is equivalent to the "conformal walk dimension" of such a process being equal to 2.

This characterisation can be used to show the stability of the EHI for symmetric diffusion processes, as a consequence of the stability of the PHI(see Proposition 2.2.5). One can ask if the same strategy works for symmetric jump processes, given that a characterisation of PHI(β) given by Chen, Kumagai and Wang[26] exists in such a setting.

Our aim in this chapter is to show that such a strategy fails, by proving that the conformal walk dimension of any geometric stable process is *infinite*. That is, geometric stable processes can not be made to satisfy the PHI with space time scaling r^β for some $\beta > 0$ even after a reparametrisation of space and time. Combining this with Theorem 1.1.2 completes the main objective of the preprint [2] i.e. showing that geometric stable processes are a "barrier" between characterisations of the EHI and the PHI.

We begin by defining the problem in Section 2.1. In Section 2.2 we mention some literature, motivate the main problem of this chapter and provide the main idea of the proof. In Section 2.3 we state a key proposition and prove our main result assuming it. The key proposition is proved in Sections 2.4 with the help of additional propositions which are proved in 2.5 and 2.6 respectively.

2.1 Main result

Throughout this subsection, we fix $\alpha \in (0, 2)$ and $d \geq 3$, and let m denote the Lebesgue measure. For defining the notion of the conformal walk dimension, we will require several preliminary definitions. We first define the jump kernel for the geometric stable processes of index α, $\alpha \in (0, 2), \{X_t^\alpha\}_{t\geq 0}$ and the corresponding Dirichlet form associated with it. Then, we will define metric-measure-Dirichlet(MMD) spaces and the notion of quasisymmetric maps. After this, we will provide descriptions of admissible measures and caloric functions. Based on these, we will provide a precise definition of the conformal walk dimension and state our main result, Theorem 2.1.4. All integration, unless specified otherwise, is performed with respect to the Lebesgue measure.

To define the Dirichlet form associated to $\{X_t^\alpha\}_{t\geq 0}$, and the corresponding metric-measure-Dirichlet spaces, we will need some preliminaries. Let (X, d) be a Polish space (i.e. a complete separable metric space). Denote the open ball of radius $r > 0$ around a point $x \in X$ by $B_d(x, r)$. In addition, we will assume that (X, d) is locally compact and *uniformly perfect* (i.e. there exists a constant $C > 0$ such that $B(x, r) \setminus B(x, \frac{r}{C}) \neq \emptyset$ for all $x \in X, r > 0$ such that $B(x, r) \neq X$).

Let μ be a Radon measure on (X, d) with full support. That is, $\mu(K) < \infty$ for all $K \subset X$ compact, and $\mu(U) > 0$ for all $U \subset X$ open. Let $L^2(X, \mu)$ be the set of all (μ-a.e. equivalence classes of) functions f such that $\int_X f(x)^2 \mathrm{d}\mu(x) < \infty$. For any $f, g \in L^2(X, \mu)$ let $\langle f, g\rangle_{L^2(X,\mu)} = \int_X f(x)g(x)d\mu(x)$ and $\|f\|_{L^2(X,\mu)} = \langle f, f\rangle^{\frac{1}{2}}_{L^2(X,\mu)}$.

Consider a non-negative definite symmetric bilinear form

$$\mathscr{E} : D(\mathscr{E}) \times D(\mathscr{E}) \to \mathbb{R},$$

with $D(\mathscr{E})$ being a dense subset of $L^2(X, \mu)$. We say that $(\mathscr{E}, D(\mathscr{E}))$ is *closed* if $D(\mathscr{E})$ is a Hilbert space under the norm

$$\|f\|_{\mathscr{E}_1} = \sqrt{\mathscr{E}(f, f) + \|f\|^2_{L^2(X,\mu)}}, \tag{2.1}$$

is *Markovian* if $\max\{f, 0\} \wedge 1 \in D(\mathscr{E})$, and

$$\mathscr{E}(\max\{f, 0\} \wedge 1, \max\{f, 0\} \wedge 1) \leq \mathscr{E}(f, f),$$

for all $f \in D(\mathscr{E})$, and is said to be *regular* if $D(\mathscr{E}) \cap C_c(X)$ is dense in both the normed spaces $(D(\mathscr{E}), \|\cdot\|_{\mathscr{E}_1})$ and $(C_c(X), \|\cdot\|_\infty)$. We call a non-negative definite symmetric bilinear form which is closed, Markovian and regular, a *symmetric regular Dirichlet form*. We refer the reader to [33] for a detailed study of Dirichlet forms.

We will now define the Dirichlet form associated to the geometric stable process of index α, $\{X_t^\alpha\}_{t\geq 0}$. Recall its jump kernel j_α from Section 1.2.2.

Definition 2.1.1 ([33, Section 1.1]) *The Dirichlet form corresponding to $\{X_t^\alpha\}_{t\geq 0}$ is defined by*

$$\mathscr{E}^\alpha(f,g) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(x)-f(y))(g(x)-g(y))j_\alpha(\| x-y \|)\mathrm{d}x\mathrm{d}y \text{ for all } f,g \in D(\mathscr{E}^\alpha),$$

where $D(\mathscr{E}^\alpha)$ is the closure of $C_c^\infty(\mathbb{R}^d)$ (compactly supported smooth functions on $\mathbb{R}^d$) in $L^2(\mathbb{R}^d, m)$ under the norm

$$\|f\|_{\mathscr{E}^\alpha} = \left(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(f(x)-f(y))^2 j_\alpha(\| x-y \|)\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^d} f(x)^2\mathrm{d}x\right)^{\frac{1}{2}}.$$

We remark that an alternate definition of $\mathscr{E}^\alpha$ can be given using the Fourier transform. Let $\mathcal{S}(\mathbb{R}^d)$ denote the space of Schwartz functions on $\mathbb{R}^d$. For $f \in \mathcal{S}(\mathbb{R}^d)$, we define

$$\mathscr{E}^\alpha(f,f) = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \log(1+\|\xi\|^\alpha)\mathrm{d}\xi,$$

where

$$\hat{f}(\xi) = (2\pi)^{-n/2}\int_{\mathbb{R}^d} e^{i\langle \xi,y\rangle} f(y)\mathrm{d}y$$

is the Fourier transform of $f \in \mathcal{S}(\mathbb{R}^d)$. This can be extended to $f \in D(\mathscr{E}^\alpha)$ by a density argument, where $D(\mathscr{E}^\alpha)$ is as in Definition 2.1.1. By construction, $(\mathscr{E}^\alpha, D(\mathscr{E}^\alpha))$ is a regular symmetric Dirichlet form. For a proof that the above definitions coincide, we refer the reader to [21, (2.2.9),(2.2.11), Section 2.2.2].

A *metric-measure-Dirichlet* (MMD) space is a uniformly perfect, locally compact Polish space (X,d) equipped with a Radon measure μ on X having full support and a regular Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ on $L^2(X,\mu)$. We denote a MMD space collectively by $(X,d,\mu,\mathscr{E},D(\mathscr{E}))$. We will now define the notion of quasisymmetry as in [50, Definition 3.1].

Definition 2.1.2 (Quasisymmetry) *Let (X,d) be a metric space and θ be another metric on X. For a homeomorphism $\eta : [0,\infty) \to [0,\infty)$, we say that θ is η-quasisymmetric to d if*

$$\frac{\theta(x,a)}{\theta(x,b)} \leq \eta\left(\frac{d(x,a)}{d(x,b)}\right) \quad \text{for all } x,a,b \in X, a \neq b.$$

We say that θ is quasisymmetric to d if there exists $\eta : [0,\infty) \to [0,\infty)$ such that θ is η-quasisymmetric to d. The set of all metrics quasisymmetric to d is called the conformal gauge of d and is denoted $\mathscr{J}(X,d)$.

We refer the reader to [44, Chapter 10] for a review on the notion of quasisymmetry.

The descriptions of admissible measures, time-changed MMD spaces and caloric functions will now be provided. We refer the reader to Definition 2.3.1, Definition 2.3.2 and Definition 2.6.1 for precise definitions of these concepts.

A measure ν is *admissible* with respect to an MMD space $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ if ν does not charge sets of 1-capacity zero and keeps the Dirichlet form $\mathscr{E}$ essentially unchanged. We denote the set of admissible measures by $\mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$.

Given an MMD space $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$, every choice of $d' \in \mathscr{J}(X,d)$ and $\nu \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ can be used to define a *changed MMD space* $(X, d', \nu, \mathscr{E}_\nu, D(\mathscr{E}_\nu))$. We refer the reader to [33, Chapter 6] for more details on transformations of Dirichlet forms.

Remark 3 *By [44, Proposition 10.6], $d \in \mathscr{J}(X,d)$ and for any $\theta \in \mathscr{J}(X,d)$, $\mathscr{J}(X,\theta) = \mathscr{J}(X,d)$. Furthermore, by [21, Theorem 5.2.11], $\mu \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ and if $\nu \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$, with the time-changed MMD space denoted by $(X, d, \nu, \mathscr{E}_\nu, D(\mathscr{E}_\nu))$ then*

$$\mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E})) = \mathscr{A}(X, d, \nu, \mathscr{E}_\nu, D(\mathscr{E}_\nu)).$$

Given $a, b \in \mathbb{R}$, $a < b$ and $U \subset \mathbb{R}^d$ open, a function $u : (a,b) \times U \to \mathbb{R}$ is *caloric* if u is a weak solution of

$$\frac{\partial u}{\partial t} = \mathscr{L} u, \quad \text{on } (a,b) \times U. \tag{2.2}$$

Here, $\mathscr{L}$ is the *infinitesimal generator* of $(\mathscr{E}, D(\mathscr{E}))$, which is a negative-definite self-adjoint operator on $D(\mathscr{L}) \subset L^2(X,\mu)$ satisfying

$$\mathscr{E}(f,g) = -\int_{\mathbb{R}^d} (\mathscr{L} f(x)) g(x) \mathrm{d}\mu(x), \qquad \text{for all } f, g \in D(\mathscr{L}).$$

Remark 4 *There are various definitions of caloric functions in the literature. For example, caloric functions are also called space-time harmonic because they can be defined in a fashion similar to how harmonic functions are defined in Definition 1.1.1. We refer to [26, Definition 1.13] for the precise details. On the other hand, our definition is an appropriate weak formulation of (2.2). Another definition of caloric functions, found in [7, (2.2), page 492] (where caloric functions are referred to as "solutions of the heat equation") converts the weak formulation into an integral equation. For more discussions on the definition of caloric functions, we refer the reader to [7, Section 2.2].*

The *parabolic* Harnack inequality for $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ is said to hold with dimension $\beta > 0$ (denoted by PHI(β)) if there exist $0 < C_1 < C_2 < C_3 < C_4 < \infty$, $C_5 > 1$ and $\delta \in (0, 1)$ such that for all $x \in X, r > 0, a \in \mathbb{R}$ and non-negative bounded function u which is caloric on $(a, a + C_4 r^\beta) \times B(x, r)$,

$$\operatorname{esssup}_{Q_-} u \le C_5 \operatorname{essinf}_{Q_+} u, \text{ where} \tag{2.3}$$
$$Q_- = (a + C_1 r^\beta, a + C_2 r^\beta) \times B(x, \delta r), \quad Q_+ = (a + C_3 r^\beta, a + C_4 r^\beta) \times B(x, \delta r).$$

Remark 5 *The definition of PHI(β) used here admits generalizations and variations in other articles. For example, one can define PHI(τ) for any increasing scale function $\tau : (0, \infty) \to (0, \infty)$ that satisfies the following scaling condition : there exists $0 < \beta_1 \le \beta_2 < \infty$ and $C > 1$ such that*

$$C^{-1}\left(\frac{R}{r}\right)^{\beta_1} \le \frac{\tau(R)}{\tau(r)} \le C\left(\frac{R}{r}\right)^{\beta_2}$$

for all $R \ge r > 0$. Our definition coincides with the definition of the weak parabolic Harnack inequality PHI(τ) defined in [5, Section 3.1]. In the same paper, the strong version of PHI(τ) is also defined, which demands that for every choice of $C_1, C_2, C_3, C_4 > 0$ and $\delta \in (0, 1)$, there is a $C_5 > 0$ such that (2.3) is satisfied for all $a \in \mathbb{R}, x \in \mathbb{R}^d, r > 0$ and u caloric on $(a, a + C_4\tau(r))$. A further restriction, PHI$^+$(τ) may be found in [26, Definition 1.15(i)]. In particular, we use the weakest form of the parabolic Harnack inequality in the literature, and Theorem 2.1.4 would continue to hold even if the definition of the inequality were changed to a stronger form.

We are now ready to define the notion of conformal walk dimension and state our main result.

Definition 2.1.3 (Conformal walk dimension) *The conformal walk dimension of an MMD space $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ is defined as*

$$\dim_{\mathrm{cw}}(X, d, \mu, \mathscr{E}, D(\mathscr{E})) = \inf\left\{\beta > 0 \;\middle|\; \begin{array}{l} \exists \theta \in \mathscr{J}(X, d), \nu \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E})), \\ (X, \theta, \nu, \mathscr{E}_\nu, D(\mathscr{E}_\nu)) \text{ satisfies PHI}(\beta) \end{array}\right\},$$

where $\inf \emptyset := +\infty$. We now state the main result of this chapter.

Theorem 2.1.4 (Conformal walk dimension of geometric stable processes) *Let m be the Lebesgue measure on $\mathbb{R}^d, d \ge 3$ and $\alpha \in (0, 2)$. Then,*

$$\dim_{\mathrm{cw}}(\mathbb{R}^d, \|\cdot\|, m, \mathscr{E}^\alpha, D(\mathscr{E}^\alpha)) = +\infty.$$

2.2 Motivation and history

We shall divide this section into three parts. In Section 2.2.1 we shall briefly review the wide literature on the EHI. In Section 2.2.2, we shall discuss the motivation for the main result. Finally, in Section 2.2.3 we conclude with an overview of Theorem 2.1.4.

2.2.1 Literature on the PHI

The PHI, like the EHI, is an important inequality in the fields of partial differential equations and probability theory. We refer the reader to [52] for a survey on the PHI. Throughout the rest of this section, PHI will refer to the inequality PHI(2), where PHI(β) is defined in Section 2.1.

The PHI was first proved to hold for the Laplacian by Pini [70] and Hadamard [42] independently. A suitable reformulation of their result is as follows : there exist constants $0 < C_1 < C_2 < C_3 < C_4 < \infty, C_5 \geq 1$, and $\delta \in (0,1)$ such that for every $x_0 \in \mathbb{R}^d, a \in \mathbb{R}, r > 0$ and bounded non-negative function $u : (a, a+C_4r^2) \times B(x_0, r) \to \mathbb{R}$ satisfying $\frac{\partial u}{\partial t} = \Delta u$, we have

$$\sup_{Q^-} u \leq C_5 \inf_{Q^+} u$$

where

$$Q^- = (a + C_1r^2, a + C_2r^2) \times B(x_0, \delta r) \quad Q^+ = (a + C_3r^2, a + C_4r^2) \times B(x_0, \delta r).$$

Note that the above formulation is equivalent to PHI(2). As with the EHI, an improvement of this result was found by Moser[68], who proved that the PHI also held for operators of the form

$$\mathcal{A} = \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(a_{ij} \frac{\partial}{\partial x_j} \right), \tag{2.4}$$

where $(a_{ij}), 1 \leq i, j \leq n$ is a uniformly elliptic bounded function on $\mathbb{R}^d$. When $(a_{ij}) \equiv I$, the operator above is the Laplacian Δ.

As an application of Moser's PHI, we state the celebrated result of Aronson [1]: for any fundamental solution $p_t(x, y)$ of the equation $\frac{\partial u}{\partial t} = \mathcal{A}u$ in $\mathbb{R}^d \times [0, T]$ where $\mathcal{A}$ is of the form (2.4), there exist constants C_1, c_1, C_2, c_2 depending only upon the ellipticity constants of $(a_{ij}), 1 \leq i, j \leq n$ and T such that

$$\frac{C_1}{t^{n/2}} \exp\left(-c_1 \frac{\|x-y\|^2}{t}\right) \leq p_t(x, y) \leq \frac{C_2}{t^{n/2}} \exp\left(-c_2 \frac{\|x-y\|^2}{t}\right)$$

for all $x, y \in \mathbb{R}^d$ and $t \in [0, T]$. That is, the heat kernel corresponding to the semigroup associated to $\mathcal{A}$ satisfies Gaussian-type estimates and the proof uses the PHI.

The task of finding characterisations of the EHI and PHI in various settings is a long-standing question. In particular, characterisations or sufficient conditions that prove stability of the inequalities are desirable. To define stability, suppose that $(\mathscr{E}, D(\mathscr{E}))$ is a Dirichlet form (possibly satisfying additional assumptions) on a metric space (X, d) which is perturbed to $(\tilde{\mathscr{E}}, D(\mathscr{E}'))$ so that there exists $C > 1$ such that

$$C^{-1}\mathscr{E}(f, g) \leq \mathscr{E}'(f, g) \leq C\mathscr{E}(f, g)$$

for all $f, g \in D(\mathscr{E})$. Suppose that $(\mathscr{E}, D(\mathscr{E}))$ satisfies either the EHI (see Remark 1) or the PHI. Then, does $(\tilde{\mathscr{E}}, D(\mathscr{E}'))$ also satisfy the same inequality? Note that Moser's results show that the EHI and PHI are stable around the Dirichlet form associated to the Laplacian.

We will now discuss some notable characterisations of the PHI. A major advancement in the understanding of the PHI came via the characterisations of Grigorýan[34] and Saloff Coste[73] for second order differential operators on smooth manifolds. It was proved that the PHI is equivalent to a family of Poincaré inequalities and a volume doubling condition on the manifold. Each of these is stable under the perturbation of Dirichlet forms, hence the stability of the PHI follows. Further extensions of the same characterisation were found on graphs by Delmotte [30] and on general metric measure spaces by Sturm [80]. In particular, a corollary of the former result is that the PHI is stable under rough isometries of graphs.

Until this point, it was unclear whether the EHI was equivalent to the PHI or not. A remark by Grigorýan[34, pages 75-76] claimed the existence of a two dimensional manifold over which a Brownian motion satisfies the EHI but not the PHI. The first explicit example of diffusions which satisfied the EHI but not the PHI were given by Barlow and Bass in [3]. These diffusions were constructed on generalized Sierpinski carpets, and it was proved that they satisfy the PHI but with a distinct space-time scaling given by $\Psi(t) = t^\beta \wedge t^2$, where $\beta > 2$. This is incompatible with the usual PHI. In particular, this work established that the questions of characterisations and stability for the EHI and PHI are distinct.

The condition on the space time scaling was relaxed further by Barlow, Grigorýan and Kumagai [5], where the inequality PHI(Ψ) is defined and proven to be equivalent to heat kernel estimates and $\Psi : \mathbb{R}_+ \to \mathbb{R}_+$ is only required to be a continuous increasing bijection such that for some $C_1, C_2 > 0$ and $\beta_2 \geq \beta_1 \geq 1$,

$$C_1 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\Psi(R)}{\Psi(r)} \leq C_2 \left(\frac{R}{r}\right)^{\beta_2} \tag{2.5}$$

for all $R > r > 0$. Under the same assumptions on Ψ in a similar setting, Grigorýan, Hu and Lau [37] prove that heat kernel estimates are equivalent to a number of conditions, some of which are stable under bounded perturbations. When combined, the results of the above papers show that PHI(Ψ) has stable characterisations on a metric measure space with volume doubling and reverse volume doubling equipped with a strongly local Dirichlet form.

The study of the PHI for non-local Dirichlet forms was first seen in the context of "stable-like" processes. The infinitesimal generators of such processes are of the form

$$\mathcal{L}f(x) = \int_{\mathbb{R}^d\setminus\{0\}} (f(x+h) - f(x) - \nabla f(x)\cdot h1_{\|h\|<1})\frac{n(x,h)}{\|h\|^{d+\alpha}}\mathrm{d}h$$

for some $\alpha \in (0,2)$, where $n(x,h) = n(x,-h)$ and $c_1 \leq n(x,h) \leq c_2$ for some $c_1, c_2 > 0$. Chen and Kumagai (see [22, Proposition 4.3]) show that PHI(α) holds, which is then used to prove heat kernel estimates. Thus, PHI(β) is stable under bounded perturbations of the Dirichlet forms associated to α-stable processes. Further inequalities of the type PHI(Ψ) were proved for mixtures of stable processes by Chen and Kumagai [23], where Ψ satisfies the assumption (2.5).

The stability of PHI(Ψ) for a large class of nonlocal Dirichlet forms on metric measure spaces was proved by Chen, Kumagai and Wang [26]. They provide many equivalent conditions for PHI(Ψ), where Ψ satisfies (2.5) with the relaxation $\beta_2 \geq \beta_1 > 0$, under the only assumptions that the space satisfies volume doubling and reverse volume doubling.

2.2.2 Motivation

The conformal walk dimension arises from the need to connect the stability of the EHI to the stability of the PHI. While much is known about the latter question as was described in the previous section, the former question was solved only recently by Barlow and Murugan[6]. We refer the reader to Section 1.2.1 for more information.

The starting point of this work is Kajino and Murugan's characterisation of the EHI for MMD spaces with strongly local Dirichlet forms. We shall now make some preliminary definitions before stating their main result.

Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ be an MMD space(recall the definition of an MMD space from Section 2.1). We first define a *strongly local* Dirichlet form as in [50, Page 12, Section 2.1]. For $f \in D(\mathscr{E})$, let $\mathrm{supp}_\mu[f]$ denote the smallest closed $F \subset X$ such that $\int_{X\setminus F}|f|d\mu = 0$ (This exists since the topology on X possesses a local countable base). We call $(\mathscr{E}, D(\mathscr{E}))$ strongly local if $\mathscr{E}(f,g) = 0$ for all $f, g \in D(\mathscr{E})$ such that $\mathrm{supp}_\mu[f]$ and $\mathrm{supp}_\mu[g]$ are compact, and $\mathrm{supp}_\mu[f - a1_X] \cap \mathrm{supp}_\mu[g] = \emptyset$ for some $a \in \mathbb{R}$.

A metric space (X, d) is said to be *metric doubling* if, for some $N \in \mathbb{N}$, for every $x_0 \in X$ and $r > 0$ we can find $x_1, \ldots, x_N \in B(x_0, r)$ such that

$$B_d(x_0, r) \subset \cup_{i=1}^N B_d(x_i, r/2).$$

We now define the EHI for MMD spaces as in [50, Definition 2.4]. For a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$, let $(\mathscr{E}_e, D(\mathscr{E}_e))$ denote the extended Dirichlet space of $(\mathscr{E}, D(\mathscr{E}))$ as in [33, Theorem 1.5.2]. A function $h \in D(\mathscr{E}_e)$ is said to be $\mathscr{E}$-harmonic on an open set $U \subset X$ if

$$\mathscr{E}(h, f) = 0 \text{ for all } f \in D(\mathscr{E}) \cap C_c(X) \text{ with } \operatorname{supp}_\mu[f] \subset U.$$

The MMD space $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies the EHI if there exist $C > 1$ and $\delta \in (0, 1)$ such that for all $x_0 \in X, r > 0$ and non-negative $h \in D(\mathscr{E}_e)$ that is $\mathscr{E}$-harmonic on $B_d(x_0, r)$ we have

$$h(x) \leq Ch(y), \quad \text{for } \mu\text{-almost all } x, y \in B_d(x_0, r).$$

We observe that every harmonic function lifts to a caloric function. More precisely, if h is harmonic on $B_d(x, r)$, then $u(t, x) = h(x)$ is caloric on $(a, b) \times B_d(x, r)$ for all $b > a$. This immediately shows that PHI(β) implies the EHI for all $\beta > 0$.

We are now ready to state the main result in [50]. Recall the definition of the conformal walk dimension d_{cw} from Definition 2.1.3.

Theorem 2.2.1 *([50, Theorem 2.10]) Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ be an MMD space where $(\mathscr{E}, D(\mathscr{E}))$ is strongly local, and let d_{cw} denote its conformal walk dimension. Then the following are equivalent :*

(a) $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies the metric doubling property and EHI.

(b) $d_{cw} < \infty$.

(c) $d_{cw} = 2$.

Note that the equivalence of (a) and (b) is a corollary of the main result in [6](see also [54, Theorem 4.9]). The techniques used by the authors to prove the above theorem heavily use the strong locality assumption on the Dirichlet form, which corresponds to a symmetric diffusion process on metric measure spaces. This leads them to ask if the above theorem continues to hold for MMD spaces with non-local Dirichlet forms, such as those corresponding to symmetric pure jump Lévy processes on $\mathbb{R}^d$. We state their version of the problem.

Problem 2.2.2 *([50, Problem 7.1]) Does $d_{cw} < \infty$ characterise the EHI for symmetric jump processes?*

The conclusions of Theorem 1.1.2 and Theorem 2.1.4, when combined, answers Problem 2.2.2 in the negative.

We shall now prove three propositions that motivate the definition of the conformal walk dimension. Recall the set of quasisymmetric metrics $\mathcal{J}(X, d)$ from Definition 2.1.2, and the set of admissible measures $\mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ and time-changed MMD spaces from Section 2.1. The first of the propositions states that the EHI is invariant under quasisymmetric changes of metric and admissible changes of measure.

Proposition 2.2.3 *([50, Lemma 4.8]) Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ be an MMD space. For any $\nu \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ and $d' \in \mathcal{J}(X, d)$, the MMD space $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies the EHI if and only if the time-changed MMD space $(X, d', \nu, \mathscr{E}_\nu, D(\mathscr{E}_\nu))$ does.*

Proof. cf. [50, Lemma 4.8]. Note that the proof does not use the strong locality of $(\mathscr{E}, D(\mathscr{E}))$. □

Recall the definition of quasi-symmetry from Definition 2.1.2, and the definition of PHI(β) from Section 2.1. The second proposition states that if an MMD space satisfies PHI(β) then there is a quasisymmetric change of metric under which the changed MMD space satisfies PHI(β') for any $\beta' > \beta$. This motivates the choice of the infimum in the definition of the conformal walk dimension (see Definition 2.1.3), and therefore the challenge in Theorem 2.2.1 lies in *reducing* the value of β for which a time-changed MMD space satisfies PHI(β).

Proposition 2.2.4 *Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfy PHI(β) and $\alpha \in (0, 1]$ be arbitrary. Then, the metric $d_\alpha(x, y) = d(x, y)^\alpha$ is quasisymmetric to d and the MMD space $(X, d_\alpha, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies PHI(β/α).*

Before we begin the proof, we note that $\alpha \in (0, 1]$ is only required to ensure that d_α is a well-defined metric on X. We will only use the assumption $\alpha > 0$ in the proof.

Proof. Let $\eta(x) = x^\alpha$. Then, $\eta : [0, \infty) \to [0, \infty)$ is a homeomorphism and by Definition 2.1.2, it is clear that d_α is η-quasisymmetric to d. Hence, $d_\alpha \in \mathcal{J}(X, d)$ and $(X, d_\alpha, \mu, \mathscr{E}, D(\mathscr{E}))$ is a well defined MMD space.

Let $C_i, i = 1, \ldots, 5$ and $\delta \in (0, 1)$ be constants such that PHI(β) holds i.e. (2.3) holds with this choice of constants for $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$. We will now show that $(X, d_\alpha, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies PHI(β/α) with the same constants $C_i, i = 1, \ldots, 5$ and $\delta' = \delta^\alpha$.

Let $x \in X, r > 0, a \in \mathbb{R}$ be arbitrary and u be caloric with respect to $(X, d_\alpha, \mu, \mathscr{E}, D(\mathscr{E}))$ on $(a, a + C_4 r^{\beta/\alpha}) \times B_{d_\alpha}(x, r)$. By the definition of caloricity (see Section 2.1 for an informal definition and Definition 2.6.1 for a rigorous definition), u remains caloric on the same domain with respect to $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$. Further, since $B_{d_\alpha}(x, r) = B_d(x, r^{1/\alpha})$, u is caloric on $(a, a + C_4 (r^{\frac{1}{\alpha}})^\beta) \times B_d(x, r^{1/\alpha})$. Applying (2.3) to u with the choice $r^{1/\alpha}$,

$$\mathrm{esssup}_{Q_-} u \leq C_5 \mathrm{essinf}_{Q^+} u,$$

where

$$Q_- = (a + C_1 r^{\beta/\alpha}, a + C_2 r^{\beta/\alpha}) \times B_d(x, \delta r^{\frac{1}{\alpha}}), Q_+ = (a + C_3 r^{\beta/\alpha}, a + C_4 r^{\beta/\alpha}) \times B_d(x, \delta r^{\frac{1}{\alpha}})$$

However, observe that $B_d(x, \delta r^{1/\alpha}) = B_{d_\alpha}(x, \delta' r)$. In particular,

$$\begin{gathered} \text{esssup}_{Q\_} u \leq C_5 \text{essinf}_{Q+} u, \text{ where} \\ Q_- = (a + C_1 r^{\beta/\alpha}, a + C_2 r^{\beta/\alpha}) \times B_{d_\alpha}(x, \delta' r) \\ Q_+ = (a + C_3 r^{\beta/\alpha}, a + C_4 r^{\beta/\alpha}) \times B_{d_\alpha}(x, \delta' r) \end{gathered}$$

Thus, we have shown that $(X, d_\alpha, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies PHI(β/α). □

The final proposition shows how Theorem 2.2.1 proves the stability of the EHI using the stability of PHI(β). A few inequalities are required for the proof which we leave the reader to refer to. These are volume doubling(see [50, Definition 1.7]), the Poincaré inequality PI(β) (see [50, Definition 4.3(i)]) and the cutoff-Sobolev inequality CS(β) (see [50, Definition 4.3(iii)]).

We are ready to state the proposition.

Proposition 2.2.5 *Let the EHI hold for $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$, where $(\mathscr{E}, D(\mathscr{E}))$ is a strongly local form. Suppose that $(\mathscr{E}', D(\mathscr{E}))$ is another strongly local Dirichlet form such that for some constants $c_1, c_2 > 0$,*

$$c_1 \mathscr{E}(f, f) \leq \mathscr{E}'(f, f) \leq c_2 \mathscr{E}(f, f)$$

for all $f \in D(\mathscr{E})$. Then, $(X, d, \mu, \mathscr{E}', D(\mathscr{E}))$ satisfies the EHI.

Proof. By (a) $\implies$ (c) of Theorem 2.2.1, there exists $\mu' \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ and $d' \in \mathscr{J}(X, d)$ such that $(X, d', \mu', \mathscr{E}, D(\mathscr{E}))$ satisfies PHI(β) for some $2 \leq \beta < \infty$. By [50, Theorem 4.5], (X, d', μ') satisfies volume-doubling and $(X, d', \mu', \mathscr{E}, D(\mathscr{E}))$ satisfies PI(β) and CS(β) . The inequalities PI(β) and CS(β) also hold for $(X, d', \mu', \mathscr{E}', D(\mathscr{E}))$ by [7, Lemma 2.18]. Hence,$(X, d', \mu', \mathscr{E}', D(\mathscr{E}))$ satisfies PHI(β) by [50, Theorem 4.5], and therefore the EHI.

Now, since $d \in \mathscr{J}(X, d')$ and $\mu \in \mathscr{A}(X, d', \mu', \mathscr{E}', D(\mathscr{E}))$ by Remark 3, the MMD space $(X, d, \mu, \mathscr{E}', D(\mathscr{E}))$ also satisfies the EHI by Proposition 2.2.3. □

2.2.3 Overview of the proof of Theorem 2.1.4

We shall now discuss the main idea behind the proof of Theorem 2.1.4. The theorem states that the conformal walk dimension of the MMD space corresponding to the geometric stable process is infinite. In other words, it is not possible to change the metric quasisymmetrically and the measure admissibly in such a manner that the resulting MMD space satisfies PHI(β) for any $\beta > 0$. Our proof technique is novel since we demonstrate the first example of a process with infinite conformal walk dimension.

Let m be the Lebesgue measure on $\mathbb{R}^d, d \geq 3$ and $\alpha \in (0,2)$ be fixed. Let $(\mathscr{E}^\alpha, D(\mathscr{E}^\alpha))$ be as in Definition 2.1.1. Recall from Section 2.1 that the MMD space corresponding to the geometric stable process of index α, $\{X_t^\alpha\}_{t\geq 0}$ is given by $(\mathbb{R}^d, \|\cdot\|, m, \mathscr{E}^\alpha, D(\mathscr{E}^\alpha))$. Note that $(\mathbb{R}^d, \|\cdot\|)$ satisfies the metric doubling property (see Section 2.2.2 for the definition).

We shall now explain the role of Proposition 2.3.3, which is the key proposition used to prove Theorem 2.1.4. The proof of Theorem 2.1.4 proceeds by contradiction. Suppose that $d_{cw}(\mathbb{R}^d, \|\cdot\|, m, \mathscr{E}^\alpha, D(\mathscr{E}^\alpha)) < \infty$. Then, by Definition 2.1.3, there exists a time-changed MMD space $(\mathbb{R}^d, d, \mu, \mathscr{E}_\mu, D(\mathscr{E}_\mu))$ (see Section 2.1 for the definition of a time-changed space) which satisfies PHI(β) for some $\beta > 0$. Now, Proposition 2.3.3 compares the volumes of balls in $(\mathbb{R}^d, \|\cdot\|)$ under the measure μ. The proof of the theorem follows by showing that this comparison contradicts the metric doubling property of $(\mathbb{R}^d, \|\cdot\|)$.

The proof of Proposition 2.3.3 uses two auxiliary propositions. In order to state these propositions, we require the definition of capacity between sets. Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ be an MMD space (see Section 2.1 for the definition). For disjoint Borel sets $A, B \subset X$ such that $\bar{A}$ is compact, define

$$\mathscr{F}_{\mathscr{E}}(A,B) = \{\phi \in D(\mathscr{E}) \mid \operatorname{supp}(\phi) \subset B^c, \phi \equiv 1 \text{ on a neighbourhood of } A\}. \tag{2.6}$$

The *capacity between A and B* is defined as

$$\operatorname{cap}_{\mathscr{E}}(A,B) = \inf\{\mathscr{E}(f,f) : f \in \mathscr{F}_{\mathscr{E}}(A,B)\}. \tag{2.7}$$

Proposition 2.4.1 is an estimate on the capacity between concentric balls in the MMD space $(\mathbb{R}^d, \|\cdot\|, m, \mathscr{E}^\alpha, D(\mathscr{E}^\alpha))$. On the other hand, Proposition 2.4.1 contains two implications of PHI(β) for an MMD space $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$. The first of these is a volume doubling condition : there exists a constant $C_{VD} > 0$ such that for every $x \in X, r > 0$ we have

$$\mu(B_d(x, 2r)) \leq C_{VD}\mu(B_d(x,r)).$$

The second is a capacity estimate between concentric balls in the MMD space. The proof of Proposition 2.3.3 follows by a computation after combining these two results.

Recall the jump kernel j_α of the geometric stable process from Section 1.2.2. The proof of Proposition 2.4.1 is a standard computation using only the estimate (1.33) for j_α. While the lower bound follows using a simple inequality, the upper bound follows by some computations after the choice of a specific cut-off function.

Prior to overviewing the proof of Proposition 2.4.2, we define the notion of a heat kernel. Recall the infinitesimal generator $\mathscr{L}$ of $(\mathscr{E}, D(\mathscr{E}))$ from Section 2.1. The associated *semigroup operator* on $L^2(X,\mu)$ is given by $\{P_t\}_{t\geq 0}, P_t = e^{-t\mathscr{L}}$. For any $V \subset X$ open, let $(D(\mathscr{E}))_V$ be the closure of $D(\mathscr{E}) \cap C_c(V)$ under $\|\cdot\|_{\mathscr{E}_1}$ (see (2.1) for the definition). By [33, Theorem 4.4.3], the tuple $(\mathscr{E}, (D(\mathscr{E}))_V)$ is also a Dirichlet form, whose infinitesimal generator and semigroup will be denoted by $\mathscr{L}^V$ and $\{P_t^V\}_{t\geq 0}$ respectively.

A family of non-negative Borel functions $\{p_t\}_{t>0}$, $p_t : X \times X \to [0, \infty)$ is called a heat kernel for the semigroup $\{P_t\}_{t\geq 0}$ if, for all $t > 0$ and $f \in L^2(X, m)$,

$$P_t f(x) = \int_X p_t(x, y) f(y) \mathrm{d}\mu(y), \quad \text{for } \mu\text{-a.e. } x \in X.$$

To prove Proposition 2.4.2, we first show the existence of a heat kernel for the semigroup $\{P_t\}_{t\geq 0}$ assuming that PHI(β) holds. Furthermore, we show that the heat kernel satisfies some lower and upper bounds (Lemma 2.6.3). The lemma is proved by first showing that if PHI(β) holds, then the semigroup $\{P_t\}_{t\geq 0}$ is ultracontractive i.e. for all $f \in L^1(X, \mu)$ and μ-a.e. x,

$$P_t f(x) \leq C \|f\|_{L^1(X,\mu)}$$

for a constant C independent of f. The existence of a heat kernel now follows by the argument of [26, Proposition 3.1], while the rest of the proposition follows using arguments from [26, Proposition 3.1] and [4, Theorem 2.3].

To show Proposition 2.4.1(a) using Lemma 2.6.3, we use a non-local version of an argument in the proof of [50, Theorem 4.5], which uses PHI(β) and the heat kernel lower bound Lemma 2.6.3(c) to compare the volumes of concentric balls. Once Proposition 2.4.1(a) is known, the proof of Proposition 2.4.1(b) follows using the numerous implications of PHI(β) proved in [26] and [25].

2.3 Proof of Theorem 2.1.4

In this section, we shall prove Theorem 2.1.4 with the help of a key proposition, Proposition 2.3.3, whose proof is deferred to Section 2.4.

Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ be an MMD space (see Section 2.1 for the definition). Recall the norm $\|\cdot\|_{\mathscr{E}_1}$ from (2.1). Given $A \subset X$ Borel, the 1-*capacity* of A is defined as

$$\mathrm{cap}_1(A) = \inf \left\{ \|f\|_{\mathscr{E}_1}^2 : \ f \geq 1 \ \mu\text{-a.e. on a neighbourhood of } A \right\}. \tag{2.8}$$

Prior to stating our key proposition, we provide precise definitions of admissible measures and time-changed MMD spaces as in [50, pages 15-16]. The notion of quasi-closed sets is required in order to define admissible measures. An increasing sequence of closed subsets $\{F_k\}_{k\geq 1}$ of X is called a *nest* if

$$\bigcup_{k\geq 1} \{f \in D(\mathscr{E}) : f \equiv 0 \ \mu\text{-a.e. on } F_k\}$$

is dense in $(D(\mathscr{E}), \|\cdot\|_{\mathscr{E}_1})$. A set $D \subset X$ is called quasi-open if there is a nest $\{F_k\}_{k\geq 1}$ such that $F_k \cap D$ is open relative to F_k for each $k \geq 1$. A set $C \subset X$ is called *quasi-closed* if C^c is quasi-open.

Definition 2.3.1 (Admissible measures) *A Radon measure ν is admissible with respect to $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ if it is smooth ($\nu(A) = 0$ for all Borel A such that $\mathrm{cap}_1(A) = 0$), and it has full quasisupport ($\mathrm{cap}_1(X \setminus F) = 0$ for all quasi-closed F such that $\nu(X \setminus F) = 0$).*

We now provide the rigorous definition of a time changed MMD space. Recall $\mathscr{J}(X, d)$ and $\mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ from Section 2.1. For a Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$, let $(\mathscr{E}_e, D(\mathscr{E}_e))$ denote the extended Dirichlet space of $(\mathscr{E}, D(\mathscr{E}))$ as in [33, Theorem 1.5.2].

Definition 2.3.2 (Time changed MMD space) *Given $\nu \in \mathscr{A}(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ and $d' \in \mathscr{J}(X, d)$, define $D(\mathscr{E}_\nu) = L^2(X, \nu) \cap D(\mathscr{E}_e)$, and define $\mathscr{E}_\nu(f, g) = \mathscr{E}(f, g)$ for all $f, g \in D(\mathscr{E}_\nu)$. The tuple $(X, d', \nu, \mathscr{E}_\nu, D(\mathscr{E}_\nu))$ is the time changed MMD space.*

We are now ready to state our key proposition. The MMD space $(\mathbb{R}^d, \|\cdot\|, m, \mathscr{E}^\alpha, D(\mathscr{E}^\alpha))$ is denoted by M_α, where $(\mathscr{E}^\alpha, D(\mathscr{E}^\alpha))$ is as in Definition 2.1.1. Recall the function L from (1.16) and the definition of $\mathrm{PHI}(\beta)$ from Section 2.1.

The volume doubling property of the underlying measure was proven to be a consequence of $\mathrm{PHI}(\beta)$ in [5, Theorem 3.2] and [50, Theorem 4.5]. Since the metric doubling property constrains the values that any measure can take on balls of various sizes, in the next proposition we present a result which estimates the volumes of balls.

Proposition 2.3.3 *Suppose that M_α is time changed to an MMD space $(\mathbb{R}^d, \theta, \mu, \mathscr{E}_\mu, D(\mathscr{E}_\mu))$ which satisfies PHI(β) for some $\beta > 0$. Then, there exist constants $R, \zeta, C_1 > 0$ and $C_2 > 1$ such that for all $\epsilon \in \left(0, \frac{1}{2}\right)$,*

$$\frac{\mu(B(x, \epsilon R))}{\mu(B(0, C_2 R))} \leq C_1 \epsilon^{d+\zeta} \frac{L(R)}{L(\epsilon R)} \quad \text{for all } x \in B(0, C_2 R). \tag{2.9}$$

We now prove Theorem 2.1.4 assuming the above proposition.

Proof of Theorem 2.1.4 : We proceed by contradiction. Suppose that

$$\dim_{\mathrm{cw}}(M_\alpha) < \infty.$$

Then, there exists a time-changed space of M_α denoted by $(\mathbb{R}^d, \theta, \mu, \mathscr{E}_\mu, D(\mathscr{E}_\mu))$, which satisfies $\mathrm{PHI}(\beta)$ for some $\beta > 0$. We now apply Proposition 2.3.3 to obtain a contradiction. Let $R, \eta, C_1 > 0$ and $C_2 > 1$ be such that (2.9) holds, and $\epsilon \in (0, \frac{1}{2})$.

By the geometry of $(\mathbb{R}^d, \|\cdot\|)$, there exist $y_1, \ldots, y_N \in B(0, C_2 R)$ such that $N \leq C\epsilon^{-d}$ for some constant $C > 0$, and

$$B(0, C_2 R) \subset \bigcup_{i=1}^{N} B(y_i, \epsilon R). \tag{2.10}$$

By (2.9), we know for $1 \le i \le N$,

$$\mu(B(y_i, \epsilon R)) \le C_1 \epsilon^{d+\zeta} \frac{L(R)}{L(\epsilon R)} \mu(B(0, C_2 R)).$$

Using the above and (2.10) we have

$$\mu(B(0, C_2 R)) \le \sum_{i=1}^{N} \mu(B(y_i, \epsilon R)) \le N C_1 \epsilon^{d+\zeta} \frac{L(R)}{L(\epsilon R)} \mu(B(0, C_2 R)).$$

Dividing both sides by $\mu(B(0, C_2 R))$ and using the fact that $N < C\epsilon^{-d}$ we have

$$1 \le C \epsilon^{\zeta} \frac{L(R)}{L(\epsilon R)}$$

for some $C > 0$. As $\epsilon \in (0, \frac{1}{2})$ was arbitrary, letting $\epsilon \to 0$ we obtain a contradiction.

Thus, the time-changed MMD space $(\mathbb{R}^d, \theta, \mu, \mathscr{E}_\mu, D(\mathscr{E}_\mu))$ cannot satisfy PHI(β) for any $\beta > 0$. Hence, $\dim_{\mathrm{cw}}(M_\alpha) = +\infty$, as desired. □

2.4 Proof of Proposition 2.3.3

In this section, we will prove Proposition 2.3.3 with the help of two additional propositions, Proposition 2.4.1 and Proposition 2.4.2. The proofs of these propositions are deferred to Sections 2.5 and 2.6 respectively.

Recall the definition of the capacity between sets from (2.7). We state our first key proposition on capacity bounds with respect to $\mathscr{E}_\alpha$(See Definition 2.1.1). Recall the function L from (1.16).

Proposition 2.4.1 ($\mathscr{E}_\alpha$ capacity bounds) *Fix $a > 1$. There exists $C > 1$ and $R_0 > 0$ such that for all $x_0 \in \mathbb{R}^d, r < R_0$,*

$$\frac{1}{C} \frac{r^d}{L(r)} \le \mathrm{cap}_{\mathscr{E}^\alpha}(B(x_0, r), B(x_0, ar)^c) \le C \frac{r^d}{L(r)}. \tag{2.11}$$

Recall PHI(β) from Section 2.1. We are now ready to state our second key proposition, on two implications of PHI(β). Given a metric space (X, d) and $A \subset X$, let

$$\mathrm{diam}(A) = \sup_{x,y \in A} d(x, y)$$

denote the *diameter* of the set A, where we set $\sup \emptyset = 0$.

Proposition 2.4.2 (Implications of PHI(β)) *Suppose that $(X, \theta, \mu, \mathscr{E}, D(\mathscr{E}))$ is an MMD space which satisfies PHI(β) for some $\beta > 0$. Then,*

(a) *there exists a constant* $C > 1$ *such that for every* $x_0 \in \mathbb{R}^d, r > 0$,

$$\mu(B_\theta(x_0, 2r)) \leq C\mu(B_\theta(x_0, r))$$

and

(b) *for every* $a > 1$, *there exists* $C > 1$ *such that for all* $R \in (0, diam(X, \theta))$, $x \in X$, *we have*

$$\frac{1}{C}\frac{\mu(B_\theta(x,R))}{R^\beta} \leq \operatorname{cap}_{\mathscr{E}}(B_\theta(x,R), B_\theta(x,aR)^c) \leq C\frac{\mu(B_\theta(x,R))}{R^\beta}. \tag{2.12}$$

The following two lemmas will be used in the proof of Proposition 2.3.3. We will first prove the proposition. This will be followed by the proof of the lemmas.

The first of the lemmas compares the capacities between two pairs of sets under time-changed Dirichlet forms. Recall the MMD space M_α from Section 2.3 and the set of admissible measures $\mathscr{A}(M_\alpha)$ from Section 2.1.

Lemma 2.4.3 *Let* $\mu \in \mathscr{A}(M_\alpha)$, *and* $(\mathscr{E}_\mu, D(\mathscr{E}_\mu))$ *be the time changed Dirichlet form as in Definition 2.3.2. Suppose that*

$$A \subset B \subset C \subset D \subset \mathbb{R}^d$$

are four bounded open sets such that $\overline{B} \subset C$. *Then, the following inequalities hold :*

$$\operatorname{cap}_{\mathscr{E}_\mu}(B, C^c) \geq \operatorname{cap}_{\mathscr{E}^\alpha}(A, D^c), \text{ and} \tag{2.13}$$

$$\operatorname{cap}_{\mathscr{E}^\alpha}(B, C^c) \geq \operatorname{cap}_{\mathscr{E}_\mu}(A, D^c), \tag{2.14}$$

where the capacity between sets $\operatorname{cap}_{\mathscr{E}}$ *is defined by* (2.7).

The second lemma contains some properties of quasisymmetry and quasisymmetric metrics.

Lemma 2.4.4 *Suppose* d_1, d_2 *are two metrics on* X, *and* $\eta : [0,\infty) \to [0,\infty)$ *is a homeomorphism such that* d_2 *is* η*-quasisymmetric to* d_1. *Let* $B_i(x,r)$ *denote the open ball centered at* $x \in X$ *of radius* $r \geq 0$ *in the metric* d_i *for* $i = 1, 2$.

(a) *For all* $a \geq 1, x \in X$ *and* $r > 0$, *there exists* $s > 0$ *such that*

$$B_2(x,s) \subset B_1(x,r) \subset B_1(x,ar) \subset B_2(x,\eta(a)s). \tag{2.15}$$

Similarly, for all $a \geq 1, x \in X$ *and* $r > 0$, *there exists* $t > 0$ *such that*

$$B_1(x,r) \subset B_2(x,t) \subset B_2(x,at) \subset B_1\left(x, \frac{1}{\eta^{-1}(a^{-1})}r\right). \tag{2.16}$$

(b) *If* $A \subset B \subset X$ *are such that* $0 < \operatorname{diam}(A, d_1) \leq \operatorname{diam}(B, d_1) < \infty$*, then* $0 < \operatorname{diam}(A, d_2) \leq \operatorname{diam}(B, d_2) < \infty$ *and*

$$\frac{1}{2\,\eta\left(\frac{\operatorname{diam}(B,d_1)}{\operatorname{diam}(A,d_1)}\right)} \leq \frac{\operatorname{diam}(A, d_2)}{\operatorname{diam}(B, d_2)} \leq \eta\left(\frac{2\operatorname{diam}(A, d_1)}{\operatorname{diam}(B, d_1)}\right).$$

(c) *If* (X, d_1) *is uniformly perfect, then there exist* $C' \geq 1$ *and* $\delta \in (0, 1]$ *such that* d_2 *is* κ*-quasisymmetric to* d_1*, where* $\kappa : [0, \infty) \to [0, \infty)$ *is given by*

$$\kappa(t) = C' \max\{t^{\delta}, t^{\frac{1}{\delta}}\}.$$

We are now ready to prove Proposition 2.3.3.

Proof of Proposition 2.3.3. We will prove the result assuming the following statement: there exist $R, \zeta, C_1, C_2 > 0$ such that for all $\epsilon \in \left(0, \frac{1}{2}\right)$ and $y \in \mathbb{R}^d$,

$$\frac{\mu(B(y, \epsilon R))}{\mu(B(y, C_2 R))} \leq C_1 \epsilon^{d+\zeta} \frac{L(R)}{L(\epsilon R)}. \tag{2.17}$$

To prove (2.9), suppose that $x \in B(0, C_2 R)$. Then, $B(x, C_2 R) \subset B(0, 2C_2 R)$. By (2.15) of Lemma 2.4.4(a), there exists $t > 0$ such that

$$B_\theta(0, tR) \subset B(0, C_2 R) \subset B(0, 2C_2 R) \subset B_\theta(0, \eta(2) t C_2 R). \tag{2.18}$$

By monotonicity of μ and (2.18),

$$\mu(B(x, C_2 R)) \leq \mu(B(0, 2C_2 R)) \leq \mu(B_\theta(0, \eta(2) t C_2 R)). \tag{2.19}$$

Let $D \in \mathbb{N}$ be such that $2^D > \eta(2) C_2$. Iterating Proposition 2.4.1(a) D times, there exists a constant $C > 0$ such that $\mu(B_\theta(0, \eta(2) t C_2 R)) \leq C \mu(B_\theta(0, tR))$. Combining this with (2.18) and using the monotonicity of μ,

$$\mu(B_\theta(0, \eta(2) t C_2 R)) \leq C \mu(B_\theta(0, tR)) \leq C \mu(B(0, C_2 R)).$$

In conjunction with the above inequality, (2.19) yields

$$\mu(B(x, C_2 R)) \leq C \mu(B(0, C_2 R)) \quad \text{for all } x \in B(0, C_2 R).$$

Using the above estimate and applying (2.17) at the point $x \in B(0, C_2 R)$, for some constant $C_1 > 0$,

$$\frac{\mu(B(x, \epsilon R))}{\mu(B(0, C_2 R))} \leq C \frac{\mu(B(x, \epsilon R))}{\mu(B(x, C_2 R))} \leq C_1 \epsilon^{d+\delta} \frac{L(R)}{L(\epsilon R)}.$$

The above is true for all $x \in B(0, C_2R)$ and $\epsilon \in (0, \frac{1}{2})$, completing the proof of the proposition.

We will now prove (2.17). Note that $(\mathbb{R}^d, \|\cdot\|)$ is a uniformly perfect space. Since θ is quasisymmetric to $\|\cdot\|$, we may apply Lemma 2.4.4(c) to obtain $C' \geq 1$ and $\delta \in (0, 1]$ such that $\eta : [0, \infty) \to [0, \infty)$ defined by

$$\eta(t) = C' \max\{t^\delta, t^{\frac{1}{\delta}}\} \tag{2.20}$$

is a homeomorphism, and θ is η-quasisymmetric to d.

Let $y \in \mathbb{R}^d, \epsilon \in (0, \frac{1}{2})$ be fixed, and $R > 0$ be a fixed constant whose value will be chosen later. We begin by applying Lemma 2.4.3 and Lemma 2.4.4(a) to obtain inequalities on the capacity between some sets. By (2.15) of Lemma 2.4.4(a) with the choices $a = 2, x = y$ and $r = \epsilon R$, there exists $s > 0$ such that

$$B_\theta(y, s) \subset B(y, \epsilon R) \subset B(y, 2\epsilon R) \subset B_\theta(y, \eta(2)s). \tag{2.21}$$

Similarly, by (2.16) of Lemma 2.4.4(a) with the choices $a = 2, x = y$ and $r = R$, there exists $S > 0$ such that

$$B(y, R) \subset B_\theta(y, S) \subset B_\theta(y, 2S) \subset B\left(y, \frac{1}{\eta^{-1}(\frac{1}{2})}R\right). \tag{2.22}$$

We apply Lemma 2.4.3 to each of the last two containments above. Applying (2.13) to (2.21) gives

$$\mathrm{cap}_{\mathscr{E}^\alpha}(B(y, \epsilon R), B(y, 2\epsilon R)^c) \geq \mathrm{cap}_{\mathscr{E}_\mu}(B_\theta(y, S), B_\theta(y, \eta(2)S)^c)), \tag{2.23}$$

and applying (2.14) to (2.22) gives

$$\mathrm{cap}_{\mathscr{E}_\mu}(B_\theta(y, S), B_\theta(y, 2S)^c) \geq \mathrm{cap}_{\mathscr{E}^\alpha}\left(B(y, R), B\left(y, \frac{1}{\eta^{-1}(\frac{1}{2})}R\right)^c\right). \tag{2.24}$$

We will now estimate the left hand side of (2.23) and the right hand side of (2.24) using Proposition 2.4.1. For this, we note that $\frac{1}{\eta^{-1}(\frac{1}{2})} > 1$. Indeed, since η is a strictly increasing function by (2.20), the inequality is equivalent to $\eta(1) > \frac{1}{2}$, which is true since $\eta(1) = C' \geq 1$. As a consequence, we may apply Proposition 2.4.1 with both $a = 2$ and $a = \frac{1}{\eta^{-1}(\frac{1}{2})}$.

Applying the upper bound of Proposition 2.4.1 for $a = 2$, there exist $C > 0$ and $R_1 > 0$ such that if $R < R_1$ and $y \in \mathbb{R}^d, \epsilon \in (0, \frac{1}{2})$, then

$$\mathrm{cap}_{\mathscr{E}^\alpha}(B(y, \epsilon R), B(y, 2\epsilon R)^c) \leq C\epsilon^d R^d L(\epsilon R)^{-1}. \tag{2.25}$$

On the other hand, applying the lower bound of Proposition 2.4.1 for $a = \frac{1}{\eta^{-1}(\frac{1}{2})}$, there exists $C > 0$ and $R_2 > 0$ such that if $R < R_2$, then

$$\mathrm{cap}_{\mathscr{E}^\alpha}\left(B(y,R), B\left(y, \frac{1}{\eta^{-1}(1/2)}R\right)^c\right) \geq CR^d L(R)^{-1}. \tag{2.26}$$

Similarly, we wish to estimate the right hand side of (2.23) and the left hand side of (2.24) using Proposition 2.4.2(b). For this, we note that $\eta(2) > \eta(1) = C' \geq 1$. Therefore, we may apply Proposition 2.4.2(b) at the point $x = y$ with the choices $a = 2$ and $a = \eta(2)$. Applying the lower bound for $a = \eta(2)$, there exists $C > 0$ such that

$$\mathrm{cap}_{\mathscr{E}_\mu}(B_\theta(y,S), B_\theta(y,\eta(2)S)^c) \geq C\mu(B_\theta(y,S))S^\beta. \tag{2.27}$$

On the other hand, applying the upper bound for $a = 2$, there exists $C > 0$ such that

$$\mathrm{cap}_{\mathscr{E}_\mu}(B_\theta(y,s), B_\theta(y,2s)^c) \leq C\mu(B_\theta(y,s))s^{-\beta}. \tag{2.28}$$

Now, we combine the above equations as follows. Combining (2.25), (2.23) and (2.27) in that order, there exists a constant $C > 0$ such that if $R < R_1$ and $y \in \mathbb{R}^d, \epsilon \in (0, \frac{1}{2})$,

$$\mu(B_\theta(y,S))S^\beta \leq C\epsilon^d R^d L(\epsilon R)^{-1}. \tag{2.29}$$

Similarly, combining (2.26), (2.24) and (2.28) in that order, there exists a constant $C > 0$ such that if $R < R_2$ and $y \in \mathbb{R}^d, \epsilon \in (0, \frac{1}{2})$,

$$\mu(B_\theta(y,s))s^{-\beta} \geq CR^d L(R)^{-1}. \tag{2.30}$$

Dividing (2.29) by (2.30), for all $y \in \mathbb{R}^d, \epsilon \in (0, \frac{1}{2})$ and $R < R_1 \wedge R_2$,

$$\frac{\mu(B_\theta(y,s))}{\mu(B_\theta(y,S))} \leq C\epsilon^d \frac{L(R)}{L(\epsilon R)}\left(\frac{s}{S}\right)^\beta. \tag{2.31}$$

We fix $R = \frac{R_1 \wedge R_2}{2}$, and will now estimate the quantity $\left(\frac{s}{S}\right)^\beta$. Combining (2.21) and (2.22), the following containments hold :

$$B_\theta(y,s) \subset B(y,\epsilon R) \subset B(y,R) \subset B_\theta(y,S).$$

As a consequence of the above,

$$\frac{s}{S} = \frac{\mathrm{diam}(B_\theta(y,s),\theta)}{\mathrm{diam}(B_\theta(y,S),\theta)} \leq \frac{\mathrm{diam}(B(y,\epsilon R),\theta)}{\mathrm{diam}(B(y,R),\theta)}. \tag{2.32}$$

Note that $\mathrm{diam}(B(y,\epsilon R)) = 2\epsilon R$ and $\mathrm{diam}(B(y,R)) = 2R$ in the metric space $(\mathbb{R}^d, \|\cdot\|)$. Thus, we may apply Lemma 2.4.4(b) to the right hand side of the above equation to obtain

$$\frac{\mathrm{diam}(B(y,\epsilon R),\theta)}{\mathrm{diam}(B(y,R),\theta)} \leq \eta\left(\frac{2\mathrm{diam}(B(y,\epsilon R),\|\cdot\|)}{\mathrm{diam}(B(y,R),\|\cdot\|)}\right) = \eta\left(2\epsilon\right)$$
$$\overset{(2.20)}{\leq} 2^{\delta}C'\epsilon^{\delta} = C\epsilon^{\delta}. \tag{2.33}$$

Combining (2.32) and (2.33),

$$\frac{s}{S} \leq C\epsilon^{\delta}. \tag{2.34}$$

By (2.31) and the above, for all $\epsilon \in (0, \frac{1}{2})$ we have

$$\frac{\mu(B_\theta(y,s))}{\mu(B_\theta(y,S))} \leq C\epsilon^{d+\zeta}\frac{L(R)}{L(\epsilon R)}. \tag{2.35}$$

To prove (2.17), we will compare its left hand side to the left hand side of (2.35). Since μ is a volume doubling measure, by (2.21), there is a constant $C > 0$ such that

$$\mu(B(y,\epsilon R)) \leq \mu(B_\theta(y,2s)) \leq C\mu(B_\theta(y,s)). \tag{2.36}$$

On the other hand, by (2.22),

$$\mu\left(B_E\left(y, \frac{1}{\eta^{-1}(\frac{1}{2})}R\right)\right) \geq \mu(B_\theta(y,S)). \tag{2.37}$$

Substituting the bounds (2.36) and (2.37) into (2.35), we obtain a constant $C_1 > 0$ such that

$$\frac{\mu(B_\theta(y,\epsilon R))}{\mu\left(B_\theta\left(y, \frac{1}{\eta^{-1}(\frac{1}{2})}R\right)\right)} \leq C_1\epsilon^{d+\delta}\frac{L(R)}{L(\epsilon R)}.$$

Note that this statement holds for all $y \in \mathbb{R}^d, \epsilon \in (0, \frac{1}{2})$. Thus, choosing $R = \frac{R_1 \wedge R_2}{2}$, $\zeta, C_1 > 0$ as above and $C_2 = \frac{1}{\eta^{-1}(\frac{1}{2})}$, we have completed the proof of (2.17). □

We will now prove Lemma 2.4.3 and Lemma 2.4.4.

Proof of Lemma 2.4.3. By the definition (2.6) of $\mathscr{F}_{\mathscr{E}}$ we have

$$\mathscr{F}_{\mathscr{E}_\alpha}(B, C^c) \subset \mathscr{F}_{\mathscr{E}_\alpha}(A, D^c).$$

By the above containement and the definition (2.7) of the capacity between sets we have

$$\mathrm{cap}_{\mathscr{E}^\alpha}(B, C^c) \geq \mathrm{cap}_{\mathscr{E}^\alpha}(A, D^c). \tag{2.38}$$

The argument in [50, (6.64),page 115] shows that $\text{cap}_{\mathscr{E}_\mu}(S_1, S_2^c) = \text{cap}_{\mathscr{E}^\alpha}(S_1, S_2^c)$ for any $S_1 \subset S_2$ Borel such that $\overline{S_1} \subset S_2$. This implies that $\text{cap}_{\mathscr{E}_\mu}(B, C^c) = \text{cap}_{\mathscr{E}^\alpha}(B, C^c)$ and $\text{cap}_{\mathscr{E}_\mu}(A, D^c) = \text{cap}_{\mathscr{E}^\alpha}(A, D^c)$.

Combining each of these equalities with (2.38), we obtain (2.13) and (2.14) respectively, completing the proof. □

Proof of Lemma 2.4.4. For the proof of (2.15) of part(a), see [61, Lemma 1.2.18]. By [44, Proposition 10.6] (also see [61, (3),Example 1.2.10]), if d_2 is η-quasisymmetric to d_1, then d_1 is η'-quasisymmetric to d_2, where $\eta'(t) = \frac{1}{\eta^{-1}(t^{-1})}$. Thus, (2.16) also follows by [61, Lemma 1.2.18].

For the proof of part(b), see [44, Proposition 10.8](also see [61, Lemma 1.2.19]). For the proof of part(c), see [44, Theorem 11.3]. □

2.5 Proof of Proposition 2.4.1

In this section, we prove Proposition 2.4.1. We first prove the lower bound in (2.11). This will be followed by the proof of the upper bound.

Proof of lower bound in (2.11) of Proposition 2.4.1 : Fix $a > 1$ and let $R_0 = \frac{1}{a+1}$. Suppose that $x_0 \in \mathbb{R}^d, r < R_0$ and $\phi \in \mathscr{F}_{\mathscr{E}}(B(x_0, r), B(x_0, ar)^c)$(see (2.6) for the definition) are given. Note that $\phi \equiv 1$ on $B(x_0, r)$ and $\phi \equiv 0$ on $B(x_0, ar)^c$. So using Definition 2.1.1 we have

$$\begin{aligned}\mathscr{E}^\alpha(\phi, \phi) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\phi(x) - \phi(y))^2 j_\alpha(\| x - y \|)\mathrm{d}y\mathrm{d}x \\ &\geq \int_{B(x_0,r)}\int_{B(x_0,ar)^c} j_\alpha(\| x - y \|)\mathrm{d}y\mathrm{d}x. \qquad (2.39)\end{aligned}$$

Fix $x \in B(x_0, r)$. Note that $B(x, (a+1)r)^c \subset B(x_0, ar)^c$. So

$$\begin{aligned}\int_{B(x_0,r)} \int_{B(x_0,ar)^c} j_\alpha(\| x - y \|)\mathrm{d}y\mathrm{d}x &\geq \int_{B(x_0,r)} \int_{B(x,(a+1)r)^c} j_\alpha(\| x - y \|)\mathrm{d}y\mathrm{d}x \\ &= \int_{B(x_0,r)} \int_{B(0,(a+1)r)^c} j_\alpha(\| y \|)\mathrm{d}y\mathrm{d}x \\ &= m(B(x_0, r)) \int_{B(0,(a+1)r)^c} j_\alpha(\| y \|)\mathrm{d}y \\ &= Cr^d \int_{B(0,(a+1)r)^c} j_\alpha(\| y \|)\mathrm{d}y \\ &\geq Cr^d \int_{B(0,(a+1)r)^c} \frac{\tilde{L}(\| y \|)}{\| y \|^d}\mathrm{d}y,\end{aligned}$$

where we have used (1.33) and $\tilde{L}$ is defined in (1.22). Now as $r(a+1) < R_0(a+1) < 1$, using the precise definition of $\tilde{L}$ from (1.22) in above inequality we have

$$\begin{aligned}\int_{B(x_0,r)} \int_{B(x_0,ar)^c} j_\alpha(\| x - y \|)\mathrm{d}y\mathrm{d}x \geq& Cr^d \int_{B(0,(a+1)r)^c} \frac{\mathrm{d}y}{\| y \|^d} \\ \geq& Cr^d \log\left(\frac{1}{(a+1)r}\right).\end{aligned}$$

Using the above in (2.39) we obtain

$$\mathscr{E}^\alpha(\phi, \phi) \geq Cr^d \log\left(\frac{1}{(a+1)r}\right), \tag{2.40}$$

for all $0 < r < R_0$. Recall the function $L(s)$ from (1.16). As $\lim_{s \to 0} L(s) \log \frac{1}{(a+1)s} = 1$, there exists $C > 0$ such that

$$L(s) \log\left(\frac{1}{(a+1)s}\right) > C$$

for all $s \in (0, 1]$. Combining this with (2.40) we have

$$\mathscr{E}^\alpha(\phi, \phi) \geq C\frac{r^d}{L(r)}.$$

for all $x_0 \in \mathbb{R}^d, r < R_0$ and $\phi \in \mathscr{F}_{\mathscr{E}}(B(x_0, r), B(x_0, ar)^c)$. By the definition of the capacity (see (2.7)), it follows that

$$\mathrm{cap}_{\mathscr{E}}(B(x_0, r), B(x_0, ar)^c) \geq C\frac{r^d}{L(r)}.$$

The result follows. □

We now prove the upper bound in (2.11).

Proof of upper bound in (2.11). Fix $a > 1$, and let $R_0 = \frac{1}{a+1}$ be as in the proof of the lower bound. Fix $x_0 \in \mathbb{R}^d$ and $r < R_0$, and define $\phi : \mathbb{R}^d \to [0,1]$ by

$$\phi(x) = 1 \wedge \frac{((a+1)r - \parallel x - x_0 \parallel)^+}{r}.$$

We will prove the following inequality :

$$\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\parallel x - y \parallel) \mathrm{d}y \leq C \frac{r^d}{L(r)}. \tag{2.41}$$

Assuming the above inequality we have $\phi \in D(\mathscr{E}^\alpha)$ by Definition 2.1.1. Further, $\mathrm{supp}(\phi) \subset B(x_0, ar)$, $\phi \equiv 1$ on $B(x_0, r)$, and $\phi \in L^2(\mathbb{R}^d, m)$ since it is compactly supported and bounded. By the definition of $\mathscr{F}_{\mathscr{E}}$ in (2.6) and the capacity between sets in (2.7) we have $\phi \in \mathscr{F}_{\mathscr{E}^\alpha}(B(x_0, r), B(x_0, ar)^c)$ and

$$\mathrm{cap}_{\mathscr{E}^\alpha}(B(x_0, r), B(x_0, ar)^c) \leq \mathscr{E}^\alpha(\phi, \phi).$$

Combining the above inequality with (2.41), the upper bound in (2.11) has been proved. It is therefore sufficient to prove (2.41).

We break the domain of the outer integral in (2.41) into three parts.

$$\begin{aligned}
&\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\parallel x - y \parallel) \mathrm{d}y = \int\limits_{B(x_0, ar+1)^c} \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\parallel x - y \parallel) \mathrm{d}y \\
&+ \int\limits_{\overline{A(x_0, ar, ar+1)}} \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\parallel x - y \parallel) \mathrm{d}y + \int\limits_{\overline{B(x_0, ar)}} \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\parallel x - y \parallel) \mathrm{d}y \\
&:= I_1 + I_2 + I_3
\end{aligned} \tag{2.42}$$

Before beginning the upper bounds of each of the three parts, we note that for all $x, y \in \mathbb{R}^d$,

$$|\phi(x) - \phi(y)| \leq \min \left\{ \frac{\parallel x - y \parallel}{r(a-1)}, 1 \right\}. \tag{2.43}$$

We will now derive an inequality which will be used to upper bound both I_1 and I_2. Fix $z \in B(x_0, ar)^c$ and $y \in B(z, \parallel z - x_0 \parallel - ar)$. By the triangle inequality,

$$\parallel x_0 - y \parallel \geq \big| \parallel x_0 - z \parallel - \parallel z - y \parallel \big| > ar,$$

therefore $y \in B(x_0, ar)^c$. Hence, $\phi(y) = \phi(z) = 0$. In particular, for all $z \in B(x_0, ar)^c$,

$$\int_{\mathbb{R}^d} (\phi(z)-\phi(y))^2 j_\alpha(\| z-y \|)\mathrm{d}y = \int_{B(z,\|z-x_0\|-ar)^c} (\phi(z)-\phi(y))^2 j_\alpha(\| z-y \|)\mathrm{d}y. \quad (2.44)$$

By (2.43) and (1.33),

$$\int_{B(z,\|z-x_0\|-ar)^c} (\phi(z) - \phi(y))^2 j_\alpha(\| z - y \|)\mathrm{d}y \leq C \int_{B(z,\|z-x_0\|-ar)^c} \| z - y \|^{-d} \tilde{L}(\| z - y \|)\mathrm{d}y,$$

where $\tilde{L}$ is defined in (1.22). Combining (2.44) and this inequality, for all $z \in B(x_0, ar)^c$ we have

$$\int_{\mathbb{R}^d} (\phi(z)-\phi(y))^2 j_\alpha(\| z-y \|)\mathrm{d}y \leq C \int_{B(z,\|z-x_0\|-ar)^c} \| z-y \|^{-d} \tilde{L}(\| z-y \|)\mathrm{d}y. \quad (2.45)$$

We shall now estimate I_1. For this, fix $x \in B(x_0, ar + 1)^c$. Note that $\| x - x_0 \| - ar > 1$. Setting $z = x$ in (2.45) and applying the definition of $\tilde{L}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\| x - y \|)\mathrm{d}y \\
\leq &C \int_{B(x,\|x-x_0\|-ar)^c} \| x - y \|^{-d} \tilde{L}(\| x - y \|)\mathrm{d}y \\
= &C \int_{B(x,\|x-x_0\|-ar)^c} \| x - y \|^{-d-\alpha} \mathrm{d}y \\
= &C \int_{\|x-x_0\|-ar}^{\infty} \| x - y \|^{-\alpha-1} \mathrm{d}y \\
= &C(\| x - x_0 \| - ar)^{-\alpha}.
\end{aligned}$$

Integrating both sides of the above inequality over $x \in B(x_0, ar + 1)^c$,

$$\begin{aligned}
I_1 = &\int_{B(x_0,ar+1)^c} \int_{\mathbb{R}^d} (\phi(x) - \phi(y))^2 j_\alpha(\| x - y \|)\mathrm{d}y \\
\leq &C \int_{B(x_0,ar+1)^c} (\| x - x_0 \| - ar)^{-\alpha}\mathrm{d}y \\
= &C \int_{B(0,1)^c} \| y \|^{-\alpha} \mathrm{d}y \\
= &C \int_1^{\infty} t^{-1-\alpha}\mathrm{d}t \leq C. \quad (2.46)
\end{aligned}$$

Thus, I_1 is bounded above by a constant.

We now upper bound I_2. Observe that

$$I_2 = \int_{0<\|x-x_0\|-ar\leq 1} \int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 j_\alpha(\| x-y \|)\mathrm{d}y,$$

since we have only removed the measure zero set $\{x : \|x-x_0\| = ar\}$ from the integration domain of I_2. We will upper bound the right hand side of this equality.

Let x be such that $0 <\| x-x_0 \| -ar \leq 1$. Applying (2.45) with $z=x$ and using the definition of $\tilde{L}$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 j_\alpha(\| x-y \|)\mathrm{d}y \\
\leq& C\int_{B(x,\|x-x_0\|-ar)^c} \| x-y \|^{-d} \tilde{L}(\| x-y \|)\mathrm{d}y \\
=& C\int_{\overline{A(x,\|x-x_0\|-ar,1)}} \| x-y \|^{-d} \mathrm{d}y + C\int_{B(x,1)^c} \| x-y \|^{-d-\alpha} \mathrm{d}y \\
\leq& C\int_{\|x-x_0\|-ar}^{1} \frac{\mathrm{d}t}{t} + C\int_1^\infty \frac{\mathrm{d}t}{t^{1+\alpha}} \\
\leq& C\log\left(\frac{1}{\| x-x_0 \| -ar}\right) + \frac{C}{\alpha}.
\end{aligned}$$

Integrating both sides of the above inequality over $x \in \overline{0 < \|x-x_0\| - ar \leq 1}$,

$$\begin{aligned}
I_2 =& \int_{0<\|x-x_0\|-ar\leq 1} \int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 j_\alpha(\| x-y \|)\mathrm{d}y \\
\leq& C\int_{0<\|x-x_0\|-ar\leq 1} \ln\left(\frac{1}{\| x-x_0 \| -ar}\right)\mathrm{d}x + Cm(\overline{A(x_0,ar,ar+1)}) \\
\leq& C\int_0^1 t^{d-1}\ln\left(\frac{1}{t}\right)\mathrm{d}y + Cr^d \leq C + CR_0^d \leq C,
\end{aligned} \tag{2.47}$$

where we used $r < R_0$ in the final inequality. Therefore, I_2 is bounded above by a constant.

Finally, we derive an upper bound for I_3. Fix $x \in \overline{B(x_0,ar)}$. We first bound the inner integral in I_3 using (1.33) :

$$\int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 j_\alpha(\| x-y \|)\mathrm{d}y \leq C\int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 \| x-y \|^{-d} \tilde{L}(\| x-y \|)\mathrm{d}y. \tag{2.48}$$

We break the integral on the right hand side of (2.48) into two parts.

$$\begin{aligned}
&\int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 \parallel x-y \parallel^{-d} \tilde{L}(\parallel x-y \parallel)\mathrm{d}y \\
=&\int_{\overline{B(x,1)}} (\phi(x)-\phi(y))^2 \parallel x-y \parallel^{-d} \tilde{L}(\parallel x-y \parallel)\mathrm{d}y \\
+&\int_{B(x,1)^c} (\phi(x)-\phi(y))^2 \parallel x-y \parallel^{-d} \tilde{L}(\parallel x-y \parallel)\mathrm{d}y. \qquad (2.49)
\end{aligned}$$

We bound the first term on the right hand side of (2.49) now. Note that $\tilde{L}(\parallel x-y \parallel)=1$ for all $y\in\overline{B(x,1)}$. Since $r<R_0=\frac{1}{a+1}$, it follows that $(a-1)r<1$. By (2.43), we have

$$\begin{aligned}
&\int_{\overline{B(x,1)}} (\phi(x)-\phi(y))^2 \parallel x-y \parallel^{-d} \tilde{L}(\parallel x-y \parallel)\mathrm{d}y \\
\leq&\frac{1}{(a-1)^2r^2}\int_{\overline{B(x,(a-1)r)}} \parallel x-y \parallel^{-d+2} \mathrm{d}y + \int_{\overline{A(x,(a-1)r,1)}} \parallel x-y \parallel^{-d} \mathrm{d}y \\
=&\frac{C}{(a-1)^2r^2}\int_0^{(a-1)r} t\mathrm{d}t + C\int_{(a-1)r}^1 \frac{\mathrm{d}t}{t} \\
=&C+C\log\frac{1}{(a-1)r} \leq C\log\frac{1}{(a-1)r}. \qquad (2.50)
\end{aligned}$$

For the second term in (2.49), note that for all $y\in B(x,1)^c$, $\tilde{L}(\parallel x-y \parallel)=\parallel x-y \parallel^{-\alpha}$. By (2.43),

$$\begin{aligned}
&\int_{B(x,1)^c} (\phi(x)-\phi(y))^2 \parallel x-y \parallel^{-d} \tilde{L}(\parallel x-y \parallel)\mathrm{d}y \\
\leq&C\int_{B(x,1)^c} \parallel x-y \parallel^{-d-\alpha} \mathrm{d}y \\
=&C\int_1^\infty \frac{dt}{t^{1+\alpha}} = C. \qquad (2.51)
\end{aligned}$$

Combining (2.48)-(2.51), for all $x\in\overline{B(x_0,ar)}$:

$$\int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 j_\alpha(\parallel x-y \parallel)\mathrm{d}y \leq C\log\left(\frac{1}{(a-1)r}\right).$$

Finally, we integrate both sides of the above inequality over all $x\in\overline{B(x_0,ar)}$:

$$\begin{aligned}
I_3=&\int_{\overline{B(x_0,ar)}}\int_{\mathbb{R}^d} (\phi(x)-\phi(y))^2 j_\alpha(\parallel x-y \parallel)\mathrm{d}y \\
\leq&C\, m(\overline{B(x_0,ar)})\ln\left(\frac{1}{(a-1)r}\right) \leq C\frac{r^d}{L(r)}, \qquad (2.52)
\end{aligned}$$

where, in the final inequality, we used the fact that $L(r)\ln\left(\frac{1}{(a-1)r}\right)$ is bounded above by a constant on $(0,R_0)$. By (2.46), (2.47) and (2.52),

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}(\phi(x)-\phi(y))^2 j_\alpha(\| x-y \|)\mathrm{d}y \le C\frac{r^d}{L(r)}.$$

This completes the proof of (2.41) and the proof of the upper bound in (2.11). □

2.6 Proof of Proposition 2.4.2

In this section, we prove Proposition 2.4.2, which is the second key proposition required for the proof of Proposition 2.3.3. We begin by defining the notion of caloric functions and a heat kernel, and prove that every MMD space satisfying PHI(β) for some $\beta > 0$ admits a heat kernel with a lower bound(Lemma 2.6.3). Using the heat kernel lower bound, we prove Proposition 2.4.2(a), namely that volume doubling holds for the metric and measure associated to the MMD space. We conclude this section with the proof of Proposition 2.4.2(b) as a consequence of volume doubling.

Let $(X,\theta,\mu,\mathscr{E},D(\mathscr{E}))$ be an MMD space (recall the definition from Section 2.1). We begin by defining caloric functions rigorously, as in [50, Definition 2.4]. Given any open interval $I\subset\mathbb{R}$, a function $u: I\to L^2(X,\mu)$ is called *weakly differentiable* at $t_0\in I$ if, for every $f\in L^2(X,\mu)$, the function $t\to\langle u(t),f\rangle_{L^2(X,\mu)}$ is differentiable at t_0. As a consequence of the uniform boundedness principle, there exists a unique function $w\in L^2(X,\mu)$ such that

$$\lim_{t\to t_0}\left\langle\frac{u(t)-u(t_0)}{t-t_0},f\right\rangle_{L^2(X,\mu)}=\langle w,f\rangle_{L^2(X,\mu)},\quad\text{for all } f\in L^2(X,\mu).$$

The function w is referred to as the *weak derivative* of u at t_0 and we write $w=u'(t_0)$.

Definition 2.6.1 (Caloric function) *For $I\subset\mathbb{R}$ open and $\Omega\subset X$ open, a function $u: I\to D(\mathscr{E})$ is called caloric in $I\times\Omega$ if u is weakly differentiable in $L^2(X,\mu)$ at any $t\in I$, and*

$$\langle u',f\rangle+\mathscr{E}(u,f)=0,\quad\textit{for all } f\in D(\mathscr{E})\cap C_c(\Omega), t\in I.$$

We will now state a result that provides us with a large class of caloric functions. Recall the infinitesimal generator $\mathscr{L}$ of $(\mathscr{E},D(\mathscr{E}))$ from Section 2.1. The associated *semigroup operator* on $L^2(X,\mu)$ is given by $\{P_t\}_{t\ge0}, P_t=e^{-t\mathscr{L}}$. For any $V\subset X$ open, let $(D(\mathscr{E}))_V$ be the closure of $D(\mathscr{E})\cap C_c(V)$ under $\|\cdot\|_{\mathscr{E}_1}$ (see (2.1) for the definition). By [33, Theorem 4.4.3], the tuple $(\mathscr{E},(D(\mathscr{E}))_V)$ is also a Dirichlet form, whose infinitesimal generator and semigroup will be denoted by $\mathscr{L}^V$ and $\{P_t^V\}_{t\ge0}$ respectively.

Lemma 2.6.2 (Canonical Caloric function) *Let $V \subset X$ be an open set and fix $g \in L^2(X,\mu) \cap (D(\mathscr{E}))_V$ and $c > 0$. The function*

$$u : \mathbb{R}_+ \to D(\mathscr{E}) \quad ; \quad u(t,\cdot) = (P^V_{ct}(g))(\cdot)$$

is a caloric function on $\mathbb{R}_+ \times V$ (and therefore on $I \times V'$ for any $I \subset \mathbb{R}$ and $V' \subset V$).

Proof. cf. [5, Example 2.1(i), Section 2.2]. Note that the proof works even when $(\mathscr{E}, D(\mathscr{E}))$ is a nonlocal Dirichlet form. □

Recall that $(\mathscr{E}, D(\mathscr{E}))$ is a regular Dirichlet form. By [33, Theorem 7.2.1], there exists a symmetric Hunt process $\{Y_t\}_{t\geq 0}$ that can start from every point outside a properly exceptional set $\mathcal{N} \subset X$(see [33, Section 4.1] for the definition of a properly exceptional set), and whose associated Dirichlet form is $(\mathscr{E}, D(\mathscr{E}))$. The semigroup $\{P_t\}_{t\geq 0}$ associated to $(\mathscr{E}, D(\mathscr{E}))$ can now be written as

$$P_t f(x) = \mathbb{E}_x[f(Y_t)], \quad \text{for all } f \in D(\mathscr{L}), t > 0, x \in X \setminus \mathcal{N}. \tag{2.53}$$

As in [26, page 3752, Section 1.1], the above definition may be extended to all bounded Borel functions f on X. Similarly, by [21, page 109,(3.3.2)] :

$$P^V_t f(x) = \mathbb{E}_x[f(Y_t); t < \tau_V], \quad \text{for all } f \in D(\mathscr{L}^V), t > 0, x \in X \setminus \mathcal{N}. \tag{2.54}$$

The above definition, as before, extends to all bounded Borel functions f on V. It is clear that $P^V_t f(x) \leq P_t f(x)$ for all $t > 0, x \in V$, and bounded Borel f on V. Furthermore, by Jensen's inequality, L^p-contractivity of the semigroup $\{P_t\}_{t\geq 0}$ holds. That is, for all $1 \leq p \leq \infty$, $t > 0$ and $f \in D(\mathscr{L}) \cap L^p(X,\mu)$,

$$\|P_t f\|_{L^p(X,\mu)} \leq \|f\|_{L^p(X,\mu)}. \tag{2.55}$$

We now define the *heat kernel* as in [50, Definition 4.1]. A family of non-negative Borel functions $\{p_t\}_{t>0}$, $p_t : X \times X \to [0,\infty)$ is called a heat kernel for the semigroup $\{P_t\}_{t\geq 0}$ if, for all $t > 0$ and $f \in L^2(X,m)$,

$$P_t f(x) = \int_X p_t(x,y) f(y) \mathrm{d}\mu(y), \quad \text{for } \mu\text{-a.e. } x \in X. \tag{2.56}$$

The heat kernel for the semigroup $\{P_t\}_{t\geq 0}$ will be denoted by $p(t,x,y)$.

Remark 6 *Fix any constant $K > 0$. By (2.56), the semigroup identity*

$$p(Kt,x,y) = \left[P_{\frac{K}{2}t}\left(p\left(\frac{K}{2}t,\cdot,y\right)\right)\right](x)$$

holds for any $x, y \in X \setminus \mathcal{N}$ and $t > 0$. By Lemma 2.6.2, for $x \in X \setminus \mathcal{N}$ fixed, $(t,y) \mapsto p(Kt,x,y)$ is caloric in $(0,\infty) \times V'$, where V' is any subset of X.

We will now prove PHI(β) holds, then the heat kernel exists for $\{P_t\}_{t\geq 0}$, and satisfies a lower bound.

Lemma 2.6.3 (Heat kernel existence and bounds) *Suppose that for some $\beta > 0$, $(X, \theta, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfies PHI(β). Then, the semigroup $\{P_t\}_{t\geq 0}$ admits a heat kernel $p : (0, \infty) \times X \times X \to \mathbb{R}_+$, such that $p(t, \cdot, \cdot)$ is jointly measurable on $X \times X$ and $p(t, \cdot, \cdot)$ is symmetric for all $t > 0$.*

Furthermore, the heat kernel satisfies the following lower bound : there exists a $C > 0$ such that for any $x_0 \in X$ and $r > 0$,

$$\operatorname{esssup}_{x,y\in B_\theta(x_0,r)} p(t, x, y) \geq \frac{e^{-Ctr^{-\beta}}}{\mu(B_\theta(x_0, r))}. \tag{2.57}$$

In order to prove this lemma, we shall use the following results. The first one provides necessary conditions for the existence of a heat kernel.

Proposition 2.6.4 *Let $(\mathscr{E}, D(\mathscr{E}))$ be a regular Dirichlet form on $L^2(X, \mu)$, and let $\{P_t\}_{t\geq 0}$ be the associated heat semigroup. Suppose that $\mathcal{S}$ is a countable family of open sets with $M = \cup_{U\in\mathcal{S}} U$ and $\phi : \mathcal{S} \times (0, \infty) \to \mathbb{R}_+$ is a function such that*

$$\|P_t f\|_{L^\infty(U,\mu)} \leq \phi(U, t)\|f\|_{L^1(X,\mu)} \tag{2.58}$$

for all $U \in \mathcal{S}, t > 0$ and $f \in L^1(X, \mu) \cap L^2(X, \mu)$. Then $\{P_t\}_{t>0}$ possesses a heat kernel $p(t, x, y)$ such that $p(t, \cdot, \cdot)$ is jointly measurable on $X \times X$ and $p(t, \cdot, \cdot)$ is symmetric for all $t > 0$.

Proof. cf.[39, Theorem 2.2]. We apply the case when $p = 1$ and $T_0 = \infty$. □

The second result is required for the proof of the lower bound.

Proposition 2.6.5 *For open $\Omega \subset X$ with $0 < \mu(\Omega) < \infty$, define*

$$\lambda_1(\Omega) = \inf_{f\in(C_c(\Omega)\cap D(\mathscr{E}))\setminus\{0\}} \frac{\mathscr{E}(f, f)}{\|f\|^2_{L^2(\Omega,\mu)}}.$$

Then, for all $t > 0$,

$$\sup_{f\in(D(\mathscr{E}))_\Omega:\|f\|_{L^1(X,\mu)}=1} \|P_t^\Omega f\|_{L^\infty(\Omega,\mu)} \geq \frac{1}{\mu(\Omega)} \exp(-t\lambda_1(\Omega)). \tag{2.59}$$

Proof. We note that the proofs of [81, Proposition 2.2] and [81, Proposition 2.3] do not use the existence of a heat kernel anywhere. Furthermore, the left hand side in the inequality [81, Proposition 2.3] may be changed from $\sup_{x\in X} p_t(x, x)$ to $\|P_t\|_{1\to\infty}$. The proof of Proposition 2.6.5 now follows by an application of [81, Proposition 2.3] to the semigroup $\{P_t^\Omega\}_{t\geq 0}$, noting that $C_c(\Omega) \cap D(\mathscr{E})$ is dense in $L^2(X, \mu) \cap L^1(X, \mu)$. □

We are now ready to prove the lemma.

Proof of Lemma 2.6.3. We will first prove the existence of the heat kernel and its properties using Proposition 2.6.4. Following this, we will use Proposition 2.6.5 to prove the lower bound (2.57).

Let $\{Y_t\}_{t\geq 0}$ be the symmetric Hunt process on X that can start from every point outside a properly exceptional set $\mathcal{N} \subset X$, and whose Dirichlet form is $(\mathscr{E}, D(\mathscr{E}))$. Let $\{P_t\}_{t\geq 0}$ be the transition semigroup given by (2.53).

Let $C_1, C_2, C_3, C_4, C_5, \delta$ be as in the definition of PHI(β). Fix $x_0 \in X$ and $T > 0$. Let $a = T - \frac{C_1+C_2}{2}\frac{1}{2^\beta}$. For any non-negative $f \in C_c(X) \cap D(\mathscr{E})$, let

$$u : \left(a, a + \frac{C_4}{2^\beta}\right) \times B_\theta(x_0, 1/2) \to \mathbb{R} \quad ; \quad u(s, x) = P_s f(x). \tag{2.60}$$

Since f is bounded and compactly supported and μ is a Radon measure, $f \in L^2(X, \mu)$. By Lemma 2.6.2, u is caloric on its domain. By (2.53) and the non-negativity of f, u is non-negative. Thus, applying PHI(β) to u with choices $a = T - \frac{C_1+C_2}{2}\frac{1}{2^\beta}$, $x = x_0$ and radius $r = \frac{1}{2}$,

$$\text{esssup}_{Q^-} u \leq C_5 \text{essinf}_{Q^+} u, \tag{2.61}$$

where $Q^- = (a + \frac{C_1}{2^\beta}, a + \frac{C_2}{2^\beta}) \times B_\theta(x_0, \frac{\delta}{2})$ and $Q^+ = (a + \frac{C_3}{2^\beta}, a + \frac{C_4}{2^\beta}) \times B_\theta(x_0, \frac{\delta}{2})$.

We will now prove that the hypothesis of Proposition 2.6.4 is satisfied, by estimating both sides of (2.61). We shall first estimate the right hand side of (2.61). Note that for some $T' \in (a+\frac{C_3}{2^\beta}, a+\frac{C_4}{2^\beta})$, it must be that $\text{essinf}_{Q^+} u \leq u(T', y)$ for μ-a.s. $y \in B_\theta(x_0, \frac{\delta}{2})$. Integrating this inequality over $B_\theta(x_0, \frac{\delta}{2})$, it follows that

$$\begin{aligned} \text{essinf}_{Q^+} u &\leq \frac{1}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \int_{B_\theta(x_0, \frac{\delta}{2})} u(T', y) \mathrm{d}\mu(y) \\ &= \frac{1}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \int_{B_\theta(x_0, \frac{\delta}{2})} P_{T'} f(y) \mathrm{d}\mu(y), \end{aligned} \tag{2.62}$$

where we used the definition (2.60) of u in the equality above. By non-negativity of $P_{T'} f$ and (2.55) we have

$$\begin{aligned} \frac{1}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \int_{B_\theta(x_0, \frac{\delta}{2})} P_{T'} f(y) \mathrm{d}\mu(y) &\leq \frac{1}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \|P_{T'} f\|_{L^1(X,\mu)} \\ &\leq \frac{1}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \|f\|_{L^1(X,\mu)}. \end{aligned}$$

Combining this with (2.62) we have

$$\text{essinf}_{Q^+} u \leq \frac{1}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \|f\|_{L^1(X,\mu)}. \tag{2.63}$$

We shall now estimate the left hand side of (2.61). Note that $\{Y_t\}_{t\geq 0}$ has right continuous paths a.s. by the definition of a Hunt process (see [33, (iv),M.6,Section A.2]). Therefore, for every sequence $t_n \downarrow T$, we have $f(Y_{t_n}) \to f(Y_T)$ a.s. by continuity of f. Since f is a bounded function, it follows by the dominated convergence theorem that $\mathbb{E}_x[f(Y_{t_n})] \to \mathbb{E}_x[f(Y_T)]$ for every $x \in X \setminus \mathcal{N}$. By the definitions (2.53) and (2.60) of $\{P_t\}_{t\geq 0}$ and u respectively,

$$u(t_n, x) \to u(t, x), \quad \text{for all } x \in B_\theta(x_0, \delta/2) \setminus \mathcal{N}. \tag{2.64}$$

For $t \in (a + \frac{C_1}{2^\beta}, a + \frac{C_2}{2^\beta})$, define the sets

$$S_t := \{x \in B\,(x_0, \delta/2) \setminus \mathcal{N} : \operatorname{esssup}_{Q^-} u < u(t, x)\}, \tag{2.65}$$

and let $g(t) = \mu(S_t)$. By Tonelli's theorem and the definition of the essential supremum,

$$\int_{a+\frac{C_1}{2^\beta}}^{a+\frac{C_2}{2^\beta}} g(t)\mathrm{d}\mu(t) = (m \otimes \mu)\{(t, x) \in Q^- : \operatorname{esssup}_{Q^-} u < u(t, x))\} = 0.$$

Since g is a non-negative function, it follows that $g(t) = 0$ m-a.s. Thus, there exists a sequence $T_n \downarrow T$ such that $g(T_n) = 0$ for all $n \geq 1$. Let $N = \mathcal{N} \cup (\cup_{n\geq 1} S_{T_n})$. Note that $\mathcal{N}$ is properly exceptional. Thus, by the definition (2.65) of S_t and (2.64), it follows that $\mu(N) = 0$, and hence $g(T) = 0$. Since $u(T, \cdot) = P_T f(\cdot)$, we obtain the pointwise estimate

$$\|P_T f\|_{L^\infty(B_\theta(x_0, \frac{\delta}{2}), \mu)} \leq \operatorname{esssup}_{Q^-} u.$$

Combining the above equation with (2.61) and (2.63) in that order, it follows for some constant $C > 0$ that

$$\|P_T f\|_{L^\infty(B_\theta(x_0, \delta/2), \mu)} \leq \frac{C}{\mu(B_\theta(x_0, \frac{\delta}{2}))} \|f\|_{L^1(X, \mu)}. \tag{2.66}$$

By writing f as the sum of its positive and negative parts, (2.66) also holds for arbitrary $f \in C_c(X) \cap D(\mathscr{E})$ with a different constant C. Finally, by regularity of $(\mathscr{E}, D(\mathscr{E}))$ we have that $C_c(X) \cap D(\mathscr{E})$ is dense in $L^1(X, \mu) \cap L^2(X, \mu)$. An approximation argument now shows that (2.66) holds for all $f \in L^1(X, \mu) \cap L^2(X, \mu)$, $x_0 \in \mathbb{R}^d$ and $T > 0$.

Let $D \subset X$ be a countable dense set, and let $\mathcal{S} = \{B_\theta(x, \delta/2) : x \in D\}$. Let $\phi : \mathcal{S} \times (0, \infty) \to \mathbb{R}_+$ be given by $\phi(B_\theta(x, \delta/2), T) = \frac{C}{\mu(B_\theta(x_0, \frac{\delta}{2}))}$. By (2.66) it follows that (2.58) holds with $\mathcal{S}$ and ϕ. Thus, by Proposition 2.6.4, a heat kernel $p(t, x, y)$ exists which satisfies the conclusions of the proposition.

We will now prove the lower bound (2.57) using Proposition 2.6.5. For this, we first upper bound the left hand side of (2.59). Let $x_0 \in \mathbb{R}^d$ and $r > 0$ be arbitrary, and let $\Omega = B_\theta(x_0, r)$. Let $f \in (D(\mathscr{E}))_\Omega$ be such that $\|f\|_{L^1(X,\mu)} = 1$. Then, for μ-a.e. $x \in \Omega$ and $t > 0$,

$$\begin{aligned}
\left|P_t^\Omega f(x)\right| &= |\mathbb{E}_x[f(Y_t); t < \tau_V]| \\
&\leq \mathbb{E}_x[|f(Y_t)|; t < \tau_V] = P_t^\Omega |f|(x) \\
&\leq P_t|f|(x) \\
&= \int_X p(t,x,y)|f(y)|\mathrm{d}\mu(y) \\
&= \int_\Omega p(t,x,y)|f(y)|\mathrm{d}\mu(y) \\
&\leq \mathrm{esssup}_{y\in\Omega} p(t,x,y),
\end{aligned}$$

where we used the definition of P_t^Ω in the first equality, the definition of the heat kernel in the third last line and the fact that $\|f\|_{L^1(X,\mu)} = 1$ in the last inequality. Therefore,

$$\sup_{f\in(D(\mathscr{E}))_\Omega : \|f\|_{L^1(X,\mu)}=1} \|P_t^\Omega f\|_{L^\infty(X,\mu)} \leq \mathrm{esssup}_{x,y\in\Omega} p(t,x,y). \tag{2.67}$$

We shall now lower bound the right hand side of (2.59). Recall the definition of $\lambda_1(\Omega)$ from Proposition 2.6.5, and the infinitesimal generator $\mathcal{L}^\Omega$ which satisfies $\mathscr{E}(f,f) = \int_\Omega f(x)\mathcal{L}^\Omega f(x)\mathrm{d}\mu(x)$ for μ-a.e. $x \in \Omega$. It follows by the spectral theorem applied to $\mathcal{L}^\Omega$ that there exists a non-negative $\Psi \in L^2(\Omega,\mu)$ such that $\mathcal{L}^\Omega\Psi(x) = \lambda_1(\Omega)\Psi(x)$ and $\|\Psi\|_{L^2(\Omega,\mu)} = 1$. Since $P_t^\Omega = e^{-t\mathcal{L}^\Omega}$ for all $t > 0$, it follows that $P_t^\Omega\Psi(x) = e^{-\lambda_1(\Omega)t}\Psi(x)$. Define

$$v : (0, C_4 r^\beta) \times \Omega \to \mathbb{R}_+ \quad ; \quad v(t,x) = P_t^\Omega \Psi(x) = e^{-\lambda_1(\Omega)t}\Psi(x).$$

By Lemma 2.6.2, v is caloric on $(0, C_4 r^\beta)\times\Omega$. It is non-negative since Ψ is non-negative. Therefore, applying PHI(β) to v with the parameters $a = 0, x = x_0$ and r we have

$$\mathrm{esssup}_{(C_1 r^\beta, C_2 r^\beta)\times B_\theta(x_0,\delta r)} e^{-\lambda_1(\Omega)t}\Psi(x) \leq C_5 \mathrm{essinf}_{(C_3 r^\beta, C_4 r^\beta)\times B_\theta(x_0,\delta r)} e^{-\lambda_1(\Omega)t}\Psi(x).$$

Let $x' \in B_\theta(x_0, \delta r)$ be such that $\mathrm{essinf}_{B_\theta(x_0,\delta r)}\Psi(x) < \Psi(x') < \mathrm{esssup}_{B_\theta(x_0,\delta r)}\Psi(x)$. The function $P_t^\Omega\Psi(x) = e^{-\lambda_1(\Omega)t}\Psi(x)$ is the product of two separate functions in t and x, hence

$$\mathrm{esssup}_{(C_1 r^\beta, C_2 r^\beta)\times B_\theta(x_0,\delta r)} e^{-\lambda_1(\Omega)t}\Psi(x) = e^{-C_1\lambda_1(\Omega)r^\beta}\mathrm{esssup}_{B_\theta}(x_0, \delta r),$$

and

$$\mathrm{essinf}_{(C_3 r^\beta, C_4 r^\beta)\times B_\theta(x_0,\delta r)} e^{-\lambda_1(\Omega)t}\Psi(x) = e^{-C_4\lambda_1(\Omega)r^\beta}\mathrm{essinf}_{B_\theta}(x_0, \delta r).$$

Combining the above observations we have

$$\Psi(x')e^{-C_1\lambda_1(\Omega)r^\beta} \le \text{esssup}_{(C_1r^\beta,C_2r^\beta)\times B_\theta(x_0,\delta r)}e^{-\lambda_1(\Omega)t}\Psi(x)$$
$$\le C_5\text{esssup}_{(C_3r^\beta,C_4r^\beta)\times B_\theta(x_0,\delta r)}e^{-\lambda_1(\Omega)t}\Psi(x) \le C_5\Psi(x')e^{-C_4\lambda_1(\Omega)r^\beta}$$

Cancelling out $\Psi(x')$,

$$e^{-C_1\lambda_1(\Omega)r^\beta} \le C_5e^{-C_4\lambda_1(\Omega)r^\beta}$$

Combining the exponential terms,

$$e^{\lambda_1(\Omega)r^\beta(C_4-C_1)} \le C_5 \implies \lambda_1(\Omega)r^\beta \le \frac{\ln C_5}{C_4-C_1}$$

Thus, for some constant $C > 0$ we have

$$\lambda_1(\Omega) \le Cr^{-\beta}. \tag{2.68}$$

Using the fact that $\Omega = B_\theta(x_0, r)$ and combining (2.59) and (2.68), followed by (2.67) in that order, we obtain the lower bound (2.57), as desired.

□

We are ready to prove Proposition 2.4.2(a).

Proof of Proposition 2.4.2(a). We will execute the same argument used in the proof of [50, Theorem 4.5]. However, the aforementioned argument only works in a setting where the Dirichlet form of the MMD space is strongly local. Indeed, we do not use continuity of the heat kernel $p(t, x, y)$ in t, which means that the essential supremum and infimum in the definition of PHI(β) cannot be converted into the usual supremum and infimum (in time) respectively. We also have to handle the case when the point in question lies in a properly exceptional set. Our proof is thus a non-local analogue of the argument in [50, Theorem 4.5].

Let $(X, d, \mu, \mathscr{E}, D(\mathscr{E}))$ satisfy PHI(β), with $\{P_t\}_{t\ge0}$ being the semigroup associated to $(\mathscr{E}, D(\mathscr{E}))$. By Lemma 2.6.3, $\{P_t\}_{t\ge0}$ admits a heat kernel $p : (0,\infty)\times X\times X \to \mathbb{R}_+$ which satisfies the lower bound (2.57). Let $\mathcal{N}$ be the properly exceptional set associated to $(\mathscr{E}, D(\mathscr{E}))$.

We will first argue that volume doubling holds for $x_0 \in X \setminus \mathcal{N}$ and $r > 0$. Then, we will argue that it holds when $x_0 \in \mathcal{N}$.

Let $x_0 \in X \setminus \mathcal{N}$ and $r > 0$ be given. Set $B = B_\theta(x_0, r)$. Let $\delta \in (0, 1)$ and $0 < C_1 < C_2 < C_3 < C_4$, $C_5 > 1$ be as in the definition of PHI(β). By (2.57) of Lemma 2.6.3, we have

$$\text{esssup}_{x,y\in B}p(t, x, y) \ge \frac{e^{-Ctr^{-\beta}}}{\mu(B)}.$$

for all $t > 0$. Note that for any $t \in (C_1\delta^{-\beta}r^\beta, C_2\delta^{-\beta}r^\beta)$ we have $\text{esssup}_{x,y\in B}p(t,x,y) \geq \frac{C}{\mu(B)}$ for some $C > 0$. In particular, for some $y_0 \in B$, we have that

$$\text{esssup}_{x\in B}p(t,x,y_0) \geq \frac{C}{2\mu(B)}.$$

It follows that for some $C > 0$,

$$\text{esssup}_{(t,x)\in(C_1\delta^{-\beta}r^\beta, C_2\delta^{-\beta}r^\beta)\times B}p(t,x,y_0) \geq \frac{C}{\mu(B)}. \tag{2.69}$$

Consider $u(t,\cdot) = p(t,y_0,\cdot)$, which is caloric on $\mathbb{R}_+ \times X$ by Remark 6 and non-negative. Applying PHI(β) to u with choices $a = 0, x = x_0$ and radius $\delta^{-1}r$, we have

$$\text{esssup}_{Q^-}u \leq C_5\text{essinf}_{Q^+}u, \tag{2.70}$$

where $Q^- = (C_1\delta^{-\beta}r^\beta, C_2\delta^{-\beta}r^\beta) \times B$ and $Q^+ = (C_3\delta^{-\beta}r^\beta, C_4\delta^{-\beta}r^\beta) \times B$ are as in (2.3).

Combining (2.70) and (2.69),

$$\text{essinf}_{Q^+}u \geq \frac{C}{\mu(B)}. \tag{2.71}$$

Let $B' = B_\theta(x_0, K^{\frac{1}{\beta}}r)$. Applying PHI($\beta$) to u with choices $a = 0, x = x_0$ and radius $K^{\frac{1}{\beta}}\delta^{-1}r$, we have

$$\text{esssup}_{Q'^-}u \leq C_5\text{essinf}_{Q'^+}u, \tag{2.72}$$

where $Q'^- = (C_1K\delta^{-\beta}r^\beta, C_2K\delta^{-\beta}r^\beta) \times B'$ and $Q'^+ = (C_3K\delta^{-\beta}r^\beta, C_4K\delta^{-\beta}r^\beta) \times B'$ are as in (2.3). Since $K > 1$, $B \subset B'$. Furthermore, the inequalities $C_1K < C_4$ and $C_3 < C_2K$ hold. In particular, the time interval $((C_1K \wedge C_3)\delta^{-\beta}r^\beta, (C_2K \wedge C_4)\delta^{-\beta}r^\beta)$ is non-empty. As a consequence, it follows that

$$((C_1K \wedge C_3)\delta^{-\beta}r^\beta, (C_2K \wedge C_4)\delta^{-\beta}r^\beta) \times B \subset Q^+ \cap Q'^-.$$

Thus, by (2.71) and (2.72) we have

$$p(t,x_0,y) \geq \text{esssup}_{Q'^-\cap Q^+}u \geq \frac{C}{\mu(B)}, \quad (m \otimes \mu)\text{-a.e. } (t,y) \in Q'^+.$$

In particular, there exists $T \in (C_3K\delta^{-\beta}r^\beta, C_4K\delta^{-\beta}r^\beta)$ such that

$$p(T,x_0,y) \geq \frac{C}{\mu(B)}, \quad \mu\text{-a.e. } y \in B'. \tag{2.73}$$

Integrating (2.73) over the variable $y \in B'$ on both sides,

$$\int_{B'} p(T, x_0, y) \mathrm{d}\mu(y) \geq \frac{C\mu(B')}{\mu(B)}. \tag{2.74}$$

Since $x \notin \mathcal{N}$, it follows that $\int_{B'} p(T, x_0, y)\mathrm{d}\mu(y) = P_t(1_{B'})(x_0)$, and by (2.55) we then have

$$\int_{B'} p(T, x_0, y)\mathrm{d}\mu(y) = P_t(1_{B'})(x_0) \leq \|P_t 1_{B'}\|_{L^\infty(X,\mu)} \leq 1.$$

Combining (2.74) with the above equation and recalling the definitions of B and B',

$$\mu(B_\theta(x_0, K^{\frac{1}{\beta}} r)) \leq C\mu(B_\theta(x_0, r)).$$

Let $D \geq 1$ be a positive integer such that $K^{\frac{D}{\beta}} > 2$. Iterating the above estimate D times, we have

$$\mu(B_\theta(x_0, 2r)) \leq \mu(B_\theta(x_0, K^{\frac{D}{\beta}} r)) \leq C^D \mu(B_\theta(x_0, r)). \tag{2.75}$$

This completes the proof of volume doubling when $x_0 \in X \setminus \mathcal{N}$. We will now argue that it holds when $x_0 \in \mathcal{N}$.

Fix any $x_0 \in \mathcal{N}$ and $r > 0$. We claim that there exist $\{x_n\}_{n\geq 1} \subset X \setminus \mathcal{N}$ such that $\theta(x_n, x_0) \to 0$. If not, then for some $\epsilon > 0$, $B_\theta(x_0, \epsilon) \subset \mathcal{N}$. Note that $\mu(B_\theta(x_0, \epsilon)) > 0$ as μ has full support. On the other hand, $\mu(\mathcal{N}) = 0$ as $\mathcal{N}$ is a properly exceptional set. This contradiction proves our claim.

Let $\{x_n\}_{n\geq 1}$ be any sequence satisfying $\theta(x_n, x_0) \to 0, x_n \in X \setminus \mathcal{N}$. Let $n \geq 1$ satisfy $\theta(x_n, x_0) < r/2$. The following containments hold as a result :

$$B_\theta(x_n, r/2) \subset B_\theta(x_0, r) \subset B_\theta(x_0, 2r) \subset B_\theta(x_n, 4r).$$

By (2.75) applied thrice, $\mu(B_\theta(x_n, 4r)) \leq C^3 \mu(B_\theta(x_n, r/2))$. It follows that

$$\mu(B_\theta(x_0, 2r)) \leq C^3 \mu(B_\theta(x_0, r)),$$

where C^3 is independent of r. Thus, we have proved that volume doubling holds when $x_0 \in \mathcal{N}$, completing the proof. □

Finally, we prove Proposition 2.4.2(b).

Proof of Proposition 2.4.2(b). : We note that volume doubling holds for μ and θ by Proposition 2.4.2(a), and (X, θ) is a uniformly perfect metric space by assumption. By [6, Remark 5.6], it follows that reverse volume doubling([50, Definition 3.17(b)]) holds for μ and θ.

In the presence of volume doubling and reverse volume doubling, we may use results from [25] and [26] since $\phi(r) = r^\beta$ satisfies conditions [25, page 6,(1.13)] and [26, page 3753,(1.11)].

We will now prove the lower bound in (2.12). This will be followed by the proof of the upper bound.

Fix $a > 1$. By [26, (1) implies (3), Theorem 1.18] and the remark at the end of the same theorem, PHI(β) implies the conservativeness of $(\mathscr{E}, D(\mathscr{E}))$ and the estimate UHK(r^β)(see page 3755 and [26, Definition 1.10,(ii)] for the definitions of conservativeness and UHK(r^β) respectively). By [25, (1) implies (3), Theorem 1.15], the condition FK(r^β)(see [26, Definition 1.8] for the definition) holds. As a consequence of FK(r^β) and [36, page 43,(8.19)],

$$C\frac{\mu(B_\theta(x_0, r))}{r^\beta} \leq \mathrm{cap}_{\mathscr{E}}(B_\theta(x_0, r), B_\theta(x_0, ar)^c) \quad \text{for all } x_0 \in X, r > 0.$$

The proof of the lower bound in (2.12) is complete.

We now prove the upper bound. Fix $a > 1$. From [26, (1) implies (7),Theorem 1.18], we know that PHI(β) implies the conditions $\mathrm{J}_{r^\beta,\leq}$ and CSJ(r^β)(see [26, Definition 1.5] and [26, Definition 1.6(ii)] for the respective definitions). Using [25, (2.7), Condition (5), Proposition 2.3] with the choice ar in place of r, we have

$$\mathrm{cap}_{\mathscr{E}}(B(x_0, r)), B(x_0, ar)^c) \leq C\frac{\mu(B_\theta(x_0, ar))}{r^\beta}, \quad \text{for all } x_0 \in X, r > 0. \tag{2.76}$$

Let $D \geq 1$ be an integer such that $2^D \geq a$. Iterating Proposition 2.4.1(a) D times,

$$\mu(B_\theta(x_0, ar)) \leq C\mu(B_\theta(x_0, r)) \quad \text{for all } x_0 \in X, r > 0.$$

Combining (2.76) and the above equation we have

$$\mathrm{cap}_{\mathscr{E}}(B(x_0, r)), B(x_0, ar)^c) \leq C\frac{\mu(B_\theta(x_0, r))}{r^\beta}, \quad \text{for all } x_0 \in X, r > 0.$$

The proof of the upper bound in (2.12) is complete. □

Chapter 3

The Martingale Problem

In this chapter, we will introduce the notion of a martingale problem and prove that it is well-posed for a class of pure-jump symmetric Lévy type operators which are perturbations of the infinitesimal generator of geometric stable processes.

Given an operator $\mathcal{A}$, the existence and uniqueness in law of a strong Markov process whose infinitesimal generator is $\mathcal{A}$ is an important question. In many situations, the laws of such processes can be characterised as solutions to martingale problems. Then, the question of uniqueness in law is in many instances equivalent to the well-posedness of the associated martingale problem.

A class of operators for which the well-posedness of the martingale problem has been studied is that of symmetric pure jump Lévy-type operators $\mathcal{L} : C_b^2(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)$. These are of the form

$$\begin{aligned}(\mathcal{L}f)(x) &= \int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+h) - f(x) - (\nabla f(x)\cdot h)1_{\|h\|<1}\right) K(x,h)\mathrm{d}h \\ &= \frac{1}{2}\int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+h) + f(x-h) - 2f(x)\right) K(x,h)\mathrm{d}h, \qquad (3.1)\end{aligned}$$

where $K : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ satisfies the following properties :

$$\begin{aligned}\int_{\mathbb{R}^d\setminus\{0\}} (1\wedge \|h\|^2)K(x,h)\mathrm{d}h < \infty \quad &\text{for all } x\in\mathbb{R}^d, \text{ and} \\ K(x,h) = K(x,-h) \quad &\text{for all } x\in\mathbb{R}^d, h\in\mathbb{R}^d\setminus\{0\}.\end{aligned}$$

For the rest of this article, we refer to K as the *kernel* corresponding to $\mathcal{L}$.

In [14], the martingale problem was proved to be well-posed for operators of the form

$$K(x,h) = \frac{A(x,h)}{\|h\|^{d+\alpha}}, \qquad (3.2)$$

for some $\alpha \in (0,2)$ and $A : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ which satisfies the following conditions : there exists $c_1, c_2 > 0$ such that

$$c_1 \leq A(x,h) \leq c_2 \quad \text{for all } x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}, \tag{3.3}$$

and there exists $\eta > 0$ such that for every $y \in \mathbb{R}^d$ and every $b > 0$,

$$\lim_{x \to y} \sup_{\|h\| \leq b} |A(x,h) - A(y,h)|(1 + (\log(1/\|x\|) \vee 0))^{1+\eta} = 0. \tag{3.4}$$

These operators are linked to the α-stable processes as follows. The α-stable process on $\mathbb{R}^d, d \geq 1$, is a pure jump Lévy process with jump kernel given by $h \mapsto \frac{C_\alpha}{\|h\|^{d+\alpha}}$ for all $h \in \mathbb{R}^d \setminus \{0\}$ and some constant $C_\alpha > 0$. Thus, the kernel K defined in (3.2) is a perturbation A of the jump kernel of an α-stable process, where A is bounded (see (3.3)) and satisfies a continuity condition (see (3.4)). The processes that solve the martingale problem for a kernel K of the form (3.2) are known as stable-like processes in the literature (see [22] for the terminology).

Recall the geometric stable process of index α, $\alpha \in (0,2)$ on $\mathbb{R}^d, d \geq 1$ which is defined by (1.5) in Section 1.1. It has a jump kernel $j_\alpha : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$, defined in Section 1.2.2. By the bound (1.33), there exist $0 < c_1 < c_2$ such that

$$\frac{c_1}{\|h\|^d} \leq j_\alpha(h) \leq \frac{c_2}{\|h\|^d} \quad \text{for all } 0 < \|h\| \leq 1.$$

Thus, if we define the kernel $K(x,h) = j_\alpha(h)$, the resulting operator does not satisfy the assumptions of [14]. To the best of our knowledge, such kernels have not been studied before in the context of the martingale problem for operators of the form $\mathcal{L}$ in (3.1).

Our aim in this paper is to prove the well-posedness of the martingale problem for operators whose kernels $K(x,\cdot) = A(x,\cdot)J(\cdot)$ are suitable perturbations A of the jump kernel J of a geometric stable process of index α for each $x \in \mathbb{R}^d$, where $\alpha \in (0,2]$. That is, for some constants $C_1, C_2 > 0$,

$$\frac{C_1}{\|h\|^d} \leq J(h) \leq \frac{C_2}{\|h\|^d}$$

for all $x \in \mathbb{R}^d, 0 < \|h\| < 1$, and A satisfies a boundedness assumption similar to (3.3) and a continuity assumption similar to (3.4). Our proof technique is a suitable modification of the proof of [10, Theorem 2.2] and covers a larger class of kernels K, as described in Assumption 3.1.1.

The chapter is organized as follows. In Section 3.1, we define the martingale problem and state the main result of this chapter, Theorem 3.1.2. In Section 3.2, we discuss the motivation behind the problem and provide an outline of the proof. In Section 3.4, we state some preliminary lemmas and key propositions, and prove the theorem. These propositions are proved subsequently in Sections 3.5, 3.6, 3.7 and 3.8.

Throughout this paper, we fix a dimension $d \geq 1$. All constants which may change from line to line are denoted by C, while those whose values are important will have a subscript e.g. C_1, C_2. All integration in this article will be performed with respect to the Lebesgue measure on $\mathbb{R}^d$. For a set $A \subset \mathbb{R}^d$, 1_A denotes the indicator function of A.

3.1 Main result

We shall first define the martingale problem. Then we state our assumptions, followed by the main result, Theorem 3.1.2. Throughout this section, let $\mathcal{L}$ be as in (3.1) with kernel K.

We begin by introducing some function spaces needed for defining the martingale problem. Let

$$L^\infty(\mathbb{R}^d) = \{f : \mathbb{R}^d \to \mathbb{R},\ \text{esssup}_{\mathbb{R}^d}|f| < \infty\}$$

be the space of all essentially bounded (with respect to the Lebesgue measure) functions on $\mathbb{R}^d$, equipped with the norm $\|f\|_\infty = \text{esssup}_{\mathbb{R}^d}|f|$. Let $C_b^k(\mathbb{R}^d)$ be the set of all bounded functions $f : \mathbb{R}^d \to \mathbb{R}$ which are k-times differentiable such that all partial derivatives up to the kth order are bounded. For a twice-differentiable function f, let H_f denote the Hessian (i.e. matrix of second partial derivatives) of f. Let $\|H_f\|_\infty$ be the spectral norm of H_f.

We will now define a norm on $C_b^k(\mathbb{R}^d)$. For $x \in \mathbb{R}^d$ let $x = (x_1, \ldots, x_d)$ denote the coordinates of x. Given a multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$ such that $\alpha_i \in \mathbb{N}$ for all $i = 1, \ldots, d$, let

$$\frac{\partial^\alpha f}{\partial \mathbf{x}^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_d}}{\partial x_d^{\alpha_d}} f,$$

and $|\alpha| = \alpha_1 + \ldots + \alpha_d$. We equip $C_b^k(\mathbb{R}^d)$ with the norm

$$\|f\|_{C_b^k(\mathbb{R}^d)} = \sum_{|\alpha| \leq k} \left\| \frac{\partial^\alpha f}{\partial \mathbf{x}^\alpha} \right\|_\infty . \tag{3.5}$$

Let $\Omega = D([0,\infty); \mathbb{R}^d)$ be the set of all cádlág functions $f : [0,\infty) \to \mathbb{R}^d$ (i.e. those which are right continuous and possess left limits at all points). The space Ω will be endowed with the Skorokhod topology (see [49, Chapter 6] for the definition). Define the canonical coordinate process $\{X_t\}_{t \geq 0}$ by $X_t(\omega) = \omega(t)$ for all $\omega \in \Omega, t \geq 0$. Let $\{\mathcal{F}_t\}_{t \geq 0}$ be the right continuous augmentation of the natural filtration of $\{X_t\}_{t \geq 0}$ and let $\mathcal{F}_\infty = \sigma(\cup_{n \geq 1} \mathcal{F}_n)$.

We are now ready to define the notion of a martingale problem associated to $\mathcal{L}$ as in (3.1). Given $x \in \mathbb{R}^d$, a probability measure $\mathbb{P}_x$ on Ω is a solution to the *martingale problem* for $\mathcal{L}$ started at x if $\mathbb{P}_x(X_0 = x) = 1$ and the process $\{M_t^f\}_{t\geq 0}$ given by

$$M_t^f = f(X_t) - f(x_0) - \int_0^t \mathcal{L}f(X_s)ds \tag{3.6}$$

is an $\{\mathcal{F}_t\}_{t\geq 0}$-martingale under $\mathbb{P}_x$ for all $f \in C_b^2(\mathbb{R}^d)$. A collection of probability measures $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ is called a *strong Markov family* of solutions to the martingale problem for $\mathcal{L}$ if $\mathbb{P}_x$ solves the martingale problem for $\mathcal{L}$ started at x for every $x \in \mathbb{R}^d$, and the strong Markov property holds for the process $\{X_t\}_{t\geq 0}$. That is,

$$\mathbb{E}_x[Y \circ \theta_T | \mathcal{F}_T] = \mathbb{E}_{X_T}[Y] \quad \mathbb{P}_x\text{-a.s.},$$

for every $x \in \mathbb{R}^d$, any finite stopping time T and bounded $\mathcal{F}_\infty$-measurable random variable Y. Here, the shift operator $\theta_T : \Omega \to \Omega$ is defined by

$$(\theta_T(\omega))(s) = \omega(s + T(\omega)). \tag{3.7}$$

We will now state our assumption on the kernel K.

Assumption 3.1.1 *There exist $A : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ and $J : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ such that*

$$K(x,h) = A(x,h)J(h) \tag{3.8}$$

for all $x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}$. Further,

(a) We have $\int_{\mathbb{R}^d\setminus\{0\}}(1 \wedge \|h\|^2)J(h)\mathrm{d}h = K_0 < \infty$.

(b) For all $x \in \mathbb{R}^d$, $h \in \mathbb{R}^d \setminus \{0\}$, $K(x,h) = K(x,-h)$.

(c) There exist $\kappa > 1$ and $l : (0,1) \to \mathbb{R}_+$ which is slowly varying at 0 and satisfies

$$\int_0^1 \frac{l(s)}{s}\mathrm{d}s = +\infty, \tag{3.9}$$

such that

$$\kappa^{-1}\frac{l(\|h\|)}{\|h\|^d} \leq J(h) \leq \kappa\frac{l(\|h\|)}{\|h\|^d}$$

for all $0 < \|h\| \leq 1$.

(d) There exist $c_1, c_2 > 0$ such that $c_1 \leq A(x,h) \leq c_2$ for all $x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}$.

(e) There exists $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ such that

(i) $\psi(s) = 1$ *for all* $s \geq 1$.

(ii) For all $b, \epsilon > 0$ *there exists* $\delta > 0$ *such that*

$$\|y - x\| < \delta \implies \sup_{\|h\|<b} |A(x,h) - A(y,h)|\psi(\|h\|) < \epsilon. \tag{3.10}$$

(iii) We have

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{J(h)}{\psi(\|h\|)} \mathrm{d}h = \mathcal{J} < \infty. \tag{3.11}$$

We now mention some examples of jump kernels $J(h)$ which satisfy Assumption 3.1.1(c). Recall the geometric stable process of index $\alpha \in (0,2]$ defined by (1.5) in Section 1.1. By (1.33), its jump kernel $j_\alpha(h)$ satisfies Assumption 3.1.1(c) with $l(h) = 1$. Another class of examples are the *iterated geometric stable processes* $\{Y_t^n\}_{t\geq 0}$, whose Lévy exponents (see (1.3) for the definition) are given by

$$\Psi_n(\lambda) = \underbrace{\log\circ\log\circ\ldots\circ\log}_{n \text{ times}}(1 + \|\lambda\|),$$

for each $n \geq 1$ and $x, \lambda \in \mathbb{R}^d$. Let j_n be the jump kernel of $\{Y_t^n\}_{t\geq 0}$ for $n \geq 1$. Then, j_n also satisfy Assumption 3.1.1(c) for all $n \geq 1$ with the choice

$$l_n(h) = \prod_{i=1}^{n-1} \left(\underbrace{\log\circ\log\circ\ldots\circ\log}_{i \text{ times}} \left(\frac{1}{h}\right)\right)^{-1}.$$

The reader is referred to [64, Theorem 4.1] for the proofs of the above estimate. We now state the main result of this paper.

Theorem 3.1.2 *There exists a unique strong Markov family of solutions* $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ *to the martingale problem for any operator* $\mathcal{L}$ *of the form* (3.1) *whose kernel* K *satisfies Assumption 3.1.1.*

3.2 Motivation, literature and overview of proof

We shall divide this section into three parts. In Section 3.2.1 we shall briefly review the literature on the martingale problem. In Section 3.2.2 we shall discuss the motivation for the main result. Finally, in Section 3.3 we conclude with an overview of Theorem 3.1.2.

3.2.1 Literature on the martingale problem

Suppose that $\{X_t\}_{t\geq 0}$ is a Feller process taking values in $\mathbb{R}^d$ with infinitesimal generator $\mathcal{A}$ defined on a domain $D(\mathcal{A})$. Then, by [31, Proposition 1.7, Chapter 4], for every $f \in D(\mathcal{A})$ the process

$$M_t^f = f(X_t) - f(X_0) - \int_0^t \mathcal{A}f(X_s)\mathrm{d}s$$

is a martingale with respect to the natural filtration of $\{X_t\}_{t\geq 0}$. Therefore, corresponding to every Feller process $\{X_t\}_{t\geq 0}$ is a family of martingales $\{M_t^f\}_{f\in D(\mathcal{A})}$. The martingale problem asks whether the reverse correspondence holds : for an operator $\mathcal{A}$, does the set of martingales $\{M_t^f\}_{f\in D(\mathcal{A})}$ correspond to a unique process $\{X_t\}_{t\geq 0}$?

The martingale problem formulation was introduced by Stroock and Varadhan[78] as a probabilistic framework to accommodate the theory of existence and uniqueness of solutions for a class of backward partial differential equations (see [60] for more details on this connection). They proved that the martingale problem is well-posed for operators of the form

$$(\mathcal{A}u)(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{ij}(u(x), x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{d} b_i(u(x), x)\frac{\partial}{\partial x_i}, \tag{3.12}$$

where $a_{ij} : \mathbb{R}^d \to \mathbb{R}$ are continuous and bounded functions for all $1 \leq i, j \leq d$ such that $[a_{ij}(x)]_{1\leq i,j\leq d}$ is positive definite for all $x \in \mathbb{R}^d$, and $b_i : \mathbb{R}^d \to \mathbb{R}$ are bounded and measurable for all $1 \leq i \leq d$ (see [78, Theorem 6.2]).

They use a weak-convergence argument to prove the existence of a solution to the martingale problem. Their technique for proving uniqueness of solutions, for $b_i \equiv 0$, involves an L^p estimate for the Green operator associated to the Brownian motion on $\mathbb{R}^d$ that is stable under perturbation of the generator. To adapt this to the case $b_i \not\equiv 0$, they change the underlying measure suitably using a Girsanov transform. We refer the reader to [78, Section 5] for the precise details.

The result of Stroock and Varadhan extended the work of Krylov[58] who considered the stochastic differential equation

$$X_t = x + \int_0^t \sigma(X_t)\mathrm{d}\xi_t + \int_0^t b(X_t)\mathrm{d}t$$

and proved existence and uniqueness of a solution in two dimensions under the assumption that the matrix valued function σ is uniformly elliptic while the function b is bounded. However, only existence could be proved in dimensions three or above, and the formulation as a martingale problem addresses the question of uniqueness in this setting for a much larger class of stochastic differential equations. Other applications of the martingale problem include an extension of the Donsker invariance theorem for Markov chains (see [78, Section 10]) and weak uniqueness for differential equations (see [11, Theorem 5.4, Chapter VI]).

3.2.2 Motivation behind Theorem 3.1.2

We will now motivate our choice of working with operators of the form (3.1). A pure-jump Lévy process $\{Y_t\}_{t\geq 0}$ on $\mathbb{R}^d, d \geq 1$ is one that has a Lévy exponent (see (1.3) for the definition) of the form

$$\phi_Y(\lambda) = \int_{\mathbb{R}^d\setminus\{0\}} (e^{i(\lambda \cdot h)} - 1 - i(\lambda \cdot h))\mathrm{d}\Pi(h) \quad \text{for all } \lambda \in \mathbb{R}^d,$$

where Π is a measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int_{\mathbb{R}^d\setminus\{0\}}(1 \wedge \|h\|^2)\mathrm{d}\Pi(h) < \infty$. We call $\Pi(h)$ the Lévy measure of $\{Y_t\}_{t\geq 0}$. If Π is absolutely continuous with respect to the Lebesgue measure, then its density $j : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ is called the jump kernel of $\{Y_t\}_{t\geq 0}$. The infinitesimal generator of $\{Y_t\}_{t\geq 0}$ is then given by

$$\mathcal{L}_Y f(x) = \int_{\mathbb{R}^d\setminus\{0\}} (f(x+h) - f(x) - 1_{\|h\|<1}(\nabla f(x) \cdot h))j(h)\mathrm{d}h$$

for all $x \in \mathbb{R}^d$, $f \in C_b^2(\mathbb{R}^d)$. Thus, operators of the form (3.1) are mixtures of the infinitesimal generators of pure-jump Lévy processes. $\{Y_t\}_{t\geq 0}$ is called symmetric if $j(h) = j(-h)$ for $h \in \mathbb{R}^d \setminus \{0\}$. Symmetry is commonly seen as an assumption in the martingale problem.

The first works on the well-posedness of the martingale problem for processes having a non-trivial jump part were those of Stroock[77] and Komatsu ([56],[57]). Of these, the assumptions on the Lévy measure in [57] are phrased in terms of the symbol of the generator as a pseudo-differential operator. Assumptions of this kind will not be imposed in this chapter. For other articles which make assumptions on the symbol of the generator and prove well-posedness of the associated martingale problem, we refer the reader to the articles of Hoh([45]), Jacob([48, Chapter 4]) and Kühn([59]).

On the other hand, [56] and [77] prove that the martingale problem is well-posed for operators of the form $\mathcal{A} + \mathcal{K}$, where $\mathcal{A}$ is as in (3.12) and $\mathcal{K}$ is of the form (3.1). The proofs focus on obtaining the same L^p estimates as derived in [78]. Another method of obtaining these estimates that uses more analytic tools is that of Oleinik and Radkevich[69], which was also used in the works of Mikulevičius and Pragarauskas ([62], [63]).

Bass[10], proved the well-posedness of the martingale problem for operators of the form (3.1) such as those with kernel

$$K(x,h) = \frac{\xi_{\zeta(x)}}{\|h\|^{1+\zeta(x)}},$$

where $\alpha : \mathbb{R} \to (0,2)$ is strictly bounded away from 0 and 2, the function $\beta(s) = \max_{\|x-y\|\leq s} |\zeta(x) - \zeta(y)|$ satisfies

$$\int_0^1 \frac{\beta(s)}{s} \mathrm{d}s < \infty \text{ and } \lim_{s\to 0} \beta(s)|\ln(s)| = 0,$$

and $\xi_{\zeta(x)}$ is a normalising constant for each $x \in \mathbb{R}$. While the proof of the existence of a solution coincides with the weak convergence technique used in the proof of [78, Theorem 4.2], the proof of uniqueness avoids having to deal with probabilistic estimates like those of the Green function and heat kernels. To explain the method, let $\{X_t\}_{t\geq 0}$ be a solution to the martingale problem started at $x_0 \in \mathbb{R}^d$ corresponding to K, and let $\{Y_t\}_{t\geq 0}$ be the pure jump symmetric Lévy process with jump kernel $j_{x_0} : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ given by $j_{x_0}(h) = K(x_0, h)$. The key idea is to estimate the semigroup corresponding to $\{X_t\}_{t\geq 0}$ by the semigroup corresponding to $\{Y_t\}_{t\geq 0}$. A demonstration of this proof technique may also be found in Bass and Perkins[13].

Recall the main result of Bass and Tang, [14, Theorem 1.2] from the introduction of this chapter. While they also used the semigroup comparison method of Bass[10], the key difference in their estimate lies in the usage of the $L^2(\mathbb{R}^d)$ norm instead of the $L^\infty(\mathbb{R}^d)$ norm for comparing semigroups. The usage of this norm requires additional estimates such as bounds on the Fourier transform of the heat kernel(see [14, Corollary 2.8]).

We note that the geometric stable processes (or its associated operator/jump kernel) do not satisfy any of the assumptions imposed in any of the papers above, insofar as uniqueness of the martingale problem is concerned. The reason is that bounds such as heat kernel bounds, semigroup ultracontractivity or Green function bounds are either not known or do not satisfy the assumptions imposed in previous papers, such as those of [10] and [14]. This is the main motivation behind this chapter.

We complete this section with an example which illustrates the usage of Theorem 3.1.2. Our example is motivated from [10, Corollary 2.3 and (2.6)].

We require the below lemma before stating our example.

Lemma 3.2.1 *Let $J, \beta : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$, $\zeta : \mathbb{R}^d \to \mathbb{R}_+$ and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be functions which satisfy the following conditions :*

I) J satisfies Assumption 3.1.1(a),(c) with some choice of κ and l.

II) $c_1 \leq \zeta(x) \leq c_2$ *and* $c_1 \leq \beta(h) \leq c_2$ *for some* $c_1, c_2 > 0$ *and every* $x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}$.

III) $J(h) = J(-h)$ *and* $\beta(h) = \beta(-h)$ *for every* $h \in \mathbb{R}^d \setminus \{0\}$.

IV) $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ *satisfies Assumption 3.1.1(e)(i),(iii) for* J.

V) $|\ln(\beta(h))|\psi(\|h\|) < c_3$ *for some constant* $c_3 > 0$ *and every* $h \in \mathbb{R}^d \setminus \{0\}$, *and* ζ *is uniformly continuous.*

Let $A(x,h) = \beta(h)^{\zeta(x)}$. *Then, the kernel* $K(x,h) = A(x,h)J(h)$ *satisfies Assumption 3.1.1 and there is a unique strong Markov family of solutions to the martingale problem for any operator of the form* (3.1) *with kernel* K.

Proof. Assumption 3.1.1(a),(c) are satisfied from I), while Assumption 3.1.1(d) follows from the definition of A and II). Assumption 3.1.1(b) follows by III), while Assumption 3.1.1(e)(i),(iii) follow from IV).

In order to verify Assumption 3.1.1(e)(ii), let $b, \epsilon > 0$ be arbitrary. Suppose that $\|h\| < b$ and $x, y \in \mathbb{R}^d$. We have

$$|A(x,h) - A(y,h)|\psi(\|h\|) = |\beta(h)^{\zeta(x)} - \beta(h)^{\zeta(y)}|\psi(\|h\|)$$
$$\leq |\zeta(x) - \zeta(y)| \max\{\beta(h)^{\zeta(x)}, \beta(h)^{\zeta(y)}\}|\ln(\beta(h))|\psi(\|h\|),$$

where we used the mean value inequality and the monotone nature of $z \to \beta(h)^z$ above. Note that $\beta(h)^{\zeta(x)} = A(x,h)$ and $\beta(h)^{\zeta(y)} = A(y,h)$ by definition. Combining Assumption 3.1.1(d) with V),

$$|\zeta(x) - \zeta(y)| \max\{\beta(h)^{\zeta(x)}, \beta(h)^{\zeta(y)}\}|\ln(\beta(h))|\psi(\|h\|) \leq C'|\zeta(x) - \zeta(y)|$$

for some constant $C' > 0$ independent of x, y and h. Combining the above inequalities and taking the supremum over $\|h\| < b$,

$$\sup_{\|h\|<b} |A(x,h) - A(y,h)|\psi(\|h\|) \leq C'|\zeta(x) - \zeta(y)| \tag{3.13}$$

for every $b > 0$.

Let $\delta > 0$ be such that $\|x - y\| < \delta \implies |\zeta(x) - \zeta(y)| < \frac{\epsilon}{C'}$. Then,

$$\|x - y\| < \delta \implies |\zeta(x) - \zeta(y)| < \frac{\epsilon}{C'} \implies \sup_{\|h\|<b} |A(x,h) - A(y,h)|\psi(\|h\|) < \epsilon$$

by (3.13). This shows that Assumption 3.1.1(e)(ii), and hence Assumption 3.1.1 is satisfied by $K(x,h)$. The result now follows by applying Theorem 3.1.2. □

Example 3.2.2 *Let J_α be the jump kernel corresponding to the geometric stable process of index $\alpha \in (0,2]$ (See Section 3.1), which satisfies $J_\alpha(h) = J_\alpha(-h)$ and Assumption 3.1.1(c) with $l \equiv 1$. Let $\zeta : \mathbb{R}^d \to \mathbb{R}_+$ be any uniformly continuous function which is bounded and bounded away from 0 (for example, $\zeta(x) = 1 + e^{-\|x\|}$) and let $\beta(h) = e^{1\wedge\|h\|^\epsilon}$ for some fixed $\epsilon > 0$. Set $\psi(s) = \frac{1}{s^\epsilon \wedge 1}$ for any $\epsilon \in (0,1)$.*

We note that all conditions I),II),III) and V) of Lemma 3.2.1 are satisfied by the above definitions, and only need to verify condition IV). For this, note that $\psi(s) = 1$ for $s \geq 1$. Thus, it is sufficient to prove that

$$\int_{\mathbb{R}^d\setminus\{0\}} \frac{J_\alpha(h)}{\psi(\|h\|)} \mathrm{d}h < \infty. \tag{3.14}$$

Clearly $\int_{\|h\|\geq 1} J_\alpha(h)\mathrm{d}h < K_0$ since J_α satisfies Assumption 3.1.1(a). Now, by Assumption 3.1.1(c) and a change of variable,

$$\int_{0<\|h\|<1} \|h\|^\epsilon J_\alpha(h)\mathrm{d}h \leq \kappa \int_{0<\|h\|<1} \|h\|^{\epsilon-d}\mathrm{d}h = C\int_0^1 r^{\epsilon-1}\mathrm{d}r = \frac{C}{\epsilon} < \infty \tag{3.15}$$

since $\epsilon > 0$. Combining this with (3.14) shows that ψ satisfies condition V) of Lemma 3.2.1. Thus, the kernel

$$K(x,h) = e^{\zeta(x)(\|h^\epsilon\|\wedge 1)} J_\alpha(h)$$

satisfies Lemma 3.2.1 for any $\epsilon > 0$. By an application of the lemma, there is a unique strong Markov family of solutions to the martingale problem for the operator associated to it via (3.1).

3.3 Overview of the proof of Theorem 3.1.2

We will now provide an overview of the proof of Theorem 3.1.2, beginning with comments on Assumption 3.1.1. In keeping with the assumptions in [14], we ensure that the kernel K is of the form $K(x,h) = A(x,h)J(h)$, where A satisfies the boundedness assumption Assumption 3.1.1(d) which is the analogue of (3.3), and the continuity assumption Assumption 3.1.1(e) which is the analogue of (3.4).

Recall the definition of a jump kernel and the formula of the infinitesimal generator of a Lévy process from the previous section. Assumptions 3.1.1(a),(b) and (c) ensure that we restrict our attention to perturbing the generators of symmetric Lévy processes that possess a jump kernel J which is slowly varying at 0. This class of processes has not been covered previously in the history of the martingale problem, hence we choose to focus on it.

We shall now discuss the proof of Theorem 3.1.2. Suppose that $\mathcal{L}$ is of the form (3.1) such that its kernel K satisfies Assumption 3.1.1. Our proof of the existence of a solution requires a key proposition (Proposition 3.4.1), namely that $\mathcal{L}f$ is uniformly continuous for each $f \in C_b^3(\mathbb{R}^d)$. The proof of the existence of the solution now follows by a suitable modification of the proof of [10, Theorem 2.1], using a weak convergence argument. A selection argument then gives the existence of a strong Markov solution to the problem.

The proof of uniqueness in Theorem 3.1.2 requires three key propositions. The first of these, Proposition 3.4.3 requires the definition of the resolvent. Given a strong Markov process $\{Y_t\}_{t\geq 0}$, its resolvent is defined by

$$(R_\lambda^Y g)(x) = \mathbb{E}_x \int_0^\infty e^{-\lambda t} g(Y_t) dt, \tag{3.16}$$

for every $g \in L^\infty(\mathbb{R}^d)$, $x \in \mathbb{R}^d$ and $\lambda \geq 0$. The first key proposition, Proposition 3.4.3 states that for any $\lambda > 0$, the resolvent $R_\lambda g$ of a solution of the martingale problem is continuous on $\mathbb{R}^d$ for any $g \in L^\infty(\mathbb{R}^d)$. The proof relies on exit time estimates and a result on the regularity of harmonic functions with respect to a solution of the martingale problem.

The second proposition, Proposition 3.4.5 is a resolvent perturbation bound. More precisely, let $\{X_t\}_{t\geq 0}$ be a solution to the martingale problem started at $x_0 \in \mathbb{R}^d$ corresponding to K, and let $\{Y_t\}_{t\geq 0}$ be the pure jump symmetric Lévy process with jump kernel $j_{x_0} : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ given by $j_{x_0}(h) = K(x_0, h)$. Let $R_\lambda^{x_0}$ and $\mathcal{M}^{x_0}$ be the resolvent and infinitesimal generator of $\{Y_t\}_{t\geq 0}$ respectively. Under an additional assumption on K, Assumption 3.4.4 which is stronger than Assumption 3.1.1(e), Proposition 3.4.5 shows a bound of the form

$$\|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)\|_\infty \leq C\|g\|_\infty$$

where it can be ensured under some conditions that $0 < C < 1$. This proposition provides an estimate of how well $\mathcal{M}^{x_0}$ approximates $\mathcal{L}$.

Finally, the last proposition Proposition 3.4.6 is a localization argument, which relaxes Assumption 3.4.4 to Assumption 3.1.1(e). It is a technical result that uses the structure of $D([0,\infty);\mathbb{R}^d)$. Informally, given a strong Markov solution to the martingale problem $\{X_t\}_{t\geq 0}$ with filtration $\{\mathcal{F}_t\}_{t\geq 0}$, define the stopping times $\tau_i : \Omega \to \mathbb{R}_+$ for $i \geq 0$ by

$$\tau_0 = 0, \tau_i = \inf\{t \geq \tau_{i-1} : \|X_t - X_{\tau_{i-1}}\| \geq \eta\} \text{ for } i \geq 1.$$

The result shows that $\tau_i \to +\infty$ a.s., and then inductively shows that any solution to the martingale problem is uniquely determined on the stopped filtration $\mathcal{F}_i, i \geq 1$. The result follows after showing that $\mathcal{F}_\infty$ is generated by $\cup_{i=1}^\infty \mathcal{F}_i$.

Given these three propositions, the key idea of the proof is as follows. If $\{\mathbb{P}_x^i\}_{x\in\mathbb{R}^d}$ are two strong Markov families of solutions to the martingale problem, then we consider the resolvents under these measures

$$R_\lambda^i f(x) = \mathbb{E}_x^i \int_0^\infty e^{-\lambda t} f(X_t)dt, \quad i = 1, 2$$

for $f \in L^\infty(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Assuming Assumption 3.4.4, we show using the resolvent continuity and perturbation bound that for large enough $\lambda > 0$, $R_\lambda^1 g = R_\lambda^2 g$ for all $g \in L^\infty(\mathbb{R}^d)$ such that $\|g\|_{L^\infty(\mathbb{R}^d)} \leq 1$, whence uniqueness follows under the assumption, which is subsequently removed by the localization argument.

3.4 Proof of Theorem 3.1.2

In this section, we will prove Theorem 3.1.2 assuming some key propositions that will be proved in subsequent sections. We will prove that a solution to the martingale problem exists in Section 3.4.1. Finally, we will prove that the solution is unique in Section 3.4.2. Throughout this section, let $\mathcal{L}$ be an operator as in (3.1) with kernel K which satisfies Assumption 3.1.1.

3.4.1 Proof of the Existence in Theorem 3.1.2

We require the following key proposition to prove the existence of a solution.

Proposition 3.4.1 *For all $f \in C_b^3(\mathbb{R}^d)$, $\mathcal{L}f : \mathbb{R}^d \to \mathbb{R}$ is uniformly continuous.*

Proposition 3.4.1 will be proved in Section 3.5. We can now prove the existence as in Theorem 3.1.2.

Proof of the existence in Theorem 3.1.2. Fix $x_0 \in \mathbb{R}^d$. Recall $\Omega = D([0,\infty); \mathbb{R}^d)$ from Section 1.1. For all $n \geq 1$, let $\mathbb{P}_n$ be a probability measure on Ω such that for all $k < n2^n$ and for all $f \in C_b^2(\mathbb{R}^d)$,

$$M_t^{f,n} = f\left(X_{t\wedge\frac{k+1}{2^n}}\right) - f\left(X_{t\wedge\frac{k}{2^n}}\right) - \mathcal{L}f(X_{\frac{k}{2^n}})\left(t \wedge \frac{k+1}{2^n} - t \wedge \frac{k}{2^n}\right) \tag{3.17}$$

is a $\mathbb{P}_n$-martingale, and $\mathbb{P}_n(X_0 = x_0) = 1$. As described in the proof of [10, Theorem 2.1], this martingale can be constructed using a stochastic differential equation driven by a Poisson point process.

Fix $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$. Note that for any $h \in \mathbb{R}^d$, $\|h\| < 1$ we have

$$|f(x+h) - f(x) - \nabla f(x) \cdot h| \leq \|H_f\|_\infty \|h\|^2.$$

On the other hand, for $\|h\| \geq 1$ we have

$$|f(x+h) - f(x)| \leq 2\|f\|_\infty.$$

Combining the two inequalities above,

$$\begin{aligned}|f(x+h) - f(x) - 1_{\|h\|<1}(\nabla f(x) \cdot h)| &\leq (2\|f\|_\infty \wedge \|H_f\|_\infty \|h\|^2) \\ &\leq 2(1 \wedge \|h\|^2)(\|f\|_\infty \vee \|H_f\|_\infty)\end{aligned}$$

for all $h \in \mathbb{R}^d \setminus \{0\}$. Thus, by (3.1) and Assumption 3.1.1(c), for all $x \in \mathbb{R}^d$ we have

$$\begin{aligned}\|\mathcal{L}f(x)\| &\leq 2(\|f\|_\infty \vee \|H_f\|_\infty) \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge \|h\|^2) K(x,h) \mathrm{d}h \\ &\leq C_f \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge \|h\|^2) J(h) \mathrm{d}h \leq 2C_f K_0 \end{aligned} \tag{3.18}$$

where we used Assumption 3.1.1(a) in the last inequality. It follows from the definition of $\mathbb{P}_n$ that $f(X_t) - f(X_0) - 2C_f K_0 t$ is a $\mathbb{P}_n$- supermartingale for all $n \geq 1$. Thus, $\{\mathbb{P}_n\}_{n\geq 1}$ satisfy the hypothesis of [10, Proposition 3.2] since C_f depends only on $\|f\|_\infty$ and $\|H_f\|_\infty$. It follows that the sequence of probability measures $\{\mathbb{P}_n\}_{n\geq 1}$ are tight on $D([0,t_0];\mathbb{R}^d)$ for all $t_0 > 0$.

Letting $\mathbb{P}_{x_0}$ be a subsequential limit of $\mathbb{P}_n$, for arbitrary fixed $f \in C_b^3(\mathbb{R}^d)$ we follow the argument of [10, Theorem 2.1] from the third paragraph onwards, and conclude that [10, (3.3)] holds for $f \in C_b^3(\mathbb{R}^d)$. That is,

$$M_t^f = f(X_t) - f(x_0) - \int_0^t \mathcal{L}f(X_s) \mathrm{d}s$$

is a $\mathbb{P}_{x_0}$-martingale for all $f \in C_b^3(\mathbb{R}^d)$. Now, given $f \in C_b^2(\mathbb{R}^d)$ arbitrary, let $f_n \in C_b^3(\mathbb{R}^d)$ be such that $f_n \to f$ in $C_b^2(\mathbb{R}^d)$.

We claim that $M_t^{f_n} \to M_t^f$ in $L^1(\mathbb{P}_{x_0})$ for each fixed $t > 0$. Since $f_n \to f$ in $C_b^2(\mathbb{R}^d)$ we have

$$f_n(x+h) - f_n(x) - 1_{\|h\|<1}(\nabla f_n(x) \cdot h) \to f(x+h) - f(x) - 1_{\|h\|<1}(\nabla f(x) \cdot h)$$

for all $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$. By (3.1), (3.18) and the dominated convergence theorem, it now follows that $\mathcal{L}f_n(x) \to \mathcal{L}f(x)$ for all $x \in \mathbb{R}^d$. Since $f_n \to f$ pointwise, it follows that $M_t^{f_n} \to M_t^f$ pointwise as random variables on Ω.

By (3.17) and (3.18),

$$|M_t^{f_n}| \leq \|f_n\|_{C_b^2(\mathbb{R}^d)}(2 + 2C_f K_0 t)$$

for all $n \geq 1$. Note that $\|f_n\|_{C_b^2(\mathbb{R}^d)}$ is a convergent, hence bounded sequence. It follows by the bounded convergence theorem that $M_t^{f_n} \to M_t^f$ in $L^1(\mathbb{P}_{x_0})$.

Thus, $\{M_t^f\}_{t\geq 0}$ is the $L^1(\mathbb{P}_{x_0})$-pointwise limit of $\{M_t^{f_n}\}_{t\geq 0}$. Since $\{M_t^{f_n}\}_{t\geq 0}$ are martingales and $\{\mathcal{F}_t\}_{t\geq 0}$ is a complete filtration, it follows that $\{M_t^f\}_{t\geq 0}$ is also a martingale. We have shown that $\mathbb{P}_{x_0}$ is a solution to the martingale problem for $\mathcal{L}$ started at x_0 for each $x_0 \in \mathbb{R}^d$.

Suppose that $\mathcal{G}_{x_0}$ is the set of all solutions to the martingale problem for $\mathcal{L}$ started at x_0. Then, by the arguments in [10, Section 4], it can be shown that $\mathcal{G}_{x_0}$ is compact in the space of probability measures $\mathbb{P}$ on $D([0,\infty);\mathbb{R}^d)$ such that $\mathbb{P}(X_0 = x_0) = 1$. Hence, by the proofs in [79, Chapter 14], we obtain the existence of $\mathbb{P}_{x_0} \in \mathcal{G}_{x_0}$ for each $x_0 \in \mathbb{R}^d$ such that $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ is a strong Markov family of solutions to the martingale problem (see the paragraph below [14, Remark 4.2] for a similar argument). □

3.4.2 Proof of Uniqueness in Theorem 3.1.2

By the previous section, we assume the existence of a strong Markov family of solutions to the martingale problem $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ with associated coordinate process $\{X_t\}_{t\geq 0}$ as described in Section 3.1.

Recall the resolvent of a strong Markov process from (3.16). The following lemma states some facts about the resolvent. We include a proof of it after the proof of uniqueness in Theorem 3.1.2.

Lemma 3.4.2 (Resolvent properties) *Let $\{Y_t\}_{t\geq 0}$ be a strong Markov process with resolvent R_λ^Y. Fix $\lambda > 0$.*

(a) For any $g \in L^\infty(\mathbb{R}^d)$, $(R_\lambda^Y g) \in L^\infty(\mathbb{R}^d)$ and

$$\|(R_\lambda^Y g)\|_\infty \leq \frac{1}{\lambda}\|g\|_\infty.$$

(b) Suppose that $\{Y_t\}_{t\geq 0}$ is a Lévy process. Then, for any $\lambda > 0$ and $g \in C_b^2(\mathbb{R}^d)$, $R_\lambda^Y g \in C_b^2(\mathbb{R}^d)$.

We shall now state our first key proposition, the continuity of the resolvent.

Proposition 3.4.3 (Continuity of resolvent) *Let R_λ be the resolvent of $\{X_t\}_{t\geq 0}$. For any $g \in L^\infty(\mathbb{R}^d)$ and $\lambda > 0$, $R_\lambda g$ is a continuous function.*

For stating our second key proposition, we require a stronger assumption on $A(x,h)$ than Assumption 3.1.1(e)(ii).

Assumption 3.4.4 *There exists $\xi > 0$ such that for every $x, y \in \mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$,*

$$|A(x,h) - A(y,h)| \leq \frac{\xi}{\psi(\|h\|)},$$

where ψ is as in Assumption 3.1.1(e).

For $z \in \mathbb{R}^d$, let $\mathcal{M}^z$ be defined by

$$\mathcal{M}^z f(x) = \int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+h) - f(x) - (\nabla f(x)\cdot h)1_{\|h\|<1}\right) K(z,h)\mathrm{d}h$$
$$= \int_{\mathbb{R}^d\setminus\{0\}} \left(f(x+h) + f(x-h) - 2f(x)\right) K(z,h)\mathrm{d}h \tag{3.19}$$

for all $f \in C_b^2(\mathbb{R}^d)$ (where we used Assumption 3.1.1(b) to derive the second equality). By [74, Theorem 31.5,Chapter 6], there exists a unique symmetric Lévy process $\{X_t^z\}_{t\geq 0}$ which solves the martingale problem corresponding to the operator $\mathcal{M}^z$.

Let R_λ^z denote the resolvent of $\{X_t^z\}_{t\geq 0}$. We are now ready to state our second key proposition, an estimate of how well $\mathcal{M}^{x_0}$ approximates $\mathcal{L}$. Note that if $f \in C_b^2(\mathbb{R}^d)$, then by Lemma 3.4.2(b), $R_\lambda^x f \in C_b^2(\mathbb{R}^d)$ for all $x \in \mathbb{R}^d, \lambda > 0$. Therefore, $\mathcal{M}^x R_\lambda^x f$ is well defined for all $x \in \mathbb{R}^d$ and $\lambda > 0$.

Proposition 3.4.5 *Suppose that K satisfies Assumptions 3.1.1 and 3.4.4. For every $f \in C_b^2(\mathbb{R}^d)$, $x_0 \in \mathbb{R}^d$ and $\lambda > 0$,*

$$\|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} f)\|_\infty \leq \frac{4\xi\mathcal{J}}{\lambda}\|f\|_\infty,$$

where $\mathcal{J}$ is as in Assumption 3.1.1(e)(iii) and ξ is as in Assumption 3.4.4.

Our third key proposition is a localization argument.

Proposition 3.4.6 (Localization) *Suppose that every operator $\mathcal{L}$ of the form (3.1) such that K satisfies Assumptions 3.1.1 and 3.4.4 admits a unique solution to the martingale problem. Then, uniqueness holds in Theorem 3.1.2.*

We are now ready to prove the main result of this section. This will be followed by the proof of Lemma 3.4.2.

Proof of Uniqueness in Theorem 3.1.2. Let $\{\mathbb{P}_x^i\}_{x\in\mathbb{R}^d}, i = 1, 2$ be two strong Markov families of solutions to the martingale problem for $\mathcal{L}$. Recall the resolvent of a process defined by (3.16). Consider the resolvents of the coordinate process $\{X_t\}_{t\geq 0}$ under each of these families of measures,

$$R_\lambda^i f(x) = \mathbb{E}_x^i \int_0^\infty e^{-\lambda t} f(X_t)dt, \quad i = 1, 2. \tag{3.20}$$

Let $R_\lambda^\Delta = R_\lambda^1 - R_\lambda^2$ denote their difference. Then, for all $f \in L^\infty(\mathbb{R}^d)$ and $i = 1, 2$, $\|(R_\lambda^i f)\|_\infty \leq \frac{1}{\lambda}\|f\|_\infty$ by Lemma 3.4.2(a). By the triangle inequality,

$$\|R_\lambda^\Delta f\|_\infty \leq \|R_\lambda^1 f\|_\infty + \|R_\lambda^2 f\|_\infty \leq \frac{2}{\lambda}\|f\|_\infty \tag{3.21}$$

for all $f \in L^\infty(\mathbb{R}^d)$. Let

$$\mathcal{B} = \{g \in L^\infty(\mathbb{R}^d) : \|g\|_\infty \le 1\}$$

be the unit ball in $L^\infty(\mathbb{R}^d)$. Define

$$\Theta = \sup_{g \in \mathcal{B}} \|R_\lambda^\Delta g\|_\infty. \tag{3.22}$$

Note that $\Theta \le \frac{2}{\lambda}$ by (3.21).

Fix $g \in C_b^2(\mathbb{R}^d) \cap \mathcal{B}$ and $x, x_0 \in \mathbb{R}^d$. Recall that $R_\lambda^{x_0}$ is the resolvent of the unique process which solves the martingale problem corresponding to the operator $\mathcal{M}^z$, where $\mathcal{M}^z$ is given by (3.19).

Following the computations in the proof of [14, Theorem 1.2, pages 1164-1165] and using the definition of the resolvent (3.16) and Lemma 3.4.2(b) where necessary, we have the identity

$$(R_\lambda^\Delta g)(x) = R_\lambda^\Delta((\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g))(x).$$

By Proposition 3.4.6, it is sufficient to prove that uniqueness holds when K also satisfies Assumption 3.4.4. We assume this from now on. Let $\lambda \ge 8\xi\mathcal{J}$, where ξ is as in Assumption 3.4.4 and $\mathcal{J}$ is as in Assumption 3.1.1(e)(iii). Using the linearity of R_λ^Δ,

$$\begin{aligned}(R_\lambda^\Delta g)(x) &= R_\lambda^\Delta((\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g))(x) \\ &= \|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)\|_\infty \left(R_\lambda^\Delta \left(\frac{(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)}{\|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)\|_\infty} \right) \right)(x)\end{aligned} \tag{3.23}$$

By Proposition 3.4.5 and noting that $\lambda \ge 8\xi\mathcal{J}$ and $\|g\| \le 1$,

$$\|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)\|_\infty \le \frac{4\xi\mathcal{J}}{\lambda}\|g\|_\infty \le \frac{1}{2}. \tag{3.24}$$

On the other hand, by the definition (3.22) of Θ,

$$\frac{(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)}{\|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)\|_\infty} \in \mathcal{B} \implies \left(R_\lambda^\Delta \left(\frac{(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)}{\|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} g)\|_\infty} \right) \right)(x) \le \Theta.$$

Combining the above estimate with (3.23) and (3.24), it follows that

$$\|R_\lambda^\Delta g\|_\infty \le \frac{1}{2}\Theta \tag{3.25}$$

for all $g \in C_b^2(\mathbb{R}^d) \cap \mathcal{B}$ and $\lambda \ge 8\xi\mathcal{J}$.

For any $g \in \mathcal{B}$, we can find a sequence $\{g_n\}_{n\geq 1} \subset C_b^2(\mathbb{R}^d) \cap \mathcal{B}$ such that $g_n(x) \to g(x)$ a.e. $x \in \mathbb{R}^d$. By (3.20) and the dominated convergence theorem, $(R_\lambda^\Delta g_n)(x) \to (R_\lambda^\Delta g)(x)$ for a.e. $x \in \mathbb{R}^d$. Applying (3.25) to each $g_n, n \geq 1$ and letting $n \to \infty$, it follows that (3.25) holds for all $g \in \mathcal{B}$. Taking the supremum over all $g \in \mathcal{B}$ in (3.25) and using the definition (3.22) of Θ, we obtain $\Theta \leq \frac{1}{2}\Theta$. Since $\Theta < \infty$, it must be that $\Theta = 0$.

Therefore, $R_\lambda^\Delta g = 0$ a.e. for all $g \in \mathcal{B}$. By Proposition 3.4.3, $R_\lambda^\Delta g$ is continuous. Hence, $R_\lambda^\Delta g = 0$ identically for all $g \in \mathcal{B}$. By definition of R_λ^Δ,

$$R_\lambda^1 g = R_\lambda^2 g \quad \text{for all } g \in \mathcal{B}, \lambda \geq 8\xi\mathcal{J}.$$

By the definition (3.20) of R_λ^i, uniqueness of the Laplace transform and the right continuity of $\{X_t\}_{t\geq 0}$, it follows that

$$\mathbb{E}_x^1 g(X_t) = \mathbb{E}_x^2 g(X_t),$$

for all $x \in \mathbb{R}^d$, $t > 0$, and continuous $g \in \mathcal{B}$. The same equality also follows for all $g \in \mathcal{B}$ by limiting arguments.

In particular, by setting $g = 1_A$ for any Borel $A \subset \mathbb{R}^d$ in the above equation, it follows that $\mathbb{P}_x^1(X_t \in A) = \mathbb{P}_x^2(X_t \in A)$ for all $x \in \mathbb{R}^d, t > 0$ i.e. the one-dimensional distributions of X_t are the same under $\mathbb{P}_x^i, i = 1, 2$ for all $x \in \mathbb{R}^d$ and $t > 0$. Thus, applying [31, (a), Theorem 4.2, Chapter 4] to the families $\{\mathbb{P}_x^i\}_{x\in\mathbb{R}^d}, i = 1, 2$, it follows that for each $x \in \mathbb{R}^d$, $\{X_t\}_{t\geq 0}$ has the same finite dimensional distributions under each of $\mathbb{P}_x^i, i = 1, 2$. Thus, uniqueness holds for the martingale problem, as desired. □

We will now prove Lemma 3.4.2.

Proof of Lemma 3.4.2. We first prove part (a). Let $g \in L^\infty(\mathbb{R}^d)$ be given. For any $x \in \mathbb{R}^d$, by the definition (3.16) of the resolvent,

$$\begin{aligned}(R_\lambda^Y g)(x) &= \mathbb{E}_x \int_0^\infty e^{-\lambda t} g(Y_t) dt \\ &\leq \|g\|_\infty \mathbb{E}_x \int_0^\infty e^{-\lambda t} dt \\ &= \frac{1}{\lambda}\|g\|_\infty.\end{aligned}$$

Part (a) follows. We now prove part(b). Suppose that $\{Y_t\}_{t\geq 0}$ is a Lévy process, and let $f \in C_b^2(\mathbb{R}^d)$ be given. For any $x \in \mathbb{R}^d$ and $h \in \mathbb{R}^d \setminus \{0\}$, by the definition (3.16) of the resolvent R_λ^Y we have

$$R_\lambda^Y f(x+h) - R_\lambda^Y f(x) = \mathbb{E}_{x+h} \int_0^\infty e^{-\lambda t} f(Y_t) \mathrm{d}t - \mathbb{E}_x \int_0^\infty e^{-\lambda t} f(Y_t) \mathrm{d}t. \qquad (3.26)$$

Since $\{Y_t\}_{t\geq 0}$ is a Lévy process, the process $\{Y_t + h\}_{t\geq 0}$ under the measure $\mathbb{P}_x$ has the same distribution as the process $\{Y_t\}_{t\geq 0}$ under the measure $\mathbb{P}_{x+h}$. Therefore,

$$\mathbb{E}_{x+h}\int_0^\infty e^{-\lambda t} f(Y_t)\mathrm{d}t = \mathbb{E}_x\int_0^\infty e^{-\lambda t} f(Y_t + h)\mathrm{d}t.$$

Combining (3.26) with the above equation and dividing by $\|h\|$,

$$\frac{R_\lambda^Y f(x+h) - R_\lambda^Y f(x)}{\|h\|} = \mathbb{E}_x\int_0^\infty e^{-\lambda t}\frac{f(Y_t+h) - f(Y_t)}{\|h\|}\mathrm{d}t. \tag{3.27}$$

By the mean value theorem,

$$\int_0^\infty e^{-\lambda t}\frac{f(Y_t+h) - f(Y_t)}{\|h\|}\mathrm{d}t \leq \|f'\|_\infty \int_0^\infty e^{-\lambda t}\mathrm{d}t \leq \frac{\|f'\|_\infty}{\lambda}.$$

By (3.27), the above inequality and the dominated convergence theorem, it follows that

$$\frac{\partial(R_\lambda^Y f)}{\partial x_i} = R_\lambda^Y\left(\frac{\partial f}{\partial x_i}\right),$$

for all $i = 1, \ldots, d$. By part(a),

$$\left\|\frac{\partial(R_\lambda^Y f)}{\partial x_i}\right\|_\infty = \left\|R_\lambda^Y\left(\frac{\partial f}{\partial x_i}\right)\right\|_\infty \leq \frac{1}{\lambda}\left\|\frac{\partial f}{\partial x_i}\right\|_\infty$$

for all $i = 1, \ldots, d$. Iterating this argument for higher derivatives, it follows that $(R_\lambda^Y f) \in C_b^2$. □

3.5 Proof of Proposition 3.4.1

In this section, we prove Proposition 3.4.1. Throughout this section, let $\mathcal{L}$ be as in (3.1) with kernel K of the form $K(x,h) = A(x,h)J(h)$, satisfying Assumption 3.1.1.

Proof of Proposition 3.4.1. Fix $\epsilon > 0$ and $f \in C_b^3(\mathbb{R}^d)$. Let

$$g(x,h) = (f(x+h) - f(x) - 1_{\|h\|<1}(\nabla f(x)\cdot h))K(x,h) \tag{3.28}$$

for all $x \in R^d, h \in \mathbb{R}^d \setminus \{0\}$. By (3.1),

$$\mathcal{L}f(y) = \int_{\mathbb{R}^d\setminus\{0\}} g(y,h)\mathrm{d}h \quad \text{for all } y \in \mathbb{R}^d. \tag{3.29}$$

Observe that by Assumption 3.1.1(d), for some constant $C > 0$ we have

$$|g(x, h)| \leq C(\|f\|_\infty \wedge \|H_f\|_\infty)(1 \wedge \|h\|^2)J(h) \tag{3.30}$$

for all $x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}$. Let $M > 0$ be a constant that will be chosen later. For any $\delta > 0$, by (3.30),

$$\left| \int_{\|h\|>\delta^{-1}} g(x, h)\mathrm{d}h \right| \leq C \int_{\|h\|>\delta^{-1}} (1 \wedge \|h\|^2)J(h)\mathrm{d}h.$$

Since $(1 \wedge \|h\|^2)K(0, h)$ is integrable over $\mathbb{R}^d \setminus \{0\}$ by Assumption 3.1.1(a), we can choose δ_M small enough depending on M such that

$$\int_{\|h\|\geq\delta_M^{-1}} |g(x, h)|\mathrm{d}h < \frac{\epsilon}{4M}$$

for all $x \in \mathbb{R}^d$.

So, for any $x, y \in \mathbb{R}^d$, by (3.29), the triangle inequality and the above inequality we have

$$\begin{aligned} |\mathcal{L}f(x) - \mathcal{L}f(y)| &\leq \int_{\|h\|\geq\delta_M^{-1}} (|g(x, h)| + |g(y, h)|)\mathrm{d}h + \int_{\|h\|<\delta_M^{-1}} |g(x, h) - g(y, h)|\mathrm{d}h \\ &\leq \frac{\epsilon}{2M} + \int_{\|h\|<\delta_M^{-1}} |g(x, h) - g(y, h)|\mathrm{d}h. \end{aligned} \tag{3.31}$$

Let $\bar{g}(x, h) = f(x + h) - f(x) - 1_{\|h\|<1}\nabla(f(x) \cdot h)$. Note that by the triangle inequality,

$$\begin{aligned} \int_{\|h\|<\delta_M^{-1}} |g(x, h) - g(y, h)|\mathrm{d}h \leq &\int_{\|h\|<\delta_M^{-1}} |\bar{g}(x, h) - \bar{g}(y, h)|\, K(x, h)\mathrm{d}h \\ &+ \int_{\|h\|<\delta_M^{-1}} |\bar{g}(y, h)|\, |K(x, h) - K(y, h)|\, \mathrm{d}h. \end{aligned} \tag{3.32}$$

We shall now bound the first term in (3.32). Observe that for some constant $C > 0$,

$$|\bar{g}(x, h) - \bar{g}(y, h)| \leq C\|f\|_{C_b^3(\mathbb{R}^d)}(1 \wedge \|h\|^2)$$

for all $x \in \mathbb{R}^d, h \in \mathbb{R}^d \setminus \{0\}$. So, by parts (a) and (d) of Assumption 3.1.1,

$$\begin{aligned} &\int_{\|h\|<\delta_M^{-1}} |\bar{g}(x, h) - \bar{g}(y, h)|\, K(x, h)\mathrm{d}h \\ \leq &\|f\|_{C_b^3(\mathbb{R}^d)}\|x - y\| \int_{\|h\|<\delta_M^{-1}} (1 \wedge \|h\|^2)K(x, h)\mathrm{d}h \\ \leq &c_2\|f\|_{C_b^3(\mathbb{R}^d)}\|x - y\| \int_{\|h\|<\delta_M^{-1}} (1 \wedge \|h\|^2)J(h)\mathrm{d}h \qquad (3.33) \\ &\leq c_2\|f\|_{C_b^3(\mathbb{R}^d)}K_0\|x - y\|. \qquad (3.34) \end{aligned}$$

We will now bound the second term in (3.32). By Assumption 3.1.1(e)(ii) applied with $b = \delta_M^{-1}$, there exists $\delta'_M > 0$ depending on M such that

$$\|x-y\| < \delta'_M \implies \sup_{\|h'\|<\delta_M^{-1}} |A(x,h') - A(y,h')|\psi(h') < \frac{\epsilon}{2M}. \tag{3.35}$$

Thus, whenever $\|x-y\| < \delta'_M$, by Assumption 3.1.1(d) and the above inequality, for all $\|h\| < \delta_M^{-1}$ we have

$$\begin{aligned}|K(x,h) - K(y,h)| &= |A(x,h) - A(y,h)|J(h) \\ &\le \left(\sup_{\|h'\|<\delta_M^{-1}} |A(x,h') - A(y,h')|\psi(h')\right)\frac{J(h)}{\psi(h)} \\ &< \frac{\epsilon}{2M}\frac{J(h)}{\psi(h)}.\end{aligned}$$

Note that $|\bar{g}(x,h)| \le 2\|f\|_{C_b^2(\mathbb{R}^d)}$. Combining this with the above estimate,

$$\int_{\|h\|<\delta_M^{-1}} |\bar{g}(y,h)||K(x,h) - K(y,h)|\mathrm{d}h \le \frac{\epsilon}{M}\|f\|_{C_b^2(\mathbb{R}^d)} \int_{\|h\|<\delta_M^{-1}} \frac{J(h)}{\psi(h)}\mathrm{d}h \le \frac{\mathcal{J}\|f\|_{C_b^2(\mathbb{R}^d)}\epsilon}{M}, \tag{3.36}$$

where we used Assumption 3.1.1(e)(iii) in the last inequality. Combining (3.32), (3.34) and (3.36) we have

$$\int_{\|h\|<\delta_M^{-1}} |g(x,h) - g(y,h)|\mathrm{d}h \le \left(\mathcal{J}\|f\|_{C_b^2(\mathbb{R}^d)}\right)\frac{\epsilon}{M} + \|f\|_{C_b^3(\mathbb{R}^d)}K_0\|x-y\|$$

Combining this with (3.31),

$$|\mathcal{L}f(x) - \mathcal{L}f(y)| \le \left(\mathcal{J}\|f\|_{C_b^2(\mathbb{R}^d)} + \frac{1}{2}\right)\frac{\epsilon}{M} + c_2\|f\|_{C_b^3(\mathbb{R}^d)}K_0\|x-y\| \tag{3.37}$$

for all $x, y \in \mathbb{R}^d$ such that $\|x-y\| \le \delta'_M$. In the previous argument, we now choose

$$M = \frac{1}{2\left(\mathcal{J}\|f\|_{C_b^2(\mathbb{R}^d)} + \frac{1}{2}\right)}$$

and obtain $\delta'_M > 0$ such that (3.35) holds for all $\|x-y\| \le \delta'_M$. Furthermore, let

$$\delta'' = \frac{\epsilon}{2c_2\|f\|_{C_b^3(\mathbb{R}^d)}K_0}.$$

Then it follows from (3.37) that

$$\|x-y\| \le \delta'_M \wedge \delta'' \implies |\mathcal{L}f(x) - \mathcal{L}f(y)| < \epsilon,$$

which proves that $\mathcal{L}f$ is uniformly continuous, as desired. □

3.6 Proof of Proposition 3.4.3

In this section, we will prove Proposition 3.4.3. Let $\mathcal{L}$ be as in (3.1) with kernel K satisfying Assumption 3.1.1. Suppose that $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ is a strong Markov family of solutions to the martingale problem and $\{X_t\}_{t\geq 0}$ is the associated coordinate process as in Section 3.1.

We will require two additional results for the proof. Define $L:(0,1)\to(0,\infty)$ by

$$L(r)=\int_r^1\frac{l(s)}{s}ds,$$

where l is as in Assumption 3.1.1(c). By the same assumption,

$$\lim_{r\to 0^+}L(r)=+\infty. \tag{3.38}$$

For $x\in\mathbb{R}^d, r>0$ let $B(x,r)=\{y\in\mathbb{R}^d:\|y-x\|<r\}$ be the Euclidean ball around x of radius r, and let $\tau_D=\inf\{t>0:X_t\notin D\}$ be the exit time of $\{X_t\}_{t\geq 0}$ from a set $D\subset\mathbb{R}^d$. We have the following upper bound on the exit time of $\{X_t\}_{t\geq 0}$ from small balls.

Lemma 3.6.1 (Exit time estimate) *There exists $C>0$ such that*

$$\mathbb{E}_z\tau_{B(x,r)}\leq\frac{C_2}{L(r)}$$

for all $r\in(0,1)$, $x\in\mathbb{R}^d$ and $z\in B(x,r)$.

Proof. cf. [53, Proposition 3.3]. □

A function $f:\mathbb{R}^d\to\mathbb{R}$ is harmonic on $D\subset\mathbb{R}^d$ if, for all $B\subset D$ open and $x\in B$ we have $\mathbb{E}_x[h(X_{\tau_B})]=h(x)$. We also have the following regularity estimate for bounded harmonic functions.

Lemma 3.6.2 (Regularity of harmonic functions) *There exists $c>0$ and $\gamma\in(0,1)$ such that for all $r\in(0,\frac{1}{2})$ and $x_0\in\mathbb{R}^d$,*

$$|u(x)-u(y)|\leq c\|u\|_\infty\frac{L(\|x-y\|)^{-\gamma}}{L(r)^{-\gamma}},\quad x,y\in B(x_0,r/4)$$

for all bounded $u:\mathbb{R}^d\to\mathbb{R}$ that are harmonic in $B(x_0,r)$.

Proof. cf. [53, Theorem 1.4]. □

We are now ready to prove the proposition.

Proof of Proposition 3.4.3. Fix $\lambda > 0$ and $h \in L^\infty(\mathbb{R}^d)$. Fix $x \in \mathbb{R}^d$ and let $r_0 \in (0, \frac{1}{2})$ be a constant that will be chosen later. We follow the argument as in [12, Proposition 4.2].

Begin with the identity

$$(R_0 h)(z) = \mathbb{E}^z \int_0^{\tau_{B(x,r_0)}} h(X_s)\mathrm{d}s + \mathbb{E}^z R_0 h(X_{\tau_{B(x,r)}})$$

which holds for all $z \in B(x, r_0)$. Let $y \in B(x, \frac{r_0}{4})$ be arbitrary. We plug in $z = x$ and $z = y$ in the above identity, take the difference and use the triangle inequality to get

$$\begin{aligned}|(R_0 h)(x) - (R_0 h)(y)| \le & \left|\mathbb{E}^x \int_0^{\tau_{B(x,r_0)}} h(X_s)\mathrm{d}s\right| + \left|\mathbb{E}^y \int_0^{\tau_{B(x,r_0)}} h(X_s)\mathrm{d}s\right| \\ & + |\mathbb{E}^x R_0 h(X_{\tau_{B(x,r_0)}}) - \mathbb{E}^y R_0 h(X_{\tau_{B(x,r_0)}})|\end{aligned}$$

The first two terms may be bounded by $\|h\|_\infty \sup_{z \in B(x,r_0)} \mathbb{E}_z \tau_{B(x,r_0)}$. Thus, we see that for any $y \in B(x, \frac{r_0}{4})$,

$$\begin{aligned}&|(R_0 h)(x) - (R_0 h)(y)| \\ \le& 2\|h\|_\infty \sup_{z \in B(x,r_0)} \mathbb{E}_z \tau_{B(x,r_0)} + \left|\mathbb{E}_x\left[(R_0 h)(X_{\tau_{B(x,r_0)}})\right] - \mathbb{E}_y\left[(R_0 h)(X_{\tau_{B(x,r_0)}})\right]\right|. \quad (3.39)\end{aligned}$$

Observe that the function $z \to \mathbb{E}^z\left[(R_0 h)(X_{\tau_{B(x,r_0)}})\right]$ is bounded and harmonic in $B(x, r_0)$. Since $r_0 \in (0, \frac{1}{2})$, we can apply Lemma 3.6.2 to this function. Thus, there exist constants $C > 0, \gamma \in (0, 1)$ such that

$$\left|\mathbb{E}_x\left[(R_0 h)(X_{\tau_{B(x,r_0)}})\right] - \mathbb{E}_y\left[(R_0 h)(X_{\tau_{B(x,r_0)}})\right]\right| \le C\|h\|_\infty \left(\frac{L(r_0)}{L(\|x - y\|)}\right)^\gamma,$$

for all $y \in B(x, \frac{r_0}{4})$. Combining the above with (3.39),

$$|(R_0 h)(x) - (R_0 h)(y)| \le 2\|h\|_\infty \sup_{z \in B(x,r_0)} \mathbb{E}_z \tau_{B(x,r_0)} + C\|h\|_\infty \left(\frac{L(r_0)}{L(\|x - y\|)}\right)^\gamma \quad (3.40)$$

for all $y \in B(x_0, \frac{r_0}{4})$.

Now, let $g \in L^\infty$ have compact support and let $h = g - \lambda R_\lambda g$. By the triangle inequality and Lemma 3.4.2(a), $\|h\|_\infty \le 2\|g\|_\infty$. Let $\{X'_t\}_{t \ge 0}$ be an independent copy of $\{X_t\}_{t \ge 0}$. We will now prove that $R_\lambda g = R_0 h$. To prove this, fix $x \in \mathbb{R}^d$ and let

$$k(s) = \int_s^\infty \mathbb{E}_x g(X_t)\mathrm{d}t.$$

We have $k(0) = R_0 g(x)$, and $k'(s) = -\mathbb{E}_x g(X_s)$. Using integration by parts,

$$\int_0^\infty \lambda e^{-\lambda s} k(s) \mathrm{d}s = -R_0 g(x) + \int_0^\infty e^{-\lambda s} \mathbb{E}_x g(X_s) \mathrm{d}s = -R_0 g(x) + R_\lambda g(x). \quad (3.41)$$

Let $\{X_t'\}_{t\geq 0}$ be an independent copy of $\{X_t\}_{t\geq 0}$. Then,

$$\begin{aligned}(R_0(h-g))(x) = (R_0(\lambda R_\lambda)g)(x) &= \mathbb{E}_x \int_0^\infty \lambda (R_\lambda g)(X_t) \mathrm{d}t \\ &= \mathbb{E}_x \int_0^\infty \lambda \mathbb{E}_{X_t} \int_0^\infty e^{-\lambda s} g(X_s') \mathrm{d}s\mathrm{d}t \\ &= \int_0^\infty \lambda e^{-\lambda s} \int_0^\infty [\mathbb{E}_x \mathbb{E}_{X_t} g(X_s')] \mathrm{d}t\mathrm{d}s \\ &= \int_0^\infty \lambda e^{-\lambda s} \int_0^\infty \mathbb{E}_x g(X_{t+s}) \mathrm{d}t\mathrm{d}s \\ &= \int_0^\infty \lambda e^{-\lambda s} k(s) \mathrm{d}s, \end{aligned} \quad (3.42)$$

where we used the Strong Markov property of $\{X_t\}_{t\geq 0}$ in the second last equality. Combining (3.41) and (3.42) it follows that $R_\lambda g = R_0 h$. Now, by (3.40),

$$|(R_\lambda g)(x) - (R_\lambda g)(y)| \leq 4\|g\|_\infty \sup_{z\in B(x,r_0)} \mathbb{E}_z \tau_{B(x,r_0)} + 2C\|g\|_\infty \left(\frac{L(r_0)}{L(\|x-y\|)}\right)^\gamma, \quad (3.43)$$

for all $y \in B(x, \frac{r_0}{4})$.

Observe that (3.43) holds for all bounded g with compact support. However, any bounded measurable g can be approximated pointwise from below by a sequence of bounded functions $g_n, n \geq 1$ which have compact support. By the dominated convergence theorem, $R_\lambda g_n \to R_\lambda g$ pointwise. Applying (3.43) to g_n for each $n \geq 1$ and letting $n \to \infty$, we see that (3.43) holds for all bounded measurable g.

Now, fix $g \in L^\infty(\mathbb{R}^d)$ and let $\epsilon > 0$ be given. By Lemma 3.6.1 and (3.38),

$$\lim_{r\to 0} \sup_{z\in B(x,r)} \mathbb{E}_z \tau_{B(x,r)} = 0.$$

We choose r_0 small enough such that

$$\sup_{z\in B(x,r_0)} \mathbb{E}_z \tau_{B(x,r_0)} < \frac{\epsilon}{8\|g\|_\infty}. \quad (3.44)$$

By (3.38), we can choose $0 < r' < r_0$ small enough such that

$$\sup_{z\in B(x,r')} \left(\frac{L(r_0)}{L(\|x-z\|)}\right)^\gamma = \left(\frac{L(r_0)}{L(r')}\right)^\gamma < \frac{\epsilon}{4C\|g\|_\infty}.$$

Combining (3.43), (3.44) and the above equation,

$$|R_\lambda g(x) - R_\lambda g(y)| < \epsilon \quad \text{for all } y \in B(x, r').$$

Thus, $R_\lambda g$ is a continuous function. □

3.7 Proof of Proposition 3.4.5

In this section, we will prove Proposition 3.4.5, which is required for the proof of uniqueness in Theorem 3.1.2. Throughout, we assume that $\mathcal{L}$ is as in (3.1) with kernel K of the form $K(x,h) = A(x,h)J(h)$ satisfying Assumptions 3.1.1 and 3.4.4.

Proof of Proposition 3.4.5. Fix $x_0 \in \mathbb{R}^d$, $\lambda > 0$ and $f \in C_b^2(\mathbb{R}^d)$. By Lemma 3.4.2(b), $R_\lambda^{x_0} f \in C_b^2(\mathbb{R}^d)$. Hence, it lies in the domains of $\mathcal{L}$ and $\mathcal{M}^{x_0}$. By (3.1) and (3.19) we have

$$\begin{aligned}&(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} f)(x)\\ =&\int_{\mathbb{R}^d\setminus\{0\}} ((R_\lambda^{x_0} f)(x+h) + (R_\lambda^{x_0} f)(x-h) - 2(R_\lambda^{x_0} f)(x))(K(x,h) - K(x_0,h))\mathrm{d}h.\end{aligned}$$

Taking absolute values on both sides and applying the triangle inequality,

$$\begin{aligned}&|(\mathcal{L} - \mathcal{M}^{x_0})(R_\lambda^{x_0} f)(x)|\\ \le&\int_{\mathbb{R}^d\setminus\{0\}} |((R_\lambda^{x_0} f)(x+h) + (R_\lambda^{x_0} f)(x-h) - 2(R_\lambda^{x_0} f)(x))(K(x,h) - K(x_0,h))|\,\mathrm{d}h\\ \le& 4\|R_\lambda^{x_0} f\|_\infty \int_{\mathbb{R}^d\setminus\{0\}} |K(x,h) - K(x_0,h)|\mathrm{d}h\\ \le&\frac{4}{\lambda}\|f\|_\infty \int_{\mathbb{R}^d\setminus\{0\}} |K(x,h) - K(x_0,h)|\mathrm{d}h,\end{aligned} \tag{3.45}$$

where we used Lemma 3.4.2(a) in the final inequality. By Assumption 3.4.4,

$$|K(x,h) - K(x_0,h)| = |A(x,h) - A(x_0,h)|J(h) \le \xi \frac{J(h)}{\psi(\|h\|)},$$

for all $h \in \mathbb{R}^d \setminus \{0\}$. Applying this bound to the right hand side of (3.45),

$$\int_{\mathbb{R}^d\setminus\{0\}} |K(x,h) - K(x_0,h)|\mathrm{d}h \le \xi \int_{\mathbb{R}^d\setminus\{0\}} \frac{J(h)}{\psi(\|h\|)}\mathrm{d}h = \xi\mathcal{J},$$

where $\mathcal{J}$ is as in Assumption 3.1.1(e)(iii). Combining (3.45) with the above inequality, the proposition follows. □

3.8 Proof of Proposition 3.4.6

In this section, we prove Proposition 3.4.6, the localization argument required to prove uniqueness in Theorem 3.1.2. Let $\mathcal{L}$ be as in (3.1) with kernel K which satisfies Assumption 3.1.1. Let $l:(0,1)\to(0,\infty)$ be as in Assumption 3.1.1(c) and $\psi:\mathbb{R}_+\to\mathbb{R}_+$ be as in Assumption 3.1.1(e). For $x\in\mathbb{R}^d, r>0$ let $B(x,r)=\{y\in\mathbb{R}^d : \|y-x\|<r\}$ be the Euclidean ball around x of radius r.

We require some preliminary lemmas before proceeding to the proof of Proposition 3.4.6. We will first create a family of localized operators for $\mathcal{L}$ which satisfy Assumption 3.4.4.

Lemma 3.8.1 *There exists $\eta>0$ and, for each $x_0\in\mathbb{R}^d$, there exists an operator $\mathcal{L}_{x_0}: C_b^2(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)$ of the form (3.1) such that :*

(a) For all $f\in C_b^2(\mathbb{R}^d)$ and $x\in B(x_0,\eta)$,

$$\mathcal{L}f(x)=\mathcal{L}_{x_0}f(x).$$

(b) For all $x_0\in\mathbb{R}^d$, the kernel of $\mathcal{L}_{x_0}$ satisfies Assumptions 3.1.1 and 3.4.4.

Proof. By Assumption 3.1.1(e)(ii) applied with $b=\epsilon=1$, there exists $\delta>0$ such that for all $x_0\in\mathbb{R}^d, 0<\|h\|\le 1$ and $y\in B(x_0,\delta)$ we have

$$|A(y,h)-A(x_0,h)|\le\frac{1}{\psi(\|h\|)}. \tag{3.46}$$

By the triangle inequality and Assumption 3.1.1(d), for all $\|h\|\ge 1$ and $y\in B(x_0,\delta)$ we have

$$|A(y,h)-A(x_0,h)|\le|A(y,h)|+|A(x_0,h)|\le 2c_2$$

By Assumption 3.1.1(e)(i), we have $\psi(s)=1$ for $s\ge 1$. Combining this with the above inequality, for all $\|h\|\le 1$ and $y\in B(x_0,\delta)$,

$$|A(y,h)-A(x_0,h)|\le 2c_2\le\frac{2c_2}{\psi(\|h\|)}$$

Combining the above inequality with (3.46), there exists $C>0$ such that for all $y\in B(x_0,\delta)$ and $h\in\mathbb{R}^d\setminus\{0\}$,

$$|A(y,h)|-|A(x_0,h)|\le\frac{C}{\psi(\|h\|)}.$$

For all $x,y\in B\left(x_0,\delta\right)$ and $h\in\mathbb{R}^d\setminus\{0\}$, we use the above inequality to see that

$$|A(y,h)-A(x,h)|\le|A(y,h)-A(x_0,h)|+|A(x_0,h)-A(x,h)|\le\frac{2C}{\psi(\|h\|)}. \tag{3.47}$$

Fix $x_0 \in \mathbb{R}^d$ and let $\eta = \frac{\delta}{2}$. We shall now define $A_{x_0} : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$. For each z such that $z \in B(x_0, \eta)^c$, let $\Lambda(z)$ be the unique point y on the line segment joining z and x_0 such that $\|y - x_0\| = \eta$. Define

$$A_{x_0}(x,h) = \begin{cases} A(x,h) & x \in B(x_0,\eta) \\ A(\Lambda(x),h) & B(x_0,\eta)^c \end{cases}.$$

Now, we define $K_{x_0} : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ by

$$K_{x_0}(x,h) = A_{x_0}(x,h)J(h).$$

By (3.47) and the definition of A_{x_0}, K_{x_0} satisfies Assumptions 3.1.1(c) with the function l, Assumption 3.1.1(e) with ψ, and Assumption 3.4.4. Furthermore,

$$K_{x_0}(x,h) = K(x,h) \quad \text{for all } x \in B\left(x_0, \eta\right), h \in \mathbb{R}^d \setminus \{0\}.$$

Let $\mathcal{L}_{x_0}$ be given by (3.1) with kernel K_{x_0}. By (3.1) and the above facts about K_{x_0}, it is clear that $\mathcal{L}_{x_0}$ satisfies condition (a) and (b), completing the proof. □

In order to state our second lemma, let $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ be a strong Markov family of solutions to the martingale problem for $\mathcal{L}$. Let $\eta > 0$ be given by Lemma 3.8.1 and define the stopping times $\tau_i : \Omega \to \mathbb{R}_+$ for $i \geq 0$ by

$$\tau_0 = 0, \tau_i = \inf\{t \geq \tau_{i-1} : \|X_t - X_{\tau_{i-1}}\| \geq \eta\} \text{ for } i \geq 1. \tag{3.48}$$

Let $\mathcal{F}_{\tau_i}$ be the filtration We require the following fact about $\mathcal{F}_{\tau_i}, i \geq 1$. Note that the space $(\Omega, \mathcal{F}_\infty)$ is sufficiently rich in that, for any $t \geq 0$ and $\omega \in \Omega$, the element $\omega' \in \Omega$ given by $\omega'(s) = \omega(t \wedge s)$ satisfies [75, (1.11)]. By [75, Theorem 6] it follows that

$$\mathcal{F}_{\tau_i} = \sigma(X_{t\wedge\tau_i} : t \geq 0) \quad \text{for all } i \geq 1. \tag{3.49}$$

Lemma 3.8.2 *$\tau_i \to \infty$ a.s. under every $\mathbb{P}_x$, $x \in \mathbb{R}^d$. Furthermore, $\sigma(\cup_{i=1}^\infty \mathcal{F}_{\tau_i}) = \mathcal{F}_\infty$.*

Proof. Let us assume that there exists a constant $C > 0$ such that for all $x_0 \in \mathbb{R}^d$ and $t > 0$,

$$\mathbb{P}_{x_0}(\tau_1 \leq t) \leq Ct/\eta^2 \tag{3.50}$$

for some constant $C > 0$ independent of x_0, t and η. We will prove that $\tau_i \to +\infty$ a.s. assuming (3.50). Then, we will prove (3.50).

Fix $x_0 \in \mathbb{R}^d$. For any $s \geq 0$, by (3.50),

$$\mathbb{P}_{x_0}(e^{-\tau_1} \geq s) = \mathbb{P}_{x_0}(\tau_1 \leq -\ln s) \leq 0 \vee (C(-\ln(s))/\eta^2 \wedge 1).$$

By the layer-cake formula and the above bound,

$$\begin{aligned}\mathbb{E}_{x_0}[e^{-\tau_1}] &= \int_0^1 \mathbb{P}_{x_0}(e^{-\tau_1} \geq s)\mathrm{d}s \\ &\leq \int_0^{e^{-\eta^2/C}} 1\mathrm{d}s + \int_{e^{-\eta^2/C}}^1 -\frac{C\ln(s)}{\eta^2}\mathrm{d}s \\ &= e^{-\eta^2/C} + \frac{1-\eta^2/C - e^{-\eta^2/C}}{\eta^2/C}.\end{aligned}$$

By increasing C as much as necessary, we can therefore ensure that

$$\mathbb{E}_{x_0}[e^{-\tau_1}] \leq \gamma < 1 \tag{3.51}$$

for some γ independent of x_0. Since $\tau_i \geq \tau_{i-1}$, we have by the Strong Markov property that

$$\mathbb{E}_{x_0}[e^{-\tau_i}] = \mathbb{E}_{x_0}[\mathbb{E}_{x_0}[e^{-\tau_{i-1}-(\tau_i-\tau_{i-1})}|\mathcal{F}_{\tau_{i-1}}]] = \mathbb{E}_{x_0}[e^{\tau_{i-1}}\mathbb{E}_{X_{\tau_{i-1}}}e^{-(\tau_i-\tau_{i-1})}]$$

Applying this inductively with (3.51) we have $\mathbb{E}_{x_0}[e^{-\tau_i}] \leq \gamma^i$ for all $x_0 \in \mathbb{R}^d, i \geq 1$.

It follows that $e^{-\tau_i} \to 0$ in $L^1(\Omega, \mathbb{P}_{x_0})$. However, $e^{-\tau_i}$ is an a.s. decreasing sequence of random variables since $\tau_i \leq \tau_{i+1}$ a.s. Hence, it follows that $e^{-\tau_i} \to 0$ a.s., and $\tau_i \to +\infty$ a.s..

We will now prove (3.50) by using [10, Proposition 3.1]. Let $f \in C_b^2(\mathbb{R}^d)$ be fixed. Note that for all $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$, we have

$$|f(x+h) - f(x) - 1_{\|h\|<1}(\nabla f(x)\cdot h)| \leq 2(\|f\|_\infty \wedge \|H_f\|_\infty).$$

Thus, by (3.1) and Assumption 3.1.1, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}\|\mathcal{L}f(x)\| \leq 2(\|f\|_\infty \wedge \|H_f\|_\infty)\int_{\mathbb{R}^d\setminus\{0\}} K(x,h)\mathrm{d}h &\leq 2C(\|f\|_\infty \wedge \|H_f\|_\infty)\int_{\mathbb{R}^d\setminus\{0\}} J(h)\mathrm{d}h \\ &\leq 2CK_0\end{aligned}$$

Therefore ,by (3.6) we have

$$M_t^f \leq f(X_t) - f(X_0) - C(\|f\|_\infty + \|H_f\|_\infty)t.$$

It follows that the right hand side is a $\mathbb{P}_{x_0}$-supermartingale. Now, (3.50) is a direct consequence of [10, Proposition 3.1].

We will now prove that $\sigma(\cup_{i=1}^\infty \mathcal{F}_{\tau_i}) = \mathcal{F}_\infty$. Clearly, $\sigma(\cup_{i=1}^\infty \mathcal{F}_{\tau_i}) \subset \mathcal{F}_\infty$ holds. Thus, we only need to show the reverse inclusion.

Let $t_1, \ldots, t_n \geq 0$ and $B_1, \ldots, B_n \in \mathcal{B}(\mathbb{R}^d)$ be arbitrary. Let $D = \cap_{1\leq i\leq n}\{X_{t_i} \in B_i\}$ and $D_j = \cap_{1\leq i\leq n}\{X_{t_i\wedge\tau_j} \in B_i\}$ for all $j \geq 1$. Since $\tau_j \to \infty$ a.s. it follows that $1_{D_j} \to 1_D$ a.s. as $j \to \infty$. However, by (3.49), $D_j \in \mathcal{F}_{\tau_j} \subset \cup_{i=1}^{\infty}\mathcal{F}_{\tau_i}$ for all $j \geq 1$. Thus, it follows that $D \subset \sigma(\cup_{i=1}^{\infty}\mathcal{F}_{\tau_i})$. However, sets of the form D clearly generate $\mathcal{F}_\infty$. The second part of the lemma follows. □

We require the following technical lemma. For this, fix $x_0 \in \mathbb{R}^d$ and let $\mathcal{L}_{x_0}$ be as in Lemma 3.8.1. Suppose that $\{\mathbb{P}'_x\}_{x\in\mathbb{R}^d}$ is a strong Markov family of solutions to the martingale problem for $\mathcal{L}_{x_0}$.

Lemma 3.8.3 *Define the measure $\mathbb{P}'_{x_0} \circ \theta_{\tau_1}(A) = \mathbb{P}'_{x_0}(\{\omega : \theta_{\tau_1}(\omega) \in A\})$, and let $Q(\omega, \cdot)$ be a regular conditional probability of $\mathbb{P}'_{x_0} \circ \theta_{\tau_1}$ given $\mathcal{F}_{\tau_1}$. For $A \in \mathcal{F}_{\tau_1}$ and $B \in \mathcal{F}_\infty$, define the event*

$$E_{A,B} = A \cap \{\omega : \theta_{\tau_1}(\omega) \in B\}. \tag{3.52}$$

Then,

1. *For every $i \geq 0$, $\mathcal{F}_{\tau_{i+1}} \subset \sigma\{E_{A,B} : A \in \mathcal{F}_{\tau_1}, B \in \mathcal{F}_{\tau_i}\}$. Further,*

$$\mathcal{F}_\infty = \sigma\{E_{A,B} : A \in \mathcal{F}_{\tau_1}, B \in \mathcal{F}_\infty\}.$$

2. *For every $x \in \mathbb{R}^d$, define the measure*

$$(\mathbb{P}_x \otimes_{\tau_1} Q)(E_{A,B}) = \mathbb{E}_{\mathbb{P}_x}[Q(\omega, B)1_{\{\omega\in A\}}] \tag{3.53}$$

for every $A \in \mathcal{F}_{\tau_1}$ and $B \in \mathcal{F}_\infty$, which can be extended to $\mathcal{F}_\infty$ by the previous part. Then, a right-continuous progressively measurable bounded process $\{M_t\}_{t\geq 0}$ is a $(\mathbb{P}_x \otimes_{\tau_1} Q)$-martingale if

   (a) $\{M_{\tau_1\wedge t}\}_{t\geq 0}$ is a $\mathbb{P}_{x_0}$- martingale, and

   (b) For all $\omega \in \Omega$, $\{M_t - M_{t\wedge\tau(\omega)}\}_{t\geq 0}$ is a $Q(\omega, \cdot)$-martingale.

Proof. We will first prove (1). By taking $B = \Omega$, the statement is clearly true for $i = 0$. Therefore, we assume that $i > 0$.

Let $f : \Omega \to \mathbb{R}$ be $\mathcal{F}_{\tau_{i+1}}$-measurable. By (3.49) and [79, Exercise 1.5.6], there exists a function $F : (\mathbb{R}^d)^{\mathbb{Z}^+} \to \mathbb{R}$ and an increasing sequence $\{t_n\}_{n\geq 1}$ such that

$$f(\omega) = F(X_{t_1\wedge\tau_{i+1}}(\omega), X_{t_2\wedge\tau_{i+1}}(\omega), \ldots).$$

For $0 \leq s \leq \tau_1(\omega)$, $s \wedge \tau_{i+1}(\omega)$ can be written as $c\tau_1(\omega)$ for some $c \in [0, 1]$. On the other hand, for $\tau_1(\omega) \leq s$, $s \wedge \tau_{i+1}(\omega)$ can be written as $c\tau_1(\omega) + (1 - c)\tau_{i+1}(\omega)$ for some $c \in [0, 1]$. Thus, it follows that

$$\mathcal{F}_{\tau_{i+1}} = \sigma(\{X_{c\tau_1} : c \in [0, 1]\} \cup \{X_{c\tau_1+(1-c)\tau_{i+1}} : c \in [0, 1]\}).$$

Clearly, for every $c \in [0,1]$, $X_{c\tau_1}$ is $\mathcal{F}_{\tau_1}$-measurable. On the other hand, for any $c \in [0,1]$ and $B \in \mathcal{B}(\mathbb{R}^d)$,

$$\{X_{c\tau_1+(1-c)\tau_{i+1}(\omega)} \in B\} = E_{\Omega,B'},$$

where $B' = \{\omega : \theta_{(1-c)\tau_i}(\omega) \in B\}$.

Note that $B' \in \mathcal{F}_{\tau_i}$. Therefore, it follows that $X_{c\tau_1+(1-c)\tau_{i+1}(\omega)}$ is also measurable with respect to $\sigma\{E_{A,B} : A \in \mathcal{F}_{\tau_1}, B \in \mathcal{F}_\infty\}$, completing the proof. The second part of (1) follows from the first part and Lemma 3.8.2.

Now, we will prove (2). Recall that $Q(\omega,\cdot)$ is the regular conditional probability of $\mathbb{P}'_{x_0} \circ \theta_{\tau_1}$ given $\mathcal{F}_{\tau_1}$. For fixed $\omega \in \Omega$, define the measure $\delta_\omega \otimes_{\tau_1} Q(\omega,\cdot)$ by

$$\delta_\omega \otimes_{\tau_1} Q(\omega,\cdot)(E_{A,B}) = 1_{\{\omega\in A\}}Q(\omega,B), \tag{3.54}$$

which extends to $\mathcal{F}_\infty$ by part (1). By (3.53), for every $A \in \mathcal{F}_{\tau_1}, B \in \mathcal{F}_\infty$,

$$\mathbb{E}_{x_0}[\delta_\omega \otimes_{\tau_1} Q(\omega,\cdot)(E_{A,B})] = \mathbb{P}_{x_0} \otimes_{\tau_1} Q(E_{A,B}).$$

Thus, by part (1) again, it follows that

$$\mathbb{P}_{x_0} \otimes_{\tau_1} Q = \mathbb{E}_{x_0}[(\delta_\omega \otimes_{\tau_1} Q(\omega,\cdot))(\cdot)] \tag{3.55}$$

Now let $\{M_t\}_{t\geq 0}$ be any right continuous progressively measurable bounded process. By boundedness, it is also $\mathbb{P}_{x_0} \otimes_{\tau_1} Q$-integrable. Observe that $\mathcal{F}_t$ is countably generated for each $t \geq 0$ since $\{X_t\}_{t\geq 0}$ has right continuous paths. Therefore, [79, Theorem 1.2.10] holds in our case.

The proof of part (2) now follows from the same argument as in the proof of [79, Theorem 6.1.2], since (3.55) holds. □

We are now ready to prove Proposition 3.4.6.

Proof of Proposition 3.4.6. We will use the proof technique of [10, Theorem 2.2, Section 6]. Recall that $\{\mathbb{P}_x\}_{x\in\mathbb{R}^d}$ is assumed to be a strong Markov family of solutions to the martingale problem for $\mathcal{L}$. Fix $x_0 \in \mathbb{R}^d$ and recall the sequence of stopping times τ_i from (3.48). We will prove that, for each $i \geq 1$, $\mathbb{P}_{x_0}(A)$ is uniquely determined for each $A \in \mathcal{F}_{\tau_i}$ by induction. By Lemma 3.8.2, $\mathcal{F}_\infty = \sigma\left(\cup_{i=1}^\infty \mathcal{F}_{\tau_i}\right)$, therefore $\mathbb{P}_{x_0}$ is uniquely determined on $\mathcal{F}_\infty$.

We will now work on the base case. Recall that $\{\mathbb{P}'_{x_0}\}_{x_0\in\mathbb{R}^d}$ is a strong Markov family of solutions to the martingale problem for $\mathcal{L}_{x_0}$, which is uniquely determined by Lemma 3.8.1(b) and our assumption. Let $Q^1_{x_0} = \mathbb{P}_{x_0} \otimes_{\tau_1} Q$ be the measure given in (3.53).

We claim that $Q^1_{x_0}$ solves the martingale problem for $\mathcal{L}_{x_0}$. To do this, let $f \in C_b^2(\mathbb{R}^d)$ be arbitrary. We must show that $\{M_t^f\}_{t\geq 0}$, given by (3.6), is a $Q^1_{x_0}$-martingale. By definition, $\{M_t^f\}_{t\geq 0}$ is right-continuous and progressively measurable. Note that for all $x \in \mathbb{R}^d$ and $f \in C_b^2(\mathbb{R}^d)$, we have

$$|f(x+h) - f(x) - 1_{\|h\|<1}(\nabla f(x) \cdot h)| \leq 2(\|f\|_\infty \wedge \|H_f\|_\infty).$$

Thus, by (3.1) and Assumption 3.1.1, for all $x \in \mathbb{R}^d$,

$$\begin{aligned}\|\mathcal{L}f(x)\| \leq 2(\|f\|_\infty \wedge \|H_f\|_\infty) \int_{\mathbb{R}^d\setminus\{0\}} K(x,h)\mathrm{d}h \leq 2C(\|f\|_\infty \wedge \|H_f\|_\infty) \int_{\mathbb{R}^d\setminus\{0\}} J(h)\mathrm{d}h \\ \leq 2CK_0 \qquad (3.56)\end{aligned}$$

Hence,

$$|M_t^f|_\infty \leq \|f\|_{C_b^2(\mathbb{R}^d)}(2 + 2CK_0 t)$$

Thus, M_t^f is bounded a.s. for each $t > 0$. We will now apply Lemma 3.8.3(2) in order to show that it is a $Q^1_{x_0}$-martingale.

Condition (i) of the theorem requires that $\{M^f_{t\wedge\tau_1}\}_{t\geq 0}$ be a $\mathbb{P}_{x_0}$ martingale. However, observe that for $t > 0$ fixed,

$$M^f_{t\wedge\tau_1} = f(X_{t\wedge\tau_1}) - f(x_0) - \int_0^{t\wedge\tau_1} \mathcal{L}_{x_0} f(X_s)\mathrm{d}s = f(X_{t\wedge\tau_1}) - f(X_0) - \int_0^{t\wedge\tau_1} \mathcal{L} f(X_s)\mathrm{d}s$$

by Lemma 3.8.1. Thus, $M^f_{t\wedge\tau_1}$ is a $\mathbb{P}_{x_0}$ martingale by the optional stopping theorem, since $\mathbb{P}_{x_0}$ is a solution to the martingale problem for $\mathcal{L}$.

Condition (ii) of the theorem requires $\{M_t^f - M^f_{t\wedge\tau_1(\omega)}\}_{t\geq 0}$ to be a $\mathbb{P}'_{X_{\tau_1}(\omega)}$ martingale for every $\omega \in \Omega$. However,

$$M_t^f - M^f_{t\wedge\tau_1(\omega)} = f(X_t) - f(X_{t\wedge\tau_1(\omega)}) - \int_{t\wedge\tau_1(\omega)}^t \mathcal{L}_{x_0} f(X_s)\mathrm{d}s$$

is a martingale since $\mathbb{P}'_{X_{\tau_1}(\omega)}$ is the solution to the martingale problem for $\mathcal{L}_{x_0}$ started at $X_{\tau_1(\omega)}$.

Applying Lemma 3.8.3(2), it follows that $\{M_t^f\}_{t\geq 0}$ is a martingale. Hence, $Q^1_{x_0}$ solves the martingale problem for $\mathcal{L}_{x_0}$. By uniqueness, it follows that $Q^1_{x_0} = \mathbb{P}'_{x_0}$. By (3.53), for any $A \in \mathcal{F}_{\tau_1}$ we have $Q^1_{x_0}(A) = \mathbb{P}_{x_0}(A) = \mathbb{P}'_{x_0}(A)$. Hence, $\mathbb{P}_{x_0}(A)$ is uniquely determined on $\mathcal{F}_{\tau_1}$ by uniqueness of $\mathbb{P}'_{x_0}$.

For the induction step, suppose that $\mathbb{P}_{x_0}$ is uniquely determined on $\mathcal{F}_{\tau_i}$ for some $i \geq 1$ and every $x_0 \in \mathbb{R}^d$. By the Strong Markov property, $(\omega, A) \to \mathbb{P}_{X_{\tau_1}(\omega)}(A)$ is the regular conditional probability distribution of $\mathbb{P}_{x_0}$ given $\mathcal{F}_{\tau_1}$. In particular, if $B \in \mathcal{F}_{\tau_1}$ and θ_{τ_1} denotes the shift operator (3.7), then

$$\mathbb{E}_{\mathbb{P}_{x_0}}\left[\{\omega' : \theta_{\tau_1}(\omega') \in B\}|\mathcal{F}_{\tau_1}\right](\omega) = \mathbb{P}_{X_{\tau_1}(\omega)}(B).$$

Thus, it follows that $\mathbb{P}_{x_0}$ is determined on all sets $A \in \mathcal{F}_{\tau_1}$, as well as all sets of the form $\{\omega : \theta_{\tau_1}(\omega) \in B\}$, where $B \in \mathcal{F}_{\tau_i}$. However, this implies that if

$$\Sigma = \sigma\left(E_{A,B} : A \in \mathcal{F}_{\tau_1}, B \in \mathcal{F}_{\tau_i}\right),$$

Then, $\mathbb{P}_{x_0}$ is determined on Σ. By Lemma 3.8.3(1) we have $\mathcal{F}_{\tau_{i+1}} \subset \Sigma$. Hence, $\mathbb{P}_{x_0}$ is determined on $\mathcal{F}_{\tau_{i+1}}$, completing the proof of the induction step and the proposition. □

Chapter 4

Future work

In this chapter, we discuss some open problems which we will consider for future work. Prior to this, we define the Elliptic Harnack inequality(EHI) in more generality.

Given a strong Markov process $\{X_t\}_{t\geq 0}$ on $\mathbb{R}^d, d \geq 1$, a function $h : \mathbb{R}^d \to \mathbb{R}$ is said to be *harmonic* on a set $D \subset \mathbb{R}^d$ with respect to $\{X_t\}_{t\geq 0}$ if, for all $x \in D$,

$$\mathbb{E}_x[h(X_{\tau_D})] = h(x),$$

where $\mathbb{E}_x$ indicates the expectation under the measure $\mathbb{P}_x$ satisfying $\mathbb{P}_x(X_0 = x) = 1$, and $\tau_D = \inf\{t > 0 : X_t \notin D\}$ is the exit time of $\{X_t\}_{t\geq 0}$ from D.

The scale-invariant Elliptic Harnack inequality is said to hold for $\{X_t\}_{t\geq 0}$ if there exists $C > 0$ such that for all $x_0 \in \mathbb{R}^d, r > 0$ and $h : \mathbb{R}^d \to \mathbb{R}$ non-negative and harmonic on $B(x_0, r)$,

$$h(x) \leq Ch(y) \quad \text{for all } x, y \in B(x_0, r/2).$$

4.1 Stability of the EHI for jump processes

Recall the definition of a Lévy process from Section 1.1. A Lévy process on $\mathbb{R}^d, d \geq 1$ is called a pure-jump Lévy process if its sample paths consist of only jumps and do not possess a diffusion or a drift part almost surely.

We call a pure-jump Lévy process $\{Y_t\}_{t\geq 0}$ on $\mathbb{R}^d, d \geq 1$ as geometric stable-like if its Lévy measure possesses a density j on $\mathbb{R}^d \setminus \{0\}$, and there exists $\alpha \in (0, 2], C_1, C_2 > 0$ such that $C_1 j_\alpha(h) \leq j(h) \leq C_2 j_\alpha(h)$ for all $h \in \mathbb{R}^d \setminus \{0\}$, where j_α is the jump kernel of an α-geometric stable process (see Section 1.2.2).

The question of whether these processes satisfy the elliptic Harnack inequality or not is open.

Question 1 *Let $\{Y_t\}_{t\geq 0}$ be a geometric stable-like process on $\mathbb{R}^d, d \geq 1$. Does $\{Y_t\}_{t\geq 0}$ satisfy the scale invariant EHI?*

This question is motivated by a number of techniques that are known for proving the elliptic Harnack inequality for similar classes of processes. There are methods such as those by Bass and Levin[12] for α-stable-like processes, which modifies Moser's method appropriately, and those by Grigoryan, Hu and Lau[37] which modify De Giorgi's method and the Landis approach appropriately. Due to the anomalous jump kernel scaling (1.33), these methods have tended to fail thus far. Furthermore, the method discussed in chapter 1 clearly fails, since it relies on the underlying process being a subordinate Brownian motion and the rotational invariance and properties of such a process, which $\{Y_t\}_{t\geq 0}$ may not possess.

As an illustration of the anomalous nature of the jump kernel, we note that the α-stable process satisfy a Krylov-Safonov type estimate (see [12, Proposition 3.4] for such an estimate) which fails for geometric stable processes due to the jump kernel scaling (see [64, Theorem 1.2]).

It is possible that probabilistic inequalities which scale differently from those seen in the α-stable case can be used. Such inequalities are considered in [53], where it is proved that harmonic functions with respect to geometric stable-like processes are continuous in the interior of their domain of harmonicity with an *a priori* modulus of continuity.

4.2 Consequences of EHI for jump processes

For diffusion processes on metric measure spaces, much is known about the EHI and its consequences, including oscillation inequalities, interior Hölder continuity of harmonic functions in a ball, and Green function estimates. We refer the reader to [38] and [6, Section 3] for some of these consequences.

For pure jump processes, in the special case that the jump kernel at each point is stable-like, [24, Theorem 1.11, Corollary 1.12] shows that the elliptic Harnack inequality is equivalent to a combination of Poincaré, Faber Krahn and cut-off Sobolev inequalities, and implies interior Hölder regularity of harmonic functions in a ball.

However, in the absence of such a scaling, much is not known about consequences of the Elliptic Harnack inequality. One such question concerns the regularity of harmonic functions.

Question 2 *Let $\{Y_t\}_{t\geq 0}$ be a strong Markov process on $\mathbb{R}^d, d \geq 1$ which satisfies the Elliptic Harnack inequality. Let $h : \mathbb{R}^d \to [0, \infty)$ be harmonic on an open set $D \subset \mathbb{R}^d$ with respect to $\{Y_t\}_{t\geq 0}$. Then, is h continuous on D? More strongly, if D is bounded and $B \subset D$ is a subset such that $\overline{B} \subset D$, then is h Hölder continuous on B?*

Hölder continuity of harmonic functions has been proved for every pure-jump process that is known to satisfy the EHI up till this point. For the geometric stable process, harmonic functions are in fact smooth in the interior of their domain (see [41, Theorem 1.7]). There are processes for which the EHI is not known, but continuity of harmonic functions has been proved : for example, the geometric stable-like processes which were discussed in the previous section.

The motivation for this question comes from noticing that the Poisson process $\{N_t\}_{t\geq 0}$ neither satisfies the EHI at small scales, nor are harmonic functions with respect to $\{N_t\}_{t\geq 0}$ necessarily continuous. This can be attributed to the fact that $\{N_t\}_{t\geq 0}$ does not possess any jumps of size smaller than 1, and does not possess any symmetry in its jumps.

On the other hand, the α-stable processes make small jumps in a symmetric fashion with a very high intensity, which are crucial in proving the EHI and Hölder continuity of harmonic functions in the interior of their domains of harmonicity.

There are many processes whose nature of jumps lie in the middle of these extremities. For example, the geometric stable-like processes discussed in the previous section, and compound Poisson processes on $\mathbb{R}^d$ whose jump kernel is fully supported on $\mathbb{R}^d \setminus \{0\}$, both make arbitrarily small and large jumps with varying intensities. Therefore, the exact relationship between small jump and large jump intensities which is required for the EHI and continuity of harmonic functions is not quite clear.

The converse question of whether continuity of harmonic functions imply the EHI has a negative answer : this follows by considering the geometric stable-like Lévy process on $\mathbb{R}^d, d \geq 1$ with jump kernel

$$j(h) = \|h\|^{-d} 1_{0<\|h\|\leq 1}, \ h \in \mathbb{R}^d \setminus \{0\}.$$

This process does not satisfy the EHI (see [41, Example 5.5]), but satisfies the hypothesis of [53, Theorem 1.4] and therefore possesses continuity of harmonic functions.

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