Piercing and Covering Results in Combinatorial Geometry

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Dedicated to My Parents

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Abstract

Combinatorial geometry is a branch of mathematics that studies the arrangement, properties and relationships of geometric objects based on their combinatorial structures. In this thesis, we have mainly studied the following two types of problems in combinatorial geometry: (I) Geometric Transversal Theory and (II) Covering Subsets of the Hypercube with Nice Geometric Objects.

(I) **Geometric Transversal Theory:** Suppose $\mathscr{F}$ is a collection of subsets of $\mathbb{R}^d$ and $\mathscr{T}$ is a family of geometric objects in $\mathbb{R}^d$. For example, $\mathscr{T}$ can be a set of points or lines or hyperplanes etc. Then $\mathscr{T}$ is said to be a *transversal* of $\mathscr{F}$ if for all $F \in \mathscr{F}$ there exists $T \in \mathscr{T}$ such that $F \cap T \neq \emptyset$. In other words, we say $\mathscr{T}$ *pierces* or *stabs* $\mathscr{F}$. For any $n \in \mathbb{N}$, $\mathscr{F}$ is said to be n-pierceable, if $\mathscr{F}$ has a transversal $\mathscr{T}$ of size at most n. Helly's theorem is a cornerstone result in geometric transversal theory. The theorem says that, if we are given a family $\mathscr{F}$ of compact convex sets in $\mathbb{R}^d$ such that every $d+1$ sets of $\mathscr{F}$ is pierceable by a point then the whole family is pierceable by a single point. Over the century numerous studies have been done by changing the framework of Helly's theorem from different aspects. Some of the most important variants of Helly's theorem are *Fractional Helly theorem, Colorful Helly theorem, (p,q)-theorem* etc. Holmsen and Lee (Israel Journal of Mathematics, 2021) showed that in $\mathbb{R}^d$, colorful Helly theorem implies fractional Helly theorem. Besides these, studying Helly-type theorems for piercing with higher dimensional transversals (for example, lines, hyperplanes or k-dimensional affine spaces, namely k-flats) or n-pierceabily $(n > 1)$ or some special class of sets (for example, axis-parallel boxes, unit disks etc.) is also a common practice. Danzer and Grünbaum (Combinatorica, 1982) gave the first Helly-type result for multiperceability of boxes. In Chapter **3** we have studied a colorful version of their result. One of the most interesting features of our findings is that there is a strict separation between the monochromatic Helly-type result by Danzer and Grünbaum and our colorful Helly-type result. Keller and Perles first extended the (p,q)-theorem to infinite settings, namely $(\aleph_0, k+2)$-theorem for k-transversals (that is, piercing with k-flats). In Chapter **4** we have studied a colorful $(\aleph_0, 2)$-theorem for axis parallel boxes piercing with

axis parallel lines and k-flats. One thing to notice here is that all these Helly-type results are extremely dependent on the dimension of the ambient space. Adiprasito et al. (Discrete & Computational Geometry, 2020) proved the first dimension independent Helly theorem. In Chapter **5** we have proved a dimension independent colorful Helly theorem for higher dimensional transversals.

(II) **Covering Subsets of the Hypercube with Nice Geometric Objects:** There is a long line of research, spanning over three decades, on problems about covering the vertices of the n-dimensional hypercube $\mathscr{Q}^n = \{0,1\}^n$ by hyperplanes. Suppose we want to cover all the vertices of $\mathscr{Q}^n$ with the minimum number of (affine) hyperplanes (a hyperplane H covers a vertex v of $\mathscr{Q}^n$ if v lies on H). Then at least 2 hyperplanes are required and sufficient also (for example, $x_i = 0$ and $x_i = 1$, for any $i \in [n]$). But what if we want to cover all but one, say the origin, vertices of $\mathscr{Q}^n$ keeping the origin as uncovered? The celebrated result of Alon and Füredi shows that at least n hyperplanes will be required. Also, observe that the hyperplanes, $x_i = 1$, for all $i \in [n]$ are sufficient. Lying in the intersection of finite geometry and extremal combinatorics, numerous variants of this covering problem have been studied. Notice that we can also ask the same question with a slight modification. What will be the minimum degree of a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ such that P vanishes at all the vertices of $\mathscr{Q}^n$ except at the origin, and at the origin P does not vanish at all? As hyperplanes are nothing but multi-linear polynomials, clearly, the size of the hyperplane cover (that is, the minimum number of hyperplanes required for the covering) serves as an upper bound for the size of polynomial cover (that is, the minimum degree of the polynomial that does the covering). Alon and Füredi showed that any polynomial that vanishes at every vertex of the hypercube $\mathscr{Q}^n$ except the origin and does not vanish at the origin has degree at least n. And hence we cannot cover the vertices with less than n hyperplanes. Since then it has been a question of interest for which forbidden set there is a separation between polynomial covering and hyperplane covering. We have shown that there is strict separation between polynomial covering and hyperplane covering when we consider covering with multiplicities. (We say $P \in \mathbb{R}[x_1, \ldots, x_n]$ covers a vertex v of $\mathscr{Q}^n$ with multiplicity t if v is a zero of P with multiplicity t.)

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Notations

We will use the following notations mainly in the first part of this thesis.

- $\mathbb{R}$ is the set of real numbers.
- $\mathbb{N} := \{1, 2, 3, \ldots\}$ is the set of natural numbers.
- For any $n \in \mathbb{N}$, $[n] := \{1, \ldots, n\}$.
- $\mathbb{Q}$ is the set of rational numbers, and

$$\mathbb{Q}^d := \underbrace{\mathbb{Q} \times \cdots \times \mathbb{Q}}_{d \text{ times}}.$$

- For all $a, b \in \mathbb{R}$ with $a \leq b$, $[a, b] := \{x \in \mathbb{R} \,:\, a \leq x \leq b\}$.
- An *axis-parallel box* B in $\mathbb{R}^d$ is a set of the form $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, where $\forall i \in [d]$, $a_i, b_i \in \mathbb{R}$ with $a_i \leq b_i$.
- For any $X \subseteq \mathbb{R}^d$, $|X|$ denotes the size of the set X.
- For any set S with $|S| \geq k$, $\binom{S}{k}$ denotes the collection of all k-sized subsets of S.
- $\mathcal{O}$ stands for the origin of $\mathbb{R}^d$.
- For all u, $v \in \mathbb{R}^d$, $\langle u, v \rangle$ denotes the *inner product* between u and v.
- For all $i \in [d]$, $e_i := (0 \ldots, 0, \underset{\substack{\uparrow \\ i-\text{th position}}}{1}, 0 \ldots, 0) \in \mathbb{R}^d$.
- For any $A \subseteq \mathbb{R}^d$ and $i \in [d]$, $\pi_i(A)$ denotes the orthogonal projection of A onto the i-th axis.
- For any $p, q \in \mathbb{R}^d$, $\overline{pq}$ denotes the closed line segment connecting p and q.
- For any p, q in $\mathbb{R}^d$, $\|p - q\|$ denotes the Euclidean distance between p and q.

- $\mathbb{S}^{d-1}$ denotes the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ centered at the origin $\mathscr{O}$, i.e,

$$\mathbb{S}^{d-1} := \{x \in \mathbb{R}^d \ : \ \|x\| = 1\}.$$

- For any $C \subseteq \mathbb{R}^d$, the diameter of C, denoted by diam(C), is defined by

$$\text{diam}(C) := \sup\left\{\|p-q\| \ : \ p,q \in C\right\}.$$

- For any $S_1, S_2 \subseteq \mathbb{R}^d$, the *distance* between S_1 and S_2, denoted by dist (S_1, S_2), is defined as

$$\text{dist}\,(S_1,S_2) := \inf\left\{\|p_1 - p_2\| \ : \ p_1 \in S_1 \text{ and } p_2 \in S_2\right\}.$$

- For all $b \in \mathbb{R}^d$ and $r > 0$, *closed* and *open balls* centered at the point b and radius r are denoted by

$$B(b,r) := \left\{p \in \mathbb{R}^d \ : \ \|p-b\| \leq r\right\}$$

and

$$B^o(b,r) := \left\{p \in \mathbb{R}^d \ : \ \|p-b\| < r\right\}$$

respectively.

- Any *affine subspace* K is $\mathbb{R}^d$ is of the form $x+L$, where $x \in \mathbb{R}^d$ and L is a linear subspace in $\mathbb{R}^d$.

- $C \subseteq \mathbb{R}^d$ is said to be a *convex set* if for any $a, b \in C$ and $\forall t \in [0,1]$, we have

$$ta + (1-t)b \in C.$$

- For any $S \subseteq \mathbb{R}^d$, *convex hull* of S, denoted by conv(S), is the smallest convex set containing S.

In the second part of this thesis, we will use the following notations.

- $\mathbb{R}$ denotes the set of all real numbers.
- $\mathbb{Z}$ denotes the set of all integers.
- $\mathbb{N}$ denotes the set of all nonnegative integers.
- $\mathbb{Z}^+$ denotes the set of all positive integers.
- $[a,b]$ denotes the closed interval of all integers between a and b; further, we denote $[n] := [1,n]$.

- $\mathbb{R}[\mathbb{X}]$ denotes the polynomial ring over the field $\mathbb{R}$ and a collection of indeterminates $\mathbb{X}$, where either there are n indeterminates $\mathbb{X} = (X_1, \ldots, X_n)$, or there are $N = n_1 + \cdots + n_k$ indeterminates partitioned into k blocks as $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_k)$ with each $\mathbb{X}_j = (X_{j,1}, \ldots, X_{j,n_j})$.
- For any $B \subsetneq \mathbb{R}$, $\mathbb{Q}(B)$ denotes the smallest subfield of $\mathbb{R}$ that contains $\mathbb{Q}$ and B.
- For any subset $S \subseteq \{0,1\}^n$, $|S| > 1$, the **index complexity** of S is the smallest positive integer $r_n(S)$ such that for some $I \subseteq [n]$, $|I| = r_n(S)$, there is a point $u \in S$ such that for each $v \in S$, $v \neq u$, we get $v_i \neq u_i$ for some $i \in I$. The index complexity of a singleton set is defined to be zero.
- Let $\mathsf{EHC}_n^{(t,\ell)}(S)$ denote the minimum size of a (t,ℓ)-exact hyperplane cover for S, and let $\mathsf{EPC}_n^{(t,\ell)}(S)$ denote the minimum degree of a (t,ℓ)-exact polynomial cover for S.
- The **Hamming weight** of any $x \in \{0,1\}^n$ is defined by $|x| = |\{i \in [n] : x_i = 1\}|$. Thus, the subset S is symmetric if and only if

$$x \in S, y \in \{0,1\}^n, |y| = |x| \quad \Longrightarrow \quad y \in S.$$

- We say a subset $S \subseteq \{0,1\}^n$ is **symmetric** if S is closed under permutations of coordinates.
- For any symmetric set $S \subseteq \{0,1\}^n$, we define $W_n(S) = \{|x| : x \in S\}$.
- We say a symmetric set S is a **layer** if $|W_n(S)| = 1$.
- For $i \in [0,n]$, let $W_{n,i} = [0, i-1] \cup [n-i+1, n]$, and we define the symmetric set $T_{n,i} \subseteq \{0,1\}^n$ by $W_n(T_{n,i}) = W_{n,i}$. Here we have $W_{n,0} = \emptyset$ and $T_{n,0} = \emptyset$.
- For any symmetric set $S \subseteq \{0,1\}^n$, define

$$\mu_n(S) = \max\{i \in [0, \lceil n/2 \rceil] : W_{n,i} \subseteq W_n(S)\},$$
$$\text{and} \quad \Lambda_n(S) = |W_n(S)| - \mu_n(S).$$

  Further, denote $\overline{\mu}_n(S) := \mu_n(\{0,1\}^n \setminus S)$ and $\overline{\Lambda}_n(S) := \Lambda_n(\{0,1\}^n \setminus S)$.
- For any $a \in [-1, n-1]$, $b \in [1, n+1]$, $a < b$, denote the set of weights $I_{n,a,b} = [0,a] \cup [b,n]$, and we say a **peripheral interval** is the symmetric set $J_{n,a,b} \subseteq \{0,1\}^n$ defined by $W_n(J_{n,a,b}) = I_{n,a,b}$. Here, we have the convention $[0,-1] = [n+1,n] = \emptyset$.

- For any symmetric set $S \subsetneq \{0,1\}^n$, the **inner interval** of S, denoted by in-int(S), is defined to be the peripheral interval $J_{n,a,b} \subseteq \{0,1\}^n$ of maximum size such that $J_{n,a,b} \subseteq S$. Further, we define in-int$(\{0,1\}^n) = J_{n,\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1}$.
- For any symmetric set $S \subsetneq \{0,1\}^n$, the **outer interval** of S, denoted by out-int(S), is defined by

$$\text{out-int}(S) = \begin{cases} J_{n,a,b} & \text{if } J_{n,a,b} \text{ is the unique minimizer of } |a+b-n|, \\ J_{n,a,b} & \text{if } J_{n,a,b}, J_{n,n-b,n-a} \text{ are minimizers of } |a+b-n|, \text{ and } a > n-b. \end{cases}$$

- $\text{in}_n(S) = (\min\{a, n-b\}+1) + |W_n(S) \setminus W_{n,\min\{a,n-b\}+1}|$, where $J_{n,a,b} = \text{in-int}(S)$, and $\text{out}_n(S) = a+n-b+1 = |I_{n,a,b}| - 1$, where $J_{n,a,b} = \text{out-int}(S)$.
- Fix a positive integer $k \geq 1$ we consider the hypercube $\{0,1\}^N$ as a product of k hypercubes $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$ (and so $N = n_1 + \cdots + n_k$).
- We define a subset $S \subseteq \{0,1\}^N$ to be a k-**wise grid** if $S = S_1 \times \cdots \times S_k$, where each $S_i \subseteq \{0,1\}^{n_i}$ is symmetric.
- We say $S = S_1 \times \cdots \times S_k$ is a k-**wise layer** if each S_i is a layer.
- We define a general k-**wise symmetric set** to be a union of an arbitrary collection of k-wise layers.
- By a **subcube** of a hypercube $\{0,1\}^n$, we mean a subset of the form $\{0,1\}^I \times \{a\}$, where $I \subseteq [n]$ and $a \in \{0,1\}^{[n] \setminus I}$.
- For any subset $S \subseteq \{0,1\}^N$, we define a (t,ℓ)-**block exact hyperplane cover** for S to be a (t,ℓ)-exact hyperplane cover $\mathscr{H}(\mathbb{X})$ (in $\mathbb{R}^N$) for S such that

$$|\mathscr{H}(a, \mathbb{X}_j)| = |\mathscr{H}(\mathbb{X})|,$$

for every $a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k}$, $j \in [k]$.

- For any subset $S \subseteq \{0,1\}^N$, we define a (t,ℓ)-**block exact polynomial cover** for S to be a nonzero polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that
  (a) the polynomial $P(\mathbb{X})$ vanishes at each point in S with multiplicity at least t,
  (b) for each $j \in [k]$, and every point $(a, \widetilde{a}) \in \{0,1\}^N \setminus S$ with

$$a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k} \text{ and}$$

$\widetilde{a} \in \{0,1\}^{n_j}$, the polynomial $P(a,\mathbb{X}_j)$ vanishes at $\widetilde{a}$ with multiplicity exactly ℓ.

- $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$ denote the minimum size of a (t,ℓ)-block exact hyperplane cover for S.
- $\mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$ denote the minimum degree of a (t,ℓ)-block exact polynomial cover for S.
- For any $S \subseteq \{0,1\}^N$ and $j \in [k]$, let $S_j \subseteq \{0,1\}^{n_j}$ denote the **projection** of S onto the j-th block.
- For any $S \subseteq \{0,1\}^N$, $W_{(n_1,\ldots,n_k)}(S) = \{(|x_1|,\ldots,|x_k|) : (x_1,\ldots,x_k) \in S\}$.
- For each $j \in [k]$, we consider an arbitrarily chosen total order $\leq_j \in \mathscr{T}$ on $W_{n_j}(S_j)$, say denoted by $W_{n_j}(S_j) = \{w_{j,0} <_j \cdots <_j w_{j,q_j}\}$, and further for each $z_j \in [0,q_j]$, define the symmetric set $[S]_{j,z_j} \subseteq \{0,1\}^{n_j}$ by $W_n([S]_{j,z_j}) = \{w_{j,0} <_j \cdots <_j w_{j,z_j}\}$.
- We define a k-symmetric set $S \subseteq \{0,1\}^N$ to be **pseudo downward closed (PDC)** if for every $(w_{1,z_1},\ldots,w_{k,z_k}) \in W_{(n_1,\ldots,n_k)}(S)$ we have $W_{n_1}([S]_{1,z_1}) \times \cdots \times W_{n_k}([S]_{k,z_k}) \subseteq W_{(n_1,\ldots,n_k)}(S)$.
- Let $\mathscr{N}(S) = \{(z_1,\ldots,z_k) \in \mathbb{N}^k : (w_{1,z_1},\ldots,w_{k,z_k}) \in W_{(n_1,\ldots,n_k)}(S)\}$.
- $\mathsf{E}^{(\text{out})}(S) := \mathsf{E}^{(\text{out})}_{\leq}(\mathscr{N}(S)) = \{(z_1,\ldots,z_k) \in \mathbb{N}^k : (w_{1,z_1},\ldots,w_{k,z_k}) \in \mathsf{E}^{(\text{out})}_{\preceq}(W_{(n_1,\ldots,n_k)}(S))\}$.
- $\mathsf{E}^{(\text{in})}(S) := \mathsf{E}^{(\text{in})}_{\leq}(\mathscr{N}(S)) = \{(z_1,\ldots,z_k) \in \mathbb{N}^k : (w_{1,z_1},\ldots,w_{k,z_k}) \in \mathsf{E}^{(\text{in})}_{\preceq}(W_{(n_1,\ldots,n_k)}(S))\}$.
- We define a nonempty PDC k-wise symmetric set S to be **outer intact** if for every $(z_1,\ldots,z_k) \in \mathsf{E}^{(\text{in})}(S)$ and $j \in [k]$, we have $J_{n_j,a_j,b_j} = \text{out-int}([S]_{j,z_j})$.
- *Hamming ball* is a symmetric set defined by a set of weights of the form $[0,w]$.
- For any $p \in \{0,1\}^n$, we denote $I_0(p) := \{i \in [n] : p_i = 0\}$,
  and $I_1(p) := \{i \in [n] : p_i = 1\}$.
- For any $I_0 \subseteq I_0(p), I_1 \subseteq I_1(p)$, we define the **separation of p with respect to** (I_0,I_1), denoted by $\text{sep}(p,I_0,I_1) \subseteq \{0,1\}^n$, to be the maximal symmetric set such that for every $x \in \text{sep}(p,I_0,I_1)$, we have $x_{I_0 \sqcup I_1} \neq p_{I_0 \sqcup I_1}$.

Chapter 1

Introduction

Combinatorial geometry is a branch of mathematics that studies the arrangement, properties, and relationships of geometric objects based on their combinatorial structures. It encompasses a wide array of topics, including the study of polytopes, tilings, and arrangements of points, lines, and planes. A central theme in combinatorial geometry is understanding how geometric configurations can be counted, enumerated, and optimized, often leading to insights that intersect with other mathematical disciplines such as graph theory, topology, and algebra. Problems in this field range from determining the maximum number of incidences between points and lines, to exploring the properties of convex hulls and Voronoi diagrams, to investigating the structure of higher-dimensional polytopes. The interplay between geometry and combinatorics in this area not only reveals deep theoretical results but also has practical applications in areas such as computer science, optimization, and the analysis of algorithms.

One of the quintessential problems in combinatorial geometry is the study of arrangements of points and lines. For instance, given a finite set of points in the plane, one might investigate how many distinct lines can be formed by connecting pairs of points. A fundamental result in this area is the **Sylvester-Gallai theorem**, which asserts that for any finite set of points in the plane, not all collinear, there is always a line passing through exactly two of the points. This theorem underpins more complex investigations into the structure and distribution of points and lines.

Going deep into the same line of study we get **Szemerédi-Trotter theorem** that provides an upper bound on the number of incidences between points and lines in the plane. Specifically, it states that for any set of n points and m lines, the number of incidences (point-line intersections) is $O(m^{\frac{2}{3}}n^{\frac{2}{3}}+m+n)$. This theorem has significant applications in solving problems related to graph drawings and embeddings. For example, the **crossing number**

problem, which seeks the minimum number of edge crossings in a graph drawing, benefits directly from this geometric tools.

Combinatorial geometry plays a crucial role in solving various graph theory problems by leveraging geometric insights to tackle complex combinatorial structures. For instance, the **Four Color Theorem**, which states that any planar graph can be colored with at most four colors such that no two adjacent vertices share the same color, was proven using geometric embeddings and combinatorial arguments. Moreover, geometric approaches are essential in understanding graph embeddings, as illustrated by **Kuratowski's Theorem**, which characterizes planar graphs through geometric substructures. Geometric arguments also underpin **separator theorems**, which uses vertex and edge distributions to divide graphs efficiently. These examples demonstrate how combinatorial geometry provides powerful methods for addressing and simplifying complex graph theory problems, showcasing the synergy between these two fields.

Another significant area of interest in combinatorial geometry is the study of convex sets and their properties. A classic problem in this domain is the **Erdős-Szekeres problem**, which seeks the smallest number N such that any set of N points in the plane in general position (no three points are collinear) contains a subset of n points that form the vertices of a convex polygon. This problem highlights the interplay between combinatorial properties and geometric configurations, demonstrating how discrete constraints influence geometric structures.

One of the closest mathematical field to combinatorial geometry is computational geometry. Computational geometry focuses on the design and analysis of efficient algorithms for solving geometric problems and combinatorial geometry results often find direct applications in solving those problems. For instance, results such as the Szemerédi-Trotter theorem, Erdős-Szekeres theorem are essential in developing algorithms for geometric intersection detection, convex hull computation, and range searching. Similarly, separator theorems or bounds on the number of edge crossings in graph drawings, inform the design of efficient algorithms for graph partitioning, planarity testing, and graph embedding. By leveraging combinatorial insights, computational geometry algorithms achieve improved efficiency and accuracy, driving advancements in diverse applications including computer graphics, robotics, geographic information systems (GIS) etc.

In this thesis we shall concentrate on two types of problems in combinatorial geometry, namely **Geometric transversals theory** and **Covering subsets of the hypercube with nice geometric objects**.

1.1 Geometric transversals theory

Suppose $\mathscr{F}$ is a family of subsets of $\mathbb{R}^d$ and $\mathscr{T}$ is a family of geometric objects in $\mathbb{R}^d$. Then $\mathscr{T}$ is said to be a transversal of $\mathscr{F}$ if $\mathscr{T}$ has non-empty intersection with every member of $\mathscr{F}$. In other words, we say $\mathscr{T}$ pierces $\mathscr{F}$. This notion is motivated by Helly's theorem [93], one of the most fundamental results in combinatorial geometry. The theorem says that, a finite family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$ is pierceable by a single point, if every $d+1$ sets from the family is pierceable by a single point. Since its discovery, Helly's theorem has been generalized, extended, and applied in various fields of mathematics and theoretical computer science, far beyond its geometric origins. Two main variants of this theorem are its fractional and colorful generalizations.

In Fractional Helly theorem, the assumption is relaxed so that not all but only a positive fraction of all the $(d+1)$-tuples of convex sets, say $\alpha\binom{|\mathscr{F}|}{d+1}$ many $(d+1)$-tuples, are pierceable by a single point and the question is whether the family have a large intersection. Katchalski and Liu [106], proved that a positive fraction of the family, say $\beta|\mathscr{F}|$ many sets, will be pierceable by a single point. In Colorful Helly theorem, some additional combinatorial restrictions are imposed to the intersection structure of the family. Suppose there are $(d+1)$ families (color classes) and every colorful $(d+1)$-tuple (that is, each member of the tuple has different color) are pierceable by a single point. Then Bárány [21] proved that at least one family must be pierceable by a single point. Holmsen and Lee [96] showed that in general *convexity spaces* with bounded *Radon number*, Colorful Helly Theorem implies Fractional Helly Theorem. In this thesis, we have studied several "Helly-type" theorem in colorful settings.

A typical "Helly-type" theorem has the form:

If every h or fewer members of a family of objects have property $\mathscr{P}$,then the entire family has property $\mathscr{P}$.

In original Helly's theorem the property $\mathscr{P}$ is 1-pierceability and $h = d+1$. Now a natural generalization is to consider the property n-pierceabilty instead of 1-pierceability. A family of sets $\mathscr{F}$ is said to be n-pierceable if there is an n-point set $\mathscr{T}$ that pierces $\mathscr{F}$. So the question is

What is the smallest number $h = h(d,n)$ ***such that the following holds?***
Suppose $\mathscr{F}$ is a family of convex sets in $\mathbb{R}^d$. If every h or fewer members of $\mathscr{F}$ is n-pierceable, then $\mathscr{F}$ is n-pierceable.

One of the first and foremost Helly-type result on multi-pierceability of families was proved by Danzer and Grünbaum [59] for families of axis parallel boxes. In Chapter 3, we have proved a colorful version of their result.

Another generalization of Helly's theorem can be thought of in terms of considering higher dimensional transversals, for example, piercing with hyperplanes or k-flats (k-dimensional affine spaces). In other words, the property $\mathscr{P}$ becomes pierceable with respect to k-transversals, $0 \leq k \leq d-1$, instead of just 0-transversals. So the question becomes

What is the smallest number $h = h(d,k)$ ***such that the following holds?***
Suppose $\mathscr{F}$ is a family of convex sets in $\mathbb{R}^d$. If every h or fewer members of $\mathscr{F}$ have a k-transversal ($0 \leq k \leq d-1$), then $\mathscr{F}$ has a k-transversal.

Aronov et al. [18], gave the sufficient conditions for 1-pierceability with respect to higher dimensional transversals. In Chapter 5, we will prove a colorful variant of their result.

One of the most significant generalizations of Helly's theorem is (p,q)-theorem, where the intersection conditions depend upon certain local or combinatorial conditions. Hadwiger and Debrunner [84] first introduced (p,q)-property. A family $\mathscr{F}$ of subsets of $\mathbb{R}^d$ is said to satisfy (p,q)-property if among every p members of $\mathscr{F}$ there are some q member that can be pierced by a single transversal. According to this definition Helly's theorem can be restated as the following: if a finite family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$ satisfies the $(d+1,d+1)$-property with respect to point transversals, then $\mathscr{F}$ is pierceable by a single point. Now using Fractional Helly theorem we already know that any finite family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$ satisfying (p,q)-property, where $p \geq q \geq d+1$, contains a large intersecting subfamily. So the question is how large the transversal size of $\mathscr{F}$ can be. Is it possible that the transversal size of $\mathscr{F}$ becomes independent of the size of $\mathscr{F}$? Alon and Kleitman [8] answered these questions positively. This result is known as (p,q)-theorem. Alon and Kalai [7] proved similar result for hyperplane transversals also. But unlike the case of hyperplane transversal, Alon et al. [11] gave an explicit construction showing the impossibility of getting a (p,q)-theorem, for k-transversals (i.e, piercing by k-flats), when $0 < k < d-1$.

Keller and Perles [110] first extended the (p,q)-theorem by weakening the assumption to $(\infty,.)$-property. A family $\mathscr{F}$ is said to satisfy $(\aleph_0,q)$-property if among every infinite subfamily of $\mathscr{F}$, we get some q members that can be pierced by a single transversal. Keller and Perles [110] proved that if $\mathscr{F}$ is a family of *nice* convex sets in $\mathbb{R}^d$ satisfying the $(\aleph_0,k+2)$-property with respect to k-transversals then $\mathscr{F}$ is pierceable by a finite number of k-flats. Since the first introduction to $(\aleph_0,k+2)$-property, a series of work has been done in this topic, see [111, 43, 42, 41, 100]. Studying "Helly-type" results for special classes of objects is a common practice and axis parallel boxes are always a suitable choice. In Chapter 4, we will present an $(\aleph_0, 2)$-theorem for *axis parallel boxes* with respect to *axis parallel k-transversals* along with a *short survey on* $(\aleph_0, k+2)$*-theorem.*

An interesting fact to consider here is that all these Helly-type results are extremely dependent on the dimension of the ambient space. A recent trend is to investigate such results independent of the dimension. Of course we expect the conclusion to be weaker also. For example, instead of having a common point, we may expect all the sets to be very close to a particular point. Adiprasito et al. [2] gave the first dimension independent Helly's theorem for point transversals. The immediate question that comes is the following:

What is the minimum distance $D = D(k,r)$ such that the following holds?
Suppose $\mathscr{F}$ is a finite family of convex sets in $\mathbb{R}^d$. If every r-tuple ($r < d$) of $\mathscr{F}$ can be pierced by a single k-flat K such that $\text{dist}(\mathscr{O}, K) < 1$ *then there is a k-flat $\hat{K}$ such that*
$\text{dist}\left(F, \hat{K}\right) < D,\ \forall F \in \mathscr{F}$.

In Chapter 5, we deal with this question.

1.2 Covering subsets of the hypercube with nice geometric objects

There is a long line of research, spanning over three decades, on problems about covering the vertices of the n-dimensional hypercube $\{0,1\}^n$ by hyperplanes. Suppose we want to cover all the vertices of $\{0,1\}^n$ with minimum number of (affine) hyperplanes (a hyperplane H covers a vertex v of $\{0,1\}^n$ if v lies on H). Then a pair of parallel hyperplanes (for example, $x_i = 0$ and $x_i = 1$, $i \in [n]$) are sufficient and necessary also, since $\{0,1\}^n$ has the full dimension. However, a small change to the problem quickly complicates matters. Now suppose we want to cover all but one, say the origin, vertices of $\{0,1\}^n$ keeping the origin as uncovered. How many hyperplanes do we require? Then the previous construction of parallel pair of hyperplanes is no longer useful. But observe that the n hyperplanes, $x_i = 1$, $\forall i \in [n]$, are sufficient to fulfill the task. Now the question is can we do better? Surprisingly, the celebrated result of Alon and Füredi [6] shows that this bound is optimal and this is far from obvious. Lying in the intersection of finite geometry and extremal combinatorics, numerous variants of this covering problem have been studied since then. But before diving into that detail, let's first explore the origin of this covering problem.

The covering problem was initially studied in the context of *blocking set problem* in finite geometry. A blocking set in $\mathbb{F}_2^n$ is a set of points that intersects every hyperplane, and the goal is to find the smallest possible blocking set. Since we can always translate a point to the origin $\mathscr{O}$, we can assume that our blocking set includes the origin $\mathscr{O}$. This reduces the blocking set problem to finding a set of points that intersects all hyperplanes that avoid the origin. And this is nothing but the dual of our original problem of covering the nonzero points of $\{0,1\}^n$ with affine hyperplanes.

Actually there is no need to limit our focus to the binary field $\mathbb{F}_2$; we can generalize the problem to determine how many hyperplanes are required to cover the nonzero points of $\mathbb{F}_q^n$. Extending this further, we can replace hyperplanes with affine subspaces of codimension d. In this broader context, Jamison [98] answered the problem in the late 1970s. He proved that at least $q^d - 1 + (n-d)(q-1)$ affine subspaces of codimension d are required to cover all nonzero points in $\mathbb{F}_q^n$ while avoiding the origin. When $q = 2$ and $d = 1$, this lower bound equals n, demonstrating that the earlier construction with n hyperplanes is optimal. Brouwer and Schrijver [34] independently gave a simpler proof for the case $d = 1$.

While the finite geometry perspective naturally leads to studying the covering problem over finite fields, it can also be explored over infinite fields $\mathbb{F}$. Clearly, infinitely many hyperplanes would be required to cover all nonzero points of $\mathbb{F}^n$. So we slightly modify the question. Here we ask how many hyperplanes are required to cover the nonzero points of the hypercube $\{0,1\}^n \subseteq \mathbb{F}^n$. This problem was originally raised by Komjáth [113] in the early 1990s, while studying some results in infinite Ramsey theory. Komjáth [113] showed that the minimum number of hyperplanes, required, must grow with n. Shortly after, Alon and Füredi [6] provided a tight bound in the more general context of covering all but one point of a finite grid. They proved that for any collection of finite subsets $S_1, S_2, \ldots, S_n$ of some arbitrary field $\mathbb{F}$, the minimum number of hyperplanes required to cover all but one point of $S_1 \times S_2 \times \ldots \times S_n$ is $\sum_i (|S_i| - 1)$. Specifically, if we take $S_i = \{0,1\}$ for all i, this result once again indicates that n hyperplanes are required to cover the nonzero points of the hypercube.

Despite these motivating applications in finite geometry and Ramsey theory, the main reason this problem has garnered so much attention is due to the proof techniques involved. These hyperplane covers have significantly contributed to the development of the polynomial method. In fact, considering Jamison's early contributions, this approach is sometimes called the Jamison method in finite geometry [35]. Notice that we can ask the same covering question with a slight modification. What if we consider covering by polynomials instead of hyperplanes? We say that a non-zero polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ covers a vertex v of $\{0,1\}^n$ if P vanishes at v. So the question is:

What will be the minimum degree of a polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ such that P vanishes at all the vertices of $\{0,1\}^n$ except at the origin, and at the origin P does not vanish at all?

As hyperplanes are nothing but *multi-linear* polynomials, clearly, size of the hyperplane cover (that is, the minimum number of hyperplanes required for the covering) serves as an upper bound for the size of polynomial cover (that is, the minimum degree of the polynomial that does the covering). Alon and Füredi [6] showed that the minimum degree of the polynomial for this covering is also n, that is, no improvement is possible in this type of covering if we consider the polynomial covering instead of the hyperplane covering. Since then the following has been a question of interest:

For which $S \subsetneq \{0,1\}^n$ there is a separation between polynomial covering and hyperplane covering?

Aaronson et al. [1] generalized this hyperplane covering problem by considering the forbidden sets of larger size. For forbidden sets of size at most 4, they gave the exact size of the hyperplane cover and for forbidden sets of size > 4, they gave an estimation for the size of the hyperplane cover. Clifton and Huang [56] introduced the notion of multiplicity of covering. They studied the following case:

What will be the minimum number of hyperplanes required to cover all the vertices of $\{0,1\}^n$, except the origin, at least t times keeping the origin as uncovered?

They gave a lower and an upper bound for their problem. Sauermann and Wigderson [134] studied this multiple covering problem in polynomial settings with a slight more generalization. Their problem of interest was the following:

What will be the minimum degree of the polynomial $P \in \mathbb{R}[x_1, \ldots, x_n]$ that covers all the vertices of $\{0,1\}^n$, except the origin, at least t times[1] and covers the origin exactly ℓ times, where $0 \le \ell < t$?

They gave a tight lower bound for this (t, ℓ) polynomial covering problem. Again note that putting $\ell = 0$ gives a lower bound for the hyperplane covering with multiplicity t, defined by Clifton and Huang. Though the lower bound we get by putting $\ell = 0$ in the bound given by Sauermann and Wigderson improves the lower bound given by Clifton and Huang but it falls short of the upper bound given by Clifton and Huang. This suggests that there might be a gap between the size of polynomial covering and that of hyperplane covering when considered with multiplicities.

In Chapter 8 of this thesis, we will give construction of a family of hyperplanes that matches the polynomial bound given in [75] , when the forbidden set is a single layer (that is, the forbidden set consists of all the points that have exactly p many 1's in their coordinates,

[1] we say a point v is covered t times by the polynomial P if P vanishes at v with multiplicity t

$0 \leq p \leq n$). In Chapter 9 we have shown that if the forbidden set is symmetric (that is, closed under permutation of coordinates) then also size of the hyperplane cover matches with that of polynomial cover. Moreover, if we consider (t, ℓ) polynomial covering with $0 \leq \ell \leq t-2$, we have given an explicit example showing that there is a gap between the size of the hyperplane cover and that of polynomial cover, even if we consider symmetric sets as forbidden set.

Part I

Geometric Transversals Theory

Chapter 2

Helly's Theorem: Different Variants and Applications

2.1 Helly's Theorem and its different variants

Helly's theorem, one of the cornerstones in combinatorial convexity, deals with the necessary and sufficient condition of a family of convex sets to have a common point. Nearly a century ago, this theorem motivated the study of geometric transversal theory. Suppose $\mathscr{F}$ is a collection of subsets of $\mathbb{R}^d$ and $\mathscr{T}$ is a family of geometric objects in $\mathbb{R}^d$. For example, $\mathscr{T}$ can be a set of points or lines or hyperplanes etc. Then $\mathscr{T}$ is said to be a *transversal* of $\mathscr{F}$ if $\forall F \in \mathscr{F}$, $\exists T \in \mathscr{T}$ such that $F \cap T \neq \emptyset$. In other words, we say $\mathscr{T}$ *pierces* or *stabs* $\mathscr{F}$. For any $n \in \mathbb{N}$, $\mathscr{F}$ is said to be n-pierceable, if $\mathscr{F}$ has a transversal $\mathscr{T}$ of size at most n. **We shall consider piercing with points, unless otherwise stated.** Helly's theorem [93], considers 1-pierceability with respect to point transversals.

Theorem 2.1 (Helly's theorem [93])**.** *Suppose $\mathscr{F}$ is a finite family of convex sets in $\mathbb{R}^d$ such that every $(d+1)$-tuple from $\mathscr{F}$ is 1-pierceable, then the whole family $\mathscr{F}$ is 1-pierceable.*

Over the years "Helly type" theorems have been studied thoroughly and have found various applications, see [14, 37, 38, 76, 95, 105]. One of the main reasons behind the enormous popularity of Helly's theorem is its versatility. Numerous studies have been done by changing the framework from different aspects. For example:

- **Types of sets:** The convexity assumption of the sets of $\mathscr{F}$ has been changed to either stronger conditions such as being axis parallel boxes or translates of a convex set or the assumption has been relaxed to the homology groups of the sets and their intersections.

- **Intersection requirements:** The original theorem concerns whether the whole family have a common point. This can be modified to whether a large subfamily have a common point.
- **Piercing objects:** The original theorem considers piercing by a point. Variations can involve piercing by higher-dimensional objects such as lines, planes, or k-dimensional affine spaces (k-flats).
- **Number of transversals:** Helly's theorem states that a single point can pierce all sets. Variations can explore the conditions under which a finite number of points or other transversals are sufficient, leading to results about multi-pierceability.
- **Combinatorial variations:** Helly-type theorems can be adapted to combinatorial frameworks, where conditions are based on combinatorial properties. For example, (p,q)-properties.
- **Dimension-independent results:** Recent trends explore versions of Helly's theorem that do not depend on the dimension of the space, aiming for more general applicability.

Fractional version: One of the most interesting and surprisingly useful variant of Helly's theorem is due to Katchalski and Liu [106]. In Helly's theorem, every $(d+1)$-tuple is 1-pierceable. A natural relaxation of the original assumption is that only a positive fraction of the $(d+1)$-tuples are 1-pierceable and the question is whether there is a large intersecting subfamily. Katchalski and Liu [106] answered this question positively. The result is known as *Fractional Helly Theorem*.

Theorem 2.2 (Fractional Helly Theorem [106])**.** *Suppose $\mathscr{F}$ is a family of convex sets in $\mathbb{R}^d$. Then $\forall\alpha \in (0,1]$, $\exists\beta = \beta(\alpha,d) \in (0,1]$ such that the following holds: if at least α fraction of all the $(d+1)$-tuples from $\mathscr{F}$ is 1-pierceable, then there exists a subfamily of $\mathscr{F}$ with size at least $\beta|\mathscr{F}|$, which is 1-pierceable.*

Kalai [101] gave the best possible value as $\beta = 1-(1-\alpha)^{\frac{1}{d+1}}$. Observe that, as α approaches to 1, β also goes to 1, giving back the original Helly's theorem.

Colorful version: Colorful variations of Helly's theorem, introduced by Lovász, come from asking additional combinatorial restrictions to the intersection structure of the family. Let $\mathscr{F}_1,\ldots,\mathscr{F}_n$ be non-empty families of convex sets in $\mathbb{R}^d$. A t-tuple $(C_1,\ldots,C_t)$ is a *colorful t-tuple* from the above families of convex sets if for each $j \in [t]$, there exists $i_j \in [n]$ such that $C_j \in \mathscr{F}_{i_j}$, and for distinct $j,k \subset [t]$ we have $i_j \neq i_k$. Bárány [21] proved the following theorem:

Theorem 2.3 (Colorful Helly Theorem [21]). *If $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{d+1}$ be families of convex sets in $\mathbb{R}^d$ such that every colorful $(d+1)$-tuple is 1-pierceable, then at least one of the families $\mathscr{F}_i$ is 1-pierceable.*

Observe that, if all the $d+1$ families coincide then we will get back the original Helly's theorem.

Holmsen and Lee [96] showed that in general *convexity spaces* with bounded *Radon number*[1], Colorful Helly Theorem implies Fractional Helly Theorem.

Multi-piercing version: A typical "Helly-type" theorem has the form:

If every h or fewer members of a family of objects have property $\mathscr{P}$,then the entire family has property $\mathscr{P}$.

The smallest such integer $h(\mathscr{P})$ is called the *Helly number* of property $\mathscr{P}$. In original Helly's theorem the property $\mathscr{P}$ is 1-pierceability and $h = d+1$. Now a natural question that comes into our mind is that, can we extend the "Helly-type" results for multi-piercing also? Unfortunately, for $n \geq 2$, no *Helly-type* theorem about the n-pierceable sets is valid for general convex sets (see [85], p. 17). But for special classes of families of convex sets, like axis-parallel boxes [59], special families of triangles [107] results similar to Helly's theorem have been proved for n-pierceable sets.

(p,q)**-theorem:** Another significant generalization of Helly's theorem is (p,q)-*theorem*. Fractional Helly theorem guarantees the existence of a large intersecting subfamily. But under what condition can we guarantee the existence of a finite sized transversal independent of the size of the given family? Hadwiger and Debrunner [84] first introduced such a condition, named (p,q)-*property*. A family $\mathscr{F}$ of subsets of $\mathbb{R}^d$ is said to satisfy (p,q)-property if among every p members of $\mathscr{F}$ there are some q member that can be pierced by a single transversal. According to this definition Helly's theorem can be restated as follows: if a finite family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$ satisfies $(d+1, d+1)$-property with respect to point transversals then $\mathscr{F}$ is pierceable by a single point. Hadwiger and Debrunner [84] proved the following fundamental generalization of the Helly's theorem.

Theorem 2.4 (Hadwiger and Debrunner [84]). *Let $p \geq q \geq d+1$ with $q > \frac{(d-1)p+d}{d}$, and $\mathscr{F}$ be a finite family of compact convex sets in $\mathbb{R}^d$ satisfying the (p,q)-property for points. Then $\mathscr{F}$ can be pierced by $p-q+1$ points.*

[1] This is the smallest integer r_2 (if it exists) such that any subset $P \subset X$ with $|P| \geq r_2$ can be partitioned into two parts P_1 and P_2 such that $\mathrm{conv}(P_1) \cap \mathrm{conv}(P_2) \neq \emptyset$.

Hadwiger and Debrunner [84] asked the following question that remained open for 35-years.

Question 2.5 (Hadwiger and Debrunner (p,q)-problem [84]). *For all $p \geq q \geq d+1$, does there exists a constant $c(p,q,d) > 0$ such that if a family $\mathscr{F}$ of compact convex sets in $\mathbb{R}^d$ satisfy (p,q)-property with respect to points then $\mathscr{F}$ has a point transversal of size at most $c(p,q,d)$?*

Alon and Kleitman in a breakthrough paper [8] resolved the above question of Hadwiger and Debrunner.

Theorem 2.6 ((p,q)-Theorem [8]). *For any three natural numbers $p \geq q \geq d+1$, $\exists c = c(p,q,d)$ such that if $\mathscr{F}$ is a collection of compact convex sets in $\mathbb{R}^d$ satisfying the (p,q)-property for points, then there exists a point transversal for $\mathscr{F}$ with size at most c.*

Later Alon and Kalai [7] proved the (p,q)-theorem for hyperplane transversal.

Theorem 2.7 ((p,q)-Theorem for hyperplane transversal [7]). *For any three natural numbers $p \geq q \geq d+1$, $\exists c' = c'(p,q,d)$ such that if $\mathscr{F}$ is a collection of compact convex sets in $\mathbb{R}^d$ satisfying the (p,q)-property with respect to piercing by hyperplanes then there exists a transversal for $\mathscr{F}$ with size at most c'.*

Unlike the case of hyperplane transversal, Alon, Kalai, Matoušek, and Meshulam [11] proved the impossibility of getting a (p,q)-theorem for k-transversal (i,e, piercing with k-dimensional affine spaces, namely k-flats), when $1 \leq k < d-1$. Since its introduction by Hadwiger and Debrunner [84], this relaxed variant of the Helly-type problems has been an active area of research in Discrete and Convex Geometry, and is now popularly known as *Hadwiger-Debrunner (p,q)-problems* or just *(p,q)-problems*.

Colorful (p,q)-theorem: Variations of the (p,q)-theorem, where the local intersection condition is modified, are also possible. Given $\mathscr{F}_1,\dots,\mathscr{F}_p$ families of convex sets in $\mathbb{R}^d$, we say $\mathscr{F}_1,\dots,\mathscr{F}_p$ satisfies *colorful (p,q)-property* with respect to k-flats if for every colorful p-tuple $(C_1,\dots,C_p)$ with $C_i \in \mathscr{F}_i, \forall i \in [p]$, contains q sets, say $(C_{i_1},\dots,C_{i_q})$, that can be pierced by a single k-flat. Bárány, Fodor, Montejano, Oliveros and Pór. [23] proved a *colorful (p,q)-theorem* for families of convex sets in $\mathbb{R}^d$, as an immediate consequence of a colorful version of the fractional Helly theorem.

Theorem 2.8 (Colorful Fractional Helly theorem [23]). *Suppose $\mathscr{F}_1,\dots,\mathscr{F}_{d+1}$ are finite families of convex sets in $\mathbb{R}^d$ and $\mathscr{F} = \bigcup_{i=1}^{[d+1]} \mathscr{F}_i$. If an α fraction of colorful $(d+1)$-tuples of $\mathscr{F}$ are intersecting then some $\mathscr{F}_i$ contains an intersecting subfamily of size $\frac{\alpha}{d+1}|\mathscr{F}_i|$.*

Theorem 2.9 (Colorful (p,q)-theorem [23]). *For any three natural numbers $p \geq q \geq d+1$, $\exists M = M(p,q,d)$ such that the following holds: if $\mathscr{F}_1,\dots,\mathscr{F}_p$ be finite families of convex* *sets in $\mathbb{R}^d$ satisfying colorful (p,q)-property with respect to points then there exists at least $q-d$ indices $i \in \{1,\ldots,p\}$ such that $\mathcal{F}_i$ has a point transversal of size at most M.*

Piercing with higher dimensional transversals: Another important generalization of Helly's theorem is to consider higher dimensional transversals, for example, piercing with hyperplanes or k-flats (k-dimensional affine spaces). Vincensini [147] asked whether there exists a number $m = m(k,d)$ such that for any family $\mathcal{F}$ of convex sets in $\mathbb{R}^d$, if every m or fewer members of $\mathcal{F}$ have a k-transversal, then $\mathcal{F}$ has a k-transversal. Santaló [133] showed that there can be no *Helly-type* theorem for arbitrary families of compact convex sets in $\mathbb{R}^3$ and line transversals. Because of Santaló's impossibility result, the follow-up works on k-transversals are for families of convex sets that satisfy additional properties on the shapes and/or the relative positions of the sets of the family, see [17, 18, 50, 60, 77, 82, 83, 97]. Hadwiger [83] gave the first necessary and sufficient conditions for the existence of a line transversal to a finite family $\mathcal{F}$ of pairwise disjoint convex sets in $\mathbb{R}^2$.

Theorem 2.10 (Hadwiger's Transversal Theorem [83])**.** *Suppose $\mathcal{F}$ is a finite family of pairwise disjoint convex sets in $\mathbb{R}^2$ such that there exists a linear ordering of $\mathcal{F}$ such that every three members of $\mathcal{F}$ can be intersected by a directed line in the given order. Then $\mathcal{F}$ has a line transversal.*

Danzer, Grünbaum and Klee [60] observed that Hadwiger's proof of Theorem 2.10 works well as long as the diameters of the sets are bounded.

Theorem 2.11 (Danzer–Grünbaum–Klee [60])**.** *Suppose $\mathcal{F}$ is a family of compact convex sets with bounded diameter in $\mathbb{R}^d$ such that $\bigcup_{F\in\mathcal{F}} F$ is unbounded. If every $(d+1)$ members of $\mathcal{F}$ have a line transversal, then the whole collection does.*

Pollack and Wenger [131] generalized Hadwiger's transversal theorem to higher dimensions and gave necessary and sufficient conditions for the existence of a hyperplane transversal to a family of compact convex sets in $\mathbb{R}^d$. Suppose $\mathcal{F}$ be a finite family of compact convex sets in $\mathbb{R}^d$ and $P \subseteq \mathbb{R}^k$. We say that F *separates consistently with* P if there exists a map $\phi : \mathcal{F} \to P$ such that for any two of subfamilies $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\mathcal{F}$ such that $\text{conv}(\mathcal{F}_1) \cap \text{conv}(\mathcal{F}_2) = \emptyset$, we have $\text{conv}\big(\phi(\mathcal{F}_1)\big) \cap \text{conv}\big(\phi(\mathcal{F}_2)\big) = \emptyset$.

Theorem 2.12 (Pollack and Wenger [131])**.** *A family $\mathcal{F}$ of compact convex sets in $\mathbb{R}^d$ has a hyperplane transversal if and only if $\mathcal{F}$ separates consistently with a set $P \subseteq \mathbb{R}^{d-1}$.*

When $\mathcal{F}$ is a family of pairwise disjoint sets in the plane, the separation condition of Theorem 2.12 is equivalent to the ordering condition in Hadwiger's theorem. Aronov, Goodman and Pollack [18] gave the sufficient conditions for 1-pierceability with respect to k-transversals $(0 \le k \le d-1)$ by generalizing Hadwiger(–Danzer–Grünbaum–Klee) theorem.

Theorem 2.13 (Aronov et al. [18])**.** *Suppose $k < d$ and $\mathscr{F}$ is a family of compact convex sets with bounded diameter in $\mathbb{R}^d$ such that $\bigcup_{F \in \mathscr{F}} F$ is unbounded in k independent directions. If every $d+1$ members of $\mathscr{F}$ have a k-transversal, then the whole collection does.*

Over the years generalizations of Helly's Theorem to general k-transversals, for $1 \le k \le d-1$, has been an active area of research, for details see the survey by Holmsen and Wenger [94].

Dimension independent versions: An interesting fact to consider here is that all these Helly-type results are extremely dependent on the dimension of the ambient space. A recent trend is to investigate such results independent of the dimension, see [2, 52, 87]. Adiprasito et al. [2] gave the first dimension independent Helly's theorem for point transversals.

2.2 Applications of Helly-type theorems in computational geometry

Helly's theorem has numerous applications, often in surprising contexts, ranging from voting theory [26] to clustering problems with incomplete data points [72], covering problems [62], and recently, in a novel algorithmic approach for interpolating data with differentiable functions [66]. Here we will primarily discuss optimization algorithms, where Helly numbers and Helly-type theorems are pivotal.

Helly's theorem is fundamental in the theory of convex optimization algorithms. In convex optimization problems, we are given n convex constraints (with feasible solutions forming convex sets), and we are interested whether all the constraints have a common solution. Helly's theorem indicates that by checking all subfamilies of size $d+1$ for a non-empty intersection (a common solution), we can either find a subset of $d+1$ constraints that proves the entire set has no common solution, or confirm that a common solution exists. Surely checking intersections among every subset of $d+1$ constraints is not the most efficient method for solving convex programs but the idea of examining only small subfamilies to determine a property of the entire set paves the way for developing randomized optimization algorithms that utilize other Helly-type theorems.

Helly-type results in linear optimization algorithms: Linear optimization is perhaps the central problem in the theory of optimization, and it is often used as a subroutine to solve much more complex problems that include non-linear or integrality constraints. For an introduction to linear programming, see [138]. A linear program consists of a pair $(\mathscr{H}, \omega)$, where $\mathscr{H}$ is a family of n linear halfspace constraints in $\mathbb{R}^d$, and ω is a function $\omega : \mathscr{F} \in 2^{\mathscr{H}} \to \Lambda$, where Λ is a linearly ordered set including an element ∞. If $\cap\mathscr{H} \neq \emptyset$, the linear programming algorithm should output a point belonging to $\cap\mathscr{H}$ minimizing ω, otherwise the output should be a set of $d+1$ constraints certifying that the intersection is empty.

Helly's theorem explains the combinatorics of linear halfspace intersections and naturally influences combinatorial random sampling algorithms, primarily developed to solve the linear programming problem. A combinatorial algorithm operates on input using primitive operations that yield discrete results, such as Boolean values or subsets of an input set, instead of floating-point numbers. Its runtime, measured in terms of the number of these primitive operations, remains unaffected by the size of the coefficients involved. Two notable combinatorial algorithms are Clarkson's random sampling algorithm, which solves small subproblems of size $O(d^2)$, and the classic simplex algorithm from linear optimization.

Clarkson observed that one can apply his algorithm to solve other problems also, such as integer programming and finding the smallest enclosing ball. Sharir and Welzl [140] introduced an abstract framework, called *LP-type problems*, which describes the necessary conditions for these algorithms to work. Since then the study of randomization and abstractions of linear programming have been an active area of research, see [73, 31, 86, 74]. These abstract frameworks are important because they allow us to demonstrate the existence of algorithmic solutions for computational challenges. This can be achieved by proving that the axioms required for an LP-type problem are satisfied, without necessarily showing that its constraints define convex, quasi-convex, or connected sets. Helly-type theorems are frequently employed in this context to facilitate such demonstrations.

Now we come to the classical simplex algorithm from linear optimization. The simplex method transitions from one basic feasible solution to an adjacent one, effectively visiting vertices of a polyhedron by traversing its one-dimensional faces. The following abstractions serve as replacements for these concepts. For LP-type problems, we define, as in [14], a *basis* $B \subseteq \mathscr{H}$ is a subset such that $\forall h \in B$ we have $\omega(B-h) < \omega(B)$. For $G \subseteq \mathscr{H}$, a basis of G is a minimal subset B of G with $\omega(B) = \omega(G)$. The size of a largest basis of an LP-type problem is called its *combinatorial dimension*, denoted by δ. For example, the combinatorial dimension of linear programs in d variables is $d+1$. Similar to the simplex algorithm, the solution found by a combinatorial algorithm for LP-type problems serves as a basis for the input set $\mathscr{H}$ of constraints. The following observation by Amenta [12] connects the combinatorial dimension of an LP-type problem with a Helly-type theorem regarding its constraint set $\mathscr{H}$. Before we present the observation by Amenta [12] we first need to define a *concrete LP-type problem.*

Definition 2.14 ([14]). *A concrete LP-type problem is a triple $(X, \mathscr{H}, \preceq)$, where X is a set linearly ordered by $\preceq$, $\mathscr{H}$ is a finite multiset whose elements are subsets of X, and for any $\mathscr{G} \subseteq \mathscr{H}$, if $\cap\mathscr{G} \neq \emptyset$, then $\cap\mathscr{G}$ has a unique minimum element with respect to $\preceq$. We call this unique minimum element $\omega(\mathscr{G})$.*

Observation 2.15 (Amenta [12]). *A concrete LP-type problem of combinatorial dimension δ is always associated with a Helly-type theorem, in which the Helly number is $\delta+1$: a finite set $\mathscr{F} \in 2^{\mathscr{H}}$ of constraints has non-empty intersection if and only if every subset $\mathscr{G} \subseteq \mathscr{F}$ of size $\delta+1$ has non-empty intersection.*

Now let us see what are the combinatorial primitives required in these frameworks. Simplex-like combinatorial algorithms for LP-type problems sometimes use a basis computation primitive. Given a subset $B+h \subseteq \mathscr{H}$ of constraints, where B is a basis and h is a *violator* of B (we say *h violates B* if $\omega(B+h) > \omega(B)$), the algorithm produces a new basis $B' \subseteq B+h$. One can always achieve these basis computations using violator tests and the usual run-time is exponential in δ, but for some LP-type problems, these can be done more quickly. A simplex algorithm moves from one basis to an adjacent basis using these basis computations and thus avoids cycling.

Finally we come to the analysis of run-times of combinatorial LP algorithms. Seidel [139] and Sharir and Welzl [140] showed that any LP-type problem can be solved using the randomized dual-simplex algorithm (also known as the random facet algorithm). The algorithm performs $O(n)$ violator tests and basis computations (where n is the number of constraints) and its run-time is at most exponential in terms of δ. This often results in an efficient algorithm, when δ is a constant, providing a certificate whether a family $\mathscr{F}$ of input objects has some Helly-type property. Again one can combine the randomized simplex algorithm in Clarkson's algorithm to solve small subproblems. Then a sophisticated analysis of this algorithm provides a sub-exponential bound for a combinatorial linear programming algorithm. We say that the running time is sub-exponential in δ if it is exponential in some function of $o(\delta)$, such as $\sqrt{\delta}$.

Theorem 2.16 (Matoušek et al. [118], Kalai [102]). *The combined algorithm requires $O(e^{O(\delta \log n)})$ basis computations and $O(ne^{O(\delta \log n)})$ violation tests.*

When both the primitive operations can be performed in sub-exponential time, we achieve an overall sub-exponential time algorithm.

Use of LP-type results to establish Helly-type theorems: As we have seen, Helly numbers, in the form of a combinatorial dimension, are important measures of the scalability of various algorithms for example LP-type problems. Amenta [12] proposed various constructions for objective functions that enable the formulation of LP-type problems based on existing Helly-type theorems. Often, these constructions involve employing lexicographic ordering or a strictly convex function, such as distance from the origin, to define an operator in the context of a concrete LP-type problem. Leveraging this approach, many Helly-type theorems can be applied directly to formulate concrete LP-type problems, demonstrating that such problems can be solved efficiently in linear time within fixed dimensions. Not only this, one can use algorithms that solve LP-type problems to prove Helly-type theorems also. For example, using Observation 2.15, Amenta [13] proved the following:

Theorem 2.17 (Amenta [13])**.** *Let $(X, \mathscr{H}, \preceq)$ be a concrete LP-type problem of combinatorial dimension δ with the property that $\preceq$ is a total order on the points of X. Let $\mathscr{F}$ be a family of subsets of X such that, for every $\mathscr{G} \subseteq \mathscr{F}$ with $\cap\mathscr{G} \neq \emptyset$, the intersection $\cap\mathscr{G}$ is the disjoint union of at most r elements of $\mathscr{H}$. Then $(X, \mathscr{F}, \preceq)$ is a concrete LP-type problem of combinatorial dimension at most $r(\delta+1)-1$.*

Using this theorem Morris [122] and Amenta [12] gave a simple proof of the following Helly-type theorem, first conjectured by Grünbaum and Motzkin [80].

Theorem 2.18. *Let $\mathscr{F}$ be a family of sets in $\mathbb{R}^d$, such that the common intersect of any non-empty finite sub-family of $\mathscr{F}$ is the disjoint union of at most r closed convex sets. Then $\mathscr{F}$ has a Helly number of at most $r(d+1)$.*

Other applications: Several geometric problems can be solved within linear time in terms of the number of constraints, given a fixed dimension. LP-type problems can often be reformulated as fixed-dimensional linear or convex programming problems, where $\mathscr{F}$ represents a collection of convex constraints and the objective function ω is either convex or lexicographic. For example, suppose a family $\mathscr{K} = \{K_1, \ldots, K_n\}$ of vertical line segments in the plane are given and we are interested in finding a line transversal for $\mathscr{K}$. Then this problem can be expressed as a linear programming problem in a two-dimensional space of lines. N. Megiddo [119] extended this concept in finding line transversals of boxes in any fixed dimension by solving a small number of linear programs.

More complex LP-type problems, akin to Helly-type theorems where sets in $\mathscr{F}$ are not necessarily convex, are often tackled using a framework similar to quasi-convex programming. For example consider the following problem [12]: Suppose P is a set of points in $\mathbb{R}^2$ such that each pair of points in P is at least unit distance apart and $\mathscr{F}$ is a family of disks in $\mathbb{R}^2$

with radius r and center at $p \in P$. Then what is the smallest value of $r < \frac{1}{2}$ such that $\mathscr{F}$ has a line transversal?

Tverberg [144] proved that the Helly number for line intersections of unit disks in the plane is five. Using this we infer that the combinatorial dimension of this problem is also five. This is despite the fact that the dimension of the set of lines in the plane is two, and the set of transversals is not necessarily connected.

We end this chapter with an application of Helly's theorem in computational geometry where no LP-type algorithms are involved. Suppose P is a set of n points in $\mathbb{R}^d$ and $q \in \mathbb{R}^d$ is any arbitrary point. We say q has *Tukey depth* α with respect to P if any closed halfspace, containing q as a boundary point, contains at least αn points of P. If $\alpha = \frac{1}{d+1}$, we call q to be a *centerpoint* of P. The centerpoint of P can be viewed as a generalization of the median to data in higher-dimensional Euclidean space. Using Helly's theorem we can show that centerpoint of P always exists. But finding or approximating centerpoints or points of maximal Tukey depth is a challenging computational task in statistics because, naively, the number of halfspaces that need to be considered is $O(n^d)$. Although by utilizing the dependencies between these halfspaces, a more efficient algorithm for finding points of maximal Tukey depth has been developed [45] with a running time of $O(n^{d-1})$. Moreover, finding approximate center points in sub-exponential time [55, 120, 123] is highly beneficial for partitioning problems in efficient parallel computation and has garnered recent interest in statistics [58].

Chapter 3

Colorful Helly Theorem for Piercing Boxes with Multiple Points

3.1 Introduction

For any natural number n, a family $\mathscr{F}$ of subsets of a space $\mathbf{X}$ is said to be *n-pierceable*, if there exists $A \subseteq \mathbf{X}$ with $|A| \leq n$ such that for any $F \in \mathscr{F}$, $F \cap A \neq \emptyset$.

Helly's theorem (Theorem 2.1) [93], one of the fundamental results in discrete geometry, says that for any finite family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$, if every $(d+1)$-tuple from $\mathscr{F}$ is 1-pierceable, then the whole family $\mathscr{F}$ is 1-pierceable. Two of its main variants are namely *Fractional Helly Theorem* (Theorem 2.2), proved by Katchalski and Liu [106], and *Colorful Helly Theorem* (Theorem 2.3), proved by Bárány [21].

A stronger form of the colorful Helly theorem was shown by Kalai and Meshulam [103].

Theorem 3.1 (Kalai and Meshulam [103])**.** *If $\mathscr{F}_1, \mathscr{F}_2, \ldots, \mathscr{F}_{d+1}$ are finite families of convex sets in $\mathbb{R}^d$ such that every colorful $(d+1)$-tuple is 1-pierceable, then there exists an $i \in [d+1]$, and for each $k \in [d+1]$ with $k \neq i$, there exists $F_k \in \mathscr{F}_k$ such that the new extended family $\mathscr{F}_i \cup \left\{F_k \mid k \in [d+1], k \neq i\right\}$ is 1-pierceable.*

This shows that not only one of the families is one pierceable, but also we can extend one such family by adding one element from each of the rest of the families so that this new family also becomes one pierceable.

One natural generalization of Helly's theorem is to investigate for the sufficient condition for multi-pierceabily of a family. One of the first and foremost Helly-type result on multi-pierceability of families was proved by Danzer and Grünbaum [59] for families of axis parallel boxes. An *axis-parallel box* B in $\mathbb{R}^d$ is a set of the form $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d]$, where $\forall i \in [d]$, $a_i, b_i \in \mathbb{R}$ with $a_i \leq b_i$.

Theorem 3.2 (Danzer and Grünbaum [59])**.** *Suppose $\mathcal{F}$ is a family of axis parallel boxes in $\mathbb{R}^d$. Let $h = h(d,n)$ be the smallest positive integer such that every h-tuple from $\mathcal{F}$ is n-pierceable implies that $\mathcal{F}$ is n-pierceable. Then following are the values of h:*

(a) $\forall d \in \mathbb{N}$, $h(d,1) = 2$,

(b) $\forall n \in \mathbb{N}$, $h(1,n) = n+1$,

(c) $h(d,2) = \begin{cases} 3d, & \text{for odd } d \\ 3d-1, & \text{for even } d \end{cases}$

(d) $h(2,3) = 16$

(e) $h(d,n) = \aleph_0$, *for* $d \geq 2, n \geq 3$ *and* $(d,n) \neq (2,3)$

Chakraborty et al. [39] showed that Fractional Helly Theorem for multi-pierceability is not true in general. They observed that for any constant $\omega > 0$ there exists a family of disks in the plane such that any subfamily of size ω is 2-pierceable but the whole family is not 2-pierceable.

3.1.1 Our results

In this chapter, we shall first prove the following stronger version of Colorful Helly Theorem, analogous to the result of Kalai and Meshulam [103] (Theorem 3.1) for the multi-piercing setting.

Theorem 3.3 (Strong Colorful Helly Theorem for Multi-Piercing Boxes)**.** *Suppose $H_c = H_c(d,n)$ is the smallest positive integer such that if we have a collection of finite families $\mathcal{F}_1, \ldots, \mathcal{F}_{H_c}$ of boxes in $\mathbb{R}^d$ with the property that every colorful H_c-tuple from the above families is n-pierceable then there exits an $i \in [H_c]$, and for all $k \in [H_c] \setminus \{i\}$, there exists $F_k \in \mathcal{F}_k$ such that the following extended family $\mathcal{F}_i \cup \{F_k \mid k \in [H_c], k \neq i\}$ is n-pierceable. Then, we have*

(a) $\forall d \in \mathbb{N}$, $H_c(d,1) = d+1$,

(b) $\forall n \in \mathbb{N}$, $H_c(1,n) = n+1$,

(c) $\forall d \in \mathbb{N}$, $H_c(d,2) = 3d$.

Note that, for any d and n, for which $H_c(d,n)$ is defined, we will always have $H_c(d,n) \geq h(d,n)$. From Theorem 3.3, we have $H_c(d,n) = h(d,n)$ unless $d > 1$ and $n = 1$ or d is even and $n = 2$.

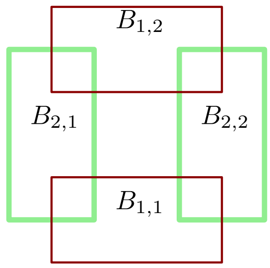

Fig. 3.1 Every colorful pair intersect but none of the families is intersecting

Observe that, for these two cases $H_c(d,n) > h(d,n)$. Since Theorem 3.3 proves a stronger variant of *colorful Helly Theorem* it may still be possible to establish the following two statements:

Let $\mathscr{F}_1, \mathscr{F}_2$ be two finite families of boxes in $\mathbb{R}^d$. If every colorful pair intersect then there exists $i \in [2]$ such that $\mathscr{F}_i$ is 1-pierceable.

or

Let $d \in \mathbb{N}$ be an even number, and also let $\mathscr{F}_1, \ldots, \mathscr{F}_{3d-1}$ be a collection of finite families of boxes in $\mathbb{R}^d$. If every colorful $(3d-1)$-tuple is 2-pierceable then there exists $i \in [3d-1]$ such that $\mathscr{F}_i$ is 2-pierceable.

We give an explicit collection of families of boxes in $\mathbb{R}^d$ disproving the second statement.

Theorem 3.4 (Extremal example)**.** *For every $d \in \mathbb{N}$, there exist non-empty families $\mathscr{F}_1, \ldots, \mathscr{F}_{3d-1}$ of axis-parallel boxes in $\mathbb{R}^d$ such that*

- *every colorful $(3d-1)$-tuple is 2-pierceable, and*
- *for each $i \in [3d-1]$, $\mathscr{F}_i$ is not 2-pierceable.*

And we disprove the first statement by a very simple counterexample. For each $i \in [d]$, consider the family $\mathscr{F}_i = \{B_{i,1}, B_{i,2}\}$, where for each $j \in [d]$, $j \neq i$, $\pi_j(B_{i,1}) = \pi_j(B_{i,2}) = [0,1]$ and $\pi_i(B_{i,1}) = [-0.25, 0.25]$ and $\pi_i(B_{i,2}) = [0.75, 1.25]$. See Figure 3.1 for the case $d = 2$. Clearly no family $\mathscr{F}_i$ is 1-pierceable.

Now consider any colorful pair of boxes B_i, B_j such that $B_i \in \mathscr{F}_i, B_j \in \mathscr{F}_j$ for some $i, j \in [d]$, $i \neq j$. For each $k \in \{i, j\}$ if $B_k = B_{k,1}$, we take $\zeta_k = 0$, else $B_k = B_{k,2}$, and we take $\zeta_k = 1$ and for each $k \in [d] \setminus \{i, j\}$ we take $\zeta_k = 0.5$. Then $\zeta = (\zeta_1, \ldots, \zeta_d) \in B_i \cap B_j$.

Observe that, this example along with Theorem 3.2 (a), give the following:

Theorem 3.5. *For every $d \in \mathbb{N}$, there exist non-empty families $\mathscr{F}_1, \ldots, \mathscr{F}_d$ of axis-parallel boxes in $\mathbb{R}^d$ such that*

- *every colorful d-tuple intersect, and*
- *for each $i \in [d]$, $\mathscr{F}_i$ is not* 1*-pierceable.*

Again, from Theorem 3.1 we get $H_c(d,1) \leq d+1$. This together with Theorem 3.5, give $H_c(d,1) = d+1$.

3.1.2 Definitions

- For any box $B = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$ in $\mathbb{R}^d$ and $j \in [d]$, $\pi_j(B)$ denotes projection of B on the j-th coordinate axis, i.e., the interval $I = [\alpha_j, \beta_j]$.
- For any two boxes $B, B' \in \mathbb{R}^d$, if $\pi_j(B) = [\alpha, \beta]$ and $\pi_j(B') = [\alpha', \beta']$ such that $\beta < \alpha'$ then we denote the distance between B and B' along j-th coordinate axis by $\text{dist}_j(B, B') = \alpha' - \beta$.
- Let $B = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$ be an axis parallel box. We say B' is a *face* of B if there exists a $j \in [d]$ such that B' is either this

$$[\alpha_1, \beta_1] \times \cdots \times [\alpha_{j-1}, \beta_{j-1}] \times \{\alpha_j\} \times [\alpha_{j+1}, \beta_{j+1}] \times \cdots \times [\alpha_d, \beta_d]$$

  or

$$[\alpha_1, \beta_1] \times \cdots \times [\alpha_{j-1}, \beta_{j-1}] \times \{\beta_j\} \times [\alpha_{j+1}, \beta_{j+1}] \times \cdots \times [\alpha_d, \beta_d].$$

- Let $B = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$ be an axis parallel box. We say $\lambda = (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d$ is a *vertex* of B if for each $j \in [d]$, $\lambda_j \in \{\alpha_j, \beta_j\}$.
- Let $B = [\alpha_1, \beta_1] \times \cdots \times [\alpha_d, \beta_d]$ be an axis parallel box. A pair of vertices (p,q) in B is called *diagonally opposite* if

$$p = (\lambda_1, \ldots, \lambda_d),\ q = (\eta_1, \ldots, \eta_d) \text{ and } \forall j \in [d], \{\lambda_j, \eta_j\} = \{\alpha_j, \beta_j\}.$$

  Clearly, for every vertex $p \in B$, there is a unique vertex $q \in B$ such that (p,q) is diagonally opposite.
- Let $I = [\alpha, \beta]$ and $I' = [\alpha', \beta']$ be two intervals on $\mathbb{R}$. We write $I < I'$ if and only if $\beta < \alpha'$.

3.2 Colorful Helly Theorem for n-piercing intervals

In this section we shall prove that $H_c(1,n) \leq n+1$.

Theorem 3.6 (Theorem 3.3: $d = 1$ case)**.** *Let $\mathscr{F}_1, \ldots, \mathscr{F}_{n+1}$ be non-empty collection of closed intervals in $\mathbb{R}$ with the property that every colorful $(n+1)$-tuple is n-pierceable. Then there exists $i \in [n+1]$ and $\forall k \in [n+1]$, $k \neq i$, $\exists F_k \in \mathscr{F}_k$ such that $\mathscr{F}_i \cup \{F_k \mid k \in [n+1], k \neq i\}$ is n-pierceable. Hence, for all $n \in \mathbb{N}$, we have $H_c(1,n) \leq n+1$.*

Proof. To prove the theorem we will consider different cases.

Case 1: Every colorful pair of intervals intersect. If every colorful pair of intervals intersect, then using Theorem 3.1, we may assume that $\mathscr{F}_1$ is one-pierceable and there exists $B_2 \in \mathscr{F}_2$ such that $\mathscr{F}_1 \cup \{B_2\}$ is one-pierceable.

Now for every $j \in [n+1]$, $j > 2$, we take any $B_j \in \mathscr{F}_j$. Then $\{B_j \mid 2 < j \leq (n+1)\}$ is $(n-1)$-pierceable, and therefore $\mathscr{F}_1 \cup \{B_j \mid 2 \leq j \leq (n+1)\}$ must be n pierceable.

Case 2: There is a disjoint colorful 2-tuple. Let r be the largest integer such that there exists a pairwise disjoint colorful r-tuple. Then $r \leq n$ because every colorful $(n+1)$-tuple is n-pierceable.

Let $\mathscr{C}_1$ be the collection of all colorful r-tuples that are not $(r-1)$-pierceable, that is, $\mathscr{C}_1$ is the collection of all pairwise disjoint colorful r-tuples and, without loss of generality, we also assume that if $(I_1, I_2, \ldots, I_r) \in \mathscr{C}_1$ then $I_1 < I_2 < \cdots < I_r$.

Observe that if $(I_1, \ldots, I_r) \in \mathscr{C}_1$ then for all $j \neq k$ we have $I_j \cap I_k = \emptyset$. Let

$$a_1 := \max\{\alpha \mid (I_1, \ldots, I_r) \in \mathscr{C}_1 \text{ and } I_1 = [\alpha, \beta]\},$$

and J_1 be one such interval corresponding to a_1. Let $\mathscr{C}_2$ be the sub-collection of $\mathscr{C}_1$ whose 1st component is J_1 and

$$a_2 := \max\{\alpha \mid (J_1, I_2, \ldots, I_r) \in \mathscr{C}_2 \text{ and } I_2 = [\alpha, \beta]\}.$$

Also, let J_2 be one such interval corresponding to a_2. Similarly, we can construct $\mathscr{C}_r \subseteq \cdots \subseteq \mathscr{C}_2 \subseteq \mathscr{C}_1$ and get the corresponding points $a_1, a_2, \ldots, a_r$ and intervals $J_1, J_2, \ldots, J_r$. Without loss of generality we may assume that for all $i \in [r]$ we have $J_i \in \mathscr{F}_i$. We will now show that $\forall i > r$, $\mathscr{F}_i$ is pierceable by the set $\{a_1, a_2, \ldots, a_r\}$.

Let's take any $p \in [n+1]$ with $p > r$ and any $I \in \mathscr{F}_p$. By assumption, the colorful $(r+1)$-tuple $(I, J_1, J_2, \ldots, J_r)$ is r-pierceable. Since $J_1, \ldots, J_r$ are pairwise disjoint, I is forced to have a non-empty intersection with at least one of the J_i's.

(a) Suppose I has a non-empty intersection with at least two J_i's. Let $j = \min\{\ell \mid I \cap J_\ell \neq \emptyset\}$ and $k = \max\{\ell \mid I \cap J_\ell \neq \emptyset\}$. Using the facts that the sets $J_1, \ldots, J_r$ are pairwise disjoint and $J_1 < \cdots < J_r$ with $J_\ell = [a_\ell, b_\ell], \forall \ell \in [r]$, we get that $a_i \in I$ for all $i \in \{j+1, \ldots, k\}$.

(b) Suppose I has a non-empty intersection with exactly one J_i's. Let J_k be the set that has a non-empty intersection with I, and also let $I = [\alpha, \beta]$. If $a_k \in I$ then we are done. Otherwise, assume that $a_k \notin I$. Observe that this implies $\alpha > a_k$ and the r-tuple $(J_1, \ldots, J_{k-1}, I, J_{k+1}, \ldots, J_r)$ is in $\mathscr{C}_k$. Note that this contradicts the fact that

$$a_k = \max\left\{\alpha_k \mid (J_1, \ldots, J_{k-1}, I_k, \ldots, I_r) \in \mathscr{C}_k \text{ and } I_k = [\alpha_k, \beta_k]\right\}.$$

So we get that $\forall i > r$, $\mathscr{F}_i$ is pierceable by the set $\{a_1, a_2, \ldots, a_r\}$. To complete the proof we need to handle the following two subcases:

- *Case 2A:* $r = n$. Then $\mathscr{F}_{n+1} \cup \{J_i \mid i \in [n]\}$ is pierceable by the set $\{a_1, a_2, \ldots, a_n\}$.
- *Case 2B:* $r < n$. Then for every $i \in [n]$, $i > r$, we take any $J_i \in \mathscr{F}_i$. As $\forall i > r$, $\mathscr{F}_i$ is pierceable by the set $\{a_1, a_2, \ldots, a_r\}$, we get $\mathscr{F}_{n+1} \cap \{J_i \mid i \in [n]\}$ is pierceable by the set $\{a_1, a_2, \ldots, a_r\}$.

□

3.3 Colorful Helly Theorem for two-piercing boxes in $\mathbb{R}^d$

We will now prove that for all $d \in \mathbb{N}$ we have $H_c(d, 2) \leq 3d$.

Theorem 3.7 (Upper bound part of Theorem 3.3 for $n = 2$ and general d)**.** *Suppose $\mathscr{F}_1, \ldots, \mathscr{F}_{3d}$ be non-empty collection of axis-parallel boxes in $\mathbb{R}^d$ with the property that every colorful $3d$-tuple is 2-pierceable. Then there is some $i \in [3d]$ and $\forall k \in [3d]$, $k \neq i$, $\exists F_k \in \mathscr{F}_k$ such that $\mathscr{F}_i \cup \{F_k \mid k \in [3d], k \neq i\}$ is 2-pierceable. Hence, for all $d \in \mathbb{N}$, we have $H_c(d, 2) \leq 3d$.*

We will prove some simple properties of axis-parallel boxes in Section 3.3.1 that will be used in the proof of Theorem 3.7. Finally, the proof of Theorem 3.7 will be given in Section 3.3.2.

3.3.1 Simple properties of axis-parallel boxes

The following result is a simple corollary of the combination of the result of Danzer and Grünbaum [59] (Theorem 3.2) and the colorful Helly theorem of Barany [21] (Theorem 2.3).

Corollary 3.8. *Let $\mathscr{F}_1, \ldots, \mathscr{F}_n$ be collections of axis parallel boxes in $\mathbb{R}^d$ (with $n \geq d+1$) such that every colorful pair of boxes have non-empty intersection, i.e., for all $B \in \mathscr{F}_i$ and $B' \in \mathscr{F}_j$, with $i \neq j$, we have $B \cap B' \neq \emptyset$. Then there exists $i \in [n]$ such that $\bigcap_{B \in \mathscr{F}_i} B \neq \emptyset$.*

Let $S \subseteq \mathbb{R}^d$ and $\mathscr{F}$ be a collection of subsets of $\mathbb{R}^d$. We say *S hits $\mathscr{F}$* if $\forall B \in \mathscr{F}$ we have $S \cap B \neq \emptyset$.

Lemma 3.9. *For each $i \in [n]$, with $n > (d+1)$, let $\mathscr{F}_i$ be a collection of axis parallel boxes in $\mathbb{R}^d$ such that every colorful pair of boxes have non-empty intersection , i.e., for all $B \in \mathscr{F}_i$ and $B' \in \mathscr{F}_j$, with $i \neq j$, we have $B \cap B' \neq \emptyset$. Then $\exists i \in [n]$ and for each $j \in [n],\ j \neq i$, $\exists B_j \in \mathscr{F}_j$ such that $\mathscr{F}_i \cup \{B_j \mid j \in [n],\ j \neq i\}$ is 1-pierceable.*

Proof. As every colorful pair of boxes intersects, then using Corollary 3.8, we get that $\exists i \in [n]$ such that $\mathscr{F}_i$ is one pierceable.

Now take a pair B, B' from the collection $\mathscr{F}' = \mathscr{F}_i \cup \{B_j \mid j \in [n],\ j \neq i\}$. Then either B, B' are a colorful pair or both of them are from the family $\mathscr{F}_i$. In the first case, B, B' have a common point by our assumption. Again, since $\mathscr{F}_i$ is 1-pierceable, in the later case also B, B' have a common point. So we get that every pair from $\mathscr{F}'$ are intersecting and hence by Theorem 3.2, $\mathscr{F}'$ is 1-pierceable. □

Lemma 3.10. *For each $i \in [n]$, with $n \geq 2$, let $\mathscr{F}_i$ be a collection of boxes in $\mathbb{R}^d$. If there exists $j \in [d]$ such that for any $B \in \mathscr{F}_\ell$ and $B' \in \mathscr{F}_k$ we have $\pi_j(B) \cap \pi_j(B') \neq \emptyset$ then there exists a subset $S \subseteq [n]$ with $|S| \geq n-1$ and there exist a hyperplane, orthogonal to the j-th coordinate axis, that hits $\cup_{i \in S} \mathscr{F}_i$.*

Proof. If each $\mathscr{F}_i$ is pierceable by a hyperplane, orthogonal to the j-th coordinate axis, that hits $\cup_{i \in S} \mathscr{F}_i$, then there is nothing to prove. So without loss of generality, let's assume that $\mathscr{F}_1$ is the collection of boxes that is not pierceble by any single hyperplane orthogonal to the j-th coordinate axis.

For every $i \in [n]$, we define $\mathscr{F}_{i,j} := \left\{\pi_j(B) \,:\, B \in \mathscr{F}_i\right\}$ and let $\mathscr{F}' = \cup_{2 \leq i \leq n} \mathscr{F}_{i,j}$. Then by our assumption, for any $I \in \mathscr{F}'$ and any $J \in \mathscr{F}_{1,j}$ we have $I \cap J \neq \emptyset$. Since $\mathscr{F}_{1,j}$ is not 1-pierceable, from Corollary 3.8 we get that

$$\bigcap_{I \in \mathscr{F}'} I \neq \emptyset,$$

that is, there exists a $c \in \mathbb{R}$ such that for all $I \in \mathscr{F}'$ we have $c \in I$. Therefore, the hyperplane $H = \left\{x \,:\, \langle x, e_j \rangle = c\right\}$, which is orthogonal to the j-th coordinate axis, hits $\cup_{i \in S} \mathscr{F}_i$. □

Lemma 3.11. *Suppose $B = \prod_{i=1}^{d}[\alpha_i, \beta_i]$ is an axis parallel box in $\mathbb{R}^d$ with a diagonally opposite pair of vertices (λ, λ') and B' is another axis parallel box in $\mathbb{R}^d$ such that $B' \cap \{\lambda, \lambda'\} = \emptyset$. Then B' is disjoint from at least two faces of B.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$ and $\lambda' = (\lambda'_1, \lambda'_2, \dots, \lambda'_d)$. Observe that (λ, λ') being diagonally opposite pair of vertices of B implies that for every $j \in [d]$, we have $\{\lambda_j, \lambda'_j\} = \{\alpha_j, \beta_j\}$.

Since $B' \cap \{\lambda, \lambda'\} = \emptyset$ there exist $j, k \in [d]$ such that $\pi_j(B') \cap \{\lambda_j\} = \emptyset$ and $\pi_k(B') \cap \{\lambda'_k\} = \emptyset$. Therefore the faces

$$\left\{x \in \mathbb{R}^d \mid \langle x, e_j \rangle = \lambda_j \right\} \cap B$$

and

$$\left\{x \in \mathbb{R}^d \mid \langle x, e_k \rangle = \lambda'_k \right\} \cap B$$

of B are disjoint from B'. □

Lemma 3.12. *Consider the axis-parallel box $B = \prod_{i=1}^{d}[\alpha_i, \beta_i]$ in $\mathbb{R}^d$, and let λ be a vertex of B, and B', B'' be two other axis-parallel boxes in $\mathbb{R}^d$ such that $\lambda \in B'$ and $\lambda \notin B''$. Then there exists a face F of B with $F \cap B'' = \emptyset$ and $F \cap B' \neq \emptyset$.*

Proof. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_d)$, where for each $j \in [d]$, $\lambda_j \in \{\alpha_j, \beta_j\}$. Since $\lambda \notin B''$ there exists $j \in [d]$ such that $\pi_j(B'') \cap \{\lambda_j\} = \emptyset$. This implies that B'' is disjoint from the face F, where $F = \left\{x \in \mathbb{R}^d \mid \langle x, e_j \rangle = \lambda_j \right\} \cap B$, of B. Again, observe that $F \cap B' \neq \emptyset$ as $\lambda \in B'$. □

Lemma 3.13. *Let $V = \prod_{i=1}^{d}[\alpha_i, \beta_i]$be an axis parallel box in $\mathbb{R}^d$ and $\mathscr{G}_1, \dots, \mathscr{G}_n$ be non-empty collections of axis parallel boxes in $\mathbb{R}^d$ such that the following conditions are satisfied:*

(i) for any $i \in [n]$ and any $B \in \mathscr{G}_i$, we have $\forall j \in [d]$, $\pi_j(B) \cap \{\alpha_j, \beta_j\} \neq \emptyset$, that is, B contains at least one vertex of V, and

(ii) every colorful n tuple is pierceable by at least one diagonally opposite pair of vertices of V.

If $(V_1, \dots, V_r)$, with $r < n$, is a colorful r-tuple such that there are at most 2^k, with $k < d$, distinct diagonally opposite pairs of vertices of V, each of which hit $(V_1, \dots, V_r)$, then one of the following two options must hold:

(I) If (λ, λ') is any diagonally opposite pair of vertices of V hitting $(V_1, \dots, V_r)$ and $\mathscr{G}_i$ is any family such that $\forall \ell \in [r]$, $V_\ell \notin \mathscr{G}_i$ then $\mathscr{G}_i$ is piercable by (λ, λ').

(II) There exists $i \in [n]$, and there exists $V_{r+1} \in \mathscr{G}_i$, such that there are at most 2^{k-1} distinct diagonally opposite pairs of vertices of V, each of which hit the colorful tuple $(V_1, \ldots, V_r, V_{r+1})$.

Proof. If $k = 0$, i.e, $(V_1, \ldots, V_r)$ is pierceable by a unique diagonally opposite pair of vertices of V, then Option 3.13 must be true. Because, otherwise we get a colorful $(r+1)$-tuple which is not pierceable by any diagonally opposite pair of vertices of V, a contradicting Condition 3.13.

Now let us consider the case when $k > 0$.

If Option 3.13 is true then there is nothing to prove. Otherwise there exists (λ, λ'), a diagonally opposite pair of vertices of V, piercing $(V_1, \ldots, V_r)$ and $\exists i \in [n]$ such that $\forall \ell \in [r]$, $V_\ell \notin \mathscr{G}_i$ and $\mathscr{G}_i$ is not piercable by (λ, λ'). So $\exists V_{r+1} \in \mathscr{G}_i$ such that $V_{r+1} \cap \{\lambda, \lambda'\} = \emptyset$. Then by Lemma 3.11, V_{r+1} must be disjoint from at least two faces of V and by Lemma 3.12, for every $\ell \in [r]$, there is a face of V meeting V_ℓ but disjoint from V_{r+1}. So by Condition 3.13 V_{r+1} contains at most 2^{d-2} vertices of V, none of which are diagonally opposite. Hence without loss of generality we may assume the following:

(a) if (μ, μ'), a diagonally opposite pair of vertices of V, hit $(V_1, \ldots, V_r)$ and $\mu = (\mu_1, \ldots, \mu_d)$ then $\forall j \in [d],\ j > k$ we have $\mu_j = \alpha_j$,

(b) $\lambda = (\lambda_1, \ldots, \lambda_d)$ such that $\forall j \in [d],\ j \geq k$ we have $\lambda_j = \alpha_j$ and

(c) if $\gamma = (\gamma_1, \ldots, \gamma_d)$ is a vertex of V contained in V_{r+1} then $\gamma_k = \beta_k$ and $\gamma_d = \alpha_d$.

So if (μ, μ') are diagonally opposite pair of vertices of V, hitting $(V_1, \ldots, V_r,\ V_{r+1})$ and $\mu = (\mu_1, \ldots, \mu_d)$ then we must have $\mu_k = \beta_k$ and $\forall j \in [d],\ j > k,\ \mu_j = \alpha_j$.

Hence $(V_1, \ldots, V_r,\ V_{r+1})$ is pierceable by at most 2^{k-1} diagonally opposite pair of vertices of V and Option 3.13 is true. □

3.3.2 Proof of Theorem 3.7

Proof. If every colorful pair of boxes intersect, then by Lemma 3.9 there exists $i \in [3d]$ and for each $j \in [3d],\ j \neq i,\ \exists B_j \in \mathscr{F}_j$ such that $\mathscr{F}_i \cup \{B_i \mid j \in [3d],\ j \neq i\}$ is two pierceable and we are done.

So from now on we assume that there is at least one non-intersecting colorful pair of boxes.

Let r be the largest integer such that we have the following: $\exists J = \{j_1 < \cdots < j_r\} \subseteq [d]$, $I = \{i_1 < \cdots < i_{2r}\} \subset [3d]$ and $\forall i \in I$, $\exists B_i \in \mathscr{F}_i$ such that the following conditions are satisfied.

(i) For each $j_k \in J$, $\pi_{j_k}(B_{i_{2k-1}}) = [\alpha_{i_{2k-1}}, \beta_{i_{2k-1}}]$, $\pi_{j_k}(B_{i_{2k}}) = [\alpha_{i_{2k}}, \beta_{i_{2k}}]$ and $\beta_{i_{2k-1}} < \alpha_{i_{2k}}$,

(ii) For any $B \in \mathscr{F}_m$, $m \notin I$ and for any $j_k \in J$, $\pi_{j_k}(B) \cap \{\beta_{i_{2k-1}}, \alpha_{i_{2k}}\} \neq \emptyset$,

(iii) For any $B \in \mathscr{F}_m$, and $B' \in \mathscr{F}_n$, such that $m, n \notin I, m \neq n$ and $\forall j \in [d]$ with $j \notin J$, we have $\pi_j(B) \cap \pi_j(B') \neq \emptyset$

By assumption there is at least one non-intersecting colorful pair of boxes, say B_1, B_2. So $\exists j \in [d]$ such that $\pi_j(B_1) \cap \pi_j(B_2) = \emptyset$. Let

$$(B_a, B_b) = \text{argmax}\{\text{dist}_j \left(B, B'\right) \mid (B, B') \text{ is a colorful pair } \}.$$

Take $J = \{j\}$ and $I = \{i_1, i_2\}$, where $B_a \in \mathscr{F}_{i_1}$ and $B_b \in \mathscr{F}_{i_2}$. Note that the above three conditions are satisfied by B_a and B_b. Thus there exists an $r > 0$ that satisfies the above conditions. Thus the r in the above definition is well-defined.

Now we only need to handle the following two cases:

- *Case 1:* $r = d$, and
- *Case 2:* $r < d$.

Case 1: $r = d$**.** In this case $J = [d]$. Without loss of generality assume that $I = [2d]$. Therefore $(B_1, \ldots, B_{2d})$, as a colorful tuple, must be pierceable by at least one diagonally opposite pair of vertices of the box $D = [\beta_1, \alpha_2] \times \cdots \times [\beta_{2d-1}, \alpha_{2d}]$. We also have the following

(a) for any $B \in \mathscr{F}_i, i > 2d$, $\exists \lambda \in B$, where (λ, λ') is a pair of diagonally opposite vertices of D pierceing $(B_1, \ldots, B_{2d})$, and

(b) every colorful d-tuple from $\mathscr{F}_i, i > 2d$, is pierceable by at least one diagonally opposite pair of vertices of D, which hit $(B_1, \ldots, B_{2d})$ also, i.e, every colorful $3d$-tuple is pierceable by at least one diagonally opposite pair of D. (In fact, every colorful pair of boxes from $\mathscr{F}_i, i > 2d$, is pierceable by at least one diagonally opposite pair of vertices of D, which hit $(B_1, \ldots, B_{2d})$ also. For otherwise, if $\exists B_i \in \mathscr{F}_i, B_j \in \mathscr{F}_j$, for some $i, j > 2d$ and $i \neq j$, such that $\{B_1, \ldots, B_{2d}, B_i, B_j\}$ can not be hit by any diagonally opposite pair of vertices of D then we can actually show that $\{B_1, \ldots, B_{2d}, B_i, B_j\}$ is not 2-pierceable, a contradiction.)

Now there is at most 2^{d-1} diagonally opposite pairs of vertices of D, such that each pair hits $(B_1,\ldots,B_{2d})$. By Lemma 3.13 we see that there is a colorful $(2d+p)$-tuple $(B_1,\ldots,B_{2d},\, B_{i_1},\ldots,B_{i_p})$ such that $p<d$ and for every diagonally opposite pair of vertices (λ,λ') of D piercing the $(2d+p)$-tuple $(B_1,\ldots,B_{2d},\, B_{i_1},\ldots,B_{i_p})$ and for every family $\mathscr{F}_i$ that has no representative in the $(2d+p)$-tuple mentioned, we have: $\mathscr{F}_i$ is pierceable by (λ,λ'), i.e, $\forall B\in\mathscr{F}_i,\ B\cap\{\lambda,\lambda'\}\neq\emptyset$.

Without loss of generality, we assume that $\forall k\in\{1,\ldots,p\}$, $i_k=2d+k$ and $B_{i_k}\in\mathscr{F}_{i_k}$. Now if we take any diagonally opposite pair of vertices (λ,λ') of D that pierce $(B_1,\ldots,B_{2d},\, B_{2d+1},\ldots,B_{2d+p})$ and any $k\in[3d-1]$, $k>2d+p$, and any $B_k\in\mathscr{F}_k$, then we have $\mathscr{F}_{3d}\cap\{B_i\mid i\in[3d-1]\}$ is pierceable by (λ,λ'). So we are done for this case.

***Case 2:* $r<d$.** Without loss of generality, we assume that, $J=[r]$ and $I=[2r]$.

Suppose r_1 is the largest integer (r_1 may be 0) such that $\exists J_1\subset[d]\setminus[r]$ with $|J_1|=r_1$ such that $\exists I_1\subset[3d]\setminus[2r]$ with $|I_1|\leq r_1$ and for each $i\in I_1$, $\exists B_i\in\mathscr{F}_i$ such that the following conditions are satisfied:

1. For each $j\in J_1$, $\exists V_j\in\{B_1,\ldots,B_{2r}\}$ and $V'_j\in\{B_1,\ldots,B_{2r}\}\cup\{B_i\mid i\in I_1\}$ such that $\pi_j(V_j)=[\alpha_j,\beta_j]$, $\pi_j(V'_j)=[\alpha'_j,\beta'_j]$ and $\beta_j<\alpha'_j$.
2. For each $j\in J_1$ and for each $B\in\mathscr{F}_i, i\notin[2r]\cup I_1$, we have $\pi_j(B)\cap\{\beta_j,\alpha'_j\}\neq\emptyset$.

Without loss of generality we assume that $J_1=\{r+1,r+2,\ldots,r+r_1\}$ and $I_1=\{2r+1,2r+2,\ldots,2r+r_1\}$. Now $\forall i\leq 2r+r_1$, let $B'_i=\pi_1(B_i)\times\cdots\times\pi_{r+r_1}(B_i)$ and for each $i>2r+r_1$, we define $\mathscr{F}'_i=\{\pi_1(B)\times\cdots\times\pi_{r+r_1}(B)\mid B\in\mathscr{F}_i\}$. We also define $D'=[\beta_1,\alpha_2]\times\cdots\times[\beta_{2r-1},\alpha_{2r}]\times[\beta_{r+1},\alpha'_{r+1}]\times\cdots\times[\beta_{r+r_1},\alpha'_{r+r_1}]$.

Then using the similar argument as in Case 1, we may assume that there is a diagonally opposite pair (λ,λ') of D' and $\forall i\in\{2r+r_1+1,\ldots,3r+2r_1-1\}$, $\exists B'_i\in\mathscr{F}'_i$ such that

$$\forall k\geq 3r+2r_1,\ \mathscr{F}'_k\cup\{B'_i\mid i<3r+2r_1\}\text{ is pierceable by }(\lambda,\lambda'). \tag{3.1}$$

Now if $r+r_1=d$, then $\forall i\leq 2r+r_1$, $B'_i=B_i$ and $\forall k\geq 3r+2r_1$, $\mathscr{F}'_k=\mathscr{F}_k$. So $\forall i\in\{3r+2r_1+1,\ldots,3d-1\}$, if we take any $B_i\in F_i$, then by Equation (3.1) we get that $\mathscr{F}_{3d}\cup\{B_i\mid i\in[3d-1]\}$ is pierceable by (λ,λ').

Otherwise, $r+r_1<d$. Then for any $j\in[d]\setminus[r+r_1]$ and any $B\in\{B_i\mid i\in[3r+2r_1]\}$ and $B'\in\{B_i\mid i\in[3r+2r_1]\}\bigcup(\cup_{i\geq 3r+2r_1}\mathscr{F}_i)$, we get that

$$\pi_j(B)\cap\pi_j(B')\neq\emptyset. \tag{3.2}$$

Now since for any $B_m\in\mathscr{F}_m, B_n\in\mathscr{F}_n$, with $m,n\geq 3r+2r_1$, $m\neq n$ and for every $k\in[d]$, $k>r+r_1$ we have, $\pi_k(B_m)\cap\pi_k(B_n)\neq\emptyset$, so by Lemma 3.10, for every $k\in[d]$, with $k>r+r_1$, $\exists$ at most one $i'(k)\in[3d]$, with $i'(k)\geq 3r+2r_1$, such that for all $i\in[3d]$, with $i\geq 3r+2r_1, i\neq i'(k)$,

$$\exists \text{ a hyperplane of the form } x_k=const.=\mu_k \text{ (say), which meets every member of } \mathscr{F}_i. \tag{3.3}$$

Now let $K=\{i'(k)\mid k\in[d]\setminus[r+r_1]\}$. Without loss of generality, we assume that $3d\notin K$. Then from Equations (3.1) and (3.3) we get that

$$\mathscr{F}_{3d} \text{ is pierceable by } (L,M), \tag{3.4}$$

where $L=(\lambda_1,\dots,\lambda_{r+r_1},\ \mu_{r+r_1+1},\dots,\mu_d)$ and $M=(\lambda'_1,\dots,\lambda'_{r+r_1},\ \mu_{r+r_1+1},\dots,\mu_d)$.

Now for each $i\in\{2r+r_1+1,\dots,3r+2r_1-1\}$, we take $B_i\in\mathscr{F}_i$ corresponding to $B'_i\in\mathscr{F}'_i$ and for each $i\in\{3r+2r_1,\dots,3d-1\}$, we take any $B_i\in\mathscr{F}_i$. We claim that $\mathscr{G}=\mathscr{F}_{3d}\cup\{B_i\mid i\in[3d-1]\}$ is two pierceable.

Take any $j\in[d]\setminus[r+r_1]$ and any $V,\tilde{V}\in\mathscr{G}$. If at least one of $V,\tilde{V}$ is in $\{B_i\mid i\in[3r+2r_1-1]\}$, then using Equation (3.2), we get $\pi_j(V)\cap\pi_j(\tilde{V})\neq\emptyset$. If $V,\tilde{V}\in\mathscr{F}_{3d}$, then using Equation (3.4), we get $\pi_j(V)\cap\pi_j(\tilde{V})\neq\emptyset$. Otherwise using Condition (*iii*), we get $\pi_j(V)\cap\pi_j(\tilde{V})\neq\emptyset$. So in any case we get $\pi_j(V)\cap\pi_j(\tilde{V})\neq\emptyset$. Then by Helly's Theorem,

$$\exists\nu_j\in\mathbb{R} \text{ such that } \forall B\in\mathscr{G},\ \nu_j\in\pi_j(B). \tag{3.5}$$

Again if we take any $B\in\mathscr{G}$ and consider $B'=\pi_1(B)\times\cdots\times\pi_{r+r_1}(B)$ then using Equation (3.1), we get

$$B'\cap\{\lambda,\lambda'\}\neq\emptyset. \tag{3.6}$$

So using Equations (3.5) and (3.6), we get that $\mathscr{G}$ is pierceable by (L_1,M_1), where

$$L_1=(\lambda_1,\dots,\lambda_{r+r_1},\ \nu_{r+r_1+1},\dots,\nu_d) \text{ and } M_1=(\lambda'_1,\dots,\lambda'_{r+r_1},\ \nu_{r+r_1+1},\dots,\mu_d).$$

□

3.4 Extremal examples: Proof of Theorem 3.4

In this section, we will prove Theorem 3.4, that is, for all $d \in \mathbb{N}$ we have $H_c(d,2) > 3d-1$.

Theorem 3.4 (Extremal example)**.** *For every $d \in \mathbb{N}$, there exist non-empty families $\mathscr{F}_1, \ldots, \mathscr{F}_{3d-1}$ of axis-parallel boxes in $\mathbb{R}^d$ such that*

- *every colorful $(3d-1)$-tuple is 2-pierceable, and*
- *for each $i \in [3d-1]$, $\mathscr{F}_i$ is not 2-pierceable.*

The above result together with Theorem 3.7, implies that for all $d \in \mathbb{N}$, we have $H_c(d,2) = 3d$.

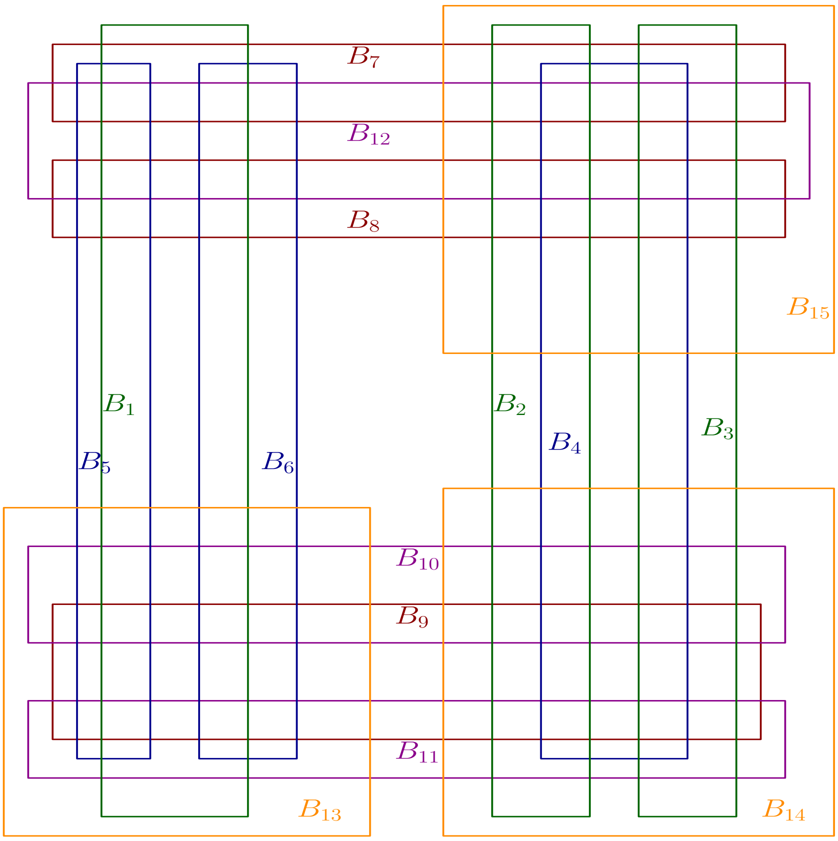

Fig. 3.2 In the figure, for each $i \in \{1, 2, 3, 4, 5\}$, $\mathscr{F}_i = \{B_{3i}, B_{3i-1}, B_{3i-2}\}$.

Before going into the proof of the above theorem, let us first consider the families $\mathscr{F}_1, \ldots, \mathscr{F}_5$ of rectangles in $\mathbb{R}^2$ given in Figure 3.2. Observe that any colorful 5-tuple is 2-pierceable but none of the families are 2-pierceable. The main difficulty in the proof of

Theorem 3.4 is to construct families $\mathscr{F}_1,\ldots,\mathscr{F}_{3d-1}$, for $d \geq 4$, satisfying both the conditions mentioned. With this in mind, we shall now give the details of the proof of Theorem 3.4.

Proof of Theorem 3.4. First observe that, if $d = 1$ and we take each of $\mathscr{F}_1,\mathscr{F}_2$ to be a collection of at least 3 pairwise disjoint intervals then all the conditions in the theorem are trivially satisfied. So we are interested, when $d \geq 2$. The main idea of the construction of families $\mathscr{F}_1,\ldots,\mathscr{F}_{3d-1}$ satisfying the two conditions of Theorem 3.4 is the following: using colorful $2d$-tuples from the first $2d$-families we will identify a box D such that every colorful $(3d-1)$-tuple will be hit by at least one pair of diagonally opposite vertices of D. The construction becomes interesting for $d \geq 4$, as in these cases after identifying D, we have to construct the remaining more than two families keeping in mind that every colorful $(d-1)$-tuple from these families are hit by at least one pair of diagonally opposite vertices of D.

For each $i \in [d]$, let $\mathscr{F}_{2i-1} = \left\{B_{2i-1,1}, B_{2i-1,2}, B_{2i-1,3}\right\}$, and $\mathscr{F}_{2i} = \{B_{2i,1}, B_{2i,2}, B_{2i,3}\}$, where

- $B_{2i-1,1} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; -4 \leq x_i \leq -2, \text{ and} \forall j \neq i, -4 \leq x_j \leq 4\right\}$
- $B_{2i-1,2} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; -1 \leq x_i \leq 0, \text{ and} \forall j \neq i, -4 \leq x_j \leq 4\right\}$
- $B_{2i-1,3} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; 1.25 \leq x_i \leq 3, \text{ and} \forall j \neq i, -4 \leq x_j \leq 4\right\}$
- $B_{2i,1} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; 2 \leq x_i \leq 4, \text{ and} \forall j \neq i, -4 \leq x_j \leq 4\right\}$
- $B_{2i,2} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; 1 \leq x_i \leq 1.5, \text{ and} \forall j \neq i, -4 \leq x_j \leq 4\right\}$
- $B_{2i,3} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; -2.5 \leq x_i \leq -1.5, \text{ and} \forall j \neq i, -4 \leq x_j \leq 4\right\}$

For each $i \in [d-1]$, let $\mathscr{F}_{2d+i} = \left\{B_{2d+i,1}, B_{2d+i,2}, B_{2d+i,3}\right\}$ where

- $B_{2d+i,1} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; -5 \leq x_1 \leq 0.25, -5 \leq x_{d-(i-1)} \leq 0.25, \text{ and} \forall j \notin \{1, d-(i-1)\}, -5 \leq x_j \leq 5\right\}$
- $B_{2d+i,2} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; -5 \leq x_1 \leq 0.25, 0.75 \leq x_{d-(i-1)} \leq 5, \text{ and} \forall j \notin \{1, d-(i-1)\}, -5 \leq x_j \leq 5\right\}$
- $B_{2d+i,3} = \left\{(x_1,\ldots,x_d) \in \mathbb{R}^d \;\middle|\; 0.75 \leq x_1 \leq 5, 0.75 \leq x_{d-(i-1)} \leq 5, \text{ and} \forall j \notin \{1, d-(i-1)\}, -5 \leq x_j \leq 5\right\}$

Observe that for all $k \in [3d-1]$ and $B, B' \in \mathscr{F}_k$, we have $B \cap B' = \emptyset$. Hence, $\mathscr{F}_k$ is not 2-pierceable.

B_{2j-1}	B_{2j}	α_j	β_j
$B_{2j-1,1}$	$B_{2j,1}$	-2	2
$B_{2j-1,1}$	$B_{2j,2}$	-2	1
$B_{2j-1,1}$	$B_{2j,3}$	-2	2
$B_{2j-1,2}$	$B_{2j,1}$	0	2
$B_{2j-1,2}$	$B_{2j,2}$	0	1
$B_{2j-1,2}$	$B_{2j,3}$	-1	1
$B_{2j-1,3}$	$B_{2j,1}$	0	2
$B_{2j-1,3}$	$B_{2j,2}$	0	1.5
$B_{2j-1,3}$	$B_{2j,3}$	-2	2

Table 3.1 Table for α_j and β_j.

Now let $(B_1, B_2, B_3, \ldots, B_{3d-1})$ be a colorful $(3d-1)$-tuple, where $B_k \in \mathscr{F}_k$ for all $k \in [3d-1]$. For each $j \in [d]$, depending upon B_{2j-1}, B_{2j}, we shall give two real numbers α_j, β_j in Table 3.1 such that for each $j \in [d]$ and $k \in [3d-1]$, $\pi_j(B_k) \cap \{\alpha_j, \beta_j\} \neq \emptyset$ and every diagonally opposite pair of vertices of the box $D = \{(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \mid \alpha_j \leq x_j \leq \beta_j, \forall j \in [d]\}$ hit $\{B_1, B_2, \ldots, B_{2d}\}$.

B_{2d+k}	$x_{d-(k-1)}$	$y_{d-(k-1)}$
$B_{2d+k,1}$	α_k	β_k
$B_{2d+k,2}$	β_k	α_k
$B_{2d+k,3}$	α_k	β_k

Table 3.2 Table gives the co-ordinates of X and Y except the first co-ordinate

Now let X, Y be two diagonally opposite vertices of the box D, where $X = (x_1, x_2, \ldots, x_d)$ and $Y = (y_1, y_2, \ldots, y_d)$ such that $x_1 = \alpha_1$ and $y_1 = \beta_1$. In Table 3.2 we define the other co-ordinates $x_{d-(k-1)}$ and $y_{d-(k-1)}$ of X and Y respectively depending upon B_{2d+k} for all $k \in [d-1]$. Observe that $\{X, Y\}$ hits the collection of boxes $\{B_1, B_2, \ldots, B_{3d-1}\}$. This completes the proof of the theorem. $\square$

Chapter 4

Piercing Boxes with Finitely-many Axis-parallel Lines and Flats

4.1 Introduction

4.1.1 A short survey on $(\aleph_0, q)$-Theorem for k-flats

Let $\mathscr{F}$ be a collection of subsets of $\mathbb{R}^d$. Recall that, a family of subsets $\mathscr{T}$ of $\mathbb{R}^d$ is said to be *transversals* for $\mathscr{F}$, if for each $F \in \mathscr{F}$ there exists a $\tau \in \mathscr{T}$ such that $F \cap \tau \neq \emptyset$. Additionally, we say $\mathscr{T}$ is a *finite-size transversal* of $\mathscr{F}$ if $\mathscr{T}$ is a transversal of $\mathscr{F}$ and the size of the set $\mathscr{T}$ is finite. $\mathscr{T}$ can be a family of points or lines or hyperplanes or k-flats[1] etc. If $\mathscr{T}$ is a collection of k-flats (or, axis-parallel k-flats) in $\mathbb{R}^d$ then $\mathscr{T}$ is called a *k-transversal* of $\mathscr{F}$ (or, axis-parallel k-transversal of $\mathscr{F}$), and additionally, if k is 0 or $d-1$, then $\mathscr{T}$ will be called a point transversal or a hyperplane transversal of $\mathscr{F}$ respectively.

A family $\mathscr{F}$ is said to satisfy the $(\aleph_0, q)$-property, where $q := q(k,d) \geq k+2$, for k-transversal if for all infinite sequence $\{C_i\}_{i\in\mathbb{N}}$ of sets from $\mathscr{F}$ there exist q sets from the sequence that can be pierced by a single k-flat. Keller and Perles [110] were the first to introduce this new infinite variant of the (p,q)-property and proved the $(\aleph_0, k+2)$-Theorem for k-transversal for family of (r,R)-fat[2] sets.

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[1] By k-flats we mean k-dimensional affine subspaces of $\mathbb{R}^d$.

[2] A family $\mathscr{F}$ of convex sets in $\mathbb{R}^d$ is said to be (r,R)-*fat* if each member of $\mathscr{F}$ contains a ball of radius r and is contained in a ball of radius R.

Theorem 4.1 (($(\aleph_0, k+2)$-Theorem for k-transversal [110]). *Let $\mathcal{F}$ be a family of (r,R)-fat compact convex sets in $\mathbb{R}^d$ and $0 \le k < d$. If $\mathcal{F}$ satisfies the $(\aleph_0, k+2)$-property for k-flats, then $\mathcal{F}$ has a finite-size k-transversal.*

An interesting observation here is that there is no universal constant c such that we can conclude that whenever $\mathcal{F}$ satisfies $(\aleph_0, k+2)$-property with respect to k-transversals, $\mathcal{F}$ is pierceable by c many k-flats. For example, for any given $c \in \mathbb{N}$, we can construct finitely many (r,R)-fat convex sets that are not pierceable by c number of k-flats and then take $\mathcal{F}$ as infinitely many copies of those sets. (For example, when $k = 0$, one can take $c+1$ pairwise disjoint balls of radius r.) Then $\mathcal{F}$ satisfies $(\aleph_0, k+2)$-property with respect to k-transversals, but $\mathcal{F}$ is not pierceable by c many k-flats.

For point transversals Keller and Perles [110] showed that the *fatness* assumption can be relaxed.

Theorem 4.2 (($(\aleph_0, 2)$-Theorem for point transversal [110]). *Suppose $\mathcal{F}$ is a family of closed balls in $\mathbb{R}^d$ (with no restriction on the radii) satisfying the $(\aleph_0, 2)$-property for points. Then $\mathcal{F}$ has a finite-size point transversal.*

In a later work, Keller and Perles [111] introduced the notion of *near-balls*. A collection $\mathcal{F}$ of subsets of $\mathbb{R}^d$ is called a *collection of near-balls* if there exists a constant $\alpha \ge 1$ such that for all $C \in \mathcal{F}$, there exist closed balls $B(p_C, r_C)$ and $B(p_C, R_C)$ satisfying the following two conditions:

$$B(p_C, r_C) \subseteq C \subseteq B(p_C, R_C), \text{ and} \tag{4.1}$$

$$R_C \le \min\{r_C + \alpha, \alpha r_C\}. \tag{4.2}$$

Keller and Perles [111] proved the following result:

Theorem 4.3 (($(\aleph_0, k+2)$-Theorem for near balls [111]). *Let $\mathcal{F}$ be a near-ball collection of compact sets from $\mathcal{F}$ satisfying the $(\aleph_0, k+2)$-property for k-flats. Then $\mathcal{F}$ has a finite-size k-transversal.*

Observe that Theorem 4.1 directly follows from Theorem 4.3, and the following corollary of Theorem 4.3 is a generalization of Theorem 4.2 to general k-flats.

Corollary 4.4 (($(\aleph_0, k+2)$-Theorem for closed balls [111]). *Suppose $\mathcal{F}$ is a family of closed balls in $\mathbb{R}^d$ (with no restriction on the radii) satisfying the $(\aleph_0, k+2)$-property for k-flats. Then $\mathcal{F}$ has a finite-size k-transversal.*

Jung and Pálvölgyi [100] developed a new framework using which one can lift (p,q)-type theorems for finite families to prove new $(\aleph_0, q)$-Theorems for infinite families, and also provided new proofs of already existing results. Chakraborty, Ghosh and Nandi [42, 41] have also given different generalizations of the above theorems with a number of interesting *no-go theorems*.

Suppose for each $n \in \mathbb{N}$, $\mathscr{F}_n$ is a family of compact convex sets in $\mathbb{R}^d$. Then an infinite sequence $\{B_n\}_{n\in\mathbb{N}}$ from $\{\mathscr{F}_n\}$ is said to be a *colorful* or *heterochromatic* one if each B_n comes from a different family and it is said to be *strictly heterochromatic* if each family $\mathscr{F}_n$ has a representative in $\{B_n\}_{n\in\mathbb{N}}$. Again an infinite sequence $\{B_n\}_{n\in\mathbb{N}}$ of convex sets in $\mathbb{R}^d$ is said to be *k-dependent* if it contains $k+2$ sets that can be pierced by a single k-flat. Otherwise, the sequence $\{B_n\}_{n\in\mathbb{N}}$ is said to be *k-independent*. A collection $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of families of convex sets in $\mathbb{R}^d$ is said to satisfy *heterochromatic* $(\aleph_0, k+2)$*-property* if every heterochromatic sequence from $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is k-dependent. Similarly, we say that $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ satisfies *weak heterochromatic* $(\aleph_0, k+2)$*-property* if every strictly heterochromatic sequence from $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is k-dependent. Chakraborty, Ghosh and Nandi [42] have proved a colorful variant of the Theorem 4.3 for families of closed convex sets satisfying heterochromatic $(\aleph_0, k+2)$-property with respect to k-flats. To state the result we will need some additional notations and definitions. For a set $S \subseteq \mathbb{R}^d$, the *out-radius* R_S is defined as the radius of the smallest closed ball containing S, and *in-radius* r_S of S is defined as the radius of the largest closed ball contained in S. The *condition number* $\sigma(S)$ of a set S is defined as $\sigma(S) := \frac{r_S}{R_S}$. For a family $\mathscr{F}$ of subsets of $\mathbb{R}^d$, *condition number* $\sigma(\mathscr{F})$ of $\mathscr{F}$ is defined as the number $\sigma(\mathscr{F}) := \min\{\sigma(S) \,:\, S \in \mathscr{F}\}$.

Theorem 4.5 (Countably Colorful $(\aleph_0, k+2)$-Theorem [42])**.** *For each $n \in \mathbb{N}$, suppose $\mathscr{F}_n$ is a family of closed convex sets in $\mathbb{R}^d$ satisfying the heterochromatic $(\aleph_0, k+2)$-property and*

$$\liminf_{n\to\infty} \sigma(\mathscr{F}_n) > 0.$$

Then there is a finite set of k-flats that pierces all but finitely many $\mathscr{F}_n$'s.

As a direct corollary of the above result we get the following generalization of Theorem 4.3 by Keller and Perles.

Corollary 4.6 (Generalization of Theorem 4.1 and Theorem 4.3 [42])**.** *Let d be a natural number and k be a non-negative integer with $0 \le k \le d-1$. If $\mathscr{F}$ be a family of closed convex sets in $\mathbb{R}^d$ with $\sigma(\mathscr{F}) > 0$ and $\mathscr{F}$ satisfy $(\aleph_0, k+2)$-property with respect to k-flats then $\mathscr{F}$ has a constant size k-transversal.*

Using a different argument, Jung and Pálvölgyi [100] have given an alternate proof of the above corollary. Note that the proof technique of Chakraborty, Ghosh and Nandi [42] uses entirely geometric arguments and builds on the approach of Keller and Perles [110].

Chakraborty, Ghosh and Nandi [42] also showed that the convexity assumption in Theorem 4.5 and Corollary 4.6 cannot be weakened.

Theorem 4.7 ([42])**.** *There is an infinite sequence $\{B_n\}_{n\in\mathbb{N}}$ of compact connected sets in $\mathbb{R}^2$ with condition number $\rho > 0$ such that $\{B_n\}_{n\in\mathbb{N}}$ satisfies the $(\aleph_0, 2)$ property, but $\{B_n\}_{n\in\mathbb{N}}$ is not pierceable by a finite number of points.*

We say that a collection $\mathscr{F}$ of compact convex sets has *bounded diameter* if there exists an $R > 0$ with $\mathrm{diam}(C) < R$ for all $C \in \mathscr{F}$. We say a collection $\mathscr{F}$ of compact convex sets is *unbounded* if $\bigcup_{C\in\mathscr{F}} C$ is unbounded. As defined in [18], we say $y \in \mathbb{S}^{d-1}$ to be a *limiting direction* of an unbounded collection $\mathscr{F}$ of compact convex sets in $\mathbb{R}^d$ with bounded diameter if there exists an infinite sequence of points $\{s_n\}_{n\in\mathbb{N}}$ in $\mathbb{R}^d$ and an infinite sequence of sets $\{S_n\}_{n\in\mathbb{N}}$ from $\mathscr{F}$ satisfying the following conditions:

- for all $n \in \mathbb{N}$, $S_n \in \mathscr{F}$,
- for all $n \neq m$ in $\mathbb{N}$, $S_n \neq S_m$,
- for all $n \in \mathbb{N}$, $s_n \in S_n$, and
- $\lim_{n\to\infty} \|s_n\| = \infty$ and $\lim_{n\to\infty} \frac{s_n}{\|s_n\|} = y$

For an unbounded collection $\mathscr{F}$ of compact convex sets with bounded diameter in $\mathbb{R}^d$, *limiting direction set LDS*$(\mathscr{F})$ of $\mathscr{F}$ is defined as

$$LDS(\mathscr{F}) := \left\{ y \in \mathbb{S}^{d-1} : y \text{ is a limiting direction of } \mathscr{F} \right\}.$$

An unbounded collection $\mathscr{F}$ of compact convex sets with bounded diameter is called *k-unbounded* if the vector space spanned by set $LDS(\mathscr{F})$ has dimension at least k. As already mentioned earlier in the introduction, Santaló [133] established the impossibility of getting a Helly Theorem for transversals of arbitrary convex sets with k-flats. Aronov, Goodman and Pollack [18] proved the Helly Theorem for k-flats for k-unbounded families of arbitrary convex compact sets. Chakraborty, Ghosh and Nandi [42] showed the impossibility of getting $(\aleph_0, k+2)$-Theorem for k-unbounded families of arbitrary convex compact bodies.

Theorem 4.8 (No-go theorem for k-unbounded families [42])**.** *There exists an infinite family $\mathscr{F}$ of compact convex sets in $\mathbb{R}^3$ satisfying the following:*

(a) $\mathscr{F}$ is one-unbounded,

(b) $\mathscr{F}$ satisfies the $(\aleph_0, 3)$-property with respect to lines, and

(c) $\mathscr{F}$ does not have a finite line transversal.

Given Theorem 4.5 it is natural to ask the following question:

Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be an infinite sequence of families of *nicely shaped* closed convex sets satisfying *weak heterochromatic* $(\aleph_0, k+2)$*-property* for $0 \leq k \leq d-1$.
Does there exist a family $\mathscr{F}_m$ from the sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$
that has a finite size k-transversal?

Chakraborty, Ghosh and Nandi [42] showed the following colorful $(\aleph_0, 2)$-Theorem for families of (R, r)-fat convex sets satisfying weak heterochromatic $(\aleph_0, 2)$-property.

Theorem 4.9 (Strongly Colorful $(\aleph_0, 2)$-Theorem [42])**.** *Let $d \in \mathbb{N}$, and for each $n \in \mathbb{N}$, suppose $\mathscr{F}_n$ is a family of (r, R)-fat closed convex sets in $\mathbb{R}^d$. If $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ satisfies the weak heterochromatic $(\aleph_0, 2)$-property then there exist at least two distinct families $\mathscr{F}_i$ and $\mathscr{F}_j$ with $i \neq j$ and each of them can be pierced by finitely many points.*

The following proposition from [42] shows that Theorem 4.9 is tight for families satisfying weak heterochromatic $(\aleph_0, 2)$-property.

Proposition 4.10 (Tightness of Strongly Colorful Theorem 4.9 [42])**.** *For each $n \in \mathbb{N}$, there exists a family $\mathscr{F}_n$ of units balls in $\mathbb{R}^2$ such that $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ satisfies the following*

(i) $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ satisfies the weak heterochromatic $(\aleph_0, 2)$-property, and

(ii) exactly two distinct families $\mathscr{F}_i$ and $\mathscr{F}_j$ from $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ are pierceable by finitely many points, and none of the other families from $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ are not pierceable by a finite number of points.

Chakraborty, Ghosh and Nandi [42] showed the impossibility of getting an analog of Theorem 4.9 for all natural numbers k satisfying $0 < k < d$ even if we restrict ourselves to families of unit balls.

Theorem 4.11 (No-go result for strongly colorful $(\aleph_0, k+2)$-Theorem)**.** *Suppose $d \in \mathbb{N}$ and $d > 1$. Then for any non-negative integer k with $1 \leq k \leq d-1$, there exists a collection $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ satisfying the following:*

(i) for all $n \in \mathbb{N}$, $\mathscr{F}_n$ is a family of closed unit balls in $\mathbb{R}^d$ such that $\mathscr{F}_n$ cannot be pierced by finitely many k-flats and

(ii) every strictly heterochromatic sequence $\{B_n\}_{n\in\mathbb{N}}$ of $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ is k-dependent.

Chakraborty, Ghosh and Nandi [41] also proved $(\aleph_0, d+1)$-Theorem for hyperplane transversal in $\mathbb{R}^d$.

Theorem 4.12 ($(\aleph_0, d+1)$-Theorem for hyperplane transversal [41])**.** *Let $\mathscr{F}$ be a family of compact connected sets in $\mathbb{R}^d$ such that any infinite subcollection contains $d+1$ members* *that can be pierced by a single hyperplane. Then the family $\mathscr{F}$ has a finite hyperplane transversal.*

Note that, contrary to Theorem 4.7, here in Theorem 4.12, instead of convexity only connectedness assumption is good enough, whereas we have already seen that in Theorem 4.5 and Corollary 4.6, the convexity assumption cannot be dropped. We get the following corollary directly from the above theorem.

Corollary 4.13 ($(\aleph_0, d+1)$-Theorem for hyperplane transversal of convex sets [41])**.** *Let $\mathscr{F}$ be a family of compact convex sets in $\mathbb{R}^d$ such that any infinite subcollection contains $d+1$ members that can be pierced by a single hyperplane. Then the family $\mathscr{F}$ has a finite hyperplane transversal.*

Chakraborty, Ghosh and Nandi [41] have also proved the following heterochromatic generalization of Theorem 4.12.

Theorem 4.14 (Heterochromatic $(\aleph_0, d+1)$-Theorem for hyperplane transversal [41])**.** *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be a sequence of families of compact connected sets such that every heterochromatic sequence with respect to $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ has $d+1$ sets that are pierceable by a single hyperplane. Then all but finitely many $\mathscr{F}_n$'s are pierceable by finitely many hyperplanes.*

Additionally, from Theorem 4.12 and Theorem 4.14, Chakraborty, Ghosh and Nandi [41] also showed the existence of a finite-size collection of hyperplanes for the whole sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ that is a transversal for all but finitely many families from the sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$.

Theorem 4.15 (Stronger variant of Theorem 4.14 [41])**.** *Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ be a sequence of families of compact connected sets such that every heterochromatic sequence with respect to $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ has $d+1$ sets that are pierceable by a single hyperplane. Then there exists a finite collection $\mathscr{H}$ of hyperplanes in $\mathbb{R}^d$ such that $\mathscr{H}$ pierces all but finitely many families from $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$.*

Chakraborty, Ghosh and Nandi [41] complement the above result by showing the following no-go theorem which establishes the optimality of the generalization of Theorem 4.12.

Theorem 4.16 (*No-go* theorem for strongly heterochromatic generalization [41])**.** *There exists a sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ of families of compact convex sets in $\mathbb{R}^d$ such that*

- *for all $n \in \mathbb{N}$, $\mathscr{F}_n$ is not pierceable by finitely many hyperplanes, and*
- *every strictly heterochromatic sequence with respect to $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ contains $d+1$ balls which can be pierced by a single hyperplane.*

The proof of the above no-go theorem shows the impossibility of getting a strongly heterochromatic $(\aleph_0, d+1)$-Theorem for hyperplanes even when the compact connected sets in question have a nice shape when $d > 1$ as closed unit balls do. Additionally, Chakraborty, Ghosh and Nandi [41] proved the following stronger impossibility result.

Theorem 4.17 (Stronger variant of Theorem 4.16 [41])**.** *There exists a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of compact convex sets in $\mathbb{R}^d$ satisfying the following two conditions:*

- *for all $n \in \mathbb{N}$, $\mathcal{F}_n$ is not pierceable by finitely many hyperplanes, and*
- *for any $m \in \mathbb{N}$ and every sequence $\{B_n\}_{n=m}^{\infty}$ of compact connected sets in $\mathbb{R}^d$, where $B_i \in \mathcal{F}_i$ for all $i \geq m$, there exists a hyperplane in $\mathbb{R}^d$ that pierces at least $d+1$ sets in the sequence.*

4.2 Our results: $(\aleph_0, 2)$-Theorem for boxes and axis-parallel k-flats

Studying geometric, combinatorial, algorithmic, and even topological properties of a collection of axis-parallel boxes has been an active area of research in Discrete and Computational Geometry [47, 143, 49, 128, 132, 63, 44, 121, 46, 71]. **Note that for the rest of the chapter unless otherwise stated explicitly when we say boxes we mean axis-parallel boxes.** More specifically, in the context of transversal theory boxes have been one of the fundamental objects of study [59, 81, 69, 3, 127, 108, 129, 5, 130, 57, 116, 124, 125, 53, 112, 48, 142]. We build on these previous works by studying the infinite Hadwiger-Debrunner (p,q)-problem for boxes and axis-parallel flats. Again observe that Theorem 4.1, Theorem 4.3, and Theorem 4.5 all require the convex bodies to have *nice* shape. In Theorem 4.1 and Theorem 4.3, we need the convex bodies to be (r,R)-fat and closed balls respectively, and in Theorem 4.5 we require the condition number of the families to be bounded away from zero. It is natural to ask if one can prove a result, similar to the above mentioned results, for some *natural* families of convex bodies that are neither fat nor do they have a lower bound on the condition number. Axis-parallel boxes are neither fat nor do they have a lower bound on the condition number away from zero.

An infinite collection $\mathcal{G}$ of subsets of $\mathbb{R}^d$ is said to satisfy the $(\aleph_0, 2)$*-property for axis-parallel k-flats* if for every infinite sequence $\{S_n\}_{n\in\mathbb{N}}$, where $S_n \in \mathcal{G}$ for all $n \in \mathbb{N}$, there exist S_i and S_j with $i \neq j$ from $\{S_n\}_{n\in\mathbb{N}}$ that can be pierced by a single axis-parallel k-flat in $\mathbb{R}^d$. In this chapter, we prove the following $(\aleph_0, 2)$-Theorem for boxes.

Theorem 4.18 ($(\aleph_0, 2)$-Theorem for boxes and axis-parallel k-flats)**.** *Let $\mathcal{F}$ be an infinite collection of boxes in $\mathbb{R}^d$ and $0 \leq k < d$. If $\mathcal{F}$ satisfies the $(\aleph_0, 2)$-property for axis-parallel k-flats, then $\mathcal{F}$ has a finite axis-parallel k-transversal.*

The following impossibility result shows that

- Theorem 4.18 is not true if the boxes are not axis-parallel.
- Assuming that the convex bodies have non-zero volumes is not sufficient to establish a $(\aleph_0, \cdot)$-theorem.
- A natural question related to Theorem 4.1 was, whether an $(\aleph_0, k+2)$-theorem can be obtained for families of convex sets without the fatness assumption. But the following impossibility result shows that this is not possible, even for $k = 0$.

Theorem 4.19 (Impossibility result)**.** *There exists an infinite family $\mathscr{F}$ of pairwise intersecting rectangles (not necessarily axis-parallel) in $\mathbb{R}^2$ with bounded diameters and non-zero volumes that cannot be pierced by any finite collection of points in $\mathbb{R}^2$.*

Jung and Pálvölgyi [100] have shown the following result about convex sets and axis-parallel hyperplanes.

Theorem 4.20 (Jung and Pálvölgyi [100]: Convex sets and hyperplane transversal)**.** *Let $\mathscr{F}$ be a collection of compact convex sets in $\mathbb{R}^d$ that satisfies the $(\aleph_0, 2)$-property for axis-parallel hyperplanes. Then $\mathscr{F}$ has a finite hyperplane transversal.*

From Theorem 4.18 we directly get the following generalization of Theorem 4.20.

Corollary 4.21. *Let $\mathscr{F}$ be a collection of compact connected sets in $\mathbb{R}^d$ that satisfies the $(\aleph_0, 2)$-property for axis-parallel hyperplanes. Then there exists a finite collection of axis-parallel hyperplanes that is a transversal for $\mathscr{F}$.*

Proof. For each compact convex set $F \in \mathscr{F}$, we consider the box

$$B_F = \pi_{x_1}(F) \times \pi_{x_2}(F) \times \cdots \times \pi_{x_d}(F)$$

(where $\pi_{x_i}(F)$ is the projection of F on the i-th axis, $1 \leq i \leq d$), see Figure 4.1. Then by substituting $k = d-1$ in Theorem 4.18, we get that the family $\mathscr{B}_{\mathscr{F}} = \{B_F \mid F \in \mathscr{F}\}$ has a constant size axis-parallel hyperplane transversal. And by construction of the family $\mathscr{B}_{\mathscr{F}}$ we can conclude that the same family of axis-parallel hyperplanes will also serve as a transversal for the family $\mathscr{F}$. □

Also note that the results of Keller and Parles [110] were either for near-balls or for closed balls (rotationally symmetric convex sets). Boxes don't have to be fat and they are definitely not rotationally symmetric. Due to these reasons, the proof techniques used in this section are distinct from the ones in Keller and Parles [110], Jung and Pálvölgyi [100] and and also Chakraborty, Ghosh and Nandi [42].

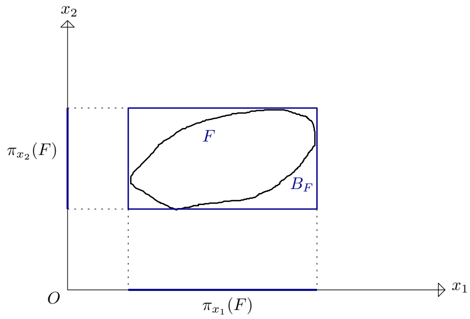

Fig. 4.1 Smallest box containing F.

An infinite sequence $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is said to satisfy the *colorful* $(\aleph_0, 2)$*-property for axis-parallel k-flats* if for every colorful sequence $\{S_n\}_{n\in\mathbb{N}}$ with respect to $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$, there exist two sets S_i and S_j with $i \neq j$ from $\{S_n\}_{n\in\mathbb{N}}$ which can be pierced by a single axis-parallel k-flat. We prove the following *colorful* generalization of Theorem 4.18.

Theorem 4.22 (Colorful generalization of Theorem 4.18). *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an infinite sequence of families of axis-parallel boxes in $\mathbb{R}^d$ and $0 \leq k < d$. If the infinite sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ satisfies the colorful $(\aleph_0, 2)$-property for axis-parallel k-flats, then there exists an $\ell \in \mathbb{N}$ such that $\mathcal{F}_\ell$ has a constant size axis-parallel k-transversal.*

Using the observations of Chakraborty, Ghosh and Nandi [42, Section 3.1], we get that the heterochromatic $(\aleph_0, q)$-theorem implies existence of finite hitting set (for details, see Section 4.5). Hence we get the following stronger result.

Corollary 4.23. *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an infinite sequence of families of axis-parallel boxes in $\mathbb{R}^d$ and let $0 \leq k < d$. If $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ satisfies the colorful $(\aleph_0, 2)$-property for axis-parallel k-flats, then there exists an $N \in \mathbb{N}$ and a finite collection $\mathcal{K}$ of axis-parallel k-flats that is a transversal for all $\mathcal{F}_n$ with $n \geq N$.*

Additionally, we can also prove the following colorful generalization of Corollary 4.21.

Corollary 4.24 (Generalization of Corollary 4.21). *Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an infinite sequence of families of compact connected sets in $\mathbb{R}^d$. If $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ satisfies the colorful $(\aleph_0, 2)$-property for axis-parallel hyperplanes, then there exists an $N \in \mathbb{N}$ and a finite collection $\mathcal{K}$ of axis-parallel hyperplanes that is a transversal for all $\mathcal{F}_n$ with $n \geq N$.*

4.2.1 Definitions

- An *axis-parallel k-flat K* in $\mathbb{R}^d$ is a k-dimensional affine space such that $\exists I \subset [d]$ with $|I| = d - k$ and $\forall x \in K$, $\forall i \in I$, $x_i = a_i$, for some fixed number $a_i \in \mathbb{R}$.

- A collection $\mathscr{F}$ of axis-parallel boxes in $\mathbb{R}^d$ is called a *k-collection* if $\mathscr{F}$ is not pierceable by any finite family $\mathscr{K}$ of axis-parallel k-flats in $\mathbb{R}^d$.
- A set $S \subset \mathbb{R}^d$ is said to be an *i-strip* if S is of the form $S = S_1 \times S_2 \times \cdots \times S_d$, where $\exists I \subseteq [d]$ with $|I| = i$ such that $\forall j \in [d] \setminus I$, $S_j = [a_j, b_j] \subset \mathbb{R}$ and $\forall j \in I, S_j = \mathbb{R}$.

  Clearly, any finite box in $\mathbb{R}^d$ is a 0-*strip* and a d-*strip* is the whole $\mathbb{R}^d$. We define a (-1)-*strip* to be the empty set.

4.3 Proof of Theorem 4.18

In this section, we will prove Theorem 4.18. We will give the proof of the hyperplane transversal case first, and then prove the general case.

Theorem 4.25 (($\aleph_0, 2$)-Theorem for boxes and axis-parallel hyperplanes)**.** *Let $\mathscr{F}$ be a collection of axis-parallel boxes in $\mathbb{R}^d$ such that $\mathscr{F}$ is not pierceable by any finite number of axis-parallel hyperplanes. Then $\mathscr{F}$ contains an infinite sequence $\{B_n\}_{n\in\mathbb{N}}$ of boxes such that no two boxes from this sequence can be hit by an axis-parallel hyperplane.*

Observe that the statement of the above theorem is in contrapositive form with respect to the statement of Theorem 4.18 (when restricted to $k = d-1$).

Proof of Theorem 4.25. By our assumption, $\mathscr{F}$ is a $(d-1)$-collection. Let $d-d'$ be the smallest value of $i \in \{0,1,\ldots,d\}$ such that there is an i-strip S and a $(d-1)$-collection $\mathscr{F}_1 \subset \mathscr{F}$ such that $\forall B \in \mathscr{F}_1$, $B \subset S$. Without loss of generality, let a $(d-d')$-strip containing $\mathscr{F}_1$ be

$$S_1 = [a_{1,1}, b_{1,1}] \times [a_{1,2}, b_{1,2}] \times \cdots \times [a_{1,d'}, b_{1,d'}] \times \mathbb{R}^{d-d'}.$$

Case-I: $d' = 0$.

In this case, $S_1 = \mathbb{R}^d$. Pick any $B_1 \in \mathscr{F}$, and let

$$\mathscr{B}_1 := \{B \in \mathscr{F} \mid \forall i \in [d], \pi_i(B) \cap \pi_i(B_1) = \emptyset\}.$$

We claim that $\mathscr{B}_1$ is a $(d-1)$-collection. If not, then $\mathscr{B}_1' := \mathscr{F} \setminus \mathscr{B}_1$ would be a $(d-1)$-collection. Then, $\forall B \in \mathscr{B}_1'$, there is a $i \in [d]$ depending on B such that $\pi_i(B) \cap \pi_i(B_1) \neq \emptyset$. Consider the d $(d-1)$-strips

$$\pi_1(B_1) \times \mathbb{R}^{d-1}, \ldots, \mathbb{R}^{j-1} \times \pi_j(B_1) \times \mathbb{R}^{d-j}, \ldots, \mathbb{R}^{d-1} \times \pi_d(B_1).$$

Since the boundaries of these $(d-1)$-strips consist of finitely many axis-parallel hyperplanes, the collection of boxes in $\mathscr{B}_1'$ that intersect the boundaries of these $(d-1)$-strips cannot be a $(d-1)$-collection. Let $\mathscr{B}_1''$ be the collection of all boxes in $\mathscr{B}_1'$ which do not intersect the boundaries of the above $(d-1)$-strips, that is,

$$\mathscr{B}_1'' := \bigcup_{0\le j\le d} \left\{B \in \mathscr{F} \mid B \subset \mathbb{R}^{j-1} \times \pi_j(B_1) \times \mathbb{R}^{d-j}\right\}.$$

As $\mathscr{B}_1''$ is a $(d-1)$-collection therefore there exists a $j \in \{0,\dots,d\}$ such that

$$\left\{B \in \mathscr{F} \mid B \subset \mathbb{R}^{j-1} \times \pi_j(B_1) \times \mathbb{R}^{d-j}\right\}$$

is a $(d-1)$-collection. But, this would mean that there exists a $(d-1)$-strip that contains a $(d-1)$-collection which is a subset of $\mathscr{F}$. Observe that this contradicts the assumption that $d' = 0$. Therefore, $\mathscr{B}_1$ is a $(d-1)$-collection and we pick $B_2 \in \mathscr{B}_1$. Clearly, $\forall i \in [d]$, we have $\pi_i(B_2) \cap \pi_i(B_1) = \emptyset$. Suppose we have constructed m boxes $B_1, B_2, \dots, B_m$ such that $B_i \in \mathscr{F}$ for all $i \in [m]$ and no two distinct B_i, B_j can be intersected by an axis-parallel hyperplane. Then, as argued above, we can choose a $B_{m+1} \in \mathscr{F}$ that does not intersect any of the finitely many $(d-1)$-strips of the form $\mathbb{R}^{j-1} \times \pi_j(B_i) \times \mathbb{R}^{d-j}$ for $j \in [d]$ and $i \in [m]$. Continuing this process we can construct an infinite sequence of boxes $\{B_n\}_{n\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ we have $B_i \in \mathscr{F}$ and any axis-parallel hyperplane can intersect at most one box from this infinite sequence.

Case-II: $d' > 0$.

Let $\forall j \in [d']$,

$$c_{1,j} := \frac{b_{1,j} + a_{1,j}}{2},\ A^1_{1,j} := \left[a_{1,j}, c_{1,j}\right],\ \text{and}\ A^2_{1,j} := \left[c_{1,j}, b_{1,j}\right].$$

Now consider the d' hyperplanes $\left\{H_j\right\}_{j\in[d']}$ where $H_j = \left\{x \in \mathbb{R}^d \mid x_j = c_{1,j}\right\}$, $\forall j \in [d']$. Suppose

$$\hat{\mathscr{F}} = \left\{B \in \mathscr{F} \mid B \cap H_j \neq \emptyset \text{ for some } j \in [d']\right\},$$

that is, $\hat{\mathscr{F}}$ is a collection of boxes from $\mathscr{F}_1$ which are pierced by at least one hyperplane from $\left\{H_j\right\}_{j\in[d']}$. As $\mathscr{F}_1$ is a $(d-1)$-collection, $\mathscr{F}_1 \setminus \hat{\mathscr{F}}$ must also be a $(d-1)$-collection. The d' hyperplanes in $\left\{H_j\right\}_{j\in[d']}$ split S_1 into $2^{d'}$ many $(d-d')$-strips. Therefore there exist a $(d-1)$-collection $\mathscr{F}_2$ and a $(d-d')$-strip S_2 such that

(a) $\mathscr{F}_2 \subseteq \mathscr{F}_1 \setminus \hat{\mathscr{F}} \subseteq \mathscr{F}_1$,

(b) $S_2 = A_{1,1}^{i_1} \times A_{1,2}^{i_2} \times \dots A_{1,d'}^{i_{d'}} \times \mathbb{R}^{d-d'}$ where $i_j \in \{1,2\}$ for all $j \in [d']$, and

(c) $\forall B \in \mathscr{F}_2$, we have $B \subseteq S_2$.

Again, as S_2 is of the form $[a_{2,1}, b_{2,1}] \times \cdots \times [a_{2,d'}, b_{2,d'}] \times \mathbb{R}^{d-d'}$ we split S_2, in the same way as S_1 was split, to obtain a $(d-d')$-strip $S_3 = [a_{3,1}, b_{3,1}] \times \cdots \times [a_{3,d'}, b_{3,d'}] \times \mathbb{R}^{d-d'} \subseteq S_2$, say, such that there is a $(d-1)$-collection $\mathscr{F}_3 \subseteq \mathscr{F}_2$ with the property $\forall B \in \mathscr{F}_3$, then $B \subseteq S_3$. Proceeding this way, we get that, for each $n \in \mathbb{N}$, there is a $(d-1)$-collection $\mathscr{F}_n$ and a $(d-d')$-strip $S_n = [a_{n,1}, b_{n,1}] \times [a_{n,d'}, b_{n,d'}] \times \mathbb{R}^{d-d'}$ such that the following hold:

(i) $\forall B \in \mathscr{F}_n$, $B \subseteq S_n$,

(ii) $\mathscr{F}_{n+1} \subseteq \mathscr{F}_n$,

(iii) $S_{n+1} \subseteq S_n$ and

(iv) $\forall i \in [d']$, $\frac{b_{n,i} - a_{n,i}}{2} = b_{n+1,i} - a_{n+1,i}$.

Therefore, for each $i \in [d']$, $\exists c_i \in \mathbb{R}$ such that

$$\lim_{n \to \infty} a_{n,i} = \lim_{n \to \infty} b_{n,i} = c_i \tag{4.3}$$

Now for each $i \in [d']$, consider the axis-parallel hyperplane $h_i = \{x \in \mathbb{R}^d \mid x_i = c_i\}$ and $\mathscr{H} = \{h_i \mid i \in [d']\}$.

As $\mathscr{F}_1$ is a $(d-1)$-collection, $\mathscr{H}$ is not a transversal of $\mathscr{F}_1$. Therefore $\exists B_1 \in \mathscr{F}_1$ such that $\forall i \in [d']$, $B_1 \cap h_i = \emptyset$. Using the fact that $\forall i \in [d']$ we have $c_i \notin \pi_i(B_1)$ and Equation (4.3), we can show that $\exists t_1 \in \mathbb{N}$ such that for $\forall i \in [d']$ and $\forall n \geq t_1$,

$$[a_{n,i}, b_{n,i}] \cap \pi_i(B_1) = \emptyset.$$

Again from the definition of d', we know that $\forall j \in [d] \setminus [d']$, the $(d-d'-1)$-strip $S_{t_1} \cap \mathbb{R}^{j-1} \times \pi_j(B_1) \times \mathbb{R}^{d-j}$ contains no $(d-1)$-collection. So, we can pick a $B_2 \in \mathscr{F}_{t_1}$ such that $\forall j \in [d]$, we have $B_2 \cap \mathbb{R}^{j-1} \times \pi_j(B_1) \times \mathbb{R}^{d-j} = \emptyset$ and $\forall h \in \mathscr{H}$, we have $B_2 \cap h = \emptyset$.

Continuing this process, suppose we have constructed m boxes $B_1, \dots, B_m$ satisfying the following properties:

Prop-1: for all $i \in [m]$ we have $B_i \in \mathscr{F}$,

Prop-2: no axis-parallel hyperplane pierces more than one box from the set $\{B_1, \dots, B_m\}$, and

Prop-3: $\forall s \in [m]$ and $\forall h_i \in \mathscr{H}$, we have $B_s \cap h_i = \emptyset$.

Since $\forall h_i \in \mathscr{H}$ and $\forall j \in [m]$, $B_j \cap h_i \neq \emptyset$, there exists a $t_m \in \mathbb{N}$ such that $\forall n \geq t_m$, $\forall r \in [m]$ and $\forall i \in [d']$, we have $[a_{n,i}, b_{n,i}] \cap \pi_i(B_r) = \emptyset$. Let $\mathscr{B}_m$ the collection of boxes in $\mathscr{F}_{t_m}$ which do not intersect the $(d-d'-1)$-strip $S_{t_m} \cap \mathbb{R}^{j-1} \times \pi_j(B_r) \times \mathbb{R}^{d-k}$ for any $j \in [d] \setminus [d']$ and any $r \in [m]$. Observe that $\mathscr{B}_m$ will be a $(d-1)$-collection. Pick $B_{m+1} \in \mathscr{B}_m$ so that B_{m+1} does not intersect h_i for any $h_i \in \mathscr{H}$. Therefore we now have $m+1$ boxes $B_1, \ldots, B_{m+1}$ that satisfy **Prop-1**, **Prop-2**, and **Prop-3** given above. Again observe that if we continue this process we will be able to construct an infinite sequence of boxes $\{B_n\}_{n\in\mathbb{N}}$ such that for all $i \in \mathbb{N}$ we have $B_i \in \mathscr{F}$ and any axis-parallel hyperplane can intersect at most one box from this infinite sequence. This concludes the proof of the theorem. □

Now we prove Theorem 4.18.

Theorem 4.18 (($\aleph_0, 2$)-Theorem for boxes and axis-parallel k-flats)**.** *Let $\mathscr{F}$ be an infinite collection of boxes in $\mathbb{R}^d$ and $0 \leq k < d$. If $\mathscr{F}$ satisfies the $(\aleph_0, 2)$-property for axis-parallel k-flats, then $\mathscr{F}$ has a finite axis-parallel k-transversal.*

Proof. We use induction on d to prove the theorem. Note that, using Theorem 4.25, we get that Theorem 4.18 is true for $d = 1$ and $k = 0$. Now we assume that Theorem 4.18 is true for all $d < m$ and $k < d$. We now prove it when $d = m$. Again using Theorem 4.25, we get that the result is true when $k = m-1$. So we assume that $k < m-1$. We shall prove the result by contradiction, that is, we shall show that if $\mathscr{F}$ is a k-collection, then $\mathscr{F}$ contains an infinite sequence of boxes such that no two of them can be pierced by an axis-parallel k-flat.

Case 1: *There is a k-collection $\mathscr{F}' \subset \mathscr{F}$ which is intersected by an axis-parallel hyperplane*

Without loss of generality, let $h = \{x \in \mathbb{R}^m \mid x_1 = a\}$ be an axis-parallel hyperplane that intersects all sets in a k-collection $\mathscr{F}'$ that is a subset of $\mathscr{F}$. Let $\widetilde{B} = B \cap h$ for all $B \in \mathscr{F}'$. Let $\widetilde{\mathscr{F}'} = \{\widetilde{B} \mid B \in \mathscr{F}'\}$. Then $\widetilde{\mathscr{F}'}$ is a k-collection in an $(m-1)$-flat, so by the induction hypothesis, there is a sequence $\{\widetilde{B_n}\}_{n=1}^{\infty}$, no two of which can be pierced by an axis-parallel k-flat on h. We claim that the corresponding sequence $\{B_n\}_{n=1}^{\infty}$ is the required sequence. If not, then let two distinct B_s, B_j be pierced by an axis-parallel k-flat K such that $x \in K$ whenever $x_{r_i} = a_i$ for all $i \in [m-k]$, where

$$r_1 < r_2 < \cdots < r_{m-k}.$$

If $r_1 = 1$, then as all sets are axis-parallel boxes, the axis-parallel k-flat K', defined as

$$K' := \{x \in \mathbb{R}^m \mid x_1 = a, \text{ and } x_{r_i} = a_i \text{ for all } i = 2, \ldots, m-k\},$$

lies on h and intersects $\widetilde{B}_s, \widetilde{B}_j$, which is a contradiction. If $r_1 > 1$, then $K \cap h$ is an axis-parallel $(k-1)$-flat on h intersecting B_s, B_j, which is a contradiction since every axis-parallel $(k-1)$-flat is a subset of an axis-parallel k-flat. Therefore, $\{B_n\}_{n=1}^{\infty}$ is the required sequence.

Case 2: *There is no k-collection that is a subset of $\mathscr{F}$ which is intersected by an axis-parallel hyperplane*

In this case, $\mathscr{F}$ is also a $(d-1)$-collection, because if $\mathscr{F}$ could be pierced by finitely many axis-parallel hyperplanes, then at least one of the hyperplanes would pierce a k-collection. Then, by Theorem 4.25, we can find a sequence $\{B_n\}_{n=1}^{\infty}$ in $\mathscr{F}$ such that no distinct pair B_i, B_j with $i, j \in \mathbb{N}$, $i > j$ can be pierced by an axis-parallel hyperplane. Since an axis-parallel k-flat is a subset of an axis-parallel hyperplane, the sequence $\{B_n\}$ satisfies the condition of Theorem 4.18.

□

4.4 Proof of Theorem 4.19

In this section, we will prove Theorem 4.19.

Theorem 4.19 (Impossibility result)**.** *There exists an infinite family $\mathscr{F}$ of pairwise intersecting rectangles (not necessarily axis-parallel) in $\mathbb{R}^2$ with bounded diameters and non-zero volumes that cannot be pierced by any finite collection of points in $\mathbb{R}^2$.*

We will first fix some notations that will be used in the proof of Theorem 4.19.

- We will denote by $\mathbb{S}^1$ as the unit circle center at the origin in $\mathbb{R}^2$.
- For all $n \in \mathbb{N}$, we will denote by θ_n the angle $\pi/2^n$.
- Let $a = (0,1)$, $b = (1,0)$, $c = (1,1)$, and $u_n := (\cos\theta_n, \sin\theta_n)$ for all $n \in \mathbb{N}$.
- For all $n \in \mathbb{N}$, let a_n and b_n be the intersection points of the tangent to the circle $\mathbb{S}^1$ at the point u_n with the lines $x_2 = 1$ and $x_1 = 1$ respectively.
- Let $\mathscr{R} := \left\{(\bar{x}_1, \bar{x}_2) \in \mathbb{R}^2 \;\middle|\; 0 \le \bar{x}_1 \le 1 \text{ and } 0 \le \bar{x}_2 \le 1\right\}$.

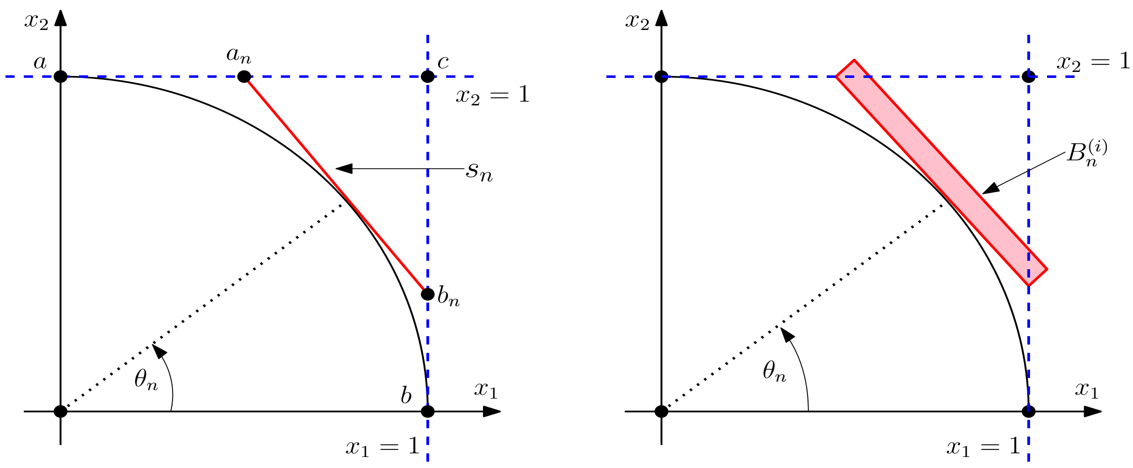

Fig. 4.2 The first figure shows the constructed line segments s_n, and the second figure shows the rectangles $B_n^{(i)}$.

- For all $n \in \mathbb{N}$, s_n denotes the closed line segment joining the points a_n and b_n. Let

$$B_n^{(i)} := \left\{ \overline{x} + \lambda u_n \mid \overline{x} \in s_n, \lambda \in \left[0, 1/2^i\right] \right\},$$

  and $\mathscr{F}_i := \left\{ B_n^{(i)} \mid n \in \mathbb{N} \right\}$. See Figure 4.2.

Consider the family of rectangles

$$\mathscr{F} := \bigcup_{i \in \mathbb{N}} \mathscr{F}_i.$$

Observe that for all m and n in $\mathbb{N}$, $s_m \cap s_n \neq \emptyset$, see Figure 4.3. Therefore, $B_n^{(i)} \cap B_m^{(j)} \neq \emptyset$. We will now show that $\mathscr{F}$ cannot be pierced by a finite set of points. To reach a contradiction suppose there exists a finite set $C \subsetneq \mathbb{R}^2$ such that for all $B_n^{(i)} \cap C \neq \emptyset$. Without loss of generality we can assume that the points $a = (0,1)$, $b = (1,0)$, and $c = (1,1)$ are in the set C.

First, we want to show that it will be sufficient to consider the case where

$$C \subseteq \mathscr{R} \setminus B^o(O,1).$$

Observe that by the construction of $\mathscr{F}$ if C contains a point $p = (p_1, p_2)$ with $p_1 < 0$ or $p_2 < 0$ then p does not hit any rectangles in $\mathscr{F}$. Therefore, we can assume that for all $p = (p_1, p_2) \in C$ we have $p_1 \geq 0$ and $p_2 \geq 0$. Now, if C contains a point $p = (p_1, p_2)$ with $p_1 > 1$ then $C \cup \{\widetilde{p}\} \setminus \{p\}$ where $\widetilde{p} = (1, p_2)$ is a point transversal of $\mathscr{F}$. Similarly, if C contains a point $q = (q_1, q_2)$ with $q_2 > 1$ then $C \cup \{\widetilde{q}\} \setminus \{q\}$ where $\widetilde{q} = (q_1, 1)$ is also a point

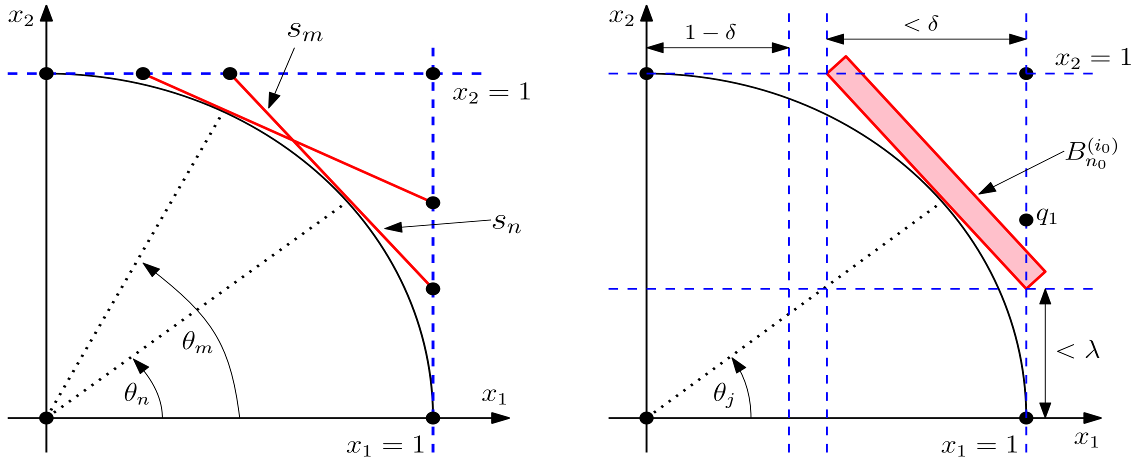

Fig. 4.3 The first figure shows that any two segments always intersect, and the second figure shows how the rectangle $B_{n_0}^{(i_0)}$ is disjoint from the set C.

transversal of $\mathscr{F}$. Observe that none of the points in $B^o(O,1)$ can hit any rectangles in $\mathscr{F}$. Therefore, combining all the above observations we can safely assume that $C \subseteq \mathscr{R} \setminus B^o(O,1)$.

Let C_1 be the subset of C obtained by intersecting C with the line $\ell_1 : x_1 - 1 = 0$, and $C_2 := C \setminus C_1$. Also, let

$$\delta := \min_{q \in C_2} \text{dist}\,(q, \ell_1) \text{ and } \lambda := \min_{p \in C_1 \setminus \{b\}} \|p - b\|.$$

Observe that as

$$\lim_{n \to \infty} a_n = (1,1) \text{ and } \lim_{n \to \infty} b_n = (1,0),$$

therefore there exists $n_0 \in \mathbb{N}$ such that

$$\|a_{n_0} - c\| < \delta \text{ and } \|b_{n_0} - b\| < \lambda.$$

See Figure 4.3.

Let q_1 be the point in the set $C_1 \setminus \{b\}$ closest to b, and let ℓ_{n_0} be the line tangent to the circle $\mathbb{S}^1$ at the point u_n. Note that there exits $i_0 \in \mathbb{N}$ such that

$$1/2^{i_0} < \text{dist}\,(q_1, \ell_{n_0}),$$

see Figure 4.3. Now observe that the rectangle $B_{n_0}^{(i_0)}$ and the set C is disjoint. Therefore, C is not piercing $\mathscr{F}$ and we have reached a contradiction. This completes the proof of Theorem 4.19.

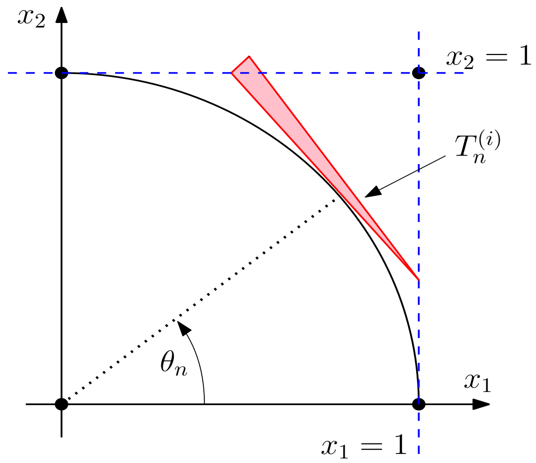

Fig. 4.4 The figure shows the triangles $T_n^{(i)}$ that can be used to establish Theorem 4.26.

Observe that the construction can also be done along similar lines for non-axis parallel right-angled triangles in $\mathbb{R}^2$, and therefore we can prove the following result.

Theorem 4.26. *There exists a family $\mathscr{F}$ of pairwise intersecting right-angled triangles in $\mathbb{R}^2$ that cannot be pierced by any finite collection of points in $\mathbb{R}^2$. Note that the triangles in $\mathscr{F}$ have bounded diameters and non-zero volumes.*

4.5 Heterochromatic $(\aleph_0, q)$-Theorem implies existence of finite hitting set

Here we show that the existence of a heterochromatic $(\aleph_0, q)$-theorem implies the existence of a finite hitting set. Let $\mathscr{B}$ and $\mathscr{S}$ be collections of nonempty sets in $\mathbb{R}^d$. For any $\mathscr{F} \subseteq \mathscr{B}$, we say $\mathscr{F}$ is *finitely pierceable by $\mathscr{S}$* if there exists a finite set $S \subseteq \mathscr{S}$ such that for all $B \in \mathscr{F}$, we have $\bigcup_{A \in S} A \cap B \neq \emptyset$. A sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ with $\mathscr{F}_n \subseteq \mathscr{B}$ $\forall n \in \mathbb{N}$ is said to have the *heterochromatic $(\aleph_0, q)$-property with respect to $\mathscr{S}$* if for every heterochromatic sequence $\{B_n\}_{n\in\mathbb{N}}$ chosen from $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$, we have a set $A \in \mathscr{S}$ such that there are q distinct B_n's for which $B_n \cap A \neq \emptyset$. We say that the *heterochromatic $(\aleph_0, q)$-theorem for $\mathscr{B}$ with respect to $\mathscr{S}$ holds* if for any $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ with $\mathscr{F}_n \subseteq \mathscr{B}$ satisfying the heterochromatic $(\aleph_0, q)$-property with respect to $\mathscr{S}$, there exists an $n \in \mathbb{N}$ for which $\mathscr{F}_n$ is finitely pierceable by $\mathscr{S}$. We say that the *infinite heterochromatic $(\aleph_0, q)$-theorem for $\mathscr{B}$ with respect to $\mathscr{S}$ holds* if for any $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ with $\mathscr{F}_n \subseteq \mathscr{B}$ satisfying the heterochromatic $(\aleph_0, q)$-property with respect to $\mathscr{S}$, all but finitely many $\mathscr{F}_n$'s are finitely pierceable by $\mathscr{S}$.

Lemma 4.27 (Heterochromatic property lemma)**.** *The heterochromatic* $(\aleph_0, q)$*-theorem for* $\mathscr{B}$ *with respect to* $\mathscr{S}$ *implies the infinite heterochromatic* $(\aleph_0, q)$*-theorem for* $\mathscr{B}$ *with respect to* $\mathscr{S}$*.*

Proof. Let the heterochromatic $(\aleph_0, q)$-theorem for $\mathscr{B}$ with respect to $\mathscr{S}$ hold. Let $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$, $\mathscr{F}_n \subseteq \mathscr{B}$, be a sequence that satisfies the heterochromatic $(\aleph_0, q)$-property with respect to $\mathscr{S}$. If possible, let $\{\mathscr{F}_{n_i}\}_{i\in\mathbb{N}}$ where $\{n_i\}_{i\in\mathbb{N}}$ is a strictly increasing sequence in $\mathbb{N}$ be such that no $\mathscr{F}_{n_i}$ is finitely pierceable by $\mathscr{S}$. As $\{\mathscr{F}_{n_i}\}_{i\in\mathbb{N}}$ also satisfies the heterochromatic $(\aleph_0, q)$-property with respect to $\mathscr{S}$, there is an $i \in \mathbb{N}$ for which $\mathscr{F}_{n_i}$ is finitely pierceable by $\mathscr{S}$. But this is a contradiction, and therefore, the result follows. □

So, we shall be using the phrases *heterochromatic* $(\aleph_0, q)$*-theorem* and *infinite heterochromatic* $(\aleph_0, q)$*-theorem* interchangeably. It is trivially seen that for some $\mathscr{B}$ and $\mathscr{S}$ satisfying the heterochromatic $(\aleph_0, q)$-theorem and any $N \in \mathbb{N}$, we can find a sequence $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ with $\mathscr{F}_n \subseteq \mathscr{B}$ that satisfy the heterochromatic $(\aleph_0, q)$ property and for N distinct values of $n \in \mathbb{N}$, $\mathscr{F}_n$ is not finitely pierceable by $\mathscr{S}$. A simple example is found by setting $\mathscr{B}$ to be the set of all unit balls in $\mathbb{R}^d$, $\mathscr{S}$ to be the set of all points in $\mathbb{R}^d$ and $\mathscr{F}_1 = \cdots = \mathscr{F}_N = \mathscr{B}$ and $\mathscr{F}_n = \{B(\mathscr{O}, 1)\}$ for all $n \in \mathbb{N} \setminus \{1, \ldots, N\}$.

We can show that the heterochromatic $(\aleph_0, k+2)$-theorem implies the existence of a finite hitting set.

Lemma 4.28 (Heterochromatic theorem and a finite-size piercing set)**.** *Let* $\mathscr{B}$ *and* $\mathscr{S}$ *be collections of sets in* $\mathbb{R}^d$ *and* $q \in \mathbb{N}$*, where* $\mathscr{B}$ *satisfies the heterochromatic* $(\aleph_0, q)$*-theorem with respect to* $\mathscr{S}$*. If* $\{\mathscr{F}_n\}_{n\in\mathbb{N}}$ *be a sequence of subsets of* $\mathscr{B}$ *that satisfies the heterochromatic* $(\aleph_0, q)$*-property with respect to* $\mathscr{S}$*, then there exists an* $N \in \mathbb{N}$ *and a finite collection* $\mathscr{K} \subset \mathscr{S}$ *such that* $\forall n > N$ *where* $n \in \mathbb{N}$*,* $\forall C \in \mathscr{F}_n$*, we have* $\bigcup_{K\in\mathscr{K}} K \cap C \neq \emptyset$*.*

Proof. By Lemma 4.27 we have that there is an $N \in \mathbb{N}$ such that every $\mathscr{F}_n$ with $n > N$ is finitely pierceable by $\mathscr{S}$. Let $\mathscr{F} = \bigcup_{n=N}^{\infty} \mathscr{F}_n$. Then for any infinite set $F \subseteq \mathscr{F}$, either F contains infinitely many elements of some $\mathscr{F}_n$ with $n \geq N$, or F contains a heterochromatic sequence with respect to the sequence $\{\mathscr{F}_n\}_{n=N}^{\infty}$. In either case, this means there is an $S \in \mathscr{S}$ that pierces q elements of F. Since the heterochromatic $(\aleph_0, q)$-theorem implies the monochromatic $(\aleph_0, q)$-theorem, $\mathscr{F}$ must be finitely pierceable by $\mathscr{S}$. □

4.6 Conclusion

Given Theorem 4.18 and Theorem 4.22, it is natural to ask if even a *stronger colorful* variant along the lines given below also holds:

Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be an infinite sequence of families of axis-parallel boxes in $\mathbb{R}^d$ and $0 \le k < d$. If for every strictly colorful sequence $\{B_n\}_{n\in\mathbb{N}}$ of $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ there exist two boxes B_i and B_i, with $i \neq j$, from $\{B_n\}_{n\in\mathbb{N}}$ and both B_i and B_j can be pierced by a single axis-parallel k-flat then there exists an $\ell \in \mathbb{N}$ such that $\mathcal{F}_\ell$ has a finite axis-parallel k-transversal.

Chakraborty et al. [40] have given an explicit sequence of families of boxes in $\mathbb{R}^d$ showing the infeasibility of the above statement.

Theorem 4.29 (Impossibility of strong colorful generalization of Theorem 4.18)**.** *For all $d \in \mathbb{N}$ and $0 \le k < d$, there exists an infinite sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of families of boxes in $\mathbb{R}^d$ satisfying the following properties:*

- *for all $n \in \mathbb{N}$, $\mathcal{F}_n$ does not have a finite size axis-parallel k-transversal, and*
- *for any $t \in \mathbb{N}$, every infinite sequence $\{B_n\}_{n\in\mathbb{N}}$, where $B_n \in \mathcal{F}_{n+t}$ for all $n \in \mathbb{N}$, contains two boxes B_i and B_j with $i \neq j$ and both B_i and B_j can be pierced by a single axis-parallel k-flat.*

In the paper by Jung and Pálvölgyi [100], it is mentioned that, Chaya Keller has given an independent construction ensuring the same impossibility.

Chapter 5

Colorful No-Dimensional Helly Theorem for Affine Transversals

5.1 Introduction

Helly Theorem is a fundamental result in discrete and convex geometry. The theorem states that given any finite collection $\mathscr{F}$ of convex sets in $\mathbb{R}^d$, if the sets in every subfamily of $\mathscr{F}$ of size $d+1$ have a point in common, then all the sets in the entire family $\mathscr{F}$ have a point in common. Note that the finiteness of $\mathscr{F}$ can be relaxed if we assume that the convex sets in $\mathscr{F}$ are compact, therefore unless otherwise stated explicitly, we will not assume anything about the cardinality of a collection. Since its discovery [93], Helly Theorem has found multiple applications and generalizations [65, 14, 61, 22].

We say a set $T \subseteq \mathbb{R}^d$ *pierces* a family $\mathscr{F}$ of subsets of $\mathbb{R}^d$ if every set in $\mathscr{F}$ has a non-empty intersection with the set T. A natural generalization of Helly Theorem would be to consider the problem of piercing a family of convex sets with k-flats[1]. Results of the above form are called *Helly-type* results. Generally, we want to show that there exists an integer $h(k,d)$ such that if any $h(k,d)$ convex sets from a collection $\mathscr{F}$ can be pierced by a k-flat then the whole collection $\mathscr{F}$ can be pierced by a k-flat. Unfortunately, Santaló [133] showed the impossibility of getting such a result for even piercing a collection of convex sets by a line. Hadwiger [82] showed that for a countable collection $\mathscr{F}$ of disjoint convex sets in $\mathbb{R}^d$ with non-empty interior and all congruent to a fixed compact convex set C, if every $d+1$ sets from $\mathscr{F}$ can be pierced by a line then the whole family can be pierced by

[1]By k-flat we mean affine subspace of $\mathbb{R}^d$ of dimension k. Note that by lines and hyperplanes, we will mean 1-flats and $(d-1)$-flats respectively.

a line. Later Danzer, Grünbaum and Klee [60] showed that the "congruent" assumption in Hadwiger's result [82] can be weakened if the convex sets have bounded diameters. Aronov, Goodman, Pollack, and Wenger [17] proved the first Helly-type result for hyperplanes about families of *well-separated* compact convex sets of arbitrary shapes in higher dimensions. Later, Aronov, Goodman, and Pollack [18] showed a Helly-type result for k-flat for families of convex bodies that are *unbounded* in k-independent directions.

Adiprasito, Bárány, Mustafa, and Terpai [2] in a breakthrough paper proved the first no-dimensional variant of the classical Helly Theorem:

Theorem 5.1 (Adiprasito, Bárány, Mustafa, and Terpai [2]). *Let $C_1,\dots,C_n$ be convex sets in $\mathbb{R}^d$ and $r\in\{1,\dots,n\}$. For $J\subseteq\{1,\dots,n\}$, let $\mathscr{C}(J):=\bigcap_{j\in J}C_j$ and there exists $b\in\mathbb{R}^d$ such that $B(b,1)$ intersects $\mathscr{C}(J)$ for all $J\subseteq\{1,\dots,n\}$ with $|J|=r$, then there is a point $q\in\mathbb{R}^d$ such that for all $i\in\{1,\dots,n\}$ we have*

$$\operatorname{dist}(q,C_i)\leq\sqrt{\frac{n-r}{r(n-1)}}. \tag{5.1}$$

Observe that for a fixed r and $n\to\infty$, the right-hand side of the above equation approaches $\frac{1}{\sqrt{r}}$.

Note that,

- $\forall q\in\mathbb{R}^d$ and $S\subseteq\mathbb{R}^d$, $\operatorname{dist}(q,S)$ denotes the distance between the point q and the set S, that is, $\operatorname{dist}(q,S):=\inf_{x\in S}\|q-x\|$ and
- $B(x,R)$ denotes the closed Euclidean ball centered at $x\in\mathbb{R}^d$ with radius R.

Adiprasito et al. [2] also proved the following colorful variant of the above theorem.

Theorem 5.2 (Adiprasito, Bárány, Mustafa, and Terpai [2]). *Let $\mathscr{F}_1,\dots,\mathscr{F}_r$ be families of convex sets in $\mathbb{R}^d$ with $r\leq d$, and $b\in\mathbb{R}^d$. Assume that for any r-tuple $(C_1,\dots,C_r)$ with $C_i\in\mathscr{F}_i$ for all $i\in\{1,\dots,r\}$, we have*

$$\left(\bigcap_{1\leq i\leq r}C_i\right)\bigcap B(b,1)\neq\emptyset.$$

Then there exists a point $q\in\mathbb{R}^d$ and $\exists i\in\{1,\dots,r\}$ such that $\forall C\in\mathscr{F}_i$ we have $\operatorname{dist}(q,C)\leq\dfrac{1}{\sqrt{r}}$.

We generalize the above results from points to k-flats. We also prove some impossibility results which establish the optimality of our generalization.

5.2 Our results

Before we can give the statements of our results we need to first introduce some definitions which will be required to state our results. We define the *central projection map* $f : \mathbb{R}^d \to \mathbb{S}^{d-1}$, where $\mathbb{S}^{d-1}$ is the unit sphere centered at O, in the following way: for all $x \in \mathbb{R}^d$

$$f(x) := \frac{x}{\|x\|}.$$

We say $y \in \mathbb{S}^{d-1}$ is a *limiting direction* of the collection $\mathscr{F}$ if there exists a infinite sequence of points $\{s_n\}_{n\in\mathbb{N}}$ and infinite sequence of sets $\{S_n\}_{n\in\mathbb{N}}$ satisfying the following properties:

- for all $n \in \mathbb{N}$, $S_n \in \mathscr{F}$
- for all $n \neq m$ in $\mathbb{N}$, $S_n \neq S_m$ -
- for all $n \in \mathbb{N}$, $s_n \in S_n$
- $\lim\limits_{n\to\infty} \|s_n\| = \infty$ and $\lim\limits_{n\to\infty} f(s_n) = y$

For a collection $\mathscr{F}$ of subsets of $\mathbb{R}^d$, *limiting direction set* $LDS(\mathscr{F})$ of $\mathscr{F}$ is defined as

$$LDS(\mathscr{F}) := \left\{ y \in \mathbb{S}^{d-1} : y \text{ is a limiting direction of } \mathscr{F} \right\}.$$

We will call the collection $\mathscr{F}$ *k-unbounded* if the vector space spanned by set $LDS(\mathscr{F})$ has dimension at least k. Note that k-unbounded definition was first introduced by Aronov, Goodman and Pollack [18]. We prove the following *colorful* Helly Theorem for k-flats using the k-unbounded framework.

Theorem 5.3 (Colorful Helly Theorem for k-flats)**.** *Suppose for each $i \in [d+1]$, $\mathscr{S}_i$ is a k-unbounded collection of compact convex sets in $\mathbb{R}^d$, and there exists $R > 0$ such that $\forall C \in \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{d+1}$ we have* $\mathrm{diam}(C) < R$. *Also, assume that for every $(d+1)$-tuple $(C_1, \ldots, C_{d+1}) \in \mathscr{S}_1 \times \cdots \times \mathscr{S}_{d+1}$ there exists a k-flat H such that H intersects C_i for all $i \in [d+1]$. Then $\exists j \in [d+1]$ and a k-flat $\widetilde{H}$ that intersects every set in $\mathscr{S}_j$.*

Note that the above result is a colorful generalization of the Helly Theorem for k-flats proved by Aronov, Goodman and Pollack [18]. The following theorem is the main technical result in this chapter, and we show that the rest of the results are a direct consequence of this result.

Theorem 5.4 (Colorful No-Dimensional Helly Theorem for k-flats)**.** *Let $\mathscr{F}_1, \ldots, \mathscr{F}_r$ be families of convex sets in $\mathbb{R}^d$, $k < r \leq d$, and the family $\mathscr{F} := \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_r$ of convex sets satisfy the following properties:*

(i) $\exists R > 0$ *such that for all $C \in \mathscr{F}$ we have* $\mathrm{diam}(C) < R$

(ii) $\exists J_k = \{j_1, \ldots, j_k\} \subset [r]$ *and* $\{y_{j_1}, \ldots, y_{j_k}\} \subset \mathbb{S}^{d-1}$ *such that*

- *for all* $i \in J_k$ *we have* $y_i \in LDS(\mathscr{F}_i)$*, and*
- *the collection of vectors* $\{y_{j_1}, \ldots, y_{j_k}\}$ *are linearly independent*

If for any r-tuple $(C_1, \ldots, C_r) \in \mathscr{F}_1 \times \cdots \times \mathscr{F}_r$ *there exists a k-flat that intersects the closed unit ball* $B(0,1)$ *and every convex set* C_i*,* $\forall i \in [r]$*, then there exists a k-flat K and* $\exists j \in [r]$ *such that,* $\forall C \in \mathscr{F}_j$*, we have*

$$\operatorname{dist}(C, K) \leq \sqrt{\frac{1}{r-k}}. \tag{5.2}$$

Observe that by substituting $\mathscr{F}_1 = \mathscr{F}_2 = \cdots = \mathscr{F}_r = \mathscr{F}$ in Theorem 5.4 we get the following theorem.

Theorem 5.5 (No-Dimensional Helly Theorem for *k*-flats)**.** *Let* $\mathscr{F}$ *be a k-unbounded family of convex sets in* $\mathbb{R}^d$ *and* $\exists R > 0$ *such that* $\forall C \in \mathscr{F}$ *we have have* $\operatorname{diam}(C) < R$ *and* $r \in \mathbb{N}$ *with* $k < r \leq d$*. If for any* $C_1, C_2, \ldots, C_r$ *in* $\mathscr{F}$ *there exists a k-flat H that intersects* $B(\mathscr{O}, 1)$ *and every* C_i*,* $\forall i \in [r]$*, then there exists a k-flat K such that, for all* $C \in \mathscr{F}$*, we have*

$$\operatorname{dist}(C, K) \leq \sqrt{\frac{1}{r-k}}.$$

If each $\mathscr{F}_i$ in Theorem 5.4 is *k*-unbounded then we get the following colorful generalization of the above Theorem 5.5.

Theorem 5.6 (Colorful generalization of Theorem 5.5)**.** *Let* $\mathscr{F}_1, \ldots, \mathscr{F}_r$ *be k-unbounded families of convex sets in* $\mathbb{R}^d$ *where* $k < r \leq d$*, and there exists* $R > 0$ *such that* $\forall C \in \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_r$ *we have* $\operatorname{diam}(C) < R$*. If for any r-tuple* $(C_1, \ldots, C_r) \in \mathscr{F}_1 \times \cdots \times \mathscr{F}_r$ *there exists a k-flat that intersects the closed unit ball* $B(\mathscr{O}, 1)$ *and every convex set* C_i *for all* $i \in \{1, \ldots, r\}$*, then there exists a k-flat K and* $j \in \{1, \ldots, r\}$ *such that, for all* $C \in \mathscr{F}_j$*, we have*

$$\operatorname{dist}(C, K) \leq \sqrt{\frac{1}{r-k}}.$$

In the above theorems we require the convex sets to have bounded diameter. Note that this condition cannot be relaxed. Consider hyperplanes in $\mathbb{R}^d$ and observe that any finite collection of hyperplanes can be pierced by a line passing through the origin $\mathscr{O}$ in $\mathbb{R}^d$. But, for any *k*-flat *K*, with $k \leq d-1$, and $\forall \Delta > 0$ there exists a hyperplane *H* such that $\operatorname{dist}(K, H) > \Delta$. The following two results will complement our results on no-dimensional Helly Theorem by showing the tightness of the bound guaranteed by our results and also show that the *k*-unboundedness condition is unavoidable.

Theorem 5.7 (On families not being k-unbounded)**.** *There exists a family $\mathscr{F}$ of convex sets in $\mathbb{R}^3$ such that*

- *there exists $R > 0$ such that* $\text{diam}(C) < R$ *for all $C \in \mathscr{F}$,*
- *$\mathscr{F}$ is 1-unbounded,*
- *any three convex sets in $\mathscr{F}$ can be pierced by a plane (2-dimensional affine space) passing through the origin $\mathscr{O}$, and*
- *for any plane K in $\mathbb{R}^3$ there exists $C_K \in \mathscr{F}$ such that* $\text{dist}(K, C_K) > 1$.

Theorem 5.8 (Tightness of the bound in Theorem 5.4)**.** *There exist families $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ of convex sets in $\mathbb{R}^3$ satisfying the following properties:*

- $\forall C \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, $\text{diam}(C) = \sqrt{2}$,
- *both $\mathscr{F}_1$ and $\mathscr{F}_2$ are 1-unbounded,*
- $\forall (C_1, C_2, C_3) \in \mathscr{F}_1 \times \mathscr{F}_2 \times \mathscr{F}_3$, *there exists a line L such that L pierces C_1, C_2, C_3 and* $\text{dist}(L, \mathscr{O}) \leq 1$, *and*
- *for every line K in $\mathbb{R}^3$ there exists $j \in [3]$ such that*

$$\max_{C \in \mathscr{F}_j} \text{dist}(C, K) \geq \frac{1}{\sqrt{2}}.$$

5.3 Proofs of the claimed results

In this section we will give the proofs of the results claimed in Section 5.2. We will first begin by proving the following colorful generalization of the Helly Theorem for k-flats proved by Aronov, Goodman and Pollack [18], namely, Theorem 5.3.

Theorem 5.3 (Colorful Helly Theorem for k-flats)**.** *Suppose for each $i \in [d+1], \mathscr{S}_i$ is a k-unbounded collection of compact convex sets in $\mathbb{R}^d$, and there exists $R > 0$ such that $\forall C \in \mathscr{S}_1 \cup \cdots \cup \mathscr{S}_{d+1}$ we have* $\text{diam}(C) < R$. *Also, assume that for every $(d+1)$-tuple $(C_1, \ldots, C_{d+1}) \in \mathscr{S}_1 \times \cdots \times \mathscr{S}_{d+1}$ there exists a k-flat H such that H intersects C_i for all $i \in [d+1]$. Then $\exists j \in [d+1]$ and a k-flat $\widetilde{H}$ that intersects every set in $\mathscr{S}_j$.*

Proof. Suppose for each $i \in [d+1]$, $\mathscr{L}_i$ is a set of k linearly independent vectors in the limiting directions set $LDS(\mathscr{S}_i)$ of $\mathscr{S}_i$. Then there exists a linearly independent set of k vectors, say $\mathscr{L} = \{z_1, \ldots, z_k\}$, such that for each $i \in [k]$, $z_i \in \mathscr{L}_i$. Suppose $\mathscr{K}$ is the k-flat generated by the linear span of $\mathscr{L}$. Now for each $i \in [k]$, since $z_i \in \mathscr{L}_i$, so there exists a sequence $\{S_{i,n}\}_{n\in\mathbb{N}}$ in $\mathscr{S}_i$ and for each $n \in \mathbb{N}$, *existsx*$_{i,n} \in S_{i,n}$ such that the sequence $\{f(x_{i,n})\}_{n\in\mathbb{N}}$ converges to z_i, as $n \to \infty$.

Now if $\exists i \in [k]$ such that $\mathscr{S}_i$ has a k-transversal then there is nothing to prove. Otherwise for each $i \in [d+1], i > k$, we take any $B_i \in \mathscr{S}_i$ and take any $n \in \mathbb{N}$. Then $B_{k+1},\dots,B_{d+1}$ together with $S_{1,n},\dots,S_{k,n}$, as a colorful tuple, is pierceable by a k-flat. It follows that $B_{k+1},\dots,B_{d+1}$ can be pierced by a k-flat arbitrarily close to the direction of $\mathscr{K}$. So by compactness of B_i's we can say that, $B_{k+1},\dots,B_{d+1}$ can be pierced by a k-flat in the direction of $\mathscr{K}$, i.e, parallel to $\mathscr{K}$.

Now for each $i \in [d+1]$, $i > k$, suppose $\mathscr{S}'_i$ is the projected family of $\mathscr{S}_i$ on the $(d-k)$ dimensional space $\mathscr{K}^{\perp}$, orthogonal to $\mathscr{K}$. Then every colorful $(d-k+1)$ tuple from $\mathscr{S}'_{k+1},\dots,\mathscr{S}'_{d+1}$ is pierceable by a point in the space $\mathscr{K}^{\perp}$. So by *Colorful Helly's Theorem*, $\exists i \in [d+1]$, $i > k$ such that $\mathscr{S}'_i$ is pierceable by a point in $\mathscr{K}^{\perp}$. Hence there exists a k-flat parallel to $\mathscr{K}$ that hits all the members of the family $\mathscr{S}_i$. □

Next we shall prove Theorem 5.4.

Theorem 5.4 (Colorful No-Dimensional Helly Theorem for k-flats)**.** *Let $\mathscr{F}_1,\dots,\mathscr{F}_r$ be families of convex sets in $\mathbb{R}^d$, $k < r \leq d$, and the family $\mathscr{F} := \mathscr{F}_1 \cup \cdots \cup \mathscr{F}_r$ of convex sets satisfy the following properties:*

(i) $\exists R > 0$ *such that for all $C \in \mathscr{F}$ we have* $\mathrm{diam}(C) < R$

(ii) $\exists J_k = \{j_1,\dots,j_k\} \subset [r]$ *and* $\{y_{j_1},\dots,y_{j_k}\} \subset \mathbb{S}^{d-1}$ *such that*

- *for all $i \in J_k$ we have $y_i \in LDS(\mathscr{F}_i)$, and*
- *the collection of vectors $\{y_{j_1},\dots,y_{j_k}\}$ are linearly independent*

If for any r-tuple $(C_1,\dots,C_r) \in \mathscr{F}_1 \times \cdots \times \mathscr{F}_r$ there exists a k-flat that intersects the closed unit ball $B(0,1)$ and every convex set C_i, $\forall i \in [r]$, then there exists a k-flat K and $\exists j \in [r]$ such that, $\forall C \in \mathscr{F}_j$, we have

$$\mathrm{dist}(C,K) \leq \sqrt{\frac{1}{r-k}}. \tag{5.2}$$

Proof. For any set $F \subset \mathbb{R}^d$, let $\bar{F}$ denote the closure of F. Since for any point $p \in \mathbb{R}^d$, $d(p,F) = d(p,\bar{F})$, it is enough to show that $\exists j \in [r]$ such that there exists a k-flat K satisfying $\forall C \in \mathscr{F}_j$,

$$\mathrm{dist}(\bar{C},K) \leq \sqrt{\frac{1}{r-k}}.$$

Now without loss of generality, we assume that $J_k = [k]$. Then for each $j \in J$, $y_j \in \text{LDS}(\mathscr{F}_j)$ implies that $\exists$ a sequence $\{F_{j,n}\}_{n\in\mathbb{N}}$ in $\mathscr{F}_j$ and for each $n \in \mathbb{N}$, $\exists x_{j,n} \in \mathscr{F}_{j,n}$ such that the sequence $\{f(x_{j,n})\}_{n\in\mathbb{N}}$ converges to y_j.

Now if $\exists j \in [r]$ such that there exists a k-flat K satisfying $\forall C \in \mathscr{F}_j$, $\text{dist}(C,K) \leq \sqrt{\frac{1}{r-k}}$, then there is nothing to prove. Otherwise, suppose K is the k-flat generated by the linear span of $\{y_1,\dots,y_k\}$. Now for each $i \in [r]$, $i > k$, take any $F_i \in \mathscr{F}_i$. Then for each $n \in \mathbb{N}$, there exists a k-flat K_n intersecting $B(\mathscr{O},1)$ and piercing $F_{1,n}, F_{2,n},\dots,F_{k,n},F_{k+1},\dots,F_r$. This implies that $F_{k+1},\dots,F_r$ can be pierced by a k-flat arbitrarily close to the direction of K. Now by compactness of $\bar{F}_i$'s, we conclude that $\exists a \in K^{\perp}$ such that the k-flat $a+K$ pierces $\bar{F}_{k+1},\dots,\bar{F}_r$. Since $a+K$ intersects $B(\mathscr{O},1)$, we must have $\|a\| \leq 1$.

Now for any set A in $\mathbb{R}^d$, let $\pi(A)$ denote the orthogonal projection of A onto the $(d-k)$-dimensional space $K^{\perp}$. Then for any $(C_{k+1},\dots C_r) \in \mathscr{F}_{k+1}\times\cdots\times\mathscr{F}_r$, we have

$$\left(\bigcap_{i=k+1}^{r} \pi\left(\bar{C}_i\right)\right)\bigcap B(\mathscr{O},1) \neq \emptyset.$$

Then, by Theorem 5.2, $\exists q \in K^{\perp}$ and $\exists i \in \{k+1,\dots,r\}$ such that $\forall C \in \mathscr{F}_i$ we have

$$\text{dist}\left(q,\pi(\overline{C})\right) < \sqrt{\frac{1}{r-k}}\,.$$

Suppose $q' \in \pi^{-1}(q)$ and consider the k-flat $K' = q'+K$. Then we have $\forall C \in \mathscr{F}_i$,

$$\text{dist}\left(K',\overline{C}\right) < \sqrt{\frac{1}{r-k}}\,.$$

□

Now we prove Theorem 5.7.

Theorem 5.7 (On families not being k-unbounded)**.** *There exists a family $\mathscr{F}$ of convex sets in $\mathbb{R}^3$ such that*

- *there exists $R > 0$ such that* $\text{diam}(C) < R$ *for all $C \in \mathscr{F}$,*
- *$\mathscr{F}$ is* 1*-unbounded,*
- *any three convex sets in $\mathscr{F}$ can be pierced by a plane (*2*-dimensional affine space) passing through the origin $\mathscr{O}$, and*

- *for any plane K in* $\mathbb{R}^3$ *there exists* $C_K \in \mathscr{F}$ *such that* $\operatorname{dist}(K, C_K) > 1$.

Proof. To establish the necessity of k-unboundedness we will be using a construction that is closely related to the one given by Aronov et al. [18].

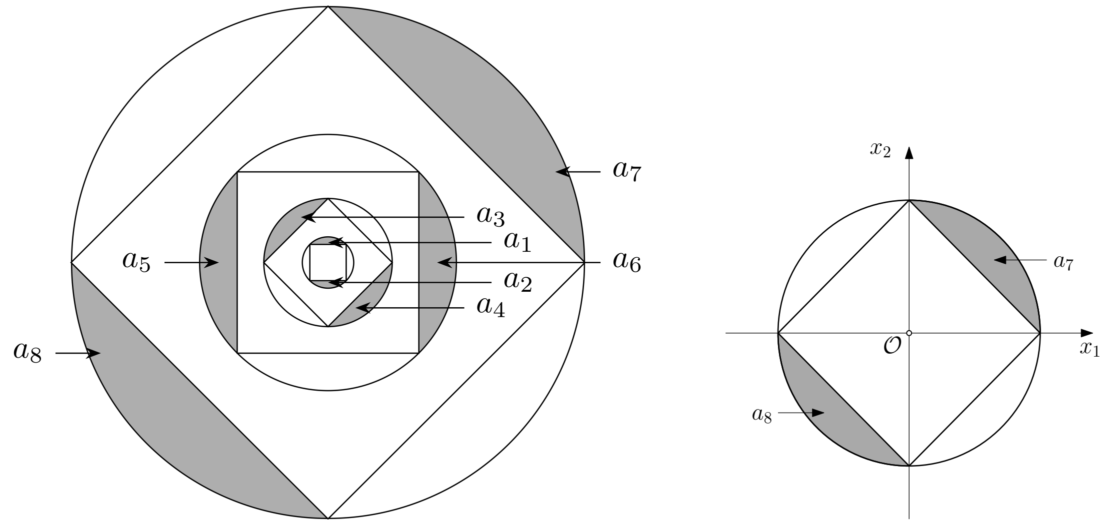

Fig. 5.1 An example demonstrating the necessity of k-unboundedness condition. This figure has been taken from [18].

Consider the eight shaded convex regions in Fig. 5.1 created by four circles and four squares centered at a point $\mathscr{O}$. Let $x_3 = 0$ be the plane that contains the Fig. 5.1, and without loss of generality assume that $\mathscr{O}$ is the origin in $\mathbb{R}^3$. We will call these eight shaded convex regions $a_1, a_2, \ldots, a_8$, respectively. Observe that any 3 of these convex sets can be intersected by a straight line passing through $\mathscr{O}$.

We will now create additional convex sets in the following way: we choose the eight convex sets in a fixed order, and in each step elevate the sets in increasing heights in that order along the x_3-axis such that for any $n \in \mathbb{N}$ there are infinitely many sets of this collection that lie outside $B(\mathscr{O}, n)$. This gives us a countably infinite sequence $\mathscr{F}$ of sets, where $\mathscr{F}$ is 1-unbounded, but not 2-unbounded. Clearly, for any three sets in $\mathscr{F}$, there exists a plane that passes through $\mathscr{O}$ and intersects these sets.

Let ℓ denote the length of the side of the smallest square in Figure 5.1. We show that it is not possible to find a plane that is at most a distance 1 unit away from all the sets in $\mathscr{F}$ when ℓ is large enough. Let K be a plane for which the maximum distance from the sets in $\mathscr{F}$ is minimized, and ℓ_K be the intersection of K with the plane $x_3 = 0$. Clearly, K must be perpendicular to the plane that contains the first 8 sets, because otherwise for any $R > 0$ we would find a set C in $\mathscr{F}$ for which $\operatorname{dist}(K, C) > R$. Consider the straight line ℓ_K that is the intersection of K and the plane given by the equation $x_3 = 0$. If ℓ_K is moved on the $x_3 = 0$ plane closer to $\mathscr{O}$ along the line perpendicular to ℓ_K from $\mathscr{O}$, the quantity $\max\{\text{dist}(a_{2n}, \ell_K), \text{dist}(a_{2n-1}, \ell_K)\}$ does not increase for $n \in \{1,2,3,4\}$. Since $\text{dist}(K, a_i) = \text{dist}(\ell_K, a_i)\ \forall i \in [8]$, we can take K to be passing through $\mathscr{O}$. Let the side-lengths of the 4 squares in Figure 5.1 be $\ell_1 (= \ell), \ell_3, \ell_5, \ell_7$, where the side of side-length ℓ_i is shared by the set a_i. Let the diagonals of the largest square in Figure 5.1 lie on the x_1-axis and the x_2-axis respectively. If ℓ_K makes an angle $\theta \in [0, \pi)$ with the x_1-axis, then we have the following: for $i \in \{1,3,5,7\}$ we have

$$\text{dist}(a_i, \ell_K) = \text{dist}(a_{i+1}, \ell_K), \tag{5.3}$$

and

$$\text{dist}((a_1, \ell_K) = \begin{cases} \frac{\ell_1}{\sqrt{2}} \sin(\pi/4 - \theta) & \text{if } 0 \le \theta \le \pi/4 \\ \frac{\ell_1}{\sqrt{2}} \sin(\pi - \theta) & \text{if } 3\pi/4 \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases} \tag{5.4}$$

$$\text{dist}(a_3, \ell_K) = \begin{cases} \frac{\ell_3}{\sqrt{2}} \sin\theta & \text{if } 0 \le \theta \le \pi/4 \\ \frac{\ell_3}{\sqrt{2}} \sin(\pi/2 - \theta) & \text{if } \pi/4 \le \theta \le \pi/2 \\ 0 & \text{otherwise} \end{cases} \tag{5.5}$$

$$\text{dist}(a_5, \ell_K) = \begin{cases} \frac{\ell_5}{\sqrt{2}} \sin(\theta - \pi/4) & \text{if } \pi/4 \le \theta \le \pi/2 \\ \frac{\ell_5}{\sqrt{2}} \sin(3\pi/4 - \theta) & \text{if } \pi/2 \le \theta \le 3\pi/4 \\ 0 & \text{otherwise} \end{cases} \tag{5.6}$$

$$\text{dist}(a_7, \ell_K) = \begin{cases} \frac{\ell_7}{\sqrt{2}} \sin(\theta - \pi/2) & \text{if } \pi/2 \le \theta \le 3\pi/4 \\ \frac{\ell_7}{\sqrt{2}} \sin(\pi - \theta) & \text{if } 3\pi/4 \le \theta \le \pi \\ 0 & \text{otherwise} \end{cases} \tag{5.7}$$

Now, observe that

$$\begin{aligned} \sum_{i=1}^{8} \text{dist}(a_i, \ell_K) &\ge \frac{\ell}{\sqrt{2}} \times \min_{\theta \in [0, \pi/4]} (\sin\theta + \sin(\pi/4 - \theta)) \\ &\ge \ell\sqrt{2} \sin(\pi/8) > 2 \end{aligned}$$

Thus, for $\ell > \sqrt{2}/\sin(\pi/8)$, there are no planes that are at most 1 distance away from each set in $\mathscr{F}$. □

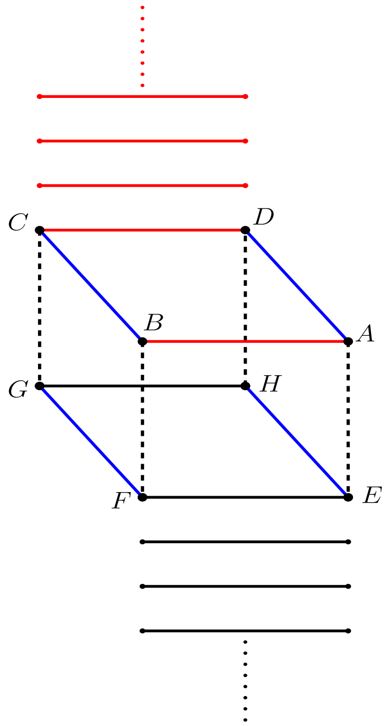

Fig. 5.2 An example demonstrating the tightness of the bound given in Theorem 5.8.

Finally we will end this section with the proof of Theorem 5.8.

Theorem 5.8 (Tightness of the bound in Theorem 5.4)**.** *There exist families $\mathscr{F}_1, \mathscr{F}_2, \mathscr{F}_3$ of convex sets in $\mathbb{R}^3$ satisfying the following properties:*

- $\forall C \in \mathscr{F}_1 \cup \mathscr{F}_2 \cup \mathscr{F}_3$, $\operatorname{diam}(C) = \sqrt{2}$,
- *both $\mathscr{F}_1$ and $\mathscr{F}_2$ are 1-unbounded,*
- $\forall (C_1, C_2, C_3) \in \mathscr{F}_1 \times \mathscr{F}_2 \times \mathscr{F}_3$, *there exists a line L such that L pierces C_1, C_2, C_3 and* $\operatorname{dist}(L, \mathscr{O}) \leq 1$, *and*
- *for every line K in $\mathbb{R}^3$ there exists $j \in [3]$ such that*

$$\max_{C \in \mathscr{F}_j} \operatorname{dist}(C, K) \geq \frac{1}{\sqrt{2}}.$$

Proof. We provide an example where the bound given in Theorem 5.6 is tight.

Let $\{A, B, C, D, E, F, G, H\}$ be the 8 vertices of a cube in $\mathbb{R}^3$ whose centroid is the origin $\mathscr{O}$ and side length is $\sqrt{2}$ and $EFGH$ is parallel to the plane $x_3 = 0$ (see Fig. 5.2). Define $\mathscr{F}_i := \{S_{i,1}, S_{i,2}, S_{i,3}, \dots\}$ for $i = 1, 2, 3$ in the following way: Set $S_{1,1} = \overline{AB}$, $S_{1,2} = \overline{CD}$ and for $n > 2$, $S_{1,n}$ is the line segment $\overline{CD}$ raised to the height $x_3 = n$. Similarly, set

$S_{2,1} = \overline{GH}$, $S_{1,2} = \overline{EF}$ and for $n > 2$, $S_{2,n}$ is the line segment $\overline{EF}$ lowered to the height $x_3 = -n$. Now set $S_{3,1} = \overline{BC}, S_{3,2} = \overline{DA}, S_{3,3} = \overline{FG}, S_{3,4} = \overline{HE}$, and let $S_{3,4n+j}$ be the set $S_{3,j}$, for all $j \in \{0,1,2,3\}$ (see Fig. 5.2). Clearly, any colorful 3-tuple (C_1,C_2,C_3), $C_i \in \mathscr{F}_i$, $i \in [3]$, can be hit by a straight line that is at most at a distance 1 away from the centroid. Let $\mathscr{L}$ denote the set of all straight line transversals ℓ of colorful 3-tuples such that ℓ is as close to $\mathscr{O}$ as possible. Let ℓ_n denote the straight line transversal in $\mathscr{L}$ that passes through $S_{1,n}, S_{2,1}, S_{3,1}$. Clearly, $\operatorname{dist}(\ell_n, \mathscr{O}) \to 1$ as $n \to \infty$. Then

$$\sup_{\ell \in \mathscr{L}} \operatorname{dist}(\mathscr{O}, \ell) = 1.$$

Now note that for the straight line L that is perpendicular to the plane on which $ABCD$ lies and passes through $\mathscr{O}$, we have

$$\inf_{i \in [3]} \sup_{C \in \mathscr{F}_i} \operatorname{dist}(\ell, C) \geq \inf_{i \in [3]} \sup_{C \in \mathscr{F}_i} \operatorname{dist}(L, C),$$

for any straight line ℓ in $\mathbb{R}^3$. This we can show in the following way: let ℓ_1 be a straight line such that

$$\sup_{C \in \mathscr{F}_1} \operatorname{dist}(\ell_1, C) = \inf_{\ell \in \mathscr{L}} \sup_{C \in \mathscr{F}_1} \operatorname{dist}(C, \ell).$$

Then ℓ_1 must be perpendicular to the plane on which $ABCD$ lies, otherwise, the supremum of its distances from sets in $\mathscr{F}_1$ would be infinity. Then, ℓ_1 must be equidistant from both $S_{1,1}$ and $S_{1,2}$, and therefore, we can take ℓ_1 to be L. Similar arguments show that

$$\sup_{C \in \mathscr{F}_2} \operatorname{dist}(L, C) = \inf_{\ell \in \mathscr{L}} \sup_{C \in \mathscr{F}_1} \operatorname{dist}(C, \ell).$$

We have

$$\inf_{i \in [3]} \sup_{C \in \mathscr{F}_i} \operatorname{dist}(L, C) = \frac{1}{\sqrt{2}}.$$

To see that

$$\inf_{\ell \in \mathscr{L}} \sup_{C \in \mathscr{F}_3} \operatorname{dist}(C, \ell) = \frac{1}{\sqrt{2}},$$

project $\overline{BC}$, $\overline{AD}$, $\overline{FG}$, $\overline{EH}$ onto the plane P that contains $ABEF$. If there is a straight line ℓ_3 such that

$$\sup_{C \in \mathscr{F}_3} \operatorname{dist}(\ell_3, C) = \inf_{\ell \in \mathscr{L}} \sup_{C \in \mathscr{F}_3} \operatorname{dist}(C, \ell),$$

then let the projection of ℓ_3 onto P be ℓ_3'. If L is not the straight line that minimizes

$$\inf_{\ell \in \mathscr{L}} \sup_{C \in \mathscr{F}_6} \operatorname{dist}(C, \ell),$$

then the perpendicular distance from ℓ_3' to A, B, E and F must be smaller than $\frac{1}{\sqrt{2}}$. Without loss of generality let A be the point from which ℓ_3' is the farthest. Then we must have another point among B, E, and F from which ℓ_3' has the same distance as A. This point then must be E, because otherwise, we could have taken L to be ℓ_3. This means that ℓ_3' passes through the centroid of the square $ABEF$ and two points from A, B, E, and F lie on each side of ℓ_3'. But this implies that ℓ_3' must be parallel to AB since ℓ_3' has the minimum distance from both A and B and is at least as close to E and F, which is a contradiction. We have

$$\inf_{i \in [3]} \sup_{C \in \mathscr{F}_i} \operatorname{dist}(L, C) = \frac{1}{\sqrt{2}},$$

which is what we get by plugging in the values of r and k in the inequality given in Theorem 5.6. $\square$

Part II

Covering Subsets of the Hypercube with Nice Geometric Objects

Chapter 6

An Introduction to the Covering Problems

We will work over the field $\mathbb{R}$, and consider the n-variate polynomial ring $\mathbb{R}[\mathbb{X}]$. A classic result by Alon and Füredi [6] states that any collection of (affine) hyperplanes[1] in $\mathbb{R}^n$, whose union contains every point of the hypercube (or Boolean cube) $\{0,1\}^n$ except the all-zeros point $0^n := (0,\ldots,0)$, must have at least n hyperplanes. This lower bound is also tight, attained by the collection of hyperplanes defined by the equations: $X_i = 1$, $i \in [n]$.[2] Further, the lower bound proof by [6] is among the early instances of the *polynomial method* in combinatorics. Note that the union of any finite collection of hyperplanes in $\mathbb{R}^n$, as a set of points, is exactly equal to the zero set of the product of the affine linear polynomials defining the individual hyperplanes. So the lower bound on the number of hyperplanes follows from a lower bound on the degree of this *product polynomial*.

An interesting point to note in the lower bound proof by Alon and Füredi [6] is that the polynomial method is oblivious to the *product structure* of the polynomials corresponding to collections of hyperplanes, or *any other structural property* of polynomials, and is only sensitive to the degree of the polynomials. In other words, we may as well consider a *polynomial covering problem* satisfying the same vanishing conditions – find the minimum degree of a polynomial, *among all unstructured polynomials*, that vanish at every point of $\{0,1\}^n$ except 0^n – and the proof by the polynomial method goes through. Therefore, in hindsight, it is amazing that the lower bound for the *weaker* polynomial covering problem is,

[1]We are interested in *affine* hyperplanes, that is, all possible translates of codimension-1 subspaces of $\mathbb{R}^n$, and not just the subspaces themselves. However, we will suppress the adjective 'affine' in the rest of this paper.

[2]This result of Alon and Füredi [6] is, in fact, true over any field $\mathbb{F}$, and not just for $\mathbb{F} = \mathbb{R}$.

in fact, tight for the *stronger* hyperplane covering problem. In this work, we are interested in further exploring this power of the polynomial method in giving tight bounds for some hyperplane covering problems by simply considering the corresponding weaker polynomial covering problems.

In order to describe our motivations as well as our results, let us first fix some terminologies and notations. We will identify a hyperplane H in $\mathbb{R}^n$ with its defining affine linear polynomial $H(\mathbb{X})$. Let $t \geq 1$, $\ell \in [0, t-1]$, and consider any subset $S \subsetneq \{0,1\}^n$. We define

- a (t,ℓ)-**exact hyperplane cover** for S to be a finite collection of hyperplanes (considered as a multiset) in $\mathbb{R}^n$ such that each point in S is contained in at least t hyperplanes, and each point in $\{0,1\}^n \setminus S$ is contained in exactly ℓ hyperplanes.
- a (t,ℓ)-**exact polynomial cover** for S to be a nonzero polynomial that vanishes at each point in S with multiplicity[3] at least t, and vanishes at each point in $\{0,1\}^n \setminus S$ with multiplicity exactly ℓ.

Let $\mathsf{EHC}_n^{(t,\ell)}(S)$ denote the minimum size of a (t,ℓ)-exact hyperplane cover for S, and $\mathsf{EPC}_n^{(t,\ell)}(S)$ denote the minimum degree of a (t,ℓ)-exact polynomial cover for S. In these notations, Alon and Füredi [6] show the following.

Theorem 6.1 ([6]). $\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus \{0^n\}) = \mathsf{EPC}_n^{(1,0)}(\{0,1\}^n \setminus \{0^n\}) = n.$

It is quite obvious from the definitions that, in general, we have $\mathsf{EHC}_n^{(t,\ell)}(S) \geq \mathsf{EPC}_n^{(t,\ell)}(S)$. But for completeness, we give a quick proof in Chapter 7.

In the present work, we are broadly concerned with the following question.

Question 6.2. *Given a proper subset $S \subsetneq \{0,1\}^n$ and integers $t \geq 1$, $\ell \in [0, t-1]$, under what conditions can we say that $\mathsf{EHC}_n^{(t,\ell)}(S) = \mathsf{EPC}_n^{(t,\ell)}(S)$?*

6.1 Motivation

Our work relies heavily on the polynomial method using Alon's Combinatorial Nullstellensatz [4] (also see Buck, Coley, and Robbins [36], and Alon and Tarsi [9]), and on a recent *multiplicity extension* of the Combinatorial Nullstellensatz given by Sauermann and Wigderson [135]. The problems of concern belong to a larger class of questions that have been of interest for a long time, and have rich literature. We mention some of these related works in Section 6.4.

[3] We say that a polynomial P vanishes at a point a with multiplicity at least t if all the derivatives of P having order at most $t-1$ vanish at a.

Let us detail the primary motivations for our present work.

- Aaronson, Groenland, Grzesik, Kielak, and Johnston [1] generalized Theorem 6.1, where the forbidden set has cardinality more than 1. They considered the problem of determining $\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus S)$ for general nonempty subsets $S \subseteq \{0,1\}^n$, and obtained the following.
  Theorem 6.3 ([1]). *(a) If $|S| \in \{2,3\}$, then $\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus S) = n-1$.*
  (b) If $|S| = 4$, then $\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus S) = n-1$, if there is a hyperplane Q with $|Q \cap S| = 3$, and $\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus S) = n-2$, otherwise.
  (c) In general, for any nonempty subset $S \subseteq \{0,1\}^n$, we have
  $$\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus S) \geq n - \lfloor \log_2 |S| \rfloor.$$
- In a remarkable development of Theorem 6.1, using techniques different from the polynomial method, Clifton and Huang [56] proved the following bounds for the hyperplane covering problem.
  Theorem 6.4 ([56]). *For all $n \geq 3, t \geq 4$, we have*
  $$n+t+1 \leq \mathsf{EHC}_n^{(t,0)}(\{0,1\}^n \setminus \{0^n\}) \leq n + \binom{t}{2},$$
  Further, for $n \geq 2, t = 2,3$, we have $\mathsf{EHC}_n^{(t,0)}(\{0,1\}^n \setminus \{0^n\}) = n + \binom{t}{2}$.
- As a multiplicity extension of Theorem 6.1 for the polynomial covering problem, Sauermann and Wigderson [135] determined the following.
  Theorem 6.5 ([135]). *For $t \geq 1$, $\ell \in [0, t-1]$, we have*
  $$\mathsf{EPC}_n^{(t,\ell)}(\{0,1\}^n \setminus \{0^n\}) = \begin{cases} n+2t-2 & \text{if } \ell = t-1, \\ n+2t-3 & \text{if } \ell < t-1 \leq \lfloor \frac{n+1}{2} \rfloor. \end{cases}$$
- We say a subset $S \subseteq \{0,1\}^n$ is **symmetric** if S is closed under permutations of coordinates. Note that the **Hamming weight** of any $x \in \{0,1\}^n$ is defined by $|x| = |\{i \in [n] : x_i = 1\}|$. Thus, the subset S is symmetric if and only if
  $$x \in S, y \in \{0,1\}^n, |y| = |x| \quad \Longrightarrow \quad y \in S.$$

For any symmetric set $S \subseteq \{0,1\}^n$, we define $W_n(S) = \{|x| : x \in S\}$. It is immediate that a symmetric set S is determined by the corresponding set $W_n(S)$. Also for $i \in [0,n]$, let $W_{n,i} = [0,i-1] \cup [n-i+1,n]$, and define the symmetric set $T_{n,i} \subseteq \{0,1\}^n$ by $W_n(T_{n,i}) = W_{n,i}$.[4]

Venkitesh [146] gave a combinatorial characterization of $\mathsf{EPC}_n^{(1,0)}(S)$ for all symmetric sets $S \subsetneq \{0,1\}^n$, as well as a partial result towards answering Question 6.2 in this setting. The characterization is in terms of a simple combinatorial measure. For any symmetric set $S \subseteq \{0,1\}^n$, define

$$\begin{aligned} \mu_n(S) &= \max\{i \in [0,\lceil n/2 \rceil] : W_{n,i} \subseteq W_n(S)\}, \\ \text{and} \quad \Lambda_n(S) &= |W_n(S)| - \mu_n(S). \end{aligned}$$

Further, denote $\overline{\mu}_n(S) := \mu_n(\{0,1\}^n \setminus S)$ and $\overline{\Lambda}_n(S) := \Lambda_n(\{0,1\}^n \setminus S)$.

Theorem 6.6 ([146]). *(a) For any symmetric set $S \subsetneq \{0,1\}^n$, we have*

$$\mathsf{EPC}_n^{(1,0)}(S) = \Lambda_n(S).$$

(b) For any symmetric set $S \subsetneq \{0,1\}^n$ such that $W_{n,2} \not\subseteq W_n(S)$, we have

$$\begin{aligned} \mathsf{EHC}_n^{(1,0)}(S) = \mathsf{EPC}_n^{(1,0)}(S) &= \Lambda_n(S) \\ &= \begin{cases} |W_n(S)| & \text{if } W_{n,1} \not\subseteq W_n(S), \\ |W_n(S)| - 1 & \text{if } W_{n,1} \subseteq W_n(S). \end{cases} \end{aligned}$$

(c) $\mathsf{EHC}_n^{(1,0)}(T_{n,2}) = \mathsf{EPC}_n^{(1,0)}(T_{n,2}) = 2 = |W_{n,2}| - \mu_n(T_{n,2}) = \Lambda_n(T_{n,2})$.

It is interesting, and important for further discussions, to note the constructions that imply the equalities in Theorem 6.6.

Example 6.7 ([146]). *(a) Let $S \subsetneq \{0,1\}^n$ be a symmetric set. By the proof of Theorem 6.6(a) [146, Proposition 6.1], for every $a \in \{0,1\}^n \setminus S$, there exists a polynomial $Q_a(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that $\deg(Q_a) \le \Lambda_n(S)$, $Q_a|_S = 0$, and $Q_a(a) = 1$. Then choose scalars $\beta_a \in \mathbb{R}$, $a \in \{0,1\}^n \setminus S$ such that the polynomial $Q(\mathbb{X}) := \sum_{a \in \{0,1\}^n \setminus S} \beta_a Q_a(\mathbb{X})$ satisfies $\deg(Q) \le \Lambda_n(S)$, $Q|_S = 0$, and $Q(b) \ne 0$ for all $b \in \{0,1\}^n \setminus S$. So the polynomial $Q(\mathbb{X})$ witnesses the equality in Theorem 6.6(a).*

[4] Here we have $W_{n,0} = \emptyset$ and $T_{n,0} = \emptyset$.

Note that the set of scalars $B := \{\beta_a : a \in \{0,1\}^n \setminus S\}$ *can always be chosen so that* $Q(\mathbb{X})$ *satisfies the above required conditions. For instance, consider a subfield of* $\mathbb{R}$ *defined by* $\widehat{\mathbb{Q}} := \mathbb{Q}\big(\{Q_a(b) : a,b \in \{0,1\}^n \setminus S\}\big)$.[5] *It then follows that* $\mathbb{R}$ *is an infinite dimensional* $\widehat{\mathbb{Q}}$*-vector space. So we can choose* B *to be any* $\widehat{\mathbb{Q}}$*-linearly independent subset of* $\mathbb{R}$ *of size* $2^n - |S|$.

(b) Let $S \subsetneq \{0,1\}^n$ *be a symmetric set such that* $W_{n,2} \not\subseteq W_n(S)$. *If* $W_{n,1} \not\subseteq W_n(S)$, *then the collection of hyperplanes* $\{H'_t(\mathbb{X}) : t \in W_n(S)\}$, *defined by* $H'_t(\mathbb{X}) := \sum_{i=1}^{n} X_i - t$, $t \in W_n(S)$ *witnesses equality in Theorem 6.6(b). If* $W_{n,1} = \{0,n\} \subseteq W_n(S)$, *note that the hyperplane* $H^*_{(1,1)}(\mathbb{X}) := \sum_{i=1}^{n-1} X_i - (n-1)X_n$ *satisfies* $H^*_{(1,1)}(x) = 0$ *for* $x \in \{0,1\}^n$ *if and only if* $x \in \{0^n, 1^n\}$, *that is,* $x \in T_{n,1}$. *Then the collection of hyperplanes* $\{H^*_{(1,1)}(\mathbb{X})\} \sqcup \{H'_t(\mathbb{X}) : t \in W_n(S) \setminus \{0,n\}\}$ *witnesses the equality in Theorem 6.6(b).*

(c) The collection of hyperplanes $\{H^*_{(2,1)}(\mathbb{X}), H^*_{(2,2)}(\mathbb{X})\}$, *where* $H^*_{(2,1)}(\mathbb{X}) := \sum_{i=1}^{n-1} X_i - (n-3)X_n + 1$ *and* $H^*_{(2,2)}(\mathbb{X}) := \sum_{i=1}^{n-2} X_i - (n-2)X_{n-1}$, *witnesses the equality in Theorem 6.6(c).*

Further, the following was conjectured in [146], appealing to Theorem 6.6(b) and Theorem 6.6(c).

Conjecture 6.8 ([146])**.** *For any symmetric set* $S \subsetneq \{0,1\}^n$ *such that* $W_{n,2} \subseteq W_n(S)$, *we have*

$$\mathsf{EHC}_n^{(1,0)}(S) = |W_n(S)| - 2,$$

and therefore, $\mathsf{EHC}_n^{(1,0)}(S) > \mathsf{EPC}_n^{(1,0)}(S)$ *if* $W_{n,2} \subsetneq W_n(S)$.

- Ghosh, Kayal and Nandi [75] improved Theorem 6.3 and bounded $\mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S)$ for all $t \geq 1$, in a more abstract sense by introducing a combinatorial measure called *index complexity*. For any subset $S \subseteq \{0,1\}^n$, $|S| > 1$, the **index complexity** of S is defined to be the smallest positive integer $r_n(S)$ such that for some $I \subseteq [n]$, $|I| = r_n(S)$, there is a point $u \in S$ such that for each $v \in S$, $v \neq u$, we get $v_i \neq u_i$ for some $i \in I$, that is, the point u can be *separated from all other points* in S in the coordinates in I. (The index complexity of a singleton set is defined to be zero.)

The improvement to Theorem 6.3 was achieved via the following two results.

Proposition 6.9 ([75])**.** *For any nonempty subset* $S \subseteq \{0,1\}^n$, *we have*

$$r_n(S) \leq \lfloor \log_2 |S| \rfloor.$$

[5] For any $B \subsetneq \mathbb{R}$, $\mathbb{Q}(B)$ denotes the smallest subfield of $\mathbb{R}$ that contains $\mathbb{Q}$ and B. This subfield exists and is unique, by elementary field theory.

Theorem 6.10 ([75]). *For any nonempty subset $S \subseteq \{0,1\}^n$ and $t \geq 1$, we have*

$$\mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) \geq n - r_n(S) + 2t - 2.$$

Returning to the context of symmetric sets, note that for any symmetric set $S \subseteq \{0,1\}^n$, the complement $\{0,1\}^n \setminus S$ is also symmetric. Further, we say a symmetric set S is a **layer** if $|W_n(S)| = 1$. Using Theorem 6.10, Ghosh, Kayal and Nandi [75] proved the following generalization of Theorem 6.5, when the restricted set is a *layer* instead of a single point.

Theorem 6.11 ([75]). *For any layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, and any $t \geq 1$, we have*

$$\mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) \geq \max\{w, n-w\} + 2t - 2.$$

In the present work, we will build upon some of the above results.

6.2 Our results: higher multiplicity hyperplane covers

As mentioned earlier, we are broadly interested in understanding when Question 6.2 has an answer in the affirmative. In the present work, we will obtain some such characterizations when $t \geq 1$, $\ell = t-1$, for some structured subsets of the hypercube; specifically, we will consider symmetric sets, as well as a *block generalization* of symmetric sets. Strictly speaking, we will also have some nondegeneracy conditions in some characterizations.

Proof technique

We also have a common proof technique for our results, which is simple and similar to the approach adopted in the earlier works [6, 135, 146]. To summarize the technique, consider a subset $S \subsetneq \{0,1\}^n$ (with a suitable structure, as we detail later), and suppose we would like to determine $\mathsf{EHC}_n^{(t,t-1)}(S)$. Via the polynomial method, we first obtain a lower bound for the *weaker* polynomial covering problem, say $\mathsf{EPC}_n^{(t,t-1)}(S) \geq L_t$ (for some $L_t \geq 1$). We then construct a hyperplane cover to obtain an upper bound $\mathsf{EHC}_n^{(t,t-1)}(S) \leq L_t$ for the *stronger* hyperplane covering problem. Thus, we immediately have the inequalities

$$L_t \geq \mathsf{EHC}_n^{(t,t-1)}(S) \geq \mathsf{EPC}_n^{(t,t-1)}(S) \geq L_t,$$

which gives a tight characterization.

Some fundamental hyperplane families

Before we detail our results, let us fix the notations for some fundamental hyperplane families which will appear repeatedly in this work.

(a) For each $t \in [0,n]$, define $H'_t(\mathbb{X}) = \sum_{i=1}^{n} X_i - t$. Further, for any $W \subseteq [0,n]$, let $\mathscr{H}'_W(\mathbb{X}) = \{H'_t(\mathbb{X}) : t \in W\}$.

(b) For each $i \in [0,\lceil n/2 \rceil]$, $j \in [i]$, we have

$$H^*_{(i,j)}(\mathbb{X}) = \sum_{k=1}^{n-j} X_k - (n-2i+j)X_{n-j+1} - (i-j).$$

Further, let $\mathscr{H}^*_i(\mathbb{X}) = \{H^*_{(i,j)}(\mathbb{X}) : j \in [i]\}$.

(c) Define $H^\circ_0(\mathbb{X}) = X_1$ and $H^\circ_1(\mathbb{X}) = X_1 - 1$. Further, let $\mathscr{H}^{\circ m}(\mathbb{X}) = \bigsqcup_{\ell=1}^{m} \{H^\circ_0(\mathbb{X}), H^\circ_1(\mathbb{X})\}$ (disjoint union, as a multiset), for any $m \geq 1$.

6.2.1 Warm-up: Index complexity of symmetric sets

We have seen that Ghosh, Kayal and Nandi [75] obtain a lower bound on the polynomial covering problem for any general subset of the hypercube, in terms of index complexity (Theorem 6.10), by employing the polynomial method. First we shall give a matching hyperplane construction when the restricted set is a single layer and disprove Conjecture 6.8 by a counterexample. As a consequence, it will show that the index complexity of a single layer can be expressed in terms of the combinatorial measure Λ_n introduced in [146] (also see Theorem 6.6). To summarise, we have the following.

Proposition 6.12. *For a layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, we have*

$$\overline{\Lambda}_n(S) = n - r_n(S) = \max\{w, n-w\}.$$

Such an equality is no longer true for general symmetric sets. We can, in fact, precisely understand the general case combinatorially. We introduce some terminology before we proceed.

For any $a \in [-1, n-1]$, $b \in [1, n+1]$, $a < b$, denote the set of weights $I_{n,a,b} = [0,a] \cup [b,n]$ [6] and we say a **peripheral interval** is the symmetric set $J_{n,a,b} \subseteq \{0,1\}^n$ defined by $W_n(J_{n,a,b}) = I_{n,a,b}$. We will consider *inner and outer approximations* of a symmetric set.

[6] Here, we have the convention $[0,-1] = [n+1,n] = \emptyset$.

Let $S \subseteq \{0,1\}^n$ be a symmetric set.

- If $S \subsetneq \{0,1\}^n$, then the **inner interval** of S, denoted by in-int(S), is defined to be the peripheral interval $J_{n,a,b} \subseteq \{0,1\}^n$ of maximum size such that $J_{n,a,b} \subseteq S$. Further, we define in-int$(\{0,1\}^n) = J_{n,\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1}$.
- Let $\mathscr{O}(S)$ be the collection of all peripheral intervals $J_{n,a,b}$ such that $S \subseteq J_{n,a,b}$ and $I_{n,a,b} = W_n(J_{n,a,b})$ has minimum size. It is easy to check that there exists either a unique peripheral interval $J_{n,a,b} \in \mathscr{O}(S)$, or exactly a pair of peripheral intervals $J_{n,a,b}, J_{n,n-b,n-a} \in \mathscr{O}(S)$ such that the quantity $|a+b-n|$ is minimum. The **outer interval** of S, denoted by out-int(S), is defined by

$$\text{out-int}(S) = \begin{cases} J_{n,a,b} & \text{if } J_{n,a,b} \text{ is the unique minimizer of } |a+b-n|, \\ J_{n,a,b} & \text{if } J_{n,a,b}, J_{n,n-b,n-a} \text{ are minimizers of } |a+b-n|, \text{ and } a > n-b. \end{cases}$$

We will elaborate the definitions in the Preliminaries (Section 7.2.2), and discuss the uniqueness (and therefore, well-definedness) of inner and outer intervals, along with some illustrations. Now define

$$\begin{aligned} \text{in}_n(S) &= (\min\{a, n-b\}+1) + |W_n(S) \setminus W_{n,\min\{a,n-b\}+1}| && \text{where } J_{n,a,b} = \text{in-int}(S), \\ \text{and out}_n(S) &= a+n-b+1 = |I_{n,a,b}| - 1 && \text{where } J_{n,a,b} = \text{out-int}(S). \end{aligned}$$

Towards understanding the index complexity of general symmetric sets, we obtain the following important relation between inner and outer intervals of symmetric sets.

Proposition 6.13. *For any nonempty symmetric set $S \subseteq \{0,1\}^n$, we have*

$$\text{in}_n(\{0,1\}^n \setminus S) + \text{out}_n(S) \geq n.$$

Further, equality holds if and only if either S or $\{0,1\}^n \setminus S$ is a peripheral interval.

We are now ready to characterize the index complexity of symmetric sets.

Proposition 6.14. *For any nonempty symmetric set $S \subseteq \{0,1\}^n$, we have $r_n(S) = \text{out}_n(S)$.*

Also, the following is trivial, by definitions.

Fact 6.15 (By definitions)**.** *For any symmetric set $S \subsetneq \{0,1\}^n$, we have $\Lambda_n(S) = \text{in}_n(S)$.*

The following is then an immediate corollary of Proposition 6.13, Proposition 6.14, and Fact 6.15.

Corollary 6.16. *For any nonempty symmetric set $S \subseteq \{0,1\}^n$, we have*

$$\overline{\Lambda}_n(S) \geq n - r_n(S).$$

Further, equality holds if and only if either S or $\{0,1\}^n \setminus S$ is a peripheral interval.

Note that if $S \subseteq \{0,1\}^n$ is a layer, then $\{0,1\}^n \setminus S$ is a peripheral interval, and hence Corollary 6.16 recovers Proposition 6.12.

6.2.2 Covering symmetric sets

We obtain a characterization of $\mathsf{EHC}_n^{(1,0)}(S)$ for all symmetric sets $S \subsetneq \{0,1\}^n$ here, and disprove Conjecture 6.8. (In particular, this answers a question of Venkitesh [146, Open Problem 36].) Our first main result extends Theorem 6.6 and Theorem 6.17, and answers Question 6.2 in the affirmative for symmetric sets, with $t \geq 1$, $\ell = t-1$. As a proof attempt, for a general symmetric set, we may directly apply Theorem 6.10 (which was obtained in [75] by the polynomial method), and then attempt to find a tight construction. This would require a precise understanding of the index complexity of symmetric sets, which we obtain in Proposition 6.14. However, the lower bound obtained in this way is *weak*. It turns out that the tight lower bound is larger, and the gap is, in fact, exactly captured by Corollary 6.16.

For convenience, we will state the result in terms of complements of symmetric sets (which are also symmetric). This will be an important distinction in an extended setting, which we consider later.

Theorem 6.17. *For any layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, and any $t \geq 1$, we have*

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) = \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) = \max\{w, n-w\} + 2t - 2.$$

Theorem 6.18. *For any nonempty symmetric set $S \subseteq \{0,1\}^n$ and $t \geq 1$, we have*

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) = \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) = \overline{\Lambda}_n(S) + 2t - 2.$$

The following construction is important throughout our discussion.

Lemma 6.19. *For $i \in [0, \lceil n/2 \rceil]$, the collection of hyperplanes $\{H^*_{(i,j)}(\mathbb{X}) : j \in [i]\}$ defined by*

$$H^*_{(i,j)}(\mathbb{X}) = \sum_{k=1}^{n-j} X_k - (n-2i+j)X_{n-j+1} - (i-j), \quad j \in [i],$$

satisfies the following.

- *For every $a \in T_{n,i}$, there exists $j \in [i]$ such that $H^*_{(i,j)}(a) = 0$.*
- *$H^*_{(i,j)}(b) \neq 0$ for every $b \in \{0,1\}^n \setminus T_{n,i}$, $j \in [i]$.*

A construction that implies the equality in Theorem 6.17 is then immediate.

Example 6.20. *Let $S \subsetneq \{0,1\}^n$ be a layer with $W_n(S) = w$, and $t \geq 1$. Let $w' = \min\{w, n-w\}$. Denote $H_0^\circ(\mathbb{X}) = X_1$, $H_1^\circ(\mathbb{X}) = X_1 - 1$. Then the collection of hyperplanes*

$$\{H^*_{(w',j)}(\mathbb{X}) : j \in [w']\} \ \sqcup \bigsqcup_{\ell \in [t-1]} \{H_0^\circ(\mathbb{X}), H_1^\circ(\mathbb{X})\} \qquad \text{(disjoint union, as a multiset)}$$

witnesses the equality in Theorem 6.17.

Interestingly, we obtain the tight bound in Theorem 6.18 since our instantiation of the polynomial method turns out to be *stronger* than that in the proof of Theorem 6.10 by [75]. This relative strength is also captured exactly by Corollary 6.16!

Remark 6.21. *The proof of Theorem 6.18 will, in fact, show that for any nonempty symmetric set $S \subseteq \{0,1\}^n$ and $t \geq 1$, we have*

$$\begin{aligned}
\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) &= \mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus S) + 2t - 2 \\
&= \overline{\Lambda}_n(S) + 2t - 2 \\
&= \mathsf{EPC}_n^{(1,0)}(\{0,1\}^n \setminus S) + 2t - 2 = \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S).
\end{aligned}$$

A simple generalization of Example 6.20 gives a construction implying the equality in Theorem 6.18.

Example 6.22. *Let $S \subseteq \{0,1\}^n$ be a nonempty symmetric set, and $t \geq 1$. Then the collection of hyperplanes*

$$\mathscr{H}^*_{\overline{\mu}_n(S)}(\mathbb{X}) \sqcup \mathscr{H}'_{W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}}(\mathbb{X}) \sqcup \mathscr{H}^{\circ(t-1)}(\mathbb{X})$$

witnesses the equality in Theorem 6.18.

6.2.3 Covering special *k-wise symmetric* sets

Fix a positive integer $k \geq 1$, and consider the hypercube $\{0,1\}^N$ as a product of k hypercubes $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$ (where $N = n_1 + \cdots + n_k$). We would like to extend the notion of symmetric sets to subsets in $\{0,1\}^N$ which also respect the structure of $\{0,1\}^N$ as a *product of k blocks*. We define a subset $S \subseteq \{0,1\}^N$ to be a k-**wise grid** if $S = S_1 \times \cdots \times S_k$, where each $S_i \subseteq \{0,1\}^{n_i}$ is symmetric. Further, we say $S = S_1 \times \cdots \times S_k$ is a k-**wise layer** if each S_i is a layer. Then we define a general k-**wise symmetric set** to be a union of an arbitrary collection of k-wise layers.

Note that every k-wise grid $S_1 \times \cdots \times S_k$ is a k-wise symmetric set, as given by

$$S = \bigsqcup_{\text{layer } L_i \subseteq S_i,\, i \in [k]} (L_1 \times \cdots \times L_k),$$

but the converse is not true. For instance, the complement of a k-wise layer $L_1 \times \cdots \times L_k$ is k-wise symmetric, as given by

$$\{0,1\}^N \setminus (L_1 \times \cdots \times L_k) = \bigsqcup_{\substack{\emptyset \neq I \subseteq [k] \\ \text{layer } \widetilde{L}_i \subseteq \{0,1\}^{n_i},\, \widetilde{L}_i \neq L_i,\, i \in I \\ \text{layer } L'_i \subseteq \{0,1\}^{n_i},\, i \notin I}} \left(\prod_{i \in I} \widetilde{L}_i \right) \times \left(\prod_{i \notin I} L'_i \right),$$

which is clearly not a k-wise grid.

Covering complements of k-wise grids

Our second main result extends Theorem 6.18 to complements of k-wise grids, Thus, answering Question 6.2 in the affirmative in this case.

Theorem 6.23. *For any nonempty k-wise grid $S = S_1 \times \cdots \times S_k \subseteq \{0,1\}^N$ and $t \geq 1$, we have*

$$\mathsf{EHC}_N^{(t,t-1)}(\{0,1\}^N \setminus S) = \mathsf{EPC}_N^{(t,t-1)}(\{0,1\}^N \setminus S) = \sum_{i=1}^{k} \overline{\Lambda}_{n_i}(S_i) + 2t - 2.$$

Remark 6.24. *The proof of Theorem 6.23 will, in fact, show that for any nonempty k-wise grid $S = S_1 \times \cdots \times S_k \subseteq \{0,1\}^N$ and $t \geq 1$, we have*

$$\begin{aligned}
\mathsf{EHC}_N^{(t,t-1)}(\{0,1\}^N \setminus S) &= \mathsf{EHC}_N^{(1,0)}(\{0,1\}^N \setminus S) + 2t - 2 \\
&= \sum_{i=1}^{k} \overline{\Lambda}_{n_i}(S_i) + 2t - 2 \\
&= \mathsf{EPC}_N^{(1,0)}(\{0,1\}^N \setminus S) + 2t - 2 = \mathsf{EPC}_N^{(t,t-1)}(\{0,1\}^N \setminus S).
\end{aligned}$$

A construction that implies the equality in Theorem 6.23 is a block extension of Example 6.22.

Example 6.25. *Let $S = S_1 \times \cdots \times S_k \subseteq \{0,1\}^N$ be a nonempty k-wise grid, and $t \geq 1$. Then the collection of hyperplanes*

$$\left(\bigsqcup_{j=1}^{k} \left(\mathscr{H}^{*}_{\overline{\mu}_{n_j}(S_j)}(\mathbb{X}_j) \sqcup \mathscr{H}'_{W_{n_j}(\{0,1\}^{n_j} \setminus S_j) \setminus W_{n_j,\overline{\mu}_{n_j}(S_j)}}(\mathbb{X}_j) \right) \right) \sqcup \mathscr{H}^{\circ(t-1)}(\mathbb{X}_1)$$

witnesses the equality in Theorem 6.23.

A special case: covering subcubes and their complements

Here we consider the special case of 2-wise grids, where one of the blocks is a *full* hypercube. The results we mention here are immediate from previous results, and hence we simply mention them without repeating the proofs.

By a **subcube** of a hypercube $\{0,1\}^n$, we mean a subset of the form $\{0,1\}^I \times \{a\}$, where $I \subseteq [n]$ and $a \in \{0,1\}^{[n]\setminus I}$. Since we are now concerned with polynomials with vanishing conditions on a subcube, without loss of generality, we will assume that the subcube is $\mathscr{Q}_m := \{0,1\}^m \times \{0^{n-m}\}$, for some $m \in [0,n]$. This is true since we can permute coordinates, as well as introduce translations of variables in any polynomial without changing the degree of the polynomial. Further, we will assume that $1 \leq m \leq n-1$. So $\mathscr{Q}_m$ is a 2-wise grid, where we consider the product $\{0,1\}^n = \{0,1\}^m \times \{0,1\}^{n-m}$.

Covering complements of subcubes. As a consequence of Theorem 6.23, we immediately get the following about covering complements of subcubes.

Corollary 6.26. *For any $1 \leq m \leq n-1$ and $t \geq 1$, we have*

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus \mathscr{Q}_m) = \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus \mathscr{Q}_m) = n-m+2t-2.$$

In this case, Example 6.25 simplifies to the following.

Example 6.27. *Let $1 \leq m \leq n-1$ and $t \geq 1$. Then the collection of hyperplanes*

$$\{X_{m+1}-1, \ldots, X_n - 1\} \sqcup \mathscr{H}^{\circ(t-1)}(\mathbb{X})$$

witnesses the equality in Corollary 6.26.

A variant of Corollary 6.26 can be obtained by considering arbitrary symmetric sets in the *second block*. For a symmetric set $S \subseteq \{0,1\}^{n-m}$, denote $\mathscr{Q}_m(S) = \{0,1\}^m \times S$.

Corollary 6.28. *For $1 \le m \le n-1$, any nonempty symmetric set $S \subseteq \{0,1\}^{n-m}$, and $t \ge 1$, we have*

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus \mathcal{Q}_m(S)) = \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus \mathcal{Q}_m(S)) = \overline{\Lambda}_{n-m}(S) + 2t - 2.$$

In this case, Example 6.25 simplifies to the following.

Example 6.29. *Let $1 \le m \le n-1$, $S \subseteq \{0,1\}^{n-m}$ be a nonempty symmetric set, and $t \ge 1$. Also denote $\mathbb{X} = (\mathbb{X}', \mathbb{X}'')$ with $\mathbb{X}' = (X_1, \dots, X_m)$, $\mathbb{X}'' = (X_{m+1}, \dots, X_n)$. Then the collection of hyperplanes*

$$\mathscr{H}^*_{\overline{\mu}_{n-m}(S)}(\mathbb{X}'') \sqcup \mathscr{H}'_{W_{n-m}(\{0,1\}^{n-m} \setminus S) \setminus W_{n-m,\overline{\mu}_{n-m}(S)}}(\mathbb{X}'') \sqcup \mathscr{H}^{\circ(t-1)}(\mathbb{X}'')$$

witnesses the equality in Corollary 6.28.

Covering subcubes. It turns out that covering subcubes is easier than covering their complements. In fact, we can even consider a more general case – with an arbitrary symmetric set in the second block, as in Corollary 6.28, as well as more general multiplicities. We give a quick proof here.

Proposition 6.30. *For $1 \le m \le n-1$, any symmetric set $S \subsetneq \{0,1\}^{n-m}$, and $t \ge 1, \ell \in [0, t-1]$, we have*

$$\mathsf{EHC}_n^{(t,\ell)}(\{0,1\}^m \times S) = \mathsf{EHC}_{n-m}^{(t,\ell)}(S).$$

In particular, for any non-empty symmetric set $S \subseteq \{0,1\}^{n-m}$ and $t \ge 1$, we have

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^m \times S) = \mathsf{EHC}_{n-m}^{(t,t-1)}(S) = \Lambda_{n-m}(S) + 2t - 2.$$

Proof. Denote the indeterminates $\mathbb{X} = (\mathbb{X}', \mathbb{X}'')$ with $\mathbb{X}' = (X_1, \dots, X_m)$, $\mathbb{X}'' = (X_{m+1}, \dots, X_n)$. Let $\mathscr{H}(\mathbb{X}) = \{h_1(\mathbb{X}), \dots, h_q(\mathbb{X})\}$ be a (t,ℓ)-exact hyperplane cover for $\{0,1\}^m \times S$ with $q = |\mathscr{H}| = \mathsf{EHC}_n^{(t,\ell)}(\{0,1\}^m \times S)$. Now let $\mathscr{H}''(\mathbb{X}'') = \mathscr{H}(0^m, \mathbb{X}'') = \{h_1(0^m, \mathbb{X}''), \dots, h_q(0^m, \mathbb{X}'')\}$. Then it is immediate that $\mathscr{H}''(\mathbb{X}'')$ is a (t,ℓ)-exact hyperplane cover for $S \subsetneq \{0,1\}^{n-m}$. This implies $\mathsf{EHC}_n^{(t,\ell)}(\{0,1\}^m \times S) \ge \mathsf{EHC}_{n-m}^{(t,\ell)}(S)$.

Conversely, let $\mathscr{H}(\mathbb{X}'') = \{h_1(\mathbb{X}''), \dots, h_q(\mathbb{X}'')\}$ be a (t,ℓ)-exact hyperplane cover for $S \subsetneq \{0,1\}^{n-m}$ with $q = |\mathscr{H}(\mathbb{X}'')| = \mathsf{EHC}_{n-m}^{(t,\ell)}(S)$. Then again, it is immediate that $\overline{\mathscr{H}}(\mathbb{X}', \mathbb{X}'') := \mathscr{H}(\mathbb{X}'')$ is a (t,ℓ)-exact hyperplane cover for $\{0,1\}^m \times S$. This implies $\mathsf{EHC}_n^{(t,\ell)}(\{0,1\}^m \times S) \le \mathsf{EHC}_{n-m}^{(t,\ell)}(S)$. Thus, we have proved the first identity.

The second identity then follows immediately from Theorem 6.18. □

6.3 Our results: higher multiplicity polynomial covers

Let us now look at a few instances where we can solve the polynomial covering problem in broader generality, but not the hyperplane covering problem. In fact, in this extended setting, we will also need some *nondegeneracy conditions* to obtain a clean combinatorial characterization.

Consider the hypercube $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$. Recall that we will now work with the indeterminates $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_k)$, where $\mathbb{X}_j = (X_{j,1}, \ldots, X_{j,n_j})$ are the indeterminates for the j-th block. Let $t \geq 1$, $\ell \in [0, t-1]$, and consider any subset $S \subseteq \{0,1\}^N$. We define

- a (t,ℓ)-**block exact hyperplane cover** for S to be a (t,ℓ)-exact hyperplane cover $\mathscr{H}(\mathbb{X})$ (in $\mathbb{R}^N$) for S such that

$$|\mathscr{H}(a, \mathbb{X}_j)| = |\mathscr{H}(\mathbb{X})|,$$

  for every $a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k}$, $j \in [k]$.

- a (t,ℓ)-**block exact polynomial cover** for S to be a nonzero polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ such that

  (a) the polynomial $P(\mathbb{X})$ vanishes at each point in S with multiplicity at least t,

  (b) for each $j \in [k]$, and every point $(a, \widetilde{a}) \in \{0,1\}^N \setminus S$ with

$$a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k} \text{ and}$$

  $\widetilde{a} \in \{0,1\}^{n_j}$, the polynomial $P(a, \mathbb{X}_j)$ vanishes at $\widetilde{a}$ with multiplicity exactly ℓ.

Let $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$ denote the minimum size of a (t,ℓ)-block exact hyperplane cover for S, and let $\mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$ denote the minimum degree of a (t,ℓ)-block exact polynomial cover for S. It is obvious from the definitions that, in general, we have $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$. For completeness, we give a quick proof in Chapter 7 (see Claim 7.5). Further, it is trivial from the definitions that $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{EHC}^{(t,\ell)}_N(S)$ and $\mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{EPC}^{(t,\ell)}_N(S)$. A blockwise variant of Question 6.2 that we will consider is the following.

Question 6.31. *Given a proper subset $S \subsetneq \{0,1\}^N$ and integers $t \geq 1$, $\ell \in [0, t-1]$, under what conditions can we say that $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) = \mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$?*

Unfortunately, we are unable to answer Question 6.31 in the generality that we consider; in fact, we suspect that the answer could be negative. Instead, we can solve simply the blockwise polynomial covering problem.

Question 6.32. *Given a proper subset* $S \subsetneq \{0,1\}^N$ *and integers* $t \geq 1, \ell \in [0,t-1]$, *under what conditions can we (combinatorially) characterize* $\mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\dots,n_k)}(S)$*?*

6.3.1 Covering *pseudo downward closed* (PDC) k-wise symmetric sets

Our proof technique extends further to a more general class of k-wise symmetric sets to give a characterization for the blockwise polynomial covering problem, that is, we answer Question 6.32. In fact, the tight polynomial construction for this characterization hints that in this generality, the answers to Question 6.2 and Question 6.31 could be negative.

Consider the two obvious total orders $\leq$ and $\leq'$ on $\mathbb{N}$ defined by

$$0 < 1 < 2 < 3 < \cdots \quad \text{and} \quad 0 >' 1 >' 2 >' 3 >' \cdots$$

Let $\mathscr{T} = \{\leq, \leq'\}$. For any $S \subseteq \{0,1\}^N$ and $j \in [k]$, let $S_j \subseteq \{0,1\}^{n_j}$ denote the **projection** of S onto the j-th block. Consider any k-wise symmetric set $S \subseteq \{0,1\}^N$. It is immediate that each S_j is symmetric, $S_1 \times \cdots \times S_k$ is a k-wise grid, and $S \subseteq S_1 \times \cdots \times S_k$. Further, denote

$$W_{(n_1,\dots,n_k)}(S) = \{(|x_1|,\dots,|x_k|) : (x_1,\dots,x_k) \in S\}.$$

Then clearly, $W_{(n_1,\dots,n_k)}(S) \subseteq W_{n_1}(S_1) \times \cdots \times W_{n_k}(S_k)$. For each $j \in [k]$, we consider an arbitrarily chosen total order $\leq_j \in \mathscr{T}$ on $W_{n_j}(S_j)$, say denoted by $W_{n_j}(S_j) = \{w_{j,0} <_j \cdots <_j w_{j,q_j}\}$, and further for each $z_j \in [0,q_j]$, define the symmetric set $[S]_{j,z_j} \subseteq \{0,1\}^{n_j}$ by $W_n([S]_{j,z_j}) = \{w_{j,0} <_j \cdots <_j w_{j,z_j}\}$.

We define a k-symmetric set $S \subseteq \{0,1\}^N$ to be **pseudo downward closed (PDC)** if for every $(w_{1,z_1},\dots,w_{k,z_k}) \in W_{(n_1,\dots,n_k)}(S)$ we have $W_{n_1}([S]_{1,z_1}) \times \cdots \times W_{n_k}([S]_{k,z_k}) \subseteq W_{(n_1,\dots,n_k)}(S)$, that is, $W_{(n_1,\dots,n_k)}(S)$ is *downward closed*[7] in $W_{n_1}(S_1) \times \cdots \times W_{n_k}(S_k)$ under the partial order induced by $\leq_1,\dots,\leq_k$. Further, let

$$\mathscr{N}(S) = \{(z_1,\dots,z_k) \in \mathbb{N}^k : (w_{1,z_1},\dots,w_{k,z_k}) \in W_{(n_1,\dots,n_k)}(S)\}.$$

It is clear that $\mathscr{N}(S)$ is downward closed in $\mathbb{N}^k$. Also let $\mathsf{E}^{(\mathrm{out})}(S)$ denote the set of all minimal elements of the complement set $\mathbb{N}^k \setminus \mathscr{N}(S)$ with respect to the natural partial order on $\mathbb{N}^k$.

[7] For any poset $(P,\leq)$, a subset $D \subseteq P$ is downward closed if for any $x \in D$ we have $y \in D$ for all $y \in P$, $y \leq x$.

It is quite easy to check that the complement of a PDC k-symmetric set is again PDC k-symmetric. Our third main result generalizes Theorem 6.23, but solves only the block polynomial covering problem, that is, answers Question 6.32. Note that in this generality, the combinatorial characterization that we have is nicer to describe in terms of complements.[8]

Theorem 6.33. *For any nonempty PDC k-wise symmetric set $S \subseteq \{0,1\}^N$ and $t \geq 1$, we have*

$$\mathsf{b\text{-}EPC}^{(t,t-1)}_{(n_1,\ldots,n_k)}(\{0,1\}^N \setminus S) = \max_{(z_1,\ldots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \left\{ \sum_{j\in[k]:z_j\geq 1} \overline{\Lambda}_{n_j}([S]_{j,z_j-1}) \right\} + 2t - 2.$$

Remark 6.34. *The proof of Theorem 6.33 will also show that for any nonempty PDC k-wise symmetric set $S \subseteq \{0,1\}^N$ and $t \geq 1$, we have*

$$\begin{aligned}\mathsf{b\text{-}EPC}^{(t,t-1)}_{(n_1,\ldots,n_k)}(\{0,1\}^N \setminus S) &= \mathsf{b\text{-}EPC}^{(1,0)}_{(n_1,\ldots,n_k)}(\{0,1\}^N \setminus S) + 2t - 2 \\ &= \max_{(z_1,\ldots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \left\{ \sum_{i\in[k]:z_i\geq 1} \overline{\Lambda}_{n_i}([S]_{i,z_i-1}) \right\} + 2t - 2.\end{aligned}$$

A construction that implies the equality in Theorem 6.33 can be adapted from Example 6.25 as follows.

Example 6.35. *For any fundamental family of hyperplanes $\mathscr{H}(\mathbb{X}) = \{H_1(\mathbb{X}),\ldots,H_p(\mathbb{X})\}$ defined in Section 6.2, let us abuse notation and also denote the corresponding product polynomial by $\mathscr{H}(\mathbb{X}) = H_1(\mathbb{X})\cdots H_p(\mathbb{X})$. Let $S \subseteq \{0,1\}^N$ be a nonempty PDC k-wise symmetric set, and $t \geq 1$. Assuming notations as in Example 6.25, for each $(z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)$, define*

$$\mathscr{H}_{S,(z_1,\ldots,z_k)}(\mathbb{X}) = \prod_{j\in[k]:z_j\geq 1} \Big(\mathscr{H}^*_{\overline{\mu}_{n_j}([S]_{j,z_j-1})}(\mathbb{X}_j) \cdot \mathscr{H}'_{W_{n_j}(\{0,1\}^{n_j}\setminus[S]_{j,z_j-1})\setminus W_{n_j,\overline{\mu}_{n_j}([S]_{j,z_j-1})}}(\mathbb{X}_j) \Big).$$

Now consider a subfield of $\mathbb{R}$ defined by $\widehat{\mathbb{Q}} = \mathbb{Q}\big(\mathscr{H}_{S,(z_1,\ldots,z_k)}(b) : b \in \{0,1\}^N, (z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)\big)$. It follows that $\mathbb{R}$ is an infinite dimensional $\widehat{\mathbb{Q}}$-vector space. Choose any $\widehat{\mathbb{Q}}$-linearly independent subset $\{\lambda_{S,(z_1,\ldots,z_k)} : (z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)\} \subseteq \mathbb{R}$. Then the polynomial

$$\left(\sum_{(z_1,\ldots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \lambda_{S,(z_1,\ldots,z_k)} \mathscr{H}_{S,(z_1,\ldots,z_k)}(\mathbb{X}) \right) \cdot \mathscr{H}^{\circ(t-1)}(\mathbb{X}_1)$$

witnesses the equality in Theorem 6.33.

---

[8]This is why, for consistency, we have retained the description in terms of complements throughout this work.

Covering k-wise grids

Note that both k-wise grids and their complements are special cases of PDC k-wise symmetric sets. So our first two main results (Theorem 6.18, and Theorem 6.23 via Corollary 6.36) are, in fact, corollaries of our third main result (Theorem 6.33). Further, the tight example of a polynomial cover mentioned in Example 6.35 specializes to the tight examples of hyperplane covers mentioned in Example 6.22 and Example 6.25.

Theorem 6.23 characterizes the hyperplane and polynomial covering problems for complements of k-wise grids. In this case, appealing to Theorem 6.33, we get the following, we see that the blockwise variants of our covering problems are equivalent to the usual *non-blockwise* covering problems.

Corollary 6.36. *For any nonempty k-wise grid* $S = S_1 \times \cdots \times S_k \subseteq \{0,1\}^N$ *and* $t \geq 1$, *we have*

$$\begin{aligned}\text{b-EPC}^{(t,t-1)}_{(n_1,\dots,n_k)}(\{0,1\}^N \setminus S) &= \text{EPC}^{(t,t-1)}_N(\{0,1\}^N \setminus S)\\ &= \sum_{j=1}^{k} \overline{\Lambda}_{n_j}(S_j) + 2t - 2\\ &= \text{EHC}^{(t,t-1)}_N(\{0,1\}^N \setminus S) = \text{b-EHC}^{(t,t-1)}_{(n_1,\dots,n_k)}(\{0,1\}^N \setminus S).\end{aligned}$$

Further, when it comes to covering k-wise grids (and not their complements), we get the following as a corollary of Theorem 6.33.

Corollary 6.37. *For any k-wise grid* $S = S_1 \times \cdots \times S_k \subsetneq \{0,1\}^N$ *and* $t \geq 1$, *we have*

$$\text{b-EPC}^{(t,t-1)}_{(n_1,\dots,n_k)}(S) = \max\left\{\Lambda_{n_j}(S_j) : j \in [k]\right\} + 2t - 2.$$

A construction that implies the equality in Corollary 6.37 is a special case of Example 6.35.

Example 6.38. *Let* $S = S_1 \times \cdots \times S_k \subsetneq \{0,1\}^N$ *be a nonempty k-wise grid, and* $t \geq 1$. *For each* $j \in [k]$, *define*

$$\mathscr{H}_{S_j}(\mathbb{X}_j) = \mathscr{H}^*_{\overline{\mu}_{n_j}(S_j)}(\mathbb{X}_j) \cdot \mathscr{H}'_{W_{n_j}(\{0,1\}^{n_j} \setminus S_j) \setminus W_{n_j,\overline{\mu}_{n_j}(S_j)}}(\mathbb{X}_j).$$

Now consider a subfield of $\mathbb{R}$ *defined by* $\widehat{\mathbb{Q}} = \mathbb{Q}\big(\mathscr{H}_{S_j}(b) : b \in \{0,1\}^{n_j},\ j \in [k]\big)$. *It follows that* $\mathbb{R}$ *is an infinite dimensional* $\widehat{\mathbb{Q}}$*-vector space. Choose any* $\widehat{\mathbb{Q}}$*-linearly independent subset*

$\{\lambda_1,\ldots,\lambda_k\} \subseteq \mathbb{R}$. *Then the polynomial*

$$\left(\sum_{j=1}^{k} \lambda_j \mathscr{H}_{S_j}(\mathbb{X}_j)\right) \cdot \mathscr{H}^{\circ(t-1)}(\mathbb{X}_1)$$

witnesses the equality in Corollary 6.37.

6.3.2 Partial results on other multiplicity polynomial covers

Let us now mention a couple of results on $(t,0)$-exact polynomial covers. The first result concerns the polynomial covering problem for the *Hamming ball*, which is a symmetric set defined by a set of weights of the form $[0,w]$.

Proposition 6.39. *For $w \in [0,n-1]$, let $S \subsetneq \{0,1\}^n$ be the symmetric set defined by $W_n(S) = [0,w-1]$. Then for any $t \in \left[2, \left\lfloor \frac{n+3}{2} \right\rfloor\right]$, we have*

$$\mathsf{EPC}_n^{(t,0)}(S) = w + 2t - 3.$$

Further, the answer to Question 6.2 is negative, in general.

The second result concerns the polynomial covering problem for a single layer. Surprisingly, in this case, our proof employs basic analytic facts about coordinate transformations of polynomials, but we do not know of a proof via the polynomial method.

Proposition 6.40. *For any layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, and $t \geq 1$, we have*

$$\mathsf{EPC}_n^{(t,0)}(S) = t.$$

6.3.3 Cool-down: Index complexity of PDC k-wise symmetric sets

We conclude this by noting that the index complexity, which is a weaker notion for the blockwise covering problems that we consider, can be characterized to a good extent, even in the generality of PDC k-wise symmetric sets. Note that for symmetric sets $S, S' \subseteq \{0,1\}^n$ with $S' \subseteq S$, if $J_{n,a,b} = \text{out-int}(S)$, then $J_{n,a,b} = \text{out-int}(S')$ if and only if $\{a,b\} \subseteq W_n(S')$. This turns out to be an important structural feature that we will work with.

Assume the block decomposition of the hypercube $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$. Now let $S \subseteq \{0,1\}^N$ be a nonempty PDC k-wise symmetric set. Further, for each $j \in [k]$, consider $S_j \subseteq \{0,1\}^{n_j}$ (the j-th projection of S) and let $J_{n_j,a_j,b_j} = \text{out-int}(S_j)$. We define S to be **outer intact** if for every $(z_1,\ldots,z_k) \in \mathsf{E}^{(\text{in})}(S)$ and $j \in [k]$, we have $J_{n_j,a_j,b_j} =$

out-int$([S]_{j,z_j})$. Equivalently, S is outer intact if and only if

$$\{a_j, b_j\} \subseteq W_{n_j}([S]_{j,z_j}) \text{ for each } j \in [k], \quad \text{for every } (z_1, \ldots, z_k) \in \mathsf{E}^{(\mathrm{in})}(S).$$

Proposition 6.41. *For any nonempty outer intact PDC k-wise symmetric set $S \subseteq \{0,1\}^N$, we have*

$$r_N(S) = \sum_{j=1}^{k} r_{n_j}(S_j) = \sum_{j=1}^{k} \mathrm{out}_{n_j}(S_j).$$

An important special case of Proposition 6.41 is for a k-wise layer, which is trivially outer intact PDC. As an immediate corollary of Proposition 6.41 and Proposition 6.12, we get the following.

Corollary 6.42. *For any k-wise layer $S = S_1 \times \cdots \times S_k \subseteq \{0,1\}^N$ with $W_{n_j}(S_j) = \{w_j\}$, $j \in [k]$, we have*

$$r_N(S) = \sum_{j=1}^{k} \min\{w_j, n_j - w_j\}.$$

Proposition 6.41 shows that the index complexity is sensitive only to the blockwise projections, but Theorem 6.33 (for any PDC k-wise symmetric set) shows that the characterization of the polynomial covering problem is more sensitive to the *specific PDC structure*. This adds to our observation that our polynomial method argument is stronger than simply giving a lower bound in terms of index complexity.

6.4 Related work

In addition to the works that motivated our results, there is a plethora of literature on hyperplane covering problems and related questions, over both the reals as well as finite fields. Even more, the polynomial method itself has been subject to intense investigation in the last few decades. We mention here a sample from this vast literature that we believe is most relevant to our present work.

Hyperplane covering problems

- Alon, Bergmann, Coppersmith, and Odlyzko studied a *balancing problem* for sets of binary vectors, which admits a simple reformulation as a hyperplane covering problem. An extension of this problem to higher order complex roots of unity, which takes the form of a polynomial covering problem, was studied by Hegedűs [88].

- Kós, Mészáros, and Rónyai [114] extended the result of Alon and Füredi [6] to the case where the vanishing constraints at every point of the hypercube have multiplicities depending on the individual coordinates of the point. The question in [6] itself was extracted by Bárány from the work of Komjáth [113].
- Linial and Radhakrishnan [117] considered the notion of an *essential hyperplane cover* for the hypercube, which is a minimal family of hyperplanes that are sufficiently *oblique*, and such that every coordinate influences at least one hyperplane. They gave an upper bound of $\lfloor n/2 \rfloor + 1$ and a lower bound of $\Omega(\sqrt{n})$. Saxton [136] gave a tight bound of $n+1$ in the special case wherein the coefficients of all the variables in the affine linear polynomials representing the hyperplanes are restricted to be nonnegative. Recent breakthroughs by Yehuda and Yehudayoff [151], and Araujo, Balogh, and Mattos [16] have improved the lower bound to $n^{5/9-o(1)}$.
- Several extensions and variants of covering problems over finite fields have appeared in the language of hyperplanes as well as in the dual language of *blocking sets*, and the proof techniques in most of these works involve the polynomial method – Jamison [98], Brouwer [34], Ball [19], Zanella [152], Ball and Serra [20], Blokhuis [32], and Bishnoi, Boyadzhiyska, Das and Mészáros [30], to name a few.

The polynomial method

- One of the simplest ways to formally encapsulate the polynomial method is via a classical algebraic object called the *finite-degree Zariski closure*. It was defined by Nie and Wang [126] in the context of combinatorial geometry over finite fields, who studied bounds on its size for arbitrary subsets of the hypercube. However, it had been studied implicitly even earlier by, for instance, Wei [149], Heijnen and Pellikaan [92], Keevash and Sudakov [109], and Ben-Eliezer, Hod, and Lovett [25]. Attempts to characterize the finite-degree Zariski closures of symmetric sets of the hypercube were done in the works of Hegedűs [88, 89], Venkitesh [146], as well as Srinivasan and Venkitesh [141] (and also implicitly in Bernasconi and Egidi [27]).
- A stronger notion than finite-degree Zariski closure is another algebraic object called the *affine Hilbert function*. The affine Hilbert functions of all layers of the hypercube over all fields were determined by Wilson [150]. Further, Bernasconi and Egidi [27] determined the affine Hilbert functions of all symmetric sets of the hypercube over the reals. This was extended to the setting of larger grids by Venkitesh [145].

- An even stronger notion than affine Hilbert functions is yet another algebraic object called the *Gröbner basis*, along with the associated collection of *standard monomials*. Anstee, Rónyai, and Sali [15], and Friedl and Rónyai [70] studied the standard monomials for any subset of the hypercube in terms of a combinatorial phenomenon called *order shattering*. Felszeghy, Ráth, and Rónyai [67] characterized the standard monomials of all symmetric sets of the hypercube via a *lex game*. Hegedűs and Rónyai [90, 91], and Felszeghy, Hegedűs, and Rónyai [68] characterized the Gröbner basis for special cases of symmetric sets of the hypercube.

Chapter 7

Preliminaries

In this chapter, we will refresh some essential preliminary notions, as well as set up terminologies and notations.

7.1 Posets

Let $(P, \leq)$ be a poset, that is, let $\leq$ is a partial order on a nonempty set P. For a subset $S \subseteq P$, we denote $\min_{\leq}(S)$ to be the set of all **minimal elements** of S, and $\max_{\leq}(S)$ to be the set of all **maximal elements**, that is,

$$\min_{\leq}(S) = \{a \in S : (b \in S, b \leq a) \implies b = a\},$$
$$\max_{\leq}(S) = \{a \in S : (b \in S, a \leq b) \implies b = a\}.$$

Further, we define the sets of **outer extremal elements** and **inner extremal elements** of S, respectively, by

$$\mathsf{E}_{\leq}^{(\mathrm{out})}(S) = \min_{\leq}(P \setminus S),$$
$$\text{and} \quad \mathsf{E}_{\leq}^{(\mathrm{in})}(S) = \max_{\leq}(S).$$

A subset $S \subseteq P$ is defined to be **downward closed** if

$$a \in S, b \in P, b \leq a \quad \implies \quad b \in P.$$

For two posets $(P_1, \leq_1)$ and $(P_2, \leq_2)$, the **product poset** is the poset $(P_1 \times P_2, \leq)$, where $\leq$ is defined by

$$(a_1, a_2) \leq (b_1, b_2) \text{ if and only if } a_1 \leq_1 b_1 \text{ and } a_2 \leq_2 b_2.$$

We also say $\leq$ is the induced order on $P_1 \times P_2$.

If we consider the obvious total order $\leq$ on $\mathbb{N}$ given by $0 < 1 < 2 < 3 < \cdots$, then the induced order on $\mathbb{N}^k$ is called the natural order on $\mathbb{N}^k$.

7.2 Symmetry preserving subsets of the hypercube

We are interested in hyperplane and polynomial covering problems for some structured subsets of the hypercube $\{0,1\}^n$, where the *structures* that we are concerned with are specified by invariance under the action of some subgroups of the symmetric group $\mathfrak{S}_n$.

Symmetric sets. Let $S \subseteq \{0,1\}^n$. We say S is symmetric if

$$(x_1, \ldots, x_n) \in S \text{ and } \sigma \in \mathfrak{S}_n \quad \Longrightarrow \quad (x_{\sigma(1)}, \ldots, x_{\sigma(n)}) \in S.$$

It follows immediately that S is symmetric if and only if

$$x \in S,\, y \in \{0,1\}^n, \text{ and } |y| = |x| \quad \Longrightarrow \quad y \in S.$$

In this case, we denote $W_n(S) = \{|x| : x \in S\} \subseteq [0,n]$. So the symmetric set S is completely determined by $W_n(S)$. If $|W_n(S)| = 1$, then we say S is a layer. It is immediate that a subset of the hypercube is symmetric if and only if it is a union of some collection of layers.

Two combinatorial measures. For any $x \in \{0,1\}^n$, the Hamming weight of x is defined by $|x| = \{i \in [n] : x_i = 1\}$. For any subset of coordinates $I \subseteq [n]$, we denote $x_I = (x_i : i \in I) \in \{0,1\}^|I|$. We require a simple combinatorial measure defined in [75]. For a subset $S \subseteq \{0,1\}^n$, the index complexity is defined by

$$r_n(S) = \min\{|I| : I \subseteq [n], \text{there exists } a \in S \text{ such that } b_I \neq a_I \text{ for all } b \in S, b \neq a\}.$$

So $r_n(S)$ is the minimum number of coordinates required to *separate some element in S from all other elements in S.*

An important symmetric set that we will need consists of elements with Hamming weights in an *initial interval* of weights or a *final interval* of weights. For $i \in [0,n]$, define $W_{n,i} = [0,i-1] \cup [n-i+1,n]$, and the symmetric set $T_{n,i} \subseteq \{0,1\}^n$ by $W_n(T_{n,i}) = W_{n,i}$. We also require another combinatorial measure, that is specific to symmetric sets, defined in [146]. For any symmetric set $S \subseteq \{0,1\}^n$, define

$$\begin{aligned} \mu_n(S) &= \max\{i \in [0,\lceil n/2\rceil] : W_{n,i} \subseteq W_n(S)\}, \\ \text{and} \quad \Lambda_n(S) &= |W_n(S)| - \mu_n(S). \end{aligned}$$

Further, we denote $\overline{\mu}_n(S) := \mu_n(\{0,1\}^n \setminus S)$ and $\overline{\Lambda}_n(S) := \Lambda_n(\{0,1\}^n \setminus S)$. We will also need a simple fact about the invariance of the above two combinatorial measures under *complementation of coordinates*. Though it follows straightforwardly from the definitions, here we will give a quick proof using inner and outer intervals.

Fact 7.1. *Let $S \subseteq \{0,1\}^n$ be a symmetric set, and $\widetilde{S}$ be the image of S under the coordinate transformation $(X_1,\ldots,X_n) \mapsto (1-X_1,\ldots,1-X_n)$.*

(a) If $S \neq \{0,1\}^n$, then $\Lambda_n(\widetilde{S}) = \Lambda_n(S)$.

(b) If $S \neq \emptyset$, then $r_n(\widetilde{S}) = r_n(S)$.

Proof. Let $S \subseteq \{0,1\}^n$ be a symmetric set, and $\widetilde{S}$ be the image of S under the coordinate transformation $(X_1,\ldots,X_n) \mapsto (1-X_1,\ldots,1-X_n)$. This implies

$$W_n(\widetilde{S}) = \{n-w : w \in W_n(S)\}.$$

So we get the following observations.

- If out-int$(S) = J_{n,a,b}$, then out-int$(\widetilde{S}) = J_{n,n-b,n-a}$.
- If in-int$(S) = J_{n,a,b}$, then in-int$(\widetilde{S}) = J_{n,n-b,n-a}$.

Then, using Proposition 6.14 and Fact 6.15 complete the proof of Fact 7.1.

□

***Blockwise* symmetric sets.** Now fix a *block decomposition* of the hypercube as $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$. Let $S \subseteq \{0,1\}^N$. We say S is k-wise symmetric if

$$\begin{aligned} & \big((x_{1,1},\ldots,x_{1,n_1}),\ldots,(x_{k,1},\ldots,x_{k,n_k})\big) \in S \text{ and } (\sigma_1,\ldots,\sigma_k) \in \mathfrak{S}_{n_1} \times \cdots \times \mathfrak{S}_{n_k} \\ \implies \quad & \big((x_{1,\sigma_1(1)},\ldots,x_{1,\sigma_1(n_1)}),\ldots,(x_{k,\sigma_k(1)},\ldots,x_{k,\sigma_k(n_k)})\big) \in S. \end{aligned}$$

It follows immediately that S is k-wise symmetric if and only if

$$(x_1,\ldots,x_k)\in S,\ (y_1,\ldots,y_k)\in\{0,1\}^N,\ \text{and } |y_i|=x_i \text{ for all } i\in[k]$$
$$\implies (y_1,\ldots,y_k)\in S.$$

In this case, we denote $W_{(n_1,\ldots,n_k)}(S)=\{(|x_1|,\ldots,|x_k|):(x_1,\ldots,x_k)\in S\}\subseteq[0,n_1]\times\cdots\times[0,n_k]$. So the k-wise symmetric set S is completely determined by $W_{(n_1,\ldots,n_k)}(S)$. For each $j\in[k]$, let $S_j\subseteq\{0,1\}^{n_j}$ denote the j-th projection of S, that is, $S_j=\{x_j\in\{0,1\}^{n_j}:(x_1,\ldots,x_k)\in S\}$. So we clearly have $W_{(n_1,\ldots,n_k)}(S)\subseteq W_{n_1}(S_1)\times\cdots W_{n_k}(S_k)$. We say S is a k-wise grid if $W_{(n_1,\ldots,n_k)}(S)=W_{n_1}(S_1)\times\cdots\times W_{n_k}(S_k)$. We say a S is a k-wise layer if $|W_{(n_1,\ldots,n_k)}(S)|=1$, or equivalently, each S_j is a layer. It is immediate that a subset of a hypercube is k-wise symmetric if and only if it is a union of some collection of k-wise layers.

Consider the two obvious total orders $\leq$ and $\leq'$ on $\mathbb{N}$ defined by

$$0<1<2<3<\cdots \quad \text{and} \quad 0>'1>'2>'3>'\cdots$$

Let $\mathscr{T}=\{\leq,\leq'\}$. Let $S\subseteq\{0,1\}^N$ be k-wise symmetric. Fix arbitrary total orders $\leq_j\in\mathscr{T}$ on $W_{n_j}(S_j)$ for each $j\in[k]$, and consider the induced partial order $\preceq$ on $W_{n_1}(S_1)\times\cdots\times W_{n_k}(S_k)$. We define S to be pseudo downward closed (PDC) if $W_{(n_1,\ldots,n_k)}(S)$ is downward closed in $W_{n_1}(S_1)\times\cdots\times W_{n_k}(S_k)$. Further, for all $j\in[k]$, enumerate $W_{n_j}(S_j)=\{w_{j,0}<_j\cdots<_j w_{j,q_j}\}$, and for each $z_j\in[0,q_j]$, define the symmetric set $[S]_{j,z_j}\subseteq\{0,1\}^{n_j}$ by $W_{n_j}([S]_{j,z_j})=\{w_{j,0}<_j\cdots<_j w_{j,z_j}\}$. Then define

$$\mathscr{N}(S)=\{(z_1,\ldots,z_k)\in\mathbb{N}^k:(w_{1,z_1},\ldots,w_{k,n_k})\in W_{(n_1,\ldots,n_k)}(S)\}.$$

It is immediate that the following are both equivalent conditions to S being PDC.

- $\mathscr{N}(S)$ is downward closed in $\mathbb{N}^k$ with respect to the natural order (also denoted by $\leq$).
- $W_{n_1}([S]_{1,z_1})\times\cdots\times W_{n_k}([S]_{k,z_k})\subseteq W_{(n_1,\ldots,n_k)}(S)$ for each $(z_1,\ldots,z_k)\in\mathscr{N}(S)$.

We will also need two simple indexing sets in our results. We denote

$$\mathsf{E}^{(\mathrm{out})}(S):=\mathsf{E}^{(\mathrm{out})}_{\leq}(\mathscr{N}(S))=\{(z_1,\ldots,z_k)\in\mathbb{N}^k:(w_{1,z_1},\ldots,w_{k,z_k})\in\mathsf{E}^{(\mathrm{out})}_{\preceq}(W_{(n_1,\ldots,n_k)}(S))\},$$
$$\mathsf{E}^{(\mathrm{in})}(S):=\mathsf{E}^{(\mathrm{in})}_{\leq}(\mathscr{N}(S))=\{(z_1,\ldots,z_k)\in\mathbb{N}^k:(w_{1,z_1},\ldots,w_{k,z_k})\in\mathsf{E}^{(\mathrm{in})}_{\preceq}(W_{(n_1,\ldots,n_k)}(S))\}.$$

7.2.1 Polynomials, multiplicities, hyperplanes, and covers

We will work with the polynomial ring $\mathbb{R}[\mathbb{X}]$, where $\mathbb{X} = (X_1, \ldots, X_n)$ are the indeterminates. We are interested in higher order vanishing properties of polynomials. Let $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, denote $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We will denote the α-th order partial derivative of $P(\mathbb{X})$ by $\partial^\alpha P(\mathbb{X})$, that is,

$$\partial^\alpha P(\mathbb{X}) := \frac{\partial^{|\alpha|} P(\mathbb{X})}{\partial X_1^{\alpha_1} \cdots \partial X_n^{\alpha_n}}.$$

For any $t \geq 0$ and $a \in \mathbb{R}^n$, we define the multiplicity of $P(\mathbb{X})$ at a as follows: we define $\text{mult}(P(\mathbb{X}), a) \geq t$ if $\partial^\alpha P(a) = 0$, for all $\alpha \in \mathbb{N}^n$, $|\alpha| < t$. Therefore, we get $\text{mult}(P(\mathbb{X}), a) = t$ if $\text{mult}(P(\mathbb{X}), a) \geq t$ and $\partial^\alpha P(a) \neq 0$ for some $\alpha \in \mathbb{N}^n$ with $|\alpha| = t$.

An affine hyperplane in $\mathbb{R}^n$ is any set of the form $K + v$, where $K \subseteq \mathbb{R}^n$ is a vector subspace with $\dim(K) = n - 1$, and $v \in \mathbb{R}^n$. In the rest of the thesis, we will drop the adjective 'affine' and simply refer to these as hyperplanes. A set $H \subseteq \mathbb{R}^n$ is a hyperplane if and only if $H = \mathscr{Z}(P) := \{a \in \mathbb{R}^n : P(a) = 0\}$ for some nonzero polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ with $\deg(P) = 1$. In fact, we will identify H with its defining affine linear polynomial, and denote $P(\mathbb{X})$ by $H(\mathbb{X})$. So according to the context (which will be obvious), $H(\mathbb{X})$ will either denote the hyperplane as a subset of $\mathbb{R}^n$ or the defining affine linear polynomial. Similarly, if $\mathscr{H}(\mathbb{X}) = \{H_1(\mathbb{X}), \ldots, H_k(\mathbb{X})\}$ is a family of hyperplanes, we may also abuse notation and denote the corresponding defining polynomial by $\mathscr{H}(\mathbb{X}) = H_1(\mathbb{X}) \cdots H_k(\mathbb{X})$. For our concern, the family $\mathscr{H}(\mathbb{X})$ will be a multiset, and $|\mathscr{H}(\mathbb{X})|$ will denote the multiset cardinality of the family, that is, the number of hyperplanes counted with repetition.

We are interested in covering[1] subsets of the hypercube $\{0,1\}^n$ by polynomials and families of hyperplanes. Let $S \subsetneq \{0,1\}^n$, and consider *multiplicity parameters* $t \geq 1$, $\ell \in [0, t-1]$. We define

- a nonzero polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ to be a (t, ℓ)-**exact polynomial cover** for S if

$$\begin{aligned} &\text{mult}(P(\mathbb{X}), a) \geq t \quad \text{for all } a \in S, \\ \text{and} \quad &\text{mult}(P(\mathbb{X}), b) = \ell \quad \text{for all } b \in \{0,1\}^n \setminus S. \end{aligned}$$

[1] We say a polynomial P covers a point a if $a \in \mathscr{Z}(P)$. Similarly, we say P covers a point a with multiplicity at least t if $a \in \mathscr{Z}^t(P)$.

- a finite multiset of hyperplanes $\mathscr{H}(\mathbb{X})$ in $\mathbb{R}^n$ to be a (t,ℓ)-**exact hyperplane cover** for S if

$$\begin{aligned} &|\{H(\mathbb{X}) \in \mathscr{H}(\mathbb{X}) : H(a) = 0\}| \geq t \quad \text{for all } a \in S, \\ \text{and} \quad &|\{H(\mathbb{X}) \in \mathscr{H}(\mathbb{X}) : H(b) = 0\}| = \ell \quad \text{for all } b \in \{0,1\}^n \setminus S. \end{aligned}$$

  This implies that $\mathscr{H}(\mathbb{X})$ is also a (t,ℓ)-exact polynomial cover for S.

Let $\mathsf{EHC}_n^{(t,\ell)}(S)$ denote the minimum size of a (t,ℓ)-exact hyperplane cover for S, and let $\mathsf{EPC}_n^{(t,\ell)}(S)$ denote the minimum degree of a (t,ℓ)-exact polynomial cover for S. The definitions immediately imply that $\mathsf{EHC}_n^{(t,\ell)}(S) \geq \mathsf{EPC}_n^{(t,\ell)}(S)$. But for completeness, we give a quick proof here.

Claim 7.2. *Consider $S \subsetneq \{0,1\}^n$, and $t \geq 1$, $\ell \in [0,t-1]$. Then* $\mathsf{EHC}_n^{(t,\ell)}(S) \geq \mathsf{EPC}_n^{(t,\ell)}(S)$.

Proof. Let $\mathscr{H}(\mathbb{X}) = \{H_1(\mathbb{X}), \ldots, H_k(\mathbb{X})\}$ be a (t,ℓ)-exact hyperplane cover for S. We have

- $|\{i \in [k] : H_i(a) = 0\}| \geq t$ for all $a \in S$.
- $|\{i \in [k] : H_i(b) = 0\}| = \ell$ for all $b \in \{0,1\}^n \setminus S$.

Now consider the polynomial $\mathscr{H}(X) = H_1(\mathbb{X}) \cdots H_k(\mathbb{X})$.

(a) Fix any $a \in S$ and $\alpha \in \mathbb{N}^n$, $|\alpha| \leq t-1$. We have by the product rule for derivatives,

$$(\partial^\alpha \mathscr{H})(a) = \sum_{\substack{\gamma^{(1)},\ldots,\gamma^{(k)} \in \mathbb{N}^n \\ \gamma^{(1)}+\cdots+\gamma^{(k)} = \alpha}} \binom{\alpha}{\gamma^{(1)} \ \cdots \ \gamma^{(k)}} (\partial^{\gamma^{(1)}} H_1)(a) \cdots (\partial^{\gamma^{(k)}} H_k)(a).$$

For each $\gamma^{(1)}, \ldots, \gamma^{(k)} \in \mathbb{N}^n$ with $\gamma^{(1)} + \cdots + \gamma^{(k)} = \alpha$, since $|\{i \in [k] : H_i(a) = 0\}| \geq t$ and $|\gamma^{(1)}| + \cdots + |\gamma^{(k)}| = |\alpha| \leq t-1$, there exists $i \in [k]$ such that $\gamma^{(i)} = 0^n$. This implies $(\partial^{\gamma^{(1)}} H_1)(a) \cdots (\partial^{\gamma^{(k)}} H_k)(a) = 0$. Thus, $(\partial^\alpha \mathscr{H})(a) = 0$.

(b) Fix any $b \in \{0,1\}^n \setminus S$. Since $|\{i \in [k] : H_i(b) = 0\}| = \ell$, by the argument above, we get $(\partial^\beta \mathscr{H})(b) = 0$ for every $\beta \in \mathbb{N}^n$, $|\beta| \leq \ell - 1$. Now recall that the collection $\mathscr{H}(\mathbb{X})$ is a multiset, and suppose we alternatively represent $\mathscr{H}(\mathbb{X}) = \{(F_1(\mathbb{X}))^{(m_1)}, \ldots, (F_v(\mathbb{X}))^{(m_v)}\}$, where $F_1(\mathbb{X}), \ldots, F_v(\mathbb{X})$ are distinct, and $(F_u(\mathbb{X}))^{(m_u)}$ (for $u \in [v]$) indicates m_u copies of $F_u(\mathbb{X})$. Then the condition $|\{i \in [k] : H_i(b) = 0\}| = \ell$ implies that there exists a subset $U \subseteq [v]$ such that $\sum_{u \in U} m_u = \ell$, and $F_u(b) = 0$ exactly when $u \in U$. Further, by definition, we also have the inequality $\ell \leq t-1 < k$. This means $U \subsetneq [v]$, and $F_{u'}(b) \neq 0$ for all $u' \in [v] \setminus U$. Without loss of generality, we may assume $U = [v']$ for some $v' \in [0, v-1]$.

So we have the following.

- $F_u(b) = 0$ if $u \in [v']$, and $F_u(b) \neq 0$ if $u \in [v'+1, v]$.
- $\sum_{u=1}^{v'} m_u = \ell$.

Now for each $u \in [v']$, since $F_u(\mathbb{X})$ is an affine linear polynomial, let $i_u \in [n]$ be the least integer such that $\mathrm{coeff}(X_{i_u}, F_u) \neq 0$. Define $\nu = \sum_{u=1}^{v'} m_u e_{i_u}$. Then we get

$$(\partial^\nu \mathscr{H})(b) = \prod_{u=1}^{v'} (\mathrm{coeff}(X_{i_u}, F_u))^{m_u} \neq 0,$$

where $|\nu| = \sum_{u=1}^{v'} m_u = \ell$.

Thus, $\mathscr{H}(\mathbb{X})$ is a (t, ℓ)-exact polynomial cover for S. This completes the proof.

□

A covering result. In the results of Alon and Füredi [6] (Theorem 6.1), as well as Sauermann and Wigderson [134] (Theorem 6.5), there is nothing special about the origin; one could instead choose to avoid any single point. We will use these version of the results, and therefore state it here.

Theorem 7.3 ([6]). *For any $a \in \{0,1\}^n$, we have*

$$\mathsf{EHC}_n^{(1,0)}(\{0,1\}^n \setminus \{a\}) = \mathsf{EPC}_n^{(1,0)}(\{0,1\}^n \setminus \{a\}) = n.$$

Theorem 7.4 ([135]). *For all $t \geq 1$, $\ell \in [0, t-1]$, and any $a \in \{0,1\}^n$, we have*

$$\mathsf{EPC}_n^{(t,\ell)}(\{0,1\}^n \setminus \{a\}) = \begin{cases} n + 2t - 2 & \text{if } \ell = t-1, \\ n + 2t - 3 & \text{if } \ell < t - 1 \leq \left\lfloor \frac{n+1}{2} \right\rfloor. \end{cases}$$

Nondegenerate polynomial and hyperplane covers for the *blockwise* hypercube. Fix a block decomposition of the hypercube $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$. We will work with the polynomial ring $\mathbb{R}[\mathbb{X}]$, where $\mathbb{X} = (\mathbb{X}_1, \ldots, \mathbb{X}_k)$, and $\mathbb{X}_j$ is the set of indeterminates for the j-th block.

We are interested in covering subsets of the hypercube $\{0,1\}^N$ by polynomials and families of hyperplanes. In this context, our proof techniques work under some nondegeneracy conditions. Let $S \subsetneq \{0,1\}^N$, and consider *multiplicity parameters* $t \geq 1$, $\ell \in [0, t-1]$. We define

- a nonzero polynomial $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ to be a (t,ℓ)-**block exact polynomial cover** for S if
  (a) for every point $a \in S$, we have $\mathrm{mult}(P(\mathbb{X}),a) \geq t$.
  (b) for each $j \in [k]$, and every point $(a,\widetilde{a}) \in \{0,1\}^N \setminus S$ with $a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k}$, $\widetilde{a} \in \{0,1\}^{n_j}$, we have $\mathrm{mult}(P(a,\mathbb{X}_j),\widetilde{a}) = \ell$.
- a finite multiset of hyperplanes $\mathscr{H}(\mathbb{X})$ in $\mathbb{R}^N$ to be a (t,ℓ)-**block exact hyperplane cover** for S if
  (a) for every $a \in S$, we have $|\{H(\mathbb{X}) \in \mathscr{H}(\mathbb{X}) : H(a) = 0\}| \geq t$.
  (b) for every $b \in \{0,1\}^N \setminus S$, we have $|\{H(\mathbb{X}) \in \mathscr{H}(\mathbb{X}) : H(b) = 0\}| = \ell$.
  (c) for each $j \in [k]$, and every $a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k}$, we have $|\mathscr{H}(a,\mathbb{X}_j)| = |\mathscr{H}(\mathbb{X})|$.

  (In other words, no two hyperplanes in the family *collapse* into one, upon restriction to any single block.)

Let $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$ denote the minimum size of a (t,ℓ)-block exact hyperplane cover for S, and let $\mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$ denote the minimum degree of a (t,ℓ)-block exact polynomial cover for S. The definitions immediately imply that every (t,ℓ)-block exact hyperplane cover is also a (t,ℓ)-block exact polynomial cover, and so $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$. For completeness, we give a quick proof here. Further, it is trivial that $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{EHC}^{(t,\ell)}_N(S)$ and $\mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{EPC}^{(t,\ell)}_N(S)$.

Claim 7.5. *Let* $\{0,1\}^N = \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_k}$. *Consider* $S \subsetneq \{0,1\}^N$, *and* $t \geq 1, \ell \in [0,t-1]$. *Then* $\mathsf{b\text{-}EHC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S) \geq \mathsf{b\text{-}EPC}^{(t,\ell)}_{(n_1,\ldots,n_k)}(S)$.

Proof. Let $\mathscr{H}(\mathbb{X}) = \{H_1(\mathbb{X}),\ldots,H_k(\mathbb{X})\}$ be a (t,ℓ)-block exact hyperplane cover for S. We have

- $|\{i \in [k] : H_i(a) = 0\}| \geq t$ for all $a \in S$.
- $|\{i \in [k] : H_i(b) = 0\}| = \ell$ for all $b \in \{0,1\}^N \setminus S$.
- for each $j \in [k]$, and every $a \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k}$, we have $|\mathscr{H}(a,\mathbb{X}_j)| = |\mathscr{H}(\mathbb{X})|$.

Now consider the polynomial $\mathscr{H}(\mathbb{X}) = H_1(\mathbb{X}) \cdots H_k(\mathbb{X})$.

(a) repeating the argument as in the proof of Claim 7.2, case(a), we can show that $(\partial^{\alpha}\mathscr{H})(a) = 0$ for any $a \in S$ and $\alpha \in \mathbb{N}^N$, $|\alpha| \leq t-1$. So $\text{mult}(\mathscr{H}(\mathbb{X}), a) \geq t$ for all $a \in S$.

(b) Fix any $b \in \{0,1\}^N \setminus S$. Again, repeating the argument as in the proof of Claim 7.2, case(b), we can show that $(\partial^{\beta}\mathscr{H})(b) = 0$ for every $\beta \in \mathbb{N}^N$, $|\beta| \leq \ell - 1$. Further, now fix any $j \in [k]$, and denote $b = (b', \widetilde{b})$, where $b' \in \{0,1\}^{n_1} \times \cdots \times \{0,1\}^{n_{j-1}} \times \{0,1\}^{n_{j+1}} \times \cdots \times \{0,1\}^{n_k}$, $\widetilde{b} \in \{0,1\}^{n_j}$. This immediately gives $\partial^{\widetilde{\beta}}\mathscr{H}(b', \mathbb{X}_j)\big|_{\widetilde{b}} = 0$ for every $\widetilde{\beta} \in \mathbb{N}^{n_j}$, $|\widetilde{\beta}| \leq \ell - 1$. Further, we have $|\mathscr{H}(b', \mathbb{X}_j)| = |\mathscr{H}(\mathbb{X})|$. This implies

$$|\{H(b', \mathbb{X}_j) \in \mathscr{H}(b', \mathbb{X}_j) : H(b', \widetilde{b}) = 0\}| = |\{H(\mathbb{X}) \in \mathscr{H}(\mathbb{X}) : H(b) = 0\}| = \ell.$$

Now repeating the argument as in Appendix 7.2(b) over the hypercube $\{0,1\}^{n_j}$, for the point $\widetilde{b} \in \{0,1\}^{n_j}$, we can show that there exists $\nu \in \mathbb{N}^{n_j}$, $|\nu| = \ell$ such that $\partial^{\nu}\mathscr{H}(b', \mathbb{X}_j)\big|_{\widetilde{b}} \neq 0$. So $\text{mult}(\mathscr{H}(b', \mathbb{X}_j), \widetilde{b}) = \ell$.

Thus, $\mathscr{H}(\mathbb{X})$ is a (t, ℓ)-block exact polynomial cover for S. This completes the proof.

□

7.2.2 Peripheral intervals, and inner and outer intervals of symmetric sets

For any $a \in [-1, n-1]$, $b \in [1, n+1]$, $a < b$, denote the set of weights $I_{n,a,b} = [0,a] \cup [b,n]$, and we say a **peripheral interval** is the symmetric set $J_{n,a,b} \subseteq \{0,1\}^n$ defined by $W_n(J_{n,a,b}) = I_{n,a,b}$. Here, we have the convention $[0,-1] = [n+1,n] = \emptyset$. In other words, a peripheral interval $J_{n,a,b}$ could be either (i) *one-sided*, that is, one or both of the weight intervals $[0,a]$, $[b,n]$ could be empty ($a = -1$ or $b = n+1$ or both), or (ii) *two-sided*, that is, both the weight intervals $[0,a]$, $[b,n]$ are nonempty ($a \geq 1$ and $b \leq n$).

Now let $S \subseteq \{0,1\}^n$ be a symmetric set.

- If $S \subsetneq \{0,1\}^n$, then the **inner interval** of S, denoted by in-int(S), is defined to be the peripheral interval $J_{n,a,b} \subseteq \{0,1\}^n$ of maximum size such that $J_{n,a,b} \subseteq S$. It is easy to check that in-int(S) is unique. Further, we define in-int$(\{0,1\}^n) = J_{n,\lfloor n/2\rfloor,\lfloor n/2\rfloor+1}$.
- Let $\mathscr{O}(S)$ be the collection of all peripheral intervals $J_{n,a,b}$ such that $S \subseteq J_{n,a,b}$ and $I_{n,a,b} = W_n(J_{n,a,b})$ has minimum size. It is easy to see that $\mathscr{O}(S)$ can contain several peripheral intervals; the following is an example.

Example 7.6. *Let n be even, and choose $W_n(S) = \{w \in [0,n] : w \text{ is even}\}$. Then for any even $w \in [0,n]$, we have $I_{n,w,w+2} = [0,w] \cup [w+2,n]$, $|I_{n,w,w+2}| = n$ and $S \subseteq J_{n,w,w+2}$. Further, for any peripheral interval $J_{n,a,b} \supseteq S$, it is immediate that $|b-a| \leq 2$, and so $|I_{n,a,b}| \geq n$. Thus, $\mathscr{O}(S) = \{J_{n,w,w+2} : w \in [0,n] \text{ is even}\}$.*

Moving on, consider the function $\lambda_S : \mathscr{O}(S) \to \mathbb{N}$ defined by

$$\lambda_S(J_{n,a,b}) = |a+b-n|, \quad \text{for all } J_{n,a,b} \in \mathscr{O}(S).$$

It is easy to check that the minimizer of λ_S is either a unique peripheral interval $J_{n,a,b}$, or exactly a pair of peripheral intervals $\{J_{n,a,b}, J_{n,n-b,n-a}\}$. The **outer interval** of S, denoted by out-int(S), is defined by

$$\text{out-int}(S) = \begin{cases} J_{n,a,b} & \text{if } J_{n,a,b} \text{ is the unique minimizer of } \lambda_S, \\ J_{n,a,b} & \text{if } \{J_{n,a,b}, J_{n,n-b,n-a}\} \text{ are minimizers of } \lambda_S \text{ and } a > n-b. \end{cases}$$

Therefore, out-int(S) is unique.

Let us discuss more on uniqueness of inner and outer intervals, and look at some illustrations.

Uniqueness of inner and outer intervals Let $S \subseteq \{0,1\}^n$ be a symmetric set. It is immediate that in-int$(\emptyset)$ = out-int$(\emptyset) = J_{n,-1,n+1} = \emptyset$. It is also immediate that in-int$(\{0,1\}^n)$ = out-int$(\{0,1\}^n) = J_{n,\lfloor n/2 \rfloor, \lfloor n/2 \rfloor+1} = \{0,1\}^n$.

Now consider a nonempty symmetric set $S \subsetneq \{0,1\}^n$. We note the following.

- There exists $w \in [0,n]$ such that $w \notin W_n(S)$, that is, $W_n(S) \subseteq [0,w-1] \cup [w+1,n]$. Let

$$a = \max\{a' \in [0,w-1] : [0,a] \subseteq W_n(S)\} \quad \text{with the convention } \max(\emptyset) = -1,$$
$$b = \min\{b' \in [w+1,n] : [b',n] \subseteq W_n(S)\} \quad \text{with the convention } \min(\emptyset) = n+1.$$

  Then it follows immediately by definition that in-int$(S) = J_{n,a,b}$, and is therefore unique.

- Recall that $\mathscr{O}(S)$ is the collection of all peripheral intervals $J_{n,a,b}$ such that $S \subseteq J_{n,a,b}$ and $I_{n,a,b} = W_n(J_{n,a,b})$ has minimum size. Now consider the function $\lambda_S : \mathscr{O}(S) \to \mathbb{N}$ defined by

$$\lambda_S(J_{n,a,b}) = |a+b-n|, \quad \text{for all } J_{n,a,b} \in \mathscr{O}(S).$$

  We observe a simple property of the minimizer of λ_S.

Observation 7.7. *The minimizer of λ_S is either a unique peripheral interval $J_{n,a,b}$, or exactly a pair of peripheral intervals $\{J_{n,a,b}, J_{n,n-b,n-a}\}$.*

Proof. Suppose the minimizer of λ_S is not unique, that is, there are two distinct minimizers $J_{n,a,b}, J_{n,a',b'} \in \mathscr{O}(S)$. Then by definition of $\mathscr{O}(S)$, we already have $|I_{n,a,b}| = |I_{n,a',b'}|$, which implies $a-b = a'-b'$. So there exists $h \in \mathbb{Z}$ such that $a' = a+h$, $b' = b+h$. Further, by the minimization of λ_S, we have $|a+b-n| = |a'+b'-n|$, which yields two cases.

(a) $a+b-n = a'+b'-n$, that is, $a+b = a'+b'$. This implies $h = 0$, and so $a' = a$, $b' = b$.

(b) $a+b-n = n-a'-b'$, that is, $a+b = 2n-(a'+b')$. This implies $h = n-(a+b)$, and so $a' = n-b$, $b' = n-a$.

This completes the proof. □

Recall that out-int(S) is defined by

$$\text{out-int}(S) = \begin{cases} J_{n,a,b} & \text{if } J_{n,a,b} \text{ is the unique minimizer of } \lambda_S, \\ J_{n,a,b} & \text{if } \{J_{n,a,b}, J_{n,n-b,n-a}\} \text{ are minimizers of } \lambda_S \text{ and } a > n-b. \end{cases}$$

Thus, by Observation 7.7, it is immediate that out-int(S) is unique.

Illustrations of inner and outer intervals Let us illustrate some examples of inner and outer intervals. Figure 7.1 shows two typical symmetric sets – *one-sided* and *two-sided* – and their inner and outer intervals.

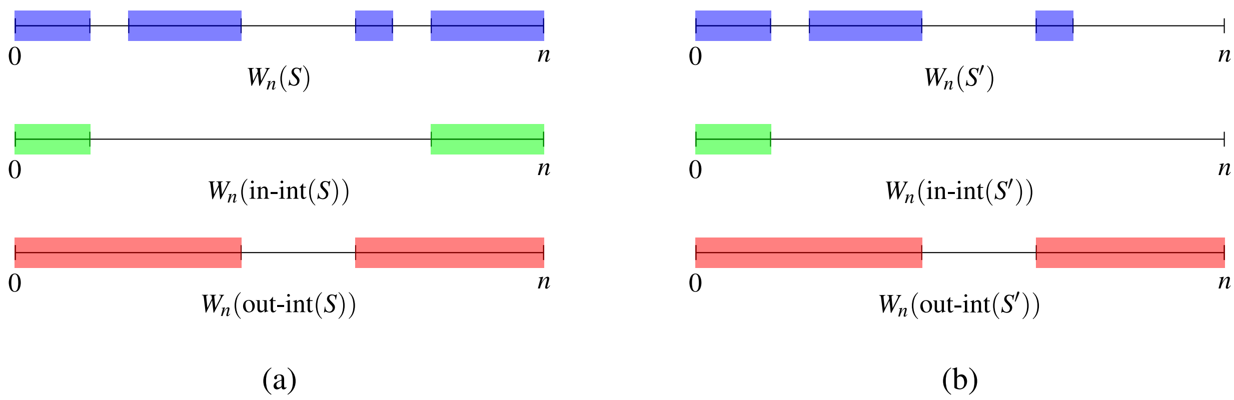

Fig. 7.1 (a) a *two-sided* symmetric set S, and (b) a *one-sided* symmetric set S'

Note that Figure 7.1 is a typical illustration. The inner and outer intervals are special when S itself is either a peripheral interval or the complement of a peripheral interval. Figure 7.2 shows the inner and outer intervals of a *two-sided* peripheral interval and its complement.

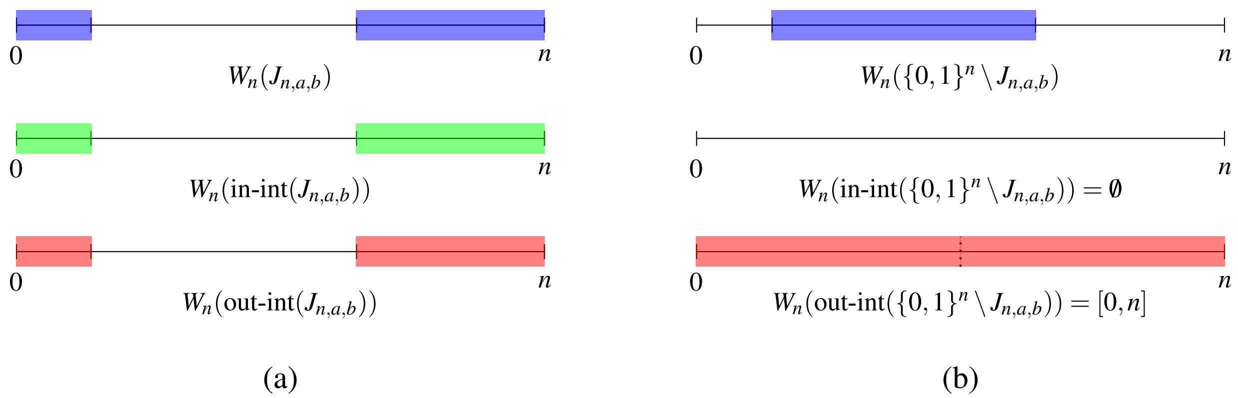

Fig. 7.2 (a) a *two-sided* peripheral interval $J_{n,a,b}$, and (b) the complement of $J_{n,a,b}$

Figure 7.3 shows the inner and outer intervals of a *one-sided* peripheral interval and its complement. Note that the complement of a *one-sided* peripheral interval is again a *one-sided* peripheral interval.

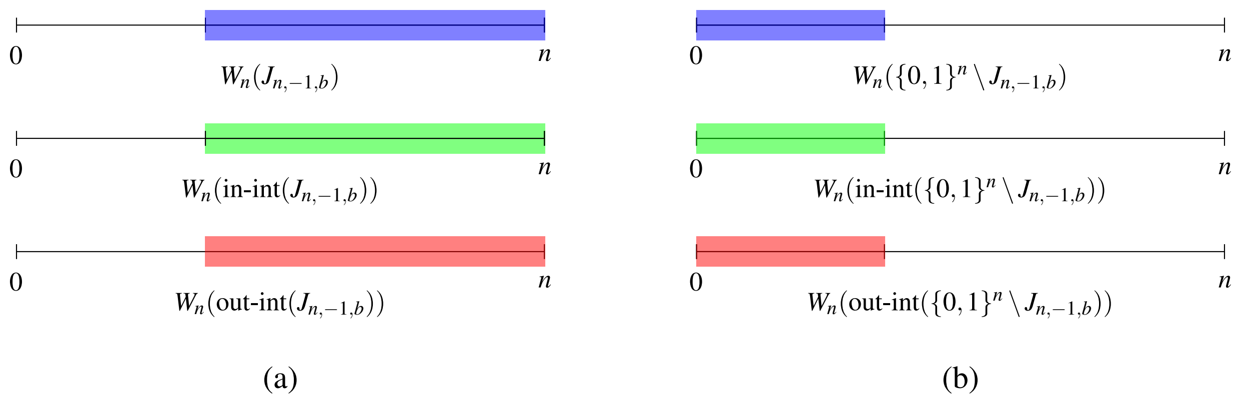

Fig. 7.3 (a) a *one-sided* peripheral interval $J_{n,-1,b}$, and (b) the complement of $J_{n,-1,b}$. Note that the complement is $\{0,1\}^n \setminus J_{n,-1,b} = J_{n,b-1,n+1}$

Now we define

$$\text{in}_n(S) = (\min\{a, n-b\} + 1) + |W_n(S) \setminus W_{n,\min\{a,n-b\}+1}| \quad \text{, where } J_{n,a,b} = \text{in-int}(S),$$

$$\text{and } \text{out}_n(S) = a + n - b + 1 = |I_{n,a,b}| - 1 \quad \text{, where } J_{n,a,b} = \text{out-int}(S).$$

Remark 7.8. *Let $J_{n,a,b} \subseteq \{0,1\}^n$ be a peripheral interval. It is trivial that* in-int$(J_{n,a,b}) =$ out-int$(J_{n,a,b}) = J_{n,a,b}$. *Further, it is easy to check that*

$$\text{in-int}(\{0,1\}^n \setminus J_{n,a,b}) = \begin{cases} \emptyset & \textit{if } a \geq 1, b \leq n, \\ J_{n,a+1,n} & \textit{if } b = n+1, \\ J_{n,0,b-1} & \textit{if } a = -1, \end{cases}$$

$$\textit{and} \quad \text{out-int}(\{0,1\}^n \setminus J_{n,a,b}) = \begin{cases} J_{n,-1,a+1} & \textit{if } a \geq n-b, \\ J_{n,b-1,n+1} & \textit{if } a < n-b. \end{cases}$$

Therefore,

$$\text{in}_n(J_{n,a,b}) = \max\{a, n-b\} + 1, \qquad \text{in}_n(\{0,1\}^n \setminus J_{n,a,b}) = b - a - 1,$$
$$\text{out}_n(J_{n,a,b}) = a + n - b + 1, \qquad \text{out}_n(\{0,1\}^n \setminus J_{n,a,b}) = \min\{n-a, b\} - 1.$$

The following interesting and important observations are immediate from Remark 7.8, and the definitions.

Observation 7.9. *(a) For any peripheral interval $J_{n,a,b} \subseteq \{0,1\}^n$, we have*

$$\text{in}_n(J_{n,a,b}) + \text{out}_n(\{0,1\}^n \setminus J_{n,a,b}) = \text{in}_n(\{0,1\}^n \setminus J_{n,a,b}) + \text{out}_n(J_{n,a,b}) = n.$$

(b) For any symmetric set $S \subseteq \{0,1\}^n$, we have $S =$ in-int$(S) =$ out-int(S) *if and only if either S or $\{0,1\}^n \setminus S$ is a peripheral interval.*

Chapter 8

Covering Symmetric Sets and its Applications in Additive Number Theory

8.1 Introduction

For any non-zero vector $a \in \mathbb{R}^n$ and $b \in \mathbb{R}$, the set of solutions to the affine equation $H(x) := \langle a,x \rangle - b = 0$[1] defines a hyperplane in $\mathbb{R}^n$. We say a point u in $\mathbb{R}^n$ is covered by a hyperplane H if u *lies* on the hyperplane H, that is, if $H(u) = 0$.

Using Combinatorial Nullstellensatz [9, 4], Alon and Füredi [6] proved the following lower bound on the natural except the origin $(0,\dots,0)$.

Theorem 8.1 (Alon and Füredi [6])**.** *Let P be a polynomial in $\mathbb{R}[x_1,\dots,x_n]$ such that P covers every point of $\{0,1\}^n$ expect the origin $(0,\dots,0)$. Then deg$(P) \geq n$.*

Now suppose that we are given an n-cube $\{0,1\}^n$, and we want to cover all its vertices using minimum number of hyperplanes. Observe that, using only two hyperplanes (namely $x_k = 0$ and $x_k - 1 = 0$ for any $k \in [n]$), one can cover all the vertices of the n-cube. Also, note that at least two hyperplanes are required to cover all the vertices of $\{0,1\}^n$.

Bárány inquired about the minimum number m such that there exists a family of m hyperplanes in $\mathbb{R}^n$ covering every point of the hypercube $\{0,1\}^n$ except the origin $(0,\dots,0)$. Bárány derived this problem from a paper by Komjáth on an infinite extension of Rado's Theorem [113]. Komjáth [113] demonstrated that $m \geq \log_2 n - \log_2 \log_2 n$, for all $n \geq 2$. Additionally, note that $m \leq n$, because the hyperplanes $H_1,\dots,H_n$, where $H_i(x) := x_i - 1 = 0$ for all $i \in [n]$, cover every point of $\{0,1\}^n$ except $(0,\dots,0)$. Alon and Füredi proved

[1] For all a, b in $\mathbb{R}^n$, $\langle a,b \rangle$ will denote the standard *inner product* between a and b.

Theorem 8.1 in the context of solving this covering problem. As a direct consequence of Theorem 8.1, we get the following celebrated result in combinatorial geometry.

Theorem 8.2 (Alon and Füredi [6]). *Let m be the least positive integer such that there exists a family of m hyperplanes covering the n-cube $\{0,1\}^n$ leaving out only the origin. Then $m = n$.*

Combinatorial Nullstellensatz [9], and Alon and Furedi's covering result [6] have found multiple extensions and applications in areas like finite geometry, coding theory, combinatorial geometry, and extremal combinatorics, see [4, 20, 115, 114, 64, 29, 24, 56, 134, 30].

We say that a polynomial $P \in \mathbb{R}[x_1,\ldots,x_n]$ has a zero of *multiplicity* at least t at a point $v \in \mathbb{R}^n$ if all derivatives of P up to order $t-1$ vanish at v and $P(v) = 0$.

Ghosh, Kayal and Nandi [75], proved the following polynomial covering result.

Theorem 6.11 ([75]). *For any layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, and any $t \geq 1$, we have*

$$\mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) \geq \max\{w, n-w\} + 2t - 2.$$

A family of hyperplanes $H_1,\ldots,H_m$ in $\mathbb{R}^n$ is said to *cover t times* a point u in $\mathbb{R}^n$ if t hyperplanes from the family cover u. Note that the following corollary is a direct consequence of the above theorem.

Corollary 8.3. *For any layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, and any $t \geq 1$, we have*

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) \geq \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) \geq \max\{w, n-w\} + 2t - 2.$$

In this chapter, we give an explicit construction of a family of hyperplanes matching the lower bound of Corollary 8.3.

Theorem 6.17. *For any layer $S \subsetneq \{0,1\}^n$ with $W_n(S) = \{w\}$, and any $t \geq 1$, we have*

$$\mathsf{EHC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) = \mathsf{EPC}_n^{(t,t-1)}(\{0,1\}^n \setminus S) = \max\{w, n-w\} + 2t - 2.$$

Using ideas from the proofs of Theorems 6.11 and 6.17, we also study a new variant of *restricted sumset* problem and properties of polynomials vanishing on a grid.

8.2 An exact hyperplane cover of a symmetric set

In this section, first we prove Lemma 6.19. Using this, we get an exact hyperplane cover for the symmetric set $T_{n,i} \subseteq \{0,1\}^n$, $i \in [0, \lceil n/2 \rceil]$.

Lemma 8.4 (Restatement of Lemma 6.19). *For $i \in [0, \lfloor n/2 \rfloor]$, the collection of hyperplanes $\{H^*_{(i,j)}(\mathbb{X}) : j \in [i]\}$ defined by*

$$H^*_{(i,j)}(\mathbb{X}) = \sum_{k=1}^{n-j} X_k - (n-2i+j)X_{n-j+1} - (i-j), \quad j \in [i],$$

satisfies the following.

- *For every $a \in T_{n,i}$, there exists $j \in [i]$ such that $H^*_{(i,j)}(a) = 0$.*
- *$H^*_{(i,j)}(b) \neq 0$ for every $b \in \{0,1\}^n \setminus T_{n,i}$, $j \in [i]$.*

Proof. For any $a \in \{0,1\}^n$, denote $I_0(a) = \{t \in [n] : a_t = 0\}$ and $I_1(a) = \{t \in [n] : a_t = 1\}$. Consider any $a \in T_{n,i}$. So $|a| \in [0, i-1] \cup [n-i+1, n]$. We have two cases.

(a) $|a| \in [0, i-1]$. Then $|I_0(a)| \geq n-i+1$. Let t_0 be the $(n-i+1)$-th element in $I_0(a)$. This means $t_0 \in [n-i+1, n]$, which implies that there exists $j \in [i]$ such that $t_0 = n-j+1$. So $a_{n-j+1} = a_{t_0} = 0$. Further, by definition of t_0, we get

$$|a_{[1,n-j]}| = |a_{[1,n-j+1]}| = (n-j+1) - (n-i+1) = i-j.$$

Thus, $H^*_{(i,j)}(a) = (i-j) - (n-2i+j) \cdot 0 - (i-j) = 0$.

(b) $|a| \in [n-i+1, n]$. Then $|I_1(a)| \geq n-i+1$. Let t_1 be the $(n-i+1)$-th element in $I_1(a)$. This means $t_1 \in [n-i+1, n]$, which implies that there exists $j \in [i]$ such that $t_0 = n-j+1$. So $a_{n-j+1} = a_{t_1} = 1$. Further, by definition of t_1, we get

$$|a_{[1,n-j]}| = |a_{[1,n-j+1]}| - 1 = (n-i+1) - 1 = n-i.$$

Thus, $H^*_{(i,j)}(a) = (n-i) - (n-2i+j) \cdot 1 - (i-j) = 0$.

Now consider any $b \in \{0,1\}^n \setminus T_{n,i}$. So $|b| \in [i, n-i]$. Fix any $j \in [i]$. We have two cases.

(a) $b_{n-j+1} = 0$. Then $|b_{[1,n-j]}| \in [i, n-i]$, and so

$$H^*_{(i,j)}(b) \in [i-(i-j), n-i-(i-j)] = [j, n-2i+j],$$

which implies $H^*_{(i,j)}(b) \geq j \geq 1$.

(b) $b_{n-j+1} = 1$. Then $|b_{[1,n-j]}| \in [i-1, n-i-1]$, and so

$$H^*_{(i,j)}(b) \in [i-1-(n-2i+j)-(i-j), n-i-1-(n-2i+j)-(i-j)] = [2i-n-1, -1],$$

which implies $H^*_{(i,j)}(b) \leq -1$.

This completes the proof. $\square$

Then, as we have already seen, the following construction immediately implies the equality in Theorem 6.17.

Example 1 (6.20). *Let $S_k \subsetneq \{0,1\}^n$ be a layer with $W_n(S_k) = w$, and $t \geq 1$. Let $w' = \min\{w, n-w\}$. Denote $H^\circ_0(\mathbb{X}) = X_1$, $H^\circ_1(\mathbb{X}) = X_1 - 1$. Then the collection of hyperplanes*

$$\{H^*_{(w',j)}(\mathbb{X}) : j \in [w']\} \ \sqcup \bigsqcup_{\ell \in [t-1]} \{H^\circ_0(\mathbb{X}), H^\circ_1(\mathbb{X})\} \qquad \text{(disjoint union, as a multiset)}$$

witnesses the equality in Theorem 6.17.

Further, the above example disproves Conjecture 6.8, pertaining to the remaining case '$W_{n,2} \subsetneq W_n(S)$'. For instance, if we consider $n = 7$ and $S \subseteq \{0,1\}^7$ with $W_7(S) = 3$, then Conjecture 6.8 would imply $\mathsf{EHC}(\{0,1\}^7 \setminus S) = 5$. But observe that from Example 6.20 we get that, there exist 4 hyperplanes that cover exactly $\{0,1\}^7 \setminus S$ and therefore disproves Conjecture 6.8.

8.3 Covering subsets of sets with product structures with polynomials

In this section, we shall obtain some results about covering subsets of sets with product structures using ideas from the proof of Theorem 6.10. Let us first recall the definition of the *zero set* of a polynomial.

Definition 8.5 (Set of zeros $\mathbb{Z}(f)$ of a polynomial f)**.** *For a polynomial $f \in \mathbb{F}[x_1,\ldots,x_n]$, zero set of f is defined as $\mathbb{Z}(f) := \{a \in \mathbb{F}^n \mid f(a) = 0\}$.*

The following celebrated theorem, conjectured by Artin (1935), was proved by Chavelley [51] and extended by Warning [148].

Theorem 8.6 (Chevalley-Warning Theorem)**.** *Let $P_1,\ldots,P_m \in \mathbb{Z}_p[x_1,\ldots,x_n]$, for a prime p. If $n > \sum_{i=1}^{m} \deg(P_i)$ and the polynomials P_i have a common zero $(c_1,\ldots,c_n)$ then they have another common zero.*

Recent studies [137, 33, 54, 28] have extended the above fundamental result in different directions. For example, Schauz [137] and Brink [33] gave nice generalizations of Theorem 8.6 which they call *restricted variable* Chevalley Theorems. Instead of considering solutions to our polynomial system $P_1 = \cdots = P_m = 0$ on all of $\mathbb{Z}_p^n$, they consider a subset $X \subset \mathbb{Z}_p^n$ and study the following *restricted zero set* of the system of polynomials:

$$\mathbb{Z}_X(P_1, P_2, \ldots, P_m) := \{a \in X \mid P_1(a) = P_2(a) = \cdots = P_m(a) = 0\}.$$

Clark, Forrow and Schmitt [54], and later Bishnoi and Clark [28] further extended this restricted variable version by allowing varying prime power moduli. Grynkiewicz [79] gave a generalization of Chevalley-Warning Theorem on a complete system of residues modulo m. Given an integer $m \geq 1$, a complete system of residues modulo m is a set $I \subset \mathbb{Z}$ with $|I| = m$ whose elements are distinct modulo m, that is, I contains exactly one representative for every residue class modulo m.

Here we shall give a generalization of the above theorem. We shall use one of the most celebrated techniques in combinatorics, now commonly called "Combinatorial Nullstellensatz", introduced by Alon and Tarsi in the context of graph coloring [9].

Theorem 8.7 (Combinatorial Nullstellensatz [9])**.** *Suppose* $\mathbb{F}$ *be an arbitrary field and* f *be a non-zero polynomial in* $\mathbb{F}[x_1, \ldots, x_n]$ *with* $\deg(f) = \sum_{i=1}^{n} t_i$*, where each* t_i *is a non-negative integer, and the coefficient of the monomial* $\Pi_{i=1}^n x_i^{t_i}$ *in* f *is non-zero. Then for any* $S_1, \ldots, S_n \subseteq \mathbb{F}$*, with* $|S_i| > t_i$*,* $\forall i \in [n]$*,* $\exists\, (s_1, \ldots, s_n) \in S_1 \times \cdots \times S_n$ *such that* $f(s_1, \ldots, s_n) \neq 0$*.*

We shall first give a lower bound on the degree of a polynomial that vanishes on a subset of a grid $S_1 \times S_2 \times \cdots \times S_n$, where S_i's are subsets of any arbitrary field $\mathbb{F}$.

Theorem 8.8. *Suppose* $\mathbb{F}$ *be any arbitrary field and* $S_1, \ldots, S_n$ *be finite subsets of* $\mathbb{F}$ *with* $T \subset S_1 \times \cdots \times S_n$*. Let* f *and* g *be polynomials in* $\mathbb{F}[x_1, \ldots, x_n]$ *such that*

(i) $S_1 \times \cdots \times S_n \setminus T \subseteq \mathbb{Z}(f)$*,*

(ii) $T \cap \mathbb{Z}(f) = \emptyset$*, and*

(iii) $|\, T \cap \mathbb{Z}(g)\, | = |\, T\, | - 1$*.*

Then $\deg(f) + \deg(g) \geq \sum_{i=1}^n (|S_i| - 1)$*.*

Proof. Let g vanishes on T except for $v = (v_1, \ldots, v_n)$, that is, $T \setminus \{v\} \subseteq \mathbb{Z}(g)$ and $g(v) \neq 0$. For each $i \in [n]$, we define $Q_i(x) = \prod_{c_i \in S_i \setminus \{v_i\}} (x_i - c_i)$ and $Q(x) = \prod_{i=1}^n Q_i(x)$. Note that $Q(v) \neq 0$, and $\forall u \in S_1 \times S_2 \times \cdots \times S_n \setminus \{v\}$ we have $Q(u) = 0$.

Consider the polynomial $P(x) = Q(x) - \lambda f(x) g(x)$ where $\lambda = Q(v)[f(v)g(v)]^{-1}$. Note that $P(v) = 0$ and $\forall u \in T \setminus \{v\}$, $P(u) = 0$. Also, observe that $\forall s \in S_1 \times \cdots \times S_n \setminus T$, $P(s) = 0 - \lambda \cdot 0 \cdot g(s) = 0$. Therefore, $\forall s \in S_1 \times \cdots \times S_n$ we have $P(s) = 0$.

To reach a contradiction, assume that $\deg(Q) > \deg(g) + \deg(f)$. Then, $\deg(P) = \deg(Q) = \sum_{i=1}^{n}(|S_i| - 1)$. Again we have, the coefficient of the monomial $x_1^{|S_1|-1} x_2^{|S_2|-1} \dots x_n^{|S_n|-1}$ in P is 1. Using Combinatorial Nullstellensatz (Theorem 8.7) we reach a contradiction, that is, $\exists s \in S_1 \times \cdots \times S_n$ such that $P(s) \neq 0$. Therefore, $\deg(Q) \leq \deg(g) + \deg(f)$, that is, $\deg(f) + \deg(g) \geq \sum_{i=1}^{n}(|S_i| - 1)$. □

This inspires us to define a quantity, namely algebraic complexity $a(S)$ for any finite subset $S = S_1 \times \cdots \times S_n$ of $\mathbb{F}^n$, where $\mathbb{F}$ is any arbitrary field, in the following way:

Definition 8.9 (Algebraic complexity)**.** *Suppose $\mathbb{F}$ be an arbitrary field and S be a subset of $\mathbb{F}^n$. We define the algebraic complexity $a(S)$ of the set S, with $|S| > 1$, to be the smallest integer r such that there exists a polynomial $g \in \mathbb{F}[x_1, \dots, x_n]$ with $\deg(g) = r$ and g vanishes on S except at one point, that is, $a(S) := \min\{\deg(g) \mid g \in \mathbb{F}[x_1, \dots, x_n]$ and $|\mathbb{Z}(g) \cap S| = |S| - 1\}$. If S is a singleton set then we set $a(S) = 0$.*

Suppose $\mathbb{F}$ is a finite field with q elements. Putting $S_1 = \cdots = S_n = \mathbb{F}$ in Theorem 8.8, we get the following:

Corollary 8.10. *Suppose $\mathbb{F}$ be a finite field with q elements and $T(\neq \emptyset) \subset \mathbb{F}^n$ with $a(T) = r$. If f be a polynomial in $\mathbb{F}[x_1, \dots, x_n]$ such that $\mathbb{F}^n \setminus T \subseteq \mathbb{Z}(f)$ and $T \cap \mathbb{Z}(f) = \emptyset$. Then* $\deg(f) + r \geq n(q-1)$.

This gives us the following generalization of Chevalley-Warning Theorem (Theorem 8.6).

Corollary 8.11 (Generalization of Chevalley-Warning theorem)**.** *Let $\mathbb{F}$ be a field with q elements and $P_1, \dots, P_m$ be polynomials in $\mathbb{F}[x_1, \dots, x_n]$ such that $T(\neq \emptyset) \subset \bigcap_{i=1}^{m} \mathbb{Z}(P_i)$ and $\sum_{i=1}^{m} \deg(P_i) < n - \frac{r}{q-1}$, where $r = a(T)$. Then P_i's have a common zero outside T.*

Proof. To reach a contradiction, assume that for all $u \in \mathbb{F}^n \setminus T$, $\exists j \in [m]$ such that $P_j(u) \neq 0$.

Consider the polynomial $f(x) = \prod_{i=1}^{m}\left(1 - P_i^{q-1}(x)\right)$. Observe that for all $v \in T$, we have $f(v) = 1$, and therefore $T \cap \mathbb{Z}(f) = \emptyset$. For any $u \in \mathbb{F}^n \setminus T$, using the fact that, there exists $j \in [m]$ with $P_j(u) \neq 0$ we get $f(u) = 0$. Using the facts that $\deg(f) \geq n(q-1) - r$ (from Corollary 8.10) and $\deg(f) \leq (q-1)\sum_{i=1}^{m} \deg(P_i)$, we get $\sum_{i=1}^{m} \deg(P_i) \geq n - \frac{r}{q-1}$. This contradicts the fact that, $\sum_{i=1}^{m} \deg(P_i) < n - \frac{r}{q-1}$. □

Remark 8.12. *If we take T such that $|T| = 1$ in Corollary 8.11 then we get back original Chevalley-Warning Theorem (Theorem 8.6).*

8.4 Restricted sumset problem

Motivated by the above results, here we introduce a new variant of *restricted sumset* problem in terms of a *forbidden set*. More formally, let $A_1,\ldots,A_n$ be subsets of an arbitrary field $\mathbb{F}$ and $S \subseteq A_1 \times \cdots \times A_n$ then we define

$$\oplus_S \sum_{i=1}^{n} A_i := \left\{ \sum_{i=1}^{n} a_i \;\middle|\; (a_1,\ldots,a_n) \in A_1 \times \cdots \times A_n \setminus S \right\}$$

We will prove some interesting lower bounds for the size of $\oplus_S \sum_{i=1}^{n} A_i$ using ideas from the proof of Theorem 6.17. We will construct some explicit extremal examples where our lower bound outperforms the lower bounds guaranteed by Alon, Nathanson, and Ruza [10] (Theorem 8.13), results on restricted sumset problems. Using our approach we will also give a simple alternate proof of Erdős-Heilbronn Conjecture.

Let h be a polynomial in $\mathbb{Z}_p[x_1,\ldots,x_n]$, where p is a prime, and $S_1,\ldots,S_n \subset \mathbb{Z}_p$. Alon, Nathanson and Ruzsa [10] considered the following *restricted sumset* :

$$\left\{ \sum_{i=1}^{n} s_i \;\middle|\; (s_1,\ldots,s_n) \in (S_1 \times \cdots \times S_n) \setminus \mathbb{Z}(h) \right\},$$

where $\mathbb{Z}(h)$ is the *zero set* of the polynomial h.

Alon, Nathanson and Ruzsa [10] proved the following general lower bound for the restricted sumset problem.

Theorem 8.13 (Alon, Nathanson and Ruzsa [10]). *Let p be a prime and h be a polynomial in $\mathbb{Z}_p[x_1,\ldots,x_n]$. Let $S_1,\ldots,S_n \subset \mathbb{Z}_p$, with $|S_i| = c_i + 1$ and define $m = \sum_{i=1}^{n} c_i - \deg(h)$. If the coefficient of $\prod_{i=1}^{n} x_i^{c_i}$ in $(\sum_{i=1}^{n} x_i)^m h(x)$ is non-zero, then*

$$\left| \left\{ \sum_{i=1}^{n} s_i \;\middle|\; (s_1,\ldots,s_n) \in (S_1 \times \cdots \times S_n) \setminus \mathbb{Z}(h) \right\} \right| \geq m+1.$$

Consider the polynomial

$$h(x) := \prod_{i=1}^{n-1} h_i(x) \in \mathbb{Z}_p[x_1,x_2,\ldots,x_n],$$

where $\forall j \in [n-2]$ we have $h_j(x) = nx_1 + \sum_{i=2}^{n} x_i - j$, and $h_{n-1}(x) = \sum_{i=2}^{n} x_i - (n-1)$.

Claim 8.14. *If $p > n$ then the coefficient of $\prod_{i=1}^{n} x_i$ in $(\sum_{i=1}^{n} x_i)\, h(x)$ is non-zero.*

Proof. Observe that the coefficient of $\prod_{i=1}^n x_i$ in $(\sum_{i=1}^n x_i)\, h(x)$ and $\tau(x)$ is the same, where

$$\begin{aligned}\tau(x) &= \left(\sum_{i=1}^n x_i\right)\left(nx_1+\sum_{i=2}^n x_i\right)^{n-2}\left(\sum_{i=2}^n x_i\right)\\ &= \left(x_1\left(\sum_{i=2}^n x_i\right)+\left(\sum_{i=2}^n x_i\right)^2\right)\left(nx_1+\sum_{i=2}^n x_i\right)^{n-2}.\end{aligned}$$

As $p>n$, the coefficient of $\prod_{i=1}^n x_i$ in $\tau(x)$ is $\binom{n-1}{1}(n-2)!+2n\binom{n-1}{2}(n-2)! = (n-1)^3(n-2)! \neq 0$. □

As $\deg(h)=n-1$, therefore using Claim 8.14 and Theorem 8.13, we get that

$$\left|\left\{\sum_{i=1}^n s_i \mid (s_1,\ldots,s_n)\in\{0,1\}^n\setminus\mathbb{Z}(h)\right\}\right| \geq 2.$$

Observe that,

$$\mathbb{Z}(h)\cap\{0,1\}^n = \left\{x\in\{0,1\}^n \,\middle|\, x_1=0 \text{ and } \exists i>1 \text{ such that } x_i=1\right\}\bigcup\left\{(1,\ldots,1)\right\},$$

which gives us,

$$\begin{aligned}&\left\{\sum_{i=1}^n s_i \,\middle|\, (s_1,\ldots,s_n)\in\{0,1\}^n\setminus\mathbb{Z}(h)\right\}\\ &\qquad = \left\{x\in\{0,1\}^n \,\middle|\, x_1=1 \text{ and } \exists i>1 \text{ such that } x_i=0\right\}\bigcup\left\{(0,\ldots,0)\right\}.\end{aligned}$$

So, we get that

$$\left|\left\{\sum_{i=1}^n s_i \mid (s_1,\ldots,s_n)\in\{0,1\}^n\setminus\mathbb{Z}(h)\right\}\right| = n.$$

Therefore, we observe that for this particular polynomial h, Theorem 8.13 gives a bound which is far from being tight. As we observe that the lower the degree of the polynomial h, the better the bound we get using Theorem 8.13, one may think of that if we can give a polynomial, say $\tilde{h}$, with $\deg(\tilde{h})<n-1$ and $\mathbb{Z}(\tilde{h})\cap\{0,1\}^n=\mathbb{Z}(h)\cap\{0,1\}^n$ then using Theorem 8.17 for the polynomial $\tilde{h}$ we may get a better bound. But this is not possible. We can show that for any polynomial $\tilde{h}$ such that $\mathbb{Z}(\tilde{h})\cap\{0,1\}^n=\mathbb{Z}(h)\cap\{0,1\}^n$ we must have $\deg(\tilde{h})\geq(n-1)$. We get this as a corollary of the following Theorem 8.15 proved by Alon and Füredi in [6]:

Theorem 8.15 (Alon and Füredi [6]). *Suppose* $\mathbb{F}$ *is any field (finite or infinite) and for each* $i \in [n]$*,* S_i *is a non-empty finite subset of* $\mathbb{F}$*. If* P *is a polynomial in* $\mathbb{F}[x_1, \ldots, x_n]$ *such that* $\exists c \in S_1 \times \cdots \times S_n$ *with* $P(c) \neq 0$ *and* $\forall \tilde{c} \in S_1 \times \cdots \times S_n \setminus \{c\}$*,* $P(\tilde{c}) = 0$ *then* $\deg(P) \geq \sum_{i=1}^{n} (|S_i| - 1)$.
Corollary 8.16. *If* f *be a polynomial in* $\mathbb{F}[x_1, \ldots, x_n]$ *such that*

$$\mathbb{Z}(f) \cap \{0,1\}^n = \left\{ x \in \{0,1\}^n \;\middle|\; x_1 = 0 \text{ and } \exists i > 1 \text{ such that } x_i = 1 \right\} \bigcup \left\{ (1, \ldots, 1) \right\},$$

then $\deg(f) \geq n - 1$

Proof. Consider the polynomial $g(x) = x_1 - 1 \in \mathbb{F}[x_1, \ldots, x_n]$ and define $P(x) \in \mathbb{F}[x_1, \ldots, x_n]$ by $P(x) = f(x)g(x)$. Now if we take $v = (0, \ldots, 0) \in \{0,1\}^n$ then we observe that $P(v) \neq 0$ and $\forall u \in \{0,1\}^n \setminus \{v\}$, $P(u) = 0$. So, by Theorem 8.15, $\deg(P) \geq n$. Since $\deg(P) = \deg(f) + \deg(g)$, we get the required result. □

The above discussion together with Theorem 6.10 naturally motivates us to define an alternative variation of restricted sumset problem, where the restriction is given on a subset of $\mathbb{Z}_p^n$, instead of a zero set of a polynomial in $\mathbb{Z}_p[x_1, \ldots, x_n]$. We will now prove the following theorem that gives a lower bound on the cardinality of $\oplus_S \sum\limits_{i=1}^{n} A_i$:
Theorem 8.17. *Let* $\mathbb{F}$ *be any field,* $A_1, \ldots, A_n$ *be finite subsets of* $\mathbb{F}$*,* $A = A_1 \times \cdots \times A_n$ *and* $S \subset A$*. Suppose* $g \in \mathbb{F}[x_1, \ldots, x_n]$ *is the lowest degree polynomial such that* $|S \cap \mathbb{Z}(g)| = |S| - 1$*. For all* $i \in [n]$*, let* $|A_i| = c_i + 1$ *and we define* $m = \sum_{i=1}^{n} c_i - \deg(g) - 1$*. If the coefficient of the monomial* $\prod_{i=1}^{n} x_i^{c_i}$ *in* $x_k g(x) \left(\sum_{i=1}^{n} x_i\right)^m$*, for some* $k \in [n]$*, is non-zero (in* $\mathbb{F}$*) then* $|\oplus_S \sum_{i=1}^{n} A_i| \geq m + 1$.

Proof. To reach a contradiction assume that $|\oplus_S \sum_{i=1}^{n} A_i| < m + 1$. Then there exists a subset B of $\mathbb{F}$ with size m containing $\oplus_S \sum\limits_{i=1}^{n} A_i$. Let $S \cap \mathbb{Z}(g) = S \setminus \{v\}$, for some $v = (v_1, v_2, \ldots, v_n) \in S$. So $\forall u \in S \setminus \{v\}$, $g(u) = 0$. Now we define

$$f_k(x) := g(x)(x_k - v_k) \prod_{b \in B} \left(\sum_{i=1}^{n} x_i - b \right).$$

For all $u = (u_1, \ldots, u_n) \in S$ we have $g(u)(u_k - v_k) = 0$, and this implies $f_k(u) = 0$. Again, for all $u = (u_1, \ldots, u_n) \in A \setminus S$ we have $\prod_{b \in B} \left(\sum_{i=1}^{n} u_i - b\right) = 0$, and therefore $f_k(u) = 0$. So we get, $\forall u \in A$, $f_k(u) = 0$.

Observe that

$$\deg(f_k) = \deg(g) + 1 + m = \sum_{i=1}^{n} c_i,$$

and the coefficient of the monomial $\prod_{i=1}^{n} x_i^{c_i}$ in f_k is same as that in $x_k g(x)(x_1 + \cdots + x_n)^m$, which is non-zero for some $k \in [n]$ by our assumption. So by Theorem 8.7, $\exists a \in A$ such that $f_k(a) \neq 0$, but this contradicts the fact that $f_k(u) = 0$ for all $u \in A$. Therefore our starting assumption that $\left| \oplus_S \sum_{i=1}^{n} A_i \right| < m+1$ is false. □

Observation 8.18. *Consider the set* $S = \left\{ x \in \{0,1\}^n \;\middle|\; x_1 = 0 \text{ and } \exists i > 1 \text{ such that } x_i = 1 \right\} \cup \{(1,\ldots,1)\}$. *Then, using Theorem 8.17 we get* $\left| \oplus_S \sum_{i=1}^{n} A_i \right| \geq n-1$.

Note that the lower bound almost matches the exact value (which is n) which is much better than the lower bound we get using Theorem 8.13.

Proof. Suppose $g(x) = x_1$. The coefficient of $\prod_{i=1}^{n} x_i$ in the polynomial $x_2 g(x) \left(\sum_{i=1}^{n} x_i\right)^{n-2}$ is $(n-2)!$. As $n < p$, $(n-2)! \neq 0$. Therefore, from Theorem 8.17 we get $|\oplus_S \sum_{i=1}^{n} A_i| \geq n-1$. □

We will now give an alternative proof of Erdős-Heilbronn Conjecture.

Theorem 8.19 (Erdős-Heilbronn Conjecture, see [10])**.** *If p is a prime and A is a non-empty subset of* $\mathbb{Z}_p$ *then* $|\{a+a' \mid a,a' \in A, a \neq a'\}| \geq \min\{p, 2|A|-3\}$.

Proof. The conjecture is trivial when $p = 2$. So from now on we take $p > 2$. First we consider the case $2|A|-3 \geq p$. Therefore, for all $u \in \mathbb{Z}_p$, we have $(u-A) \cap \left(A \setminus \{2^{-1}u\}\right) \neq \emptyset$. This implies that $\left\{a+a' \;\middle|\; a,a' \in A, a \neq a'\right\} = \mathbb{Z}_p$. So, we are done in this case.

Next we consider the case $2|A|-3 < p$. Suppose $S = \{(a,a) \mid a \in A\} \cup \{(a',a'')\}$, for some $\{a',a''\} \subset A$ and $g \in \mathbb{Z}_p[x_1,x_2]$ be defined by $g(x) = x_1 - x_2$. Then $\forall u \in S \setminus \{(a',a'')\}$, $g(u) = 0$, that is, $|\mathbb{Z}(g) \cap S| = |S| - 1$.

Let $m = 2|A|-4$. Then the coefficient of $(x_1x_2)^{|A|-1}$ in $x_2 g(x)(x_1+x_2)^m$ is

$$\binom{2|A|-4}{|A|-2} - \binom{2|A|-4}{|A|-1} = \frac{(2q-2)(2q-3)(2q-4)\ldots(q+1)}{(q-1)!},$$

where $q = |A|-1$. So by our assumption $2q-2 < p$ and hence the coefficient is non-zero in $\mathbb{Z}_p$. Therefore, using Theorem 8.17, we get $|\oplus_S(A+A)| \geq m+1$. As $|\{a+a' \mid a,a' \in A, a \neq a'\}| = |\oplus_S (A+A)|$, so we get $\left|\{a+a' \;\middle|\; a,a' \in A, a \neq a'\}\right| \geq 2|A|-3$. □

Chapter 9

Covering Symmetry Preserving Subsets of the Hypercube

9.1 Introduction

In this chapter we study Question 6.2 for symmetry preserving subsets of the hypercube, that is,

If $S \subseteq \{0,1\}^n$ is a symmetry preserving subsets of the hypercube and $t \geq 1$, $\ell \in [0, t-1]$, then can we show that $\mathsf{EHC}_n^{(t,\ell)}(S) = \mathsf{EPC}_n^{(t,\ell)}(S)$?

To obtain the lower bound of the polynomial covering $\mathsf{EPC}_n^{(t,\ell)}(S)$ we shall study the index complexity of a symmetry preserving subset $S \subseteq \{0,1\}^n$ in detail.

9.2 Index complexity of symmetric and PDC k-wise symmetric sets

9.2.1 Inner and outer intervals of symmetric sets

Let us first prove Proposition 6.13, which relates the inner and outer intervals of symmetric sets. We will give two proofs, one combinatorial and another via the polynomial method. We restate the result, for convenience.

Proposition 1 (Restatement of Proposition 6.13)**.** *For any nonempty symmetric set $S \subseteq \{0,1\}^n$, we have*

$$\mathrm{in}_n(\{0,1\}^n \setminus S) + \mathrm{out}_n(S) \geq n.$$

Further, equality holds if and only if either S or $\{0,1\}^n \setminus S$ is a peripheral interval.

First proof. We note that the assertion is immediately true, by Observation 7.9(a), if either S or $\{0,1\}^n \setminus S$ is a peripheral interval.

Now suppose $S \subseteq \{0,1\}^n$ is some nonempty symmetric set. Let $J_{n,a,b} = \text{out-int}(S)$. So by definition, we get $\text{out}_n(S) = \text{out}_n(J_{n,a,b})$. It is, therefore, enough to prove $\text{in}_n(\{0,1\}^n \setminus S) \geq \text{in}_n(\{0,1\}^n \setminus J_{n,a,b})$.

Let $J_{n,a',b'} = \text{in-int}(\{0,1\}^n \setminus S)$, and

$$M_n(S) := \{w \in W_n(\{0,1\}^n \setminus S) : \min\{a', n-b'\} + 1 \leq w \leq \max\{n-a', b'\} - 1\}.$$

So, by definition, $\text{in}_n(\{0,1\}^n \setminus S) = \min\{a', n-b'\} + 1 + |M_n(S)|$. Also, since $J_{n,a,b} = \text{out-int}(S)$, we get

$$[0,n] \setminus I_{n,a,b} \subseteq I_{n,\min\{a',n-b'\},\max\{n-a',b'\}} \sqcup M_n(S).$$

This immediately gives

$$\begin{aligned} \text{in}_n(\{0,1\}^n \setminus J_{n,a,b}) &\leq |I_{n,\min\{a',n-b'\},\max\{n-a',b'\}}| + |M_n(S)| \\ &= \min\{a', n-b'\} + 1 + |M_n(S)| \\ &= \text{in}_n(\{0,1\}^n \setminus S). \end{aligned} \tag{9.1}$$

It is clear that equality is attained exactly when $\text{in}_n(\{0,1\}^n \setminus S) = \text{in}_n(\{0,1\}^n \setminus J_{n,a,b})$. By (9.1), this means equality is attained exactly when $\text{in}_n(\{0,1\}^n \setminus J_{n,a,b}) = \min\{a', n-b'\} + 1 + |M_n(S)|$. This happens exactly when $S = J_{n,a,b} = J_{n,a',b'}$, that is, $S = \text{in-int}(S) = \text{out-int}(S)$. By Observation 7.9(b), this is equivalent to either S or $\{0,1\}^n \setminus S$ being a peripheral interval. □

Second proof. Let $J_{n,a,b} = \text{out-int}(S)$. By the minimality of size of $I_{n,a,b}$, we have $\{a,b\} \subseteq W_n(S)$. Without loss of generality, assume $a \geq n-b$. Define $P(\mathbb{X}) = X_1 \cdots X_a (X_{a+1} - 1) \cdots (X_{a+n-b+1} - 1)$. We clearly have $1^a 0^{n-a} \in S$ and $P(1^a 0^{n-a}) \neq 0$. Now consider any $x \in S, x \neq 1^a 0^{n-a}$. We have three cases.

(C1) $|x| = a$ and $x \neq 1^a 0^{n-a}$. Then there exists $i \in [1,a]$ such that $x_i = 0$, so $P(x) = 0$.

(C2) $|x| < a$. Then there exists $i \in [1,a]$ such that $x_i = 0$, so $P(x) = 0$.

(C3) $|x| > a$, which means $|x| \geq b$, since $S \subseteq J_{n,a,b}$. So $|\{i \in [n] : x_i = 0\}| < n - b + 1$. This implies that there exists $i \in [a+1, a+n-b+1]$ such that $x_i = 1$, and so $P(x) = 0$.

Consider the family of hyperplanes

$$\mathsf{h}(\mathbb{X}) := \mathscr{H}^{*}_{\overline{\mu}_n(S)}(\mathbb{X}) \sqcup \mathscr{H}'_{W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}}(\mathbb{X}).$$

By Lemma 6.19, we have $\mathscr{H}^{*}_{\overline{\mu}_n(S)}(x) = 0$ if and only if $|x| \in W_{n,\overline{\mu}_n(S)}$. Further, by definition, we have $\mathscr{H}'_{W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}}(x) = 0$ if and only if $|x| \in W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}$. Thus, we have $\mathsf{h}(x) = 0$ if and only if $x \in \{0,1\}^n \setminus S$.

So we conclude that the polynomial $P\mathsf{h}(\mathbb{X})$ satisfies $P\mathsf{h}(1^a 0^{n-a}) \neq 0$, and $P\mathsf{h}(x) = 0$ for all $x \in \{0,1\}^n, x \neq 1^a 0^{n-a}$. Therefore, by Theorem 7.3, we get $\deg(P\mathsf{h}) \geq n$. Now by the definitions, we also have

$$\begin{aligned} &\deg(P) = a + n - b + 1 = \mathrm{out}_n(S), \\ \text{and} \quad &\deg(\mathsf{h}) = \overline{\mu}_n(S) + |W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}| = \mathrm{in}_n(\{0,1\}^n \setminus S). \end{aligned}$$

Hence,

$$\mathrm{in}_n(\{0,1\}^n \setminus S) + \mathrm{out}_n(S) = \deg(\mathsf{h}) + \deg(P) = \deg(P\mathsf{h}) \geq n.$$

By the definition above, $\deg(P) = a + n - b + 1 = \mathrm{out}_n(S)$, and therefore we have shown that $\deg(\mathsf{h}) \geq b - a - 1$. But, again by the definition above, we have $\deg(\mathsf{h}) = \overline{\mu}_n(S) + |W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}|$. Thus, we have equality exactly when $\overline{\mu}_n(S) + |W_n(\{0,1\}^n \setminus S) \setminus W_{n,\overline{\mu}_n(S)}| = b - a - 1$, where $J_{n,a,b} = \text{out-int}(S)$. This is true if and only if $S = J_{n,a,b}$. By Observation 7.9(b), this is equivalent to either S or $\{0,1\}^n \setminus S$ being a peripheral interval. $\square$

9.2.2 Index complexity of symmetric sets

We will now proceed to prove Proposition 6.14 which characterizes the index complexity of symmetric sets. We will need a definition and a technical lemma. Let $p \in \{0,1\}^n$ and denote $I_0(p) := \{i \in [n] : p_i = 0\}, I_1(p) := \{i \in [n] : p_i = 1\}$. So $|I_0(p)| = n - |p|$ and $|I_1(p)| = |p|$. For any $I_0 \subseteq I_0(p), I_1 \subseteq I_1(p)$, we define the **separation of p with respect to** (I_0, I_1), denoted by $\mathrm{sep}(p, I_0, I_1) \subseteq \{0,1\}^n$, to be the maximal symmetric set such that for every $x \in \mathrm{sep}(p, I_0, I_1)$, we have $x_{I_0 \sqcup I_1} \neq p_{I_0 \sqcup I_1}$. We will refer to the special case $\mathrm{sep}(p) := \mathrm{sep}(p, I_0(p), I_1(p))$ as simply the separation of p.

Remark 9.1. *It follows by definition that $|p| \notin W_n(\mathrm{sep}(p, I_0, I_1))$, for any $I_0 \subseteq I_0(p), I_1 \subseteq I_1(p)$.*

Lemma 9.2. *For any* $p \in \{0,1\}^n$*, and* $I_0 \subseteq I_0(p)$, $I_1 \subseteq I_1(p)$*, we have*

$$\mathrm{sep}(p, I_0, I_1) = J_{n,|I_1|-1,n-|I_0|+1}.$$

In particular, we have $\mathrm{sep}(p) = J_{n,|p|-1,|p|+1}$.

Proof. Without loss of generality, assume $p = 1^a 0^{n-a}$, and $I_1 = [1,u] \subseteq [1,a]$, $I_0 = [n-v+1,n]$, for some $u \in [0,a]$, $v \in [0,n-a]$. So $|I_1| = u$, $|I_0| = v$. We observe the following.

(P1) For any $x = 1^{a'}y$ with $a' \geq u$ and $y \in \{0,1\}^{n-a'}$, we have $x_{I_1} = p_{I_1} = 1^u$.

(P2) For any $x = y0^{b'}$ with $b' \geq v$ and $y \in \{0,1\}^{n-b'}$, we have $x_{I_0} = p_{I_0} = 0^v$.

Combining the above two observations, we get that for any $x = 1^{a'}y0^{b'}$ with $a' \geq u$, $b' \geq v$ and $y \in \{0,1\}^{n-a'-b'}$, we have $x_{I_0 \sqcup I_1} = p_{I_0 \sqcup I_1}$. Since $\mathrm{sep}(p, I_0, I_1)$ is a symmetric set, this implies that

$$[u, n-v] \cap W_n(\mathrm{sep}(p, I_0, I_1)) = \emptyset, \quad \text{that is,} \quad \mathrm{sep}(p, I_0, I_1) \subseteq J_{n,u-1,n-v+1}.$$

Now consider any $x \in J_{n,u-1,n-v+1}$. We have two cases.

(C1) $|x| \leq u-1$. Since $|I_1| = u$, there exists $i \in I_1$ such that $x_i = 0$, but $p_i = 1$.

(C2) $|x| \geq n-v+1$. Since $|I_0| = v$, there exists $i \in I_0$ such that $x_i = 1$, but $p_i = 0$.

Hence, we conclude that $J_{n,u-1,n-v+1} \subseteq \mathrm{sep}(p, I_0, I_1)$. □

We are now ready to prove Proposition 6.14. We restate the result, for convenience.

Proposition 2 (Restatement of Proposition 6.14)**.** *For any nonempty symmetric set* $S \subseteq \{0,1\}^n$*, we have* $r_n(S) = \mathrm{out}_n(S)$.

Proof. Let $J_{n,a,b}$ be the outer interval of S. So $\mathrm{out}_n(S) = a+n-b+1$. By the minimality of size of $I_{n,a,b}$, we have $\{a,b\} \subseteq W_n(S)$. So, in particular, $p := 1^a 0^{n-a} \in S$. Without loss of generality, assume $a \geq n-b$. Now consider any $x \in S$, $x \neq 1^a 0^{n-a}$. We have three cases.

(C1) $|x| = a$, $x \neq 1^a 0^{n-a}$. Then there exists $i \in [1,a]$ such that $x_i = 0$, but $p_i = 1$.

(C2) $|x| < a$. Then there exists $i \in [1,a]$ such that $x_i = 0$, but $p_i = 1$.

(C3) $|x| > a$, which means $|x| \geq b$, since $S \subseteq J_{n,a,b}$. So $|\{i \in [n] : x_i = 0\}| < n-b+1$. This implies there exists $i \in [a+1, a+n-b+1]$ such that $x_i = 1$, but $p_i = 0$.

Thus, in all three cases, there exists $i \in [1, a+n-b+1]$ such that $x_i \neq p_i$. Hence, we conclude that $r_n(S) \leq a+n-b+1 = \text{out}_n(S)$.

Now, in order to prove the reverse inequality, let $p \in S$ and $I \subseteq [n]$, $r_n(S) = |I|$ such that for every $x \in S, x \neq p$, we have $x_I \neq p_I$. Further, let $I_0 = I \cap I_0(p)$ and $I_1 = I \cap I_1(p)$. By definition of index complexity, Lemma 9.2, and Remark 9.1, we get

$$W_n(S) \setminus \{|p|\} \subseteq W_n(\text{sep}(p, I_0, I_1)) = I_{n,|I_1|-1,n-|I_0|+1}. \tag{9.2}$$

Also, trivially, we have $|I_1| \leq |p| \leq n - |I_0|$. Since $J_{n,a,b}$ is the outer interval of S, we have exactly one of the two cases, by (9.2) and the minimality of size of $I_{n,a,b}$.

(C1') $|p| = a$ and $W_n(S) \setminus \{p\} \subseteq I_{n,a-1,b} \subseteq I_{n,|I_1|-1,n-|I_0|+1}$. So $|I_1| \geq a$, $|I_0| \geq n-b+1$.

(C2') $|p| = b$ and $W_n(S) \setminus \{p\} \subseteq I_{n,a,b+1} \subseteq I_{n,|I_1|-1,n-|I_0|+1}$. So $|I_1| \geq a+1$, $|I_0| \geq n-b$.

In either of the two cases, we finally get

$$r_n(S) = |I| = |I_0| + |I_1| \geq a+n-b+1 = \text{out}_n(S). \qquad \square$$

Remark 9.3. *We also note from the proof of Proposition 6.14, that if $J_{n,a,b}$ is the outer interval of the symmetric set S, with $a \geq n-b$, then the set $I = [1, a+n-b+1]$ satisfies $|I| = r_n(S)$, and the point $p = 1^a0^{n-a}$ is such that for every $x \in S, x \neq 1^a0^{n-a}$, we have $x_I \neq (1^a0^{n-a})_I$. On the other hand, if $a < n-b$, then these choices change to $I = [b-a, n]$ and $p = 1^b0^{n-b}$.*

9.2.3 Index complexity of PDC k-wise symmetric sets

Let us now proceed to prove Proposition 6.41, which characterizes the index complexity of PDC k-wise symmetric sets. We restate the result, for convenience.

Proposition 3 (6.41)**.** *For any nonempty outer intact PDC k-wise symmetric set $S \subseteq \{0,1\}^N$, we have*

$$r_N(S) = \sum_{j=1}^{k} r_{n_j}(S_j) = \sum_{j=1}^{k} \text{out}_{n_j}(S_j).$$

Proof. For each $j \in [k]$, let J_{n_j,a_j,b_j} be the outer interval of S_j, and further, as indicated in Remark 9.3, let

$$\left(p^{(j)}, I^{(j)}\right) = \begin{cases} \left(1^{a_j}0^{n_j-a_j}, [1, a_j+n_j-b_j+1]\right) & \text{if } a_j \geq n_j - b_j, \\ \left(1^{b_j}0^{n_j-b_j}, [b_j-a_j, n_j]\right) & \text{if } a_j < n_j - b_j, \end{cases}$$

satisfy the definition of index complexity $r_{n_j}(S_j)$. Now consider any $(z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{in})}(S)$. Since S is outer intact, for each $j \in [k]$, we have the following.

- J_{n_j,a_j,b_j} is the outer interval of $[S]_{j,z_j}$.
- $p^{(j)} \in [S]_{j,z_j}$.
- $p^{(j)}$ and $I^{(j)}$ satisfy the definition of index complexity $r_{n_j}([S]_{j,z_j})$, as indicated in Remark 9.3.

Define $p = (p^{(1)},\dots,p^{(k)}) \in S$ and $I = I^{(1)} \sqcup \cdots \sqcup I^{(k)}$. Now consider any $x = (x^{(1)},\dots,x^{(k)}) \in S$, $x \neq p$. So there exists $j \in [k]$ such that $x^{(j)} \neq p^{(j)}$. Since $x \in S$, there exists $(z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{in})}(S)$ such that $x \in [S]_{1,z_1} \times \cdots \times [S]_{k,z_k}$, and so $x^{(j)} \in [S]_{j,z_j}$. Then by the choice of $I^{(j)}$, we get $x^{(j)}_{I^{(j)}} \neq p^{(j)}_{I^{(j)}}$. Thus, we have $r_N(S) \leq |I| = \sum_{j=1}^{k} I^{(j)}| = \sum_{j=1}^{k} r_{n_j}(S_j)$.

To prove the reverse inequality, now suppose $p = (p^{(1)},\dots,p^{(k)}) \in S$ and $I = I^{(1)} \sqcup \cdots \sqcup I^{(k)} \subseteq [N]$ satisfy the definition of index complexity $r_N(S)$. Let $(z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{in})}(S)$ such that $p \in [S]_{1,z_1} \times \cdots \times [S]_{k,z_k}$. Fix any $j \in [k]$. Consider any $y \in [S]_{j,z_j}$, $y \neq p^{(j)}$, and let $x = (p^{(1)},\dots,p^{(j-1)},y,p^{(j+1)},\dots,p^{(k)})$. Then $x \in [S]_{1,z_1} \times \cdots \times [S]_{k,z_k} \subseteq S$ and $x \neq p$. This implies $x_I \neq p_I$, which means $y_{I^{(j)}} \neq p^{(j)}_{I^{(j)}}$. Thus, we get $|I^{(j)}| \geq r_{n_j}([S]_{j,z_j}) = r_{n_j}(S_j)$. Hence, $r_N(S) = |I| = \sum_{j=1}^{k} |I^{(j)}| \geq \sum_{j=1}^{k} r_{n_j}(S_j)$.

The final equality in the statement is then immediate from Proposition 6.14. □

9.3 Covering PDC k-wise symmetric sets

Let us now prove our third main result (Theorem 6.33). We restate the result, for convenience. Recall that we work with the indeterminates $\mathbb{X} = (\mathbb{X}_1,\dots,\mathbb{X}_k)$, where $\mathbb{X}_j = (X_{j,1},\dots,X_{j,n_j})$ are the indeterminates for the j-th block.

Theorem 1 (6.33)**.** *For any nonempty PDC k-wise symmetric set $S \subseteq \{0,1\}^N$ and $t \geq 1$, we have*

$$\mathsf{b\text{-}EPC}^{(t,t-1)}_{(n_1,\dots,n_k)}(\{0,1\}^N \setminus S) = \max_{(z_1,\dots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \left\{ \sum_{j\in[k]:z_j\geq 1} \overline{\Lambda}_{n_j}([S]_{j,z_j-1}) \right\} + 2t - 2.$$

Proof. Let us first prove the lower bound. Let $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ be a $(t,t-1)$-block exact polynomial cover for $\{0,1\}^N \setminus S$. Fix any $(z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)$. Note that for any $j \in [k]$, we have $z_j \geq 1$ if and only if $[S]_{j,z_j-1} \neq \emptyset$. So, without loss of generality, we assume $z_j \geq 1$

for all $j \in [k]$. It is now enough to show that

$$\deg(P) \geq \sum_{j=1}^{k} \overline{\Lambda}_{n_j}([S]_{j,z_j-1}) + 2t - 2.$$

Consider any $j \in [k]$, and let

$$\overline{\mu}_j := \overline{\mu}_n([S]_{j,z_j-1}) = \max\{i \in [0, \lceil n/2 \rceil] : W_{n_j,i} \subseteq W_n(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1})\}.$$

So either $\overline{\mu}_j \in W_n([S]_{j,z_j-1})$ or $n_j - \overline{\mu}_j \in W_n([S]_{j,z_j-1})$. Without loss of generality, suppose $\overline{\mu}_j \in W_n([S]_{j,z_j-1})$. Then clearly, $|W_{n_j}([S]_{j,z_j-1}) \setminus \{\overline{\mu}_j\}| = n_j - |W_n(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1})|$. Also clearly, $1^{\overline{\mu}_j} 0^{n_j - \overline{\mu}_j} \in [S]_{j,z_j-1}$. Define

$$Q(\mathbb{X}) = P(\mathbb{X}) \cdot \prod_{j=1}^{k} \mathscr{H}'_{W_{n_j}([S]_{j,z_j-1}) \setminus \{\overline{\mu}_j\}}(\mathbb{X}_j).$$

Recall that we have $W_{(n_1,\ldots,n_k)}(S) = \{(|x^{(1)}|, \ldots, |x^{(k)}|) : x \in S\}$. Further, recall that we have $W_{(n_1,\ldots,n_k)}([S]_{1,z_1-1} \times \cdots \times [S]_{k,z_k-1}) := W_{n_1}([S]_{1,z_1-1}) \times \cdots \times W_{n_k}([S]_{k,z_k-1})$. Consider any $x = (x^{(1)}, \ldots, x^{(k)}) \in \{0,1\}^N$. We have the following cases.

(C1) $(|x^{(1)}|, \ldots, |x^{(k)}|) = (\overline{\mu}_1, \ldots, \overline{\mu}_k)$. So we have

- $\mathrm{mult}(P(x'_{(j)}, \mathbb{X}_j), x^{(j)}) = t - 1$, where $x = (x'_{(j)}, x^{(j)})$, for every $j \in [k]$.
- $\mathscr{H}'_{W_{n_j}([S]_{j,z_j-1}) \setminus \{\overline{\mu}_j\}}(x^{(j)}) \neq 0$, for every $j \in [k]$.

This implies $\mathrm{mult}(Q(\mathbb{X}), x) = t - 1$. Note that this is where we need $P(\mathbb{X})$ to be a $(t, t-1)$-block exact polynomial cover and not just a $(t, t-1)$-exact polynomial cover for $\{0,1\}^N \setminus S$.

(C2) $(|x^{(1)}|, \ldots, |x^{(k)}|) \in W_{(n_1,\ldots,n_k)}([S]_{1,z_1-1} \times \cdots \times [S]_{k,z_k-1}) \setminus \{(\overline{\mu}_1, \ldots, \overline{\mu}_k)\}$. So we have

- $\mathrm{mult}(P(x'_{(j)}, \mathbb{X}_j), x^{(j)}) = t - 1$, where $x = (x'_{(j)}, x^{(j)})$, for every $j \in [k]$.
- There exists $j \in [k]$ such that $|x^{(j)}| \neq \overline{\mu}_j$, and so $\mathscr{H}'_{W_{n_j}([S]_{j,z_j-1}) \setminus \{\overline{\mu}_j\}}(x^{(j)}) = 0$

This implies $\mathrm{mult}(Q(\mathbb{X}), x) \geq t$.

(C3) $(|x^{(1)}|, \ldots, |x^{(k)}|) \notin W_{(n_1,\ldots,n_k)}([S]_{1,z_1-1} \times \cdots \times [S]_{k,z_k-1})$. So $\mathrm{mult}(P(\mathbb{X}), x) \geq t$, and this implies $\mathrm{mult}(Q(\mathbb{X}), x) \geq t$.

Thus, $Q(\mathbb{X})$ is a $(t,t-1)$-exact polynomial cover for $\{0,1\}^N \setminus L$, where $L = L_1 \times \cdots \times L_k$ is a k-wise layer given by $W_{n_j}(L_j) = \{\overline{\mu}_j\}$, $j \in [k]$. So Theorem 6.10 and Corollary 6.42 imply

$$\deg(Q) \geq N - r_N(L) + 2t - 2 = N - \sum_{j=1}^{k} \overline{\mu}_j + 2t - 2. \tag{9.3}$$

Further, by construction, we have

$$\begin{aligned}\deg(Q) &= \deg(P) + \sum_{j=1}^{k} \left(n_j - |W_n(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1})|\right) \\ &= \deg(P) + N - \sum_{j=1}^{k} |W_{n_j}(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1})|. \end{aligned} \tag{9.4}$$

From (9.3) and (9.4), we get

$$\deg(P) \geq \sum_{j=1}^{k} \left(|W_{n_j}(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1})| - \overline{\mu}_j\right) + 2t - 2 = \sum_{j=1}^{k} \overline{\Lambda}_{n_j}([S]_{j,z_j-1}) + 2t - 2.$$

This completes the proof of the lower bound.

Let us now show that the construction in Example 6.35 attains the lower bound we just proved. Recall that Example 6.35 defines a polynomial

$$\mathsf{h}_S(\mathbb{X}) := \left(\sum_{(z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)} \lambda_{S,(z_1,\ldots,z_k)} \mathscr{H}_{S,(z_1,\ldots,z_k)}(\mathbb{X})\right) \cdot \mathscr{H}^{\circ(t-1)}(\mathbb{X}_1),$$

where, for each $(z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)$, we have

$$\mathscr{H}_{S,(z_1,\ldots,z_k)}(\mathbb{X}) = \prod_{j \in [k]: z_j \geq 1} \left(\mathscr{H}^*_{\overline{\mu}_n([S]_{j,z_j-1})}(\mathbb{X}_j) \cdot \mathscr{H}'_{W_{n_j}(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1}) \setminus W_{n_j,\overline{\mu}_n([S]_{j,z_j-1})}}(\mathbb{X}_j)\right),$$

and further, $\{\lambda_{S,(z_1,\ldots,z_k)} : (z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)\} \subseteq \mathbb{R}$ is a $\widehat{\mathbb{Q}}$-linearly independent subset of $\mathbb{R}$, with respect to the subfield $\widehat{\mathbb{Q}} := \mathbb{Q}\big(\mathscr{H}_{S,(z_1,\ldots,z_k)}(b) : b \in \{0,1\}^N,\ (z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)\big)$. Firstly, note that since $\mathscr{H}^{\circ(t-1)}(\mathbb{X}_1) = X_1^{t-1}(X_1-1)^{t-1}$, we clearly get $\mathrm{mult}(\mathscr{H}^{\circ(t-1)}(\mathbb{X}_1), x) = t-1$ for all $x \in \{0,1\}^N$. Now fix any $(z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)$, and consider any $x \in \{0,1\}^N$.

We observe

$$
\begin{aligned}
& \mathscr{H}_{S(z_1,\dots,z_k)}(x) \neq 0 \\
\iff \quad & \mathscr{H}^{*}_{\overline{\mu}_n([S]_{j,z_j-1})}(x^{(j)}) \cdot \mathscr{H}'_{W_n(\{0,1\}^{n_j}\setminus[S]_{j,z_j-1})\setminus W_{n_j,\overline{\mu}_n([S]_{j,z_j-1})}}(x^{(j)}) \neq 0, \quad \text{for all } j\in[k] : z_j \geq 1 \\
\iff \quad & x^{(j)} \notin \{0,1\}^{n_j} \setminus [S]_{j,z_j-1}, \qquad \text{for all } j\in[k] : z_j \geq 1 \\
\iff \quad & x \in \left(\prod_{j\in[k]:z_j\geq 1} [S]_{j,z_j-1} \right) \times \left(\prod_{j\in[k]:z_j=0} \{0,1\}^{n_j} \right).
\end{aligned}
$$

Now it is easy to check that

$$
\{0,1\}^N \setminus S = \bigcap_{(z_1,\dots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \left(\{0,1\}^N \Big\backslash \left(\prod_{j\in[k]:z_j\geq 1} [S]_{j,z_j-1} \right) \times \left(\prod_{j\in[k]:z_j=0} \{0,1\}^{n_j} \right) \right).
$$

So by the $\widehat{Q}$-linear independence of $\{\lambda_{S,(z_1,\dots,z_k)} : (z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)\} \subseteq \mathbb{R}$, we get

$$
\begin{aligned}
& \sum_{(z_1,\dots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \lambda_{S,(z_1,\dots,z_k)} \mathscr{H}_{S,(z_1,\dots,z_k)}(x) = 0 \\
\iff \quad & \mathscr{H}_{S,(z_1,\dots,z_k)}(x) = 0, \qquad \text{for all } (z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S) \\
\iff \quad & x \notin \left(\prod_{j\in[k]:z_j\geq 1} [S]_{j,z_j-1} \right) \times \left(\prod_{j\in[k]:z_j=0} \{0,1\}^{n_j} \right), \qquad \text{for all } (z_1,\dots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S) \\
\iff \quad & x \in \bigcap_{(z_1,\dots,z_k)\in \mathsf{E}^{(\mathrm{out})}(S)} \left(\{0,1\}^N \Big\backslash \left(\prod_{j\in[k]:z_j\geq 1} [S]_{j,z_j-1} \right) \times \left(\prod_{j\in[k]:z_j=0} \{0,1\}^{n_j} \right) \right) \\
\iff \quad & x \in \{0,1\}^N \setminus S.
\end{aligned}
$$

Thus, we have

- $\mathrm{mult}(\mathsf{h}_S(\mathbb{X}), x) \geq t$ if $x \in \{0,1\}^N \setminus S$.
- $\mathrm{mult}(\mathsf{h}_S(x_{(j)}, \mathbb{X}_j), x^{(j)}) = t-1$, where $x = (x_{(j)}, x^{(j)})$, for each $j \in [k]$.

Hence, $\mathsf{h}_S(\mathbb{X})$ is a $(t, t-1)$-block exact polynomial cover for $\{0,1\}^N \setminus S$.

Now for any $(z_1,\ldots,z_k) \in \mathsf{E}^{(\mathrm{out})}(S)$, we have

$$\begin{aligned}\deg(\mathscr{H}_{S,(z_1,\ldots,z_k)}) &= \sum_{j\in[k]:z_j\geq 1} \left(\overline{\mu}_n([S]_{j,z_j-1}) + |W_{n_j}(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1}) \setminus W_{n_j,\overline{\mu}_n([S]_{j,z_j-1})}|\right)\\ &= \sum_{j\in[k]:z_j\geq 1} \left(W_{n_j}(\{0,1\}^{n_j} \setminus [S]_{j,z_j-1})| - \overline{\mu}_n([S]_{j,z_j-1})\right)\\ &= \sum_{j\in[k]:z_j\geq 1} \overline{\Lambda}_{n_j}([S]_{j,z_j-1}).\end{aligned}$$

Hence,

$$\deg(\mathsf{h}_S) = \max_{(z_1,\ldots,z_k)\in\mathsf{E}^{(\mathrm{out})}(S)} \left\{\sum_{j\in[k]:z_j\geq 1} \overline{\Lambda}_{n_j}([S]_{j,z_j-1})\right\} + 2t-2,$$

that is, $\mathsf{h}_S(\mathbb{X})$ attains the lower bound. □

9.4 Partial results for $(t,0)$-exact polynomial covers

Let us have a few notations before we proceed. For any $a \in \mathbb{R}^n$, we define the polynomial $(\mathbb{X}-a)^\alpha := (X_1-a_1)^{\alpha_1}\cdots(X_n-a_n)^{\alpha_n}$. Therefore, for any $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ and $a \in \mathbb{R}^n$, the Taylor's expansion of P about the point a is

$$P(\mathbb{X}) = \sum_{0\leq|\alpha|\leq\deg(P)} \frac{(\partial^\alpha P)(a)}{\alpha_1\cdots\alpha_n}(\mathbb{X}-a)^\alpha.$$

Let us first prove Proposition 6.39. We restate the result, for convenience.

Proposition 4 (Restatement of Proposition 6.39)**.** *For $w \in [1,n-1]$, let $S \subsetneq \{0,1\}^n$ be the symmetric set defined by $W_n(S) = [0,w-1]$. Then for any $t \in \left[2, \left\lfloor\frac{n+3}{2}\right\rfloor\right]$, we have*

$$\mathsf{EPC}_n^{(t,0)}(S) = w+2t-3.$$

Further, the answer to Question 6.2 is negative, in general.

Proof. Let $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ be a $(t,0)$-exact polynomial cover for S. Consider the restricted polynomial $\widetilde{P}(X_1,\ldots,X_w) := P(X_1,\ldots,X_w,0^{n-w})$. Then $\widetilde{P}(1^w) = P(1^w0^{n-w}) \neq 0$, since $1^w0^{n-w} \notin S$. Further, for any $x \in \{0,1\}^w$, $x \neq 1^w$, we have $|x| \leq w-1$, which means $x0^{n-w} \in S$, and so $\mathrm{mult}(\widetilde{P}(X_1,\ldots,X_w),x) \geq \mathrm{mult}(P(\mathbb{X}),x0^{n-w}) \geq t$. Thus, $\widetilde{P}(X_1,\ldots,X_w)$ is a $(t,0)$-exact polynomial cover for $\{0,1\}^w \setminus \{1^w\}$. So by Theorem 7.4, we get $\deg(\widetilde{P}) \geq w+2t-3$. Hence, $\deg(P) \geq \deg(\widetilde{P}) \geq w+2t-3$.

In order to show that the lower bound is tight, let $Q(X_1,\ldots,X_w) \in \mathbb{R}[\mathbb{X}]$ be a $(t,0)$-exact polynomial cover of the symmetric set $\{0,1\}^w \setminus \{1^w\}$, with $\deg(Q) = w+2t-3$, as ensured by Theorem 7.4. Further, let $\alpha = Q(1^w) \neq 0$. Now define a polynomial

$$\widetilde{Q}(\mathbb{X}) = \sum_{1\le i_1<\cdots<i_w\le n} Q(X_{i_1},\ldots,X_{i_w}).$$

Then clearly, $\deg(\widetilde{Q}) = \deg(Q) = w+2t-3$. Now consider any $x \in \{0,1\}^n$. We observe the following.

- If $|x| \le w-1$, then for any $1 \le i_1 < \cdots < i_w \le n$, we have

$$\mathrm{mult}(\widetilde{Q}(\mathbb{X}),x) \ge \mathrm{mult}(Q(X_{i_1},\ldots,X_{i_w}),(x_{i_1},\ldots,x_{i_w})) \ge t.$$

- If $|x| \ge w$, let $\{j_1,\ldots,j_u\} = \{i \in [n] : x_i = 1\}$. Then we have

$$\widetilde{Q}(x) = \sum_{\substack{1\le i_1<\cdots<i_w\le n\\ \{i_1,\ldots,i_w\}\subseteq\{j_1,\ldots,j_u\}}} \alpha \;=\binom{u}{w}\alpha \neq 0.$$

Thus, $\widetilde{Q}(\mathbb{X})$ is a $(t,0)$-exact polynomial cover for S. Hence, $\widetilde{Q}(\mathbb{X})$ attains the lower bound.

Let us now show an example that illustrates that the answer to Question 6.2 is negative in general, that is, $\mathsf{EHC}_n^{(t,0)}(S) > \mathsf{EPC}_n^{(t,0)}(S)$ in general. Let $t=2, n=3>2t-3$ and $S \subseteq \{0,1\}^3$ be the symmetric set defined by $W_n(S) = \{0,1\}$.

So $S = \{(0,0,0),(1,0,0),(0,1,0),(0,0,1)\}$. We have $\mathsf{EPC}_3^{(2,0)}(S) = 3$ (since $w=1$ in this case). Now suppose there are three hyperplanes $\{h_1,h_2,h_3\}$ that form a $(2,0)$-exact hyperplane cover for S. First, observe that no single hyperplane can cover whole S. So there exist at least 2 hyperplanes that cover exactly 3 points from S. Without loss of generality, let $h_1(0,0,0) = h_2(0,0,0) = 0$. So $h_1(X_1,X_2,X_3) = a_1X_1+a_2X_2+a_3X_3$, $h_2(X_1,X_2,X_3) = b_1X_1+b_2X_2+b_3X_3$, for some $a_1,a_2,a_3,b_1,b_2,b_3 \in \mathbb{R}$. Further, at least one of h_1,h_2 must cover 2 more points of S. Without loss of generality, suppose $h_1(1,0,0) = h_1(0,1,0) = 0$. This gives $h_1(1,1,0) = h_1(1,0,0)+h_1(0,1,0) = 0$, which contradicts $\{h_1,h_2,h_3\}$ being a $(2,0)$-exact hyperplane cover for S. So we conclude that

$$|\mathscr{Z}(h_j)\cap\{(1,0,0),(0,1,0),(0,0,1)\}| \le 1$$

for all $j \in [2]$. This implies that it is impossible for each point in $\{(1,0,0),(0,1,0),(0,0,1)\}$ to be covered 2 times by $\{h_1,h_2,h_3\}$. Hence $\mathsf{EHC}_3^{(2,0)}(S) > 3$. □

Now let us prove Proposition 6.40. We restate the result, for convenience.

Proposition 5 (Restatement of Proposition 6.40)**.** *For any layer* $S \subsetneq \{0,1\}^n$ *with* $W_n(S) = \{w\}$*, and* $t \geq 1$*, we have*

$$\mathsf{EPC}_n^{(t,0)}(S) = t.$$

Proof. Let $P(\mathbb{X}) \in \mathbb{R}[\mathbb{X}]$ be a $(t,0)$-exact polynomial cover for S. Fix any $a \in S$. So there exists an invertible linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ (or equivalently, an invertible change of coordinates) such that the polynomial $\widetilde{P}(\mathbb{X}) := P(L(\mathbb{X}-a)+a)$ is a $(t,0)$-exact polynomial cover for $\{a\}$. Also, $\deg(\widetilde{P}) = \deg(P)$. Now we have the Taylor's expansion of $\widetilde{P}$ about a as

$$\widetilde{P}(\mathbb{X}) = \sum_{0 \leq |\alpha| \leq \deg(P)} \frac{(\partial^\alpha \widetilde{P})(a)}{\alpha_1 \cdots \alpha_n} (\mathbb{X}-a)^\alpha.$$

Since $\widetilde{P}(\mathbb{X})$ is a $(t,0)$-exact polynomial cover for $\{a\}$, we have $(\partial^\alpha \widetilde{P})(a) = 0$ for $|\alpha| < t$. This gives

$$\widetilde{P}(\mathbb{X}) = \sum_{t \leq |\alpha| \leq \deg(P)} \frac{(\partial^\alpha \widetilde{P})(a)}{\alpha_1 \cdots \alpha_n} (\mathbb{X}-a)^\alpha,$$

which implies $\deg(P) \geq t$. Further, this lower bound is tight; for instance, the polynomial $P(\mathbb{X}) := \left(\sum_{i=1}^n X_i - w\right)^t$ witnesses the lower bound. $\square$

Chapter 10

Conclusion and Open Questions

In Chapter 3, we have studied a clorful *Helly-type* result for multi-pierceability of family of axis-parallel boxes. Given a convex body $\mathscr{K}$ in $\mathbb{R}^d$, let $h_{\mathscr{K}}(n)$ denote the smallest integer such that any finite collection of translates of $\mathscr{K}$ in $\mathbb{R}^d$ is n-pierceable if and only if every $h_{\mathscr{K}}(n)$ many translates of $\mathscr{K}$ from the collection is n-pierceable. Danzer and Grünbaum [59] showed the existence of centrally symmetric convex body $\mathscr{K}$ in $\mathbb{R}^2$ for which $h_{\mathscr{K}}(2) = \aleph_0$. They also conjectured the following interesting result:

Conjecture 10.1 (Danzer and Grünbaum [59]). *Suppose $\mathscr{K}$ is a convex polytope in $\mathbb{R}^d$. Then $h_{\mathscr{K}}(2) < \aleph_0$.*

We believe the above result can be proved for the case of polytopes in $\mathbb{R}^2$.

Grünbaum [78] conjectured that

Conjecture 10.2 (Grünbaum [78]). *Let $\mathscr{F}$ be a finite collection of translates of a compact symmetric convex set in $\mathbb{R}^2$. If every pair of the collection intersects then $\mathscr{F}$ is 3-pierceable.*

The above conjecture remained open for nearly forty years till it was resolved by Karasev [104].

Theorem 10.3 (Karasev [104]). *Suppose K be any convex set in plane and $\mathscr{F}$ be a family of pairwise intersecting translates of K. Then $\mathscr{F}$ is 3-pierceable.*

Dolńikov[1] conjectured a colorful version of the above result.

Conjecture 10.4 (Dolńikov 2011). *Let K be a convex set in the plane and F_1, F_2, F_3 be finite families of translates of K. If any two translates of K from different families intersect, then there is an index i such that F_i is 3 pierceable.*

[1] In 2011 at an Oberwolfach workshop in Discrete Geometry.

Till now the conjecture have been verified for triangles or *centrally symmetric* convex bodies, see [99]. It will be interesting to investigate Conjecture 10.4.

In the second part of this thesis, we have proved Theorem 6.33, which also subsumes our other main results (Theorem 6.18 and Theorem 6.23). We note that Theorem 6.33 characterizes the tight bound for the $(t,t-1)$-block exact polynomial cover, but in the special cases of Theorem 6.18 and Theorem 6.23, our tight example specializes to the tight example for the $(t,t-1)$-exact hyperplane cover. Therefore, as seen in the earlier work of Alon and Füredi [6] as well as initial attempts in [146, 75], solving the *weaker* polynomial covering problem by the polynomial method indeed solves the *stronger* hyperplane covering problem in these settings.

Some of the obvious questions that seem beyond the proof technique employed in this work are the following.

Question 10.5. *In the broad generality of the polynomial covering problem considered for PDC blockwise symmetric sets, is the* degeneracy condition *necessary? More precisely, for any nonempty PDC k-wise symmetric set $S \subseteq \{0,1\}^N$ and $t \geq 1$, is it true that* $\mathsf{b\text{-}EPC}^{(t,t-1)}_{(n_1,\dots,n_k)}(S) = \mathsf{EPC}^{(t,t-1)}_{N}(S)$*?*

We believe this could be true.

Question 10.6. *How do we solve the hyperplane covering problem considered for PDC blockwise symmetric sets? In other words, do Theorem 6.18 and Theorem 6.23 extend to the setting of Theorem 6.33 for the exact hyperplane cover version? More precisely, for any nonempty PDC k-wise symmetric set $S \subseteq \{0,1\}^N$ and $t \geq 1$, is it true that* $\mathsf{b\text{-}EPC}^{(t,t-1)}_{(n_1,\dots,n_k)}(S) = \mathsf{b\text{-}EHC}^{(t,t-1)}_{(n_1,\dots,n_k)}(S)$*?*

Our work shows that our proof technique can not possibly extend to prove this. We therefore believe this may not be true.

Question 10.7. *Characterize the index complexity of all nonempty PDC k-wise symmetric sets. In other words, obtain the characterization without requiring the* outer intact *condition in Proposition 6.41.*

List of publications

List of works the thesis is based on:

Published

- **Covering Almost All the Layers of Hypercube with Multiplicities**
  Co-authors: Arijit Ghosh and Chandrima Kayal
  Published in the journal Discrete Mathematics (DM), 346(7): 113397, 2023.
  https://www.sciencedirect.com/science/article/pii/S0012365X23000833
  Only a part of this work is included in the thesis.

- **Stabbing Boxes with Finitely many Axis-parallel Lines and Flats**
  Co-authors: Sutanoya Chakraborty and Arijit Ghosh
  Current status: Accepted for publication to the journal Discrete Mathematics (DM).
  Manuscript number: DM-34081, submitted on April 01, 2024.

  Only a part of this work is included in the thesis.

Submitted

- **Colorful Helly Theorem for Piercing Boxes with Multiple Points**
  Co-authors: Sourav Chakraborty and Arijit Ghosh
  Current status: Submitted to the journal Computational Geometry: Theory and Applications (CGTA) and the final communication is going on.
  Manuscript number: CGTA-D-24-00013, submitted on February 01, 2024.

- **Heterochromatic Geometric Transversal of Convex Sets**
  Co-authors: Sutanoya Chakraborty and Arijit Ghosh

Current status: Under submission to the journal Combinatorics, Probability and Computing (CPC)
Paper ID: 240103-Ghosh, submitted on January 03, 2024.

Only a part of this work is included in the thesis.

- **Colorful No-Dimensional Helly Theorem for Affine Transversal**
  Co-authors: Sutanoya Chakraborty and Arijit Ghosh
  Current status: Under submission to the journal Computational Geometry: Theory and Applications (CGTA)
  Manuscript number: CGTA-D-24-00026, submitted on March 27, 2024

- **On Higher Multiplicity Hyperplane and Polynomial Covers for Symmetry Preserving Subsets of the Hypercube**
  Co-authors: Arijit Ghosh, Chandrima Kayal and S. Venkitesh
  Current status: Under submission to the journal SIAM Journal on Discrete Mathematics (SIDMA)
  SIDMA manuscript #M159156, submitted on August 02, 2023.

Work done other than the thesis:

- **Countably Colorful Hyperplane Transversal**
  Co-authors: Sutanoya Chakraborty and Arijit Ghosh
  Current status: Under submission to the journal Discrete & Computational Geometry (DCG)
  Manuscript Number: DCGE-D-24-00044, submitted on February 26, 2024.

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