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A REMARK ON SPIN CORRELATIONS

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SUMMARY. Bell (1964) showed that for any three random variables ξ_i, $i = 1, 2, 3$ assuming only the values ± 1 the inequality $1 - E\xi_1\xi_3 > |E\xi_2(\xi_1 - \xi_3)|$ holds but for three quantum mechanical observables S_i, $i = 1, 2, 3$ which are selfadjoint operators with spectrum $\{-1, 1\}$ and a nonnegative selfadjoint operator ρ of unit trace in a Hilbert space it is possible that $1 - \text{tr}\, \rho\, S_1 S_3 < |\text{tr}\, \rho\, S_2(S_1 - S_3)|$. Here we show that given any positive definite kernel $K(x, y)$, $x, y \in X$ such that $K(x, x) \equiv 1$ there always exists a unit vector Ω, a family $\{U_x, x \in X\}$ of unitary operators and a selfadjoint operator S with spectrum $\{-1, 1\}$ in a Hilbert space such that $< \Omega, S_x \Omega > = 0$, $< \Omega, S_x S_y \Omega > = K(x, y)$ for all $x, y \in X$ where $S_x = U_x^{-1} S U_x$. In other words, any preassigned correlation structure can be achieved by a process of spin observables.

1. Introduction

Let X be a set, $A, B, C \subset X$ be three subsets and let A', I_A denote respectively the complement of A and the indicator function of A. Then

$$I_B(1 - I_A - I_C) + I_A I_C = I_{BA'C' \cup B'AC} \qquad \dots \quad (1.1)$$

where AB denotes the interesection of A and B. In particular, if P is a probability distribution over X and A, B, C are events then (1.1) implies

$$P(B) - P(AB) - P(BC) + P(AC) \geqslant 0. \qquad \dots \quad (1.2)$$

A random variable assuming only the values ± 1 is called a *spin random variable*. If ξ is a spin random variable then $\frac{1}{2}(1+\xi)$ is the indicator of an event. From this observaton and (1.2) one obtains for any three spin random variables ξ_i, $i = 1, 2, 3$

$$1 - E\xi_1 \xi_3 \geqslant |E\xi_2(\xi_1 - \xi_3)| \qquad \dots \quad (1.3)$$

which is Bell's inequality (Bell, 1964).

In the context of quantum probability (cf. Meyer, 1984) consider the Hilbert space $\mathcal{H} = \mathbb{C}^2$, the density matrix $\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ and the Hermitian matrices

$$H = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

$$S_t = e^{itH} S e^{-itH} = \begin{pmatrix} 0 & e^{it} \\ e^{-it} & 0 \end{pmatrix}, \; t \in \mathcal{R}.$$

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Then S_t is Hermitian with eigenvalues ± 1 and

$$\text{tr}\,\rho\, S_t = 0,\ \text{tr}\,\rho\, S_t S_s = \cos(t-s).$$

There exist t_1, t_2, t_3 such that

$$1-\cos(t_1-t_3) < |\cos(t_2-t_1)-\cos(t_2-t_3)|.$$

For example one may choose $t_2-t_1 = \alpha,\ t_3-t_2 = \pi+\alpha,\ 0 < \alpha < \pi$. This is usually called a violation of Bell's inequality (1.3) by quantum observables with spectrum $\{-1, 1\}$.

We shall generalise the above mentioned example as follows. A selfadjoint operator in a complex Hilbert space with spectrum $\{-1, 1\}$ is called a *spin observable*. If S is a spin observable and U is a unitary operator then $U^{-1}S\,U$ is also a spin observable. If X is a set, a map $\mathcal{K} : X\times X \to \mathbb{C}$ such that $\mathcal{K}(x, x) \equiv 1$ and the matrix $((\mathcal{K}(x_i, x_j)))$, $1 \leqslant i, j \leqslant n$ is positive semidefinite for any finite set $\{x_1, x_2, \ldots, x_n\} \subset X$ is called a *correlation kernel*. The aim of the present article is to show by means of very elementary arguments that to any preassigned correlation kernel on X there exists a family of unitary operators $\{U_x,\ x \in X\}$, a spin observable S and a pure state determined by a unit vector Ω in some Hilbert space such that the family of spin observables $S_x = U_x^{-1}SU_x,\ x \in X$ satisfies the following : $< \Omega, S_x\Omega > = 0$, $< \Omega, S_xS_y\Omega > = \mathcal{K}(x, y)$ for all $x, y \in X$. In particular, each S_x has the Bernoulli distribution with probability $\frac{1}{2}$ for ± 1 in the state Ω.

We adopt the convention that Hilbert spaces are complex and inner products $< .,. >$ are conjugate linear in the first variable and linear in the second variable.

2. The main result

Let $\mathcal{K}$ be a correlation kernel on X. Enlarge the set x by adding a point ε, put $\widetilde{X} = X \cup \{\varepsilon\}$ and define $\widetilde{\mathcal{K}}$ on $\widetilde{X}\times\widetilde{X}$ by

$$\widetilde{\mathcal{K}}(\varepsilon, \varepsilon) = 1,\ \widetilde{\mathcal{K}}(\varepsilon, x) = \widetilde{\mathcal{K}}(x, \varepsilon) = 2^{-1/2}$$

$$\widetilde{\mathcal{K}}(x, y) = \tfrac{1}{2}\,(1+\mathcal{K}(x, y) \text{ for all } x, y \in X. \qquad \ldots \quad (2.1)$$

Lemma 2.1 : *$\widetilde{\mathcal{K}}$ is a correlation kernel on $\widetilde{X}$.*

Proof : It suffices to show that for any $x_1, x_2, \ldots, x_n \in X$ the matrix

$$M = \left[\begin{array}{c|c} 1 & 2^{-1/2} \ldots 2^{-1/2} \\ \hline 2^{-1/2} & \\ \vdots & \\ . & ((\tfrac{1}{2}\,(1+\mathcal{K}(x_i,x_j))) \\ 2^{-1/2} & \end{array}\right]$$

is positive semidefinite. This is immediate from the observation that

$$M = \begin{bmatrix} 1 \\ 2^{-1/2} \\ \vdots \\ 2^{-1/2} \end{bmatrix} (1\,2^{-1/2} \ldots 2^{-1/2}) + \tfrac{1}{2} \left[\begin{array}{c|c} 0 & 0 \quad 0 \ldots 0 \\ \hline 0 & \\ \vdots & ((\mathcal{K}(x_i, x_j))) \\ 0 & \end{array} \right]$$

Lemma 2.2 : (Gelfand, Neumark, Segal theorem) : *There exists a Hilbert space $\mathcal{H}$ and a total family $\{v(x), x \in \widetilde{X}\}$ of unit vectors in $\mathcal{H}$ such that*

$$< v(x), v(y) > = \widetilde{\mathcal{K}}(x, y) \text{ for all } x, y \in \widetilde{X} \qquad \ldots \quad (2.2)$$

Proof : In view of Lemma 2.1 this is immediate from the fact that there exists a complex valued Gaussian family $\{v(x), x \in \widetilde{X}\}$ of random variables with mean 0 and covariance $E\overline{v(x)}v(y) = \widetilde{\mathcal{K}}(x, y)$ for all $x, y \in X$. We may choose $\mathcal{H}$ to be the closed linear span of the Gaussian variables $\{v(x), x \in \widetilde{X}\}$. □

Lemma 2.3 : *Let $\{v(x), x \in \widetilde{X}\}$ be as in Lemma 2.2. Suppose P_x is the orthogonal projection on the one dimensional subspace $\mathbb{C}v(x)$, $S_x = 2P_x - 1$ and $\Omega = v(\varepsilon)$. Then S_x is a spin observable and*

$$< \Omega, S_x \Omega > = 0, < \Omega, S_x S_y \Omega > = \mathcal{K}(x, y)$$

for all $x, y \in X$.

Proof : By definition $P_x \Omega = < v(x), v(\varepsilon) > v(x)$. The lemma is now immediate from (2.1) and (2.2). □

Lemma 2.4 : *Let $\Omega, v(x), S_x$ be as in Lemma 2.2—2.3. Let $S_\varepsilon = S$. Then there exists a family $\{V_x, x \in X\}$ of unitary operators such that*

$$V_x \Omega = v(x)$$

$$V_x u = u + (2^{1/2} - 2) < \Omega + v(x), u > \Omega$$

$$+ 2^{1/2} < \Omega + (1 - 2^{1/2}) v(x), u > v(x) \qquad \ldots \quad (2.3)$$

for all $u \in \mathcal{H}$. Furthermore $V_x S V_x^{-1} = S_x$ for all $x \in X$.

Proof : Consider the two dimensional subspace $\mathcal{H}_x$ spanned by Ω and $v(x)$ in $\mathcal{H}$ for each fixed $x \in X$. In this sub-space $\mathcal{H}_x$ there is a unitary operator V_x^0 defined by

$$V_x^0 \Omega = v(x), \; V_x^0 v(x) = -\Omega + 2^{1/2} v(x).$$

Indeed, this follows from the fact that Ω and $v(x)$ are unit vectors such that $< \Omega, v(x) > = 2^{-1/2}$. Define $V_x = V_x^0 \oplus I$ in the direct sum decomposition $\mathcal{H} = \mathcal{H}_x \oplus \mathcal{H}_x^\perp$. An easy computation shows that $V_x u$ is given by (2.3). Since $V_x \Omega = v(x)$, $S = 2P_\varepsilon - 1$, $S_x = 2P_x - 1$ it follows that $V_x S V_x^{-1} = S_x$. □

Theorem 2.5 : *Let $\mathcal{K}(x, y)$ be a correlation kernel on a set X. Then there exists a Hilbert space $\mathcal{H}$, a unit vector Ω, a family $\{U_x, x \in X\}$ of unitary operators and a spin observable S in $\mathcal{H}$ such the family $\{S_x = U_x^{-1} S U_x, x \in X\}$ of spin observables satisfy the relations*

$$< \Omega, S_x \Omega > = 0, < \Omega, S_x S_y \Omega > = \mathcal{K}(x, y) \text{ for all } x, y \in X.$$

If X is a topological space and $\mathcal{K}$ is continuous on $X \times X$ then the family $\{U_x, x \in X\}$ can be chosen to be strongly continuous.

Proof: The first part is immediate from Lemma 2.3, 2.4 if we put $U_x = V_x^{-1}$. To prove the second part we observe that $< v(x), v(y) > = \frac{1}{2}(1 + \mathcal{K}(x, y))$ is a continuous function on $X \times X$. Hence the map $x \to v(x)$ is strongly continuous. Equation (2.3) shows that the map $x \to V_x$ is strongly continuous and hence the map $x \to U_x$ is strongly continuous. □

Corollary : *Suppose G is a group of transformations acting on X such that the map $x \to gx$ is bijective on X for each $g \in G$ and the correlation kernel $\mathcal{K}$ is G-invariant in the sense that $\mathcal{K}(x, y) \equiv \mathcal{K}(gx, gy)$. Then in Theorem 2.5 one has a unitary representation $g \to W_g$ of G in $\mathcal{H}$ such that $W_g S_x W_g^{-1} = S_{gx}$ for all $x \in X$, $g \in G$. If X is a topological space, G is a topological group acting continuously on X and $\mathcal{K}$ is a G-invariant continuous correlation kernel then the representation $g \to W_g$ can be chosen to be strongly continuous.*

Proof: This is immediate from the proof of Theorem 2.5 if we observe that there exists a unitary operator W_g satisfying $W_g v(x) = v(gx)$ if $x \in X$ and $W_g \Omega = \Omega = v(\epsilon)$. □

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