SOME CHARACTERISTIC AND NONCHARACTERISTIC PROPERTIES OF THE WISHART DISTRIBUTION

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SUMMARY. If the elements of a matrix S follow a central Wishart distribution $W_k(\nu, \Sigma)$ and $a'\Sigma a \neq 0$ it is well known that $a'Sa/a'\Sigma a$ is distributed as a chisquare on ν d.f. Further $a'\Sigma a = 0 \Longrightarrow a'Sa = 0$ with probability one. The object of this paper is to show through a counter-example that the converse of this result is not necessarily true unless Σ is of rank one. It is shown, however, that if for every matrix L, such that $L\Sigma L' = I$, the diagonal elements of LSL' are distributed as independent chisquare variables on ν d.f., then S has a central Wishart distribution $W_k(\nu, \Sigma)$.

1. Notation

We shall refer to a family of distributions by a symbol such as N, B, W etc. and to a particular member of the family by specifying its parameters in addition. Thus $N(\mu, \sigma^2)$ denotes a normal distribution with mean μ and variance σ^2, $N_k(\mu, \Sigma)$ a k-variate normal distribution with mean vector μ and dispersion matrix Σ, $\chi^2(n)$ denotes a central chisquare distribution on n d.f., $B(m, n)$ a beta distribution with parameters m and n and $W_k(n, \Sigma)$ represents a central Wishart distribution where k is the order of the Wishart matrix $S = (S_{ij})$, n is the d.f. and $\Sigma = (\sigma_{ij})$ is the associated dispersion matrix. We use the notation $X \sim N(\mu, \sigma^2)$ to indicate that X is a random variable distributed as $N(\mu, \sigma^2)$.

2. A property of $W_k(n, \Sigma)$.

Rao (1965, p. 452) proves the following proposition :

Lemma 2.1 (Rao) : *If* $S \sim W_k(\nu, \Sigma)$ *and* $a'\Sigma a \neq 0$ *then* $a'Sa/a'\Sigma a \sim \chi^2(\nu)$. *If* $a'\Sigma a = 0$, *then* $a'Sa = 0$ *with probability one.*

The object of this paper is to show that the converse of this proposition is not necessarily true, unless Rank $\Sigma = 1$. To be more specific let T be a symmetric matrix of random variables and Σ a nonstochastic nonnegative-definite matrix both of order k. Assume further that

$$a'\Sigma a \neq 0 \Longrightarrow a'Ta/a'\Sigma a \sim \chi^2(\nu) \qquad \dots \quad (2.1)$$

and

$$a'\Sigma a = 0 \Longrightarrow a'Ta = 0 \text{ with probability one.} \qquad \dots \quad (2.2)$$

We show that these facts are not sufficient to establish a Wishart distribution for T.

3. A counter-example

The counter-example is based on the following well-known result which we state without proof.

Lemma 3.1 : (a_1) $V \sim \chi^2(n)$, (a_2) *for some integer* $\nu, 0 < \nu < n$, $p \sim B\left(\frac{\nu}{2}, \frac{n-\nu}{2}\right)$, ($a_3$) V *and* p *are independently distributed* $\Longrightarrow$ (b_1) pV *and* $(1-p)V$ *are independently distributed*, (b_2) $pV \sim \chi^2(\nu)$, $(1-p)V \sim \chi^2(n-\nu)$.

Let $S \sim W_k(n, \Sigma)$ for some integer $n > \nu$ and p be independently distributed of S as a beta variable $B\left(\frac{\nu}{2}, \frac{n-\nu}{2}\right)$. Write

$$T = pS. \qquad \dots \quad (3.1)$$

Straightforward application of Lemmas 2.1 and 3.1 will show that (T, Σ) obey (2.1) and (2.2). There are several ways of verifying that T cannot possibly have a Wishart distribution. Firstly, note that (2.1) and (2.2) imply $E(T/\nu) = \Sigma$. Hence if T has at all a Wishart distribution, then $T \sim W_k(\nu, \Sigma)$. Since Rank $\Sigma > 1$ there clearly exists $i, j, i \neq j$ such that $\rho_{ij} = \sigma_{ij}/(\sigma_{ii}\sigma_{jj})^{\frac{1}{2}} \neq \pm 1$. Consider $r_{ij} = T_{ij}/(T_{ii}T_{jj})^{\frac{1}{2}}$. It is well known that if $T \sim W_k(\nu, \Sigma)$, the distribution of r_{ij} will be same as that of the product moment correlation coefficient in a sample of size $\nu+1$ drawn from the bivariate normal distribution. Since r_{ij} is also equal to $S_{ij}/(S_{ii}S_{jj})^{\frac{1}{2}}$ and $S \sim W_k(n, \Sigma)$ this is clearly a contradiction.

4. The case rank $\Sigma = 1$

We shall prove

Lemma 4.1 : *If Rank* $\Sigma = 1$, (2.1) *and* (2.2) $\Longrightarrow T \sim W_k(\nu, \Sigma)$.

Proof : Let $P = (P_1 : P_2 : \dots : P_k)$ be an orthogonal matrix of order k such that $P'\Sigma P = D_\lambda$ is diagonal. Assume $P_1'\Sigma P_1 = \lambda_1 > 0$, $P_i'\Sigma P_i = 0$, $i = 2(1)k$. (2.1) $\Longrightarrow \frac{P_1'TP_1}{\lambda_1} \sim \chi^2(\nu)$. (2.2) $\Longrightarrow P_i'TP_i = 0$ with probability one for $i = 2(1)k$. Let $p_1, p_2, \dots, p_{\nu-1}$ denote independent beta variables where $p_j \sim B\left(\frac{1}{2}, \frac{\nu-j}{2}\right)$ and let $e_1, e_2, \dots, e_\nu$ be the outcome of ν independent Bernoulli trials, each trial capable of producing $+1$ or -1 with equal probability. Define

$$V_1 = p_1(P_1'TP_1)$$

$$V_j = p_j(1-p_1)(1-p_2)(1-p_{j-1})(P_1'TP_1), \quad j = 2(1)(\nu-1)$$

$$V_\nu = (1-p_1)(1-p_2)\dots(1-p_\nu)(P_1'TP_1).$$

Check $P_1'TP_1 = \Sigma V_j$. Clearly the $\lambda_1^{-1}V_j$'s are distributed as independent chisquare variables each with 1 d.f. Write $X_j = e_j\sqrt{V_j}$ and let Y_j be the solution of equations

$$P_1'Y_j = X_j$$

$$P_i'Y_j = 0, \quad i = 2(1)k.$$

Observe that the X_j's are independent normal variables each distributed as $N(0, \lambda_i)$ implying thereby that Y_j's shall be independent k-dimensional variables each distributed as $N_k(0, \Sigma)$. Straightforward algebra will establish

$$T = \sum_{j=1}^{\nu} Y_j Y_j'.$$

Hence $$T \sim W_k(\nu, \Sigma).$$

5. A class of distributions obeying (2.1) and (2.2)

Let $S \sim W_k(n, \Sigma)$, v be independently distributed of S as a chisquare on r.d.f. and p be independently distributed of both S and v as a beta variable $B\left(\frac{\nu}{2}, \frac{n+r-\nu}{2}\right)$, ν an integer $0 < \nu \leqslant n+r$. Denote the distribution of $p(S+v\Sigma)$ by $U_k(n, r, \nu, \Sigma)$. The following results are easy to prove : (i) If $T \sim U_k(n, r, \nu, \Sigma)$ then T satisfies (2.1), (2.2), (ii) If $r \neq 0$ or $r = 0$ and $\nu < n$, then $U_k(n, r, \nu, \Sigma) \neq W_k(\nu, \Sigma)$, and (iii) $U_k(\nu, 0, \nu, \Sigma) = W_k(\nu, \Sigma)$. Let $U_k^*(q, \{n_i\}, \{r_i\}, \{\nu_i\}, \Sigma) = U_k(n_1, r_1, \nu_1, \Sigma) * U_k(n_2, r_2, \nu_2, \Sigma) * \ldots * U_k(n_q, r_q, \nu_q, \Sigma)$ be the convolution of a finite number q of distributions of the type $U_k(n, r, \nu, \Sigma)$. It is seen that the family U_k^* which includes the Wishart distribution as a special case also satisfies (2.1) and (2.2). It is interesting to speculate if the family U_k^* can be characterised by these two conditions.

6. A characterisation of the Wishart distribution

It would also be of interest to determine a minimal set of conditions which taken in addition to (2.1) and (2.2) characterises the Wishart distribution. A characterisation of the Wishart distribution is contained in Lemma 6.1.

Lemma 6.1 : *If (2.2) holds and for every matrix L(not necessarily square) with the property $L\Sigma L' = I$, the diagonal elements of LTL' are distributed as independent chi-square variables on ν d.f., then $T \sim W_k(\nu, \Sigma)$.*

Proof : The proof consists in showing that the premises suffice to determine the characteristic function of the elements of T. Observe that

$$\Phi(\theta) = E[\exp \sqrt{-1}\{\Sigma\theta_{ii}T_{ii} + 2\sum_{i<j} \theta_{ij}T_{ij}\}] = E[\exp \sqrt{-1}\{\text{tr}\, \theta T\}],$$

where θ is the symmetric matrix $\{\theta_{ij}\}$.

We first consider the case where Σ is positive definite. Here, since Σ^{-1} is also positive definite and θ is symmetric, there exists a matrix L such that $\theta = L'DL$ and $\Sigma^{-1} = L'L$ where $D = \text{diag}\,(\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a diagonal matrix the diagonal elements of which are the roots of the equation $|\theta - \lambda\Sigma^{-1}| = 0$. Check $L\Sigma L' = I$. Hence

$$\begin{aligned}\Phi(\theta) &= E[\exp \sqrt{-1}\{\text{tr}\, L'DLT\}] = E[\exp \sqrt{-1}\{\text{tr}\, DLTL'\}] \\ &= \prod_{i=1}^{k} (1-2\lambda_i\sqrt{-1})^{-\nu/2} \\ &= \frac{|\Sigma^{-1}|^{\nu/2}}{|\Sigma^{-1} - 2\sqrt{-1}\theta|^{\nu/2}}\end{aligned}$$

which is indeed the expression given by Anderson (1958, p. 101), Rao (1965, p. 506).

Consider now the case where Σ is positive semi-definite, of Rank $r < k$. Here it is well known that there exists a matrix M such that $M\Sigma M' = I_r$. Put $T_0 = MTM'$ and use the result obtained in the preceding paragraph to establish that $T_0 \sim W_r(\nu, I)$. The proof is complete, once it is noted that with probability one, $T = BT_0B'$, where $BB' = \Sigma$, and one uses result 8b.2(vi) of Rao (1965, p. 455).

Q.E.D.

The author feels that Lemma 6.1 would still be true if the 'independence' condition is replaced by an weaker condition of 'pairwise independence'. This, however, he has been unable to establish so far.

7. Characteristic function of the singular Wishart

Let $T \sim W_k(\nu, \Sigma)$ and suppose Rank $\Sigma = r < k$. Let P be an orthogonal matrix such that

$$\Omega = P\Sigma P' = \begin{matrix} r \\ k-r \end{matrix} \overset{\begin{matrix} r & k-r \end{matrix}}{\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}},$$

where D is a diagonal matrix of nonnull eigen values of Σ.

Observe that if

$$S = PTP' = \begin{matrix} r \\ k-r \end{matrix} \overset{\begin{matrix} r & k-r \end{matrix}}{\begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}},$$

then $S_{12} = S_{21}' = 0$, $S_{22} = 0$ with probability one and $S_{11} \sim W_r(\nu, D)$. Write

$$\Lambda = P\theta P' = \begin{matrix} r \\ k-r \end{matrix} \overset{\begin{matrix} r & k-r \end{matrix}}{\begin{pmatrix} \Lambda_{11} & \Lambda_{12} \\ \Lambda_{21} & \Lambda_{22} \end{pmatrix}}$$

and check

$$\begin{aligned} \Phi(\theta) &= E[\exp \sqrt{-1}(\text{tr }\theta T)] = E[\exp \sqrt{-1}(\text{tr }\Lambda S)] \\ &= \frac{|D^{-1}|^{\nu/2}}{|D^{-1}-2\sqrt{-1}\,\Lambda_{11}|^{\nu/2}} = \frac{1}{|I_r-2\sqrt{-1}\,D\Lambda_{11}|^{\nu/2}} \\ &= |I_k-2\sqrt{-1}\,\Omega\Lambda|^{-\nu/2} = |I_k-2\sqrt{-1}\,\Sigma\theta|^{-\nu/2}. \end{aligned}$$

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References

Anderson, T. W. (1958) : *An Introduction to Multivariate Statistical Analysis*, John Wiley and Sons, New York.

Rao, C. R. (1965) : *Linear Statistical Inference and its Applications*, John Wiley and Sons, New York.

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