Sankhyā : The Indian Journal of Statistics
1989, Volume 51, Series A, Pt. 1, pp. 73-78.

INFINITE DIVISIBILITIY OF MULTIVARIATE GAMMA DISTRIBUTIONS AND M-MATRICES

By R. B. BAPAT
Indian Statistical Institute

SUMMARY. Suppose $X = (X_1, \ldots, X_p)'$, has the Laplace transform $\psi(t) = |I+VT|^{-\frac{1}{2}}$, where V is a positive definite matrix and $T = \text{diag}(t_1, \ldots, t_p)$. It is shown that $\psi(t)$ is infinitely divisible if and only if $DV^{-1}D$ is an M-matrix for some diagonal matrix D with $\pm$ 1's along the diagonal.

1. Introduction

Let $Y = (Y_1, \ldots, Y_p)'$ be a multivariate normal random vector with zero mean and a positive semidefinite covariance matrix V. If $X_i = \frac{Y_i^2}{2}$, $i = 1, 2, \ldots, p$, then $X = (X_1, \ldots, X_p)'$ has the Laplace transform

$$\psi(t) = |I+VT|^{-\frac{1}{2}} \qquad \ldots \quad (1)$$

where T is a diagonal matrix with diagonal elements $t_1, \ldots, t_p$.

The question of characterizing matrices V for which (1) is infinitely divisible has been considered in the literature. (Recall that $\psi(t)$ is said to be infinitely divisible if for any $\alpha > 0$, $[\psi(t)]^\alpha$ is a Laplace transform of some distribution). Certain partial results were obtained by Moran and Vere-Jones (1969), Griffiths (1970) and Paranjape (1978). Finally, an interesting necessary and sufficient condition was given by Griffiths (1984) for (1) to be infinitely divisible. The condition involves the concept of cycle products of a matrix which we define next.

Definition 1. Let A be a $p \times p$ matrix. If $(i_1, \ldots, i_k)$, $k \geq 2$, is a subset of $\{1, 2, \ldots, p\}$, then $a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}$ is called a cycle product of A. The number k will be called the length of the cycle product.

Note that according to this definition, a diagonal element of A by itself is not considered a cycle product.

The purpose of this note is to give another necessary and sufficient condition for (1) to be infinitely divisible which uses the concept of an M-matrix. The condition is simpler than that of Griffiths in the sense that it does not make any reference to cycle products. We remark here that Griffiths (1984) considers the infinite divisibility of

$$|I+VT|^{-1}$$

AMS (1980) *subject classification :* 60E05, 62H05.
Key words and phrases : Infinite divisibility, multivariate gamma distributions, M-matrix.

rather than that of (1), but the two problems are clearly equivalent.

If A is a matrix with $a_{ij} \geqslant 0$ for all i, j, we say that A is nonnegative and write $A \geqslant 0$.

Definition 2. A $p \times p$ matrix A is said to be an M-matrix if $a_{ij} \leqslant 0$ for all $i \neq j$ and if any one of the following (equivalent) conditions is satisfied :

(a) A is nonsingular and $A^{-1} \geqslant 0$;

(b) $A = \lambda I - B$, where $B \geqslant 0$ and λ is greater than the absolute value of any eigenvalue of B ;

(c) all principal minors of A are positive.

There is an extensive literature on M-matrices. We refer to Berman and Plemmons (1979) for a survey of M-matrices and also for a proof that (a), (b), (c) are equivalent under the assumption that $a_{ij} \leqslant 0$, $i \neq j$.

Definition 3. A diagonal matrix is called a signature matrix if every diagonal entry is either 1 or -1.

Definition 4. A $p \times p$ matrix A is called reducible if there exists a permutation matrix P such that

$$PAP' = \begin{bmatrix} B & O \\ C & D \end{bmatrix}$$

where B, D are square matrixes. If A is not reducible, then it is called irreducible. A 1×1 nonzero matrix is by definition, irreducible.

Note that if A is a symmetric, reducible matrix, then after identical permutations of rows and columns it can be expressed as a direct sum of irreducible matrices.

We now state the main result in the next section.

2. Main result

Theorem 1 : *Let* $X = (X_1, \ldots, X_p)'$ *have the Laplace transform*

$$\psi(t) = |I + VT|^{-\frac{1}{2}}$$

where V *is a* $p \times p$ *positive definite matrix*, $T = diag\,(t_1, \ldots, t_p)$, *and let* $W = V^{-1}$. *Then the following conditions are equivalent* :

(i) $\psi(t)$ *is infinitely divisible* ;

(ii) *for any* $\{i_1, \ldots, i_k\} \subset \{1, 2, \ldots, p\}$, $k \geqslant 3$, $(-1)^k w_{i_1 i_2} w_{i_2 i_3} \cdots w_{i_k i_1} \geqslant 0$;

(iii) *there exists a signature matrix* D *such that* DWD *is an* M*-matrix.*

Proof: (i) ⇒ (ii) : The proof of this is contained in Griffiths (1984).

(ii) ⇒ (iii) : this may be deduced from certain results in qualitative matrix theory (see, for example, Theorem 2 in Bassett, Maybee and Quirk (1968)). However, since W is also symmetric, a different self-contained proof can be given which we present here.

If A is a $p \times p$ matrix and if I, J are subsets of $\{1, 2, ..., p\}$ then $A[I, J]$ will denote the submatrix of A formed by rows indexed by I and columns indexed by J.

The result is trivial if $p = 1$. Let $p > 1$ and suppose the result is true for matrices of order $p-1$. By induction assumption there exists a signature matrix E such that EUE is an M-matrix, where U is the leading $(p-1) \times (p-1)$ principal minor of W. For convenience, we assume, without loss of generality, that U itself is an M-matrix.

Let $\{1, 2, ..., p-1\}$ be partitioned into sets $G_1, G_2, ..., G_m$ such that U is the direct sum of $U[G_i, G_i]$, $i = 1, 2, ..., m$ and that each $U[G_i, G_i]$ is irreducible.

We claim that if $j, l \in G_i$ for some i, then w_{jp}, w_{lp} cannot have opposite signs. Suppose, for example, that $w_{jp} > 0$ and $w_{lp} < 0$. By a well-known property of irreducible matrices (Berman and Plemmons, 1979, p. 30) there exist $k_1, ..., k_r$ in G_i such that $k_1 = l, k_r = j$ and

$$w_{k_1k_2}, w_{k_2k_3}, ..., w_{k_{r-1}k_r}$$

are all nonzero and hence negative, since U is an M-matrix.

Now for the cycle product

$$w_{jp}\, w_{pl}\, w_{k_1k_2} \cdots w_{k_{r-1}k_r}$$

condition (ii) is violated since all the terms in the product are negative except w_{jp}, which is positive. This contradiction proves the claim.

Now for $i \in \{1, 2, ..., m\}$ call G_i of type I if $w_{jp} \geqslant 0$ for all $j \in G_i$ and of type II if $w_{jp} \leqslant 0$ for all $j \in G_i$ and $w_{jp} < 0$ for at least one $j \in G_i$. Define a $p \times p$ signature matrix $D = \text{diag}\ (d_1, ..., d_p)$ as

$$d_j = \begin{cases} 1 \text{ if } j \in G_i \text{ and } G_i \text{ is of type I.} \\ -1 \text{ if } j \in G_i \text{ and } G_i \text{ is of type II} \\ 1 \text{ if } j = p. \end{cases}$$

Then it follows that DWD is an M-matrix,

(iii) ⇒ (i) : We will follow the technique in Griffiths (1984) according to which it is sufficient to show that

$$P(s) = |I + Va(I-S)|^{-\frac{1}{2}} \quad \ldots \quad (2)$$

is an infinitely divisible probability generating function for all $a > 0$, where $S = \text{diag}(s_1, \ldots, s_p)$. Furthermore, the infinite divisibility of (2) follows if we show that log $P(s)$ has a power series expansion with all coefficients nonnegative except the constant term.

Since (iii) holds, there exists a signature matrix D such that DWD is an M-matrix and then by (b) of Definition 2,

$$DWD = \lambda I - B$$

where $B \geqslant 0$ and λ is greater than the absolute value of any eigenvalue of B.

Now

$$\begin{aligned} |I+Va(I-S)| &= |V(W+a(I-S))| \\ &= |V|\,|\lambda I - B + aI - aS| \\ &= |V|(\lambda+a)\left|I - \frac{1}{\lambda+a}(B+aS)\right|. \end{aligned}$$

Therefore, as in Griffiths (1984),

$$\log P(s) = -\frac{1}{2}\log|V| - \frac{1}{2}\log(\lambda+a) + \frac{1}{2}\sum_{n=1}^{\infty} \text{tr}\,\frac{(B+aS)^n}{n(\lambda+a)^n}.$$

Since $B \geqslant 0$, the coefficients in the expansion above must all be nonnegative except the constant term and the proof is complete.

Remark 1 : The implication (iii) ⇒ (ii) in Theorem 1 is easy to prove. This may be combined with the proof of (ii) ⇒ (i) given by Griffiths (1984) to produce another proof of (iii) ⇒ (i). However the proof given here is simpler in the sense that it avoids the use of induction and the introduction of the auxiliary matrix Q as in Griffiths (1984).

Remark 2 : If V is assumed to be positive semidefinite in Theorem 1, then W should be replaced by the matrix of cofactors of V. The proof can be given by approaching V by a sequence of positive definite matrices and using the continuity theorem for Laplace transforms.

A necessary condition for (1) to be infinitely divisible can be given as follows :

Corollary 1 : *Let* $X = (X_1, \ldots, X_p)'$, *have the Laplace transform*

$$\psi(t) = |I + VT|^{-\frac{1}{2}}$$

where V is a positive definite matrix and suppose that $\psi(t)$ is infinitely divisible. Then there exists a signature matrix D such that $DVD \geqslant 0$.

Proof: By Theorem 1 there exists a signature matrix D such that $DV^{-1}D$ is an M-matrix. Since

$$DVD = (DV^{-1}D)^{-1},$$

The result follows by (a) of Definition 2.

Let $X = (X_1, \ldots, x_p)'$, have the Laplace transform

$$\psi(t) = |I+VT|^{-k}$$

where V is a $p \times p$ positive definite matrix and suppose we want to determine whether $\psi(t)$ is infinitely divisible. The following procedure is suggested by a combination of Theorem 1 and Corollary 1 and it is better than verifying Griffiths' condition given in Theorem 1 (ii).

First determine whether there is a signature matrix D such that DVD is a nonnegative matrix. For small p this can be done by inspection. If such a D does not exist then by Corollary 1, $\psi(t)$ is not infinitely divisible. If there exists D such that $DVD \geqslant 0$, then we must only check whether $(DVD)^{-1}$ is an M-matrix. Since $(DVD)^{-1}$ is positive definite, this amounts to only checking whether all its off-diagonal entries are nonpositive. By Theorem 1, $\psi(t)$ is infinitely divisible if and ony if $(DVD)^{-1}$ is an M-matrix. The next two examples illustrate this method.

Example (a): Let $p = 3$ and suppose V has the following sign pattern

$$\begin{bmatrix} + & + & - \\ + & + & + \\ - & + & + \end{bmatrix}$$

Then it is easy to see that there does not exist a signature matrix D such that $DVD \geqslant 0$, and we conclude that $\psi(t)$ is not infinitely divisible. Note that in this example only the sign pattern of V was used to arrive at a conclusion. Also, there was no need to invert a matrix. In general if V has the property that v_{ij}, v_{ji} are the only negative entries of V for some i, j, the rest being positive, then we can conclude that $\psi(t)$ is not infinitely divisible.

Example (b): Let $p = 4$ and suppose

$$V = \begin{bmatrix} 3 & -2 & 2 & -1 \\ & 5 & -4 & 2 \\ & & 5 & -2 \\ & & & 4 \end{bmatrix}$$

Then if $D = \text{diag}(1, -1, 1, -1)$ it can be verified that $DVD \geqslant 0$. Furthermore

$$(DVD)^{-1} = \frac{1}{50} \begin{bmatrix} 28 & -6 & -6 & -1 \\ & 35 & -24 & -4 \\ & & 35 & -4 \\ & & & 10 \end{bmatrix}.$$

which is an M-matrix and we conclude that $\psi(t)$ is infinitely divisible.

We finally remark that if $W = V^{-1}$ has a row with no zero entires then condition (iii) of Theorem 1 is very easily verified. This can be seen as follows. Suppose the first row of W has no zeros. Let E be the signature matrix with with its i-th diagonal entry equal to 1 if and only if $w_{1i} < 0$. Then note that (iii) Theorem 1 is satisfied if and only if-EWE is an M-matrix, and thus the verification takes only one step.

References

Bassett, L., Maybee J. and Quirk, J. (1968): Qualitative economics and the scope of the correspondence principle. *Econometrica*, 36, No. 3–4, 544–563.

Berman, Abraham and Plemmons, Robert J. (1979): *Nonnegative Matrices in the Mathematical Sciences*, Academic Press, Inc. New York.

Griffiths, R. C. (1970): Infinite divisible multivariate gamma distributions. *Sankhyā*, Series A, 32, 393-404.

Griffiths, R. C. (1984): Characterization of infinitely divisible multivariate gamma distributions. *J. Multivariate Anal.*, 15, 13-20.

Moran, P. A. P. ahd Vere-Jones, D. (1969): The infinite divisibility of multivariate gamma distributions. *Sankhyā*, Series A, 31, 191-194.

Paranjape, S. R. (1978): Simpler proofs for the infinite divisibility of multivariate gamma distributions. *Sankhyā*, Series A, 40, 393-398.

Paper received: *July*, 1987.
Revised: *November*, 1987.